Problem1(Schwartz 2.8)   Solutions to Homework Set No. 4

For the eqn.

$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$

Substitute by the following in the RHS

$P_n = \rho P_0$

$\therefore \lambda\rho^{n-1}P_0 + \mu\rho^{n+1}P_0 = (\lambda\rho^{n-1} + \mu\rho^{n+1})P_0$

$= P_0\rho^n(\lambda\rho^{-1} + \mu\rho) = P_0\rho^n(\mu+\lambda)$

= L.H.S.

$\therefore P_n = \rho P_0$ satisfies the equation.

For the eqn.

$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$

substitute by

$P_{n+1} = \rho P_n$ i.e. $(P_n = P_{n+1}/\rho)$ in the R.H.S. of the equation

$\therefore \lambda P_n/\rho + \mu\rho P_n = \mu P_n + \lambda P_n = P_n(\lambda+\mu)$ = L.H.S.

$\therefore P_{n+1} = \rho P_n$ satisfies the equation.

For the M/M/1 queue

$$P_0 = 1 - \sum_{n=1}^{N} P_n$$

$$= 1 - \frac{P_1\left(1 - \left(\frac{P_n}{P_{n-1}}\right)^N\right)}{1 - \left(\frac{P_n}{P_{n-1}}\right)}$$

$$= 1 - \frac{P_0\rho(1-\rho^N)}{1-\rho}$$

$$\therefore P_0\left[1 + \frac{(\rho - \rho^{N+1})}{1-\rho}\right] = 1$$

$$\therefore P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

and $\therefore P_n = P_0\rho^n = \dfrac{(1-\rho)\rho^n}{1-\rho^{N+1}}$

Problem 2 (Schwartz 2.9)

For finite m/m/1 queue

$\gamma = \lambda(1-P_B) = \mu(1-P_0)$

$\therefore 1-P_B = \dfrac{\mu}{\lambda}(1-P_0)$

so $P_B = 1-\dfrac{\mu}{\lambda}(1-P_0)$

$\because P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$

$\therefore P_B = 1-\dfrac{1}{\rho}+\dfrac{1}{\rho}\dfrac{1-\rho}{(1-\rho^{N+1})} = (1-\rho)\rho^N[\dfrac{-1}{\rho^{N+1}}+\dfrac{1}{1-\rho^{N+1}}]$

$= (1-\rho)\rho^N[\dfrac{\rho^{N+1}-1+1}{\rho^{N+1}(1-\rho^{N+1})}] = \dfrac{(1-\rho)\rho^N}{1-\rho^{N+1}}$

$= P_N$

Problem 3 (Schwartz 2.10)

$$P_B = \frac{(1-\rho)\rho^N}{1-\rho^{N+1}}$$

1. For $\rho = 0.5$, $P_B = 10^{-3}$
    a. We can get the length of the buffer by trial and error
       N= 9 customers

    b. For $\rho = 0.5$, $P_B = 10^{-6}$
       N=19 customers

2. For $\rho = 0.8$,
    a. $P_B = 10^{-3}$
       N=23 customers

    b. For $\rho = 0.8$, $P_B = 10^{-6}$
       N= 29 customers

From the results we conclude that, in order to decrease the blocking probability of the system or to increase the utilization of the system we have to use a longer buffer.

Problem 4 (Shwartz 2-11)

$$E[n] = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} nP_0\rho^n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = (1-\rho)\rho\sum_{n=1}^{\infty} n\rho^{n-1}$$

$$\because \sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}$$

$$\therefore \frac{d}{dn}\sum_{n=1}^{\infty} \rho^n = \frac{1}{(1-\rho)^2}$$

and so, $E[n] = \dfrac{\rho}{1-\rho}$

Problem 5

1. For M / M / ∞

$\lambda_n = \lambda$, $\mu = n\mu$

$$\therefore P_n = P_0 \frac{\rho^n}{n!}$$

From the property of probability, we get $\sum_{i=0}^{\infty} P_i = 1$

$$\therefore \sum_{n=0}^{\infty} P_0 \frac{\rho^n}{n!} = 1$$

$$P_0 = e^{-\rho}$$

$\therefore P_n = e^{-\rho}\dfrac{\rho^n}{n!}$ which is the Poisson's distribution

$E[n] = \rho$; $\qquad E[T] = \dfrac{\rho}{\lambda} = \dfrac{1}{\mu}$

$E[w] = E[T] - \dfrac{1}{\mu} = 0$; $\quad E[q] = 0$

2. For queue with discouragement

$\lambda_n = \dfrac{\lambda}{n+1}$, $\quad \mu_n = \mu$

$$\therefore P_n = \frac{\lambda^n P_0}{n!\mu^n} = \frac{P_0\rho^n}{n!}$$

Thus we get the value of $P_0 = e^{-\rho}$ which is the same as the M/M/$\propto$ case

Similarly $P_n = \dfrac{e^{-\rho}\rho^n}{n!}$ is the Poisson's distribution

.

.

Problem 6 (Galleger 3.8)

a. Given λ=10packets/sec
   τ(Transmission Time)=20msec=.02sec

   $\because$ Probability [Inter-arrival time < Transmission Time] = 1-Prob.[no arrival of Packets]
   $$= 1 - e^{-\lambda\tau}$$

   $\therefore$ Probability of no collision with the predecessor $= 1 - (1 - e^{-\lambda\tau})$
   Also Probability of no collision with the predecessor and successor
   $$= e^{-\lambda\tau} * e^{-\lambda\tau}$$
   $$= e^{-.(0.2+0.2)} = e^{-0.4} = 0.67$$
   Also Prob. that a packet does-not collide with another Packet will be the same = 0.67

Problem 7 (Galleger 3.9)

λ = 150 packets/min/session
μ = $50*10^3$ bits/sec / 1000 bits = 50 packets / sec = 3000 packets / min
= 300 packets/min/session
$\rho = \lambda/\mu$ =150/300 = 0.5

a) For TDM with 10 channels (m=10)
   E[n] = mρ/(1-ρ) = 10*0.5/(1-0.5) = 10 packets
   E[T] = m / (μ-λ) = 10 / (300-150)=$6.67*10^{-3}$ min = 0.4 sec
   E[q] = E[n]-mρ=10-10*0.5=5 packets

   For Statistical Multiplexing
   E[n] = ρ/(1-ρ) =0.5 / (1-0.5) = 1 packet
   E[T] = 1 / (μ-λ) = 1/(3000-1500) = $6.67*10^{-4}$ min = 0.04 sec
   E[q] = E[n]-ρ=1-0.5=0.5 packet

b) In case of five sessions have rates 250 packets/min and other five have rates of 50 packets/min

For the 1st five sessions	For the 2nd five sessions
λ =250*5=1250 packets/min	λ =50*5=250 packets/min
μ = 3000*5/10=1500 packets/min	μ =3000*5/10=1500packets/min
$\rho = \lambda/\mu$ =1250/1500=0.833	$\rho = \lambda/\mu$ =1250/1500=0.166
E[n]=5*0.83/(1-0.83)=24.9 packets	E[n]=5*0.166/(1-0.166)=1.1packets

E[n] total = 24.9+1.1=26 packets

E[T]=m/(μ-λ)=5/ (1500-1250)=0.02min E[T] =5/(1500-250)=4*10^{-3} min

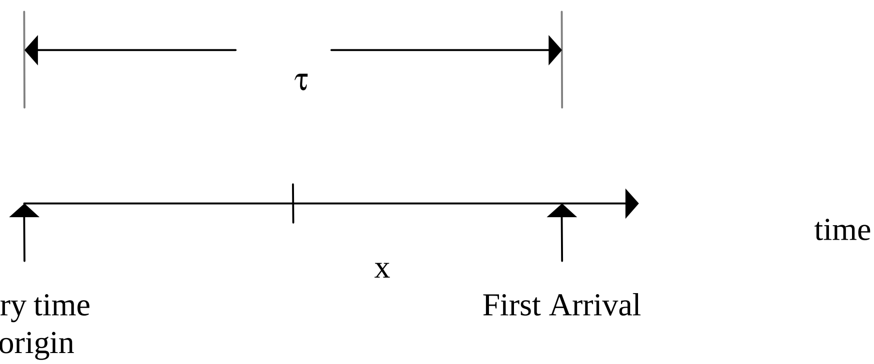
E[T] total = (0.02+0.004)*60=1.44 sec

E[q]=E[n]-mρ=24.9-5*0.83=20.735 packets E[q]=1.1-5*0.16=0.27 packets

E[q] total =20.735+0.27=21 packets

Problem 8 (Galleger 3.10)

Consider the time diagram shown



a. Let τ representing the time the first arrival after some arbitary time origin.
Take any value x.
No arrival occur in the interval (0,x) if and only if τ > x.
The probability that no arrivals ocuur in (0,x); i.e.
P(τ > x)=prob.(no arrivals in (o,x))

For Poisson's distribution
$p(k) = (\lambda t)^k e^{-\lambda t} / k!$
here k=0
$\therefore\ p(\tau > x) = e^{-\lambda x}$
Then the prob. that $\tau \le x = 1 - e^{-\lambda x}$

On the basis, the probability of packet arrivals in the small time interval (t , t+δt) is just λδt+o(δt)
Let $N^{(i)}(t, t+\varphi\delta t)$ be the number of events in Poisson process I, I=1,2,….m in the interval $(t, t+\delta t)$.
Let $N(t, t+\delta t)$ be the total number of events from the whole stream.

Then $\Pr ob.[N(t, t+\delta t) = 0] = \prod_{i=1}^{n} prob.[N^{(i)}(t, t+\delta t) = 0]$

Since the probability of no packet arrival in the time interval $(t.t+\delta t)$ is $(1-\lambda_i \delta t + o(\delta t))$

$$\therefore\ prob.[N(t,t+\delta t)=0] = \prod_{i=1}^{n}(1-\lambda_i \delta t + o(\delta t))$$

$$= 1-\lambda\delta t + o(\delta t) + \text{ higher powers of } (\delta t) \text{ that goes to zero}$$

Where $\lambda = \sum_{i=1}^{m} \lambda_i$, since the individual process are independent

Now $prob.[N(t,t+\delta t)=1] = \prod_{i=1}^{n} \lambda_i \delta t + o(\delta t)$

$$= \lambda\delta t + o(\delta t) + \text{ higher powers of } (\delta t) \text{ which goes to zero}$$

$prob.[N(t,t+\delta t) > 2]$

$$= 1 - prob.[N(t,t+\delta t)=0] - prob.[N(t,t+\delta t)=1]$$

$$= o(\delta t)$$

b. For a R .V. z with Poisson distribution

$P_z[(N_1+N_2)=n]$ is given by

$$P_z(n) = \sum_{k=0}^{n} P[N_1 = k]P[N_2 = n-k]$$

$$= \sum_{k=0}^{n} P_x(k)P_y(n-k)$$

$$= \sum \frac{1}{k!}\frac{1}{(n-k)!}e^{-(\lambda_1+\lambda_2)}\lambda_1{}^k \lambda_2{}^{n-k}$$

Recall from the Binomial theorem

$$\sum_{k=0}^{n}\binom{n}{k}\lambda_1{}^k \lambda_2{}^{n-k} = (\lambda_1+\lambda_2)^n$$

Then $P_z(n) = \frac{1}{n!}e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\binom{n}{k}\lambda_1{}^k \lambda_2{}^{n-k}$

$$= \frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)} \text{ for } n \geq 0$$

Then the number of arrivals in the union of the intervals is Poisson distributed with parameter $\lambda_1+\lambda_2$

c. P[1 arrival from A_1 prior to t |1 occurred]

$$= \frac{p[1 arrival from A_1 prior to t, 0 from A_2}{P[1 arrival occured]}$$

$$= \frac{\lambda_1 t e^{-\lambda_1 t}e^{-\lambda_2 t}}{\lambda t e^{-\lambda t}} = \frac{\lambda_1}{\lambda}$$

d. For all $s \in [t_1,t_2]$

$$= \frac{P[1 arrival occured in [t_1, s), 0 arrivals occured in [s, t_2]]}{P[1 arrival occured]}$$

$$= \frac{\lambda(s-t_1)e^{-\lambda(s-t_1)}e^{-\lambda(t_2-s)}}{\lambda(t_2-t_1)e^{-\lambda(t_2-t_1)}} = \frac{s-t_1}{t_2-t_1}$$

Hence conditional on the knowledge that only one arrival occurred, the time of this arrival is uniformly distributed in $[t_1,t_2]$.