ECE 528 – Introduction to Random Processes in ECE

Lecture 9 Annex: Random Number Generators

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OUTLINE

- Introduction
- Exponential Random Number Generation
- Bernoulli
- Binomial
- Normal
- Lognormal

Exponential Generator

$$X \sim EXP(\mu) \quad \text{pdf}: \quad f(x \mid \mu) = \mu e^{-\mu x}, \quad 0 \le x \le \infty, \quad \mu > 0$$

$$\text{mean}: \quad E[X] = 1/\mu$$

$$\text{Variance}: \quad Var(X) = 1/\mu^2$$

$$\boxed{X\sim\text{Uniform }(0,1)} \xrightarrow{-1/\mu \ln(X)} \boxed{\text{Exp }(\mu)}$$

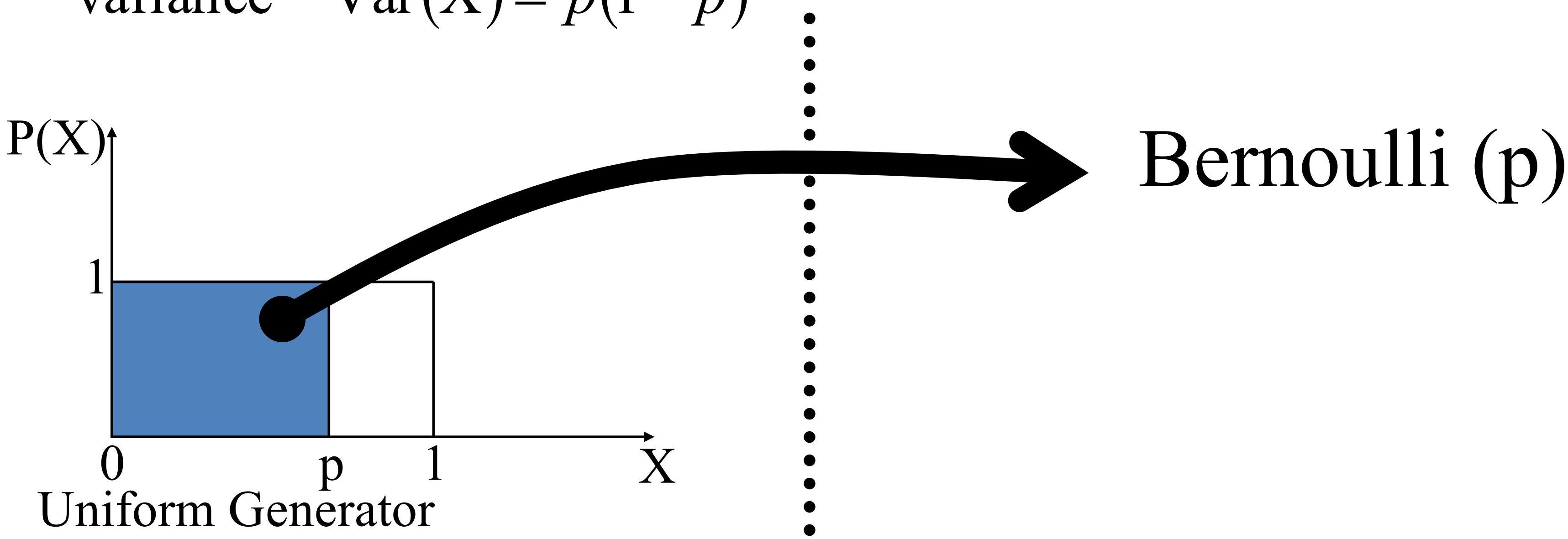
Bernoulli Generator

- Bernoulli(p)

pmf $\quad P(X = x \mid p) = p^x (1-p)^x; \quad x = 0,1; \quad 0 \le \mathrm{p} \le 1$

mean $\quad \mathrm{E}(\mathrm{X}) = p$

variance $\quad \mathrm{Var}(\mathrm{X}) = p(1-p)$



Binomial Generator

- Binomial(n,p)

pmf $\quad P\left(X = x \mid n, p\right) = \binom{n}{x} p^{x} (1-p)^{n-x}; \quad x = 0,1,2,\ldots,n; \quad 0 \le \mathrm{p} \le 1$

mean $\quad \mathrm{E}\left(\mathrm{X}\right) = np$

variance $\quad \mathrm{Var}\left(\mathrm{X}\right) = np\left(1-p\right)$

$$\begin{tikzcd}[column sep=huge]
\boxed{\text{Bernoulli } (p)} \arrow[r, shift left, "\sum_{i=1}^{n} X_i"] & \boxed{\text{Binomial } (n,p)} \arrow[l, shift left, "n=1"]
\end{tikzcd}$$

Normal Distribution

$$N\left(\mu,\sigma^2\right) \xrightarrow{\text{def}} f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

- Approach: For U_1 and U_2 uniform random variables in the range (0,1)

$$Z_1 \xrightarrow{def} \left(-2\ln U_1\right)^{1/2} \cos\left(2\pi \cdot U_2\right)$$

$$Z_2 \xrightarrow{def} \left(-2\ln U_2\right)^{1/2} \sin\left(2\pi \cdot U_1\right)$$

- The two numbers Z_1 and Z_2 follow a normal random distribution with zero mean and variance 1 (i.e., N(0,1))

LogNormal Distribution

- The following illustrates the method that can be used to generate LogNormal random variables:

$$
\begin{tikzcd}
\boxed{X \sim N(\mu, \sigma^2)} \arrow[r, shift left, "e^X"] & \boxed{\text{LogNormal}} \arrow[l, shift left, "\log(X)"]
\end{tikzcd}
$$

Log-Normal Distribution

log-normal distribution

$$f_L(x) = \frac{1/\ln 2}{x\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}\left(\log_2(x) - \mu\right)^2\right], \quad \mathrm{x} > 0$$

Cumulative distribution of a lognormal random variable

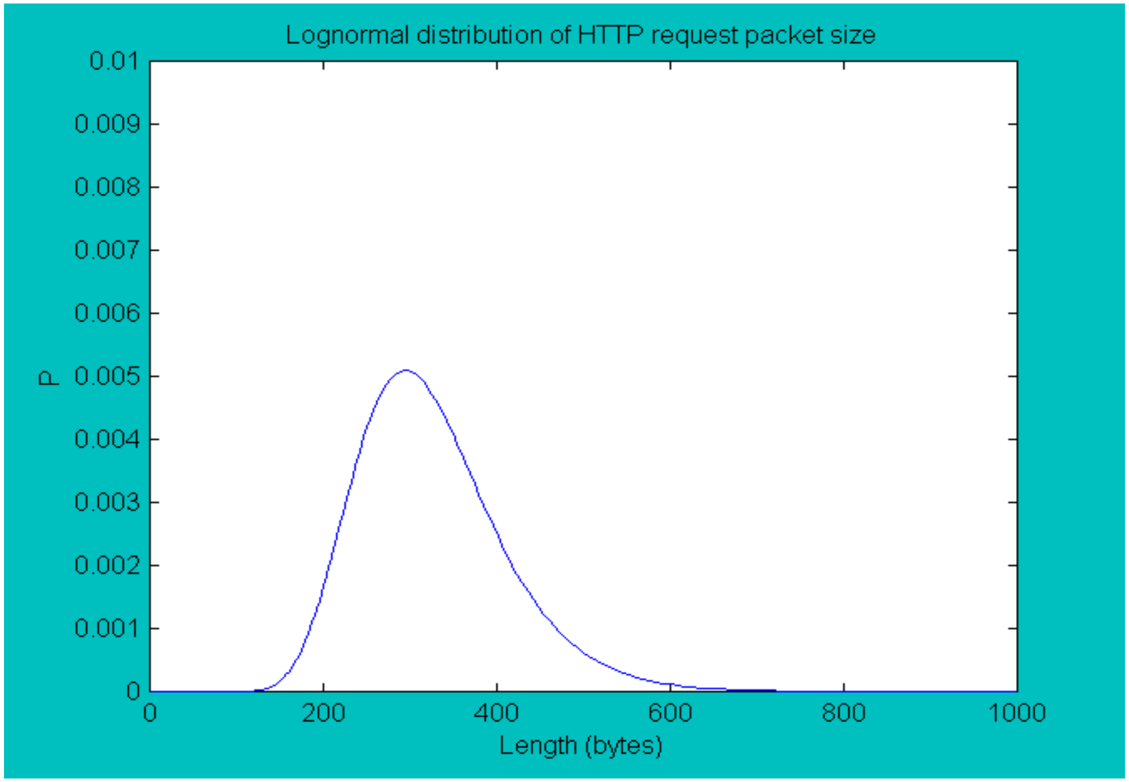
$$P(x < \tau) = 1 - Q\left(\frac{\ln\tau - \mu}{\sigma}\right)$$

Lognormal Distribution Relation to Normal Distribution

- The parameters (μ,σ) of lognormal (**ln**) distribution can be obtained as

$$\mu = \ln \frac{m}{\sqrt{1 + s^2 / m^2}}$$

$$\sigma^2 = \ln\left(1 + \frac{s^2}{m^2}\right)$$



Normal and Log-normal Random Variables

- PDF of Log-normal RV:

$$f_X(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln(x)-\mu)}{\sigma^2}}$$

Log-N(μ,σ^2)

$$m = E\{X\}$$

$$s^2 = \mathrm{var}(X)$$

- Few points:
  - If X~N(μ,σ^2), then exp(X)~Log-N(μ,σ^2).
  - Equivalently, if X~Log-N(μ,σ^2), then ln(X) ~N(μ,σ^2)

$$\mathrm{N}(\mu,\sigma^2) \xrightarrow{\boxed{\text{Exp(.)}}} \text{Log-N}(\mu,\sigma^2) \qquad \text{Log-N}(\mu,\sigma^2) \xrightarrow{\boxed{\text{Ln(.)}}} \mathrm{N}(\mu,\sigma^2)$$

Cont'd

- The PDF of a Log-normal RV is completely characterized either by (μ,σ^2) (The parameters of the corresponding Normal RV) or by its mean and variance (m,s^2).
- (m,s^2) and (μ,σ^2) are related as following:

$$\begin{cases} m = e^{\mu+\frac{\sigma^2}{2}} \\ s^2 = e^{2\mu}\left(e^{2\sigma^2} - e^{\sigma^2}\right) \end{cases}$$

$$\begin{cases} \mu = \ln\left(\sqrt{\dfrac{m^4}{s^2+m^2}}\right) \\ \sigma^2 = 2\ln\left(\sqrt{1+\dfrac{s^2}{m^2}}\right) \end{cases}$$