CNL
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# ECE 528 – Introduction to Random Processes in ECE
## Lecture 15: Examples of Random Processes

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# Random Walk Process

- Let $D_n$ be the iid step process of $\pm 1$ RVs.
- Let $S_n$ be the corresponding sum process.
- $S_n$, the position of the particle at time n, is an example of a **one-dimensional random walk**.

# Bernoulli & Step Random Processes

- Let $D_n = 2I_n - 1$, where $I_n$ is the Bernoulli random process, then

$$D_n = \begin{cases} 1 & \text{if } I_n = 1 \\ -1 & \text{if } I_n = 0 \end{cases}$$

$$\begin{aligned} E[D_n] &= 1P[I = 1] - 1P[I = -1] \\ &= p - (1 - p) = 2p - 1 \\ VAR[D_n] &= E[D_n{}^2] - E[D_n]^2 \\ &= 1 - (2p - 1)^2 = 4p - 4p^2 = 4p(1 - p) \end{aligned}$$

# Increments in Random Walk Process

- Random walk has independent and stationary increments because it is a sum process.
- Let k be the number of successes in n steps, then the net increment is: $\Delta S_n = k - (n - k) = 2k - n$
- If n is even, then increment must be even.
- If n is odd, then increment must be odd.

# Gaussian Random Process

- X(t) is a **Gaussian random process** if the samples $X_1 = X(t_1)$, $X_2 = X(t_2)$,…, $X_k = X(t_k)$ are jointly Gaussian random variables for all k, and all choices of $t_1$, …, $t_k$

$$f_{X_1,X_2,\ldots,X_k}(x_1,x_2,\ldots,x_k) = \frac{e^{-1/2(\mathbf{x}-\mathbf{m})^T K^{-1}(\mathbf{x}-\mathbf{m})}}{(2\pi)^{k/2}\left|K\right|^{1/2}}$$

# Properties of Gaussian Random Process

- Joint pdfs of Gaussian random processes are specified by
  - mean of the of process
  - the covariance function
- Linear operation on a Gaussian process (e.g., sum, derivative, integral) results in another Gaussian process.

# Wiener Process

- Let the symmetric random walk process ($p = ½$) takes steps of magnitude h every δ seconds.

$$X_{\delta}(t) = n(D_1 + D_2 + \ldots + D_n) = hS_n$$

- The mean and variance of $X_{\delta}(t)$ are:

$$E[X_{\delta}(t)] = hE[S_n] = 0$$

and

$$VAR[X_{\delta}(t)] = h^2 n VAR[D_n] = h^2 n 4p(1-p) = h^2 n$$

# Wiener Process (Cont'd)

- At time t it will have taken n = [t/δ] jumps.
- We obtain a continuous-time process by letting

$$h = \sqrt{\alpha\delta} \rightarrow 0$$

- The resulting limiting process X(t) has mean and variance

$$E[X(t)] = 0 \quad \text{and } \mathrm{VAR}[X(t)] = h^2 n = \alpha\delta(t/\delta) = \alpha t$$

# Wiener Process & Brownian Motion

- X(t) called **Wiener process**, with the following properties:
  - Begins at the origin
  - Zero mean for all time
  - Variance increases linearly with time.

- Models Brownian Motion, the motion of particles suspended in a fluid that move under the rapid and random impact of neighboring particles.

# Properties of Wiener Process

$$X(t) = \lim_{\delta \to 0} hS_n = \lim_{n \to 0} \sqrt{\alpha t}\,\frac{S_n}{\sqrt{n}}$$

- By the Central Limit Theorem, $X(t)$ approaches a Gaussian RV with zero mean and variance $\alpha t$.
- Because $X(t)$ is the limit of a sum process:
  - $C_X(t_1,t_2)=\alpha \min(t_1,t_2)$
  - Independent & stationary increments with Gaussian pdf with zero mean and variance $\alpha t$

# The Wiener Process is Gaussian

$$X(t_1) = X(t_1)$$

$$X(t_2) = X(t_1) + (X(t_2) - X(t_1))$$

$$X(t_3) = X(t_1) + (X(t_2) - X(t_1)) + (X(t_3) - X(t_2))$$

$$\cdots$$

$$X(t_n) = X(t_1) + (X(t_2) - X(t_1)) + \ldots + (X(t_n) - X(t_{n-1}))$$

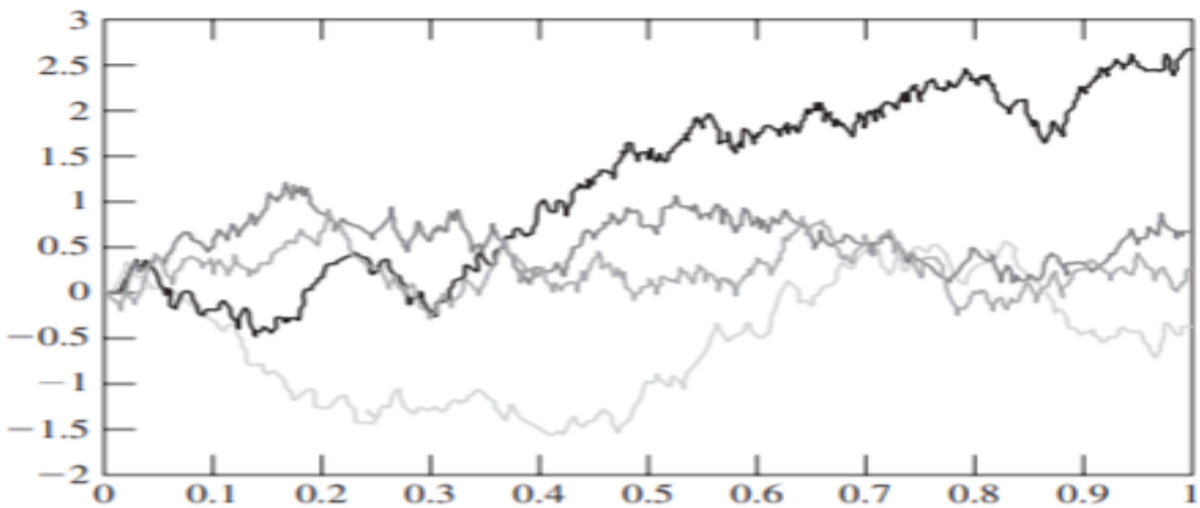


**FIGURE 9.12**
Four sample functions of the Wiener process.

# Random Telegraph Signal

- Let X(t) be $\pm 1$.
- Suppose that X(0) = $\pm 1$ with probability ½.
- X(t) changes polarity with exponentially distributed inter-event times with rate α.

$$P[X(t) = \pm 1]$$

# Transition Probabilities

$$P[X(t) = \pm 1 \mid X(0) = \pm 1] =$$

$$P[X(t) = \pm 1 \mid X(0) = \mp 1] =$$

$$P[X(t) = 1] =$$

$$P[X(t) = -1] =$$

# Mean and Autocovariance Functions

$$m_X(t) = 1\,P[X(t)=1] + (-1)P[X(t)=-1] = 0$$

$$\mathrm{VAR}[X(t)] = E[X(t)^2] = (1)^2 P[X(t)=1] + (-1)^2 P[X(t)=-1] = 1$$

$$C_X(t_1,t_2) = E[X(t_1)X(t_2)]$$

# Lecture Summary

- The random walk process is a sum process obtained as the sum of iid steps of size $\pm 1$.
- The Wiener process is a Gaussian random process that has zero mean and variance that increases linearly with time.
- The Wiener process has independent and stationary increments.

# Lecture Summary (Cont'd)

- Random telegraph process changes polarity according to exponential interevent times.
- Random telegraph process has stationary transition probabilities.