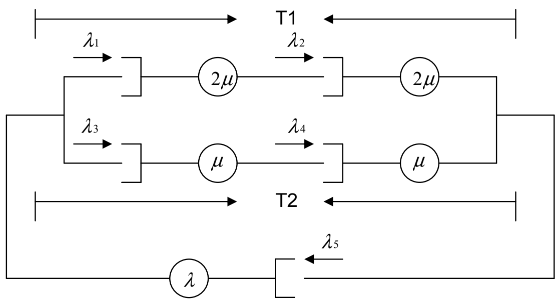
ECE 642 – Fall 2006
Excerpts from lecture 12

Part I Example 1



Given $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, and $q_{51} = q_{53} = 1/2$, we know that:

$$\rho_1 = \rho_2 = \frac{\lambda_1}{2\mu}$$

$$\rho_3 = \rho_4 = \frac{\lambda_3}{\mu} = \frac{\lambda_1}{\mu}$$

If $\rho_1 = \rho_2 = \frac{\lambda_1}{2\mu} = 1$, then:

$$\rho_1 = \rho_2 = \frac{\lambda_1}{\mu} = 2$$

$$\begin{aligned} \rho_5 &= \frac{\lambda_5}{\lambda} = \frac{2\lambda_1}{\lambda} \\ &= \frac{2\lambda_1}{\mu}\frac{\mu}{\lambda} = \frac{4}{\rho} \end{aligned}$$

By using Buzen's algorithm, we can get the following Table:

n/m	1	2	3	4	5
0	1	1	1	1	1
1	1	2	4	6	$6 + 4/\rho$
2	1	3	13	23	$23 + 4/\rho(6 + 4/\rho)$
3	1	4	26	72	$72 + 4/\rho(23 + 4/\rho(6 + 4/\rho))$
4	1	5	57	201	$201 + 4/\rho(72 + 4/\rho(23 + 4/\rho(6 + 4/\rho)))$

The throughput is calculated as follows:

$$\frac{\gamma_i}{\mu_i} = \rho_i \left[\frac{g(N-1, M)}{g(N, M)}\right]$$

The expected number of packets in the system is calculated as:

$$E(n_i) = \sum_{k=1}^{N} \rho_i^k \left[\frac{g(N-k, M)}{g(N, M))} \right]$$

The expected time delay is obtained as:

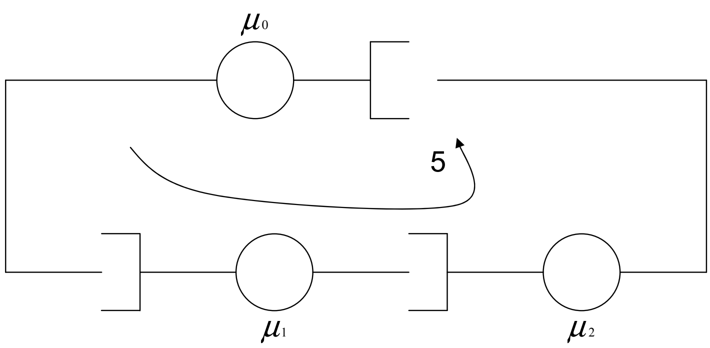
$$E(T_1) = \frac{E(n_1) + E(n_2)}{\gamma_1}$$

$$E(T_2) = \frac{E(n_3) + E(n_4)}{\gamma_3}$$

For $\rho = 1$, calculate the normalized throughput and normalized mean delay:

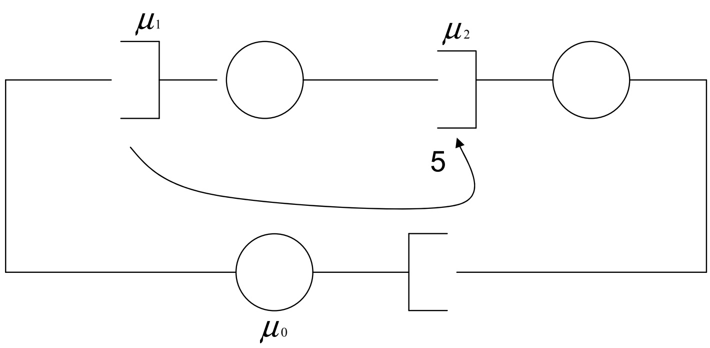
N	γ_1/μ	γ_3/μ	$\mu\overline{T_1}$	$\mu\overline{T_2}$
1	0.2	0.2	1	2
2	0.317	0.317	1.1	2.4
3	0.39	0.39	1.18	2.76
4	0.433	0.433	1.23	3.07

Part II Example 2



$\mu_1 = \mu_2 = 2$ and $\mu_0 = 1$. Calculate the percentage of time that bottom line is full of 5 packets?

- Method 1: Apply the Buzen's algorithm since the servers have different average service times.
- Method 2: Flip the graph and apply the Norton's algorithm.



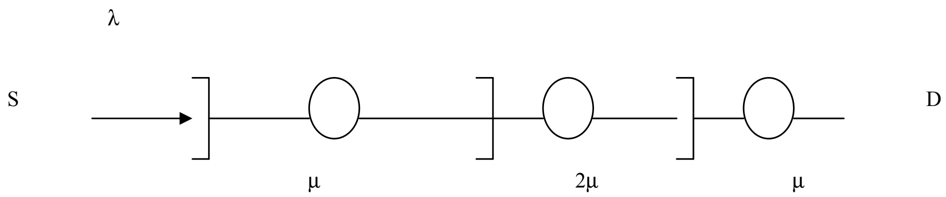
$$p_0 = 0.05$$

$$p_1 = 0.083$$

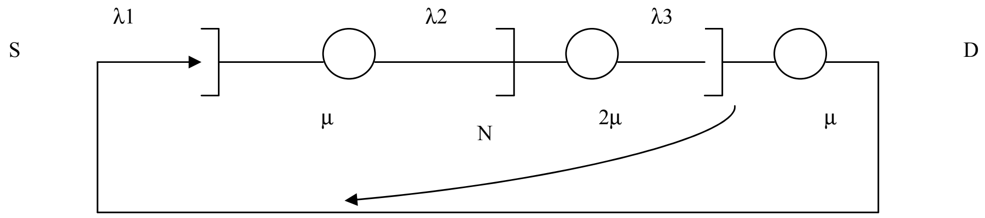
$$p_2 = 0.133$$

$$p_4 = p_5 = 0.266$$

Part III Example 3



For the case $\lambda \gg 2\mu$ the equivalent circuit becomes Using Buzen's Method:



$$\rho_1 = \frac{\lambda_1}{\mu} = \rho_3$$

$$\rho_2 = \frac{\rho_1}{2}$$

Let $\rho_2 = 1$, we get $\rho_1 = \rho_3 = 2$, the matrix obtained by using the Buzen's algorithm is shown below:

n/m	1	2	3
0	1	1	1
1	2	3	5
2	4	7	17
3	8	15	49
4	16	31	129
5	32	63	321
6	64	127	769

For throughput, time delay calculations:

$$E(n_i) = \sum_{k=1}^{N} \rho_i^k \left[\frac{g(N-k, M)}{g(N, m)}\right]$$

$$\gamma_i = \lambda_i \left[\frac{g(N-1, M)}{g(N, M)}\right]$$

$$E(t_i) = \frac{E(n_i)}{\gamma_i}$$

Using the above formulas we have:
$N = 2, \frac{\gamma}{\mu} = 2\frac{g(1,3)}{g(2,3)} = 2\frac{5}{17} = 0.588$
$N = 3, \frac{\gamma}{\mu} = 2\frac{g(2,3)}{g(3,3)} = 2\frac{17}{49} = 0.693$
$N = 4, \frac{\gamma}{\mu} = 2\frac{g(3,3)}{g(4,3)} = 2\frac{49}{129} = 0.759$
$N = 5, \frac{\gamma}{\mu} = 2\frac{g(4,3)}{g(5,3)} = 2\frac{129}{321} = 0.803$
$N = 6, \frac{\gamma}{\mu} = 2\frac{g(5,3)}{g(6,3)} = 2\frac{321}{769} = 0.834$

$E[n] = E[n_1] + E[n_2] + E[n_3]$, and $E[n_i] = \sum_{k=1}^{N} \rho_i^k \frac{g(N-k,M)}{g(N,M)}$.

For $N = 1$:
$E[n] = E[n_3] = 2\frac{g(0,3)}{g(1,3)} = 2.\frac{1}{5} = \frac{2}{5}$, $E[n_2] = \frac{1}{2}E[n_1] = \frac{1}{5}$, $E[n] = E[n_1] + E[n_2] + E[n_3] = 1$ and $\frac{E[n]}{\gamma/\mu} = \frac{1}{0.4} = 2.5$.

For $N = 2$:
$E[n_1] = E[n_3] = 2.\frac{g(1,3)}{g(2,3)} + 2^2\frac{g(0,3)}{g(2,3)} = \frac{14}{17}$, $E[n_2] = 1.\frac{5}{17} + 1.\frac{1}{17} = \frac{6}{17}$, $E[n] = E[n_1] + E[n_2] + E[n_3] = 2$ and $\frac{E[n]}{\gamma/\mu} = \frac{2}{0.58} = 3.4$.

For $N = 3$:
$E[n_1] = E[n_3] = 2.\frac{g(2,3)}{g(3,3)} + 2^2\frac{g(1,3)}{g(3,3)} + 2^3\frac{g(0,3)}{g(3,3)} = \frac{62}{49}$, $E[n_2] = 1.\frac{17}{49} + 1.\frac{5}{49} + 8.\frac{1}{49} = \frac{23}{49}$, $E[n] = E[n_1] + E[n_2] + E[n_3] = 3$, and $\frac{E[n]}{\gamma/\mu} = \frac{3}{0.693} = 4.32$.

For $N = 4$:
$E[n_1] = E[n_3] = 2.\frac{g(3,3)}{g(4,3)} + 2^2\frac{g(2,3)}{g(4,3)} + 2^3\frac{g(1,3)}{g(4,3)} + 2^4\frac{g(0,3)}{g(4,3)} = \frac{222}{129}$, $E[n_2] = \frac{72}{129}$, $E[n] = E[n_1] + E[n_2] + E[n_3] = 3$,$E[n] = E[n_1] + E[n_2] + E[n_3] = 4$, and $\frac{E[n]}{\gamma/\mu} = \frac{4}{0.759} = 5.27$.

For $N = 5$:
$E[n_1] = E[n_3] = 2.\frac{g(4,3)}{g(5,3)} + 2^2\frac{g(3,3)}{g(5,3)} + 2^3\frac{g(2,3)}{g(5,3)} + 2^4\frac{g(4,3)}{g(5,3)} + 2^5\frac{g(0,3)}{g(6,3)} = \frac{702}{321}$, $E[n_2] = \frac{201}{321}$, $E[n] = E[n_1] + E[n_2] + E[n_3] = 5$, and $\frac{E[n]}{\gamma/\mu} = \frac{5}{0.803} = 6.22$.

For $N = 6$:
$E[n_1] = E[n_3] = 2.\frac{g(5,3)}{g(6,3)} + 2^2\frac{g(4,3)}{g(6,3)} + 2^3\frac{g(3,3)}{g(6,3)} + 2^4\frac{g(2,3)}{g(6,3)} + 2^5\frac{g(1,3)}{g(6,3)} + 2^6\frac{g(0,3)}{g(6,3)} = \frac{2046}{769}$, $E[n_2] = \frac{522}{769}$, $E[n] = E[n_1] + E[n_2] + E[n_3] = 6$, $\frac{E[n]}{\gamma/\mu} = \frac{6}{0.834} = 7.19$.

N	γ/μ	$\mu E(T)$
1	0.4	2.5
2	0.58	3.4
3	0.694	4.32
4	0.76	5.265
5	0.80	6.22
6	0.83	7.19