ECE 528: Introduction to Random Processes in ECE

Fall 2019

Solutions to Homework Set No. 3

2.29 Each transmission is equivalent to tossing a fair coin. If the outcome is heads, then the transmission is successful. If it is tails, then another transmission is required. As in Example 2.11 the probability that j transmissions are required is:

$$P[j] = \left(\frac{1}{2}\right)^j$$

$$P[A] = P[j \text{ even}] = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k} = \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k} - 1 = \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3}.$$

$$P[B] = P[j \text{ multiple of } 3] = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{3k} = \frac{1}{1-\frac{1}{8}} - 1 = \frac{1}{7}.$$

$$P[C] = \sum_{k=1}^{6}\left(\frac{1}{2}\right)^{k} = \frac{1}{2}\sum_{k=0}^{5}\left(\frac{1}{2}\right)^{k} = \frac{1}{2}\frac{1-(\frac{1}{2})^6}{1-\frac{1}{2}} = \frac{63}{64}.$$

$$P[C^c] = 1 - P[C] = \frac{1}{64}.$$

$$P[A \cap B] = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{6k} = \frac{1}{1-\frac{1}{64}} - 1 = \frac{1}{63}$$ since a multiple of 2 and 3 is a multiple of 6.

$$P[A - B] = P[A] - P[A \cap B] = \frac{1}{3} - \frac{1}{63} = \frac{20}{63}$$ since

$$A = (A-B) \cup (A \cap B)] \text{ and } (A-B) \cap (A \cap B) = \phi.$$

$$P[A \cap B \cap C] = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$ since $A \cap B \cap C = \{6\}$.

2.54a The number of ways of choosing M out of 100 is $\binom{100}{M}$. This is the total number of equiprobable outcomes in the sample space.

We are interested in the outcomes in which m of the chosen items are defective and $M-m$ are nondefective.

The number of ways of choosing m defectives out of k is $\binom{k}{m}$.

The number of ways of choosing $M-m$ nondefectives out of $100-k$ is $\binom{100-k}{M-m}$.

The number of ways of choosing m defectives out of k *and* $M-m$ non-defectives out of $100-k$ is

$$\binom{k}{m}\binom{100-k}{M-m}$$

$$P[m \text{ defectives in } M \text{ samples}] = \frac{\# \text{ outcomes with } k \text{ defective}}{\text{Total } \# \text{ of outcomes}} = \frac{\binom{k}{m}\binom{100-k}{M-m}}{\binom{100}{M}}$$

This is called the Hypergeometric distribution.

$$\text{b) } P[\text{lot accepted}] = P[m=0 \text{ or } m=1] = \frac{\binom{100-k}{M}}{\binom{100}{M}} + \frac{k\binom{100-k}{M-1}}{\binom{100}{M}}.$$

2.77

Assume X is the input and Y is the output:

(a) $P(Y=0) = p*(1-\varepsilon_1) + (1-p)*\varepsilon_2$;

(b) $P(X=0|Y=1) = p*\varepsilon_1/[p*\varepsilon_1 + (1-p)*(1-\varepsilon_2)]$;

$P(X=1|Y=1) = (1-p)*(1-\varepsilon_2)/[p*\varepsilon_1 + (1-p)*(1-\varepsilon_2)]$;

Then: if $p*\varepsilon_1>(1-p)*(1-\varepsilon_2)$, the first one has a higher probability, otherwise the second one has a higher probability.

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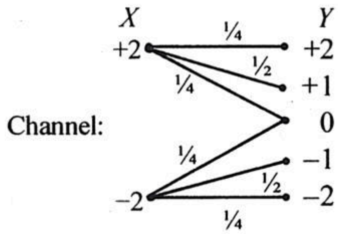
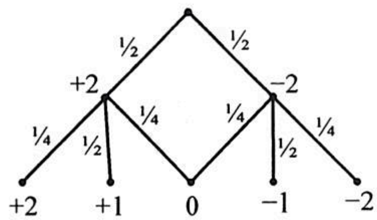
2.78 a)





b) $P[X=+2,Y=+2]=P[Y=2\,|\,X=2]P[X=2]$
$=\frac{1}{4}\frac{1}{2}=\frac{1}{8}$

$P[X=2,Y=1]=\frac{1}{2}\frac{1}{2}=\frac{1}{4}$

$P[X=2,Y=0]=\frac{1}{2}\frac{1}{4}=\frac{1}{8}$

$P[X=-2,Y=0]=\frac{1}{2}\frac{1}{4}=\frac{1}{8}$

$P[X=-2,Y=1]=\frac{1}{2}\frac{1}{2}=\frac{1}{4}$

$P[X=-2,Y=-2]=\frac{1}{2}\frac{1}{4}=\frac{1}{8}$

c) $P[Y=+2]=\frac{1}{2}\frac{1}{4}=\frac{1}{8}=P[Y=-2]$

$P[Y=+1]=\frac{1}{2}\frac{1}{2}=\frac{1}{4}=P[Y=-1]$

$P[Y=0]=2\left(\frac{1}{2}\frac{1}{4}\right)=\frac{2}{8}=P[Y=0]$

d) $$P[X=2\,|\,Y=k]=\frac{P[Y=k\,|\,X=2]P[X=2]}{P[Y=k]}$$

$$=\begin{cases}\frac{1}{8}\big/\frac{1}{8}=1 & k=2\\ \frac{1}{4}\big/\frac{1}{4}=1 & k=1\\ \frac{1}{8}\big/\frac{1}{4}=\frac{1}{2} & k=0\\ 0 & \text{other } k\end{cases}$$

2.81

(a) P(Y=0) = 1/3*(1-ε) + 1/3*ε = 1/3; P(Y=1)= P(Y=2)= P(Y=0)=1/3;

(b) P(X=0|Y=1)= ε, P(X=1|Y=1)= 1-ε, P(X=2|Y=1)=0.

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2.86

Proof: If $P(A|B)=P(A|B^C)$, then we know $P(A\cap B)/P(B) = P(A\cap B^C)/P(B^C)$. Therefore, $P(A\cap B)* P(B^C)= P(A\cap B^C) * P(B)=[P(A)-P(A\cap B)]* P(B)$. after rearranging $P(A\cap B)*[P(B^C)+ P(B)] = P(A\cap B) = P(A)*P(B)$. Hence, A and B are independent.

2.97 a) $P[0 \text{ or } 1 \text{ errors}] = (1-p)^{100} + 100(1-p)^{99} p \qquad p = 10^{-2}$

$$= 0.3660 + 0.3697$$
$$= 0.7357$$

b) $p_R = P[\text{retransmission required}] = 1 - P[0 \text{ or } 1 \text{ errors}] = 0.2642$

$$P[M \text{ retransmissions in total}] = (1-p_R)p_R^M \qquad M = 0,1,2,\ldots$$

$$P[M \text{ or more retransmissions required}] = \sum_{j=M}^{\infty}(1-p_R)p_R^j = p_R^M\sum_{j=0}^{\infty}(1-p_R)p_R^j$$
$$= p_R^M$$

2.105

(a)P(k)=(1-1/2)^(k-1)*1/2=(1/2)^k;

(b)When k<=5, P(k)=(1/2)^k. When k=6,P(k dollars paid)=\sum_{k=6}^{+∞} P(k) = 1-P(k=1)-P(k=2) -P(k=3)-P(k=4)-P(k=5) = 1/32.

2.106 $P[k$ tosses required until heads comes up three times$] = P[$heads in kth toss $|$ 2 heads in $k-1$ tosses$]$ $P[2$ heads in $k-1$ tosses$] = P[A \mid B]\, P[B]$.

$$\text{Now } P[A \mid B] = P[2 \text{ heads in first } k-1 \text{ tosses}] = \binom{k-1}{2} p^2 (1-p)^{k-3}.$$

$$\text{Thus } P[A \mid B]P[B] = P[A \mid B]p = \binom{k-1}{2} p^3 (1-p)^{k-3} \qquad k = 3, 4, \ldots$$

2.128

(a)P(obtaining an ace)=4/52=1/13;

(b)Define event A=ace in first draw, event B=ace in second draw.

$P(A)=1/13$, $P(A^C)=12/13$. Then $P(B|A)=3/51$, $P(B|A^C)=4/51$. Hence, $P(B)= P(B|A)*P(A)+ P(B|A^C)* P(A^C)=1/13$, which is the same as in first draw. The answer does not change.

(c)P(3 ace)=C(4,3)*C(48,4)/C(52,7)=0.00582;

P(2 kings)= C(4,2)*C(48,5)/C(52,7)=0.07679;

P(3 aces and 2 kings) = C(4,2)*C(4,3)*C(44,2)/C(52,7)=0.00017;

P(3 aces or 2 kings)= P(3 ace)+ P(2 kings)- P(3 aces and 2 kings)=0.0824;

(d) Each player gets 13 cards in which an ace is contained. The probability is

$$P = \frac{4! * \frac{48!}{(12!)^4}}{\frac{52!}{(13!)^4}} = 0.1055$$