Homeowrk Set No. 2

ECE 642
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Problem 1

Refer to the Figure 1 below. Calculate the probability of k independent events in the m intervals Δt units long, if the probability of one event in any interval is p, while the probability of no events is $q = 1 - p$. Show how one obtains the binomial distribution of

$$p(k) = \left(\begin{array}{c} m \\ k \end{array} \right) p^k q^{m-k} \tag{1}$$

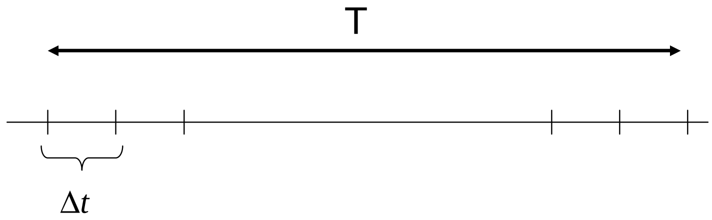

Figure 1: figure for problem 1

Problem 2

In problem 1 let $p = \Delta t$, λ a proportionality factor. This then relates the binomial distribution to the Poisson process. Let $\Delta t \to 0$, with $T = m\Delta t$ fixed. Show that in the limit one gets the Poisson distribution. Show that the mean value $E(k)$ and the variance are both equal to λT What is the probability that no arrival occurs in the interval T? Sketch this as a function of T. Repeat the probability that at least one arrival occurs in T.

Problem 3

Calculate and plot the Poisson distribution for the three cases $\lambda T = 0.1, 1, 10$. In the third case try to carry the calculation and plot out to at least $k = 20$. (Stirlings approximation for the factorial may be useful here.) Does the distribution begin to crowd in and peak about $E(k)$ as predicted by the ratio

$$\frac{\sigma_k}{E(k)} = \frac{1}{\sqrt{\lambda T}} \tag{2}$$

Problem 4

Carry out the details of the analysis leading to equations

i

$$\begin{aligned} prob.[N(t,t+\Delta t)=0] &= \prod_{i=1}^{n} prob.[N^{(i)}(t,t+\Delta t)=0] \\ &= \prod_{i=1}^{m}[1-\lambda_i\Delta t+o(\Delta t)] \\ &= 1-\lambda\Delta t+O(\Delta t) \end{aligned}$$

where

$$\lambda=\sum_{i=1}^{m}\lambda_i \tag{3}$$

ii

$$prob.[N(t,t+\Delta t)=1]=\lambda\Delta t+o(\Delta t) \tag{4}$$

Showing that the sums of Poisson processes are Poisson as well.

Problem 5

Refer to the following time-dependent equation governing the operation of the M/M/1 queue.

$$p_n(t+\Delta t)=[1-(\lambda+\mu)\Delta t]p_n(t)+\lambda\Delta t p_{n-1}(t)+\mu\Delta t p_{n+1}(t) \tag{5}$$

Start at time $t=0$ with the queue empty. (What are then the values $p_n(0)$) Let $\lambda/\mu=0.5$ for simplicity, take $\Delta t=1$, and pick $\lambda\Delta t$ and $\mu\Delta t$ very small so that the term of $(\Delta t)^2$ and higher can be ignored. Write a program that calculates $p_n(t+\Delta t)$ recursively as t is incremented by Δt and show that $p_n(t)$ does settle down eventually to the steady-state set of probabilities $\{p_n\}$. Pick the maximum value of n to be 5. the set of steady-state probabilities obtained should then agree with the following equation:

$$p_0=(1-\rho)\rho^n/(1-\rho^{N+1}) \tag{6}$$

Note: Equation (5) must be modified slightly in calculating $p_0(t+\Delta t)$ and $p_5(t+\Delta t)$. You may want to set the problem up in matrix-vector form.

Problem 6

$$(\lambda+\mu)p_n=\lambda p_{n-1}+\mu p_{n+1} \tag{7}$$

where $n\geq 1$. Derive the above equation governing the steady-state (stationary) probabilities of state of the n M/M/1 queue, in two ways:

1. from the initial generating equation

$$\begin{aligned} p_n(t+\Delta t)=p_n(t)[(1-\lambda\Delta t)(1-\mu\Delta t)+\mu\Delta t*\lambda\Delta t+o(\Delta t) &+ p_{n-1}(t)[\lambda\Delta t(1-\mu\Delta t)+o(\Delta t)] \\ &+ p_{n+1}[\mu\Delta t(1-\lambda\Delta t)+o(\Delta t)] \end{aligned}$$

2. from flow balance arguments involving transitions between states n-1, n, and n+1 as indicated in the figure below

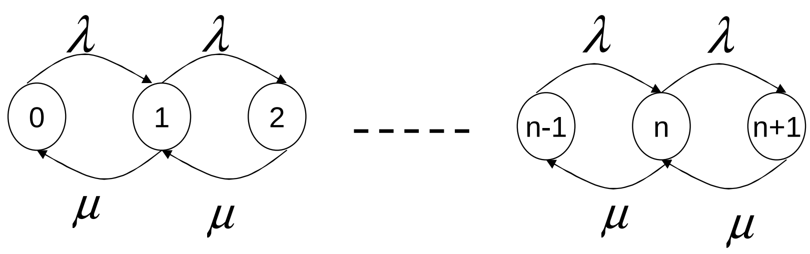

Figure 2: figure for problem 6

Problem 7

As a generalization of the M/M/1 queue analysis, consider a birth-death process with state-dependent arrivals λ_n and state-dependent departures μ_n. Show by applying balance arguments, that the equation governing the stationary state probabilities is given by

$$(\lambda_n + \mu_n)p_n = \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1} \tag{8}$$

Show that the solution to this equation is given by $p_n/p_0 = \prod_{i=1}^{n-1} \lambda_i / \prod_{i=1}^{n} \mu_i$.