

ECE 528 – Introduction to Random Processes in ECE

Lecture 6: Continuous Random Variables and Functions of a Random Variable

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Note

- These slides cover material partially presented in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-set derivation and detailed discussions were need.

Outline

- Important Continuous Random Variables
- Functions of a Random Variable
- The Markov and Chebyshev Inequalities
- Transform Methods
- Computer Methods for Generating Random Variables

Characteristic Function

- The **characteristic function** of X , when X is continuous, is defined by $\Phi_X(\omega) = E[e^{j\omega X}]$
$$= \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \quad \text{where } j = \sqrt{-1}$$
- $\Phi_X(\omega)$ is the Fourier transform of the pdf $f_X(x)$ (with reversal of sign in the exponent).
- With the Fourier transform inversion formula we get

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} d\omega$$

The Characteristic Function (2)

- Properties:

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega) \Big|_{\omega=0}$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) e^{-j\omega x} d\omega$$

The Characteristic Function (3)

- For discrete random variables,

$$\begin{aligned}\Phi_x(\omega) &= E[e^{j\omega X}] \\ &= \sum_{\forall k} p_X(x_k) e^{j\omega x_k}\end{aligned}$$

- For integer valued random variables,

$$\Phi_x(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

Note: $p_X(k) = \text{Probability mass function of the random variable } X \text{ when } (X = k) = P(X = k)$

The Characteristic Function (4)

- Properties

$$p_X(k) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_x(\omega) e^{-j\omega k} d\omega$$

for $k=0, \pm 1, \pm 2, \dots$

Example: Characteristic Function of an Exponentially Distributed RV

Example 4.41 Exponential Random Variable

The characteristic function for an exponentially distributed random variable with parameter λ is given by

$$\begin{aligned}\Phi_X(\omega) &= \int_0^{\infty} \lambda e^{-\lambda x} e^{j\omega x} dx = \int_0^{\infty} \lambda e^{-(\lambda - j\omega)x} dx \\ &= \frac{\lambda}{\lambda - j\omega}.\end{aligned}$$

Mean and Variance of Exponential RV

Example 4.43

To find the mean of an exponentially distributed random variable, we differentiate $\Phi_X(\omega) = \lambda(\lambda - j\omega)^{-1}$ once, and obtain

$$\Phi'_X(\omega) = \frac{\lambda j}{(\lambda - j\omega)^2}.$$

The moment theorem then implies that $E[X] = \Phi'_X(0)/j = 1/\lambda$.

If we take two derivatives, we obtain

$$\Phi''_X(\omega) = \frac{-2\lambda}{(\lambda - j\omega)^3},$$

so the second moment is then $E[X^2] = \Phi''_X(0)/j^2 = 2/\lambda^2$. The variance of X is then given by

$$\text{VAR}[X] = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Uniform Random Variables

- Realizations of the R.V. can take values from the interval $[a, b]$
- pdf $f_X(x) = 1/(b-a) \quad a \leq x \leq b$
- $E[X] = (a+b)/2, \quad \text{Var}[X] = (b-a)^2/12$
- $\Phi_X(\omega) = [e^{j\omega b} - e^{j\omega a}]/(j\omega(b-a))$

Characteristic Function of Integer RVs

If X is an integer-valued random variable:

$$\Phi_X(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

This Fourier series is periodic since

$$e^{j(\omega+2\pi)k} = e^{j\omega k} e^{jk2\pi} \text{ and } e^{jk2\pi} = 1$$

The Fourier transform inversion formula gives:

$$p_X(k) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_X(\omega) e^{-j\omega k} d\omega \quad k = 0, \pm 1, \pm 2, \dots$$

Moment Theorem

Moment theorem:

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_X(\omega) \Big|_{\omega=0}$$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) \left\{ 1 + j\omega X + \frac{(j\omega X)^2}{2!} + \dots \right\} dx$$

$$\Phi_X(\omega) = 1 + j\omega E[X] + \frac{(j\omega)^2 E[X^2]}{2!} + \dots + \frac{(j\omega)^n E[X^n]}{n!} + \dots$$

Probability Generating Function

- A matter of convenience – compact representation
- The same as the z-transform
- If N is a non-negative integer-valued random variable, the probability generating function is defined as

$$G_N(z) = E[z^N]$$

$$= \sum_{k=0}^{\infty} p_N(k) z^k$$

$$= p_N(0) + p_N(1)z + p_N(2)z^2 + \dots$$

Note: $p_N(k)$ = Probability mass function of the random variable N when $(N = k) = P(N = k)$

Probability Generating Function (2)

- Properties:

- 1

$$p_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(z) \Big|_{z=0}$$

- 2

$$E[N] = G'_N(1)$$

- 3

$$Var[N] = G''_N(1) + G'_N(1) - [G'_N(1)]^2$$

Probability Generating Function (3)

- For non-negative continuous random variables, let us define the Laplace transform* of the PDF

$$\begin{aligned} X^*(s) &= \int_0^{\infty} f_X(x) e^{-sx} dx \\ &= E[e^{-sx}] \end{aligned}$$

Properties:

$$E[X^n] = (-1)^n \left. \frac{d^n}{ds^n} X^*(s) \right|_{s=0}$$

* Useful in dealing with queueing theory (i.e. service time, waiting time, delay, ...)

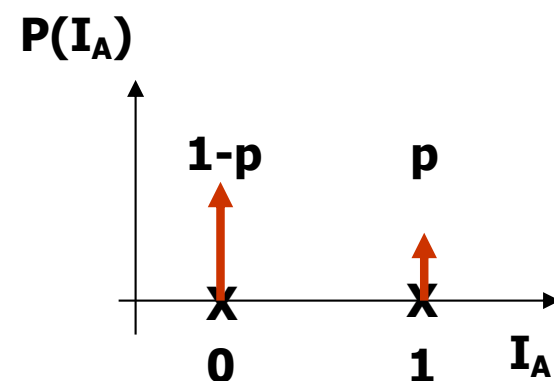
Some Important Random Variables – Discrete RVs

- Bernoulli
- Binomial
- Geometric
- Poisson

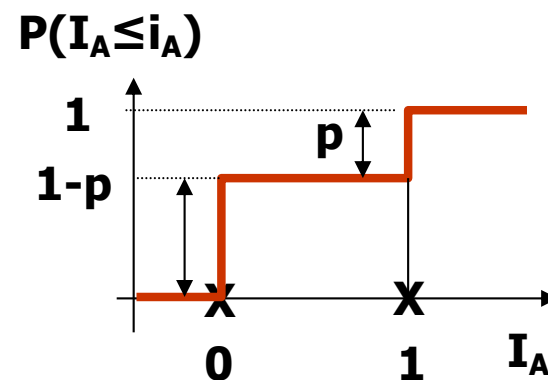
Bernoulli Random Variable

- Let A be an event related to the outcomes of some random experiment. The indicator function for A is defined as:

$$\begin{aligned} I_A(\zeta) &= 0 && \text{if } \zeta \text{ not in } A \text{ (i.e. if } A \text{ doesn't occur)} \\ &= 1 && \text{if } \zeta \text{ is in } A \text{ (i.e. if } A \text{ occurs)} \end{aligned}$$



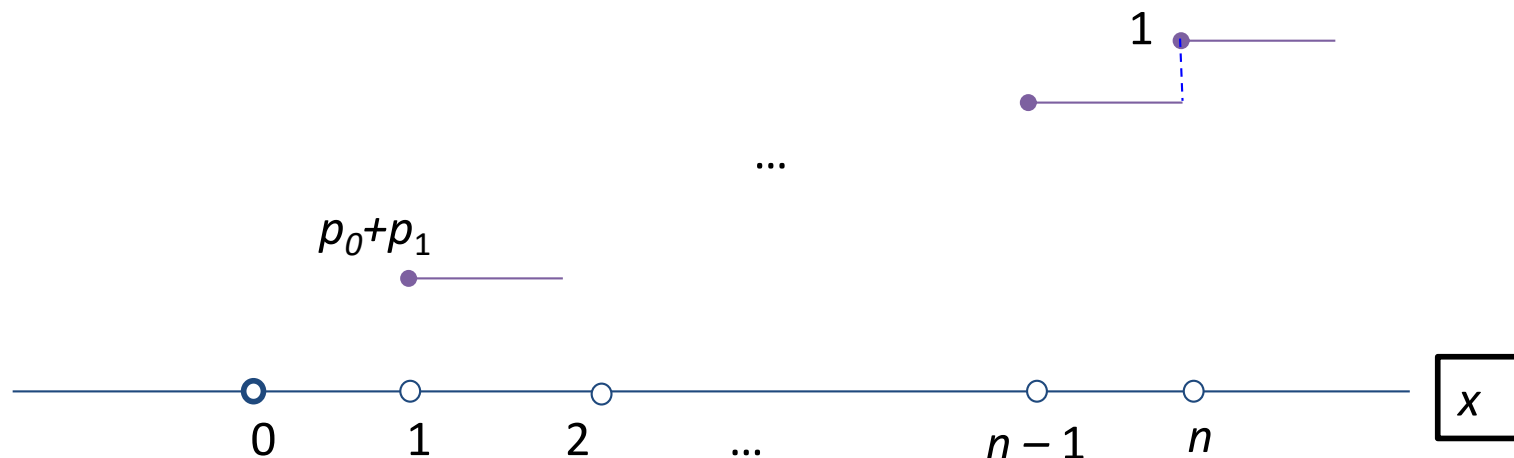
- I_A is random variable since it assigns a number $p_i(0) = 1-p$, $p_i(1) = p$ where
- $P\{A\} = p$ describes the outcome of a Bernoulli trial
- Note: $p_i(0) + p_i(1) = 1$
- $E[X] = p$, $\text{VAR}[X] = p(1-p)$
- $G_X(z) = (1-p+pz)$



Binomial Random Variable

- The Binomial random variable is an example of a **discrete random variable**, where the CDF is a staircase function of x .
- The discontinuities in the CDF are given by the probability mass function.

$$F_X(x) = P[\xi: X(\xi) \leq x] = P[\xi: \# \text{ heads} \leq x]$$



Binomial Random Variable

- Suppose a random experiment is repeated n independent times; let X be the number of times a certain event A occurs in these n trials

$$X = I_1 + I_2 + \dots + I_n$$

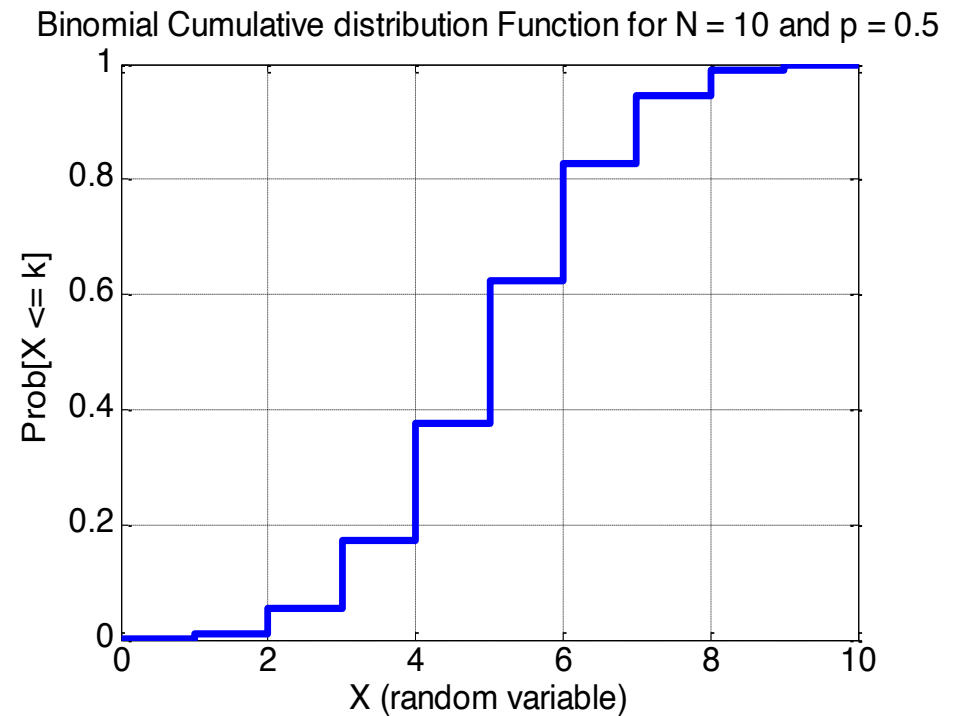
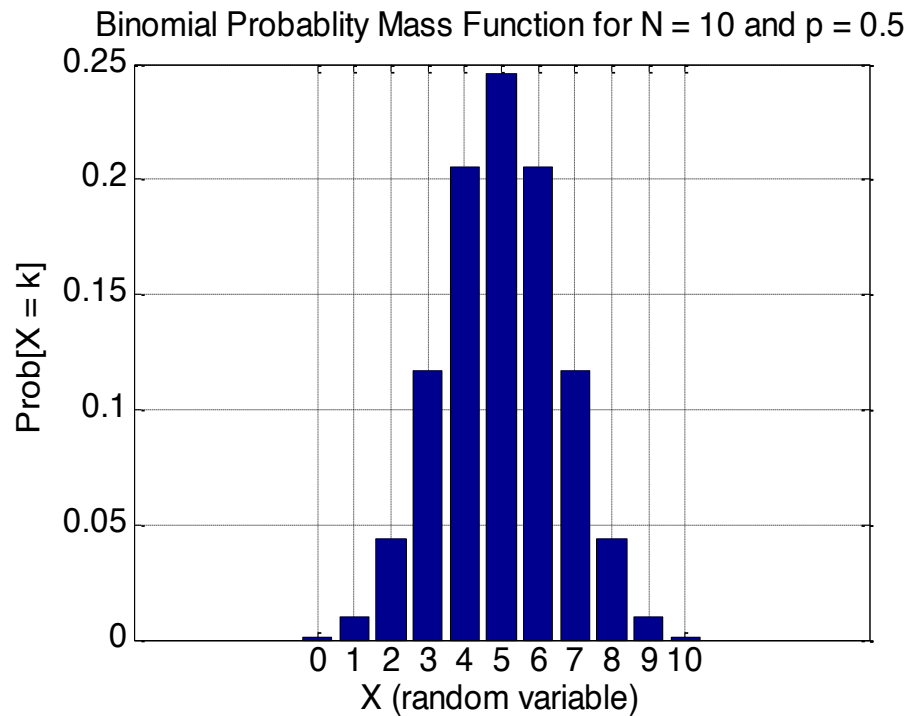
i.e., X is the sum of Bernoulli trials (X 's range = $\{0, 1, 2, \dots, n\}$)

- X has the following pmf for $k=0, 1, 2, \dots, n$

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

- $E[X] = np, \quad \text{Var}[X] = np(1-p)$
- $G_X(z) = (1-p + pz)^n$

Binomial Random Variable



Geometric Random Variable

- Suppose a random experiment is repeated - We count the number of M of independent Bernoulli trials UNTIL the first occurrence of a success
- M is called geometric random variable
 - Range of $M = 1, 2, 3, \dots$
- M has the following pmf

$$\Pr[M = k] = (1 - p)^{k-1} p \quad \text{for } k=1, 2, 3, \dots$$

- $E[X] = 1/p, \quad \text{Var}[X] = (1-p)/p^2$
- $G_X(z) = pz/(1-(1-p)z)$

Geometric Random Variable (2)

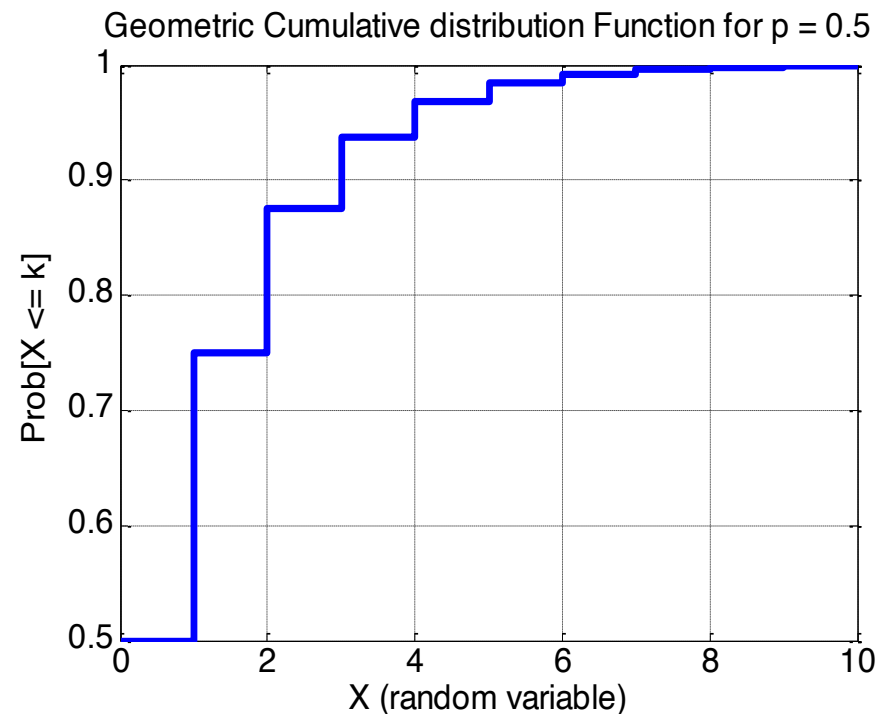
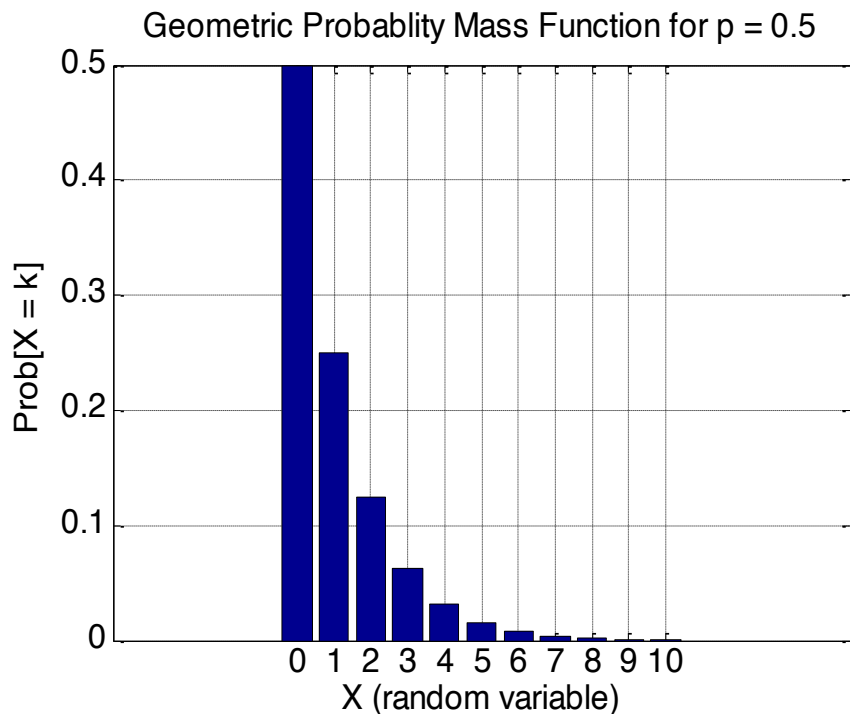
- Suppose a random experiment is repeated - We count the number of M' of independent Bernoulli trials BEFORE the first occurrence of a success
- M' is called geometric random variable
 - Range of $M' = 0, 1, 2, 3, \dots$
- M has the following pmf

$$\Pr[M = k] = (1-p)^k p \quad \text{for } k=0, 1, 2, 3, \dots$$

- $E[X] = (1-p)/p, \quad \text{Var}[X] = (1-p)/p^2$
- $G_X(z) = pz/(1-(1-p)z)$

Geometric Random Variable – cont'd

- Example: $p = 0.5$; X is number of failures BEFORE a success (2nd type)
- Note Matlab's version of geometric distribution is the 2nd type



Poisson Random Variable

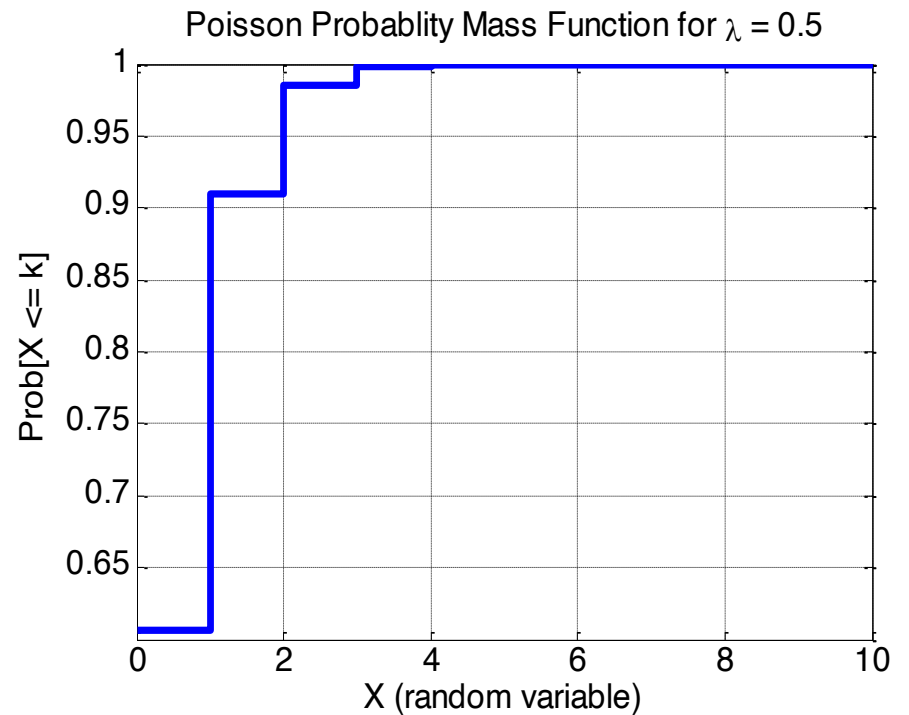
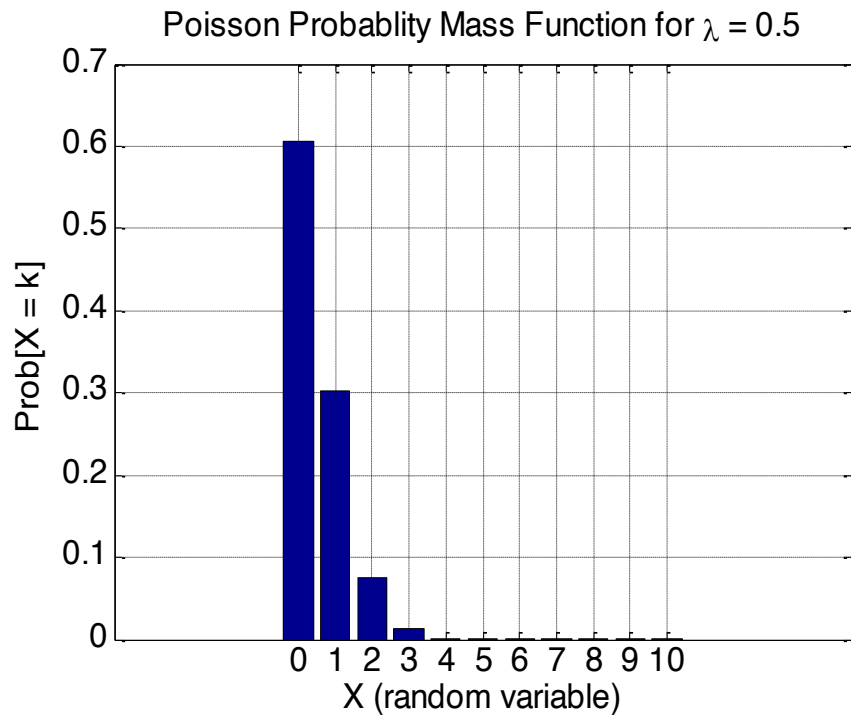
- In many applications we are interested in counting the number of occurrences of an event in a certain time period
- The pmf is given by $\Pr[X = k] = \frac{\alpha^k}{k!} e^{-\alpha}$

For $k=0, 1, 2, \dots$; α is the average number of event occurrences in the specified interval

- $E[X] = \alpha, \quad \text{Var}[X] = \alpha$
- $G_X(z) = e^{\alpha(z-1)}$
- Remember: time between events is exponentially distributed! (continuous r.v. !)
- Poisson is the limiting case for Binomial as $n \rightarrow \infty, p \rightarrow 0$, such that $np = \alpha$

Poisson Random Variable (cont'd)

- **Example:**



MATLAB Code to Plot Distributions

```
0001 % plot distributions
0002 % see "help stats"
0003 clear all
0004 FontSize = 14;
0005 LineWidth = 3;
0006 % Binomial
0007 N = 10; X = [0:1:N]; P = 0.5;
0008 ybp = binopdf(X, N, P); % get PMF
0009 ybc = binocdf(X, N, P); % get CDF
0010 figure(1); set(gca, 'FontSize', FontSize);
0011 bar(X, ybp);
0012 title(['Binomial Probability Mass Function for
        N = ' ...
        num2str(N) ' and p = ' num2str(P)]);
0013 xlabel('X (random variable)');
0014 ylabel('Prob[X = k]'); grid
0015 figure(2); set(gca, 'FontSize', FontSize);
0016 stairs(X, ybc, 'LineWidth', LineWidth);
0017 title(['Binomial Cumulative distribution
        Function for N = ' ...
        num2str(N) ' and p = ' num2str(P)]);
0018 xlabel('X (random variable)');
0019 ylabel('Prob[X <= k]'); grid
0020 % Geometric
0021 P = 0.5; X = [0:10];
0022 ygp = geopdf(X, P); % get pdf
0023 ygc = geocdf(X, P); % get cdf
```

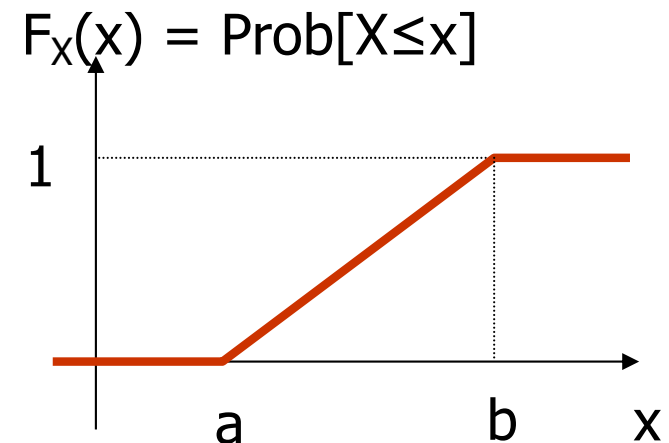
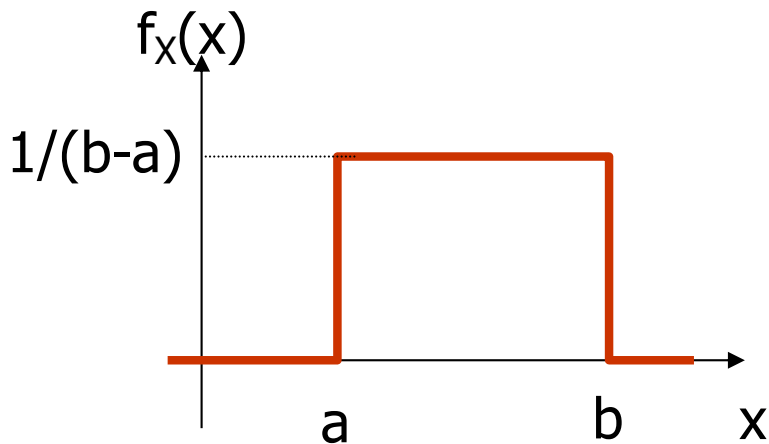
```
0026 figure(3); set(gca, 'FontSize', FontSize);
0027 bar(X, ygp);
0028 title(['Geometric Probability Mass Function for
        p = ' num2str(P)]);
0029 xlabel('X (random variable)');
0030 ylabel('Prob[X = k]'); grid
0031 figure(4); set(gca, 'FontSize', FontSize);
0032 stairs(X, ygc, 'LineWidth', LineWidth);
0033 title(['Geometric Cumulative distribution
        Function for p = ' num2str(P)]);
0034 xlabel('X (random variable)');
0035 ylabel('Prob[X <= k]'); grid
0036 % Poisson
0037 Lambda = 0.5; X = [0:10];
0038 ypp = poisspdf(X, Lambda);
0039 ypc = poisscdf(X, Lambda);
0040 figure(5); set(gca, 'FontSize', FontSize);
0041 bar(X, ypp);
0042 title(['Poisson Probability Mass Function for
        \lambda = ' num2str(Lambda)]);
0043 xlabel('X (random variable)');
0044 ylabel('Prob[X = k]'); grid
0045 figure(6); set(gca, 'FontSize', FontSize);
0046 stairs(X, ypc, 'LineWidth', LineWidth);
0047 title(['Poisson Cumulative distribution
        Function for \lambda = ' num2str(Lambda)]);
0048 xlabel('X (random variable)');
0049 ylabel('Prob[X <= k]'); grid
```

Some Important Random Variables – Continuous RVs

- Uniform
- Exponential
- Gaussian (Normal)
- Rayleigh
- Gamma
- M-Erlang

Uniform Random Variable

- A **continuous Uniform** RV for the interval $[a, b]$ is defined as
Pdf: $f_X(x) = 1/(b-a)$ $a \leq x \leq b$
- Mean and variance are: $E[X] = (a+b)/2$, $\text{Var}[X] = (b-a)^2/12$
- $\Phi_X(\omega) = [e^{j\omega b} - e^{j\omega a}]/(j\omega(b-a))$



Exponential Random Variables

- The exponential RV X with parameter λ has pdf

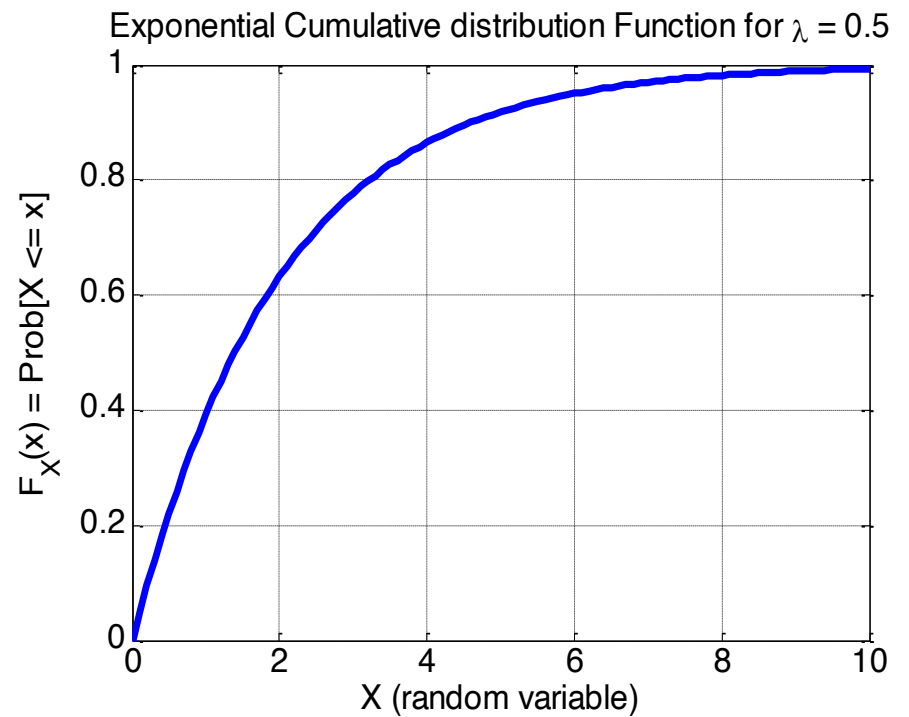
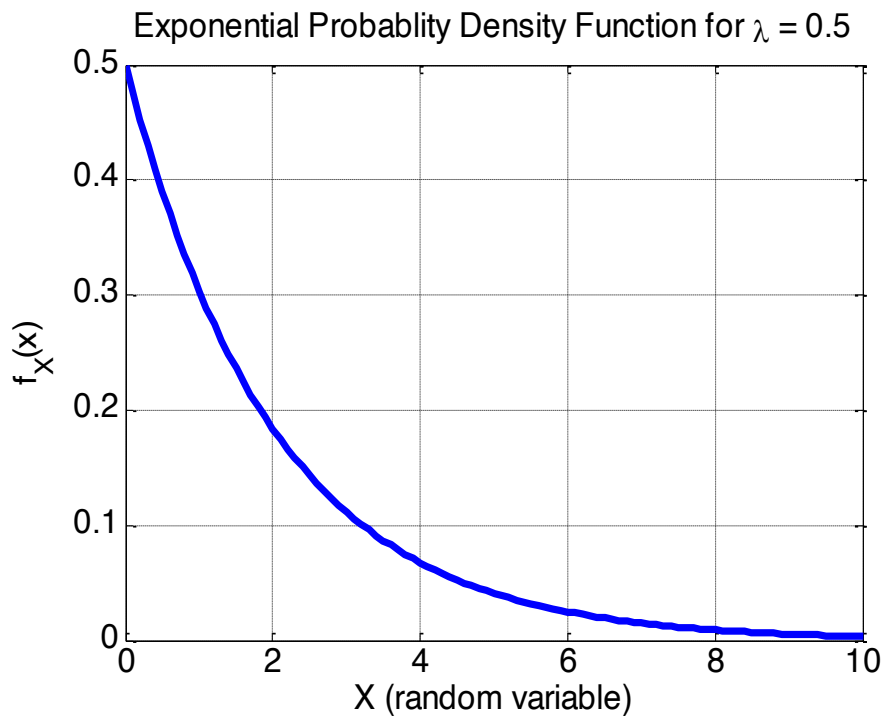
$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

- And CDF given by
- i.e., $\text{Prob}[X \leq x] = 1 - e^{-\lambda x}$, or $\text{Prob}[X > x] = e^{-\lambda x}$
- Range of X : $[0, \infty)$
- $E[X] = 1/\lambda$, $\text{Var}[X] = 1/\lambda^2$
- $\Phi_X(\omega) = \lambda/(\lambda - j\omega)$

Exponential Random Variables (2)

- **Example:**
 - Note the mean is $1/\lambda = 2$



Gaussian (Normal) Random Variable

- For situations where a random variable X is the sum of a large number of “small” random variables – central limit theorem

- pdf
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

For $-\infty < x < \infty$; μ and $\sigma > 0$ are real numbers

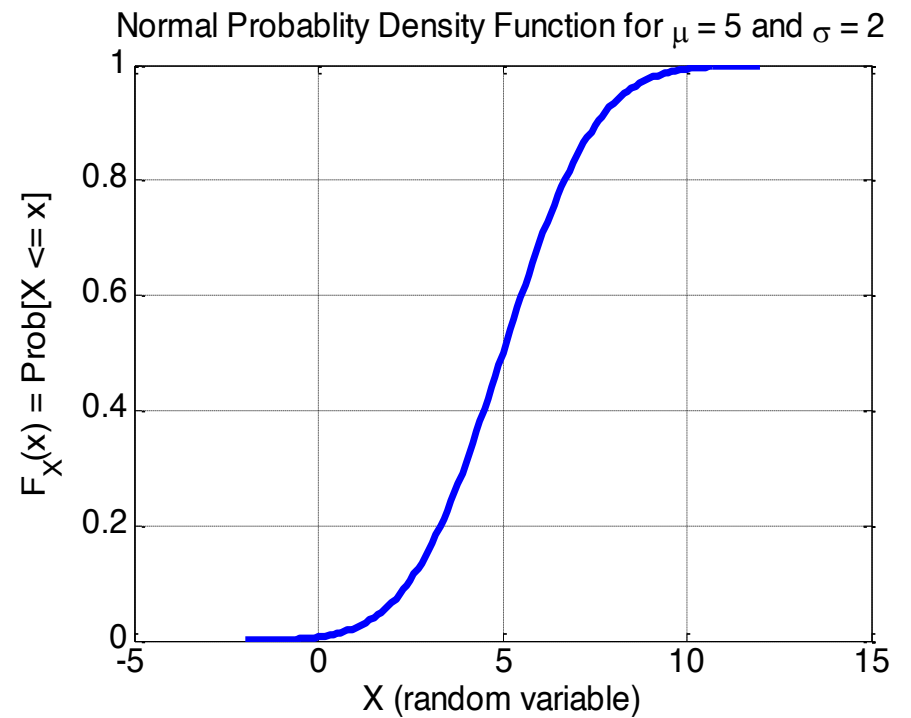
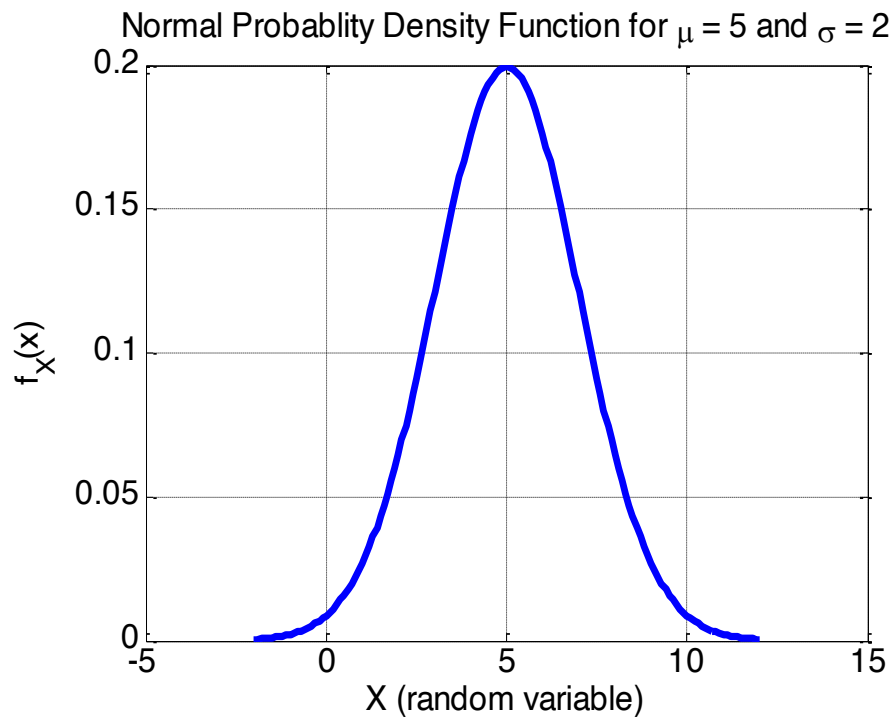
- $E[X] = \mu, \quad \text{Var}[X] = \sigma^2$

$$\Phi_X(\omega) = e^{j\mu\omega - \sigma^2\omega^2/2}$$

- Under wide range of conditions X can be used to approximate the sum of a large number of independent random variables

Gaussian (Normal) Random Variable (2)

- **Example:**



Rayleigh Random Variable

- Arises in modeling of mobile channels
- Range: $[0, \infty)$

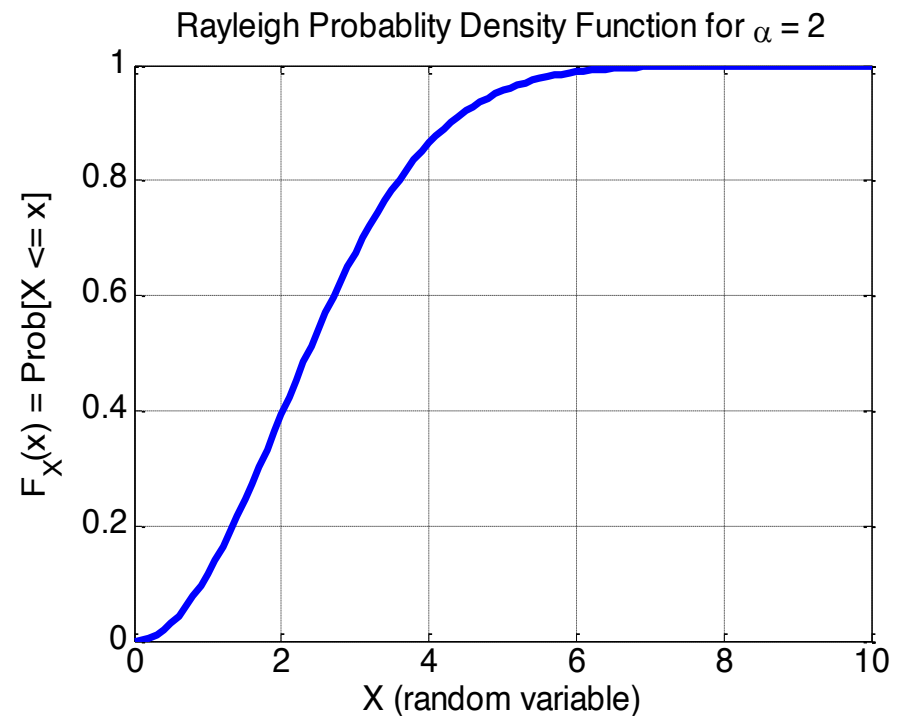
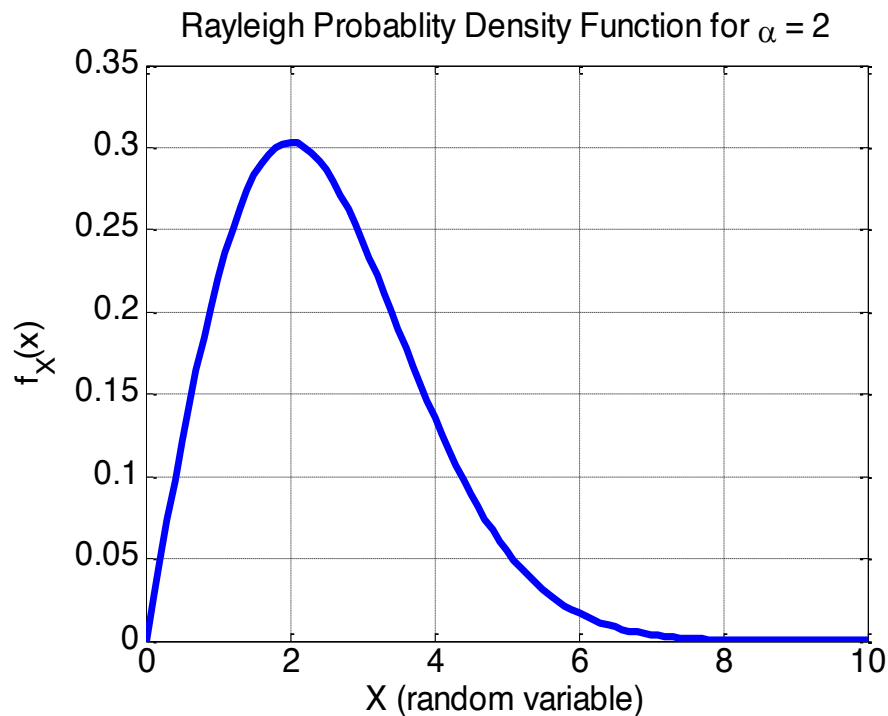
- pdf:
$$f_X(x) = \frac{x}{\alpha^2} e^{-x^2/(2\alpha^2)}$$

- For $x \geq 0, \alpha > 0$

- $E[X] = \alpha\sqrt{\pi/2}, \quad \text{Var}[X] = (2-\pi/2)\alpha^2$

Rayleigh Random Variable (2)

- **Example:**
 - **Note that for Alpha = 2, the mean is $2\sqrt{\pi}/2$**



Matlab Code to Plot Distributions

```
0001 % plot distributions
0002 % see "help stats"
0003 clear all
0004 FontSize = 14;
0005 LineWidth = 3;
0006 % exponential
0007 X = [0:.1:10]; Lambda = 0.5;
0008 yep = exppdf(X, 1/Lambda); % get PDF
0009 yec = expcdf(X, 1/Lambda); % get CDF
0010 figure(1); set(gca,'FontSize', FontSize);
0011 plot(X, yep, 'LineWidth', LineWidth);
0012 title(['Exponential Probability Density
        Function for \lambda = ' ...
0013         num2str(Lambda)]);
0014 xlabel('X (random variable)');
0015 ylabel('f_X(x)'); grid
0016 figure(2); set(gca,'FontSize', FontSize);
0017 plot(X, yec, 'LineWidth', LineWidth);
0018 title(['Exponential Cumulative Distribution
        Function for \lambda = ' ...
0019         num2str(Lambda)]);
0020 xlabel('X (random variable)');
0021 ylabel('F_X(x) = Prob[X <= x]'); grid
0022 % normal
0023 X = [-2:.1:12]; Mu = 5; Sigma = 2;
0024 ynp = normpdf(X, Mu, Sigma); % get PDF
0025 ync = normcdf(X, Mu, Sigma); % get CDF
```

```
0026 figure(3); set(gca,'FontSize', FontSize);
0027 plot(X, ynp, 'LineWidth', LineWidth);
0028 title(['Normal Probability Density Function
        for \mu = ' ...
0029         num2str(Mu) ' and \sigma = '
0030         num2str(Sigma)]);
0031 xlabel('X (random variable)');
0032 ylabel('f_X(x)'); grid
0033 figure(4); set(gca,'FontSize', FontSize);
0034 plot(X, ync, 'LineWidth', LineWidth);
0035 title(['Normal Probability Density Function
        for \mu = ' ...
0036         num2str(Mu) ' and \sigma = '
0037         num2str(Sigma)]);
0038 % Rayleigh
0039 X = [0:.1:10]; Alpha = 2;
0040 yrp = raylpdf(X, Alpha); % get PDF
0041 yrc = raylcdf(X, Alpha); % get CDF
0042 figure(5); set(gca,'FontSize', FontSize);
0043 plot(X, yrp, 'LineWidth', LineWidth);
0044 title(['Rayleigh Probability Density Function
        for \alpha = ' ...
0045         num2str(Alpha)]);
0046 xlabel('X (random variable)');
0047 ylabel('f_X(x)'); grid
0048 figure(6); set(gca,'FontSize', FontSize);
0049 plot(X, yrc, 'LineWidth', LineWidth);
0050 title(['Rayleigh Probability Density Function
        for \alpha = ' ...
0051         num2str(Alpha)]);
0052 xlabel('X (random variable)');
0053 ylabel('F_X(x) = Prob[X <= x]'); grid
```

Gamma Random Variable

Versatile distribution ~ appears in modeling of lifetime of devices and systems

Has two parameters: $\alpha > 0$ and $\lambda > 0$

PDF For $0 < x < \infty$

$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

The quantity $\Gamma(z)$ is the gamma function and is specified by

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

The gamma function has the following properties:

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(z+1) = z\Gamma(z) \text{ for } z > 0$$

$$\Gamma(m+1) = m! \text{ For } m \text{ nonnegative integer}$$

$$E[X] = \alpha/\lambda, \quad \text{Var}[X] = \alpha/\lambda^2$$

$$\Phi_X(\omega) = 1/(1-j\omega/\lambda)^\alpha$$

If $\alpha = 1 \rightarrow$ gamma R.V. becomes exponential

Functions of a Random Variable

- Very often we are interested in a function of a random variable: $Y = g(X)$

- Since $Y = g(X(\xi))$, Y is also a random variable

- Examples:

$$Y = aX + b$$

$$Y = X^2$$

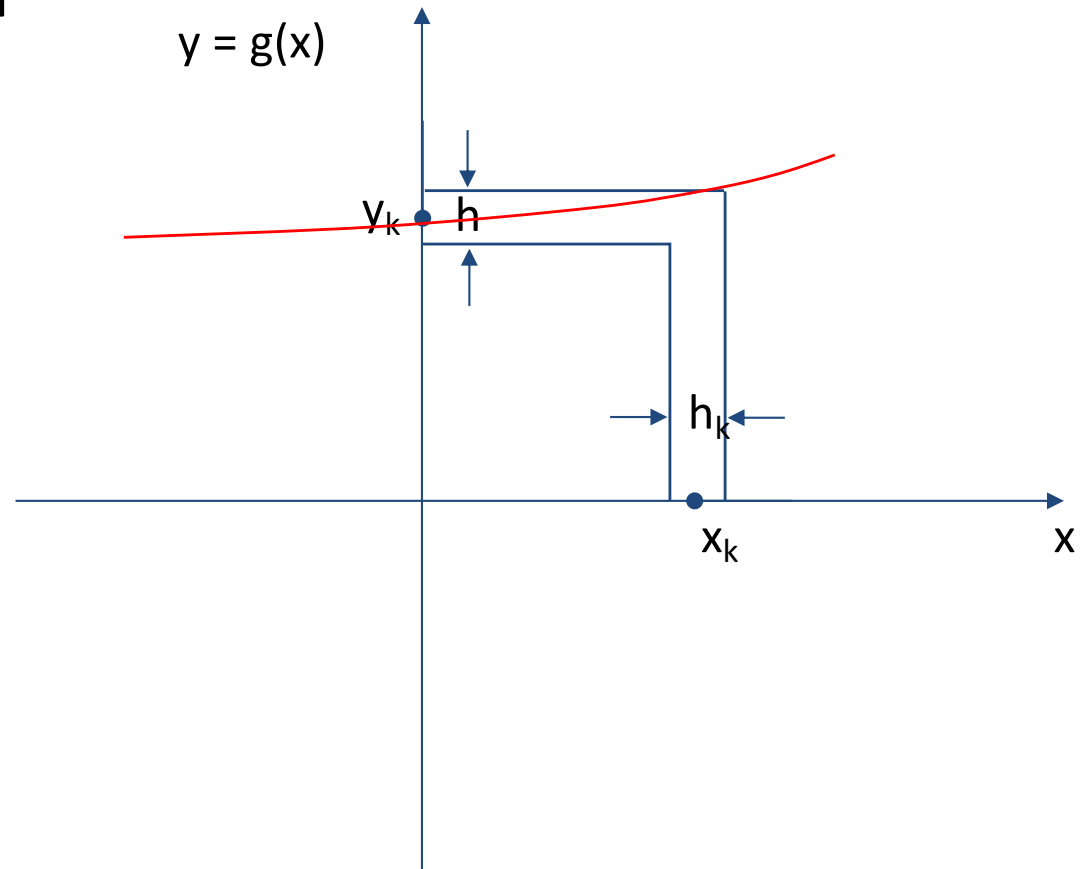
$$Y = \ln X$$

- Given CDF/pdf/pmf of X , find CDF/pdf/pmf of Y .

Expected Value of $Y = g(X)$

Can find $E[Y]$ directly in terms of the pdf of X .

$$E[Y] = \int_{-\infty}^{+\infty} g(x)f_X(x)dx$$



Finding the Distributions of Y

- Use the notion of equivalent event:

$$P[Y \text{ in } C] = P[g(X) \text{ in } C] = P[X \text{ in } B]$$

- Three types of equivalent events are useful in determining the cdf and pdf of $Y = g(X)$
- Three equivalent events:
 1. $\{g(X) = y_k\}$ Magnitude of Jump at y_k
 2. $\{g(X) \leq y\}$ CDF of Y (directly)
 3. $\{y < g(X) \leq y + h\}$ Useful for finding pdf

Expectation of a Function of the Random Variable

- Let $g(x)$ be a function of the random variable x , the expectation of $g(x)$ is given by

$$E[g(x)] = \sum_{\forall i} g(x_i) P[X = x_i]$$

for discrete variables, or

$$E[g(x)] = \int_{-\infty}^{\infty} g(t) f_x(t) dt$$

for continuous variables.

Mean, Variance, & Probabilities

- Suppose X is non-negative and $E[X]$ is small
 - We expect that X usually takes on small values
 - Can we quantify this in terms of $P[X > t]$?
- Suppose X is tightly packed about $E[X]$
 - We expect that X usually close to $E[X]$
 - Can we quantify $P[|X - E[X]| > t]$?

Markov Inequality

- Markov Inequality states that:

$$P[X \geq a] \leq \frac{E[X]}{a} \quad \text{for } X \text{ nonnegative.}$$

We obtain Eq. (4.75) as follows:

$$\begin{aligned} E[X] &= \int_0^a t f_X(t) dt + \int_a^\infty t f_X(t) dt \geq \int_a^\infty t f_X(t) dt \\ &\geq \int_a^\infty a f_X(t) dt = a P[X \geq a]. \end{aligned}$$

Chebyshev Inequality

- The Chebyshev Inequality stated that

$$P[|X - m| \geq a] \leq \sigma^2 / a^2$$

- Although this provides a good bound for probability of deviations around the mean, it does not provide a tight bound for the tail of a distribution

Chebyshev Inequality (2)

The Chebyshev inequality is a consequence of the Markov inequality. Let $D^2 = (X - m)^2$ be the squared deviation from the mean. Then the Markov inequality applied to D^2 gives

$$P[D^2 \geq a^2] \leq \frac{E[(X - m)^2]}{a^2} = \frac{\sigma^2}{a^2}.$$

Equation (4.76) follows when we note that $\{D^2 \geq a^2\}$ and $\{|X - m| \geq a\}$ are equivalent events.

Suppose that a random variable X has zero variance; then the Chebyshev inequality implies that

$$P[X = m] = 1, \tag{4.77}$$

that is, the random variable is equal to its mean with probability one. In other words, X is equal to the constant m in almost all experiments.

Examples

Example 4.38

The mean response time and the standard deviation in a multi-user computer system are known to be 15 seconds and 3 seconds, respectively. Estimate the probability that the response time is more than 5 seconds from the mean.

The Chebyshev inequality with $m = 15$ seconds, $\sigma = 3$ seconds, and $a = 5$ seconds gives

$$P[|X - 15| \geq 5] \leq \frac{9}{25} = .36.$$

Example 4.39

If X has mean m and variance σ^2 , then the Chebyshev inequality for $a = k\sigma$ gives

$$P[|X - m| \geq k\sigma] \leq \frac{1}{k^2}.$$

Now suppose that we know that X is a Gaussian random variable, then for $k = 2$, $P[|X - m| \geq 2\sigma] = .0456$, whereas the Chebyshev inequality gives the upper bound .25.

Example 4.40 Chebyshev Bound Is Tight

Let the random variable X have $P[X = -v] = P[X = v] = 0.5$. The mean is zero and the variance is $\text{VAR}[X] = E[X^2] = (-v)^2 0.5 + v^2 0.5 = v^2$.

Note that $P[|X| \geq v] = 1$. The Chebyshev inequality states:

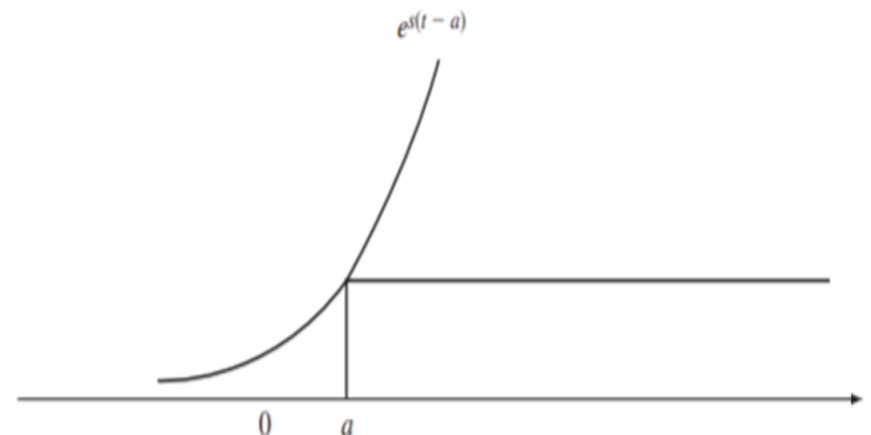
$$P[|X| \geq v] \leq 1 - \frac{\text{VAR}[X]}{v^2} = 1.$$

We see that the bound and the exact value are in agreement, so the bound is tight.

Chernoff Bound

$$P[X \geq a] = \int_0^{\infty} I_A(t) f_X(t) dt \leq \int_0^{\infty} \frac{t}{a} f_X(t) dt = \frac{E[X]}{a}.$$

By changing the upper bound on $I_A(t)$, we can find different bounds in $P(x \geq a)$. Consider the bound $I_A(t) \leq e^{s(t-a)}$, also shown below where $s > 0$. The resulting bound is:



Bounds on Indicator Function for $A = \{t \geq a\}$

$$\begin{aligned} P[X \geq a] &= \int_0^{\infty} I_A(t) f_X(t) dt \leq \int_0^{\infty} e^{s(t-a)} f_X(t) dt \\ &= e^{-sa} \int_0^{\infty} e^{st} f_X(t) dt = e^{-sa} E[e^{sX}]. \end{aligned}$$

Chernoff Bound for Gaussian

Example 4.44 Chernoff Bound for Gaussian Random Variable

Let X be a Gaussian random variable with mean m and variance σ^2 . Find the Chernoff bound for X .

The Chernoff bound (Eq. 4.78) depends on the moment generating function:

$$E[e^{sX}] = \Phi_X(-js).$$

In terms of the characteristic function the bound is given by:

$$P[X \geq a] \leq e^{-sa} \Phi_X(-js) \quad \text{for } s \geq 0.$$

The parameter s can be selected to minimize the upper bound.

The bound for the Gaussian random variable is:

$$P[X \geq a] \leq e^{-sa} e^{ms + \sigma^2 s^2 / 2} = e^{-s(a-m) + \sigma^2 s^2 / 2} \quad \text{for } s \geq 0.$$

We minimize the upper bound by minimizing the exponent:

$$0 = \frac{d}{ds}(-s(a-m) + \sigma^2 s^2 / 2) \quad \text{which implies } s = \frac{a-m}{\sigma^2}.$$

The resulting upper bound is:

$$P[X \geq a] = Q\left(\frac{a-m}{\sigma}\right) \leq e^{-(a-m)^2 / 2\sigma^2}.$$