ECE 528 – Introduction to Random Processes in ECE
Lecture 6: Continuous Random Variables and Functions of a Random Variable

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Note

- These slides cover material partially presented in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-set derivation and detailed discussions were need.

Outline

- Important Continuous Random Variables
- Functions of a Random Variable
- The Markov and Chebyshev Inequalities
- Transform Methods
- Computer Methods for Generating Random Variables

Characteristic Function

- The **characteristic function** of *X*, when *X* is continuous, is defined by $\Phi_X(\omega) = E[e^{j\omega X}]$
$$= \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \quad \text{where } j = \sqrt{-1}$$
- $\Phi_X(\omega)$ is the Fourier transform of the pdf $f_X(x)$ (with reversal of sign in the exponent).
- With the Fourier transform inversion formula we get

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} d\omega$$

The Characteristic Function (2)

- **Properties:**

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega) \bigg|_{\omega=0}$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) e^{-j\omega x} d\omega$$

The Characteristic Function (3)

- For discrete random variables,

$$\Phi_x(\omega) = E[e^{j\omega X}]$$
$$= \sum_{\forall k} p_X(x_k) e^{j\omega x_k}$$

- For integer valued random variables,

$$\Phi_x(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

Note: $p_X(k)$ = Probability mass function of the random variable X when $(X = k)$ = $P(X = k)$

The Characteristic Function (4)

- Properties

$$p_X(k) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_x(\omega) e^{-j\omega k} d\omega$$

for k=0, ±1, ±2, …

Example: Characteristic Function of an Exponentially Distributed RV

Example 4.41 Exponential Random Variable

The characteristic function for an exponentially distributed random variable with parameter λ is given by

$$\Phi_X(\omega) = \int_0^{\infty} \lambda e^{-\lambda x} e^{j\omega x}\, dx = \int_0^{\infty} \lambda e^{-(\lambda - j\omega)x}\, dx$$

$$= \frac{\lambda}{\lambda - j\omega}.$$

Characteristic Function of Integer RVs

If X is an integer-valued random variable:

$$\Phi_X(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

This Fourier series is periodic since

$$e^{j(\omega+2\pi)k} = e^{j\omega k} e^{jk2\pi} \text{ and } e^{jk2\pi} = 1$$

The Fourier transform inversion formula gives:

$$p_X(k) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_X(\omega) e^{-j\omega k} d\omega \quad k = 0, \pm 1, \pm 2, \ldots$$

Moment Theorem

Moment theorem:

$$E[X^n] = \frac{1}{j^n}\frac{d^n}{d\omega^n}\Phi_X(\omega)\bigg|_{\omega=0}$$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\left\{1 + j\omega X + \frac{(j\omega X)^2}{2!} + \ldots\right\} dx$$

$$\Phi_X(\omega) = 1 + j\omega E[X] + \frac{(j\omega)^2 E[X^2]}{2!} + \ldots + \frac{(j\omega)^n E[X^n]}{n!} + \ldots$$

Mean and Variance of Exponential RV

Example 4.43

To find the mean of an exponentially distributed random variable, we differentiate $\Phi_X(\omega) = \lambda(\lambda - j\omega)^{-1}$ once, and obtain

$$\Phi'_X(\omega) = \frac{\lambda j}{(\lambda - j\omega)^2}.$$

The moment theorem then implies that $E[X] = \Phi'_X(0)/j = 1/\lambda$.

If we take two derivatives, we obtain

$$\Phi''_X(\omega) = \frac{-2\lambda}{(\lambda - j\omega)^3},$$

so the second moment is then $E[X^2] = \Phi''_X(0)/j^2 = 2/\lambda^2$. The variance of X is then given by

$$\text{VAR}[X] = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Example: Moments of the Gaussian RV

Find the mean and variance of a zero-mean Gaussian random variable *X*.

Example 4.43

To find the mean of an exponentially distributed random variable, we differentiate $\Phi_X(\omega) = \lambda(\lambda - j\omega)^{-1}$ once, and obtain

$$\Phi'_X(\omega) = \frac{\lambda j}{(\lambda - j\omega)^2}.$$

The moment theorem then implies that $E[X] = \Phi'_X(0)/j = 1/\lambda$.

If we take two derivatives, we obtain

$$\Phi''_X(\omega) = \frac{-2\lambda}{(\lambda - j\omega)^3},$$

so the second moment is then $E[X^2] = \Phi''_X(0)/j^2 = 2/\lambda^2$. The variance of X is then given by

$$\text{VAR}[X] = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Uniform Random Variables

- Realizations of the R.V. can take values from the interval [a, b]
- pdf $f_X(x) = 1/(b-a) \qquad a \leq x \leq b$
- $E[X] = (a+b)/2, \quad Var[X] = (b-a)^2/12$
- $\Phi_X(\omega) = [e^{j\omega b} - e^{j\omega a}]/(j\omega(b-a))$

Probability Generating Function

- A matter of convenience – compact representation
- The same as the z-transform
- If N is a non-negative integer-valued random variable, the probability generating function is defined as

$$G_N(z) = E[z^N]$$

$$= \sum_{k=0}^{\infty} p_N(k) z^k$$

$$= p_N(0) + p_N(1)z + p_N(2)z^2 + \ldots.$$

<u>Note:</u> $p_N(k)$ = Probability mass function of the random variable N when $(N = k) = P(N = k)$

Probability Generating Function (2)

- **Properties:**
- **1**

$$p_N(k) = \frac{1}{k!}\frac{d^k}{dz^k}G_N(z)\Big|_{z=0}$$

- **2**

$$E[N] = G'_N(1)$$

- **3**

$$Var[N] = G''_N(1) + G'_N(1) - \left[G'_N(1)\right]^2$$

Probability Generating Function (3)

- For non-negative continuous random variables, let us define the Laplace transform* of the PDF

$$X^*(s) = \int_0^\infty f_X(x) e^{-sx} dx$$
$$= E[e^{-sx}]$$

Properties:

$$E[X^n] = (-1)^n \frac{d^n}{ds^n} X^*(s) \Big|_{s=0}$$

* Useful in dealing with queueing theory (i.e. service time, waiting time, delay, …)

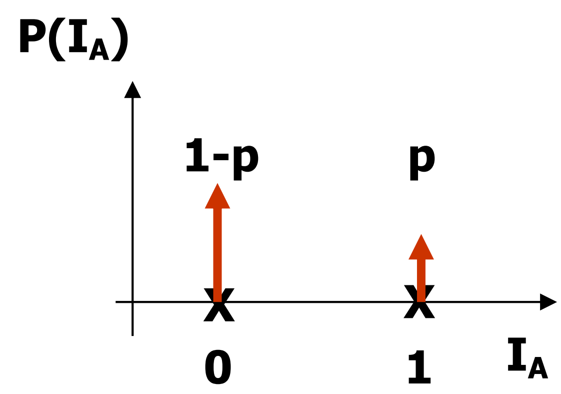
Some Important Random Variables – Discrete RVs

- Bernoulli
- Binomial
- Geometric
- Poisson

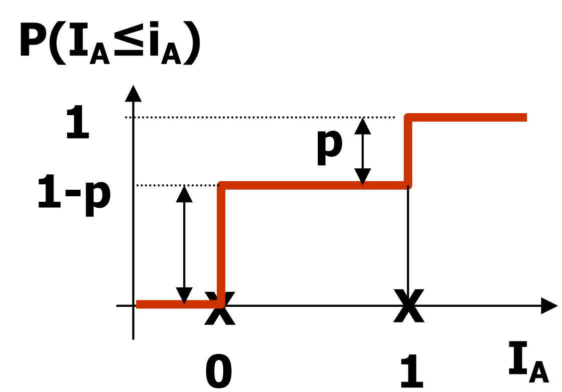
Bernoulli Random Variable

- Let A be an event related to the outcomes of some random experiment. The indicator function for A is defined as:

$I_A(\zeta) = 0$ if ζ not in A (i.e. if A doesn't occur)
$= 1$ if ζ is in A (i.e. if A occurs)



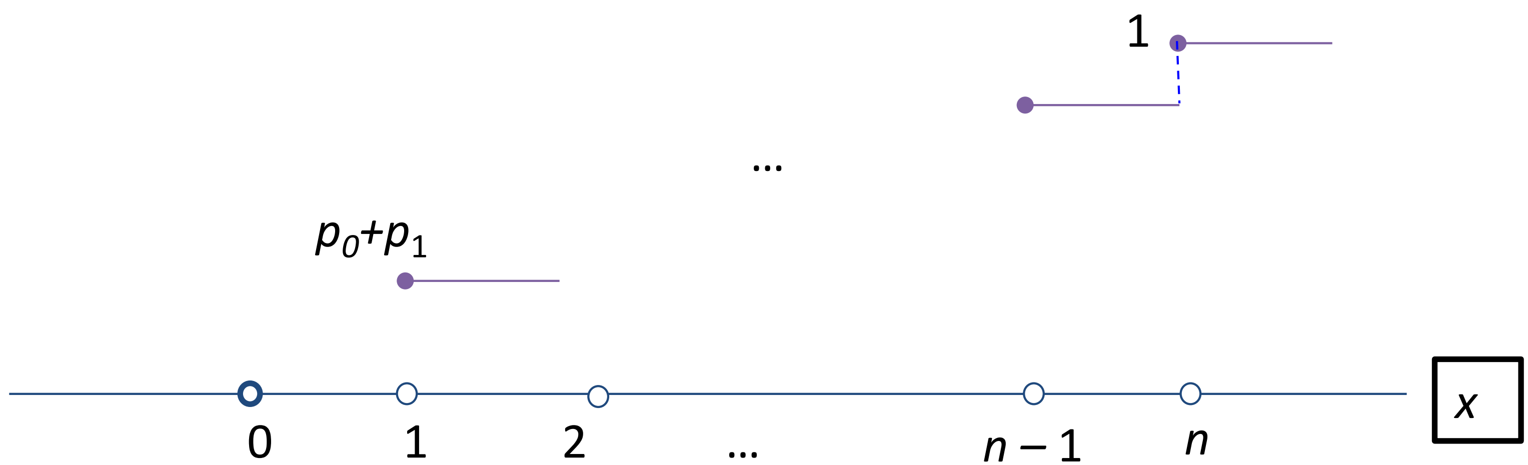
- I_A is random variable since it assigns a number $p_I(0) = 1-p$, $p_I(1) = p$ where
- $P\{A\} = p$ describes the outcome of a Bernoulli trial
- Note: $p_I(0) + p_I(1) = 1$
- $E[X] = p$, $VAR[X] = p(1-p)$
- $G_X(z) = (1-p+pz)$



Binomial Random Variable

- The Binomial random variable is an example of a **discrete random variable**, where the CDF is a staircase function of *x*.
- The discontinuities in the CDF are given by the probability mass function.

$$F_X(x) = P[\xi : X(\xi) \leq x] = P[\xi : \#\text{ heads} \leq x]$$



Binomial Random Variable

- Suppose a random experiment is repeated n independent times; let X be the number of times a certain event A occurs in these n trials

$$X = I_1 + I_2 + \ldots + I_n$$

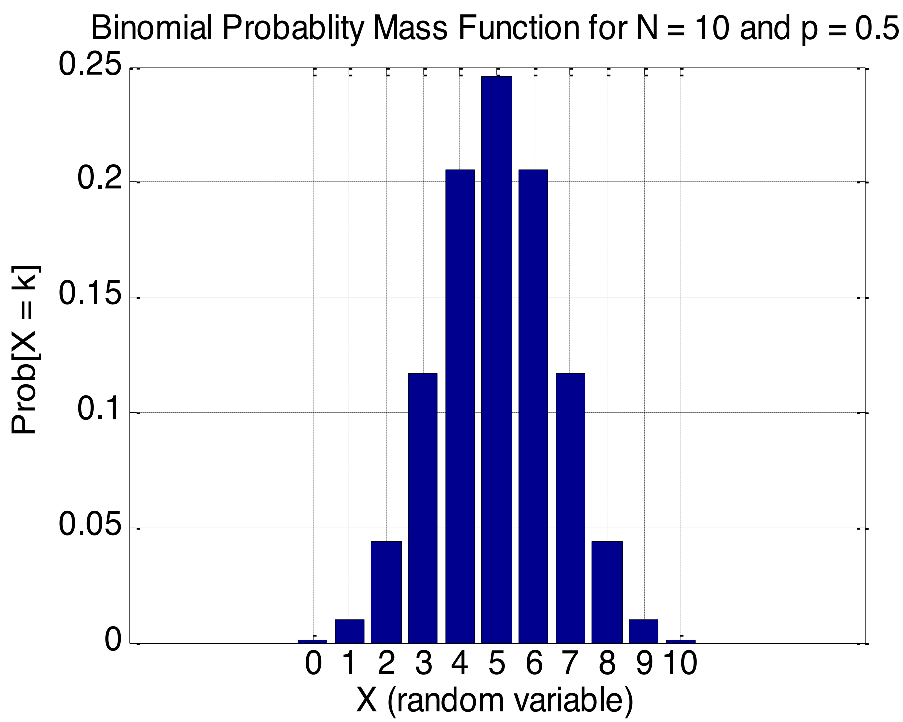
i.e., X is the sum of Bernoulli trials (X's range = {0, 1, 2, …, n})

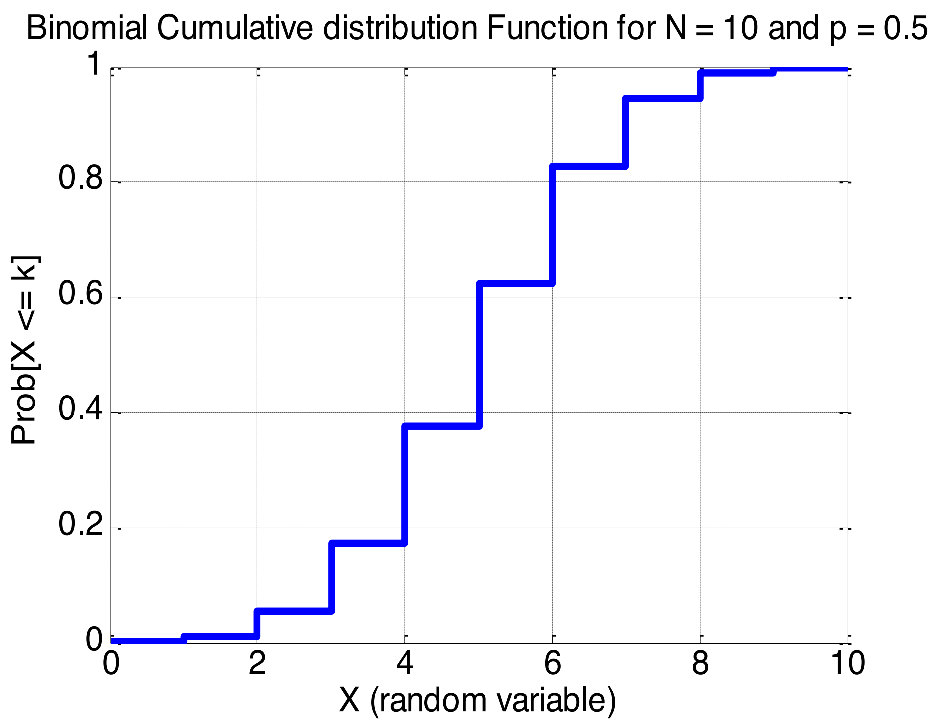
- X has the following pmf for k=0, 1, 2, …, n

$$\Pr[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}$$

- $E[X] = np$, $Var[X] = np(1-p)$
- $G_X(z) = (1-p + pz)^n$

Binomial Random Variable





Geometric Random Variable

- Suppose a random experiment is repeated - We count the number of M of independent Bernoulli trials UNTIL the first occurrence of a success
- M is called geometric random variable
  - Range of M = 1, 2, 3, …
- M has the following pmf

$$\Pr[M = k] = (1-p)^{k-1} p \quad \text{for } k=1, 2, 3, \ldots$$

- E[X] = 1/p, Var[X] = (1-p)/p^2
- $G_X(z) = pz/(1-(1-p)z)$

Geometric Random Variable (2)

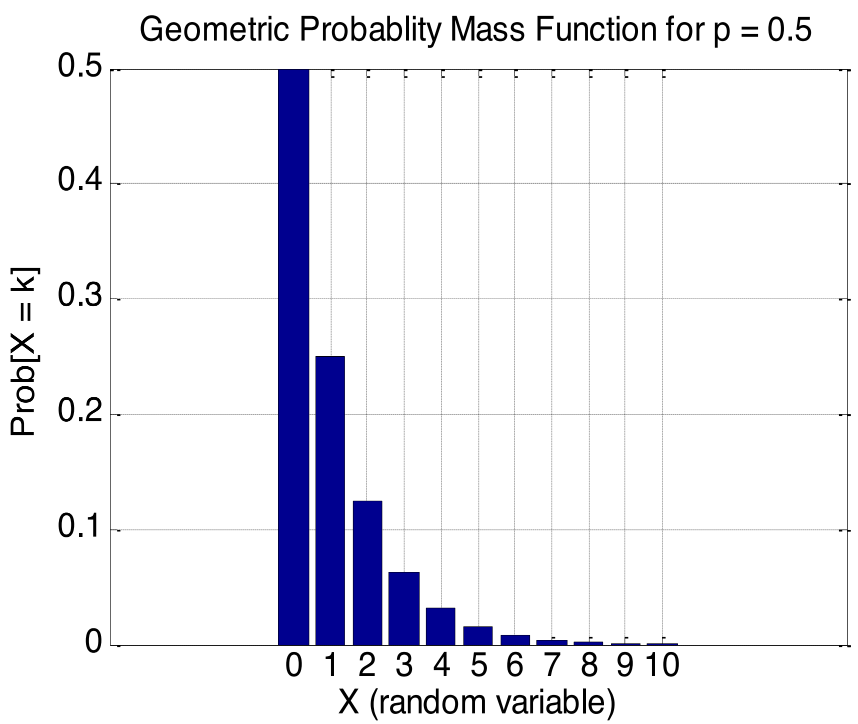
- Suppose a random experiment is repeated - We count the number of M' of independent Bernoulli trials BEFORE the first occurrence of a success
- M' is called geometric random variable
  - Range of M' = 0, 1, 2, 3, …
- M has the following pmf

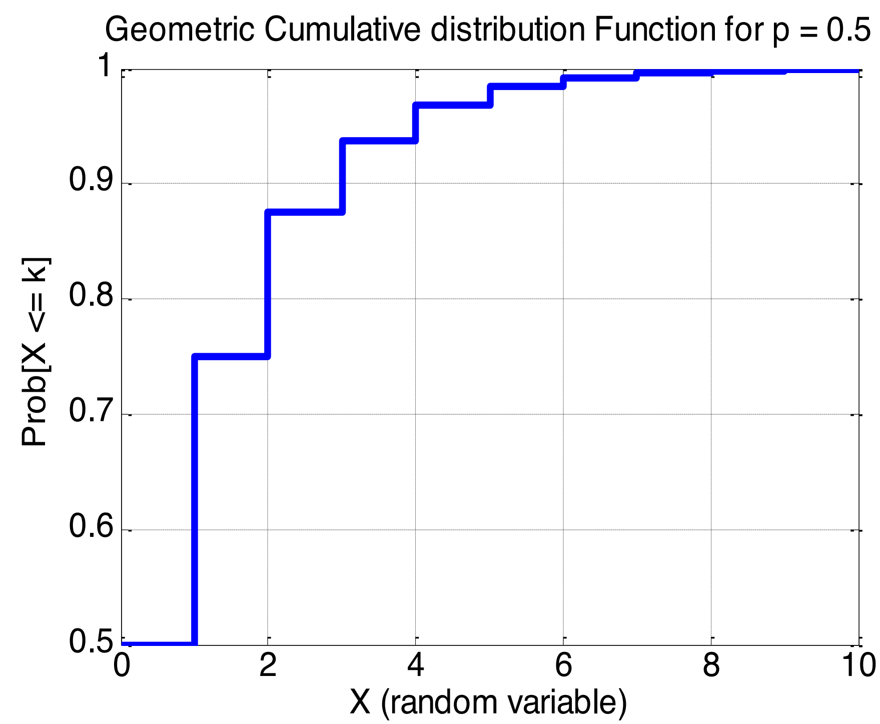
$$\Pr[M = k] = (1-p)^k p \qquad \text{for } k\text{=0, 1, 2, 3, …}$$

- $E[X] = (1-p)/p$, $\quad Var[X] = (1-p)/p^2$
- $G_X(z) = pz/(1-(1-p)z))$

Geometric Random Variable – cont'd

- Example: p = 0.5; X is number of failures BEFORE a success (2nd type)
- Note Matlab's version of geometric distribution is the 2nd type





Poisson Random Variable

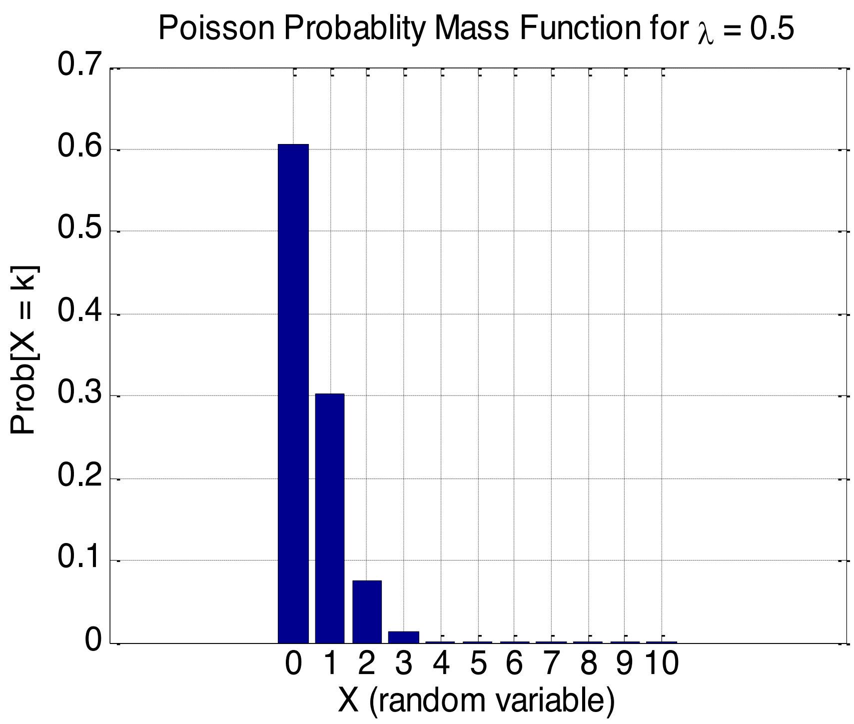
- In many applications we are interested in counting the number of occurrences of an event in a certain time period
- The pmf is given by $$\Pr[X = k] = \frac{\alpha^k}{k!} e^{-\alpha}$$

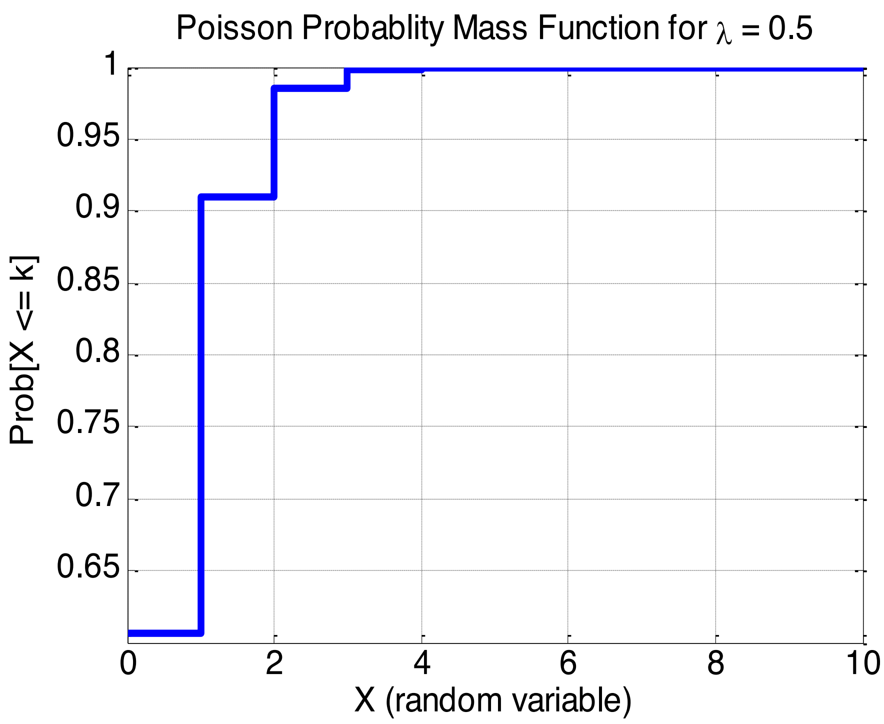
For k=0, 1, 2, … ; α is the average number of event occurrences in the specified interval

- $E[X] = \alpha$, $Var[X] = \alpha$
- $G_X(z) = e^{\alpha(z-1)}$
- Remember: time between events is exponentially distributed! (continuous r.v. !)
- Poisson is the limiting case for Binomial as $n \rightarrow \infty$, $p \rightarrow 0$, such that $np = \alpha$

Poisson Random Variable (cont'd)

- **Example:**





MATLAB Code to Plot Distributions

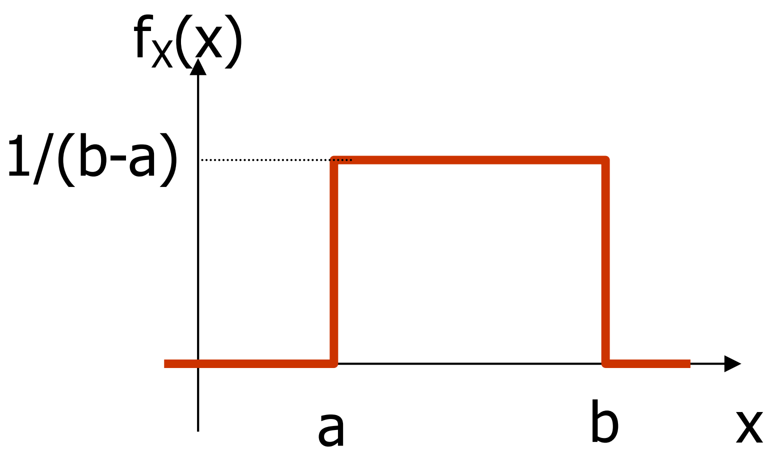
```matlab
% plot distributions
% see "help stats"
clear all
FontSize = 14;
LineWidth = 3;
% Binomial
N = 10; X = [0:1:N]; P = 0.5;
ybp = binopdf(X, N, P); % get PMF
ybc = binocdf(X, N, P); % get CDF
figure(1); set(gca,'FontSize', FontSize);
bar(X, ybp);
title(['Binomial Probablity Mass Function for N = ' ...
    num2str(N) ' and p = ' num2str(P)]);
xlabel('X (random variable)');
ylabel('Prob[X = k]'); grid
figure(2); set(gca,'FontSize', FontSize);
stairs(X, ybc,'LineWidth', LineWidth);
title(['Binomial Cumulative distribution Function for N = ' ...
    num2str(N) ' and p = ' num2str(P)]);
xlabel('X (random variable)');
ylabel('Prob[X <= k]'); grid
% Geometric
P = 0.5; X = [0:10];
ygp = geopdf(X, P); % get pdf
ygc = geocdf(X, P); % get cdf
figure(3); set(gca,'FontSize', FontSize);
bar(X, ygp);
title(['Geometric Probablity Mass Function for p = ' num2str(P)]);
xlabel('X (random variable)');
ylabel('Prob[X = k]'); grid
figure(4); set(gca,'FontSize', FontSize);
stairs(X, ygc,'LineWidth', LineWidth);
title(['Geometric Cumulative distribution Function for p = ' num2str(P)]);
xlabel('X (random variable)');
ylabel('Prob[X <= k]'); grid
% Poisson
Lambda = 0.5; X      = [0:10];
ypp    = poisspdf(X, Lambda);
ypc    = poisscdf(X, Lambda);
figure(5); set(gca,'FontSize', FontSize);
bar(X, ypp);
title(['Poisson Probablity Mass Function for \lambda = ' num2str(Lambda)]);
xlabel('X (random variable)');
ylabel('Prob[X = k]'); grid
figure(6); set(gca,'FontSize', FontSize);
stairs(X, ypc,'LineWidth', LineWidth);
title(['Poisson Probablity Mass Function for \lambda = ' num2str(Lambda)]);
xlabel('X (random variable)');
ylabel('Prob[X <= k]'); grid
```

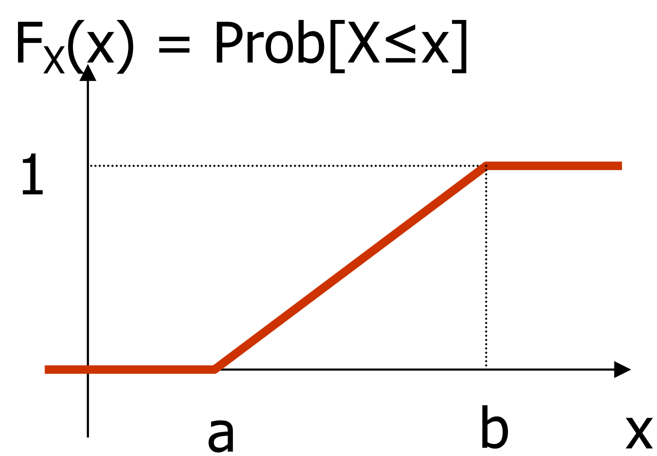
Some Important Random Variables – Continuous RVs

- Uniform
- Exponential
- Gaussian (Normal)
- Rayleigh
- Gamma
- M-Erlang

Uniform Random Variable

- A **continuous Uniform** RV for the interval [a, b] is defined as
  Pdf: $f_X(x) = 1/(b-a) \qquad a \le x \le b$
- Mean and variance are: $E[X] = (a+b)/2$, $\quad Var[X] = (b-a)^2/12$
- $\boldsymbol{\Phi_X(\omega) = [e^{j\omega b} - e^{j\omega a}]/(j\omega(b-a))}$





Exponential Random Variables

- The exponential RV X with parameter λ has pdf

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

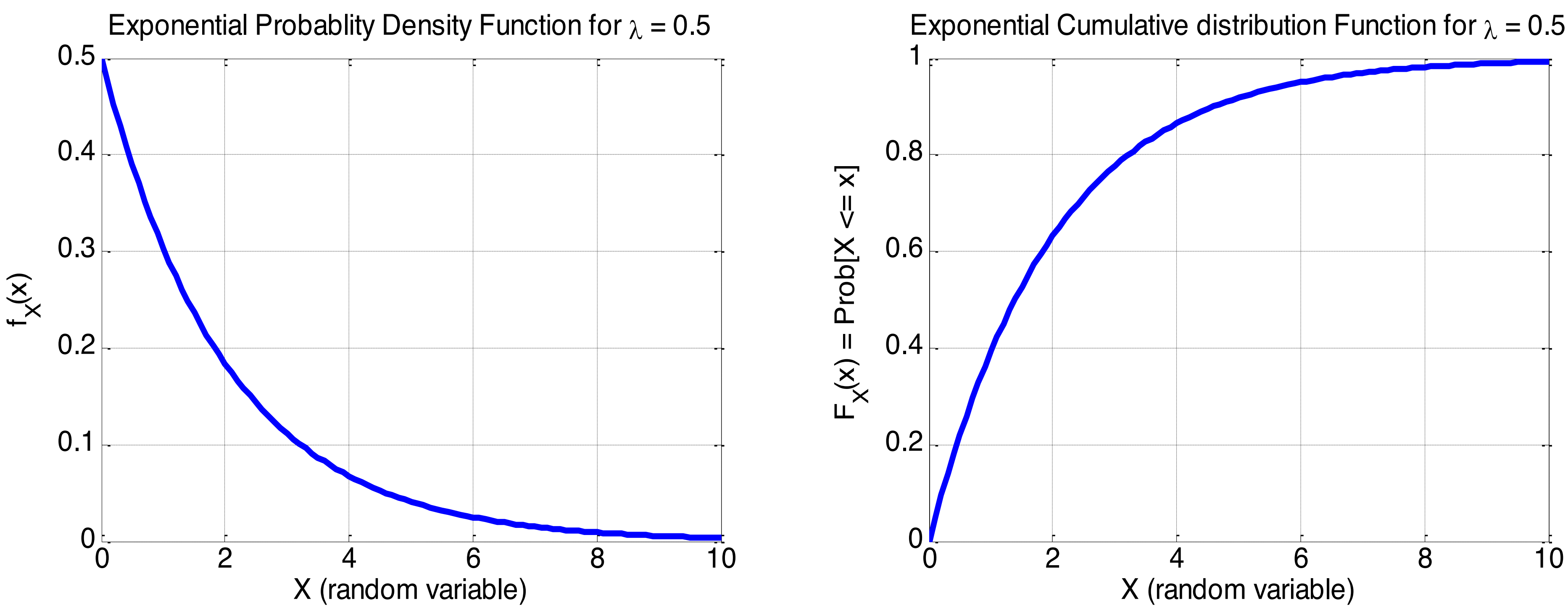
- And CDF given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

- i.e., $\mathrm{Prob}[X \leq x] = 1 - e^{-\lambda x}$, or $\mathrm{Prob}[X > x] = e^{-\lambda x}$
- Range of X: $[0, \infty)$
- $E[X] = 1/\lambda$, $\mathrm{Var}[X] = 1/\lambda^2$
- $\Phi_X(w) = \lambda/(\lambda - jw)$

Exponential Random Variables (2)

- **Example:**
  - **Note the mean is $1/\lambda = 2$**



Gaussian (Normal) Random Variable

- For situations where a random variable X is the sum of a large number of "small" random variables – central limit theorem

- pdf $$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

For $-\infty < x < \infty$; μ and $\sigma > 0$ are real numbers

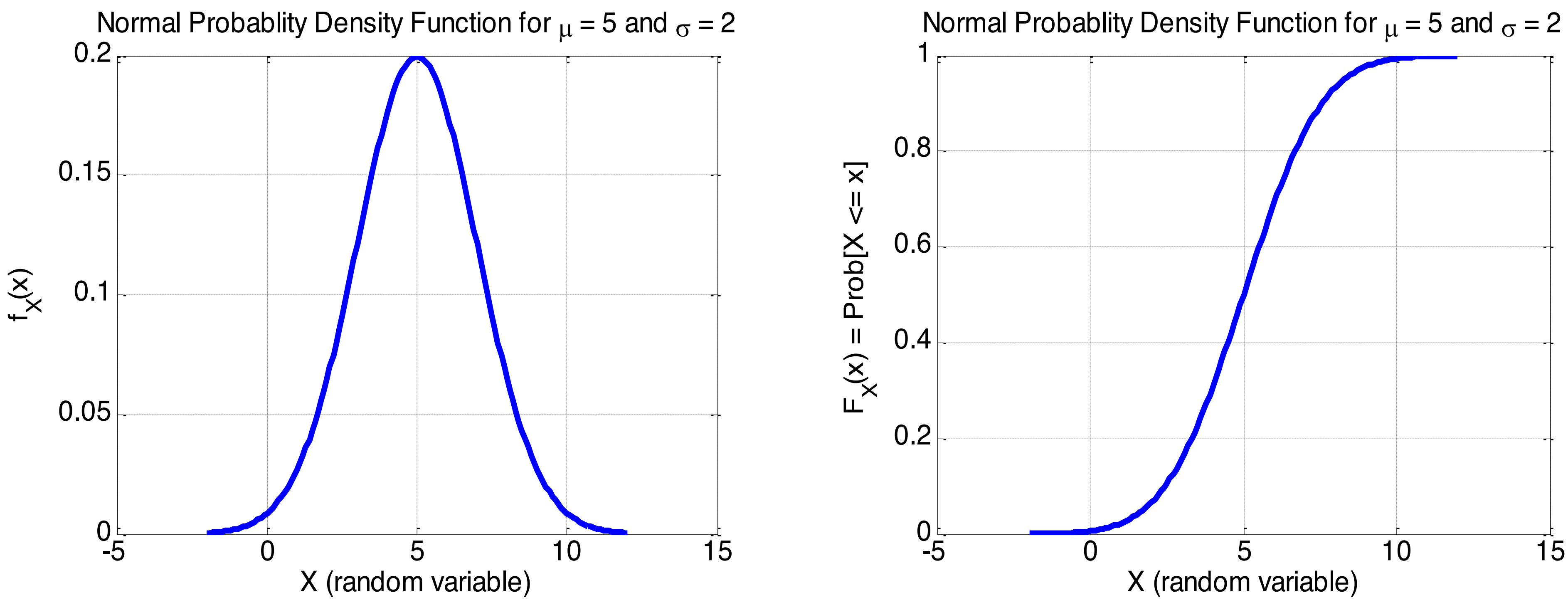
- $E[X] = \mu$, $Var[X] = \sigma^2$

$$\Phi_X(\omega) = e^{j\mu\omega - \sigma^2\omega^2/2}$$

- Under wide range of conditions *X* can be used to approximate the sum of a large number of independent random variables

Gaussian (Normal) Random Variable (2)

- **Example:**

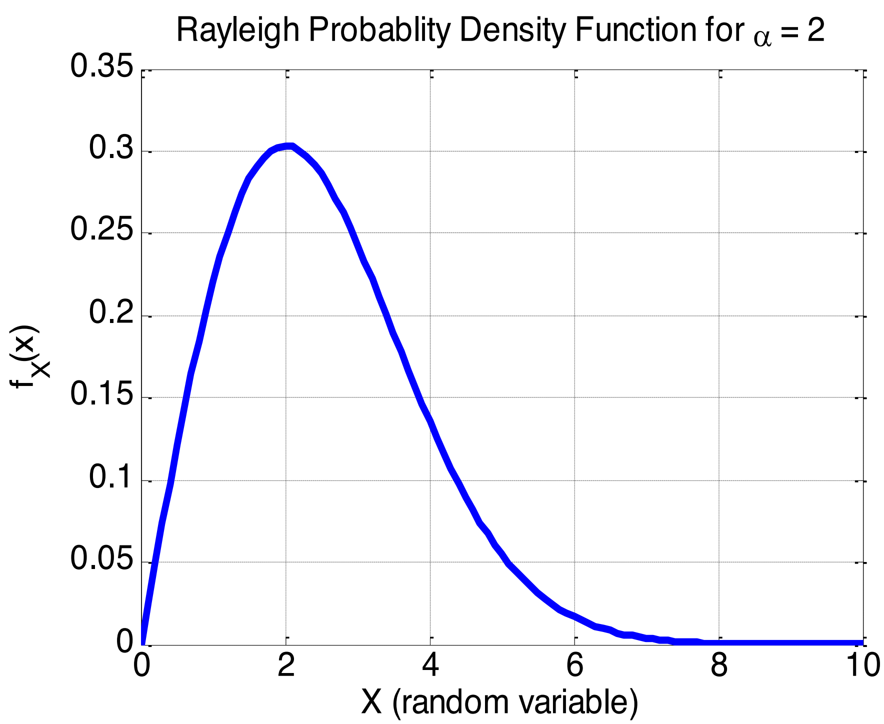


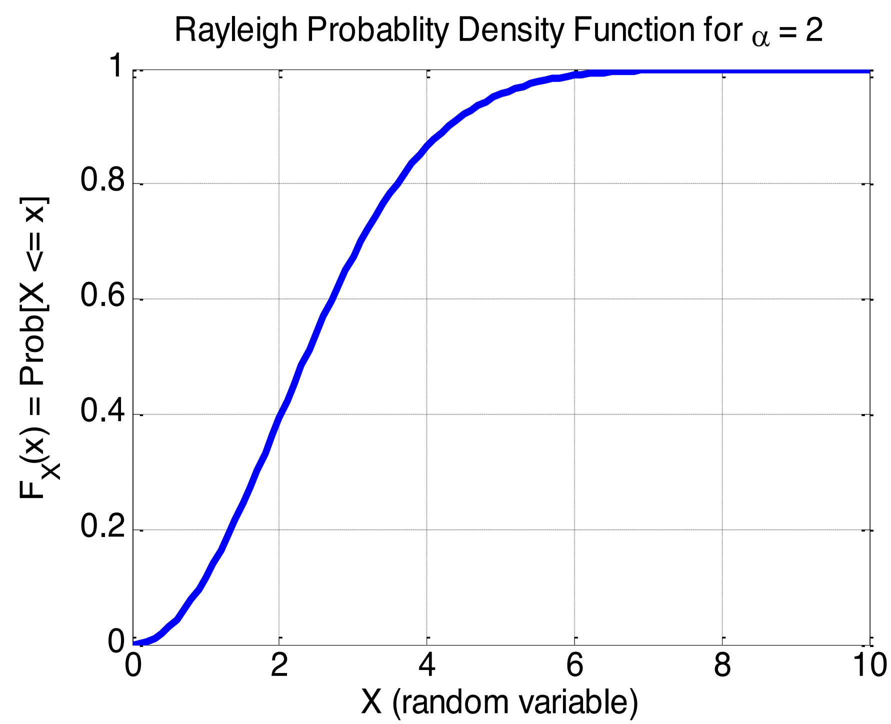
Rayleigh Random Variable

- Arises in modeling of mobile channels
- Range: $[0, \infty)$
- pdf: $$f_X(x) = \frac{x}{\alpha^2} e^{-x^2/(2\alpha^2)}$$
- For $x \geq 0$, $\alpha > 0$
- $E[X] = \alpha\sqrt{(\pi/2)}$, $Var[X] = (2-\pi/2)\alpha^2$

Rayleigh Random Variable (2)

- **Example:**
  - **Note that for Alpha = 2, the mean is 2√(π/2)**





Matlab Code to Plot Distributions

```matlab
% plot distributions
% see "help stats"
clear all
FontSize = 14;
LineWidth = 3;
% exponential
X = [0:.1:10]; Lambda = 0.5;
yep = exppdf(X, 1/Lambda); % get PDF
yec = expcdf(X, 1/Lambda); % get CDF
figure(1); set(gca,'FontSize', FontSize);
plot(X, yep, 'LineWidth', LineWidth);
title(['Exponential Probablity Density Function for \lambda = ' ...
    num2str(Lambda)]);
xlabel('X (random variable)');
ylabel('f_X(x)'); grid
figure(2); set(gca,'FontSize', FontSize);
plot(X, yec,'LineWidth', LineWidth);
title(['Exponential Cumulative Distribution Function for \lambda = ' ...
    num2str(Lambda)]);
xlabel('X (random variable)');
ylabel('F_X(x) = Prob[X <= x]'); grid
% normal
X = [-2:.1:12]; Mu = 5; Sigma = 2;
ynp = normpdf(X, Mu, Sigma); % get PDF
ync = normcdf(X, Mu, Sigma); % get CDF
figure(3); set(gca,'FontSize', FontSize);
plot(X, ynp, 'LineWidth', LineWidth);
title(['Normal Probablity Density Function for \mu = ' ...
    num2str(Mu) ' and \sigma = ' num2str(Sigma)]);
xlabel('X (random variable)');
ylabel('f_X(x)'); grid
figure(4); set(gca,'FontSize', FontSize);
plot(X, ync,'LineWidth', LineWidth);
title(['Normal Probablity Density Function for \mu = ' ...
    num2str(Mu) ' and \sigma = ' num2str(Sigma)]);
xlabel('X (random variable)');
ylabel('F_X(x) = Prob[X <= x]'); grid
% Rayleigh
X = [0:.1:10]; Alpha = 2;
yrp = raylpdf(X, Alpha); % get PDF
yrc = raylcdf(X, Alpha); % get CDF
figure(5); set(gca,'FontSize', FontSize);
plot(X, yrp, 'LineWidth', LineWidth);
title(['Rayleigh Probablity Density Function for \alpha = ' ...
    num2str(Alpha)]);
xlabel('X (random variable)');
ylabel('f_X(x)'); grid
figure(6); set(gca,'FontSize', FontSize);
plot(X, yrc,'LineWidth', LineWidth);
title(['Rayleigh Probablity Density Function for \alpha = ' ...
    num2str(Alpha)]);
xlabel('X (random variable)');
ylabel('F_X(x) = Prob[X <= x]'); grid
```

Gamma Random Variable

- **Versatile distribution ~ appears in modeling of lifetime of devices and systems**
- **Has two parameters: $\alpha > 0$ and $\lambda > 0$**

- **PDF:**

$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

- **For $0 < x < \infty$**
- **The quantity $\Gamma(z)$ is the gamma function and is specified by**

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

- **The gamma function has the following properties:**
  - **$\Gamma(1/2) = \sqrt{\pi}$**
  - **$\Gamma(z+1) = z\Gamma(z)$ for $z>0$**
  - **$\Gamma(m+1) = m!$ For m nonnegative integer**
- **$E[X] = \alpha/\lambda$, $Var[X] = \alpha/\lambda^2$**
- **$\Phi_X(\omega) = 1/(1-j\omega/\lambda)^\alpha$**

If $\alpha = 1$ ➔ gamma r.v. becomes exponential

Functions of a Random Variable

- Very often we are interested in a function of a random variable: $Y = g(X)$
- Since Y = g(X(ξ)), Y is also a random variable
- Examples:

$$Y = aX + b$$

$$Y = X^2$$

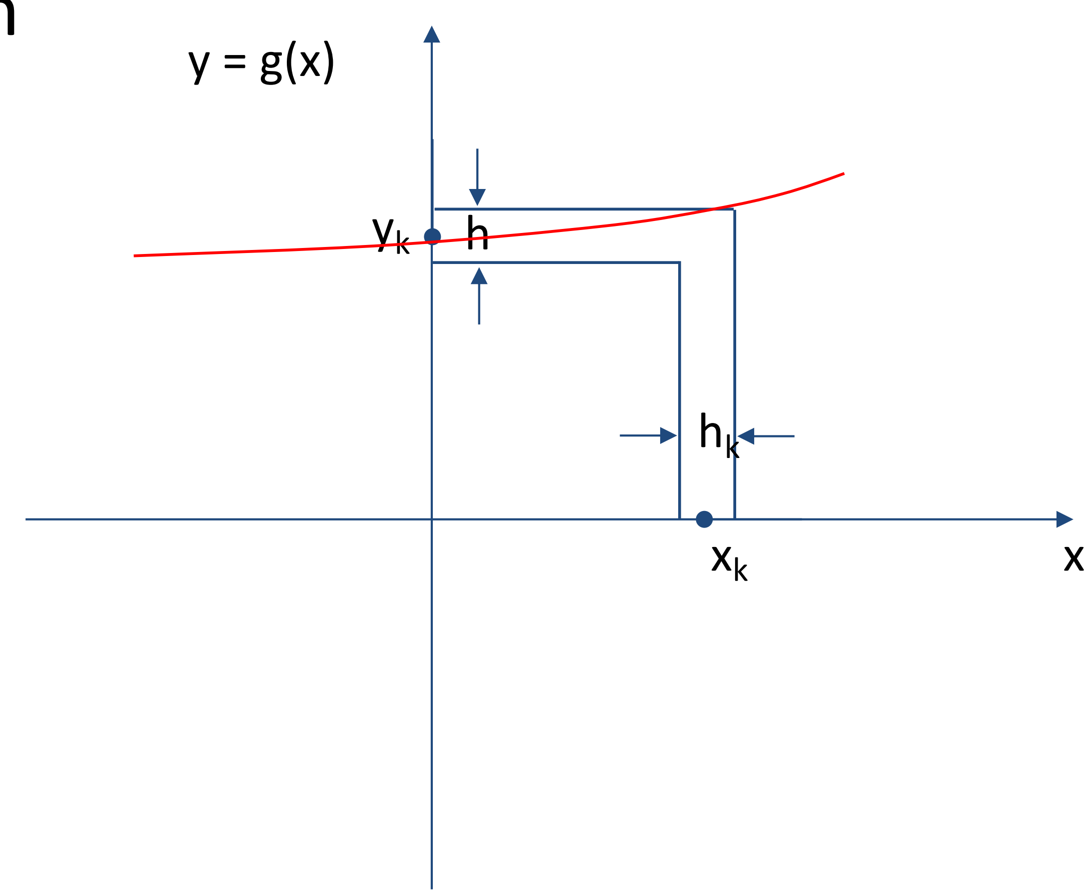
$$Y = \ln X$$

- Given CDF/pdf/pmf of X, find CDF/pdf/pmf of Y.

Expected Value of $Y = g(X)$

Can find $E[Y]$ directly in terms of the pdf of X.

$$E[Y] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$



Finding the Distributions of *Y*

- Use the notion of equivalent event:

$$P[Y \text{ in } C] = P[g(X) \text{ in } C] = P[X \text{ in } B]$$

- Three types of equivalent events are useful in determining the cdf and pdf of Y= $g(X)$

- Three equivalent events:

1. $\{g(X) = y_k\}$ Magnitude of Jump at y_k
2. $\{g(X) \leq y\}$ CDF of Y (directly)
3. $\{y < g(X) \leq y + h\}$ Useful for finding pdf

Expectation of a Function of the Random Variable

- Let g(x) be a function of the random variable x, the expectation of g(x) is given by

$$E[g(x)] = \sum_{\forall i} g(x_i) P[X = x_i]$$

for discrete variables, or

$$E[g(x)] = \int_{-\infty}^{\infty} g(t) f_x(t) dt$$

for continuous variables.

Mean, Variance, & Probabilities

- Suppose X is non-negative and $E[X]$ is small
  - We expect that X usually takes on small values
  - Can we quantify this in terms of $P[X > t]$?

- Suppose X is tightly packed about E[X]
  - We expect that X usually close to E[X]
  - Can we quantify $P[|X - E[X]| > t]$?

Markov Inequality

- Markov Inequality states that:

$$P[X \geq a] \leq \frac{E[X]}{a} \qquad \text{for } X \text{ nonnegative.}$$

We obtain Eq. (4.75) as follows:

$$E[X] = \int_0^a t f_X(t)\, dt + \int_a^\infty t f_X(t)\, dt \geq \int_a^\infty t f_X(t)\, dt$$

$$\geq \int_a^\infty a f_X(t)\, dt = aP[X \geq a].$$

Chebyshev Inequality

The **Chebyshev inequality** states that

$$P[|X - m| \geq a] \leq \frac{\sigma^2}{a^2}.$$

Chebyshev Inequality

The Chebyshev inequality is a consequence of the Markov inequality. Let $D^2 = (X - m)^2$ be the squared deviation from the mean. Then the Markov inequality applied to D^2 gives

$$P[D^2 \geq a^2] \leq \frac{E[(X - m)^2]}{a^2} = \frac{\sigma^2}{a^2}.$$

Equation (4.76) follows when we note that $\{D^2 \geq a^2\}$ and $\{|X - m| \geq a\}$ are equivalent events.

Suppose that a random variable X has zero variance; then the Chebyshev inequality implies that

$$P[X = m] = 1, \tag{4.77}$$

that is, the random variable is equal to its mean with probability one. In other words, X is equal to the constant m in almost all experiments.

Examples

Example 4.38

The mean response time and the standard deviation in a multi-user computer system are known to be 15 seconds and 3 seconds, respectively. Estimate the probability that the response time is more than 5 seconds from the mean.

The Chebyshev inequality with $m = 15$ seconds, $\sigma = 3$ seconds, and $a = 5$ seconds gives

$$P[|X - 15| \geq 5] \leq \frac{9}{25} = .36.$$

Example 4.39

If X has mean m and variance σ^2, then the Chebyshev inequality for $a = k\sigma$ gives

$$P[|X - m| \geq k\sigma] \leq \frac{1}{k^2}.$$

Now suppose that we know that X is a Gaussian random variable, then for $k = 2$, $P[|X - m| \geq 2\sigma] = .0456$, whereas the Chebyshev inequality gives the upper bound .25.

Example 4.40 Chebyshev Bound Is Tight

Let the random variable X have $P[X = -v] = P[X = v] = 0.5$. The mean is zero and the variance is $\text{VAR}[X] = E[X^2] = (-v)^2 0.5 + v^2 0.5 = v^2$.

Note that $P[|X| \geq v] = 1$. The Chebyshev inequality states:

$$P[|X| \geq v] \leq 1 - \frac{\text{VAR}[X]}{v^2} = 1.$$

We see that the bound and the exact value are in agreement, so the bound is tight.

Chernoff Bound

$$P[X \geq a] = \int_0^{\infty} I_A(t) f_X(t)\, dt \leq \int_0^{\infty} \frac{t}{a} f_X(t)\, dt = \frac{E[X]}{a}.$$

By changing the upper bound on $I_A(t)$, we can obtain different bounds on $P[X \geq a]$. Consider the bound $I_A(t) \leq e^{s(t-a)}$, also shown in Fig. 4.15, where $s > 0$. The resulting bound is:

$$P[X \geq a] = \int_0^{\infty} I_A(t) f_X(t)\, dt \leq \int_0^{\infty} e^{s(t-a)} f_X(t)\, dt$$

$$= e^{-sa} \int_0^{\infty} e^{st} f_X(t)\, dt = e$$

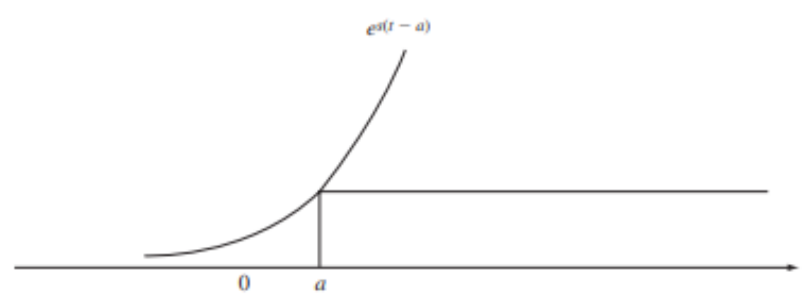


FIGURE 4.15
Bounds on indicator function for $A = \{t \geq a\}$.

Chernoff Bound

$$P[X \geq a] = \int_0^{\infty} I_A(t) f_X(t)\, dt \leq \int_0^{\infty} \frac{t}{a} f_X(t)\, dt = \frac{E[X]}{a}.$$

By changing the upper bound on $I_A(t)$, we can find different bounds in P(x>a)…..

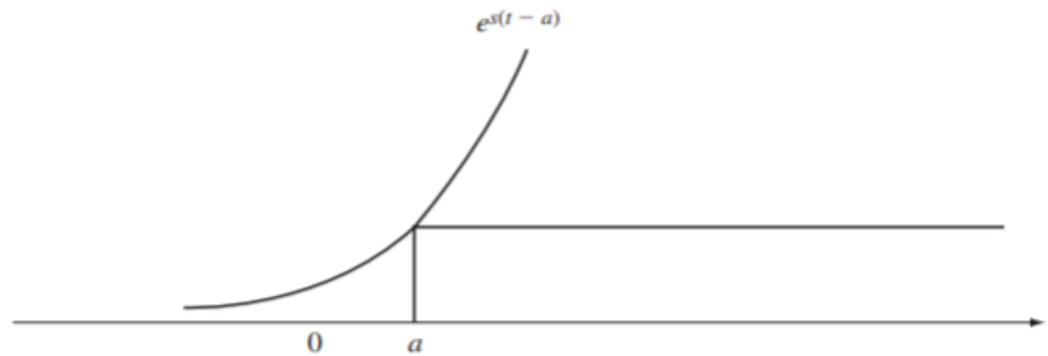


FIGURE 4.15
Bounds on indicator function for $A = \{t \geq a\}$.

$$P[X \geq a] = \int_0^{\infty} I_A(t) f_X(t)\, dt \leq \int_0^{\infty} e^{s(t-a)} f_X(t)\, dt$$

$$= e^{-sa} \int_0^{\infty} e^{st} f_X(t)\, dt = e^{-sa} E[e^{sX}].$$

Chernoff Bound for Gaussian

Example 4.44 Chernoff Bound for Gaussian Random Variable

Let X be a Gaussian random variable with mean m and variance σ^2. Find the Chernoff bound for X.

The Chernoff bound (Eq. 4.78) depends on the moment generating function:

$$E[e^{sX}] = \Phi_X(-js).$$

In terms of the characteristic function the bound is given by:

$$P[X \geq a] \leq e^{-sa}\Phi_X(-js) \quad \text{for} \quad s \geq 0.$$

The parameter s can be selected to minimize the upper bound.

The bound for the Gaussian random variable is:

$$P[X \geq a] \leq e^{-sa}e^{ms+\sigma^2 s^2/2} = e^{-s(a-m)+\sigma^2 s^2/2} \quad \text{for} \quad s \geq 0.$$

We minimize the upper bound by minimizing the exponent:

$$0 = \frac{d}{ds}(-s(a-m) + \sigma^2 s^2/2) \quad \text{which implies } s = \frac{a-m}{\sigma^2}.$$

The resulting upper bound is:

$$P[X \geq a] = Q\left(\frac{a-m}{\sigma}\right) \leq e^{-(a-m)^2/2\sigma^2}.$$