George Mason University

Department of Electrical and Computer Engineering

ECE 528: Introduction to Random Processes in ECE — Fall Semester

Homework Set 9 Solutions

1. (P5.37) Let X be the number of full pairs and let Y be the remainder of the number of dots observed in a toss of a fair die. Are X and Y independent random variables?

   Solutions:

Toss Outcome	1	2	3	4	5	6
Full Pairs X	0	1	1	2	2	3
Remainder Y	1	0	1	0	1	0
$p(X,Y)$	1/6	1/6	1/6	1/6	1/6	1/6

	$Y=0$	$Y=1$
$X=0$	0	1/6
$X=1$	1/6	1/6
$X=2$	1/6	1/6
$X=3$	1/6	0

   Hence,

$$
\begin{aligned}
&P[Y=0]=P[Y=1]=1/2;\\
&P[X=0]=P[X=3]=1/6;\\
&P[X=1]=P[X=2]=1/6+1/6=1/3.
\end{aligned}
$$

$$p(X,Y) \neq p(X)\cdot p(Y).$$

   X and Y are not independent.

2. (P5.48) Let X and Y be independent random variables that are uniformly distributed in [-1,1]. Find the probability of the following events:

   (a) $P[X^2 < 1/2, |Y| < 1/2]$.

   (b) $P[4X < 1, Y < 0]$.

   (c) $P[XY < 1/2]$.

   (d) $P[\max(X,Y) < 1/3]$.

   Solutions:

(a)
$$\begin{aligned}P[X^2 < 1/2, |Y| < 1/2] &= P[X^2 < 1/2] \cdot P[|Y| < 1/2] \\ &= P[|X| < \sqrt{2}/2] \cdot P[Y < 1/2] \\ &= \sqrt{2}/2 \cdot 1/2 = \sqrt{2}/4.\end{aligned}$$

(b)
$$P[4X < 1, Y < 0] = P[X < 1/4] \cdot P[Y < 0] = 5/8 \cdot 1/2 = 5/16.$$

(c) In the first quadrant,

$$\begin{aligned}P_1[XY < 1/2] &= 1/2 + \int_{1/2}^{1} \int_{0}^{1/2x} 1 \cdot dydx \\ &= 1/2 + \int_{1/2}^{1} \frac{1}{2x} dx \\ &= 1/2 + 1/2(\ln 1 - \ln \frac{1}{2}) \\ &= 0.85.\end{aligned}$$

Thus, $P[X < 1/2] = (P_1[XY < 1/2] * 2 + 2)/4 = 0.923$.

(d)
$$P[\max(X, Y) < 1/3] = P[X < 1/3]P[X < 1/3] = 4/9.$$

3. (P5.49) Let X and Y be random variables that take on values from the set $\{-1, 0, 1\}$.

(a) Find a joint pmf for which X and Y are independent.

(b) Are X^2 and Y^2 independent random variables for the pmf in part a?

(c) Find a joint pmf for which X and Y are not independent, but for which X^2 and Y^2 are independent.

Solutions:

(a) The joint pmf could be:

	Y	-1	0	1
X		P(Y=-1) = 1/3	P(Y=0) = 1/3	P(Y=1) = 1/3
-1	P(X=-1) = 1/3	P(-1,-1) = 1/9	P(-1,0) = 1/9	P(-1,1) = 1/9
0	P(Y=0) = 1/3	P(0,-1) = 1/9	P(0,0) = 1/9	P(0,1) = 1/9
1	P(Y=1) = 1/3	P(1,-1) = 1/9	P(1,0) = 1/9	P(1,1) = 1/9

in which X and Y are independent.

(b) The joint pmf of X^2 and Y^2 is given by:

	$P(Y^2=0)=1/3$	$P(Y^2=1)=2/3$
$P(X^2=0)=1/3$	P(0,0) = 1/9	P(0,1) = 2/9
$P(X^2=1)=2/3$	P(1,0) = 1/9	P(0,-1) = 4/9

Clearly, $P(X,Y) = P(X) \cdot P(Y)$. X^2 and Y^2 are independent.

(c) The joint pmf for X and Y could be:

	Y	-1	0	1
X		P(Y=-1) = 1/3	P(Y=0) = 1/3	P(Y=1) = 1/3
-1	P(X=-1) = 1/3	P(-1,-1) = 0	P(-1,0) = 2/9	P(-1,1) = 2/9
0	P(Y=0) = 1/3	P(0,-1) = 2/9	P(0,0) = 1/9	P(0,1) = 0
1	P(Y=1) = 1/3	P(1,-1) = 2/9	P(1,0) = 0	P(1,1) = 0

in which X and Y are apparently not independent. The corresponding joint pmf for X^2 and Y^2 is:

	$P(Y^2=0)=1/3$	$P(Y^2=1)=2/3$
$P(X^2=0)=1/3$	P(0,0) = 1/9	P(0,1) = 2/9
$P(X^2=1)=2/3$	P(1,0) = 1/9	P(0,-1) = 2/9+2/9=4/9

which is the same as in (b). Thus, X^2 and Y^2 are independent.

4. (P5.56)

   (a) Find $E[(X+Y)^2]$.

   (b) Find the variance of $X+Y$.

   (c) Under what condition is the variance of the sum equal to the sum of the individual variances?

   Solutions:

   (a)
$$E[(X+Y)^2] = E[X^2 + 2XY + Y^2] = E[X^2] + 2E[XY] + E[Y^2].$$

   (b)
$$\begin{aligned} Var[X+Y] &= E[(X+Y)^2] - (E[X+Y])^2 \\ &= E[X^2] + 2E[XY] + E[Y^2] - E[X]^2 - 2E[X]E[Y] + E[Y]^2 \\ &= Var[X] + Var[Y] + 2E[XY] - 2E[X]E[Y]. \end{aligned}$$

   (c) $Var[X+Y] = Var[X] + Var[Y]$ if $E[XY] = E[X]E[Y]$, i.e., X and Y are uncorrelated.

5. (P5.74) Use the fact that $E[(tX+Y)^2] \geq 0$ for all t to prove the Cauchy-Schwarz inequality:

$$(E[XY])^2 \leq E[X^2]E[Y^2].$$

Solutions:

We know that:

$$E[(tX+Y)^2] = t^2E[X^2] + 2tE[XY] + E[Y^2] \geq 0.$$

Thus, the quadratic equation $t^2E[X^2] + 2tE[XY] + E[Y^2] = 0$ has at most one double real root. And the discriminant satisfies:

$$4t^2(E[XY])^2 - 4t^2E[X^2]E[Y^2] \leq 0,$$

$$\rightarrow (E[XY])^2 \leq E[X^2]E[Y^2].$$

6. (P 5.75)

(a) Find $p_Y(y|x)$ and $p_X(x|y)$ in Problem 5.1 assuming fair coins are used.
(b) Find $p_Y(y|x)$ and $p_X(x|y)$ in Problem 5.1 assuming Carlos uses a coin with $p = 3/4$.
(c) What is the effect on of Carlos using a biased coin?
(d) Find $E[Y|X=x]$ and $E[X|Y=y]$ in part a; then find E[X] and E[Y].
(e) Find $E[Y|X=x]$ and $E[X|Y=y]$ in part b; then find E[X] and E[Y].

Solutions:

(a) From P5.1, we have:

$$\begin{aligned}
P[X=0,Y=0] &= P[\{00\}] = 1/16,\\
P[X=0,Y=1] &= P[\{01,10\}] = 1/8+1/8 = 1/4,\\
P[X=0,Y=2] &= P[\{02,20\}] = 1/16+1/16 = 1/8,\\
P[X=1,Y=1] &= P[\{11\}] = 1/4,\\
P[X=1,Y=2] &= P[\{21,12\}] = 1/8+1/8 = 1/4;\\
P[X=2,Y=2] &= P[\{22\}] = 1/16.
\end{aligned}$$

Hence,

$$\begin{aligned}
&p_Y(y=0|x=0) = 1/7, p_Y(y=1|x=0) = 4/7, p_Y(y=2|x=0) = 2/7;\\
&p_Y(y=1|x=1) = p_Y(y=2|x=1) = 1/2;\\
&p_Y(y=2|x=2) = 1;\\
&\\
&p_X(x=0|y=0) = 1;\\
&p_X(x=0|y=1) = p_X(x=1|y=1) = 1/2;\\
&p_X(x=0|y=2) = 2/7, p_X(x=1|y=2) = 4/7, p_X(x=2|y=2) = 1/7;
\end{aligned}$$

(b) From P5.1, we have:

$$
\begin{aligned}
&P[X=0,Y=0] = P[\{00\}] = 1/64,\\
&P[X=0,Y=1] = P[\{01,10\}] = 1/32 + 6/64 = 1/8,\\
&P[X=0,Y=2] = P[\{02,20\}] = 9/64 + 1/64 = 5/32,\\
&P[X=1,Y=1] = P[\{11\}] = 3/16,\\
&P[X=1,Y=2] = P[\{21,12\}] = 6/64 + 9/32 = 3/8;\\
&P[X=2,Y=2] = P[\{22\}] = 9/64.
\end{aligned}
$$

$$P[X=Y] = P[\{00,11,22\}] = 1/64 + 3/16 + 9/64 = 11/32.$$

Hence,

$$
\begin{aligned}
&p_Y(y=0|x=0) = 1/19, p_Y(y=1|x=0) = 8/19, p_Y(y=2|x=0) = 10/19;\\
&p_Y(y=1|x=1) = 1/3, p_Y(y=2|x=1) = 2/3;\\
&p_Y(y=2|x=2) = 1;
\end{aligned}
$$

$$
\begin{aligned}
&p_X(x=0|y=0) = 1;\\
&p_X(x=0|y=1) = 8/20, p_X(x=1|y=1) = 12/20;\\
&p_X(x=0|y=2) = 10/43, p_X(x=1|y=2) = 24/43, p_X(x=2|y=2) = 9/43;
\end{aligned}
$$

(c) $p_X(x|y)$ increases when x grows yet decreases when x goes to zero under the same y.

(d)

$$
\begin{aligned}
&E[X|y=0] = 0;\\
&E[X|y=1] = 0*1/2 + 1*1/2 = 1/2;\\
&E[X|y=2] = 0 + 4/7 + 2*1/7 = 6/7;\\
&E[X] = 0 + 1/2*8/16 + 6/7*7/16 = 5/8;
\end{aligned}
$$

$$
\begin{aligned}
&E[Y|x=0] = 0 + 4/7 + 2*2/7 = 8/7;\\
&E[Y|x=1] = 1*1/2 + 2*1/2 = 3/2;\\
&E[Y|x=2] = 2*1 = 2;\\
&E[Y] = 8/7*7/16 + 3/2*8/16 + 2*1/16 = 11/8;
\end{aligned}
$$

(e)

$$
\begin{aligned}
&E[X|y=0]=0;\\
&E[X|y=1]=0*1/2+1*12/20=12/20;\\
&E[X|y=2]=0+24/43+2*9/43=42/43;\\
&E[X]=0+43/64*42/43+12/20*20/64=54/64;\\
\\
&E[Y|x=0]=0+8/19+2*10/19=28/19;\\
&E[Y|x=1]=1*1/3+2*2/3=5/3;\\
&E[Y|x=2]=2*1=2;\\
&E[Y]=28/19*19/64+5/3*36/64+2*9/64=106/64;
\end{aligned}
$$

7. (P 5.81)

   (a) Find $f_Y(y|x)$ in Problem 5.28(i).
   (b) Find $E[Y|X=x]$ and $E[Y]$.
   (c) Repeats parts a and b for 5.28(ii).
   (d) Repeats parts a and b for 5.28(iii).

   Solutions:

   (a)
$$f_Y(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}=\frac{1}{2\sqrt{1-x^2}},-1\le x\le 1,-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}.$$

   (b)
$$
\begin{aligned}
&E[Y|X=x]=\frac{1}{2\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ydy=0,\\
&\to E[Y]=0.
\end{aligned}
$$

   (c)
$$f_Y(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}=\frac{1}{2(1-|x|)},-1\le x\le 1,-(1-|x|)\le y\le 1-|x|.$$

$$
\begin{aligned}
&E[Y|X=x]=\frac{1}{2(1-|x|)}\int_{-(1-|x|)}^{1-|x|} ydy=0,\\
&\to E[Y]=0.
\end{aligned}
$$

   (d)
$$f_Y(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}=\frac{1}{1-x},0\le x\le 1,0\le y\le 1-x.$$

$$
\begin{aligned}
&E[Y|X=x]=\frac{1}{1-x}\int_0^{1-x} ydy=\frac{1-x}{2},\\
&E[Y]=\int_0^1\frac{1-x}{2}2(1-x)dx=1/3.
\end{aligned}
$$

8. (P 5.83) Find $f_Y(y|x)$ and $f_X(x|y)$ for the jointly Gaussian distribution pdf in Problem 5.34.

Solutions:

$$f_Y(y|x) = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right)+\left(\frac{y-m_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right\}}{\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-m_1)^2}{2\sigma_1^2}\right\}}$$

$$\begin{aligned} f_Y(y|x) &= \frac{\exp\left\{-\frac{\left(\frac{y-m_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{y-m_2}{\sigma_2}\right)\left(\frac{x-m_1}{\sigma_1}\right)+\rho^2\left(\frac{x-m_1}{\sigma_1}\right)^2}{2(1-\rho^2)}\right\}}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} \\ &= \frac{\exp\left\{-\frac{\left\{\left(\frac{y-m_2}{\sigma_2}\right)-\rho\left(\frac{x-m_1}{\sigma_1}\right)\right\}^2}{2(1-\rho^2)}\right\}}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} \\ &= \frac{\exp\left\{-\frac{(y-m_2-\rho\frac{\sigma_2}{\sigma_1}(x-m_1))^2}{2\sigma_2^2(1-\rho^2)}\right\}}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} \end{aligned}$$

which is a Gaussian distribution pdf with mean $m_2 + \rho\frac{\sigma_2}{\sigma_1}(x - m_1)$ and variance $\sigma_2^2(1 - \rho^2)$. Similarly, $f_X(x|y)$ is Gaussian with mean $m_1 + \rho\frac{\sigma_1}{\sigma_2}(y - m_2)$ and variance $\sigma_1^2(1 - \rho^2)$.

9. (P 5.90) Two toys are started at the same time each with a different battery.The first battery has a lifetime that is exponentially distributed with mean 100 minutes; the second battery has a Rayleigh-distributed lifetime with mean 100 minutes.

   (a) Find the cdf to the time T until the battery in a toy first runs out.
   (b) Suppose that both toys are still operating after 100 minutes. Find the cdf of the time T_2 that subsequently elapses until the battery in a toy first runs out.
   (c) In part b, find the cdf of the total time that elapses until a battery first fails.

   Solutions:

   (a) Let the lifetime of the two batteries be X_1 and X_2, respectively. Hence $T = \min\{X1, X2\}$.

$$P[T > t] = P[X_1 > t]P[X_2 > t] = e^{-\lambda t}e^{-\alpha t^2};$$
$$P[T \le t] = 1 - P[T > t] = 1 - e^{-\lambda t - \alpha t^2}.$$

(b)
$$P[T > t + t_0 | T > t_0] = \frac{P[T > t + t_0]}{P[T > t_0]} = e^{-\lambda t - \alpha(t+t_0)^2 + \alpha t_0^2}, t_0 = 100, t > 0.$$

The cdf is $P[T < t + t_0 | T > t_0] = 1 - e^{-\lambda t - \alpha(t+t_0)^2 + \alpha t_0^2}$.

(c) The total time is $t_t = t + t0$, thus the cdf is $P[T < t_t | T > t_0] = 1 - e^{-\lambda(t_t - t_0) - \alpha t_t^2 + \alpha t_0^2}$.

10. (P 6.5) An urn contains one black ball and two white balls. Three balls are drawn from the urn. Let $I_k = 1$ if the outcome of the kth draw is the black ball and let $I_k = 0$ otherwise. Define the following three random variables:

$$X = I_1 + I_2 + I_3,$$
$$Y = \min\{I_1 + I_2 + I_3\},$$
$$Z = \max\{I_1 + I_2 + I_3\},$$

(a) Specify the range of values of the triplet (X,Y,Z) if each ball is put back into the urn after each draw; find the joint pmf for (X,Y,Z).

(b) In part a, are X, Y, and Z independent? Are X and Y independent?

(c) Repeat part a if each ball is not put back into the urn after each draw.

Solutions:

(a) The outcomes in any draw could be a black ball. Thus, $X \in \{0, 1, 2, 3\}, Y \in \{0, 1\}, Z \in \{0, 1\}$

I	X	Y	Z
000	0	0	0
001	1	0	1
010	1	0	1
100	1	0	1
011	2	0	1
101	2	0	1
110	2	0	1
111	3	1	1

$$P[X = 0] = 1/8, P[X = 1] = 3/8, P[X = 2] = 3/8, P[X = 3] = 1/8;$$
$$P[Y = 0] = 7/8, P[Y = 1] = 1/8;$$
$$P[Z = 0] = 1/8, P[Z = 1] = 7/8;$$

(b)

$$P[Y=1,Z=1] \neq P[Y=1]P[Z=1];$$
$$P[X=0,Y=0] \neq P[X=0]P[Y=0];$$
$$P[X=0,Z=0] \neq P[X=0]P[Z=0];$$

Hence, X and Y, X and Z, Y and Z are not inpendent.

(c) $X \in \{1\}, Y \in \{0\}, Z \in \{1\}$

I	X	Y	Z
001	1	0	1
010	1	0	1
100	1	0	1

It is easy to verify that $P[X,Y] = P[X]P[Y], P[Y,Z] = P[Y]P[Z], P[X,Z] = P[X]P[Z]$. Therefore, X, Y and Z are mutually independent.

11. (P 6.7) Let X,Y,Z have joint pdf

$$f_{X,Y,Z}(x,y,z) = k(x+y+z) \quad for \quad 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1.$$

(a) Find k.

(b) Find $f_X(x|y,z)$ and $f_Z(z|x,y)$.

(c) Find $f_X(x), f_Y(y), f_Z(z)$.

Solutions:

(a) The integral of the pdf should be 1. Thus

$$\int_0^1 \int_0^1 \int_0^1 k(x+y+z)dxdydz = k\frac{3}{2} = 1, k = 2/3.$$

(b)

$$f_{Y,Z}(y,z)\frac{2}{3}\int_0^1 (x+y+z)dx = \frac{2}{3}(\frac{1}{2}+y+z);$$
$$f_X(x|y,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)} = \frac{x+y+z}{1/2+y+z}.$$

Similarly, we have:

$$f_Z(z|x,y) = \frac{x+y+z}{x+y+1/2}.$$

(c)

$$f_X(x) = \frac{2}{3}\int_0^1 (x+y+1/2)dy = \frac{2}{3}(x+1).$$

Similarly, we have:

$$f_Y(y) = \frac{2}{3}(y+1), f_Z(z) = \frac{2}{3}(z+1).$$