George Mason University

Department of Electrical and Computer Engineering

ECE 528: Introduction to Random Processes in ECE — Fall Semester

Homework Set 8 Solutions

1. (P 5.1) Let X be the maximum and let Y be the minimum of the number of heads obtained when Carlos and Michael each flip a fair coin twice.

   (a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).

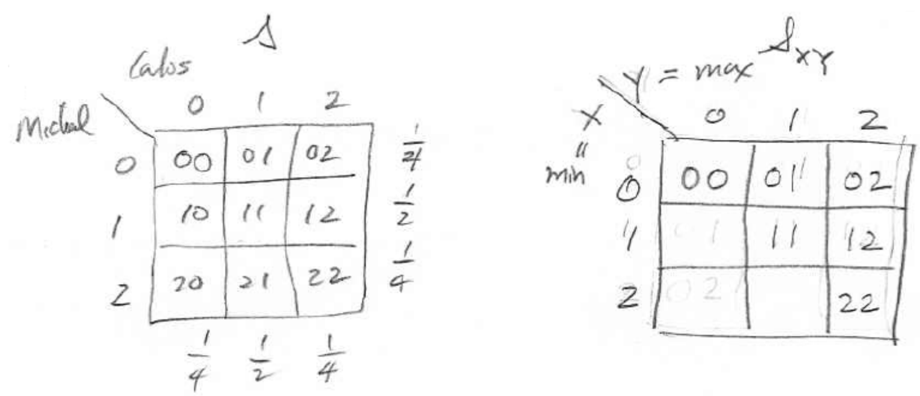
   (b) Find the probabilities for all values of (X, Y).

   (c) Find $P[X = Y]$.

   (d) Repeat parts b and c if Carlos uses a biased coin with $P[\text{heads}] = 3/4$.

   Solutions:

   (a) The mapping from S to S_{XY} and the range of (X, Y) are given by:



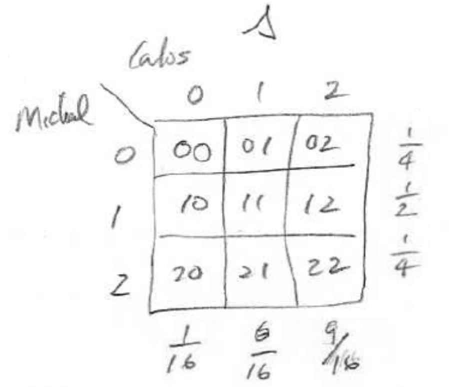
   (b)

$$
\begin{aligned}
P[X = 0, Y = 0] &= P[\{00\}] = 1/16,\\
P[X = 0, Y = 1] &= P[\{01, 10\}] = 1/8 + 1/8 = 1/4,\\
P[X = 0, Y = 2] &= P[\{02, 20\}] = 1/16 + 1/16 = 1/8,\\
P[X = 1, Y = 1] &= P[\{11\}] = 1/4,\\
P[X = 1, Y = 2] &= P[\{21, 12\}] = 1/8 + 1/8 = 1/4;\\
P[X = 2, Y = 2] &= P[\{22\}] = 1/16.
\end{aligned}
$$

   (c)

$$P[X = Y] = P[\{00, 11, 22\}] = 1/16 + 1/4 + 1/16 = 3/8.$$

(d) The new matrix for S is given by:



Hence, the probabilities are given as follows:

$$\begin{aligned}
P[X=0,Y=0] &= P[\{00\}] = 1/64, \\
P[X=0,Y=1] &= P[\{01,10\}] = 1/32 + 6/64 = 1/8, \\
P[X=0,Y=2] &= P[\{02,20\}] = 9/64 + 1/64 = 5/32, \\
P[X=1,Y=1] &= P[\{11\}] = 3/16, \\
P[X=1,Y=2] &= P[\{21,12\}] = 6/64 + 9/32 = 3/8; \\
P[X=2,Y=2] &= P[\{22\}] = 9/64.
\end{aligned}$$

$$P[X=Y] = P[\{00,11,22\}] = 1/64 + 3/16 + 9/64 = 11/32.$$

2. (P 5.8) For the pair of random variables (X,Y) sketch the region of the plane corresponding to the following events. Identify which events are of product form.

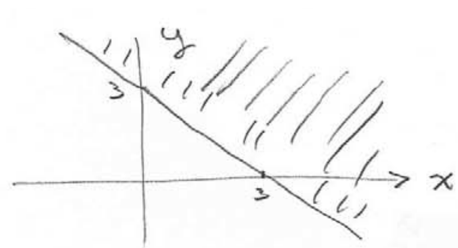
   (a) $\{X+Y>3\}$

   (b) $\{e^X > Ye^3\}$

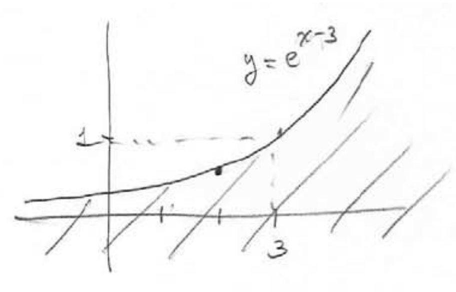
   (c) $\{\min(X,Y)>0\} \cup \{\max(X,Y)<0\}$

   Solutions:

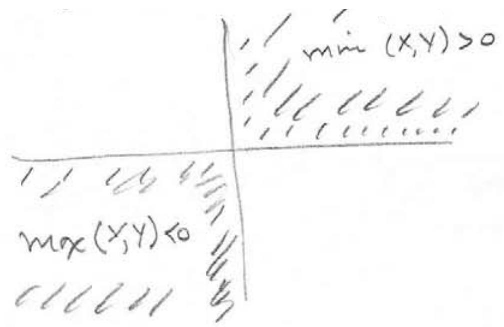
   (a) $\{X+Y>3\} = \{Y > 3-X\}$. The sketch is not product form.



(b) $\left\{e^X > Ye^3\right\} = \left\{Y < e^{X-3}\right\}$. The sketch is not product form.



(c) $\{\min(X,Y) > 0\} \cup \{\max(X,Y) < 0\}$. The sketch is not product form.

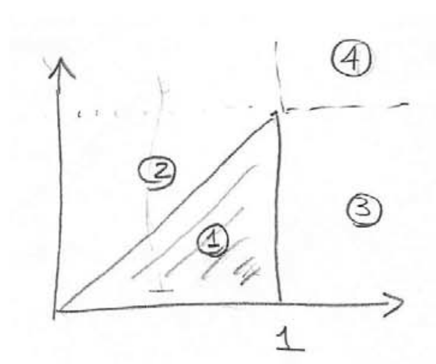


3. (5.17) A point (X,Y) is selected at random inside a triangle defined by $\{(x,y) : 0 \le x \le y \le 1\}$. Assume the point is equally likely to fall anywhere in the triangle.

   (a) Find the joint cdf of X and of Y.
   (b) Find the marginal cdf of X and of Y.
   (c) Find the probabilities of the following events in terms of the joint cdf:
   $A = \{X \le 1/2, Y \le 3/4\}\,; B = \{1/4 < X \le 3/4, 1/4 < Y \le 3/4\}\,.$

   Solutions:

   (a) The triangular area is depicted as follows and we divide the whole plane into 4 non-overlapping regions.



It is easy to verify that the area of the shaded triangle is 1/2. We further discuss the joint CDF $F_{X,Y}(x, y)$ in the four regions:

i. Region 1: $0 < y < x < 1$

$$F_{X,Y}(x, y) = P[X \leq x, Y \leq y] = \frac{y^2/2 + y(x - y)}{1/2} = 2xy - y^2$$

ii. Region 2: $0 < x < y$

$$F_{X,Y}(x, y) = P[X \leq x, Y \leq y] = \frac{x^2/2}{1/2} = x^2$$

iii. Region 3: $y < x, x > 1$

$$F_{X,Y}(x, y) = P[X \leq x, Y \leq y] = \frac{y^2/2 + y(1 - y)}{1/2} = 2y - y^2$$

iv. Region 4: $x > 1, y > 1$

$$F_{X,Y}(x, y) = P[X \leq x, Y \leq y] = 1$$

(b) The marginal cdf of X and Y are given by:

$$F_X(x) = P[X \leq x] = \begin{cases} 0, & x < 0; \\ x^2, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$$

$$F_Y(y) = P[Y \leq y] = \begin{cases} 0, & y < 0; \\ 2y - y^2, & 0 \leq y \leq 1; \\ 1, & x > 1. \end{cases}$$

(c) For event A, as $(\frac{1}{2}, \frac{3}{4})$ is in region 2, we have:

$$P[A] = (\frac{1}{2})^2 = \frac{1}{4}.$$

For event B, we know $(\frac{1}{4}, \frac{3}{4})$ is in region 2 and $(3/4, 1/4)$ is in region 1. Thus,

$$\begin{aligned} P[B] &= F_{X,Y}(\frac{3}{4}, \frac{3}{4}) - F_{X,Y}(\frac{1}{4}, \frac{3}{4}) - F_{X,Y}(\frac{3}{4}, \frac{1}{4}) + F_{X,Y}(\frac{1}{4}, \frac{1}{4}) \\ &= (\frac{3}{4})^2 - 2 \cdot [\frac{3}{4} \cdot \frac{1}{4} - \frac{1}{2} \cdot (\frac{1}{4})^2] - (\frac{1}{4})^2 + (\frac{1}{4})^2 \\ &= \frac{1}{4} \end{aligned}$$

4. (5.28) The random vector (X, Y) is uniformly distributed (i.e., $f(x, y) = k$) in the regions shown in Fig. P5.1 and zero elsewhere.

(a) Find the value of k in each case.

(b) Find the marginal pdf for X and for Y in each case.

(c) Find $P[X > 0, Y > 0]$

Solutions:

(a) i. $k \cdot \pi \cdot 1^2 = 1, k = 1/\pi;$

ii. $k \cdot (\sqrt{2})^2 = 1, k = 1/2;$

iii. $k \cdot 1^2/2 = 1, k = 2.$

(b) i.

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi}, -1 < x < 1;$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-y^2}}{\pi}, -1 < y < 1;$$

ii.

$$f_X(x) = \int_{|x|-1}^{1-|x|} \frac{1}{2} dy = 1 - |x|, -1 < x < 1;$$

$$f_Y(y) = \int_{|y|-1}^{1-|y|} \frac{1}{2} dx = 1 - |y|, -1 < y < 1;$$

iii.

$$f_X(x) = \int_0^{1-x} 2dy = 2(1-x), 0 < x < 1;$$

$$f_Y(y) = \int_0^{1-y} 2dx = 2(1-y), 0 < y < 1;$$

(c) i.

$$P[X > 0, Y > 0] = \frac{(\frac{1}{2})^2 \pi}{\pi} = 1/4;$$

ii.

$$P[X > 0, Y > 0] = \frac{1^2/2}{(\sqrt{2})^2} = 1/4;$$

iii.

$$P[X > 0, Y > 0] = \frac{1^2/2}{1^2/2} = 1.$$

5. (*5.20) The pair (X, Y) has joint cdf given by:

$$F_{X,Y}(x,y) = \begin{cases} (1-1/x^2)(1-1/y^2) & x > 1, y > 1 \\ 0 \quad \text{elsewhere.} \end{cases}$$

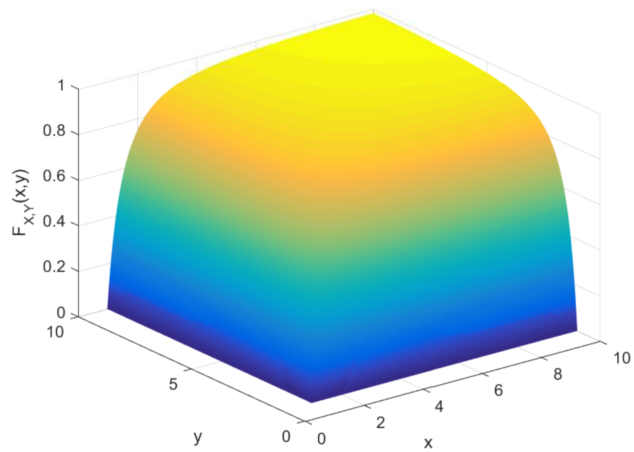
(a) Sketch the joint cdf.

(b) Find the marginal cdf of Y and of Y.

(c) Find the probability of the following events: $\{X < 3, Y \leq 5\}, \{X > 4, Y > 3\}$.

Solutions:

(a) The joint cdf sketch is as follows:



(b)

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1 - 1/x^2, & x > 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 - 1/y^2, & y > 1 \\ 0, & \text{otherwise.} \end{cases}$$

(c)

$$P[X < 3, Y \leq 5] = F_{X,Y}(3, 5) = (1 - 1/3^2) * (1 - 1/5^2) = 64/75;$$

$$P[X > 4, Y > 3] = 1 - F_{X,Y}(4, \infty) - F_{X,Y}(\infty, 3) + F_{X,Y}(4, 3) = 1/144$$

6. (*5.25) The amplitudes of two signals X and Y have joint cdf:

$$f_{X,Y}(x, y) = e^{-x/2} y e^{-y^2}, \quad for x > 0, y > 0.$$

(a) Find the joint cdf.

(b) Find $P[X^{1/2} > Y]$.

(c) Find the marginal pdfs.

Solutions:

(a) For $x > 0, y > 0$, we have:

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_0^x \int_0^y e^{-x/2} y e^{-y^2} dx dy \\
&= \int_0^x \int_0^y \frac{1}{2} e^{-x/2} \cdot 2y e^{-y^2} dx dy \\
&= (1 - e^{-x/2})(1 - e^{-y^2}).
\end{aligned}$$

(b)

$$\begin{aligned}
P[X^{1/2} > Y] = P[Y < \sqrt{x}] &= \int_0^x \int_0^y e^{-x/2} y e^{-y^2} dx dy \\
&= \int_0^{+\infty} \int_0^{\sqrt{x}} 2y e^{-y^2} dy \cdot \frac{1}{2} e^{-x/2} dx \\
&= \int_0^{+\infty} (1 - e^{-x}) \frac{1}{2} e^{-x/2} dx \\
&= \int_0^{+\infty} \frac{1}{2} e^{-x/2} dx - \int_0^{+\infty} \frac{1}{2} e^{-3x/2} dx \\
&= 1 - 1/3 \\
&= 2/3.
\end{aligned}$$

(c)

$$F_X(x) = F_{X,Y}(x, \infty) = 1 - e^{-x/2}, x > 0.$$
$$f_X(x) = \frac{\partial F_X(x)}{\partial x} = \frac{1}{2} e^{-x/2}.$$

$$F_Y(y) = F_{X,Y}(\infty, y) = 1 - e^{-y^2}, y > 0.$$
$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = 2y e^{-y^2}.$$