CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE
Lecture 4: Discrete Random Variables

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September 23, 2020

Note

- These slides cover material partially presented in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-set derivation and detailed discussions were need.

Outline

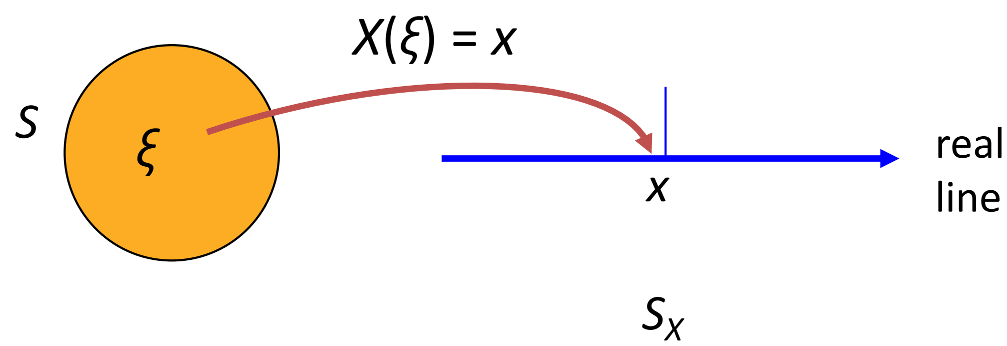
- The Notion of a Random Variable
- Discrete Random Variables and Probability Mass Function
- Expected Value and Moments of Discrete Random Variable
- Conditional Probability Mass Function
- Important Discrete Random Variables
- Generation of Discrete Random Variables

Outcomes of Random Experiment

- Outcomes need not be a number
- Usually interested in a numerical attribute of outcome
- Two step process:
  1. Perform experiment to obtain outcome ξ
  2. Do measurement to obtain $X(\xi)$
- Examples:
  - Toss coin 3 times, count number of heads
  - Select person at random, measure height
- Randomness in outcome induces randomness in measurement
- How to calculate probabilities involving measurement?

Notion of a Random Variable

A **random variable** X is a function that assigns a real number, $X(\xi)$, to each outcome ξ of a chance experiment.



- The domain of X is the sample space S.
- The range of X is the set S_X of all values taken on by X.
- S_X is a subset of the set of all real numbers.

Example: Three Coin Tosses

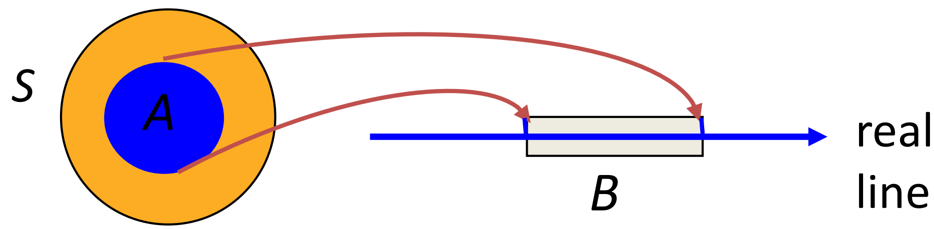
- *Let X be the number of heads in the 3 independent tosses of a fair coin. The sample space for this experiment is S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }*
- *X assigns each outcome ξ in S a number from the set S_X= {$0, 1, 2, 3$}. The Table below lists the 8 outcomes of S and the corresponding value of X.*

ξ	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$X(\xi)$	3	2	2	2	1	1	1	0

- X *is then a Random Variable taking on values in the set* S_X= {$0, 1, 2, 3$}.
- The function X is deterministic, but the randomness in the random experiment induces randomness in the $X(\xi)$.

Finding Probabilities of Random Variables

Let A be set of outcomes ξ in S that lead to values $X(\xi)$ in B,



$A = \{\xi : X(\xi) \text{ in } B\}$

then B in S_X occurs when A in S occurs, and

$$P[B] = P[A] = P[\{\xi : X(\xi) \text{ in } B\}]$$

A and B are **equivalent events**.

Example: Three Coin Tosses

Let B = two heads that occur in three tosses.

$$A = \{\xi : X(\xi) = 2\}$$

Let X be the number of heads in the three independent tosses of a fair coin. Find the probability of the event {X=2}. Find the probability that the player in example 3.2 wins $8.

Note that X(ξ) = 2 if and only if ξ is in {HHT, HTH, THH}. Therefore,

P[X=2] = P[{HHT, HTH, THH}]

= P[{HHT}] + P[{HTH}] + P[{THH}] = 3/8

The event {Y = 8} occurs if and only if the outcome ξ is HHH, therefore, P[Y = 8] = P[{HHH}] = 1/8

Equivalent Events of Special Interest

$$\{\xi: X(\xi) = x\}$$

$$\{\xi: X(\xi) \le x\}$$

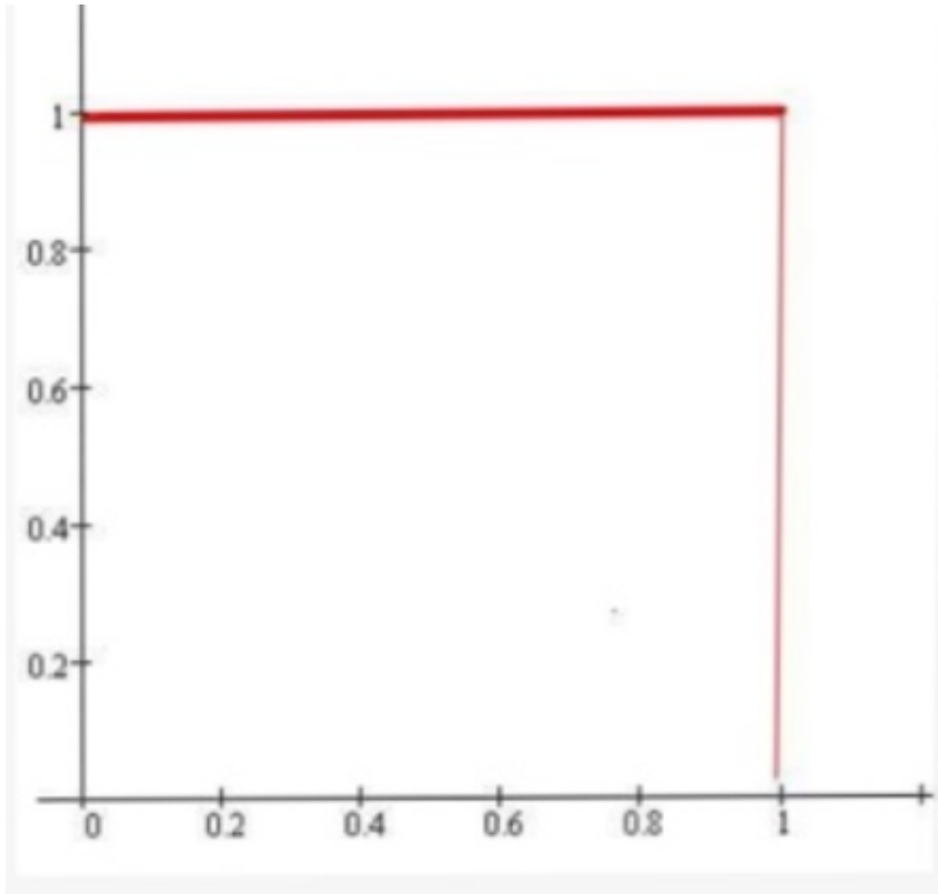
$$\{\xi: X(\xi) \le x\}$$

$$\{\xi: x < X(\xi) \le x + dx\}$$

Example: Random Number from [0,1]

$A = \{\xi : X(\xi) \leq 0.5\}$

$A = \{\xi : X(\xi) \leq x\}$



Empirical Distribution Function

Repeat an experiment *n* times, and count how many times the value of the random variable *X* is less than a constant *x*:

$$\hat{F}_X(x) = \frac{\#\text{trials for which } \left[X \le x\right]}{n}$$

$$= \frac{I_1\left[X \le x\right] + I_2\left[X \le x\right] + \ldots + I_n\left[X \le x\right]}{n}$$

Cumulative Distribution Function

- The **cumulative distribution function** (CDF) of *X* is*:*

  $F_X(x) = P[X \leq x]$ for all $-\infty < x < \infty$

- The CDF is the probability of the event $\{\xi : X(\xi) \leq x\}$
- i.e., it is equal to the probability that the variable X takes on a value in the set (- ∞,x]
- $F_X(x)$ is a function of x.
- The CDF allows us to specify probability of all semi-infinite intervals of the real line, including their complements, unions and intersections.

Properties of CDF

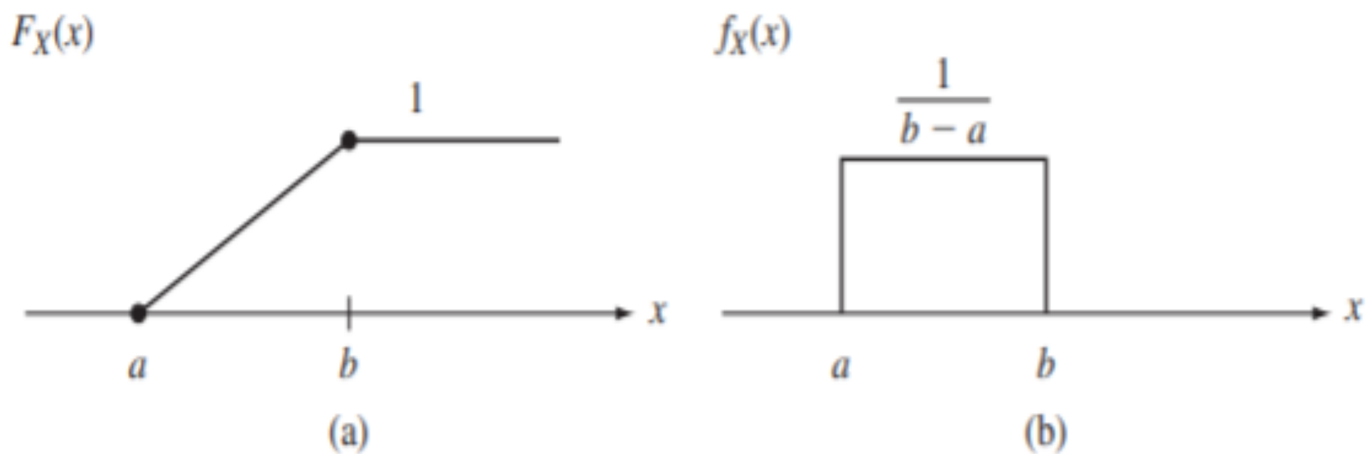
i. $0 \leq F_x(x) \leq 1$ Since $F_X(x)$ is a probability

ii. $\lim\limits_{x \to \infty} F_x(x) = 1$

iii. $\lim\limits_{x \to -\infty} F_x(x) = 0$

iv. $F_X(x)$is a non-decreasing function of x,

i.e., if $a < b$, then $F_X(a) \leq F_X(b)$.



Properties of CDF (cont'd)

v. $F_X(x)$ is continuous from the right, i.e., for $h < 0$,

$$F_X(b) = \lim_{h \to 0} F_x\,(b + h) = F_X(b^+)$$

Properties of CDF (cont'd)

vi. $P[a < X \leq b] = F_X(b) - F_X(a)$

vii. $P[X = b] = F_X(b) - F_X(b^-)$

If CDF is continuous at b then $P[X = b] = 0$.

viii. $P[X > x] = 1 - F_X(x)$

Property (vii) states that the probability that $X = b$ is given by the magnitude of the jump of the cdf at the point b. This implies that *if the cdf is continuous at a point b, then $P[X = b] = 0$.* Properties (vi) and (vii) can be combined to compute the probabilities of other types of intervals. For example, since $\{a \leq X \leq b\} = \{X = a\} \cup \{a < X \leq b\}$, then

$$\begin{aligned} P[a \leq X \leq b] &= P[X = a] + P[a < X \leq b] \\ &= F_X(a) - F_X(a^-) + F_X(b) - F_X(a) = F_X(b) - F_X(a^-). \quad (4.4) \end{aligned}$$

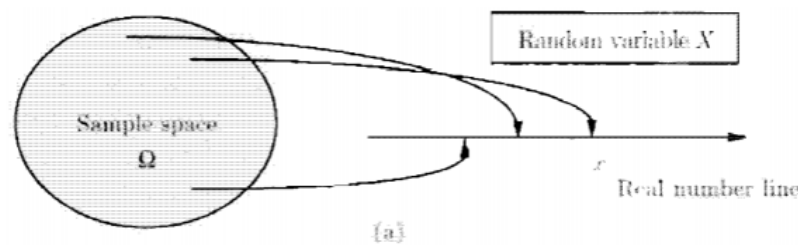
If the cdf is continuous at the endpoints of an interval, then the endpoints have zero probability, and therefore they can be included in, or excluded from, the interval without affecting the probability.

Discrete Random Variables

- X is a **discrete** random variable if $F_X(x)$ is a staircase increasing function with jumps at a countable set of points $x_0, x_1, x_2, \ldots$.

- **Probability mass function (pmf) of X** is

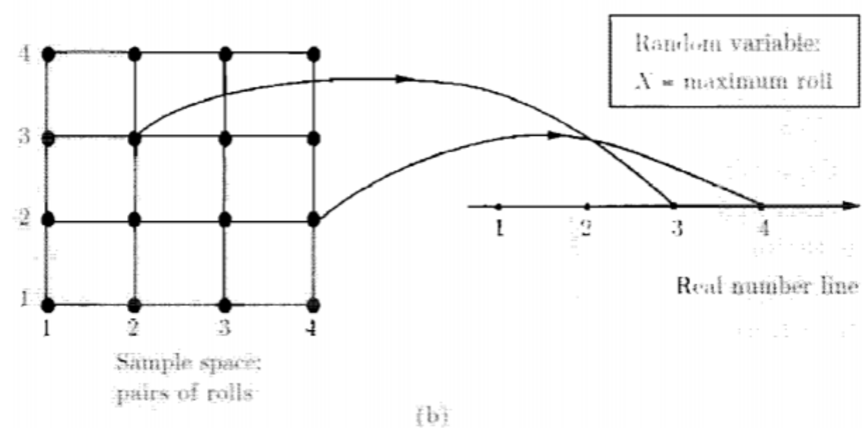
$$p_X(x_k) = P[X = x_k]$$



- Unit step function: $u(x)$

- CDF of a discrete random variable:

$$F_X(x) = \sum_k p_X(x_k) u(x - x_k)$$



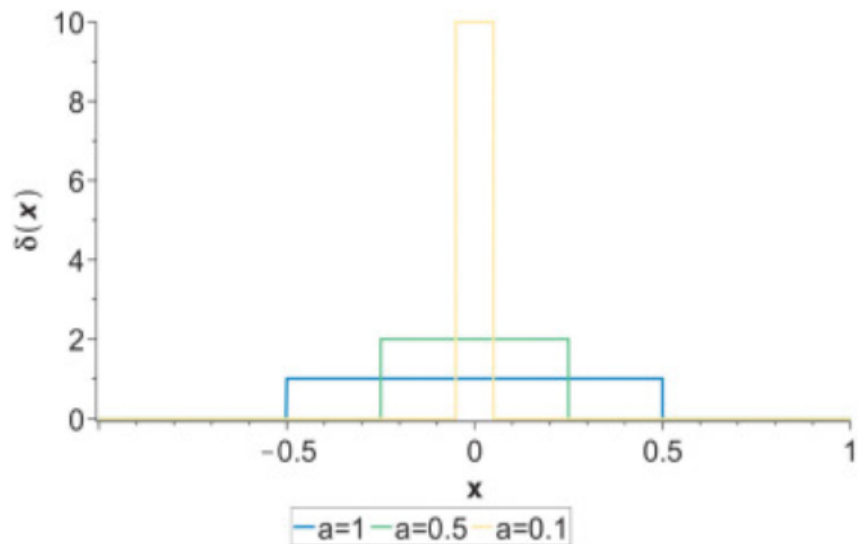
Delta Function

- The **delta function** $\delta(t)$ is defined in terms of the unit step function

$$u(x) = \int_{-\infty}^{x} \delta(t)\, dt$$

- A staircase function can be expressed by integral of time-shifted delta functions:

$$F_X(x) = \int_{-\infty}^{x} \sum_{k} p_X(x_k)\delta(t - x_k)\, dt$$



pdf of Discrete Random Variable

- Since cdf is given by:

$$F_X(x) = \sum_k p_X(x_k) u(x - x_k)$$

- Can define the pdf for a discrete random variable by:

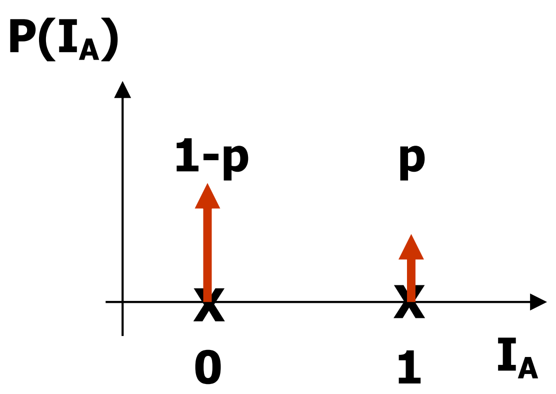
$$f_X(x) = \sum_k p_X(x_k) \delta(x - x_k)$$

Example: Bernoulli Random Variable

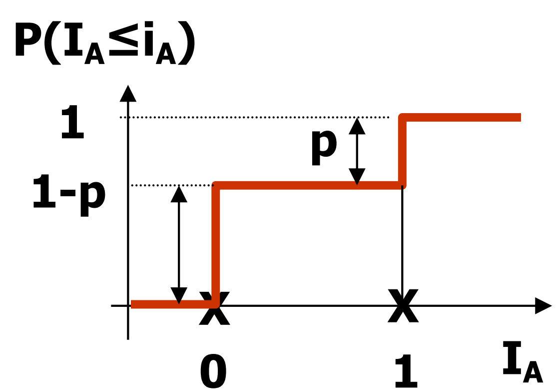
- Let A be an event related to the outcomes of some random experiment. The indicator function for A is defined as:

$I_A(\zeta) = 0$ if ζ not in A (i.e. if A doesn't occur)

$\quad\quad\; = 1$ if ζ is in A (i.e. if A occurs)



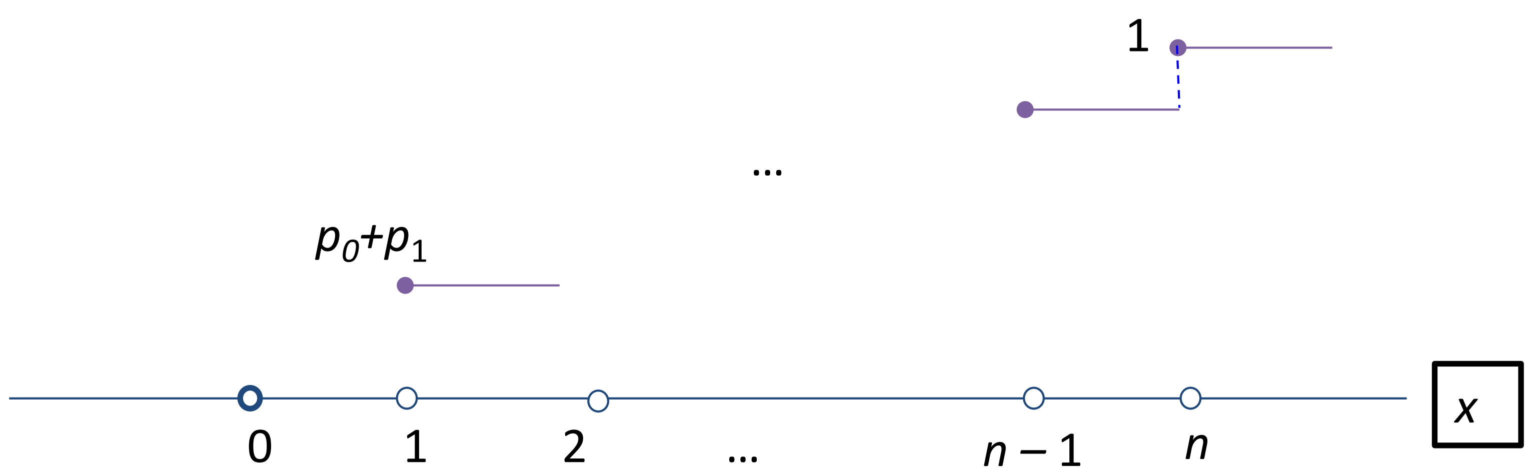
- I_A is random variable since it assigns a number $p_I(0) = 1\text{-}p$, $p_I(1) = p$ where
- $P\{A\} = p$ describes the outcome of a Bernoulli trial
- Note: $p_I(0) + p_I(1) = 1$



Example: Binomial Random Variable

- The Binomial random variable is an example of a **discrete random variable**, where the CDF is a staircase function of *x*.
- The discontinuities in the CDF are given by the probability mass function.

$$F_X(x) = P[\xi : X(\xi) \leq x] = P[\xi : \#\ \text{heads} \leq x]$$



Example: Binomial Random Variable

- Suppose a random experiment is repeated *n* independent times; let X be the number of times a certain event A occurs in these *n* trials

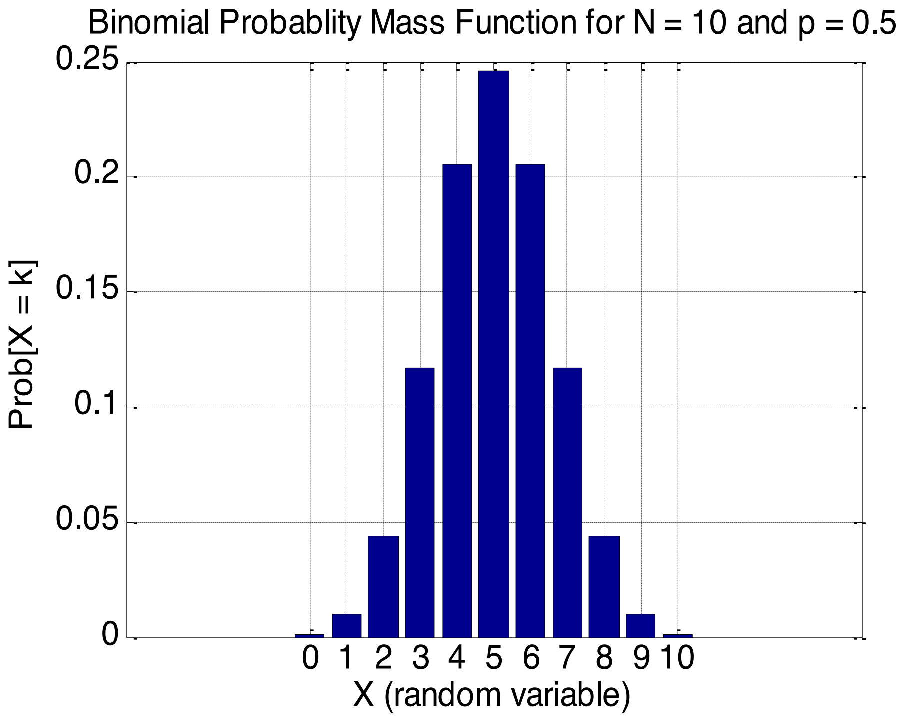
$$X = I_1 + I_2 + \ldots + I_n$$

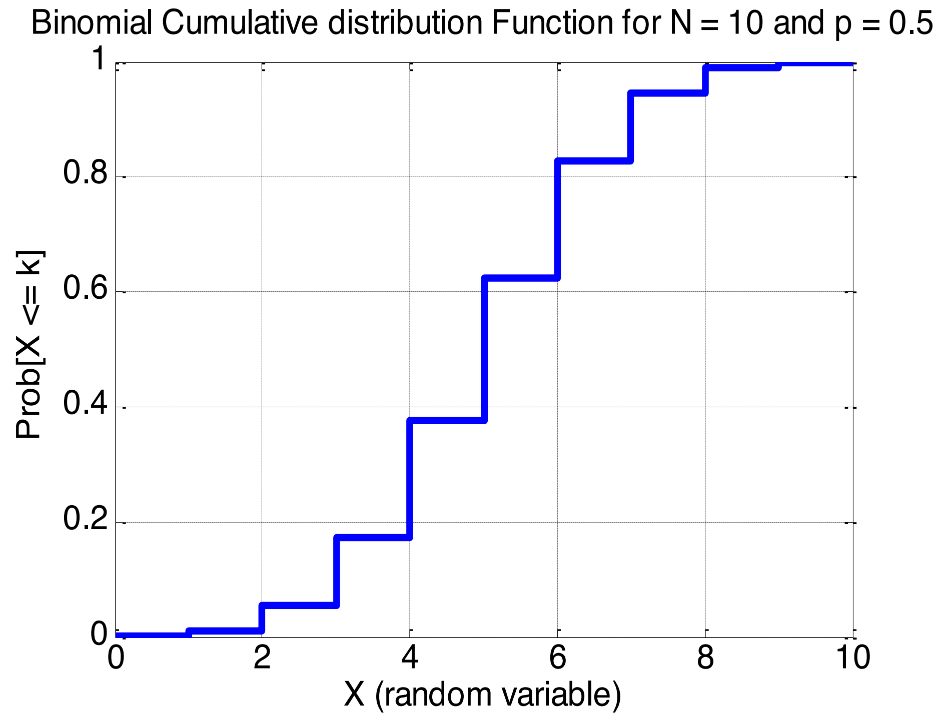
i.e., X is the sum of Bernoulli trials (X's range = {0, 1, 2, …, *n*})

- X has the following pmf for *k*=0, 1, 2, …, *n*

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

Example: Binomial Random Variable





Poisson Random Variable

- Count the number of occurrences of an event in a time period or in a region of space.
  - Packet arrivals, failure events, faults in devices

- The Poisson random variable has pmf:

$$P[N = k] = \frac{\alpha^k}{k!} e^{-\alpha} \quad \text{for } k = 0,1,2,\ldots,$$

where α is average number of event occurrences.

$$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} = e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{-\alpha} \mathrm{e}^{\alpha} = 1$$

Conditional pdf and CDF

- If some event A concerning X is given, then conditional CDF of X given A is defined by

$$F_X(x|A) = \quad P\{[X \leq x] \cap A\}/ P\{A\} \qquad \text{if } P\{A\} > 0$$

The conditional pdf of X given A is then defined by

$$f_X(x|A) = d\, F_X(x|A)/dx$$

Mixed Type Random Variables

- Consider the waiting time of customer in a queueing system which is 0 when system is idle, and exponentially distributed if system is busy.
- This is an example of a **mixed** random variable which whose CDF has continuous portions as well as jump discontinuities
- A random variable of mixed type is a RV with a CDF $F_X(x)$ that has jumps on a countable set of points x_0, x_1, x_2,…but that also increases continuously over at least one interval of values x. The CDF has the form $F_X(x) = pF_1(x) + (1 - p)F_2(x)$

Mixed Random Variables (2)

- X is **mixed** random variable if $F_X(x)$ is continuous except at a countable number of values x_0, x_1, x_2,… .

$$F_X(x) = pF_1(x) + (1-p)F_2(x)$$

- Can view mixed random variables as the result of a two-step process:
  1. Select type according to probability p.
  2. Perform experiment to produce either discrete or continuous random variable.

Example: Random Delay

- The waiting time W of a customer in a queue is zero if the system is idle, and an exponentially distributed random length of time if the system busy.

Question: What is the CDF of W, if the probabilities of finding the system idle or busy are p and 1-p, respectively.

Answer: The CDF of *W* is found as follows:

$F_X(x) = \text{Prob}\{W \le x\}$

$\quad = \text{Prob}\{W \le x | \text{idle}\}\text{p} + \text{Prob}\{W \le x | \text{busy}\}(1-p)$

Note $\text{Prob}\{W \le x \, | \text{idle}\} = 1$ for any $x \ge 0$, and 0 otherwise

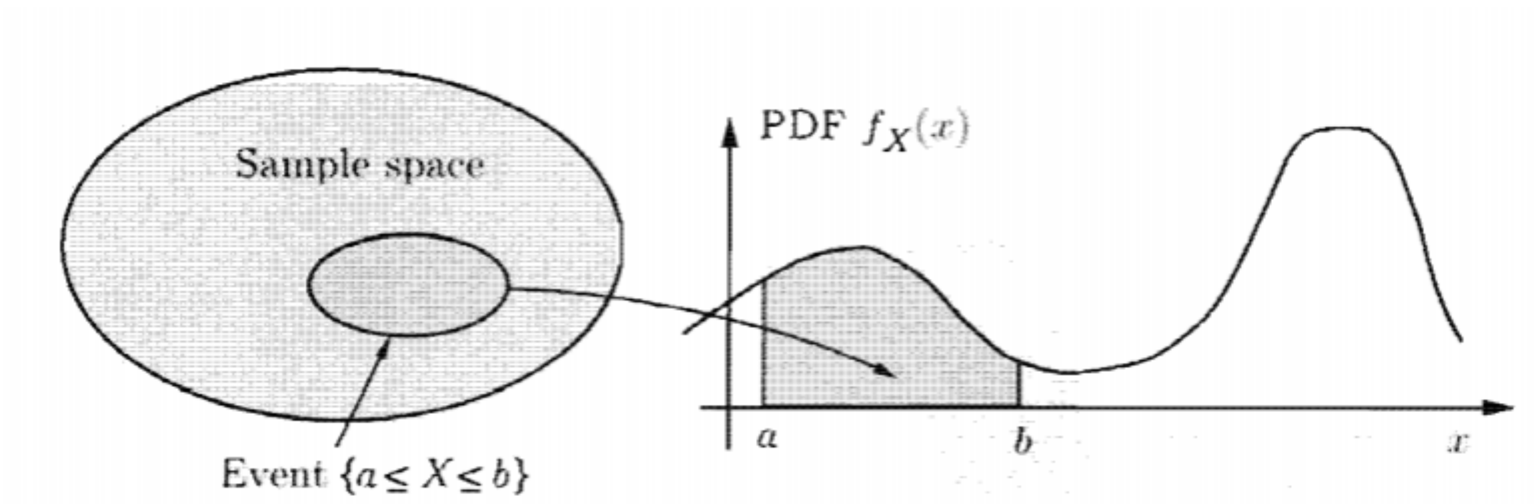
$F_X(x) = 0 \qquad x < 0$

$\quad = p+(1- p)(1- e^{-\lambda x}) \quad x \ge 0$

Continuous Random Variables

- X is a **continuous** random variable if $F_X(x)$ is continuous for all x and if it can be written as an integral of a nonnegative function.

$$F_X(x) = \int_{-\infty}^{x} f(t)dt$$

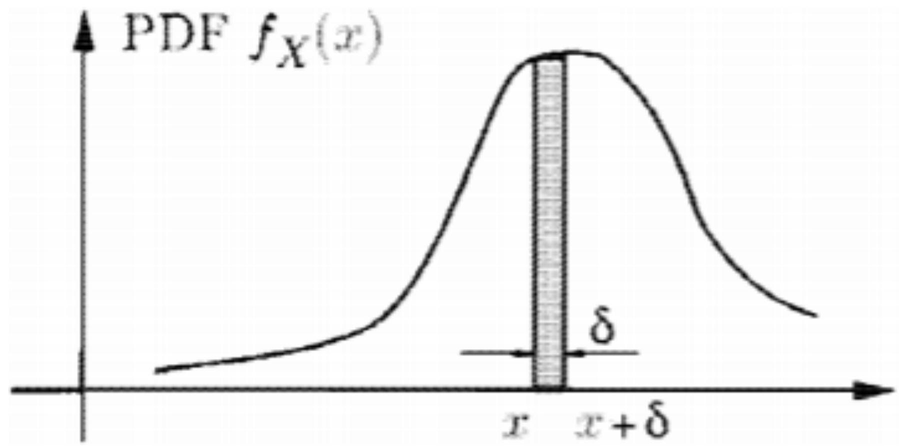


- By property vii, $P[X = x] = 0$ for all x.

Probability Density Function

- Probability **density function (pdf)** of *X* (if it exists) is:

$$f_X(x) = \frac{dF_X(x)}{dx}$$



- The pdf can be viewed as the density of probability mass at *x*:

$$P[x < X \leq x + h] = F_X(x+h) - F_X(x) = \frac{F_X(x+h) - F_X(x)}{h} h$$

$$P[x < X \leq x + h] \approx f_X(x)h$$

Properties of pdf

i. $f_X(x) \geq 0$

ii. $$P[a < X \leq b] = \int_b^a f_X(x)dx$$

Summary of PDF Properties

Let X be a continuous random variable with PDF f_X.

- $f_X(x) \geq 0$ for all x.
- $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$
- If δ is very small, then $\mathbf{P}\big([x, x+\delta]\big) \approx f_X(x) \cdot \delta$.
- For any subset B of the real line,

$$\mathbf{P}(X \in B) = \int_B f_X(x)\,dx.$$

- The probability of an interval is the area under $f_X(x)$

Properties of pdf (cont'd)

iii. $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$ CDF can be obtained from the pdf

iv. $1 = \int_{-\infty}^{+\infty} f_X(t)dt$ Normalization condition

- A valid pdf can be formed from any nonnegative, piecewise continuous function $g(x)$ that has a finite integral.

Summary: Properties of the CDF

- $0 \le F_X(x) \le 1$
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = 0$
- $F_X(x)$ is a nondecreasing function $\Rightarrow$ if $a < b \Rightarrow F_X(a) \le F_X(b)$
- $F_X(x)$ is continuous from the right $\Rightarrow$ for $h > 0$,
  $$F_X(b) = \lim_{h \to 0} F_X(b+h) = F_X(b^+)$$
- $\text{Prob}\,[a < X \le b] = F_X(b) - F_X(a)$
- $\text{Prob}\,[X = b] = F_X(b) - F_X(b^-)$

Exponential Random Variable

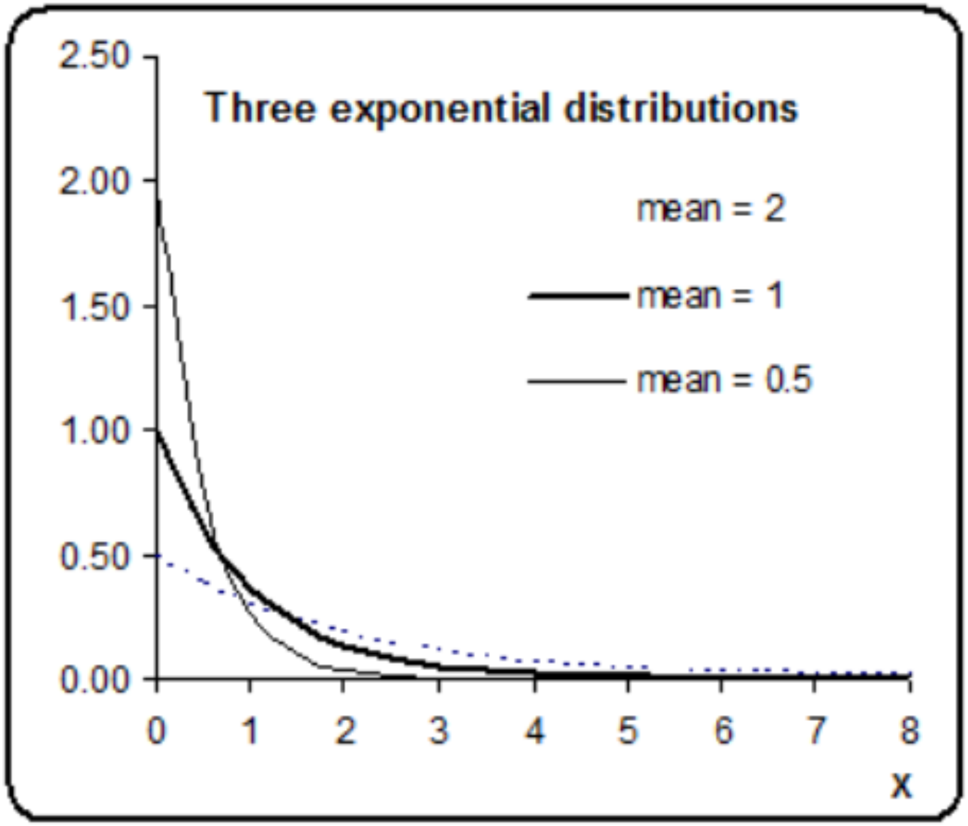
CDF of Exponential

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

pdf of Exponential

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

Plotting pdf of Exponential



Example: Exponential Random Variable

Q: The transmission time X of a message in a communication system follows $P[X > x] = e^{-\lambda x}$ for $x \geq 0$

What is P $[T < X \leq 2T]$ where $T = 1/\lambda$?

Answer: The CDF of X is found as follows

$F_X(x) = P[X \leq x] = 1 - P[X > x] = 1 - e^{-\lambda x} \quad x \geq 0$

$\quad = 0 \quad\quad x < 0$

Which is the exponential distribution with parameter λ, that is,

$$F_X(x) = P\{X \leq x\} = 1 - e^{-\lambda x} \quad x \geq 0$$

Prob $\{T < X \leq 2T\} = F_X(2T) - F_X(T) = (1-e^{-2}) - (1-e^{-1}) = 0.233$

More on Exponential or Surprises

- Consider the customer checking out at a cashier of a supermarket
- Assume the time it takes to serve customer is exponentially distributed with mean of 10 minutes

Q1: What is the probability that customers spend mote than 15 minutes?

Answer:

E[x}=1/λ = 10 min λ= 1/10

$P[X > x] = 1 - F_X(x)$= $1 - P[X \leq x]$ *

$P[X > 15] = 1 - P[X \leq 15]$ Note $P[X \leq x]$= $F_X(x)$ =1- $e^{-\lambda x}$ for $x \geq$0

$P[X > 15]$=1- (1- $e^{-15/10}$)= $e^{-15/10}$= $e^{-3/2}$=0.22

* So that $P[X > x] = e^{-\lambda x}$ for $x \geq 0$

More on Exponential or Surprises(2)

Q2: Now assume that the customer has spent already 10 minutes at the cashier. What is the probability that the customer spends more than 15 minutes?

Answer:

- Note that the exponential is memoryless.

$P[X>15 \mid X>10] = P[X > 15 \cap X>10]/P[X>10]=$

$= P[X > 15]/P[X > 10]$

$= P[X > 5]P[X > 10]/P[X > 10]$

$= P[X > 5]$

$P[X > 5] = e^{-5/10} = e^{-1/2} = 0.64$

- So given that the customer has spent already 10 minutes, the chances that he/she will spend additional 5 minutes (i.e., the total more than 15 minutes) is 64%!

Memoryless Property of Exponential Distribution

- A continuous random variable *X* is said to be memoryless if for every real *s* and *t*, $P[X >$ s+t | X>t] = P[X > s]
- We show that this leads to $P[X >$ s+t] =P[X > t]P[X > s]

$P[X >$ s+t | X>t] = P[X > s]

$P[X >$ s+t | X>t] = $P[X >$ s+t ∩ $X >$ t]/$P[X >$ t]= P[X > s]

[$X >$ s+t ∩ $X >$ t]= $P[X >$ s+t] = P[X > t]P[X > s]

- The Exponential distribution meets this property
- Note that $P[X > x]$=1 $- F_X(x) = e^{-\lambda x}$ for $x \geq 0$ for Exponential

$e^{-\lambda(s+t)} = e^{-\lambda s}\, e^{-\lambda t}$

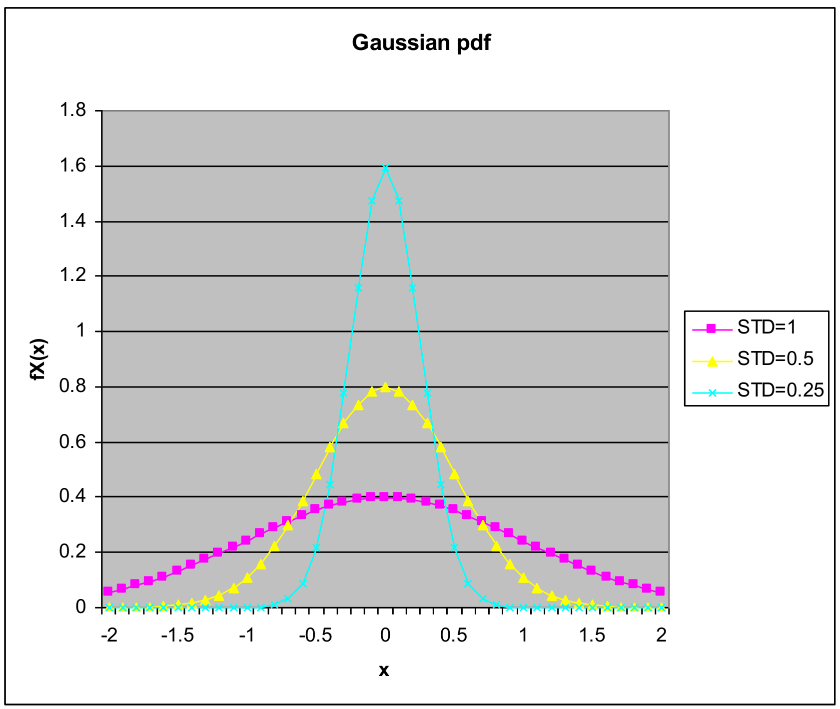
- That is, $P[X >$ s+t] =P[X > t]P[X > s]

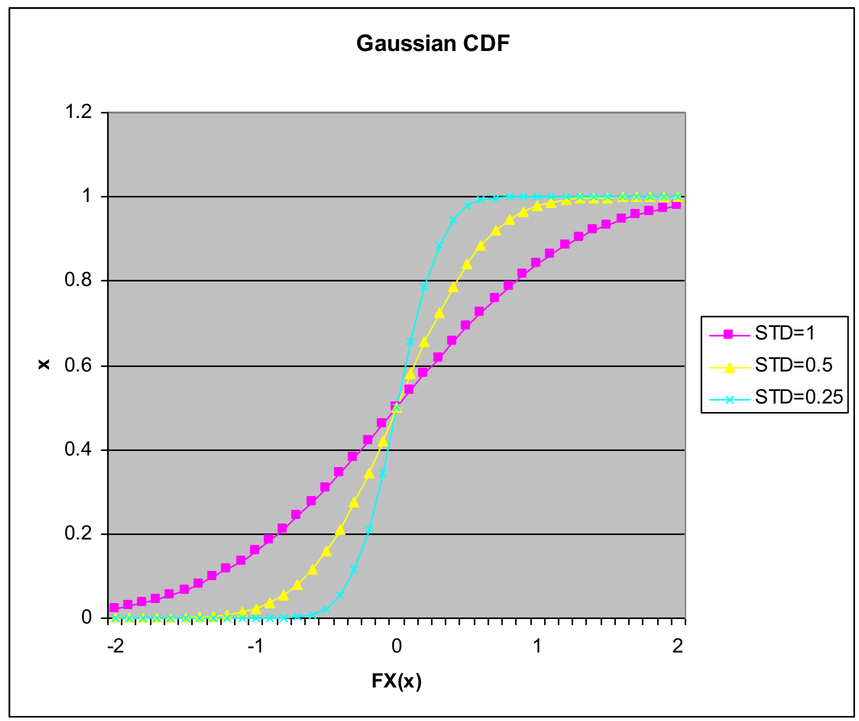
Gaussian (Normal) Random Variable

- "Bell curve" arises in problems where there is a sum of large number of "small" random variables

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$$

where m is the mean, σ > 0 is the standard deviation





Q-Function

- Probability of the "tail" of the Gaussian pdf commonly used in electrical engineering

$$Q(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt$$

- Where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$ is the standard Normal distribution with Mean of *m* = 0 and variance of $\sigma_X^2 = 1$

$$Q(0) = 1/2 \text{ and } Q(-x) = 1 - Q(x)$$

- The general CDF can be expressed in terms of the standard integral:

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(t-m)^2/2\sigma^2} dt$$

Summary

- A random variable is a function that assigns a real number to each outcome of a random experiment. A random variable is defined if the outcome of a random experiment is a number, or if a numerical attribute of an outcome is of interest.
- The notion of an equivalent event enables us to derive the probabilities of events involving a random variable in terms of the probabilities of events involving the underlying outcomes.
- A random variable is discrete if it assumes values from some countable set. The probability mass function is sufficient to calculate the probability of all events involving a discrete random variable.
- The probability of events involving discrete random variable X can be expressed as the sum of the probability mass function $p_X(x)$
- If X is a random variable, then $Y = g(X)$ is also a random variable.
- The mean, variance, and moments of a discrete random variable summarize some of the information about the random variable X. These parameters are useful in practice because they are easier to measure and estimate than the pmf.
- The conditional pmf allows us to calculate the probability of events given partial information about the random variable X.
- There are a number of methods for generating discrete random variables with prescribed pmf's in terms of a random variable that is uniformly distributed in the unit interval