CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE
Lecture 12: Random Processes

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Note

- These slides cover material partially presented in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A. Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-step derivation and detailed discussions were needed.

Family of Random Variables

- In many random experiments, the outcome is a function of time or space.
  - Voltage waveform corresponding to speech utterance
  - Number of customers in queueing system
  - Temperature in a city and demand placed on local electric power utility
- An indexed family of random variables (or even random vectors)

Definition of a Random Process

- Consider a random experiment specified by the outcomes ξ from some sample space S, by the events defined on S, and by the probabilities on these events.
- A random process is defined by the mapping of every outcome ξ ϵ S to a function of time (or space) according to some rule:

Definition of a Random Process (Cont'd)

$$\{X(t,\xi), t \in I\}$$

- Random process is **discrete-time** if the index set I is a countable set

$$X(t_k,\xi)$$

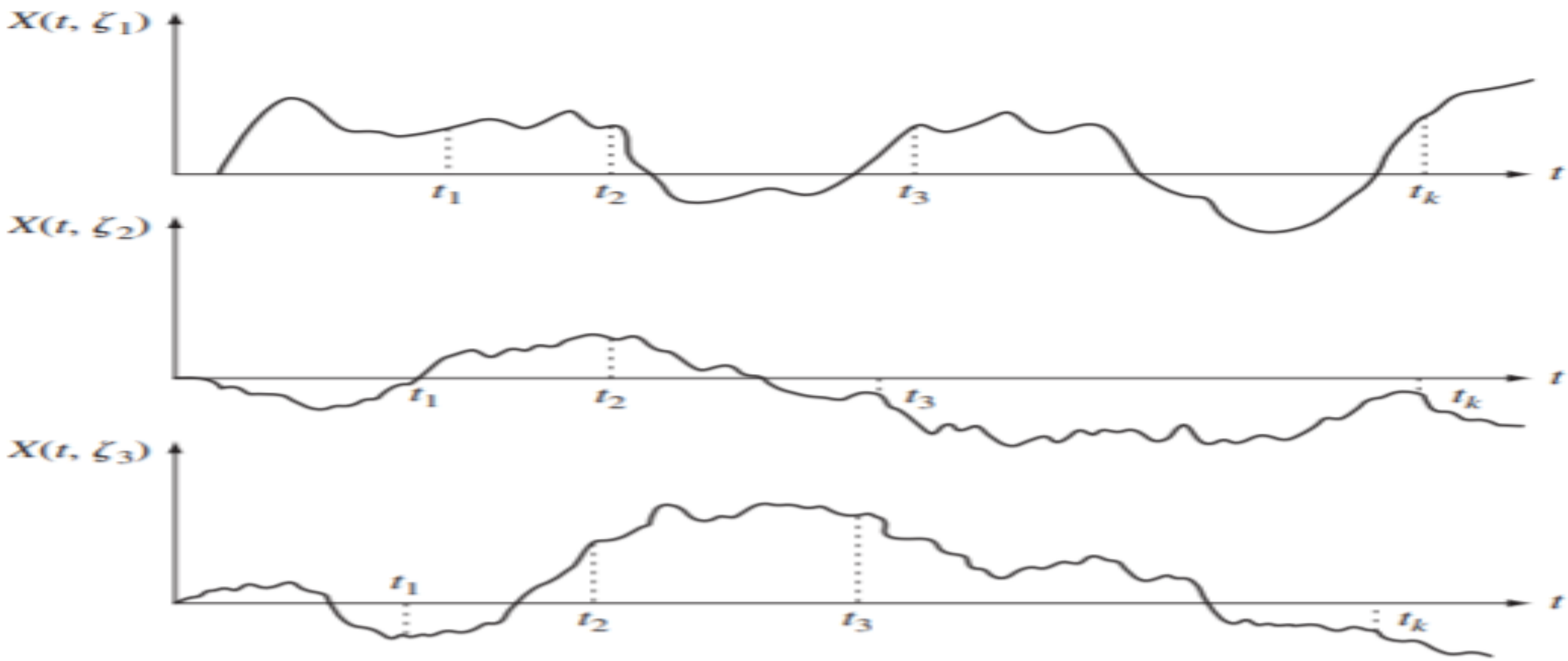
- Random process is **continuous-time**

$$X(t,\xi)$$

Different Views of a Random Process

$$\{X(t,\xi), t \in I\}$$

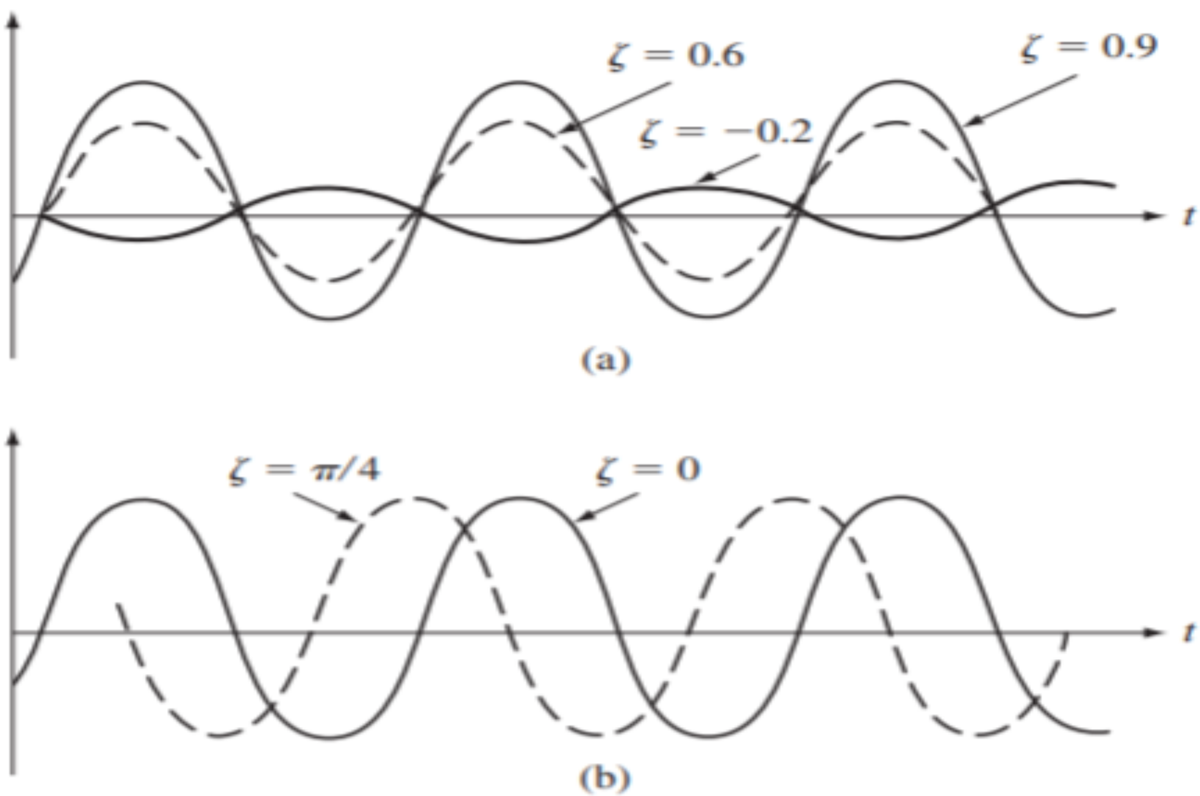
- A random process can be viewed as a function of two variables t and ξ .
- For fixed values of t and ξ, we simply have a number.



Different Views of a Random Process (Cont'd)

$$\{X(t,\xi), t \in I\}$$

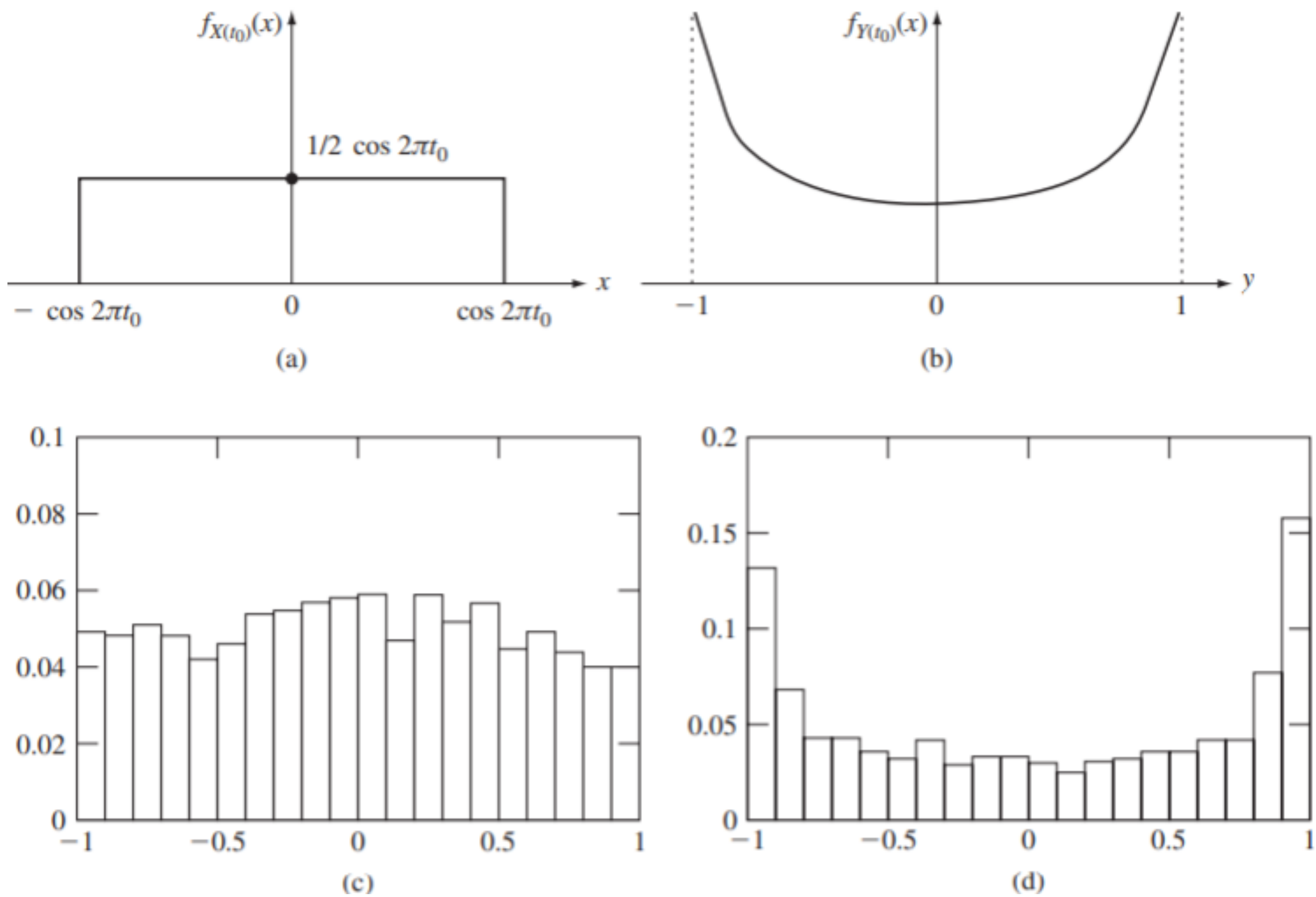
- For a fixed value of ξ, the variation vs. t is simply a function of time, which is called the **sample path** of the random process.



Different Views of a Random Process (Cont'd)

$$\{X(t,\xi), t \in I\}$$

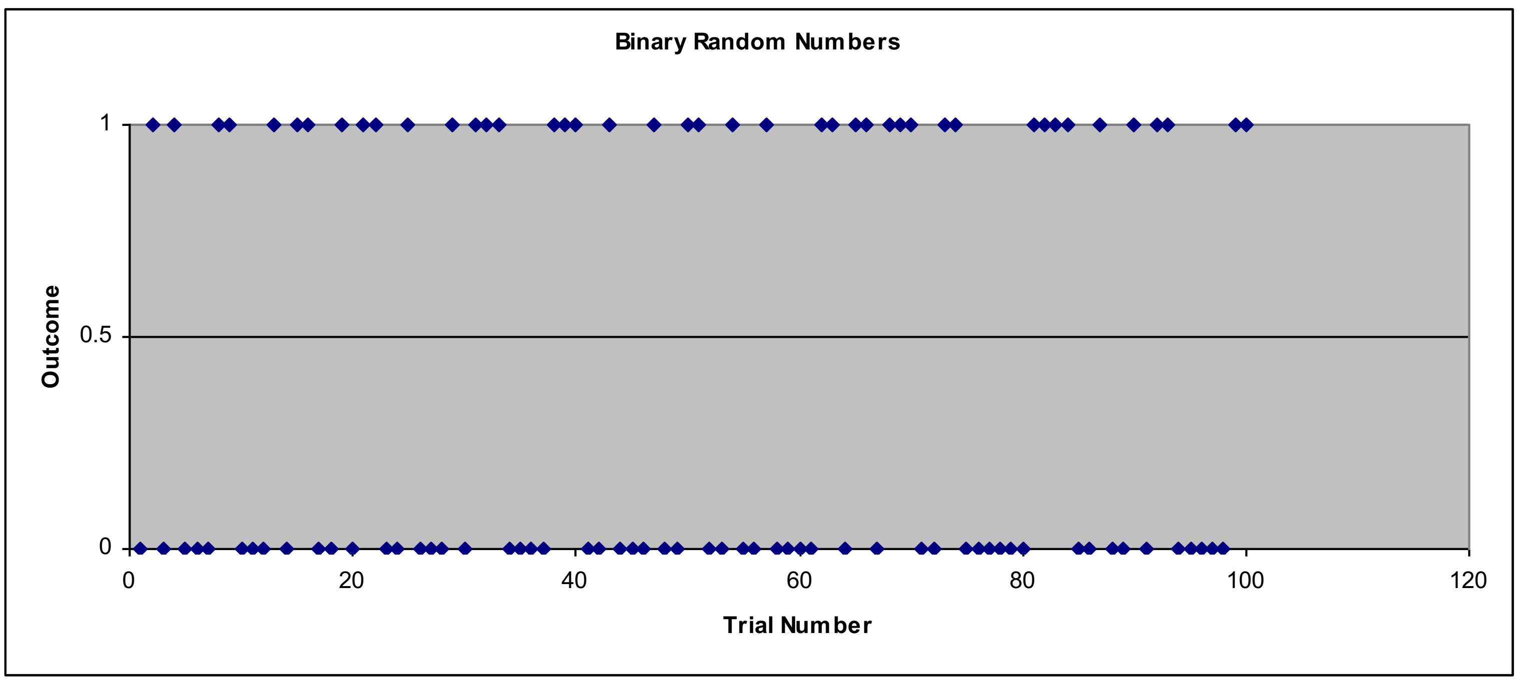
- For a fixed value of t, the variation vs. ξ is a random variable over the entire ensemble of values ξ.


$f_{X(t_0)}(x)$
1/2 cos $2\pi t_0$
x
− cos $2\pi t_0$
0
cos $2\pi t_0$
(a)
$f_{Y(t_0)}(x)$
y
−1
0
1
(b)
0.1
0.08
0.06
0.04
0.02
0
−1
−0.5
0
0.5
1
(c)
0.2
0.15
0.1
0.05
0
−1
−0.5
0
0.5
1
(d)

Example: Binary Random Process

- Let ξ be a number selected at random from the interval $S = [0,1]$ and let $b_1 b_2 \ldots$ be the binary expansion of :

$$\xi = \sum_{i=1}^{\infty} b_i 2^{-i} \quad \text{where } b_i \in \{0,1\}$$



Example: Binary Random Process (Cont'd)

- Find

$P[X(1,\xi)=0]$ and $P[X(1,\xi)=0$ and $X(2,\xi)=1]$

Example 9.3

Find the following probabilities for the random process introduced in Example 9.1: $P[X(1,\zeta)=0]$ and $P[X(1,\zeta)=0 \text{ and } X(2,\zeta)=1]$.

The probabilities are obtained by finding the equivalent events in terms of ζ:

$$P[X(1,\zeta)=0]=P\left[0\le\zeta<\frac{1}{2}\right]=\frac{1}{2}$$

$$P[X(1,\zeta)=0 \text{ and } X(2,\zeta)=1]=P\left[\frac{1}{4}\le\zeta<\frac{1}{2}\right]=\frac{1}{4},$$

since all points in the interval $[0\le\zeta\le 1]$ begin with $b_1=0$ and all points in $[1/4,1/2)$ begin with $b_1=0$ and $b_2=1$. Clearly, any sequence of k bits has a corresponding subinterval of length (and hence probability) 2^{-k}.

Example: Sinusoid w Random Amplitude

- Let ξ be selected at random from [−1, 1].

$$X(t,\xi) = \xi\cos(2\pi t) \qquad -\infty < t < \infty$$

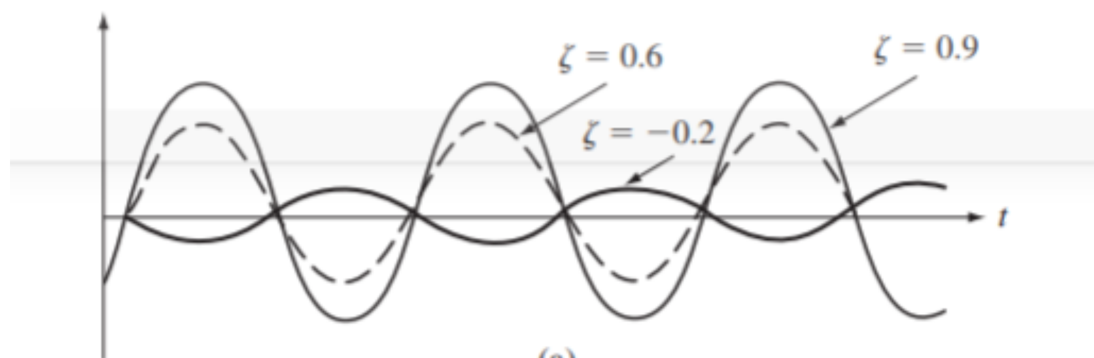
Example 9.2 Random Sinusoids

Let ζ be selected at random from the interval $[-1, 1]$. Define the continuous-time random process $X(t, \zeta)$ by

$$X(t,\zeta) = \zeta\cos(2\pi t) \qquad -\infty < t < \infty.$$

The realizations of this random process are sinusoids with amplitude ζ, as shown in Fig. 9.2(a).

Let ζ be selected at random from the interval $(-\pi, \pi)$ and let $Y(t, \zeta) = \cos(2\pi t + \zeta)$. The realizations of $Y(t, \zeta)$ are phase-shifted versions of $\cos 2\pi t$ as shown in Fig 9.2(b).

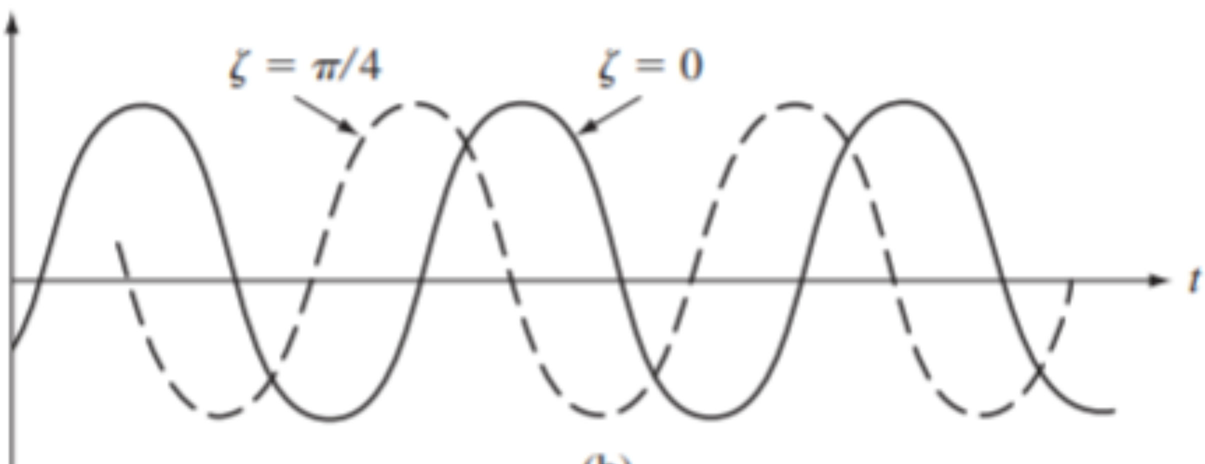


(a)

Example: Sinusoids w Random Phase

- Let ξ be selected at random [−π, π].

$$Y(t, \xi) = \cos(2\pi t + \xi).$$

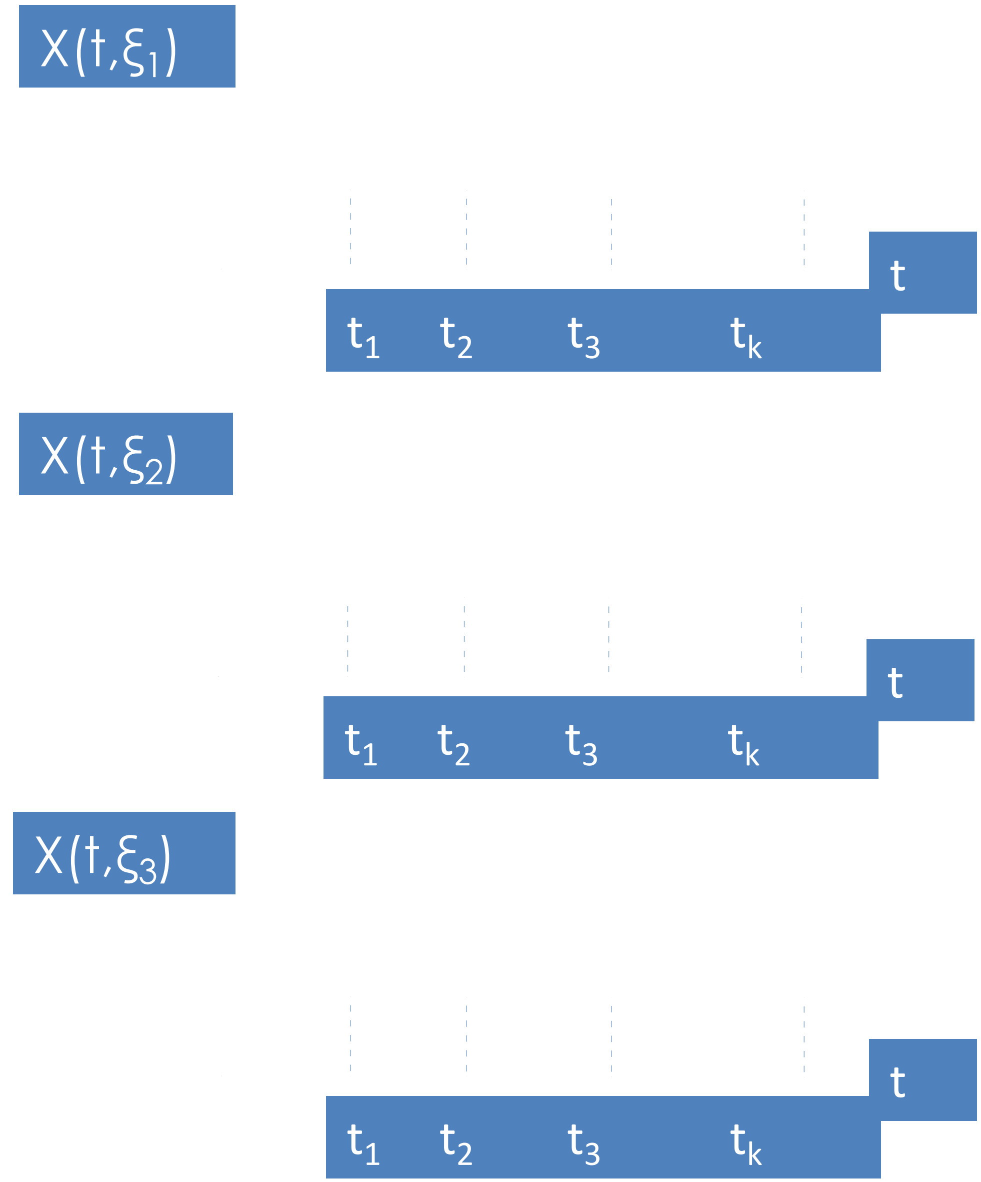


(b)

Joint Distribution of Time Samples

- Let X_1, X_2,…, X_k be the k random variables obtained by sampling the random process $X(t, \xi)$ at times $t_1, t_2, \ldots, t_k$:

$$
\begin{aligned}
X_1 &= X(t_1, \xi), \\
X_2 &= X(t_2, \xi), \\
&\vdots \\
X_k &= X(t_k, \xi)
\end{aligned}
$$



Specifying a Random Process

- A random (stochastic) process is specified by a the collection of kth-order joint cumulative distribution functions for any k and any choice of sampling instants t_1, …, t_k:

$$F_{X_1,\ldots,X_k}(x_1,x_2,\ldots,x_k) = P[X_1 \le x_1, X_2 \le x_2,\ldots,X_k \le x_k]$$

- If the process is discrete-valued, we use the pmf

$$p_{X_1,\ldots,X_k}(x_1,x_2,\ldots,x_k) = P[X_1 = x_1, X_2 = x_2,\ldots,X_k = x_k]$$

- If the process is continuous-valued, we use the pdf

$$f_{X_1,\ldots,X_k}(x_1,x_2,\ldots,x_k)$$

Example: Bernoulli Sequences

- Find the joint pmf for X_n, iid Bernoulli random variables with p = ½.

Example 9.5 iid Bernoulli Random Variables

Let X_n be a sequence of independent, identically distributed Bernoulli random variables with $p = 1/2$. The joint pmf for any k time samples is then

$$P[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k] = P[X_1 = x_1] \ldots P[X_k = x_k] = \left(\frac{1}{2}\right)^k$$

where $x_i \in \{0, 1\}$ for all i. This binary random process is equivalent to the one discussed in Example 9.1.

Mean and Variance Functions of a Random Process

- **Mean $m_X(t)$** of $X(t)$ is defined by:

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$$

- **Variance** of X(t)

$$VAR[X(t)] = E\left[\left(X(t) - m_X(t)\right)^2\right]$$

Autocorrelation & Autocovariance Functions of a Random Process

- **Autocorrelation $R_X(t_1, t_2)$**

$$R_X(t_1,t_2) = E[X(t_1),X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1),X(t_2)}(x,y)\,dx\,dy$$

- **Autocovariance $C_X(t_1, t_2)$** of

$$\begin{aligned} C_X(t_1,t_2) &= E\left[\{X(t_1)-m_X(t_1)\}\{X(t_2)-m_X(t_2)\}\right] \\ &= R_X(t_1,t_2) - m_X(t_1)m_X(t_2) \end{aligned}$$

- Note that:

$$VAR[X(t)] = E\left[\left(X(t)-m_X(t)\right)^2\right] = C_X(t,t)$$

Example: Sinusoid with Random Amplitude

Example 9.9 Sinusoid with Random Amplitude

Let $X(t) = A \cos 2\pi t$, where A is some random variable (see Fig. 9.2a). The mean of $X(t)$ is found using Eq. (4.30):

$$m_X(t) = E[A \cos 2\pi t] = E[A] \cos 2\pi t.$$

Note that the mean varies with t. In particular, note that the process is always zero for values of t where $\cos 2\pi t = 0$.

The autocorrelation is

$$\begin{aligned} R_X(t_1, t_2) &= E[A \cos 2\pi t_1 \, A \cos 2\pi t_2] \\ &= E[A^2] \cos 2\pi t_1 \cos 2\pi t_2, \end{aligned}$$

and the autocovariance is then

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) - m_X(t_1) m_X(t_2) \\ &= \{E[A^2] - E[A]^2\} \cos 2\pi t_1 \cos 2\pi t_2 \\ &= \text{VAR}[A] \cos 2\pi t_1 \cos 2\pi t_2. \end{aligned}$$

Example: Sinusoid with Random Phase

Example 9.10 Sinusoid with Random Phase

Let $X(t) = \cos(\omega t + \Theta)$, where Θ is uniformly distributed in the interval $(-\pi, \pi)$ (see Fig. 9.2b). The mean of $X(t)$ is found using Eq. (4.30):

$$m_X(t) = E[\cos(\omega t + \Theta)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta = 0.$$

The autocorrelation and autocovariance are then

$$\begin{aligned}
C_X(t_1, t_2) = R_X(t_1, t_2) &= E[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\{\cos(\omega(t_1 - t_2)) + \cos(\omega(t_1 + t_2) + 2\theta)\}\, d\theta \\
&= \frac{1}{2}\cos(\omega(t_1 - t_2)),
\end{aligned}$$

where we used the identity $\cos(a)\cos(b) = 1/2\cos(a + b) + 1/2\cos(a - b)$. Note that $m_X(t)$ is a constant and that $C_X(t_1, t_2)$ depends only on $|t_1 - t_2|$. Note as well that the samples at time t_1 and t_2 are uncorrelated if $\omega(t_1 - t_2) = k\pi$ where k is any integer.

Lecture Summary

- A random process is a mapping that assigns a function of time (or space) to each outcome ξ of a random experiment
- Random processes can be viewed as an ensemble of sample functions or as an indexed family of random variables.
- Random processes are specified in terms of the joint distribution of its values for an arbitrary number of arbitrary time instants.
- The mean, variance, correlation, and covariance functions provide partial information about a random process.