George Mason University

Department of Electrical and Computer Engineering

ECE 528: Introduction to Random Processes in ECE — Fall 2019

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Examples 9.1 and 9.3 Details - Random Binary Sequence

Let ξ be a number selected at random from the interval $S = [0, 1]$ and let $b_1 b_2 ...$ be the binary expansion of ξ:

$$\xi = \sum_{i=1}^{\infty} b_i 2^{-i} \qquad \text{where } b_i \in \{0, 1\}$$

Define the discrete-time random process $X(n, \xi)$ by

$$X(n, \xi) = b_n \qquad n = 1, 2, ...$$

The resulting process is sequence of binary numbers, with $X(n, \xi)$ equal to the nth number in the binary expansion of ξ.

For example, assume ξ is selected as 0.75. The binary expansion of ξ is:

$$\xi = 0.75 = 0.5 + 0.25 = b_1 2^{-1} + b_2 2^{-2} \qquad \text{where } b_1 = 1, b_2 = 1, b_i = 0 \ \ for\ i = 3, 4, ...$$

So, for $\xi = 0.75$, $X(1, \xi)$ =$b_1 = 1$. $X(2, \xi)$ =$b_2 = 1$. and $X(3, \xi)$ =$b_3 = 0$. Using alternate notation, since $X(n, \xi)$ is a discrete-time random process, it can be denoted as $X_n = X(n, \xi)$, that is, $X_1 = X(1, \xi)$.

Now let's find the probability of the event $X_1 = 1$. Note that when $X_1 = 1$ that means $b_1 = 1$ which implies $\xi \to 1/2$, when $b_i = 0$ for i =2,3,... and $\xi \to 1$, when $b_i = 1$ for i =2,3,...Therefore, for the event $X_1 = 1$, 1/2 <ξ <1.

Similarly, we can find the probability of the event $X(2, \xi) = 0$ i.e, when $X_2 = 0$ which means $b_2 = 0$ and in turn implies $\xi = 0$, when $b_i = 0$ for i =1,2,3,... and $\xi = 3/4$, when $b_i = 1$ for i =1,3,...Therefore, for the event $X_2 = 0$, 0 <ξ <3/4. Similar to the above, we can find the probability of the event $X(2, \xi) = 1$ (note that for $X(2, \xi) = 1$, $b_2 = 1$) as 0 <ξ <1.

Now let's find the following probabilities for the random process:

1. $P[X(1, \xi) = 0]$
2. $P[X(1, \xi) = 0$ and $X(2, \xi) = 1]$

The probabilities are obtained by finding the equivalent events in terms of ξ:

$$P[X(1,\xi) = 0] = P[0 \le \xi \le \frac{1}{2}] = \frac{1}{2}$$
$$P[X(1,\xi) = 0 \text{ and } X(2,\xi) = 1] = P[\frac{1}{4} \le \xi \le \frac{1}{2}] = \frac{1}{4},$$

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$$P[X(1,\xi) = 0 \text{ and } X(2,\xi) = 1] = P[\frac{1}{4} \le \xi \le \frac{1}{2}] = \frac{1}{4},$$

since all points in the interval $0 \le \xi \le 1$ begin with $b_1 = 0$ and all points in $[1/4, 1/2)$ begin with $b_1 = 0$ and $b_2 = 1$. Clearly, any sequence of k bits has a corresponding subinterval of length (and hence probability) 2^{-k}.

In class we also found the joint pmf of X_1 and X_2. That is, we found the probabilities of the events $X_1 = i$, $X_2 = j$ in terms of the probabilities of the equivalent events in terms of ξ. We further showed that X_1 and X_2 are independent random variables,