

Solutions to Homework Set No. 3

2.29 Each transmission is equivalent to tossing a fair coin. If the outcome is heads, then the transmission is successful. If it is tails, then another transmission is required. As in Example 2.11 the probability that j transmissions are required is:

$$P[j] = \left(\frac{1}{2}\right)^j$$

$$P[A] = P[j \text{ even}] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k - 1 = \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3}.$$

$$P[B] = P[j \text{ multiple of 3}] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{3k} = \frac{1}{1-\frac{1}{8}} - 1 = \frac{1}{7}.$$

$$P[C] = \sum_{k=1}^6 \left(\frac{1}{2}\right)^k = \frac{1}{2} \sum_{k=0}^5 \left(\frac{1}{2}\right)^k = \frac{1}{2} \frac{1 - (\frac{1}{2})^6}{1 - \frac{1}{2}} = \frac{63}{64}.$$

$$P[C^c] = 1 - P[C] = \frac{1}{64}.$$

$$P[A \cap B] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{6k} = \frac{1}{1-\frac{1}{64}} - 1 = \frac{1}{63} \text{ since a multiple of 2 and 3 is a multiple of 6.}$$

$$P[A - B] = P[A] - P[A \cap B] = \frac{1}{3} - \frac{1}{63} = \frac{20}{63} \text{ since}$$

$$A = (A - B) \cup (A \cap B) \text{ and } (A - B) \cap (A \cap B) = \phi.$$

$$P[A \cap B \cap C] = \left(\frac{1}{2}\right)^6 = \frac{1}{64} \text{ since } A \cap B \cap C = \{6\}.$$

2.54a

The number of ways of choosing M out of 100 is $\binom{100}{M}$. This is the total number of equiprobable outcomes in the sample space.

We are interested in the outcomes in which m of the chosen items are defective and $M - m$ are nondefective.

The number of ways of choosing m defectives out of k is $\binom{k}{m}$.

The number of ways of choosing $M - m$ nondefectives out of $100 - k$ is $\binom{100 - k}{M - m}$.

The number of ways of choosing m defectives out of k
and $M - m$ non-defectives out of $100 - k$ is

$$\binom{k}{m} \binom{100 - k}{M - m}$$

$$\begin{aligned} P[m \text{ defectives in } M \text{ samples}] &= \frac{\# \text{ outcomes with } k \text{ defective}}{\text{Total } \# \text{ of outcomes}} \\ &= \frac{\binom{k}{m} \binom{100 - k}{M - m}}{\binom{100}{M}} \end{aligned}$$

This is called the Hypergeometric distribution.

$$\text{b) } P[\text{lot accepted}] = P[m = 0 \text{ or } m = 1] = \frac{\binom{100 - k}{M}}{\binom{100}{M}} + \frac{k \binom{100 - k}{M - 1}}{\binom{100}{M}}.$$

2.77

Assume X is the input and Y is the output:

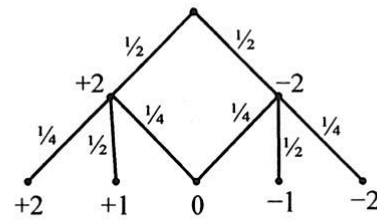
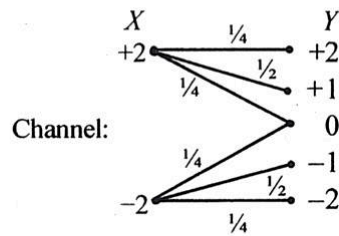
$$\text{(a) } P(Y=0) = p \cdot (1 - \epsilon_1) + (1 - p) \cdot \epsilon_2;$$

$$\text{(b) } P(X=0 | Y=1) = p \cdot \epsilon_1 / [p \cdot \epsilon_1 + (1 - p) \cdot (1 - \epsilon_2)];$$

$$P(X=1 | Y=1) = (1 - p) \cdot (1 - \epsilon_2) / [p \cdot \epsilon_1 + (1 - p) \cdot (1 - \epsilon_2)];$$

Then: if $p \cdot \epsilon_1 > (1 - p) \cdot (1 - \epsilon_2)$, the first one has a higher probability, otherwise the second one has a higher probability.

2.78 a)



$$\begin{aligned} \text{b) } P[X = +2, Y = +2] &= P[Y = 2 | X = 2]P[X = 2] \\ &= \frac{1}{4} \frac{1}{2} = \frac{1}{8} \end{aligned}$$

$$P[X = 2, Y = 1] = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$$

$$P[X = 2, Y = 0] = \frac{1}{2} \frac{1}{4} = \frac{1}{8}$$

$$P[X = -2, Y = 0] = \frac{1}{2} \frac{1}{4} = \frac{1}{8}$$

$$P[X = -2, Y = 1] = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$$

$$P[X = -2, Y = -2] = \frac{1}{2} \frac{1}{4} = \frac{1}{8}$$

$$\text{c) } P[Y = +2] = \frac{1}{2} \frac{1}{4} = \frac{1}{8} = P[Y = -2]$$

$$P[Y = +1] = \frac{1}{2} \frac{1}{2} = \frac{1}{4} = P[Y = -1]$$

$$P[Y = 0] = 2\left(\frac{1}{2} \frac{1}{4}\right) = \frac{2}{8} = P[Y = 0]$$

$$\text{d) } P[X = 2 | Y = k] = \frac{P[Y = k | X = 2]P[X = 2]}{P[Y = k]}$$

$$= \begin{cases} \frac{1/8}{1/8} = 1 & k = 2 \\ \frac{1/4}{1/4} = 1 & k = 1 \\ \frac{1/8}{1/4} = \frac{1}{2} & k = 0 \\ 0 & \text{other } k \end{cases}$$

2.81

$$\text{(a) } P(Y=0) = 1/3 * (1-\epsilon) + 1/3 * \epsilon = 1/3; P(Y=1) = P(Y=2) = P(Y=0) = 1/3;$$

$$\text{(b) } P(X=0 | Y=1) = \epsilon, P(X=1 | Y=1) = 1-\epsilon, P(X=2 | Y=1) = 0.$$

2.86

Proof: If $P(A|B)=P(A|B^C)$, then we know $P(A \cap B)/P(B) = P(A \cap B^C)/P(B^C)$. Therefore, $P(A \cap B) * P(B^C) = P(A \cap B^C) * P(B) = [P(A) - P(A \cap B)] * P(B)$. after rearranging $P(A \cap B) * [P(B^C) + P(B)] = P(A \cap B) = P(A) * P(B)$. Hence, A and B are independent.

$$\begin{aligned} \text{2.97 a) } P[0 \text{ or } 1 \text{ errors}] &= (1-p)^{100} + 100(1-p)^{99}p & p &= 10^{-2} \\ &= 0.3660 + 0.3697 \\ &= 0.7357 \end{aligned}$$

$$\text{b) } p_R = P[\text{retransmission required}] = 1 - P[0 \text{ or } 1 \text{ errors}] = 0.2642$$

$$P[M \text{ retransmissions in total}] = (1 - p_R) p_R^M \quad M = 0, 1, 2, \dots$$

$$\begin{aligned} P[M \text{ or more retransmissions required}] &= \sum_{j=M}^{\infty} (1 - p_R) p_R^j = p_R^M \sum_{j=0}^{\infty} (1 - p_R) p_R^j \\ &= p_R^M \end{aligned}$$

2.105

$$\text{(a) } P(k) = (1 - 1/2)^{k-1} * 1/2 = (1/2)^k;$$

$$\text{(b) When } k \leq 5, P(k) = (1/2)^k. \text{ When } k=6, P(k \text{ dollars paid}) = \sum_{k=6}^{+\infty} P(k) = 1 - P(k=1) - P(k=2) - P(k=3) - P(k=4) - P(k=5) = 1/32.$$

$$\text{2.106 } P[k \text{ tosses required until heads comes up three times}] = P[\text{heads in } k\text{th toss} \mid 2 \text{ heads in } k-1 \text{ tosses}] P[2 \text{ heads in } k-1 \text{ tosses}] = P[A \mid B] P[B].$$

$$\text{Now } P[A \mid B] = P[2 \text{ heads in first } k-1 \text{ tosses}] = \binom{k-1}{2} p^2 (1-p)^{k-3}.$$

$$\text{Thus } P[A \mid B] P[B] = P[A \mid B] p = \binom{k-1}{2} p^3 (1-p)^{k-3} \quad k = 3, 4, \dots$$

2.128

$$\text{(a) } P(\text{obtaining an ace}) = 4/52 = 1/13;$$

(b) Define event A=ace in first draw, event B=ace in second draw.

$$P(A) = 1/13, P(A^C) = 12/13. \text{ Then } P(B \mid A) = 3/51, P(B \mid A^C) = 4/51. \text{ Hence, } P(B) = P(B \mid A) * P(A) + P(B \mid A^C) * P(A^C) = 1/13, \text{ which is the same as in first draw. The answer does not change.}$$

$$\text{(c) } P(3 \text{ ace}) = C(4,3) * C(48,4) / C(52,7) = 0.00582;$$

$$P(2 \text{ kings}) = C(4,2) * C(48,5) / C(52,7) = 0.07679;$$

$$P(3 \text{ aces and } 2 \text{ kings}) = C(4,2) * C(4,3) * C(44,2) / C(52,7) = 0.00017;$$

$$P(3 \text{ aces or } 2 \text{ kings}) = P(3 \text{ ace}) + P(2 \text{ kings}) - P(3 \text{ aces and } 2 \text{ kings}) = 0.0824;$$

(d) Each player gets 13 cards in which an ace is contained. The probability is

$$P = \frac{4! * \frac{48!}{(12!)^4}}{\frac{52!}{(13!)^4}} = 0.1055$$