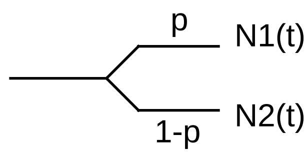
Homework Set No. 5

ECE 642

## Problem 3.11(Gallager)

**a**) The arrivals are split as follows:




Packet arrivals in $[0, t] = N(t) = N_1(t) + N_2(t)$.

$$\Pr\{N_1(t)=n, N_2(t)=m\}$$

$$=\sum_{k=0}^{\infty}\Pr\{N_1(t)=n, N_2(t)=m \mid N(t)=k\}\Pr\{N(t)=k\}$$

Note that $\Pr\{N_1(t)=n, N_2(t)=m \mid N(t)=k\}=0$ for $k \neq n+m$

$$=\Pr\{N_1(t)=n, N_2(t)=m \mid N(t)=n+m\}\Pr\{N(t)=n+m\}$$

$$=\Pr\{N_1(t)=n, N_2(t)=m \mid N(t)=n+m\}\frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}$$

Given n+m arrivals the probability of $N_1$ and $N_2$ is binomial, therefore:

$$=\binom{n+m}{n}p^n(1-p)^m\frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}$$

$$\boxed{=\frac{e^{-\lambda tp}(\lambda tp)^n}{n!}\frac{e^{-\lambda t(1-p)}\left(\lambda t(1-p)\right)^m}{m!}}$$

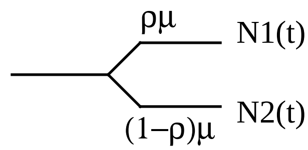
To prove that they are independent we need to find out the distribution of $N_1$:

$$\Pr\{N_1(t)=n\}=\sum_{m=0}^{\infty}\Pr\{N_1(t)=n, N_2(t)=m\}$$

$$=\frac{e^{-\lambda tp}(\lambda tp)^n}{n!}\sum_{m=0}^{\infty}\frac{e^{-\lambda t(1-p)}\left(\lambda t(1-p)\right)^m}{m!}\ \boxed{=\frac{e^{-\lambda tp}(\lambda tp)^n}{n!}}$$

Which is a Poisson process with rate $\lambda p$, and the other is $\lambda(1-p)$.

**b**) This system can be modeled as follows:




From the above problem, and knowing that interarrival times of the Poisson process are exponential we have the parameter of the Poisson as $(1-\rho)\mu = \mu - \lambda$, therefore the distribution is:

$$\boxed{P\{\mathrm{T}_i \geq \tau\} = e^{-(\mu-\lambda)\tau}}$$

## Problem 3.14(Gallager)

**a)** From the initial information we can determine the service time:

$$1/\mu = L/C$$

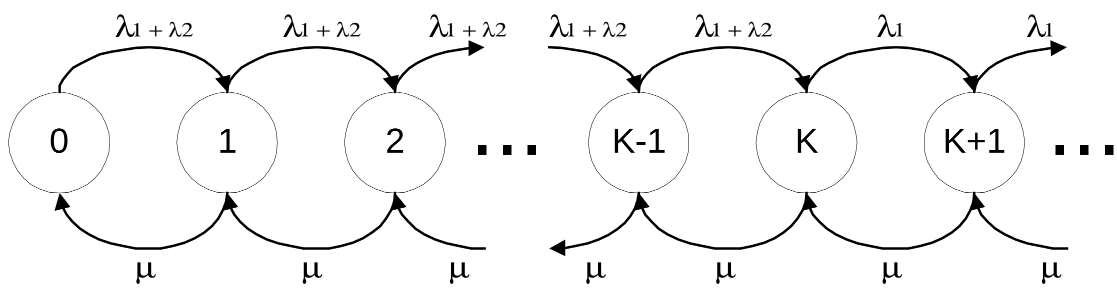
and the state dependent arrival rates:

$$\lambda_i = \begin{cases} \lambda_1 + \lambda_2 & i < K \\ \lambda_1 & i \geq K \end{cases}$$

The bounds are easily determined: $0 \leq \lambda_2$ Bounded by K packets in system

$0 \leq \lambda_1 < \mu$ Must keep $\rho < 1$ as system approaches $\infty$

**b)** The Markov state chain diagram is as follows:




Writing the state equations yields: $P_n = \begin{cases} P_0\rho^n & n < K \\ P_0\rho^K \rho_1^{\,n-K} & n \geq K \end{cases}$ $\qquad \rho = (\lambda_1 + \lambda_2)/\mu$, $\quad \rho_1 = \lambda_1/\mu$

To solve for the value of $P_0$ we proceed in the following way:

$$P_0^{-1} = \left[\frac{\rho^K - 1}{\rho - 1}\right] + \rho^K \rho_1^{-K}\left[\frac{-\rho_1^K}{\rho_1 - 1}\right]$$

$$\sum_{i=0}^{\infty} P_i = 1 \Rightarrow P_0\left[\sum_{n=0}^{K-1}\rho^n + \sum_{n=K}^{\infty}\rho_1^{n-K}\rho^K\right] = 1$$

$$= \frac{\rho^K - 1}{\rho - 1} - \frac{\rho^K}{\rho_1 - 1}$$

$$P_0^{-1} = \frac{\rho^K \rho_1 - \rho^K - \rho_1 + 1 - \rho^K \rho + \rho^K}{(\rho - 1)(\rho_1 - 1)}$$

$$P_0 = \frac{(1-\rho)(1-\rho_1)}{\left[1 - \rho_1 - \rho^K(\rho - \rho_1)\right]} \qquad \rho < 1$$

For ρ=1 we have:

$$P_0^{-1} = K + \rho^K \rho_1^{-K} \left[\frac{-\rho_1^K}{\rho_1 - 1}\right] = K + \frac{\rho^K}{1-\rho_1} = K + \frac{1}{1-\rho_1}$$

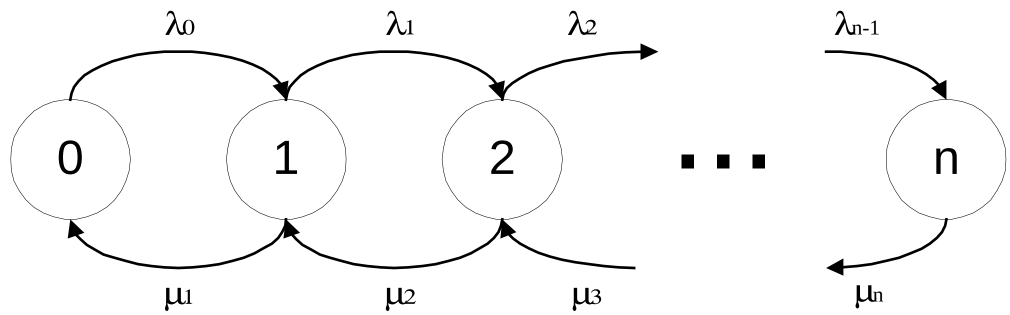
$$P_0 = \frac{1-\rho_1}{1 + K(1-\rho_1)} \qquad \rho = 1$$

In calculating the average number in the system geometric distribution's derivative will be used:

$$\sum_{k=M}^{N} r^k = \frac{r^{N+1} - r^M}{r-1} \Rightarrow \sum_{k=M}^{N} k r^k = \frac{r^{N+1}\left[(N+1)r + r - (N+1)\right] - r^{M-1}\left[Mr + r - M\right]}{(r-1)^2}$$

## Problem 3.16(Gallager)

The Markov chain and steady state diagram is as follows:




Writing the conservation equations yields: $\rho_0 p_0 = p_1$

$$\rho_1 p_1 = p_2 = \rho_0 \rho_1 p_0$$

$$\boxed{p_{n+1} = (\rho_0 \ldots \rho_n) p_0}$$

To solve for $P_0$ we use the following property:

$$\sum_{k=0}^{\infty} p_k = 1 \Rightarrow p_0 + p_0\rho_0 + p_0\rho_0\rho_1 + p_0\rho_0\rho_1\rho_2 + \ldots$$

$$\boxed{p_0 = \left[1 + \sum_{k=0}^{\infty} (\rho_0 \rho_1 \ldots \rho_k)\right]^{-1}}$$

Problem 3-23(Gallager)

The probability that the servers from 1 ... n are busy is equal to the probability that there are n customers being served on n servers:

$$P_n = \frac{\rho^n}{n!} P_o$$

Let $\lambda_n$ be the arrival rate at the n^th^ server.

Let $H_n$ be the arrival rate at the servers greater than 'n' (which happen if there are 'n' customers in system)

$$H_n = P_n \lambda$$

So, $\lambda_n = H_{n-1} - H_n = (P_{n-1} - P_n)\lambda$

The fraction of time in which all servers are busy is $\rho = \lambda/\mu$

Therefore the fraction of time the n^th^ server is busy is

$$= \frac{\lambda_n}{\mu} = (P_{n-1} - P_n) \lambda/\mu$$

Problem 3-30(Gallager)

(a) $P_0$= Pr{the system is empty} $=1 - \rho$

$$= 1 - \lambda\overline{x}$$

where $\rho = \lambda/\mu$ (utilization factor) and

$\overline{x} =$

$1/\mu$ is the average service time .

(b) The length of an idle period is the interarrival time between two typical customer arrivals. It has an exponential distribution with parameter $\lambda$. Its average length is $I=1/\lambda$.

(c) Let B = average length of busy period. Then:

$$\Pr\{system\ is\ busy\} = \lambda\overline{X} = \frac{B}{B+I} \Rightarrow B = \frac{\overline{X}}{1-\lambda\overline{X}}$$

(d) Avg. # of customers served in busy periods $= \dfrac{\text{Avg. length of a busy period}}{Avg\ service\ time} = \dfrac{1}{1-\lambda\overline{X}}$

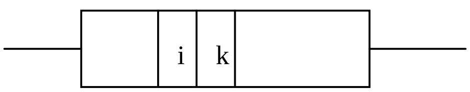
Problem 3-32(Gallager)

M/G/1 System with arbitrary order of service.
P-K Formula

$$W = \frac{\lambda E[x^2]}{2(1-\rho)}$$

We have to prove that this formula remains valid in the case the relative order of customer chosen is independent of the service time of the customer in service.
Let $N_Q(t)$ be the number of customers in the queue at time *t*
Is independent of the service time.

U=R+ρw
i.e. U = average steady state delay
The p-k formula is independent of the order of the customer service




If i , k exchange positions then expected service time of i is

$$U_i = R_i + \sum_{k=1-N_i}^{i-1} X_k$$

Take expectation and using independence of R.v. $N_I$ and $x_{i-1}$

$$E[U_i] = E[R_i] + E[\sum_{k=1-N_i}^{i-1} E[X_k / N_i]]$$

$$E[U_i] = E[R_i] + \overline{X}E[N_i]$$

As i goes to ∞

$U=R+N_Q/\mu$

$N_Q=\lambda W$

Therefore

U=R+ρW

Therefore the mean Residual time doesn't depend on the relative order of the customer service.