CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE

Lecture 3: Conditional Probability & Independent Events

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Note

- These slides cover material partially presented in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-set derivation and detailed discussions were need.

Additional Stuff Covered

- Chapter 2
- Problems 2.4 and 2.29 and 2.54
- The Balls and Boxes
- Tree diagram for Conditional probability
- BSC using conditional

Conditional Probability

- Are events *A* & *B* interrelated?
- If we know that *B* occurred, how does probability of *A* change?

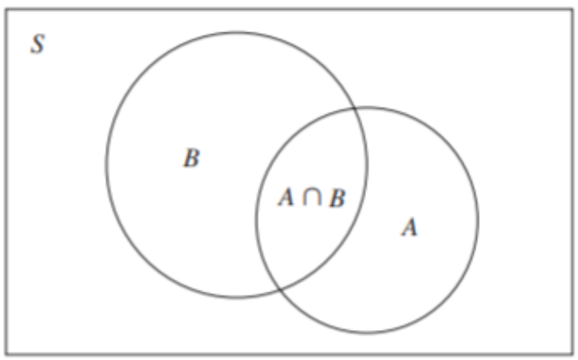


FIGURE 2.9
If *B* is known to have occurred, then *A* can occur only if $A \cap B$ occurs.

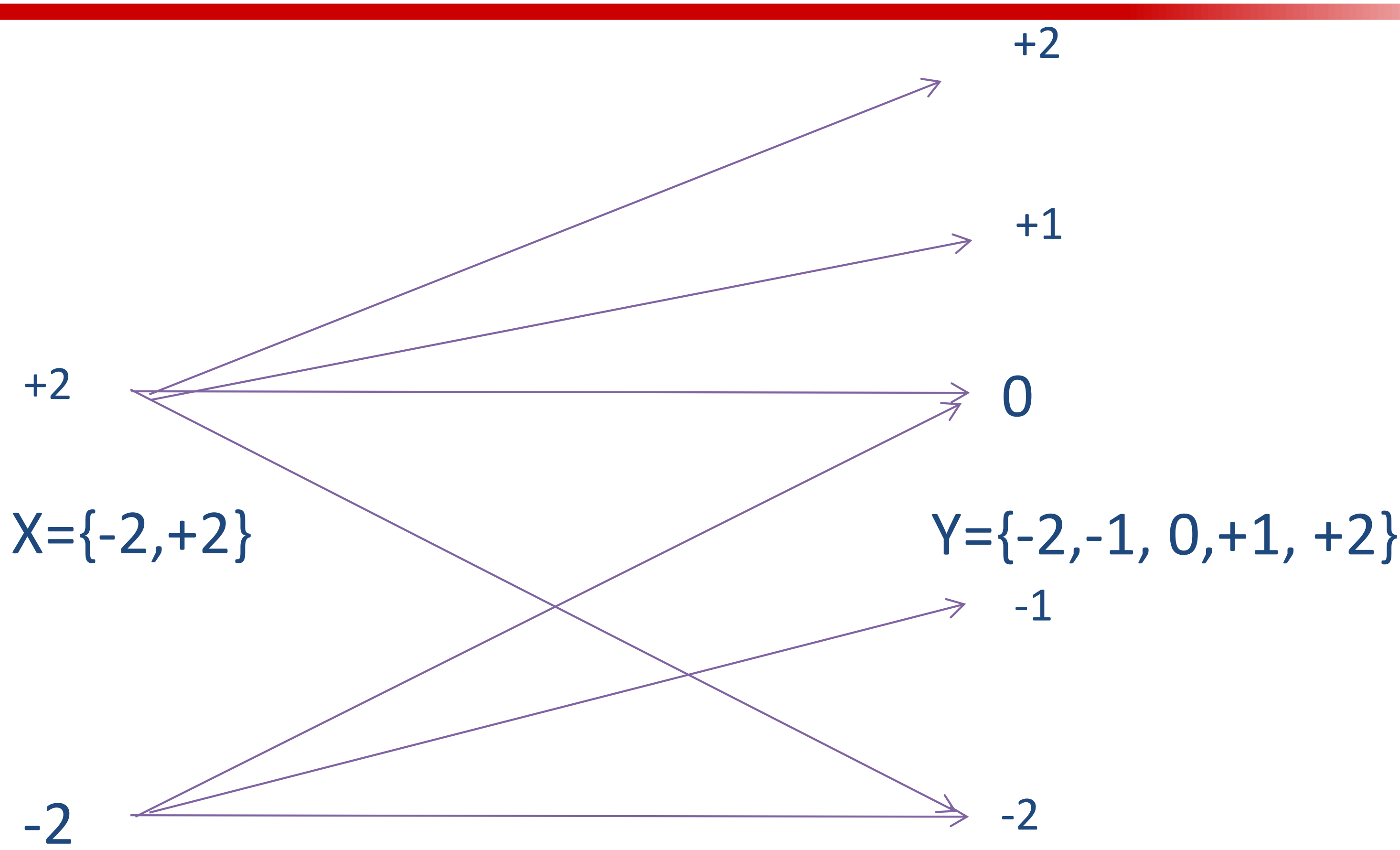
$$\frac{n_{A\cap B}}{n_B} = \frac{n_{A\cap B}/n}{n_B/n} \rightarrow \frac{P[A\cap B]}{P[B]}$$

$$P[A|B] = \frac{P[A\cap B]}{P[B]} \text{ for } P[B] > 0.$$

Problem 2.4

- A binary communication system transmits a signal X that is either a voltage signal or a voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal.
- (a) Find the sample space.
- (b) Find the set of outcomes corresponding to the event "transmitted signal was definitely +2."
- (c) Describe in words the event corresponding to the outcome

Binary Communication System



- X={-2, +2} and Y ={-2, -1, 0, +1, +2}

Problem 2.4 Solution

- a)

X \ Y	-2	-1	0	1	2
+2	--	--	(2,0)	(2,1)	(2,2)
-2	(-2,-2)	(-2,-1)	(-2,0)	--	--

- b) "X definitely + 2" : {(2,1),(2,2)}

- c) {Y=0} = {(2,0),(-2,0)}

  Observed output is Zero. Cannot determine Input

Problem 2.29

- Let M be the number of message transmissions in Problem 2.7. Find the probabilities of the events A, B,C,Cc…… Assume the probability of successful transmission is 1/2.
- Problem 2.7: Let M be the number of message transmissions in Experiment E6. (a) What is the set A corresponding to the event “M is even”? (b) What is the set B corresponding to the event “M is a multiple of 3”? (c) What is the set C corresponding to the event “6 or fewer transmissions are required”? (d) Find the sets and describe the corresponding events in words

Problem 2.29 Solution

- Each Transmission is equivalent to tossing a fair coin. If outcome is heads, the transmission is successful. If tails, another transmission is required. Let's find the probability that j transmissions are required:

$$P[j] = \left(\frac{1}{2}\right)^j$$

$$P[A] = P[j \text{ even}] = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k} = \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k} - 1 = \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3}.$$

$$P[B] = P[j \text{ multiple of } 3] = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{3k} = \frac{1}{1-\frac{1}{8}} - 1 = \frac{1}{7}.$$

Problem 2.29 Solution

$$P[C]=\sum_{k=1}^{6}\left(\frac{1}{2}\right)^{k}=\frac{1}{2}\sum_{k=0}^{5}\left(\frac{1}{2}\right)^{k}=\frac{1}{2}\frac{1-(\frac{1}{2})^{6}}{1-\frac{1}{2}}=\frac{63}{64}.$$

$$P[C^{c}]=1-P[C]=\frac{1}{64}.$$

$$P[A\cap B]=\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{6k}=\frac{1}{1-\frac{1}{64}}-1=\frac{1}{63}$$ since a multiple of 2 and 3 is a multiple of 6.

$$P[A-B]=P[A]-P[A\cap B]=\frac{1}{3}-\frac{1}{63}=\frac{20}{63}$$ since

$$A=(A-B)\cup(A\cap B)] \text{ and } (A-B)\cap(A\cap B)=\phi.$$

$$P[A\cap B\cap C]=\left(\frac{1}{2}\right)^{6}=\frac{1}{64}$$ since $A\cap B\cap C=\{6\}$.

Problem 2.54

- A lot of 100 items contains k defective items. M items are chosen at random and tested. (a) What is the probability that m are found defective? This is called the hypergeometric distribution. (b) A lot is accepted if 1 or fewer of the M items are defective. What is the probability that the lot is accepted?

Problem 2.54 Solution

2.54a The number of ways of choosing M out of 100 is $\binom{100}{M}$. This is the total number of equiprobable outcomes in the sample space.

We are interested in the outcomes in which m of the chosen items are defective and $M - m$ are nondefective.

The number of ways of choosing m defectives out of k is $\binom{k}{m}$.

The number of ways of choosing $M - m$ nondefectives out of 100 k is $\binom{100-k}{M-m}$.

The number of ways of choosing m defectives out of k <u>*and*</u> $M - m$ non-defectives out of $100 - k$ is

$$\binom{k}{m}\binom{100-k}{M-m}$$

$$P[m \text{ defectives in } M \text{ samples}] = \frac{\#\text{ outcomes with } k \text{ defective}}{\text{Total }\#\text{ of outcomes}} = \frac{\binom{k}{m}\binom{100-k}{M-m}}{\binom{100}{M}}$$

This is called the <u>Hypergeometric</u> distribution.

(b) $P[\text{lot accepted}] = P[m=0 \text{ or } m=1] = \dfrac{\binom{100-k}{M}}{\binom{100}{M}} + \dfrac{k\binom{100-k}{M-1}}{\binom{100}{M}}.$

The number of ways of choosing m defectives out of k is $\binom{k}{m}$.

The number of ways of choosing $M - m$ nondefectives out of $100 - k$ is $\binom{100-k}{M-m}$.

The number of ways of choosing m defectives out of k
<u>and</u> $M - m$ non-defectives out of $100 - k$ is

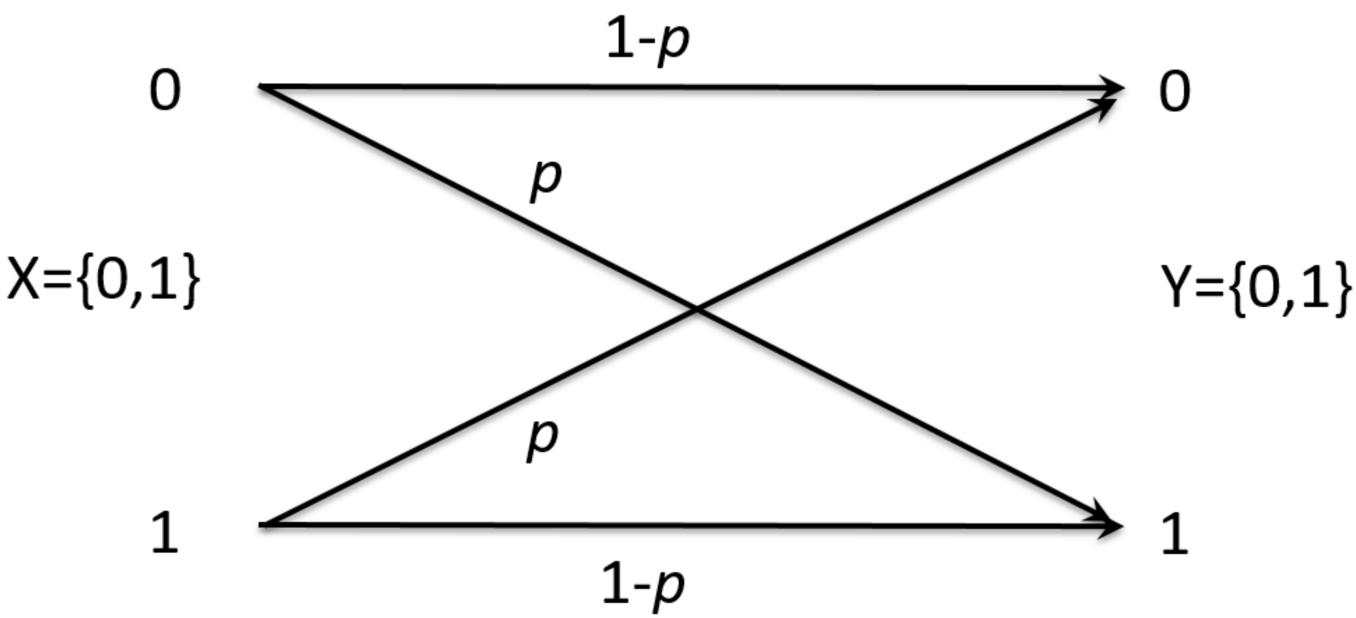
$$\binom{k}{m}\binom{100-k}{M-m}$$

$$\begin{aligned} P[m \text{ defectives in } M \text{ samples}] &= \frac{\#\text{ outcomes with } k \text{ defective}}{\text{Total }\#\text{ of outcomes}} \\ &= \frac{\binom{k}{m}\binom{100-k}{M-m}}{\binom{100}{M}} \end{aligned}$$

This is called the Hypergeometric distribution.

b) $$P[\text{lot accepted}] = P[m=0 \text{ or } m=1] = \frac{\binom{100-k}{M}}{\binom{100}{M}} + \frac{k\binom{100-k}{M-1}}{\binom{100}{M}}.$$

Binary Communication Channel



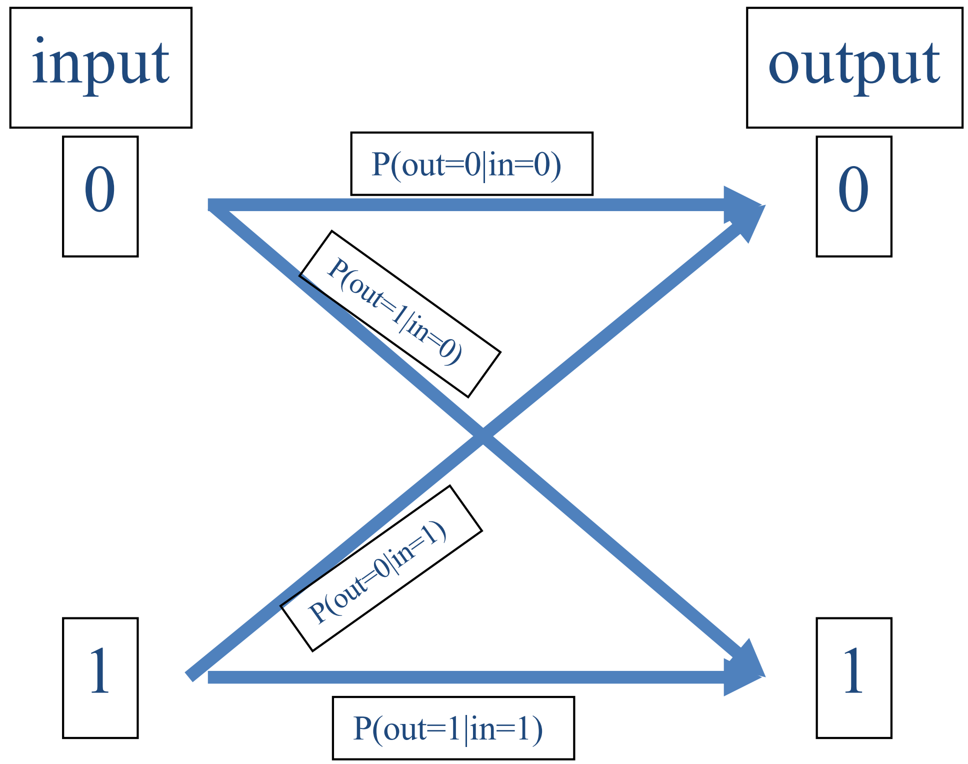
- Binary Symmetric Channel (BSC) model and noisy channel
- binary {0,1}
- symmetric means prob (0→1)= prob (1→0)

Example 1: Binary (Symmetric) Channel

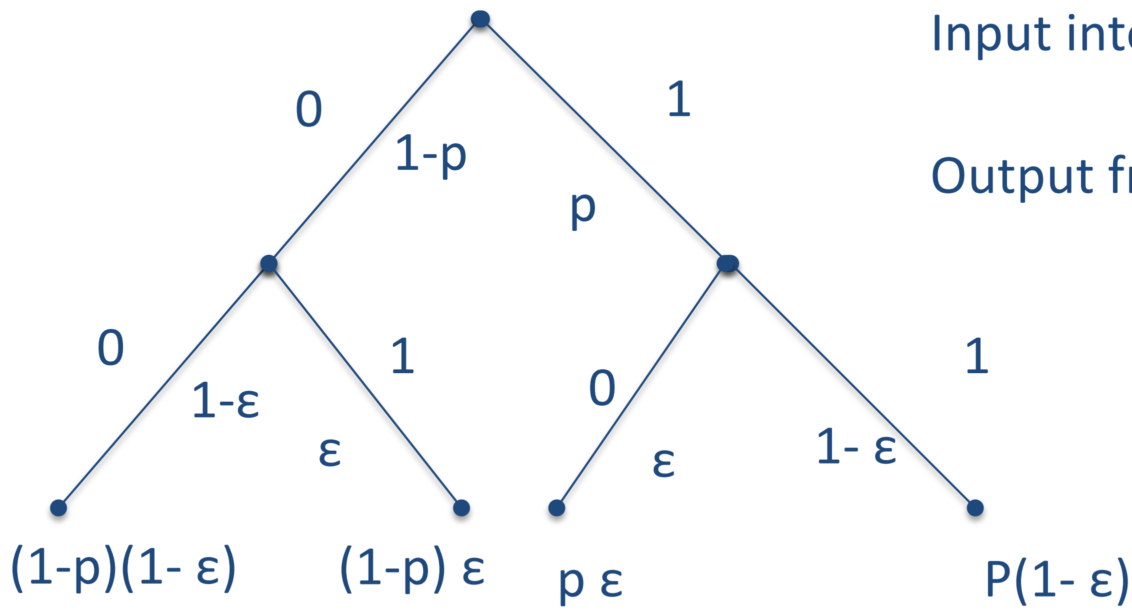
- Given the binary symmetric channel depicted in figure, find P(input = j | output = i); i,j = 0,1. Given that P(input = 0) = 0.4, P(input = 1) = 0.6.

Solution:

Refer to examples 2.23 and 2.26 of Garcia's textbook



Binary Tree Diagram



Input into binary channel

Output from binary channel

Example: Random Pair from Unit Square

Exercise 2.32 Page 55

If $x > y$, what is probability that $x > 0.5$?

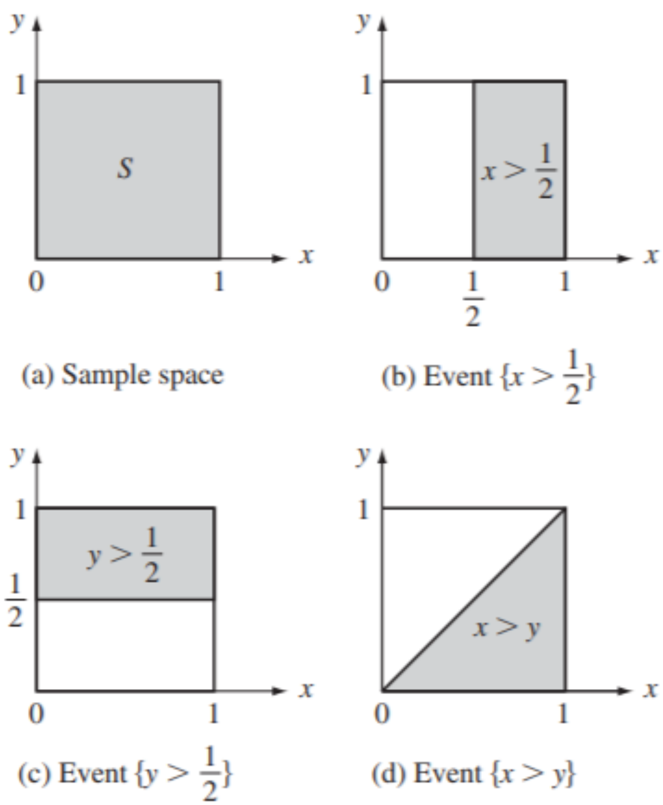


FIGURE 2.7
A two-dimensional sample space and three events.

Probability of Joint Occurrence

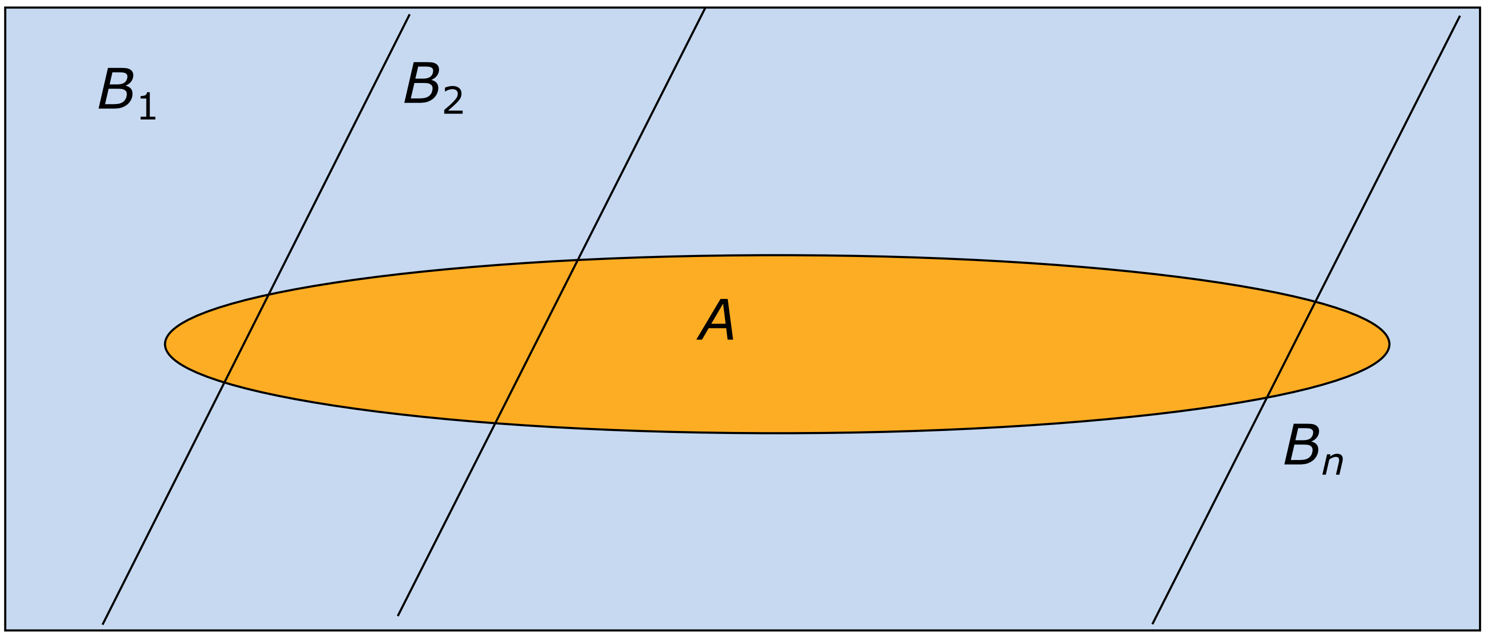
$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} \text{ for } P[B] > 0.$$

$$\begin{aligned} P[A \cap B] &= P[A \mid B]\, P[B] \\ &= P[B \mid A]\, P[A] \end{aligned}$$

$$\begin{aligned} P[A \cap B \cap C] &= P[A \mid B \cap C]\, P[B \cap C] \\ &= P[A \mid B \cap C]\, P[B \mid C]\, P[C] \end{aligned}$$

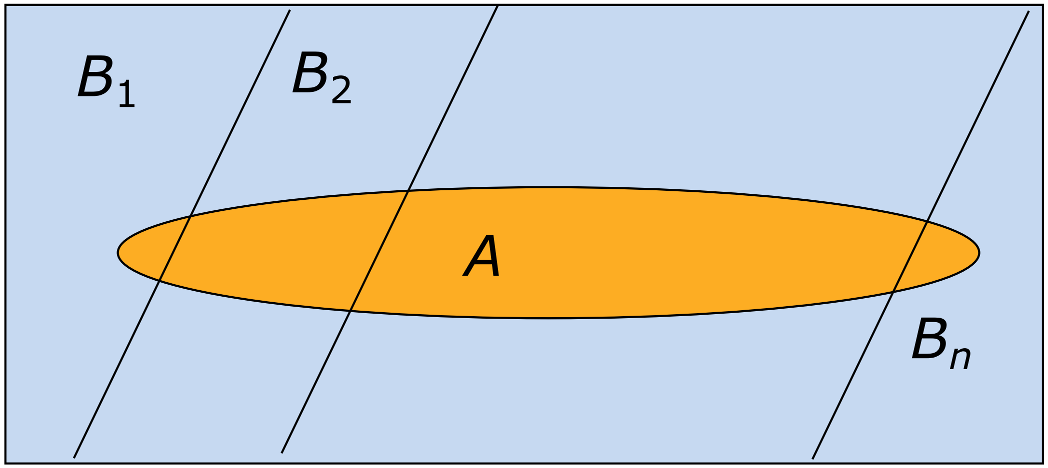
Theorem on Total Probability

$$P[A] = \sum_{i=1}^{n} P[A \cap B_i] = \sum_{i=1}^{n} P[A \mid B_i] P[B_i]$$



Bayes' Rule

Suppose A occurs, what is the probability of B_j?



$$P\left[B_j \mid A\right] = \ ???$$

$$P\left[B_j \mid A\right] = \frac{P\left[B_j \cap A\right]}{P[A]} = \frac{P\left[A \mid B_j\right]P\left[B_j\right]}{\sum_{i=1}^{n} P[A \mid B_i]P[B_i]}$$

Event Independence

- Intuition: *Knowledge that* A *occurred does not change the probability of* B.

$$P[A \cap B] = P[A]P[B]$$

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A]$$

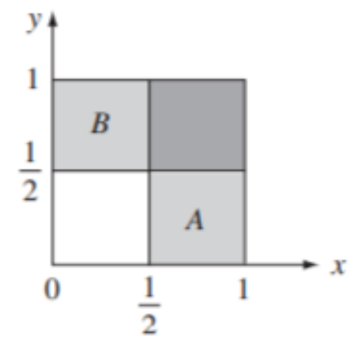
$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[A]P[B]}{P[A]} = P[B]$$

Example: Random Pair in Unit Square

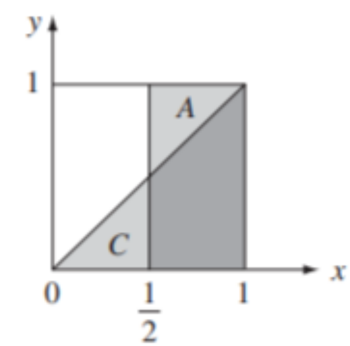
$$A = \{x > y\}, \;\; B = \{x > 0.5\} \;\; C = \{y < 0.5\}$$

$$P[A \mid B] =$$

$$P[B \mid C] =$$



(a) Events A and B are independent.



(b) Events A and C are not independent.

FIGURE 2.13
Examples of independent and nonindependent events.

Independence of Three Events

- Definition: *A*, *B*, & *C* are independent if they are pairwise independent

$$P[A \cap B] = P[A]P[B], \quad P[A \cap C] = P[A]P[C],$$
$$\text{and} \quad P[B \cap C] = P[B]P[C]$$

- and if knowledge of 2 of them does not affect the probability of the 3rd

$$P[C \mid A \cap B] = \frac{P[A \cap B \cap C]}{P[A \cap B]} = P[C]$$

- Therefore, *A*, *B*, & *C* are independent if probability of ∩ of pairs & triplets = product of individual probabilities:

$$P[A \cap B \cap C] = P[A \cap B]P[C] = P[A]P[B]P[C]$$

Independence of Multiple Events

- Similarly, A_1, ..., A_n are independent if for k = 2, ..., n:

$$P\left[A_{i_1} \cap \cdots \cap A_{i_k}\right] = P\left[A_{i_1}\right] \cdots P\left[A_{i_k}\right]$$

Exercise: Random Pair in Unit Square

$A = \{x < 0.5\}$, $B = \{y > 0.5\}$

$F = \{x < 0.5 \text{ and } y < 0.5\} \text{ U } \{x > 0.5 \text{ and } y > 0.5\}$

Sequences of Independent Experiments

- Definition: Two experiments are independent if all of their respective events are independent.

- Suppose that a random experiment E involves performing n subexperiments: $E_1, E_2, E_3, \ldots, E_n$.

- S is Cartesian product of individual sample spaces:
  - $S = S_1 \times S_2 \times S_3 \times \ldots \times S_n$

- An outcome of random experiment consists of an n-tuple

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n) \text{ where } \xi_i \text{ is an outcome of } E_i$$

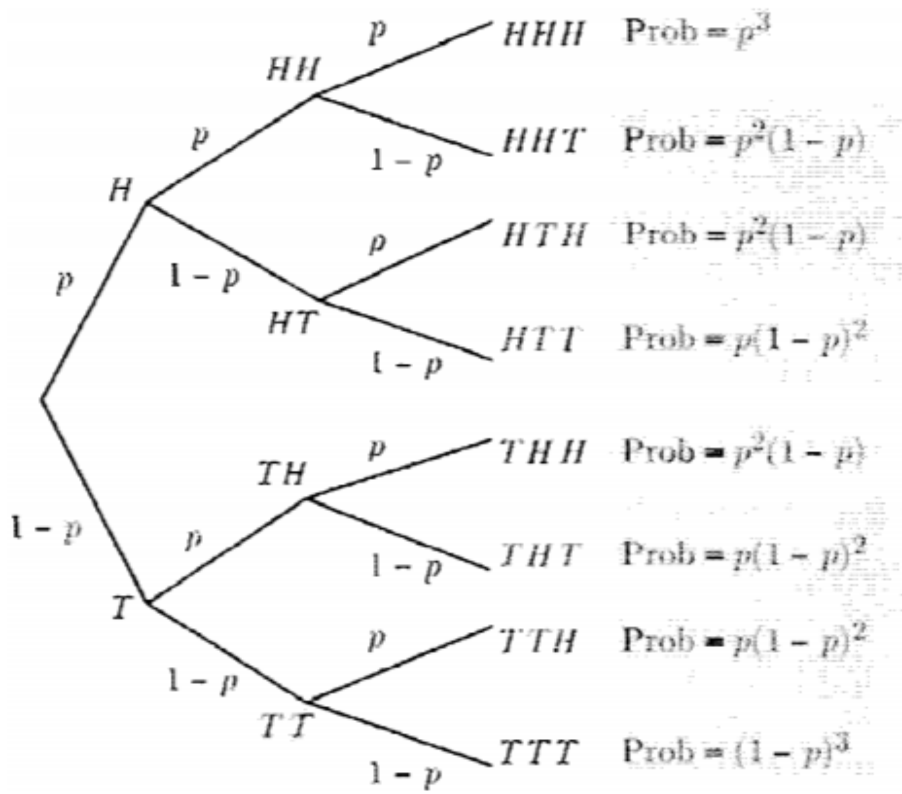
- If the subexperiments are independent, and if A_k only concerns the outcome E_k, then probabilities of events involving intersections of A_k are given by:

$$P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2]\ldots P[A_n]$$

Example: Independent Coin Tosses

- Toss a fair coin three times.
- *Assume* tosses are independent.

$$P[\{(h,t,h)\}] =$$



Example: Sequence of Bernoulli Trials

- Bernoulli trial involves performing an experiment once and noting whether an event A occurred.
  - "Success" or "1" if A occurs;
  - "Failure" or "0" otherwise
  - Suppose $P[A] = p$
- Perform n independent Bernoulli trials, what is probability of k successes in n trials?

Example: Sequence of Bernoulli Trials II

- The probability of a sequence with exactly *k* 1s in *n* trials:

$$p_n(k) = \binom{n}{k} p^k (1 - p)^{n-k} \qquad \text{for} \qquad k = 0, \ldots, n,$$

- The number of distinct sequences with *k* 1s and (*n* – *k*) 0s is:

$$p_n(k) = N_n(k) p^k (1 - p)^{n-k}.$$

- The probability of *k* successes in *n* trials is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Approximating Binomial Probabilities

- If n is large and p is small, then for $\alpha = np$

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{\alpha^k}{k!} e^{-\alpha} \quad \text{for } k = 0, 1, \ldots$$

$$\frac{p_{k+1}}{p_k} = \frac{\binom{n}{k+1} p^{k+1} q^{n-k-1}}{\binom{n}{k} p^k q^{n-k}} = \frac{k!(n-k)!p}{(k+1)!(n-k-1)!q}$$

$$= \frac{(n-k)p}{(k+1)q} = \frac{(1-k/n)\alpha}{(k+1)(1-\alpha/n)} \to \frac{\alpha}{k+1} \quad \text{as } n \to \infty$$