CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE
Lecture 2: The Axioms of Probability

Bijan Jabbari, PhD
Dept. of Electrical and Computer Eng.
George Mason University
Fairfax, VA 22030-4444, USA
bjabbari@gmu.edu
http://cnl.gmu.edu/bjabbari

September 9, 2019

Note

- These slides cover material partially covered in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-set derivation and detailed discussions were need.

Random Experiments

- A random experiment is an experiment in which the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.
- A random experiment is specified by stating:
  - An experimental procedure
  - A set of 1 or more measurements and/or observations.

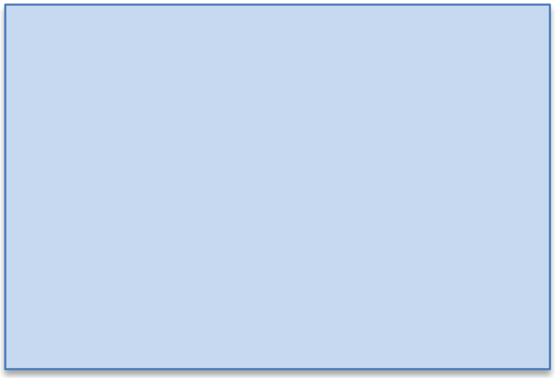
Examples of Random Experiments

E_1	A coin is tossed once; observe the outcome of the toss
E_2	A coin is tossed 3 times; note the sequence of heads and tails
E_3	The number of phone calls initiated by a community in 1 hour is counted
E_4	The round-trip time of an Internet PING packet is noted
E_5	A number in the unit interval is selected at random
E_6	The amplitudes of an audio signal at times t_0 and t_1 are measured
E_7	The amplitude signal of an entire audio signal is recorded

Sample Space

- An **outcome** or **sample point** ξ of a random experiment is a result that cannot be decomposed into other results.
  - Each performance of a random experiment results in one and only one outcome.
  - Outcomes are mutually exclusive.
- The **sample space** S is defined as the set of all possible outcomes:

$$S = \{ \xi \}$$



Sample Space (con't)

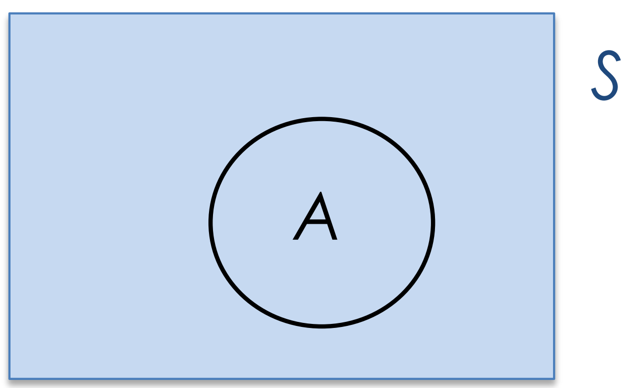
- Each performance of a random experiment can be viewed as the selection at random of a ξ from S.
- The sample space is **discrete** if S is a countable set.
- The sample space if **continuous** if S is not countable.

Examples of Sample Spaces

E_1	A coin is tossed once; observe the outcome of the toss
E_2	A coin is tossed 3 times; note the sequence of heads and tails
E_3	The number of phone calls initiated by a community in 1 hour is counted
E_4	The round-trip time of an Internet PING packet is noted
E_6	The amplitudes of an audio signal at times t_0 and t_1 are measured

Events

- Did an event occur when we conducted a random experiment?
- Did the outcome satisfy some set of conditions?
- An **event** *A* is a collection of outcomes for a random experiment *E.*
  - An event *A* is a **subset** of *S*.



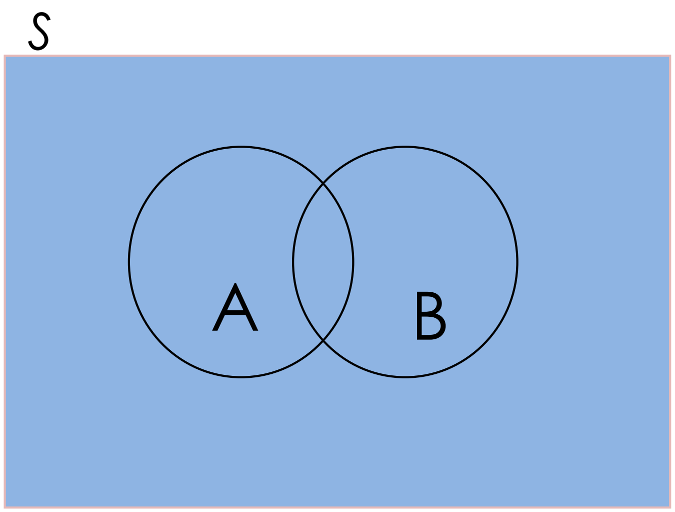
Examples of Events

- An **elementary event** is a singleton subset of a discrete sample space.

E_3	*Toss a coin three times:* *A* = more heads than tails
E_3	*B* = equal number of heads & tails
E_3	*C* = number of heads & tails are not equal

Events & Set Operations

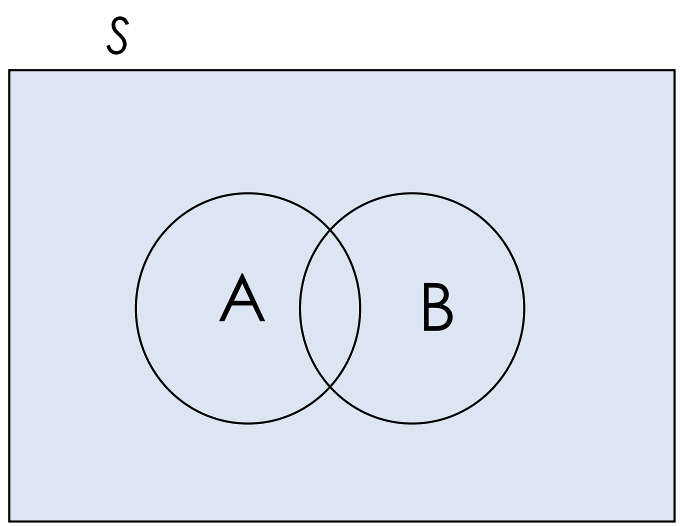
- Events can be expressed through set operations
- Union: $A \cup B = \{\xi : \xi \in A \text{ or } \xi \in B\}$



$$\bigcup_{i=1}^{n} A_i \triangleq A_1 \cup A_2 \cup \ldots \cup A_n = \{\xi : \xi \in A_i \text{ for some } i\}$$

Events & Set Operations (cont'd)

- Intersection: $A \cap B = \{\xi : \xi \in A \text{ and } \xi \in B\}$



Mutually exclusive: $A \cap B = \varnothing$

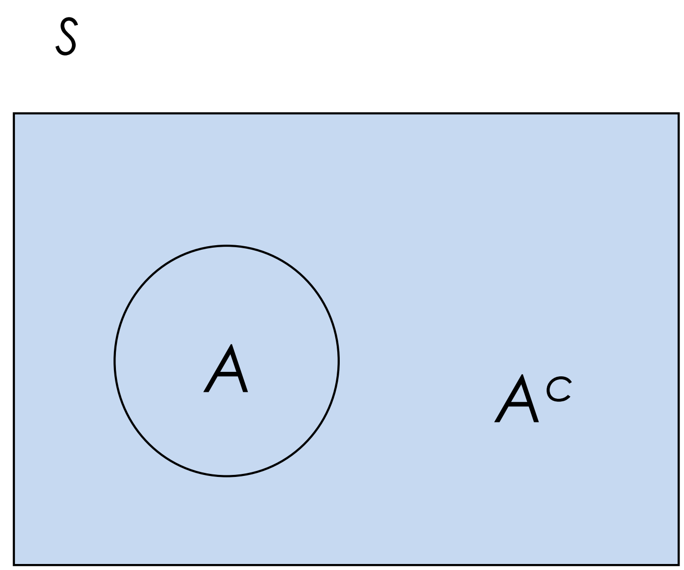
$$\bigcap_{i=1}^{n} A_i \triangleq A_1 \cap A_2 \cap \ldots \cap A_n = \{\xi : \xi \in A_i \text{ for all } i\}$$

Events & Set Operations (cont'd)

- Complementation:

$$A^c = \{\xi \in S : \xi \notin A\}$$

$$S^c = \varnothing; \ \varnothing^c = S$$



Sample Space, Events, Outcomes

- Random Experiment E:
  - Experimental procedure & set of observations and measurements
  - Outcome ξ of random experiment
- Sample Space S:
  - Set of all possible outcomes
- Events:
  - Subset A of S
  - Events obtained through set operations

Axioms of Probability

Let E be a random experiment with sample space S.

A probability law for E is a rule that assigns to each event a number $P[A]$, the probability of A, that satisfies the following axioms:

Axiom 1: $0 \leq P[A]$

Axiom 2: $P[S] = 1$

Axiom 3: If $A \cap B = \varnothing$, then $P[A \cup B] = P[A] + P[B]$

Axioms of Probability

Let E be a random experiment with sample space S.

A probability law for E is a rule that assigns to each event a number $P[A]$, the probability of A, that satisfies the following axioms:

Axiom 1: $0 \leq P[A]$

Axiom 2: $P[S] = 1$

Axiom 3: If $A \cap B = \emptyset$, then $P[A \cup B] = P[A] + P[B]$

Axioms of Probability (cont'd)

The first three axioms are sufficient to deal with finite sample spaces. For infinite sample spaces we need an additional axiom.

Axiom 3′: If $A_1, A_2, \ldots$ is a sequence of events such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P[A_k]$$

Corollaries 1, 2, & 3

Corollary 1: $P[A^c] = 1 - P[A]$

Proof: Since an event A and its complement A^c are mutually exclusive, $A \cap A^c = \varnothing$, we have from Axiom III that

$$P[A \cup A^c] = P[A] + P[A^c].$$

Since $S = A \cup A^c$, by Axiom II,

$$1 = P[S] = P[A \cup A^c] = P[A] + P[A^c].$$

The corollary follows after solving for $P[A^c]$.

Corollary 2: $P[A] \leq 1$

Corollary 3: $P[\varnothing] = 0$

Corollary 4

Corollary 4: If $A_1, A_2, \ldots A_n$ are pairwise mutually exclusive, then

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P[A_k] \text{ for } n \geq 2.$$

Proof: We use mathematical induction. Axiom III implies that the result is true for $n = 2$. Next we need to show that if the result is true for some n, then it is also true for $n + 1$. This, combined with the fact that the result is true for $n = 2$, implies that the result is true for $n \geq 2$.

Suppose that the result is true for some $n > 2$; that is,

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P[A_k], \tag{2.9}$$

and consider the $n + 1$ case

$$P\left[\bigcup_{k=1}^{n+1} A_k\right] = P\left[\left\{\bigcup_{k=1}^{n} A_k\right\} \cup A_{n+1}\right] = P\left[\bigcup_{k=1}^{n} A_k\right] + P[A_{n+1}], \tag{2.10}$$

where we have applied Axiom III to the second expression after noting that the union of events A_1 to A_n is mutually exclusive with A_{n+1}. The distributive property then implies

$$\left\{\bigcup_{k=1}^{n} A_k\right\} \cap A_{n+1} = \bigcup_{k=1}^{n} \{A_k \cap A_{n+1}\} = \bigcup_{k=1}^{n} \varnothing = \varnothing.$$

Corollary 5

Corollary 5: $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

Proof: First we decompose $A \cup B$, A, and B as unions of disjoint events. From the Venn diagram in Fig. 2.3,

$$P[A \cup B] = P[A \cap B^c] + P[B \cap A^c] + P[A \cap B]$$

$$P[A] = P[A \cap B^c] + P[A \cap B]$$

$$P[B] = P[B \cap A^c] + P[A \cap B]$$

By substituting $P[A \cap B^c]$ and $P[B \cap A^c]$ from the two lower equations into the top equation, we obtain the corollary.

Corollary 6

Corollary 6:

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P[A_j] - \sum_{j<k}^{n} P[A_j \cap A_k] + \ldots$$

$$+ (-1)^{n+1} P[A_1 \cap \ldots \cap A_n]$$

Corollary 7

Corollary 7: If $A \subset B$, then $P[A] \leq P[B]$

Proof: In Fig. 2.4, B is the union of A and $A^c \cap B$, thus

$$P[B] = P[A] + P[A^c \cap B] \geq P[A],$$

since $P[A^c \cap B] \geq 0$.

Exercise: Random Number from the Unit Interval

- Sample Space

- Probability Law

- Events

  A = {Outcome is > 0.5}

  B = {Outcome is within 0.1 of 0.6}

  C = {Outcome = 0.3}

Lecture Summary

- A random experiment is specified by an experimental procedure and a set of measurements/observations.
- An outcome or sample point of a random experiment is a result that cannot be decomposed into other results.
- The sample space specifies set of all possible outcomes.
- Events describe conditions of interest and are specified as subsets of *S.*
- When *S* is discrete, events consist of the union of elementary events.
- When *S* is continuous, events consist of the union or intersection of subsets of the real time (or plane).