Homework Solution Set No. 1

ECE 642
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Problem 1

Since $G_x[z] = \sum_{n=0}^{\infty} P_n z^n$
$\therefore G_x[z] = q + pz$ for Bernoulli distribution.
Since Binomial process is the summation for n Bernoulli processes
$\therefore G_x[z] = (q + pz)^n$ for Binomial distribution.
Since $dG_x[z]/dz|_{z=1} = E[X]$
$\therefore E[X] = n(q + pz)^{n-1}p|_{z=1} = n(1)^{n-1}p = np$
Since $d^2G_x[z]/dz^2|_{z=1} = E[x^2] - E[x]$
$\therefore E[x^2] = n(n-1)(q + pz)^{n-2}p^2|_{z=1} + E[x] = (n^2 - n)p^2 + np$
Since $\sigma_x^2 = E[x^2] - E^2[x]$
$\therefore \sigma_x^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$

Problem 2

Consider the time diagram shown:

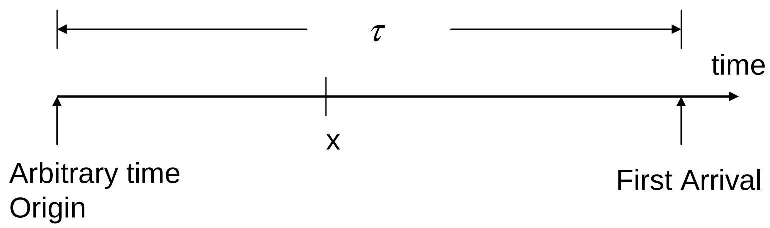

Figure 1: figure for problem 2

Let τ represent the time until the first arrival after some arbitrary time origin. Take any value x. No arrivals occur in the interval $(0, x)$ if and only if $\tau > x$.

The probability that no arrivals occur in $(0, x)$; i.e.,$P(\tau > x) = Prob.$(no arrivals in $(0, x)$).

For Poisson distribution $p(k) = (\lambda t)^k e^{-\lambda t}/k!$
here $k = 0$, $p(\tau > x) = e^{-\lambda x}$

Then the probability that $\tau \leq x$, i.e., $P(\tau \leq x) = 1 - e^{-\lambda x}$. But this is just the cumulative distribution $F_\tau(x)$ of the $r.v.\tau$ Hence we have $F_\tau(x) = 1 - e^{-\lambda x}$ from which the probability density distribution is found to be $f_\tau(x) = dF_\tau(x)/dx = \lambda e^{-\lambda x}$, which is exponential distribution.

Problem 3

For a *r.v.* z with Poisson distribution

$$P_z(n) = \sum_{k=0}^{n} P_X(k)P_Y(n-k) = \sum_{k=0}^{n} \frac{1}{k!}\frac{1}{(n-k)!}e^{-(\lambda_1+\lambda_2)}\lambda_1^k\lambda_2^{(n-k)} \tag{1}$$

Recall from binomial

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \lambda_1^k\lambda_2^{n-k} = (\lambda_1+\lambda_2)^n \tag{2}$$

then

$$P_z(n) = \frac{1}{n!}e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \lambda_1^k\lambda_2^{(n-k)} = \frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)} \tag{3}$$

where $n \geq 0$,which is Poisson with parameter $\lambda_1 + \lambda_2$. Thus, the sum of two independent r.v.s with Poisson distribution with parameter λ_1 and λ_2 is Poisson with parameter $\lambda_1 + \lambda_2$.

Problem 4

With $\varphi_x(\omega)$ denoting the characteristic function of X, and $\varphi_y(\omega)$ denoting the characteristic function of Y, we have $\varphi_z(\omega) = \varphi_x(\omega) * \varphi_y(\omega)$. However since the two r.v.s are independent and the characteristic function of a Gaussian r.v. is given as:

$$\varphi_x(\omega) = e^{\frac{-\omega^2-2j\omega\mu_1}{2\sigma_1}}, \tag{4}$$

$$\varphi_y(\omega) = e^{\frac{-\omega^2-2j\omega\mu_2}{2\sigma_2}}, \tag{5}$$

$$\therefore \varphi_z(\omega) = e^{\frac{-\omega^2(\sigma_1+\sigma_2)-2j\omega(\mu_1+\mu_2)}{2\sigma_1\sigma_2}}, \tag{6}$$

To obtain $f_z(z)$ we use the Fourier inverse $f_z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_z(\omega)e^{-j\omega z}d\omega$

$$f_z(z) = \frac{1}{\sqrt{2\pi(\sigma_1+\sigma_2)}}e^{\frac{-1}{2(\sigma_1+\sigma_2)}(z-(\mu_1+\mu_2))^2} \tag{7}$$

Hence $f_z(z)$ is indeed Gaussian.