Solution to Homework Set No. 2

ECE 642

Problem 1

Let the probability of one event (arrival) in any interval Δt to be p=λΔt, while the probability of 0 events is q=1-λΔt. Using the memoryless (independent) relation, it is then apparent that Probability of one arrival in the interval $T = m\Delta t$ is $m\lambda\Delta t(1-\lambda\Delta t)^{m-1} = mpq^{m-1}$ and probability of two arrivals in the interval T= m Δt is $\frac{m(m-1)}{2}(\lambda\Delta t)^2(1-\lambda\Delta t)^{m-2} = \frac{m(m-1)}{2}p^2q^{m-2}$ and so on

$\therefore p(k) = C_k^m p^k q^{m-k}$

Problem 2

Let Δt→0 but with T=mΔt fixed

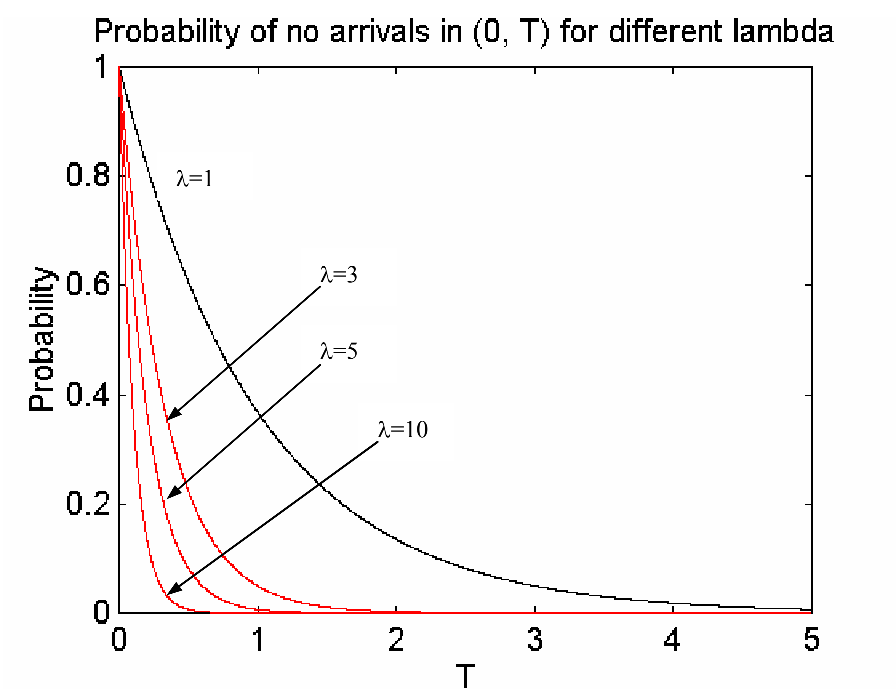
$$\lim p(k) = \lim \frac{m!}{(m-k)!k!}(\lambda\Delta t)^k(1-\lambda\Delta t)^{m-k}$$

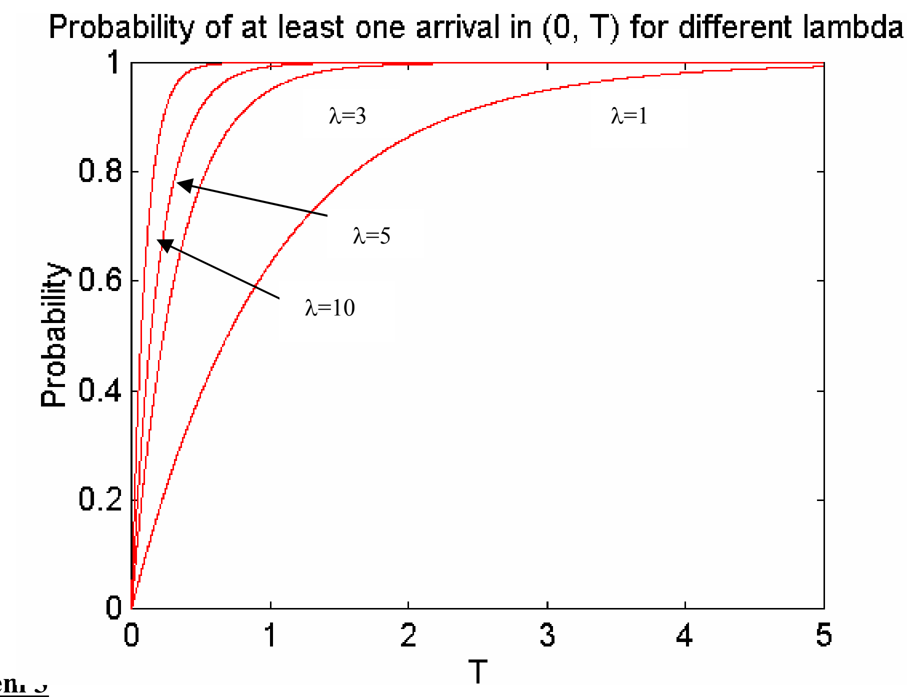
$$= \lim(1-\frac{\lambda T}{k}\Delta t)^{k/\Delta t}\frac{(\lambda T)^k}{k!}$$

since $\lim(1+at)^{k/t} = e^{ak}$

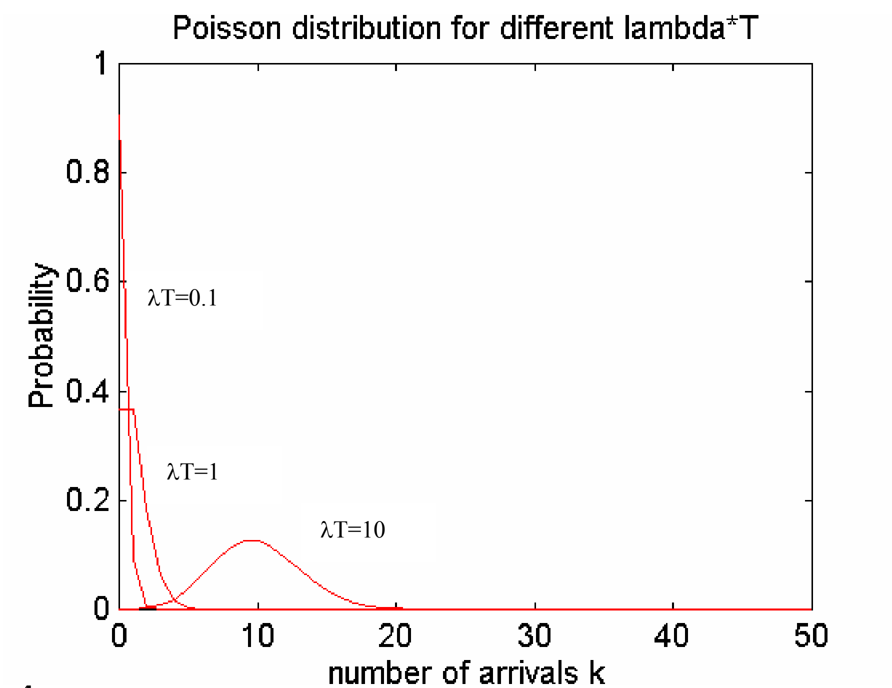
$$\therefore \lim p(k) = e^{-\lambda T}\frac{(\lambda t)^K}{K!}$$

which is the Poisson distribution. Note that all the limit is when Δt→0





Problem 3



Problem 4

On the basis, the probability of packet arrivals in the small time interval (t,t+Δt) is just λt+o(Δt)

Let $N^{(i)}(t, t+\Delta t)$ be the number of events in Poisson process I, I =1,2,….m in the interval (t,t+Δt).
Let $N(t, t+\Delta t)$ be the total number of events from the whole stream. Then

$$\Pr ob.[N(t,t+\Delta t)=0]=\prod_{i=1}^{n} prob.[N^{(i)}(t,t+\Delta t)=0]$$

Since the probability y of no packet arrival in the time interval $(t,t+\Delta t)$ is $(1-\lambda\Delta t+o(\Delta t))$

$$\therefore \Pr ob.[N(t,t+\Delta t)=0]=\prod_{i=1}^{n}(1-\lambda\Delta t+o(\Delta t))$$

$=1-\lambda\Delta t+o(\Delta t))$ +terms containing (Δt) of higher powers which goes to zero

$\lambda=\sum_{i=1}^{m}\lambda_i$, since the individual processes are independent.

$$prob.[N(t,t+\Delta t)=1]=\prod_{i=1}^{n}(\lambda_i\Delta t+o(\Delta t))$$

$=\lambda\Delta t+o(\Delta t)$ + terms containing (Δt) of high power which goes to zero.

This shows that the sum of individual Poisson processes is also a Poisson process.

$P_n(t+\Delta t) = [\,1-(\lambda+\mu)\Delta t]\,P_n(t) + \lambda\Delta t\,P_{n-1}(t) + \mu\Delta t\,P_{n+1}(t)$

At $t = 0$, the queue is empty, $P_n(0) = 0$ for $n \neq 0$ and $P_0(0) = 1$.

We'll set the problem up in a matrix-vector form. Let **P(t)** denote a vector of state probabilities $P_n(t)$ at instant t. The initial conditions can be expressed as:

$\mathbf{P}^t\mathbf{(0)}$ = [1 0 …0].

Our maximum value of n is 5. Putting $\Delta t = 1$, we have:

$$P_0(t+1) = (1-\lambda)\,P_0(t) + \mu P_1(t)$$

$n=1$ $\quad P_1(t+1) = [1-(\lambda+\mu)]\,P_1(t) + \lambda P_0(t) + \mu\Delta t\,P_2(t)$

$n=2$ $\quad P_2(t+1) = [1-(\lambda+\mu)]\,P_2(t) + \lambda P_1(t) + \mu\Delta t\,P_3(t)$

$\vdots$

$n=5$ $\quad P_5(t+1) = (1-\mu)\,P_5(t) + \lambda P_4(t),$

which can be written as:

$$\begin{bmatrix} P_0(t+1) \\ P_1(t+1) \\ P_2(t+1) \\ P_3(t+1) \\ P_4(t+1) \\ P_5(t+1) \end{bmatrix} = \begin{bmatrix} 1-\lambda & \mu & 0 & 0 & 0 & 0 \\ \lambda & 1-(\lambda+\mu) & \mu & 0 & 0 & 0 \\ 0 & \lambda & 1-(\lambda+\mu) & \mu & 0 & 0 \\ 0 & 0 & \lambda & 1-(\lambda+\mu) & \mu & 0 \\ 0 & 0 & 0 & \lambda & 1-(\lambda+\mu) & \mu \\ 0 & 0 & 0 & 0 & \lambda & 1-\mu \end{bmatrix} \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ P_3(t) \\ P_4(t) \\ P_5(t) \end{bmatrix}$$

or:

$$\mathbf{P(t+1)} = \Pi\ \mathbf{P(t)}$$

Now set $\lambda/\mu = 0.5$ (say $\lambda = 0.01$ and $\mu = 0.02$) and iteratively calculate vector **P(t +1)** using MATLAB. We could see that the steady state possibilities given by equation 2-20:

$$P_n = (1-\rho)\rho^n/(1-\rho^6) \quad n = 0,1,\ldots,5$$

are reached after sufficiently large number of iterations.

(1) From (2-12), after incorporating $(\Delta t)^2$ terms into $o(\Delta t)$, we have:

$P_n(t+\Delta t) = P_n(t)[1-(\lambda+\mu)\Delta t + o(\Delta t)] + P_{n-1}(t)[\lambda\Delta t + o(\Delta t)] + P_{n+1}(t)[\mu\Delta t + o(\Delta t)$, $n \geq 1$.

neglecting $o(\Delta t)$ and using (2 –13), we obtain:

$$\frac{dP_n(t)}{dt}\Delta t + P_n(t) = (1-(\lambda+\mu)\Delta t)P_n(t) + \lambda\Delta t P_{n-1}(t) + \mu\Delta t P_{n+1}(t)$$

and:

$$\frac{dP_n(t)}{dt} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t); n \geq 1$$

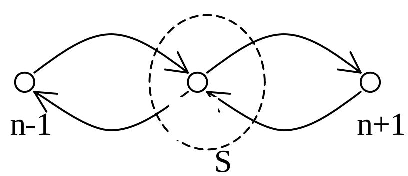
Assuming steady state exists,

$$\frac{dP_n(t)}{dt} = 0$$

hence (2-15)=>

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}; n \geq 1$$

(2)



Using the flow balance argument, The " probability flux " leaving and entering surface **S** must be equal to,

$$\therefore (\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$$

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Using the flow balance argument (2-38) =>

$$\therefore (\lambda_n + \mu_n)P_n = \lambda_{n-1}P_{n-1} + \mu_{n-1}P_{n+1}; n \geq 1$$

It is readily shown that

$$\lambda_n P_n = \mu_{n+1}P_{n+1}$$

is an equivalent equation.

Then:

$$\frac{P_n}{P_{n-1}} = \frac{\lambda_{n-1}}{\mu_n}$$

$$\frac{P_n}{P_n} = \frac{\lambda_{n-1}\lambda_{n-2}}{\mu_n \mu_{n-1}}$$

(2-40) =>

$$\frac{P_n}{P_0} = \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i}$$

Class drill problems

$$G_x(z) = E(z^x) = \sum_{j=0}^{\infty} p_j.z^j$$

a.)1.

$$G_x(1) = \sum_{j=0}^{\infty} p_j.1 = 1$$

2.

$$\left.\frac{dG_x(z)}{dz}\right|_{z=1} = \sum_{j=0}^{\infty} j.p_j. = E(x)$$

3.

$$\left.\frac{dG_x(z)}{dz^2}\right|_{z=1} = \sum_{j=0}^{\infty} j(j-1).p_j. = E(x^2) - E(x)$$

4.

$$G_y(z) = E(z^y) = E(z^{x_1} \ldots . z^{x_n})$$

Since x_I 's are independent :

$$E(z^y) = E(z^{x_1}) \ldots . E(z^{x_n}) = \prod_{i=1}^{n} Gx_i(z)$$

b.) 1. Poisson distribution

$$G_x(z) = \sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!}.z^j = e^{-\lambda} e^{\lambda z} = e^{-\lambda(1-z)}$$

$$E(x) = \left.\frac{dG_x(z)}{dz}\right|_{z=1} = \lambda e^{-\lambda(1-z)} \Big|_{z=1} = \lambda$$

$$\left.\frac{dG_x(z)}{dz^2}\right|_{z=1} = \lambda^2 e^{-\lambda(1-z)} \Big|_{z=1} = \lambda^2$$

$$\Rightarrow E(x^2) = \lambda^2 + \lambda \Rightarrow \sigma_x{}^2 = \lambda$$

2. Geometric distribution

$$G_x(z) = \sum_{j=1}^{\infty} pq^{j-1}.z^j = \frac{pz}{1-qz}$$

$$E(x) = \left.\frac{dG_x(z)}{dz}\right|_{z=1} = \left.\frac{p(1-qz)+qpz}{(1-qz)^2}\right|_{z=1} = \frac{1}{p}$$

$$\left.\frac{dG_x(z)}{dz^2}\right|_{z=1} = \left.\frac{p(-2)(-q)}{(1-qz)^3}\right|_{z=1} = \frac{2q}{p^2} = E(x^2) - E(x) \Rightarrow E(x^2) = \frac{1+q}{p^2}$$

$$\Rightarrow \sigma_x{}^2 = \frac{q}{p^2}$$

3. Bernoulli distribution

$$G_x(z) = p_0.z^0 + p_1 z^1 = q + pz$$

$$E(x) = \left.\frac{dG_x(z)}{dz}\right|_{z=1} = p$$

$$\left.\frac{dG_x(z)}{dz^2}\right|_{z=1} = 0 \Rightarrow E(x^2) = E(x) = p$$

4. Binomial distribution

Binomial distribution is a sum of independent Bernoulli distributions.

$$x = \sum_{i=1}^{n} x_i$$

$$G_x(z) = \prod_{i=1}^{n} G_{x_i}(z) = \prod_{i=1}^{n} (q + pz) = (q + pz)^n$$

$$E(x) = \left.\frac{dG_x(z)}{dz}\right|_{z=1} = n(pz+q)^{n-1}.p\Big|_{z=1} = np$$

$$\left.\frac{dG_x(z)}{dz^2}\right|_{z=1} = n(n-1)p^2(pz+q)^{n-2}\Big|_{z=1} = n(n-1)p^2; n \geq 2$$

$$\Rightarrow \sigma_x{}^2 = npq$$