APPENDIX B: SUMMARY OF RESULTS

Notation

p_n = Steady-state probability of having n customers in the system

λ = Arrival rate (inverse of average interarrival time)

μ = Service rate (inverse of average service time)

N = Average number of customers in the system

N_Q = Average number of customers waiting in queue

T = Average customer time in the system

W = Average customer waiting time in queue (does not include service time)

$\overline{X}$ = Average service time

$\overline{X^2}$ = Second moment of service time

Little's Theorem

$$N = \lambda T$$

$$N_Q = \lambda W$$

Poisson distribution with parameter m

$$p_n = \frac{e^{-m} m^n}{n!}, \quad n = 0, 1, \dots$$

$$\text{Mean} = \text{Variance} = m$$

Exponential distribution with parameter λ

$$P\{\tau \le s\} = 1 - e^{-\lambda s}, \qquad s \ge 0$$

$$\text{Density: } p(\tau) = \lambda e^{-\lambda \tau}$$

$$\text{Mean} = \frac{1}{\lambda}$$

$$\text{Variance} = \frac{1}{\lambda^2}$$

Summary of $M/M/1$ system results

1. Utilization factor (proportion of time the server is busy)

$$\rho = \frac{\lambda}{\mu}$$

2. Probability of n customers in the system

$$p_n = \rho^n (1 - \rho), \qquad n = 0, 1, \dots$$

3. Average number of customers in the system

$$N = \frac{\rho}{1 - \rho}$$

4. Average customer time in the system

$$T = \frac{\rho}{\lambda(1 - \rho)} = \frac{1}{\mu - \lambda}$$

5. Average number of customers in queue

$$N_Q = \frac{\rho^2}{1 - \rho}$$

6. Average waiting time in queue of a customer

$$W = \frac{\rho}{\mu - \lambda}$$

Summary of $M/M/m$ system results

1. Ratio of arrival rate to maximal system service rate

$$\rho = \frac{\lambda}{m\mu}$$

2. Probability of n customers in the system

$$p_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!(1-\rho)}\right]^{-1}, \qquad n = 0$$

$$p_n = \begin{cases} p_0 \dfrac{(m\rho)^n}{n!}, & n \leq m \\ p_0 \dfrac{m^m \rho^n}{m!}, & n > m \end{cases}$$

3. Probability that an arriving customer has to wait in queue (m customers or more in the system)

$$P_Q = \frac{p_0 (m\rho)^m}{m!(1-\rho)} \qquad \text{(Erlang C Formula)}$$

4. Average waiting time in queue of a customer

$$W = \frac{\rho P_Q}{\lambda(1-\rho)}$$

5. Average number of customers in queue

$$N_Q = \frac{\rho P_Q}{1-\rho}$$

6. Average customer time in the system

$$T = \frac{1}{\mu} + W$$

7. Average number of customers in the system

$$N = m\rho + \frac{\rho P_Q}{1-\rho}$$

Summary of *M/M/m/m* system results

1. Probability of m customers in the system

$$p_0 = \left[\sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right]^{-1}$$

$$p_n = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}, \qquad n = 1, 2, \ldots, m$$

2. Probability that an arriving customer is lost

$$p_m = \frac{(\lambda/\mu)^m / m!}{\sum_{n=0}^{m} (\lambda/\mu)^n / n!} \qquad \text{(Erlang B Formula)}$$

Summary of *M/G/1* system results

1. Utilization factor

$$\rho = \frac{\lambda}{\mu}$$

2. Mean residual service time

$$R = \frac{\lambda \overline{X^2}}{2}$$

3. Pollaczek–Khinchin formula

$$W = \frac{R}{1-\rho} = \frac{\lambda \overline{X^2}}{2(1-\rho)}$$

$$T = \frac{1}{\mu} + W$$

$$N_Q = \frac{\lambda^2 \overline{X^2}}{2(1-\rho)}$$

$$N = \rho + \frac{\lambda^2 \overline{X^2}}{2(1-\rho)}$$

4. Pollaczek–Khinchin formula for $M/G/1$ queue with vacations

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}}$$

$$T = \frac{1}{\mu} + W$$

where $\overline{V}$ and $\overline{V^2}$ are the first two moments of the vacation interval.

Summary of reservation/polling results

1. Average waiting time (m-user system, unlimited service)

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{(m-\rho)\overline{V}}{2(1-\rho)} + \frac{\sigma_V^2}{2\overline{V}} \qquad \text{(exhaustive)}$$

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{(m+\rho)\overline{V}}{2(1-\rho)} + \frac{\sigma_V^2}{2\overline{V}} \qquad \text{(partially gated)}$$

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{(m+2-\rho)\overline{V}}{2(1-\rho)} + \frac{\sigma_V^2}{2\overline{V}} \qquad \text{(gated)}$$

where $\rho = \lambda/\mu$, and $\overline{V}$ and σ_V^2 are the mean and variance of the reservation intervals, respectively, averaged over all users

$$\overline{V} = \frac{1}{m}\sum_{\ell=0}^{m-1} \overline{V}_\ell$$

$$\sigma_V^2 = \frac{1}{m}\sum_{\ell=0}^{m-1} \left(\overline{V_\ell^2} - \overline{V}_\ell^2\right)$$

2. Average waiting time (m-user system, limited service)

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho-\lambda\overline{V})} + \frac{(m+\rho)\overline{V}}{2(1-\rho-\lambda\overline{V})} + \frac{\sigma_V^2(1-\rho)}{2\overline{V}(1-\rho-\lambda\overline{V})} \qquad \text{(partially gated)}$$

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho-\lambda\overline{V})} + \frac{(m+2-\rho-2\lambda\overline{V})\overline{V}}{2(1-\rho-\lambda\overline{V})} + \frac{\sigma_V^2(1-\rho)}{2\overline{V}(1-\rho-\lambda\overline{V})} \qquad \text{(gated)}$$

3. Average time in the system

$$T = \frac{1}{\mu} + W$$

Summary of priority queueing results

1. *Nonpreemptive priority.* Average waiting time in queue for class k customers

$$W_k = \frac{\sum_{i=1}^{n} \lambda_i \overline{X_i^2}}{2(1-\rho_1-\cdots-\rho_{k-1})(1-\rho_1-\cdots-\rho_k)}$$

2. *Nonpreemptive priority.* Average time in the system for class k customers

$$T_k = \frac{1}{\mu_k} + W_k$$

3. *Preemptive resume priority.* Average time in the system for class k customers

$$T_k = \frac{(1/\mu_k)(1-\rho_1-\cdots-\rho_k) + R_k}{(1-\rho_1-\cdots-\rho_{k-1})(1-\rho_1-\cdots-\rho_k)}$$

where

$$R_k = \frac{\sum_{i=1}^{k} \lambda_i \overline{X_i^2}}{2}$$

Heavy traffic approximation for the $G/G/1$ system

Average waiting time in queue satisfies

$$W \leq \frac{\lambda(\sigma_a^2 + \sigma_b^2)}{2(1-\rho)}$$

where

σ_a^2 = Variance of the interarrival times

σ_b^2 = Variance of the service times

λ = Average interarrival time

ρ = Utilization factor λ/μ, where $1/\mu$ is the average service time