Homework Solution Set No. 7

ECE 642
Dr. Bijan Jabbari

Problem 5.7

(a) In this problem we are given:

$$\lambda_n = \lambda$$

Start with the state-dependent service characteristic μ_n given by Eq. (5-3):

$$\mu_n = \frac{n\mu}{n+M-1}$$

By using the state-dependent queueing equation [for example, (2-40) in Mischa Schwartz's book), we get:

$$\frac{p_n}{p_0} = \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} = \rho^n \binom{M+n-1}{n}$$

By applying the condition that the state probabilities sum up to 1:

$$\sum_{n=0}^{N} p_n = 1$$

we can get the initial state probability p_0:

$$p_0 = \frac{1}{\sum_{n=0}^{N} \rho^n \binom{M+n-1}{n}}$$

(b) The normalized throughput is:

$$\begin{aligned}
\frac{\gamma}{\mu} &= \frac{\sum_{n=0}^{N} \mu(n) p_n}{\mu} \\
&= p_0 \rho \sum_{n=0}^{N-1} \rho_n \binom{M+n-1}{n} \\
&= \rho \left[\frac{B(N-1)}{B(N)} \right]
\end{aligned}$$

where

$$B(N) = \sum_{n=0}^{N} \rho^n \binom{M+n-1}{n}$$

(c) For $M = 3$ and $N = 1$, the normalized throughput is:

$$\frac{\gamma}{\mu} = \frac{\rho}{1 + 3\rho}$$

For $M = 3$ and $N = 2$, the normalized throughput is:

$$\frac{\gamma}{\mu} = \frac{\rho(1 + 3\rho)}{1 + 3\rho + 6\rho^2}$$

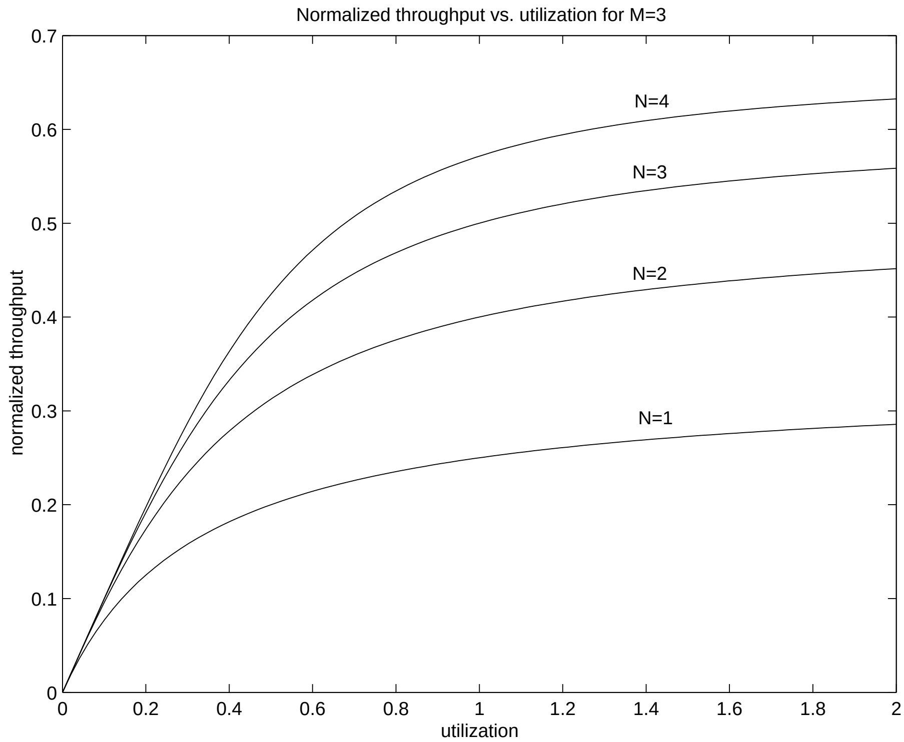
For $M = 3$ and $N = 3$, the normalized throughput is:

$$\frac{\gamma}{\mu} = \frac{\rho(1 + 3\rho + 6\rho^2)}{1 + 3\rho + 6\rho^2 + 10\rho^3}$$

For $M = 3$ and $N = 4$, the normalized throughput is:

$$\frac{\gamma}{\mu} = \frac{\rho(1 + 3\rho + 6\rho^2 + 10\rho^3)}{1 + 3\rho + 6\rho^2 + 10\rho^3 + 15\rho^4}$$

In the following we plot normalized throughput vs. utilization for $M = 3$, and $N = 1, 2, 3, 4$.



(d) From part(b) we get the normalized throughput as follows:

$$\begin{aligned}\frac{\gamma}{\mu} &= \rho\left[\frac{B(N-1)}{B(N)}\right] \\ &= \frac{\rho + \mu\rho^2 + \frac{(M+1)\mu}{2}\rho^3 + \ldots + \frac{(M+N-2)}{(N-1)!}\rho^{N-1}}{1 + \mu\rho + \frac{(M+1)\mu}{2}\rho^2 + \ldots + \frac{(M+N-1)}{N!}\rho^N}\end{aligned}$$

when $\lambda \to \infty$, which also means $\rho \to \infty$

$$\frac{\gamma}{\mu} = \frac{N}{M+N-1}$$

Another way of deriving it is like this: when $\rho \to \infty$, there are always N packets in VC, hence

$$p_n = \begin{cases} 1 & \text{if } n = N \\ 0 & \text{if } n \neq N. \end{cases}$$

Thus

$$\gamma = \mu(n) = \frac{N\mu}{M+N-1}$$

and

$$\begin{aligned} E(T) &= \frac{E(n)}{\gamma} \\ &= N\left(\frac{N\mu}{M+N-1}\right)^{-1} \\ &= \frac{1}{\mu}(N+M+1) \end{aligned}$$

For $M = 3$ and $\rho \to \infty$

$$\frac{\gamma}{\mu} = \begin{cases} 0.33 & \text{if } N = 1 \\ 0.5 & \text{if } N = 2 \\ 0.6 & \text{if } N = 3 \\ 0.667 & \text{if} N = 4. \end{cases}$$

(e) The blocking probability is the Nth state probability:

$$P_B = p_n = \rho^N \binom{M+N-1}{N} p_0$$

p_0 is given in part (a),then:

$$\frac{\gamma}{\mu} = \frac{\lambda}{\mu}(1-P_B) = \rho\left[\frac{B(N-1)}{B(N)}\right]$$

$B(N)$ is given in part (b).

Problem 5.8

As $\lambda \to \infty$,

$$\gamma = \mu(N) = \frac{N\mu}{[N+(M-1)]}$$

$$E(T) = \frac{N}{\gamma} = \frac{[M+N-1]}{\mu}$$

We have to calculate the ratio of the normalized throughput to the normalized expected delay:

$$\frac{\gamma/\mu}{\mu E(T)} = \frac{N}{(N+M-1)^2}$$

Take derivative w.r.t N and set it equal to 0:

$$\Rightarrow \quad (N+M-1)^2 - 2N(N+M-1) = 0$$

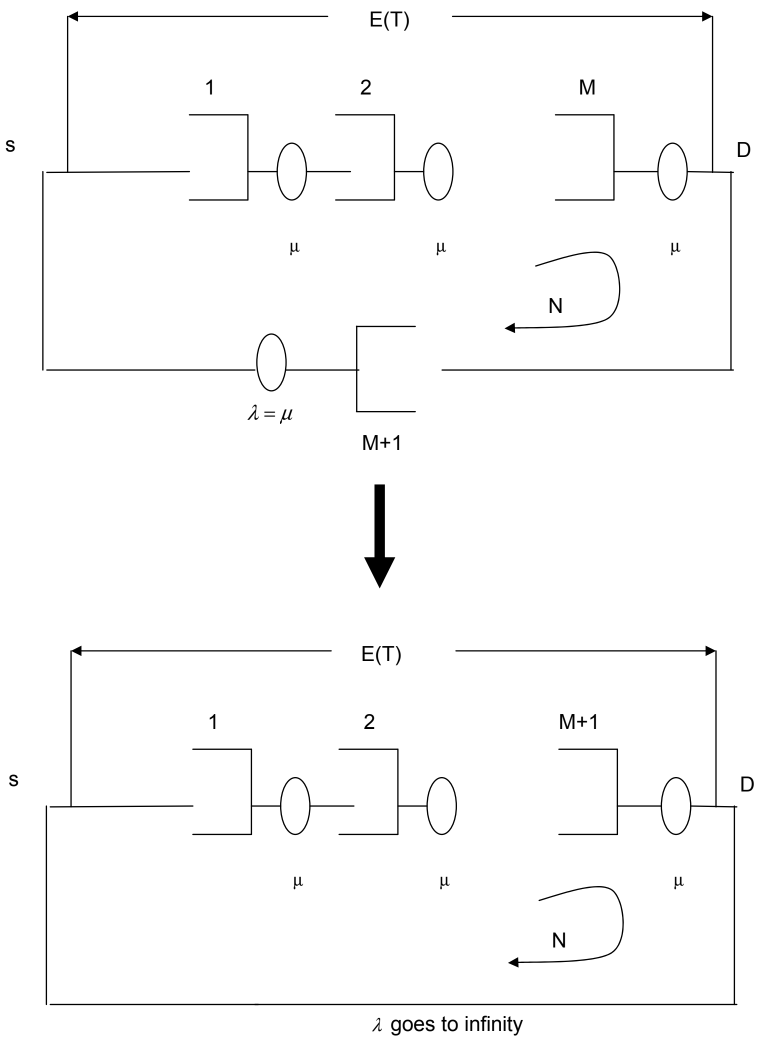
Hence,

$$N = M - 1$$

The reason we want to maximize the "power" is as follows: we would like to maximize both γ and $1/E(T)$. But it is usually not possible with a single N. Thus, as a compromise, we choose N which maximize the product of the two, i.e. the power. For $M = 3$, $N = M - 1 = 2$, refer to fig.5-15, this corresponds to the point where the curve begins to rise quickly, to an appropriate operating point.

Problem 5.9

For $\lambda = \mu$, the equivalent model takes the form of $(M+1)$ queues:



(a) When $\lambda = \mu$, the sliding window control model looks like the $\lambda \to \infty$ case for M+1 queues. Using $M+1$ instead of M in (5-9), we have

$$\gamma = \frac{N\mu}{N+M}$$

From (5-10)

$$E(T) = \frac{M+N}{\mu}$$

Hence

$$E(T) = \frac{M}{M+1} E(T) = \frac{1}{\mu}(M+N)\frac{M}{M+1}$$

(b)

$$\frac{\gamma/\mu}{\mu E(T)} = \frac{(M+1)N}{(M+N)^2 M}$$

Take the derivative w.r.t N and set it to 0

$$\Rightarrow \quad (M+N)^2 - 2N(M+N) = 0$$

Hence

$$N = M$$

Problem 5.10

λ goes to infinity, we have $M = 4$ hops

$$\because \quad \frac{\gamma}{\mu} = \frac{N}{[N+M-1]}$$

$$\mu E(T) = M + N - 1$$

At N=M, the point that maximizes the power, it appears as a good operating point because we don't gain a considerable throughput when we increase N, however, the delay increases significantly. Plot the normalized expected delay vs. normalized throughput for $M = 3, 4$ while $\lambda \to \infty$ and $\lambda = \mu$.