Analysis of Project No. 3 Part II

ECE 642
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Consider the departure time of a particular packet in a finite M/M/1 queue with N buffers. Upon the departure of this packet, there will be two different cases:

- Case 1: There are unserved packets in the system which start being served just upon the departure of the packet. The inter-departure time in this case is just the service time which is an exponential random variable with parameter μ.
- Case 2: There is no unserved packet in the system and server remains idle until a new packet packet arrives. Due to the memoryless property of the poisson process, this idle period is an exponential random variable with parameter λ. After the new packet arrives, it takes another exponential random variable with parameter μ for it to depart the system. Note that these random variables are independent. The inter-departure time in this case is sum of independent exponential random variables with parameters λ and μ respectively.

Cases 1 and 2 happen with probabilities $1 - P_0 = \frac{\rho - \rho^{N+1}}{1-\rho^{N+1}}$ and $P_0 = \frac{1-\rho}{1-\rho^{N+1}}$ respectively. Denoting the inter-departure time as Z, we can write

$$Z = \begin{cases} X + Y, & \text{with probabilty } P_0 \\ X, & \text{with probabilty } 1 - P_0 \end{cases} \tag{1}$$

where X is an exponential random variable with parameter μ, Y is an exponential random variable with parameter λ and X and Y are independent. The CDF of Z can be calculated as

$$\begin{aligned} F_Z(z) &= (1 - P_0)\Pr\{X \le z\} + P_0 \Pr\{X + Y \le z\} \\ &= (1 - P_0)(1 - e^{-\mu z}) + P_0 \int_0^z \Pr\{X + Y \le z | Y = y\} f_Y(y)\ dy \\ &= (1 - P_0)(1 - e^{-\mu z}) + P_0 \int_0^z (1 - e^{-\mu(z-y)})\lambda e^{-\lambda y}\ dy \\ &= \frac{\rho - \rho^{N+1}}{1 - \rho^{N+1}}(1 - e^{-\mu z}) + (\frac{1-\rho}{1-\rho^{N+1}}) \left[1 - e^{-\lambda z} - \frac{\lambda}{\mu - \lambda}(e^{-\lambda z} - e^{-\mu z})\right] \\ &= 1 - e^{-\lambda z}\left(\frac{1}{1-\rho^{N+1}}\right) + e^{-\mu z}\left(\frac{\rho^{N+1}}{1-\rho^{N+1}}\right). \end{aligned} \tag{2}$$

Note that the upper bound of the integral is z as X is an exponential random variable and only takes positive values. We now, consider a special case of M/M/1/N when $N \to \infty$ and $\rho < 1$, i.e. the M/M/1 queue. In this case, we have

$$F_Z(z) = 1 - e^{-\lambda z}. \tag{3}$$

This is the CDF of an exponential random variable with parameter λ. In other words, the departure process in an M/M/1 queue is a Poisson process (as the inter-departures are exponential). This is also expectable from the Burke's theorem [1]. The result we obtained here, is general and the Burke's theorem can be considered as a special case.

References

[1] D. Bertsekas and R. Gallager, *Data Networks*, 2nd edition, Prentice Hall, 1992.