CNL
Communications & Networking Laboratory

ECE 528 – Introduction to Random Processes in ECE
Lecture 5: Continuous Random Variables

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Note

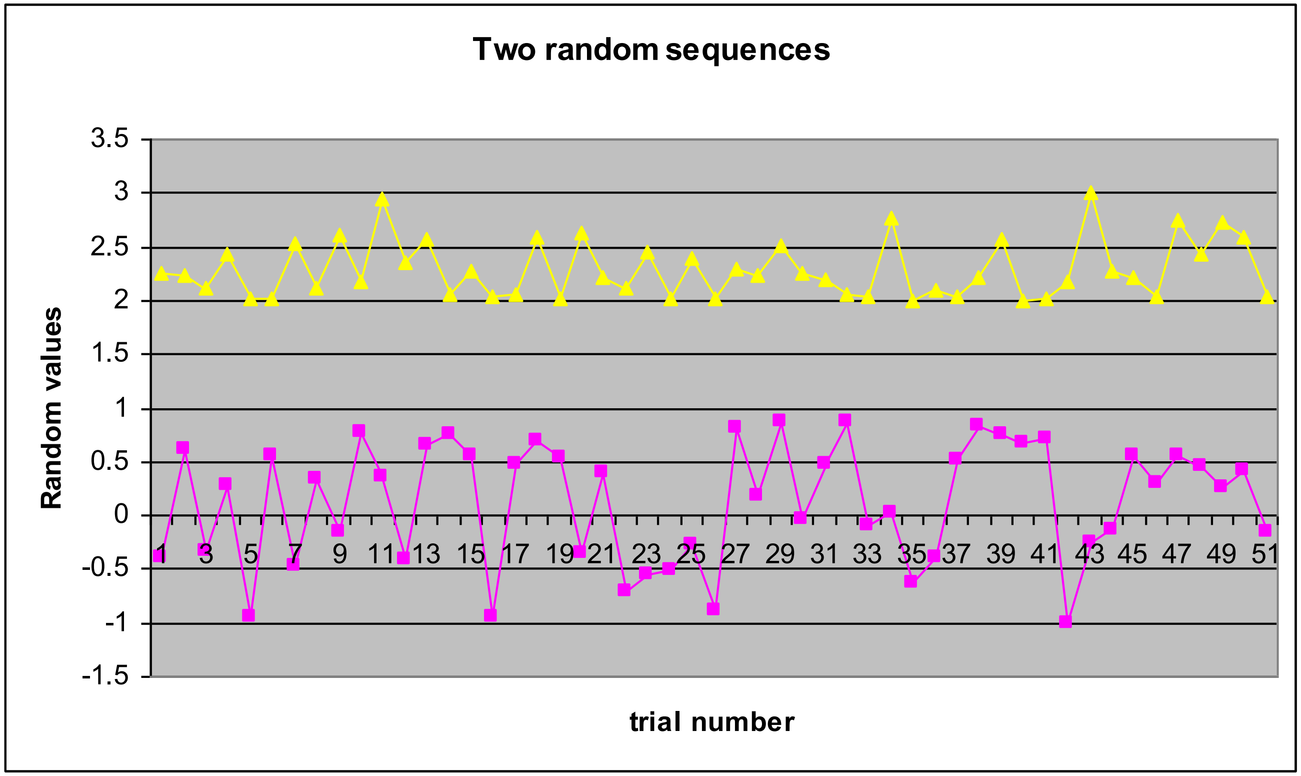
- These slides cover material partially presented in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-set derivation and detailed discussions were need.

Outline

- The Cumulative Distribution Function (CDF)
- The Probability Density Function (pdf)
- The Expected Value of X
- Important Continuous Random Variables
- Functions of a Random Variable
- The Markov and Chebyshev Inequalities
- Transform Methods
- Computer Methods for Generating Random Variables

Expected Value of Random Variables

- Expected values ("Averages") summarize information contained in the CDF, pdf, pmf.
- Mean, variance, standard deviation, skewness



Arithmetic Averages & Means

- General expression for arithmetic average for outcomes of a discrete random variable

$$\frac{n_0 x_0 + n_1 x_1 + \ldots + n_k x_k + \ldots}{n} = \sum_k x_k \frac{n_k}{n} = \sum_k x_k f_k \rightarrow \sum_k x_k p_k$$

- Expected value of a discrete RV

$$E[X] = \sum_k x_k p_X(x_k) \triangleq m_X$$

- E[X] is defined if:

$$A \cap B = \{\xi : \xi \in A \text{ and } \xi \in B\}$$

Expected Value of *X*

- Expected value or mean of a continuous random variable *X* is defined by:

$$E[X] = \int_{-\infty}^{+\infty} t f_X(t)\, dt = \mathrm{mx}$$

E[X] is defined if: $E[|X|] = \int_{-\infty}^{+\infty} |t| f_X(t)\, dt \quad < \infty$

E[X] is the center of mass of the pdf

- Expectation of the discrete random variable X can be computed by

$$E[X] = \sum_{\forall i} x_i P[X = x_i]$$

Second Moment of *X*

- Second moment of *X* is given by

$$E[X^2] = \int_{-\infty}^{+\infty} t^2 f_X(t) dt$$

Variance of *X*

- Useful to know how spread *X* is about *E*[*X*]
- Deviation D = X – E[X]
- **Variance** of X is defined as mean-squared variation E[D^2]:

$$\sigma_X^2 = \text{VAR}[X] = E[(X - E[X])^2]$$

- **Standard deviation** is the spread about the mean:

$$\sigma_X = \sqrt{\text{VAR}[X]}$$

$$\text{VAR}[X] = E[(X-m)^2] = E[X^2] - m^2$$

Exercise: Mean of Geometric RV

- Find the mean of a geometric RV:

$$E[N] = \sum_{k=1}^{\infty} k p_k = \sum_{k=1}^{\infty} k p q^{k-1}$$

Properties of Expected Value

Let *c* be a constant

$$E[c] = \int_{-\infty}^{\infty} c f_X(x)dx = c\int_{-\infty}^{\infty} f_X(x)dx = c$$

$$g(X) = c$$

$$E[cX] = \int_{-\infty}^{\infty} cx f_X(x)dx = c\int_{-\infty}^{\infty} x f_X(x)dx = cE[X]$$

$$g(x) = \sum_{k=1}^{n} g_k(x)$$

$$g(x) = \sum_{k=1}^{n} a_k X^k$$

Some Properties of Variance

- The Variance VAR [*X*] of a random Variable *X* is defined by
$$\mathrm{VAR}[X] = E[(X - E[X])^2]$$

- And can be calculated as

$$\mathrm{VAR}[X] = \sum_x (X - E[x])^2 p_x\,(x)$$

$$\mathrm{VAR}[c] = 0$$

$$\mathrm{VAR}[X + c] = \mathrm{VAR}[X]$$

$$\mathrm{VAR}[cX] = c^2\mathrm{VAR}[X]$$

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$$

Moment of a Random Variable

- Mean and variance are the 2 most important parameters for summarizing the pdf of X.

- Skewness, which measures the degree of asymmetry about the mean, is also used.

$$\text{Skewness} = \frac{E[(X - E[X])^3]}{\sigma_X^3}$$

$$\text{Curtosis} = \frac{E[(X - E[X])^4]}{\sigma_X^4}$$

Moment of a Random Variable (cont’d)

- The **nth moment of X** is defined by:

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$$

- Under some conditions, knowledge of all the moments of X is equivalent to knowing the pdf.
- For discrete random variables,

$$E[X^n] = \sum_{\forall i} x^n{}_i P[X = x_i]$$

Mean of Exponential RV

- Time X between customer arrivals has an exponential pdf with parameter λ.
- Find the mean arrival time:

Exponential Random Variable

$S_X = [0, \infty)$

$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0 \quad \text{and} \quad \lambda > 0$

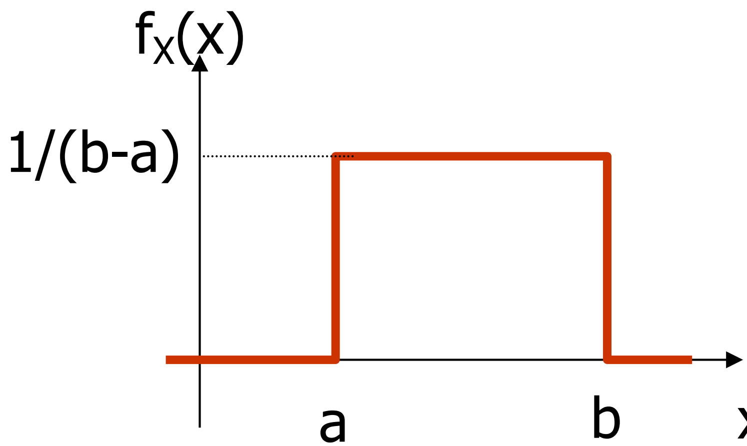
$E[X] = \frac{1}{\lambda} \qquad \text{VAR}[X] = \frac{1}{\lambda^2} \qquad \Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$

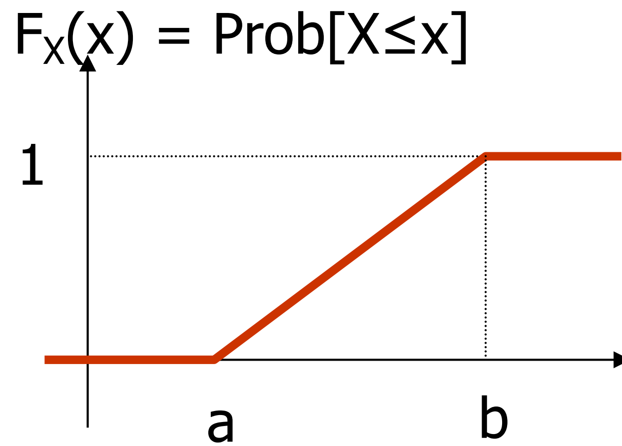
Uniform Random Variable

- A **continuous Uniform** RV for the interval [a, b] is defined as

  Pdf: $f_X(x) = 1/(b-a)$ $\quad a \le x \le b$

- Mean and variance are: $E[X] = (a+b)/2$, $\quad Var[X] = (b-a)^2/12$





- The **discrete Uniform** RV X is defined over the set {0, 1, 2, … ,M-1} with $P_X(j) = 1/M$ for j=0,1 ….M- 1

$$E[X] = \sum_{k=0}^{M-1} k p_k = \sum_{k=1}^{M-1} k \, 1/M \quad = 1/M \sum_{k=1}^{M-1} k$$

$$= 1/M\,(0+1+2 \;\ldots\ldots + M-1) = (M-1)/2$$

Where we have used 1+2 ……+N) = N(N+1)/2

Uniform Random Variable: Derivation of Mean and Variance

- The mean of the RV is written as

$$E[X] = \int_a^b 1/(b-a)\ t\,dx = 1/(b-a) \int_a^b t\,dt = (a+b)/2$$

- The second moment of *X* is given by

$$E[X^2] = \int_{-\infty}^{+\infty} t^2 f_X(t)dt = \int_a^b 1/(b-a)\ t^2 dt$$

$$= 1/(b-a) \int_a^b t^2 dt = 1/(b-a)\ t^3/3\ |^b_a = 1/(b-a)\ (b^3 - a^3)/3$$

$$= 1/3(b^2+ab+a^2)$$

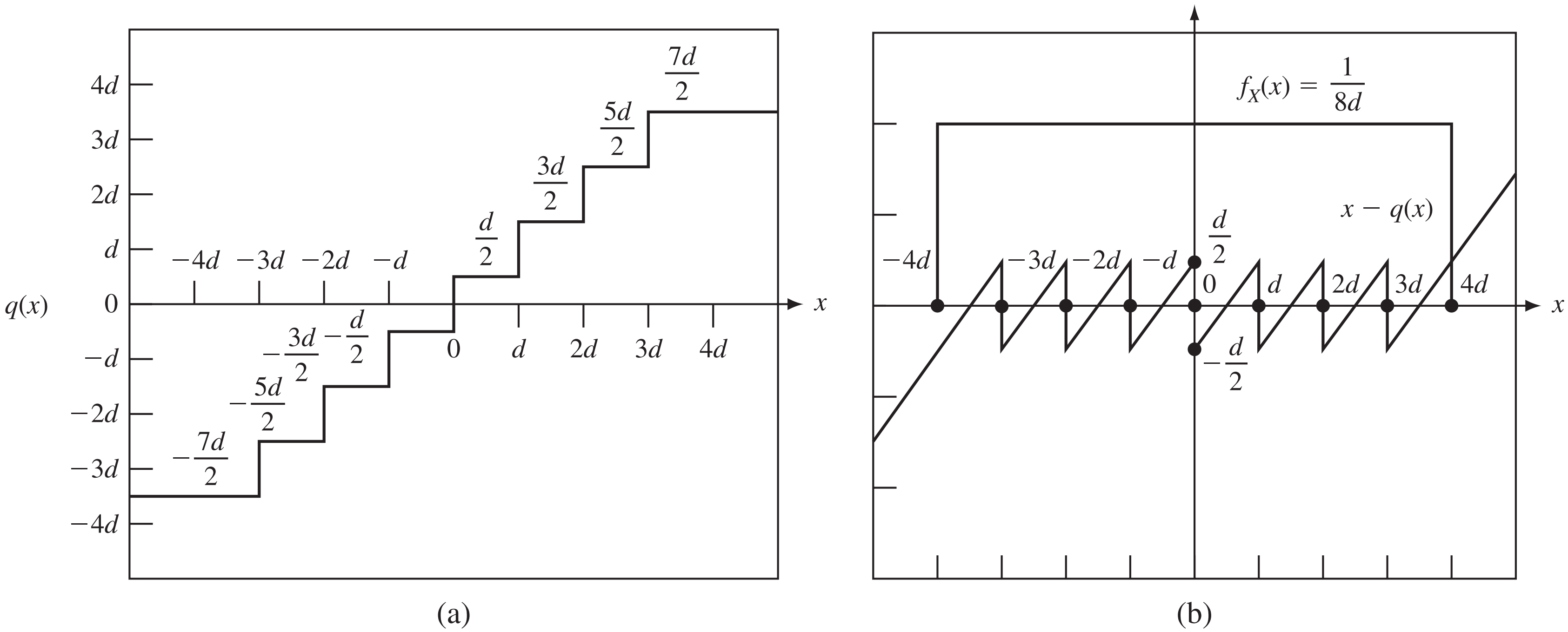
- The variance of *X* is given by

$$Var(X) = E[X^2] - (E[X])^2 = 1/3(b^2+ab+a^2)-((a+b)/2)2$$

$$Var(X) = (b^2 - a^2)/12$$

Uniform Quantizer

- In ECE 462 we saw the performance of a uniform quantizer of *n* bits as SNR=6*n*+7.3 in dB



(a) A uniform quantizer maps the input x into the closest point from the set {+_d/2, +_3d/2, +_5d/2, +_7d/2}.

(b) The uniform quantizer error for the input x is x – q(x2).

Cauchy & Pareto Distribution

- Mean and variance may not exist
- See Problems 4.26, 4.34

Example 4.26

Let the function $h(x) = (x)^+$ be defined as follows:

$$(x)^+ = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

For example, let X be the number of active speakers in a group of N speakers, and let Y be the number of active speakers in excess of M, then $Y = (X - M)^+$. In another example, let X be a voltage input to a halfwave rectifier, then $Y = (X)^+$ is the output.

Example 4.34

Example 4.34 A Chi-Square Random Variable

Let X be a Gaussian random variable with mean $m = 0$ and standard deviation $\sigma = 1$. X is then said to be a standard normal random variable. Let $Y = X^2$. Find the pdf of Y.

Substitution of Eq. (4.68) into Eq. (4.69) yields

$$f_Y(y) = \frac{e^{-y/2}}{\sqrt{2y\pi}} \qquad y \geq 0. \tag{4.70}$$

From Table 4.1 we see that $f_Y(y)$ is the pdf of a *chi-square random variable with one degree of freedom.*