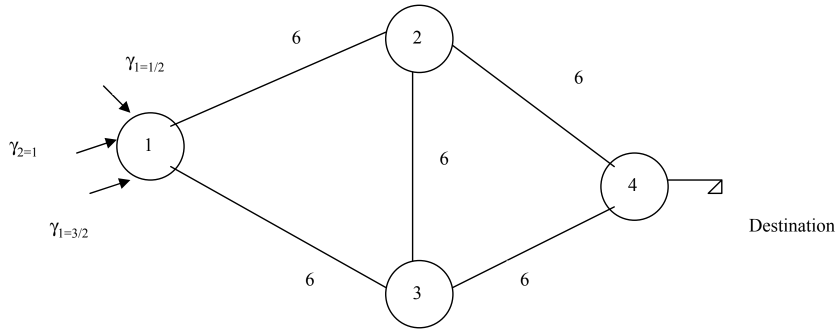
ECE 642
HW Set 9 Solutions

Problem 5.22



(a) The equivalent arrival rate is shown below:

$$\gamma = \sum_{i=1}^{3} \gamma_i = 1 + 1/2 + 3/2 = 3$$

The average time delay from node 1 to 4 through node 2 is:

$$\begin{aligned} E(T_{1,4})\text{through 2} &= \sum_{i=1}^{2} \frac{1}{\mu_i - \lambda_i} \\ &= \frac{1}{\mu_1 - \lambda_1} + \frac{1}{\mu_2 - \lambda_2} \\ &= \frac{1}{6-3} + \frac{1}{6-3} \\ &= 2/3\, sec \end{aligned}$$

(b) The average time delay from node 1 to 4 through node 2,3 is:

$$\begin{aligned} E(T_{1,4})\text{through 2 and 3} &= \sum_{i=1}^{3} \frac{1}{\mu_i - \lambda_i} \\ &= \frac{1}{\mu_1 - \lambda_1} + \frac{1}{\mu_2 - \lambda_2} + \frac{1}{\mu_3 - \lambda_3} \\ &= \frac{1}{6-3} + \frac{1}{6-3} + \frac{1}{6-3} \\ &= 1\, sec \end{aligned}$$

(c) Given the probabilities $q_{13} = 1/3$, $q_{23} = 3/4$, $q_{34} = 1$, we have the flows on each link as follows:

$$\lambda_{12} = 3 \cdot \frac{2}{3} = 2$$

$$\lambda_{24} = 2 \cdot \frac{1}{4} = 1/2$$

$$\lambda_{13} = 3 \cdot \frac{1}{3} = 1$$

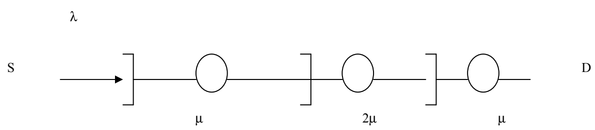
$$\lambda_{34} = (\frac{3}{2} + 1) \cdot 1 = 5/2$$

$$\lambda_{23} = 2 \cdot \frac{3}{4} = 3/2$$

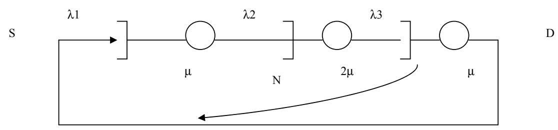
The world-wide average time delay is then calculated as:

$$\begin{aligned} E(T) &= \frac{1}{3} \sum_{i=1}^{5} \frac{\lambda_i}{\mu_i - \lambda_i} \\ &= \frac{1}{3} \left[\frac{2}{6-2} + \frac{1/2}{6-1/2} + \frac{3/2}{6-3/2} + \frac{1}{6-1} + \frac{5/2}{6-5/2} \right] \\ &= 0.61\, sec \end{aligned}$$

Problem 5.26



For the case $\lambda \gg 2\mu$ the equivalent circuit becomes



Using Buzen's Method:

$$\rho_1 = \frac{\lambda_1}{\mu} = \rho_3$$

$$\rho_2 = \frac{\rho_1}{2}$$

Let $\rho_2 = 1$, we get $\rho_1 = \rho_3 = 2$, the matrix get by using the Buzen's algorithm is shown below:

n/m	1	2	3
0	1	1	1
1	2	3	5
2	4	7	17
3	8	15	49
4	16	31	129
5	32	63	321
6	64	127	769

For throughput, time delay calculations:

$$E(n_i) = \sum_{k=1}^{N} \rho_i^k \left[\frac{g(N-k,M)}{g(N,m)}\right]$$

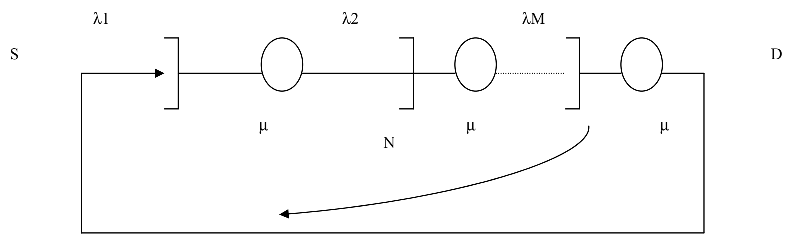
$$\gamma_i = \lambda_i \left[\frac{g(N-1,M)}{g(N,M)}\right]$$

$$E(T) = \frac{E(n_i)}{\gamma_i}$$

Using above formula:

N	γ/μ	$\mu E(T)$
1	0.4	2.5
2	0.68	3.4
3	0.694	4.32
4	0.76	5.265
5	0.80	6.22
6	0.83	7.19

Problem 5.27



$$u(n) = \mu Prob[\text{A queue is nonempty}]$$

By using the Buzen's algorithm and given $\rho_1 = \rho_2 = \rho_3 = 1$, we can get the matrix as follows:

n/m	1	2	3
0	1	1	1
1	1	2	3
2	1	3	6
3	1	4	10
4	1	5	15

The initial state probability is given as follows:

$$\begin{aligned} P(n_i = 0) &= \frac{1}{g(N,M)}(g(N,M) - \rho_i g(N-1,M)) \\ &= \frac{1}{15}(15 - 1 \cdot 10) \\ &= \frac{1}{3} \end{aligned}$$

The $u(n)$ is then calculated:

$$\begin{aligned} u(n) &= \mu p_0 \\ &= (1/3)\mu \end{aligned}$$

By using the formula:

$$u(n) = \frac{n\mu}{n+M-1}$$

For $M = 3$ and $N = 3$:

$$u(n) = \mu/3$$

Using the formula:

$$u(n) = \frac{n\mu}{n+2}$$

and

$$E(n_i) = \sum_{k=1}^{N} \rho_i^k \left[\frac{g(N-k, M)}{g(N, M)}\right]$$

we can get

$$\begin{aligned} E(n_1) &= 1^1 \cdot \frac{1}{3} \\ &= 1/3 \end{aligned}$$

By using the formula $E(n) = n/M$, we can get the same answer:

$$E(n_1) = 1/3$$

Problem 5.29

(a) For $N = 1$, using the Eq.(5.72)

$$P_B = \frac{\rho^{'N}(1-\rho^{'})}{1-\rho^{'N+1}}$$

where

$$\rho^{'} = \lambda/\mu^{'} = \rho/(1-P_B)$$

For N=1:

$$\begin{aligned} P_B &= \frac{\rho^{'}(1-\rho')}{1-\rho'^2} \\ &= \frac{\rho^{'}}{1+\rho^{'}} \end{aligned}$$

Hence

$$P_B = \frac{\rho}{1+\rho-P_B}$$

So

$$P_B = \begin{cases} \rho & \text{if } \rho < 1 \\ 1 & \text{if } \rho \geq 1. \end{cases}$$

For $\rho < 1$

$$\begin{aligned}\frac{\gamma}{\mu} &= \rho(1-P_B)\\ &= \rho(1-\rho)\end{aligned}$$

$$\begin{aligned}\mu E(T) &= \frac{E(n)}{\gamma/\mu}\\ &= \frac{1}{1-\rho}\end{aligned}$$

(b) For N=5,

$$P_B = \frac{\rho'^5(1-\rho')}{1-\rho'^6}$$

where

$$\rho' = \rho/(1-P_B)$$

$$\begin{aligned}P_B &= \frac{(\frac{\rho}{1-P_B})^5\left(1-\frac{\rho}{1-P_B}\right)}{1-\left(\frac{\rho}{1-P_B}\right)^6}\\ &= \frac{\rho^5(1-P_B-\rho)}{1-\rho^6}\end{aligned}$$

(c) For $\rho' = 1$

$$\begin{aligned}P_B &= \frac{\rho'^N(1-\rho')}{1-\rho'^{N+1}}\\ &= \frac{1}{N+1}\end{aligned}$$

$$\begin{aligned}\gamma/\mu &= \rho(1-P_B)\\ &= \rho^2\\ &= \left(\frac{N}{N+1}\right)^2\end{aligned}$$

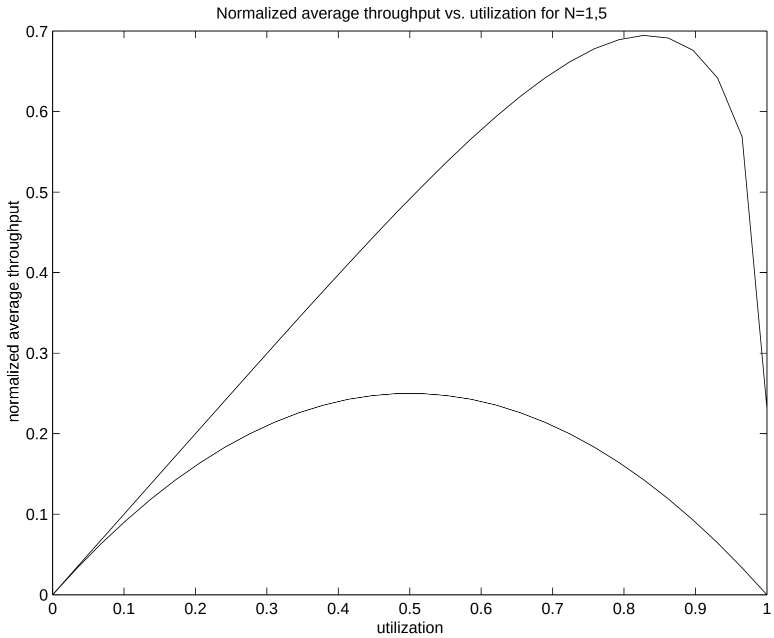
for $\rho = \frac{N}{N+1}$. And for $\rho' \ll 1$, we can get:

$$P_B = 0$$

and

$$\begin{aligned}\frac{\gamma}{\mu} &= \rho(1-P_B)\\ &= \rho\end{aligned}$$

The graph for part (a) and (b) is listed below:



Problem 5.30

The program is listed below:

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%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%          Problem 5.30       %%%%%%%%%%%%%%%%%%%%%%%%%%%%
%N=2, N_i=1, write a computer program to solve P_bt and P_bi iteratively. Use this
%to find the average throughput gamma_i=lambda_i(1-p_bi) as a function of lambda_i
%for serveral values of lambda_i. Take p_t=0.5 and compare the result.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

rho=linspace(0,1,20); pt=1/2;

for i=1:length(rho),
    % initialization for each rho
    p_btold = 0;
    p_biold = 0;
    % get the first pair of p_bt and p_bi
    rho_i = rho(i)/(1-p_btold);
    rho_t = rho(i)*(1-p_biold)*(1-pt)/(pt*(1-p_btold)^2);
    s = 1+rho_i+rho_t+2*rho_i*rho_t+rho_t^2;
    p02 = rho_t^2/s;
    p11 = 2*rho_i*rho_t/s;
    p10 = rho_i/s;
    p_bt = p02 + p11;
    p_bi = p_bt + p10;

    % iterations
    while(sqrt((p_bt-p_btold)^2 + (p_bi-p_biold)^2) > 0.0001 )
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        p_btold = p_bt;
        p_biold = p_bi;
        rho_i = rho(i)/(1-p_btold);
        rho_t = rho(i)*(1-p_biold)*(1-pt)/(pt*(1-p_btold)^2);
        s = 1+rho_i+rho_t+2*rho_i*rho_t+rho_t^2;
        p02 = rho_t^2/s;
        p11 = 2*rho_i*rho_t/s;
        p10 = rho_i/s;
        p_bt = p02 + p11;
        p_bi = p_bt + p10;
    end;
    P_BT(i) = p_bt;
    P_BI(i) = p_bi;
end;

gamma_i = rho.*(1-P_BI); plot(rho,gamma_i);
xlabel('utilization');
ylabel('normalized average throughput');
title('Normalized average throughput vs. utilization for N=2 & NI=1');
```