George Mason University

Department of Electrical and Computer Engineering

ECE 528: Introduction to Random Processes in ECE — Fall Semester

Homework Set 10 Solutions

1. (P9.2) A discrete-time random process X_n is defined as follows.A fair die is tossed and the outcome k is observed.The process is then given by $X_n = k$ for all n.

   (a) Sketch some sample paths of the process.

   (b) Find the pmf for X_n.

   (c) Find the joint pmf for X_n and X_{n+k}.

   (d) Find the mean and autocovariance functions of X_n.

   Solutions:

   (a) The sample paths are:

Outcome	X_n
1	...111...
2	...222...
3	...333...
4	...444...
5	...555...
6	...666...

   (b) The pmf is:

$$P[X_n = i] = P[k = i] = 1/6, i = 1, 2, 3, 4, 5, 6.$$

   (c) The joint pmf is:

$$P[X_n = i, X_{n+k} = j] = \begin{cases} 1/6 & \text{if} \quad \text{i} = \text{j}, \text{i}, \text{j} \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$

   (d)

$$E[X_n] = \sum_{i=1}^{6} i \cdot P[X_n = i] = 21/6;$$

$$E[X_n X_{n+k}] = \sum_{i=1}^{6} i^2 \cdot P[X_n = i, X_{n+k} = i] = 91/6;$$

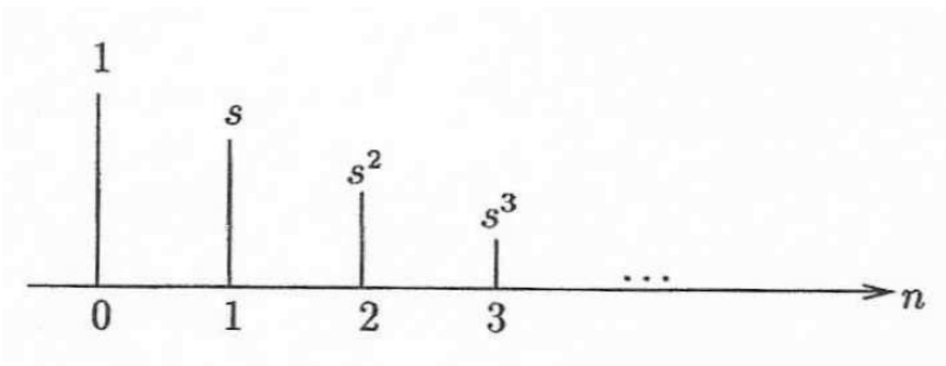
$$C_X(n, n+k) = E[X_n X_{n+k}] - E[X_n]E[X_{n+k}] = 91/6 - (21/6)^2 = 2.9167.$$

2. (P9.4) A discrete-time random process is defined by $X_n = s^n$ for $n \geq 0$ where s is selected at random from the interval (0,1).

   (a) Sketch some sample paths of the process.
   (b) Find the cdf for X_n.
   (c) Find the joint cdf for X_n and X_{n+1}.
   (d) Find the mean and autocovariance functions of X_n.
   (e) Repeat part a,b,c,d if s is uniform in (1,2).

   Solutions:

   (a) The sample path is given by:



   (b) The cdf is:

$$P[X_n \leq x] = P[s^n \leq x] = P[s < x^{1/n}] = x^{1/n}, 0 < x < 1.$$

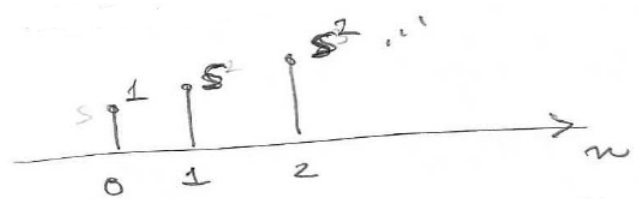
   (c) The joint cdf is:

$$\begin{aligned} P[X_n \leq x, X_{n+1} \leq y] &= P[s^n \leq x, s^{n+1} \leq y] \\ &= P[s \leq \min\left\{x^{1/n}, y^{1/(n+1)}\right\}] \\ &= \min\left\{x^{1/n}, y^{1/(n+1)}\right\} \end{aligned}$$

   (d)

$$\begin{aligned} E[X_n] = E[s^n] &= \int_0^1 s^n ds = \frac{1}{n+1} \\ E[X_n X_{n+k}] = E[s^n s^{n+k}] &= E[s^{2n+k}] = \frac{1}{2n+k+1} \\ C_X(n, n+k) = E[X_n X_{n+k}] - E[X_n]E[X_{n+k}] &= \frac{1}{2n+k+1} - \frac{1}{n+1} \cdot \frac{1}{n+k+1}. \end{aligned}$$

(e) Since $1 < s < 2$, we have:



The cdf becomes:

$$P[X_n \leq x] = P[s^n \leq x] = P[s < x^{1/n}] = x^{1/n} - 1, 1 < x^{1/n} < 2.$$

The joint pdf becomes:

$$\begin{aligned} P[X_n \leq x, X_{n+1} \leq y] &= P[s^n \leq x, s^{n+1} \leq y] \\ &= P[s \leq \min\left\{x^{1/n}, y^{1/(n+1)}\right\}] \\ &= \min\left\{x^{1/n}, y^{1/(n+1)}\right\} - 1 \end{aligned}$$

The mean and autocovariance become:

$$\begin{aligned} E[X_n] = E[s^n] = \int_1^2 s^n ds &= \frac{2^{n+1} - 1}{n+1} \\ E[X_n X_{n+k}] = E[s^n s^{n+k}] = E[s^{2n+k}] &= \frac{2^{2n+k+1} - 1}{2n+k+1} \\ C_X(n, n+k) = E[X_n X_{n+k}] - E[X_n]E[X_{n+k}] &= \frac{2^{2n+k+1} - 1}{2n+k+1} - \frac{2^{n+1} - 1}{n+1} \cdot \frac{2^{n+k+1} - 1}{n+k+1}. \end{aligned}$$

3. (P9.5) Let $g(t)$ be the rectangular pulse shown in Fig. P9.1.The random process $X(t)$ is defined as $X(t) = Ag(t)$, where A assumes the values ± 1 with equal probability.

(a) Find the pmf of $X(t)$.

(b) Find $m_X(t)$.

(c) Find the joint pmf of $X(t)$ and $X(t+d)$.

(d) Find $C_X(t, t+d), d > 0$.

Solutions:

(a) For $t \in [0, 1]$, we have:

$$P[X(t) = 1] = P[X(t) = -1] = 1/2.$$

Otherwise, we have $P[X(t) = 0] = 1$.

(b) If $t \in [0, 1]$, $m_X(t) = 1/2 * 1 + 1/2 * (-1) = 0$. Otherwise $m_X(t)$ is always 0. Hence, $m_X(t) = 0, \forall t$.

(c) For $t \in [0,1], t+d \in [0,1]$, $X(t) = X(t+d)$. Hence,

$$P[X(t) = \pm 1, X(t+d) = \pm 1] = 1/2.$$

For $t \in [0,1], t+d \notin [0,1]$, $X(t+d) = 0$. Hence,

$$P[X(t) = \pm 1, X(t+d) = 0] = 1.$$

For $t \notin [0,1], t+d \notin [0,1]$, $X(t) = X(t+d) = 0$. Hence,

$$P[X(t) = 0, X(t+d) = 0] = 1.$$

(d)

$$\begin{aligned} C_X[X(t), X(t+d)] &= E[X(t), X(t+d)] - E[X(t)]E[X(t+d)] \\ &= E[X(t)X(t+d)] \\ &= 1 * 1 * 1/2 + (-1) * (-1) * 1/2 \\ &= 1, \quad \text{if} \quad \mathrm{t} \in [0,1], \mathrm{t}+\mathrm{d} \in [0,1]. \end{aligned}$$

4. (P9.6) A random process is defined by $Y(t) = g(T-t)$, where g(t) is the rectangular pulse of Fig. P9.1, and T is a uniformly distributed random variable in the interval (0,1).

   (a) Find the pmf of $Y(t)$.

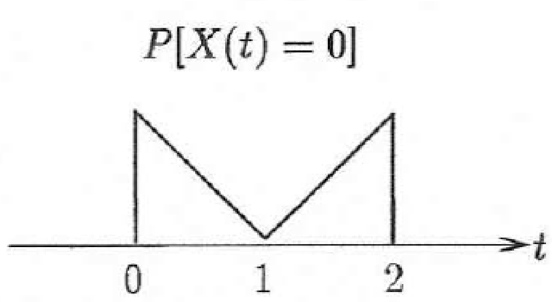
   (b) Find $m_Y(t)$ and $C_Y(t_1, t_2)$.

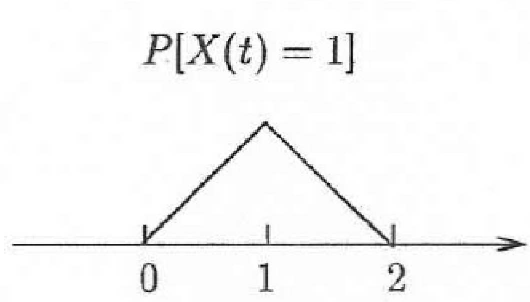
   Solutions:

   (a) The pmf is:

$$P[Y(t) = 1] = P[g(t-T) = 1], P[Y(t) = 0] = P[g(t-T) = 0].$$

If $0 < t < 1$, $P[Y(t) = 1] = 1 - t, P[Y(t) = 0] = t$; If $1 < t < 2$, $P[Y(t) = 1] = 2 - t, P[Y(t) = 0] = t - 1$.

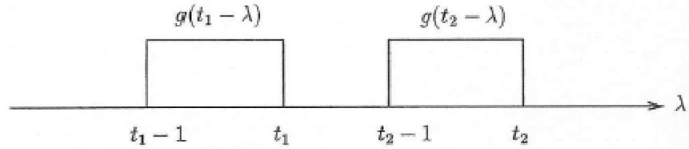




(b) The mean and autocovariance can be calculated as follows:

b) $\mathcal{E}[Y(t)] = 1 \cdot P[Y(t) = 1] = P[Y(t) = 1]$

$$\begin{aligned} E[Y(t_1)Y(t_2)] &= \int_0^1 E[g(t_1 - T)g(t_2 - T)|T = \lambda] f_T(\lambda) d\lambda \\ &= \int_0^1 g(t_1 - \lambda) g(t_2 - \lambda) d\lambda \end{aligned}$$



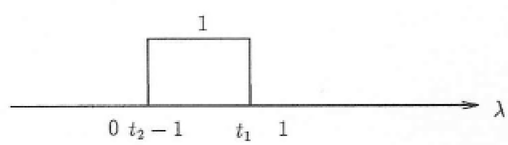
$$g(t_1 - \lambda) g(t_2 - \lambda) = 0 \quad \text{for} \quad t_1 < t_2 - 1$$

$$\Rightarrow R_Y(t_1, t_2) = 0 \quad \text{for} \quad t_2 - t_1 > 1$$

If $t_2 - 1 < t$, then

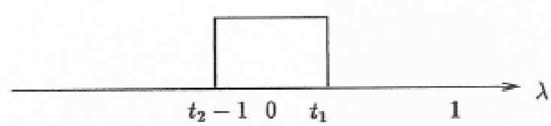
$$g(t_1 - \lambda) g(t_2 - \lambda) = \begin{cases} 1 & t_2 - 1 < \lambda < t_1 \\ 0 & \text{elsewhere} \end{cases}$$

Case 1



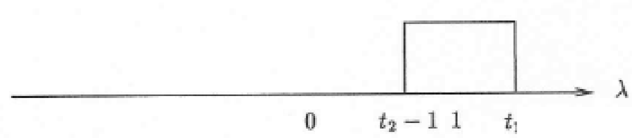
$$R_Y(t_1, t_2) = t_1 - (t_2 - 1) = 1 - (t_2 - t_1) \qquad t_1 < 1, \ \ 0 < t_2 - 1, \ \ t_2 - t_1 < 1$$

Case 2



$$R_Y(t_1, t_2) = t_1 \qquad t_1 < 1, \ \ t_2 < 1, \ \ t_2 - t_1 < 1$$

Case 3



$$R_Y(t_1, t_2) = 2 - t_2 \qquad t_1 > 1, \ \ t_2 < 2$$

5. (P9.10) Find an expression for $E[|X_{t_2} - X_{t_1}|^2]$ in terms of autocorrelation function.

Solutions:

$$\begin{aligned} E[|X_{t_2} - X_{t_1}|^2] &= E[X_{t_2}^2 - 2X_{t_1}X_{t_2} + X_{t_1}^2] \\ &= E[X_{t_2}^2] - 2E[X_{t_1}X_{t_2}] + E[X_{t_1}^2] \\ &= R_X(t_2,t_2) - 2R_X(t_2,t_1) + R_X(t_1,t_1). \end{aligned}$$

6. (P9.11) The random process $H(t)$ is defined as the hard-limited version of $X(t)$:

$$H(t) = \begin{cases} +1 & \text{if} \quad \mathrm{X(t)} \geq 0 \\ -1 & \text{if} \quad \mathrm{X(t)} < 0 \end{cases}$$

(a) Find the pdf, mean, and autocovariance of $H(t)$ if $X(t)$ is the sinusoid with a random amplitude presented in Example 9.2.

(b) Find the pdf, mean, and autocovariance of $H(t)$ if $X(t)$ is the sinusoid with a random phase presented in Example 9.9.

(c) Find a general expression for the mean of $H(t)$ in terms of the cdf of $X(t)$.

Solutions:

(a) The pdf is:

$$P[H(t) = 1] = P[X(t) \geq 0] = 1/2 = P[H(t) = -1].$$

The mean is:

$$E[H(t)] = 1 * 1/2 + (-1) * (1/2) = 0;$$

The autovariance is:

$$\begin{aligned} C_H[t, t+\tau] &= E[H(t)H(t+\tau)] - E[H(t)]E[H(t+\tau)] \\ &= E[H(t)H(t+\tau)] \\ &= 1 * P[H(t)H(t+\tau) = 1] + (-1) * P[H(t)H(t+\tau) = -1] \\ &= \begin{cases} 1, & \cos(2\pi t)\cos(2\pi(t+\tau)) = 1 \\ -1, & \cos(2\pi t)\cos(2\pi(t+\tau)) = -1 \end{cases} \end{aligned}$$

(b) The pdf is:

$$P[H(t) = 1] = P[X(t) \geq 0] = 1/2 = P[H(t) = -1].$$

The mean is:

$$E[H(t)] = 1 * 1/2 + (-1) * (1/2) = 0;$$

The autovariance is:

$$\begin{aligned} C_H[t, t+\tau] &= E[H(t)H(t+\tau)] - E[H(t)]E[H(t+\tau)] \\ &= E[H(t)H(t+\tau)] \\ &= 1 * P[X(t)X(t+\tau) \geq 0] + (-1) * P[X(t)X(t+\tau) < 0] \\ &= 1 - 2P[X(t)X(t+\tau) < 0] \end{aligned}$$

where:

$$P[X(t)X(t+\tau) < 0] = P[\cos(\omega t + \Theta)\cos(\omega(t+\tau)+\Theta) < 0]$$
$$= P[\cos(2\omega t + \omega\tau + 2\Theta) < \cos(\omega\tau)]$$

(c) The mean can be written as:

$$P[H(t) = 1] = P[X(t) \geq 0] = 1 - F_{X(t)}(0) = 1 - P[H(t) = -1]$$
$$E[H(t)] = 1 * P[H(t) = 1] + (-1) * (1 - P[H(t) = 1]) = 1 - 2F_{X(t)}(0)$$

7. (P9.14) Let $H(t)$ be the output of the hard limiter in Problem 9.11.

   (a) Find the cross-correlation and cross-covariance between $H(t)$ and $X(t)$ when the input is the sinusoid with a random amplitude presented as in Problem 9.11a.

   (b) Repeat if the input is a sinusoid with random phase as in Problem 9.11b.

   (c) Are the input and output processes uncorrelated? Orthogonal?

Solutions:

(a) The cross-correlation and cross-covariance are:

a) $E[H(t)X(t)] = E[|X(t)|]$ since $H(t)X(t) = \begin{cases} X(t) & , X(t) \geq 0 \\ -X(t) & , X(t) < 0 \end{cases}$

$E[|X(t)|] = E[|\xi \cos(2\pi t)|] = E[|\xi||\cos(2\pi t)|] = |\cos 2\pi t|\, E[|\xi|] = \frac{1}{2}|\cos 2\pi t|$

My cos is cs , My sin is sin ← Note!

$E[H(t)X(t)] - E[H(t)]E[X(t)] = E[|X(t)|] - 0 \times E[X(t)] = E[|X(t)|] = \frac{1}{2}|\cos 2\pi t|$

Not Uncorrelated, Not Orthogonal

(b) The cross-correlation and cross-covariance are:

b)

Again: $E[H(t)X(t)] = E[|X(t)|]$

$E[H(t)] = +1\, P\{X(t)>0\} - 1\times P\{X(t)<0\} = 0$

$C_V(X,H) = E[X(t)H(t)] - E[X(t)]E[H(t)] = E[|X(t)|]$

$E[|X(t)|] = E[|\cos(2\pi t+\xi)|] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\cos(2\pi t+\xi)|\, d\xi = \frac{2}{\pi}$

So $C_V(X,H) = \frac{2}{\pi}$

Not Uncorrelated, Not Orthogonal

8. (P9.21)

(a) Let Y_n be the process that results when individual 1s in a Bernoulli process are erased with probability α. Find the pmf S'_n of the counting process for Y_n. Does Y_n have independent and stationary increments?

(b) Repeat part a if in addition to the erasures, individual 0s in the Bernoulli process are changed to 1s with probability β.

Solutions:

(a) The pmf is given by:

$$\begin{aligned} P[Y_n = 1] &= P[I_n \text{ is not erased } | I_n = 1] P[I_n = 1] \\ &= (1-\alpha)p \text{ where } I_n \text{ is Bernoulli process} \end{aligned}$$

The Y_n are then a Bernoulli process with success probability

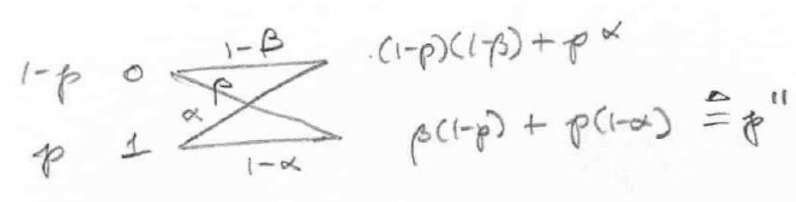
$$(1-\alpha)p \triangleq p'.$$

S'_n is then the binomial counting process with

$$P[S'_n = k] = \binom{n}{k} p'^k (1-p')^{n-k}$$

S'_n has independent and stationary increments.

(b) The pmf is given by:



Thus, the S''_n is a Bernoulli process with probability p''. S''_n has stationary and independent increments.

9. (P9.24) Consider the following moving average processes:

$$\begin{aligned} Y_n &= 1/2(X_n + X_{n-1}) \quad X_0 = 0 \\ Z_n &= 2/3X_n + 1/3X_{n-1} \quad X_0 = 0 \end{aligned}$$

   (a) Find the mean, variance, and covariance of Y_n and Z_n if X_n is a Bernoulli random process.
   (b) Repeat part a if X_n is the random step process.
   (c) Generate 100 outcomes of a Bernoulli random process X_n and find the resulting Y_n and Z_n. Are the sample means of Y_n and Z_n in part a close to their respective means?
   (d) Repeat part c with X_n given by the random step process.

   Solutions:

   (a) The calculations are as follows:

$$\begin{aligned}
\mathcal{E}[Y_n] &= \frac{1}{2}\mathcal{E}[X_n] + \frac{1}{2}\mathcal{E}[X_{n-1}] = \frac{1}{2}p + \frac{1}{2}p = p \\
\mathcal{E}[Y_n^2] &= \mathcal{E}\left[\frac{1}{4}X_n^2 + \frac{2}{4}X_nX_{n-1} + \frac{1}{4}X_{n-1}^2\right] \\
&= \frac{1}{4}p\,\underbrace{\mathcal{E}[X_n^2]}_{p} + 2\underbrace{\mathcal{E}[X_n]}_{p}\underbrace{\mathcal{E}[X_{n-1}]}_{p} + \underbrace{\mathcal{E}[X_{n-1}^2]}_{p} \\
&= \frac{1}{2}p(1+p) \\
\mathcal{E}[Y_nY_{n+1}] &= \frac{1}{4}\mathcal{E}[X_nX_{n+1} + X_n^2 + X_{n-1}X_{n+1} + X_{n-1}X_n] \\
&= \frac{1}{4}[p + 3p^2] \\
\mathcal{E}[Y_nY_{n+1}] &= \mathcal{E}\left[\frac{1}{4}(X_n + X_{n-1})(X_{n+1} + X_n)\right] = \frac{1}{4} + \frac{3}{4}(2p-1)^2
\end{aligned}$$

For $k > 1$ $\quad \mathcal{E}[Y_n Y_{n+k}] = \mathcal{E}[Y_n]\mathcal{E}[Y_{n+k}] = p^2$

$$\therefore C_Y(n, n+k) = \begin{cases} \frac{p}{2} - \frac{p^2}{2} & k = 0 \\ \frac{p}{4} - \frac{p^2}{4} & k = 1 \\ 0 & k > 1 \end{cases}$$

$$\begin{aligned}
\mathcal{E}[Z_n] &= \frac{2}{3}\mathcal{E}[X_n] + \frac{1}{3}\mathcal{E}[X_{n-1}] = p \\
\mathcal{E}[Z_n^2] &= \frac{1}{9}\mathcal{E}[(4X_n^2 + 4X_nX_{n-1} + X_{n-1}^2)] = \frac{5}{9}p + \frac{4}{9}p^2 \\
\mathcal{E}[Z_nZ_{n+1}] &= \frac{1}{9}\mathcal{E}[4X_nX_{n+1} + 2X_n^2 + 2X_{n+1}X_{n-1} + X_nX_{n-1}] = \frac{7}{9}p^2 + \frac{2}{9}p \\
\mathcal{E}[Z_nZ_{n+k}] &= \mathcal{E}[Z_n]\mathcal{E}[Z_{n+k}] = p^2 \text{ for } k > 1
\end{aligned}$$

$$\therefore C_Z(n, n+k) = \begin{cases} \frac{5}{9}p - \frac{5}{9}p^2 & k = 0 \\ \frac{2}{4}p - \frac{2}{9}p^2 & k = 1 \\ 0 & k > 1 \end{cases}$$

(b) The calculations are as follows:

$$\begin{aligned}
\mathcal{E}[Y_n] &= \frac{1}{2}\mathcal{E}[X_n] + \frac{1}{2}\mathcal{E}[X_{n-1}] = 2p - 1 \\
\mathcal{E}[Z_n] &= \frac{2}{3}\mathcal{E}[X_n] + \frac{1}{3}\mathcal{E}[X_{n-1}] = 2p - 1 \\
\mathcal{E}[Y_n^2] &= \mathcal{E}\left[\frac{1}{4}(X_n^2 + 2X_nX_{n-1} + X_{n-1}^2)\right] \\
&= \frac{1}{4}\{1 + 2(2p-1)^2 + 1\} = \frac{1}{2} + \frac{(2p-1)^2}{2} \\
\mathcal{E}[Z_n^2] &= \frac{1}{9}\mathcal{E}[4X_n^2 + 4X_nX_{n-1} + X_{n-1}^2] \\
&= \frac{1}{9}[5 + 4(2p-1)^2] = \frac{5}{9} + \frac{4}{9}(2p-1)^2
\end{aligned}$$

For $k > 1$

$$\begin{aligned}
\mathcal{E}[Y_nY_{n+k}] &= \mathcal{E}[Y_n]\mathcal{E}[Y_{n+k}] = (2p-1)^2 \\
\mathcal{E}[Z_nZ_{n+1}] &= \mathcal{E}\left[\frac{1}{9}(2X_n + X_{n-1})(2X_{n+1} + X_n)\right] = \frac{2}{9} + \frac{7}{9}(2p-1)^2
\end{aligned}$$

For $k > 1$

$$\mathcal{E}[Z_n Z_{n+1}] = \mathcal{E}[Z_n]\mathcal{E}[Z_{n+1}] = (2p-1)^2$$

$$\therefore C_Y(n, n+k) = \begin{cases} \frac{1}{2} - \frac{1}{2}(2p-1)^2 & k = 0 \\ \frac{1}{4} - \frac{1}{4}(2p-1)^2 & k = 1 \\ 0 & k > 1 \end{cases}$$

$$C_Z(n, n+k) = \begin{cases} \frac{5}{9} - \frac{5}{9}(2p-1)^2 & k = 0 \\ \frac{2}{9} - \frac{2}{9}(2p-1)^2 & k = 1 \\ 0 & k > 1 \end{cases}$$

In all cases, the sample functions are close to the mean of the processes.

10. (P9.25) Consider the following autoregressive processes:

$$W_n = 2W_{n-1} + X_n \quad W_0 = 0$$
$$Z_n = 3/4 Z_{n-1} + X_n \quad Z_0 = 0$$

(a) Suppose that X_n is a Bernoulli process.What trends do the processes exhibit?

(b) Express W_n and Z_n in terms of $X_n, X_{n-1}, ..., X_1$ and then find $E[W_n]$ and $E[Z_n]$. Do these results agree with the trends you expect?

(c) Do W_n or Z_n have independent increments? stationary increments?

(d) Generate 100 outcomes of a Bernoulli process. Find the resulting realizations of W_n and Z_n. Is the sample mean meaningful for either of these processes?

(e) Repeat part d if X_n is the random step process.

Solutions:

(a) W_n is exponentially increasing without bound as $n \to \infty$ and has meaningless sample mean. Z_n is exponentially decreasing unless X_n is 1 and the sample mean is about twice of the sample mean of X_n.

(b) The calculations are as follows:

$$
\begin{aligned}
W_n &= 2W_{n-1} + X_n \qquad n > 1 \\
&= 2(W_{n-2} + S_{n-1}) + X_n \\
&= X_n + 2X_{n-1} + 4X_{n-2} + \ldots + 2^{n-1}X_1 \\
\mathcal{E}[W_n] &= \mathcal{E}[X]\{1 + 2 + \ldots + 2^{n-1}\} = \mathcal{E}[X]\frac{1-2^n}{1-2} = (2^n - 1)\mathcal{E}[X] \\
Z_n &= \frac{3}{4}Z_{n-1} + X_n = \frac{3}{4}\left(\frac{3}{4}|Z_{n-2} + X_{n-1}\right) + X_n \\
&= X_n + \frac{3}{4}X_{n-1} + \ldots + \left(\frac{3}{4}\right)^{n-1} X_1 \\
\mathcal{E}[Z_n] &= \mathcal{E}[X]\left\{1 + \frac{3}{4} + \ldots + \left(\frac{3}{4}\right)^n\right\} = \mathcal{E}[X]\frac{1-\left(\frac{3}{4}\right)^{n-1}}{1-\frac{3}{4}}
\end{aligned}
$$

(c) Since $W_n - W_{n-1} = W_{n-1} + X_n$, W_n does not have independent or stationary increments. Similarly, since $Z_n - Z_{n-1} = X_n - 1/2Z_{n-1}$, Z_n does not have independent or stationary increments.

(d) W_n has meaningless sample mean. Z_n has a sample mean of about twice of that of X_n.