CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE
Lecture 2: The Axioms of Probability

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Note

- These slides cover material partially covered in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-set derivation and detailed discussions were need.

Random Experiments

- A random experiment is an experiment in which the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.
- A random experiment is specified by stating:
  - An experimental procedure
  - A set of 1 or more measurements and/or observations.

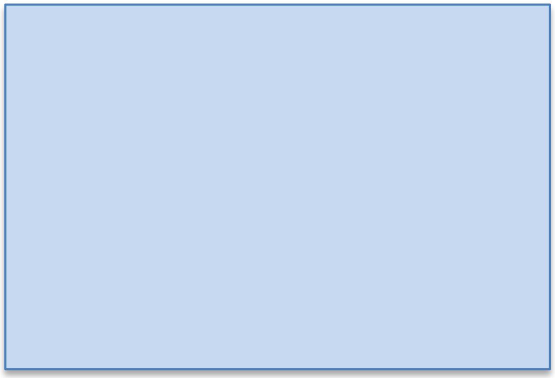
Examples of Random Experiments

E_1	A coin is tossed once; observe the outcome of the toss
E_2	A coin is tossed 3 times; note the sequence of heads and tails
E_3	The number of phone calls initiated by a community in 1 hour is counted
E_4	The round-trip time of an Internet PING packet is noted
E_5	A number in the unit interval is selected at random
E_6	The amplitudes of an audio signal at times t_0 and t_1 are measured
E_7	The amplitude signal of an entire audio signal is recorded

Sample Space

- An **outcome** or **sample point** ξ of a random experiment is a result that cannot be decomposed into other results.
  - Each performance of a random experiment results in one and only one outcome.
  - Outcomes are mutually exclusive.
- The **sample space** S is defined as the set of all possible outcomes:

$$S = \{ \xi \}$$



Sample Space (con't)

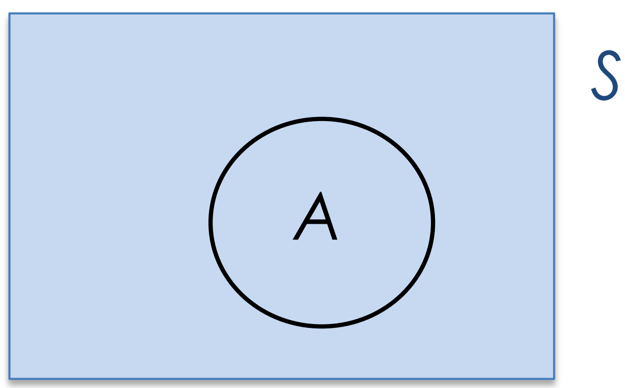
- Each performance of a random experiment can be viewed as the selection at random of a ξ from S.
- The sample space is **discrete** if S is a countable set.
- The sample space is **continuous** if S is not countable.

Examples of Sample Spaces

E_1	A coin is tossed once; observe the outcome of the toss
E_2	A coin is tossed 3 times; note the sequence of heads and tails
E_3	The number of phone calls initiated by a community in 1 hour is counted
E_4	The round-trip time of an Internet PING packet is noted
E_6	The amplitudes of an audio signal at times t_0 and t_1 are measured

Events

- Did an event occur when we conducted a random experiment?
- Did the outcome satisfy some set of conditions?
- An **event** *A* is a collection of outcomes for a random experiment *E.*
  - An event *A* is a **subset** of *S*.



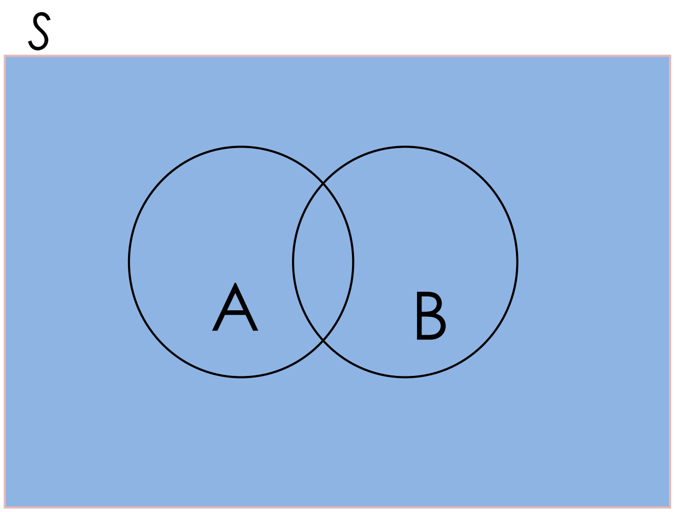
Examples of Events

- An **elementary event** is a singleton subset of a discrete sample space.

E_3	*Toss a coin three times:* A = more heads than tails
E_3	B = equal number of heads & tails
E_3	C = number of heads & tails are not equal

Events & Set Operations

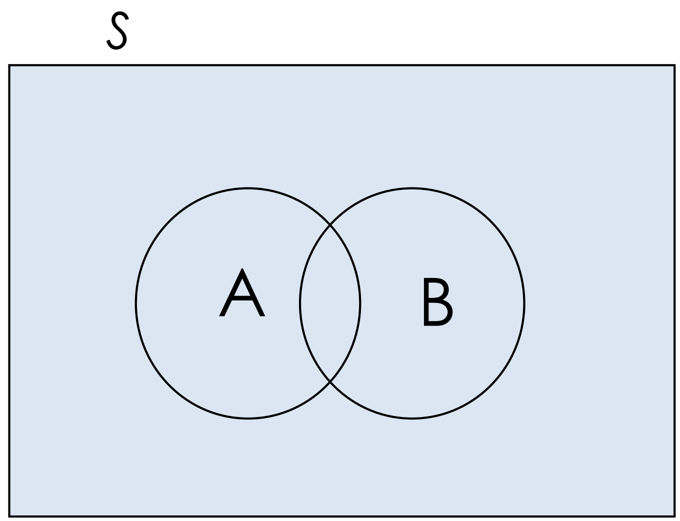
- Events can be expressed through set operations
- Union: $A \cup B = \{\xi : \xi \in A \text{ or } \xi \in B\}$



$$\bigcup_{i=1}^{n} A_i \triangleq A_1 \cup A_2 \cup \ldots \cup A_n = \{\xi : \xi \in A_i \text{ for some } i\ \}$$

Events & Set Operations (cont'd)

- Intersection: $A \cap B = \{\xi : \xi \in A \text{ and } \xi \in B\}$



Mutually exclusive: $A \cap B = \varnothing$

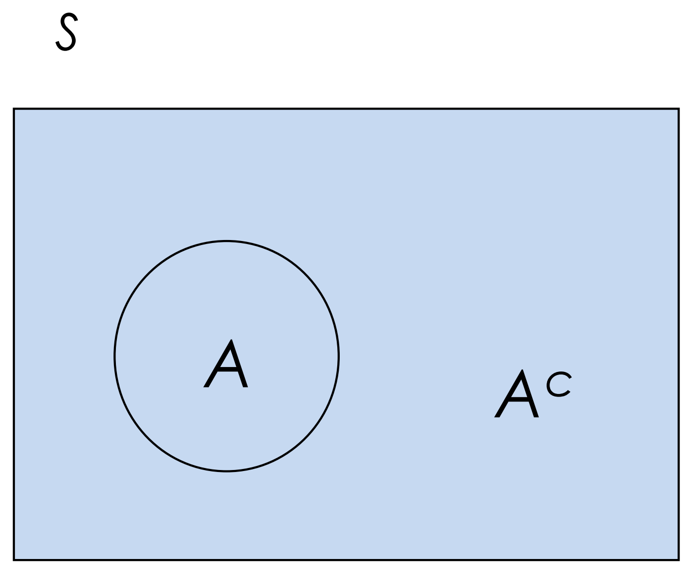
$$\bigcap_{i=1}^{n} A_i \triangleq A_1 \cap A_2 \cap \ldots \cap A_n = \{\xi : \xi \in A_i \text{ for all } i\}$$

Events & Set Operations (cont'd)

- Complementation:

$$A^c = \{\xi \in S : \xi \notin A\}$$

$$S^c = \varnothing; \ \varnothing^c = S$$



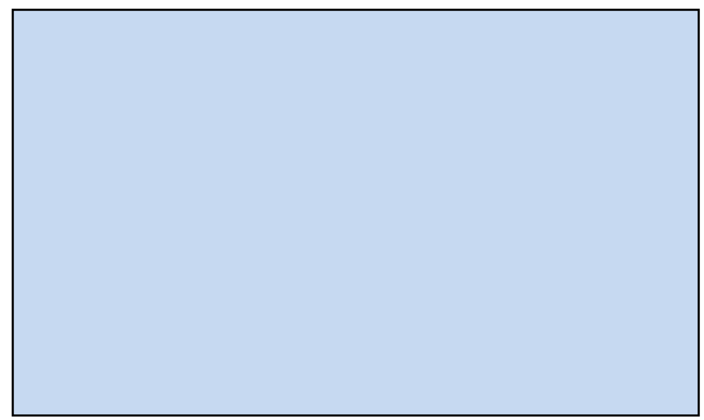
Sample Space, Events, Outcomes

- Random Experiment E:
  - Experimental procedure & set of observations and measurements
  - Outcome ξ of random experiment
- Sample Space S:
  - Set of all possible outcomes
- Events:
  - Subset A of S
  - Events obtained through set operations

Axioms of Probability

Let E be a random experiment with sample space S.

A probability law for E is a rule that assigns to each event a number $P[A]$, the probability of A, that satisfies the following axioms:



Axiom 1: $0 \leq P[A]$

Axiom 2: $P[S] = 1$

Axiom 3: If $A \cap B = \emptyset$, then $P[A \cup B] = P[A] + P[B]$

Axioms of Probability (cont’d)

The first three axioms are sufficient to deal with finite sample spaces. For infinite sample spaces we need an additional axiom.

Axiom 3′: If $A_1, A_2, \ldots$ is a sequence of events such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P[A_k]$$

Corollaries 1, 2, & 3

Corollary 1: $P[A^c] = 1 - P[A]$

Proof: Since an event A and its complement A^c are mutually exclusive, $A \cap A^c = \emptyset$, we have from Axiom III that

$$P[A \cup A^c] = P[A] + P[A^c].$$

Since $S = A \cup A^c$, by Axiom II,

$$1 = P[S] = P[A \cup A^c] = P[A] + P[A^c].$$

The corollary follows after solving for $P[A^c]$.

Corollary 2: $P[A] \leq 1$

Corollary 3: $P[\emptyset] = 0$

Corollary 4

Corollary 4: If $A_1, A_2, \ldots A_n$ are pairwise mutually exclusive, then

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P[A_k] \text{ for } n \geq 2.$$

Proof: We use mathematical induction. Axiom III implies that the result is true for $n = 2$. Next we need to show that if the result is true for some n, then it is also true for $n + 1$. This, combined with the fact that the result is true for $n = 2$, implies that the result is true for $n \geq 2$.

Suppose that the result is true for some $n > 2$; that is,

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P[A_k], \tag{2.9}$$

and consider the $n + 1$ case

$$P\left[\bigcup_{k=1}^{n+1} A_k\right] = P\left[\left\{\bigcup_{k=1}^{n} A_k\right\} \cup A_{n+1}\right] = P\left[\bigcup_{k=1}^{n} A_k\right] + P[A_{n+1}], \tag{2.10}$$

where we have applied Axiom III to the second expression after noting that the union of events A_1 to A_n is mutually exclusive with A_{n+1}. The distributive property then implies

$$\left\{\bigcup_{k=1}^{n} A_k\right\} \cap A_{n+1} = \bigcup_{k=1}^{n} \{A_k \cap A_{n+1}\} = \bigcup_{k=1}^{n} \varnothing = \varnothing.$$

Corollary 5

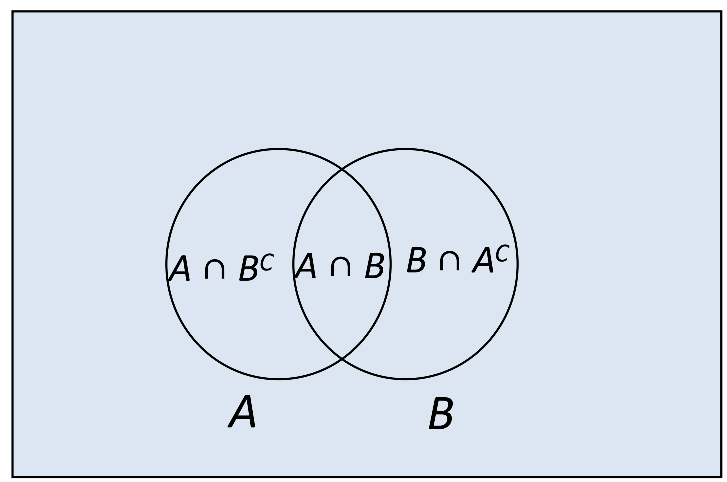
Corollary 5: $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

Proof: Decompose $A \cup B$*, A, and B as union of three* disjoint *events. (see the Venn Diagram below)*

$$P[A \cup B] = P[A \cap B^C] + P[B \cap A^C] + P[A \cap B]$$

$$P[A] = P[A \cap B^C] + P[A \cap B] \qquad P[B] = P[B \cap A^C] + P[A \cap B]$$

By substituting $P[A \cap B^C]$ and $P[B \cap A^C]$ from the two equations into the top equation we obtain the corollary.



Corollary 6

Corollary 6:

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P[A_j] - \sum_{j<k}^{n} P[A_j \cap A_k] + \ldots$$
$$+ (-1)^{n+1} P[A_1 \cap \ldots \cap A_n]$$

Corollary 7

Corollary 7: If $A \subset B$, then $P[A] \leq P[B]$

Proof: In Fig. 2.4, B is the union of A and $A^c \cap B$, thus

$$P[B] = P[A] + P[A^c \cap B] \geq P[A],$$

since $P[A^c \cap B] \geq 0$.

Exercise: Random Number from the Unit Interval

- Sample Space
- Probability Law
- Events
  A = {Outcome is > 0.5}
  B = {Outcome is within 0.1 and 0.6}
  C = {Outcome = 0.3}

Lecture Summary

- A random experiment is specified by an experimental procedure and a set of measurements/observations.
- An outcome or sample point of a random experiment is a result that cannot be decomposed into other results.
- The sample space specifies set of all possible outcomes.
- Events describe conditions of interest and are specified as subsets of *S.*
- When *S* is discrete, events consist of the union of elementary events.
- When *S* is continuous, events consist of the union or intersection of subsets of the real time (or plane).

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Lecture 2 Annex- Fine Points

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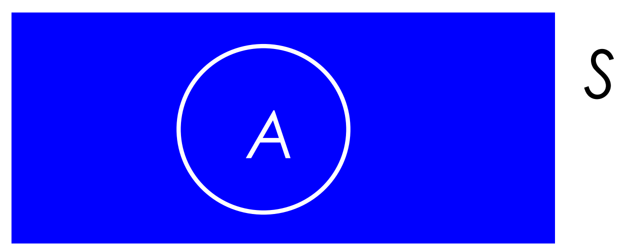
Fine Points: Event Classes and Probabilities of Sequences of Events

Sections 2.8 and 2.9

(Advanced Topics)

Sample Space and Events

- An **outcome** or **sample point** ξ of a random experiment is a result that cannot be decomposed into other results.
- The **sample space** ***S*** is defined as the set of all possible outcomes: $S = \{ \xi \}$
- An event A is a collection of outcomes for a random experiment E.
  - An event A is a subset of S
  - Not all subsets of A need be events
- Union (intersection) of events is an event
- S and $\varnothing$ are events



Event Classes

- The Axioms of Probability require a *class* F of events.
  - Only events are assigned probabilities.
  - Any set operation on events in F will produce a set that is also an event in F.
  - Complements, countable unions and intersections of events in F
- If S is finite or countable, we can let F consist of all subsets of S.
- If S is the real line R, we ***cannot*** let F be all possible subsets of R and still satisfy the Axioms of Probability.

Fine Points on Event Classes[1]

Let F be the class of events of interest in a random experiment. We require that F be a **field**.

A collection of sets F is called a **field** if it satisfies the following conditions:

$$\text{i)}\quad \varnothing \in \mathrm{F}$$

$$\text{ii) if } A \in \mathrm{F} \text{ and } B \in \mathrm{F}, \text{ then } A \cup B \in \mathrm{F}$$

$$\text{iii) if } A \in \mathrm{F}, \text{ then } A^c \in \mathrm{F}$$

DeMorgan's rule and above imply:

$$\text{if } A \in \mathrm{F} \text{ and } B \in \mathrm{F}, \text{ then } A \cap B \in \mathrm{F}$$

Also, any finite union or intersection of events of F is also in F

Example

Let S = {T, H}. Find the field F generated by set operations on the class consisting of elementary events of S: C ={{H},{T}}.

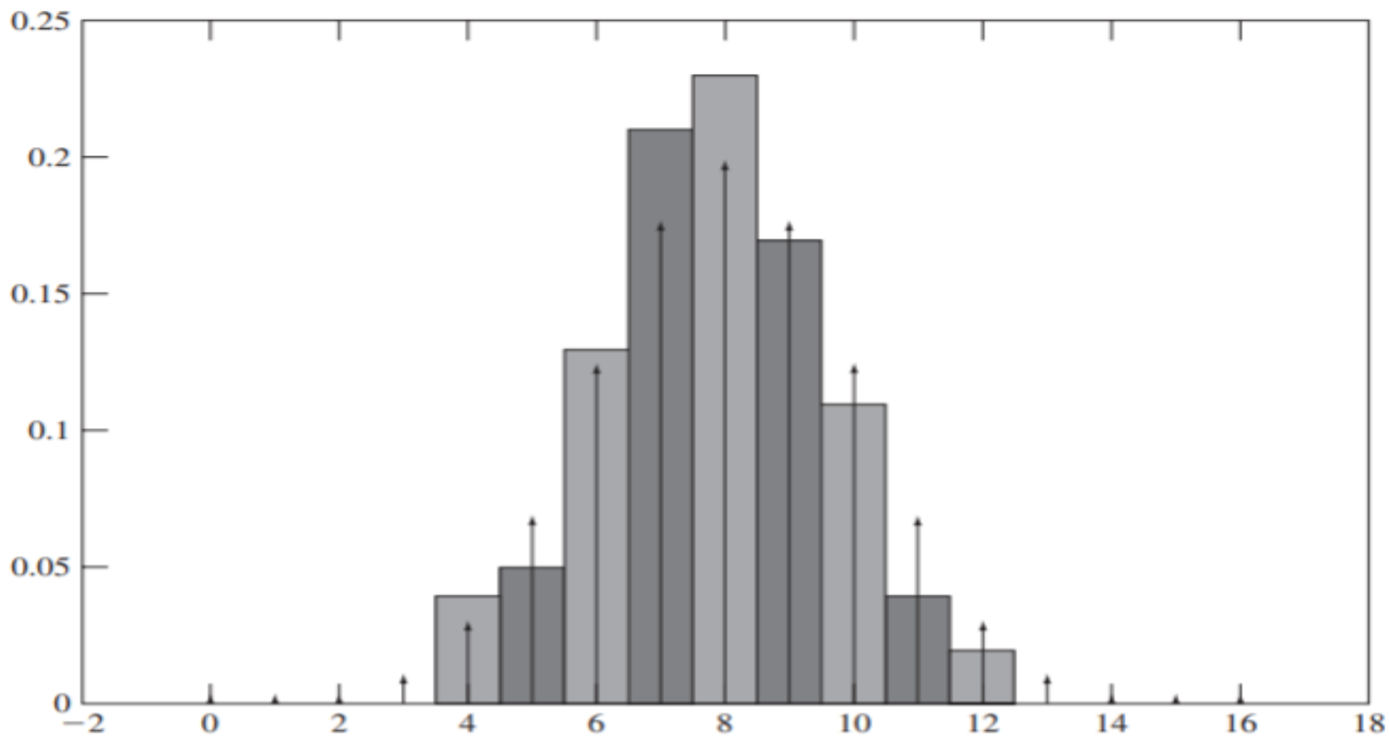
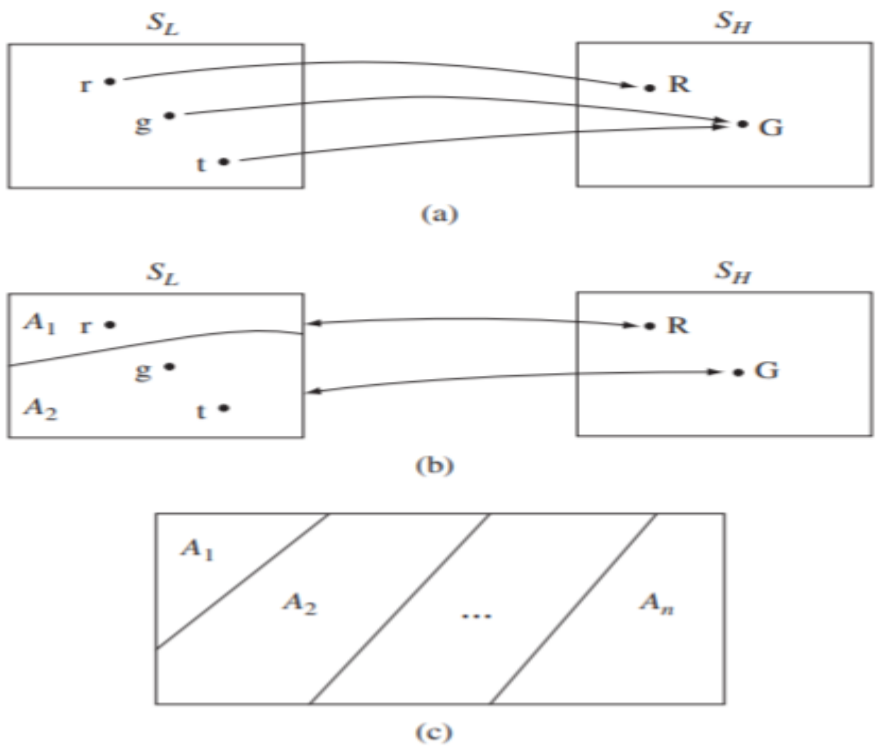


FIGURE 2.18
Relative frequencies from 100 binomial experiments and corresponding binomial probabilities.

Example: Lisa and Homer’s Urn Experiment

An urn contains three white balls. One ball has a red dot. Another ball has a green dot, and the third ball has a teal dot. Experiment: pick a ball at random & note color.

Lis



Homer

FIGURE 2.19
(a) Homer’s mapping; (b) Partition of Lisa’s sample space;
(c) Partitioning of a sample space.

Lisa figures that Homer field is generated by a partition of S_L

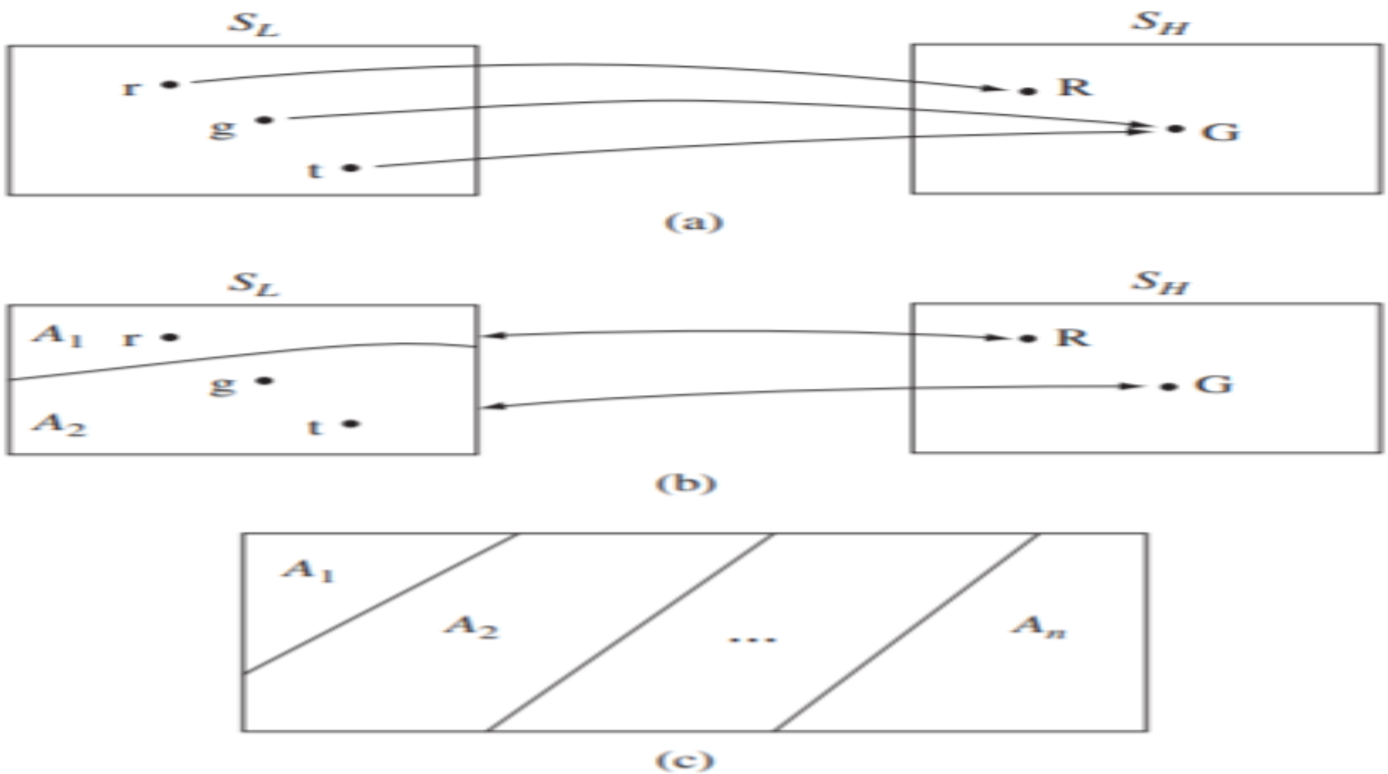


FIGURE 2.19
(a) Homer's mapping; (b) Partition of Lisa's sample space; (c) Partitioning of a sample space.

Moral: Not all subsets are events!

Any combination of unions, intersections, and complements of events in Homer's experiment result in events in the field:

$$F = \{ \ , \{r\}, \{r,g\}, \{r,g,t\}\}.$$

- F does not contain all of the events in Lisa's power set S_L.
- F suffices to address events that only involve the outcomes in S_H.
- Questions that distinguish between teal and green lead to subsets of S_L such as {r, t} that are not events in F and hence outside the scope of the experiment.

Measurable Sets

- The sets in the field F that specifies the events of interest are said to be *measurable*. Any subset of *S* that is not in F is not measurable.
- In Homer's example, the set {r, t} is not measurable with respect to F.
- In modeling, we frequently make decisions that restrict the scope of questions about a random experiment.
- In general case, the sample space *S* in the original random experiment is partitioned into mutually exclusive events $A_1,\ldots, A_n$, where $A_i \cap A_j = \emptyset$ for $i \neq j$ and

$$S = A_1 \cup A_2 \ldots \cup A_n$$

Sigma Fields

If S is countably infinite, we require countable unions of events to be events:

$$\text{i)} \quad \varnothing \in F$$

$$\text{ii) if } A_1, A_2, \ldots \in F, \text{ then } \bigcup_{n=1}^{\infty} A_n \in F$$

$$\text{iii) if } A \in F, \text{ then } A^c \in F$$

A class of sets F that satisfies above equations is called a **sigma field**.

DeMorgan's rule imply that countable intersections of events are also in F:

$$\text{if } A_1, A_2, \ldots \in F, \text{ then } \bigcap_{n=1}^{\infty} A_n \in F$$

The Borel Field of Events

- Let S be the real line R. Consider events that are open intervals of the real line:

  $(-\infty, b] = \{x: -\infty < x \leq b\}$.
- **Borel field** B, is the sigma field generated by countable unions, countable intersections and complements of class of events of the form $(-\infty, b]$.
- We can show that events of the form below are in B:

  (a,b), [a,b], (a,b], [a,b),

  $[a,\infty)$, (a,∞), $(-\infty,b)$, {b}.
- Note: the Borel field can also be generated by starting with other classes of intervals, for example, (a,b).
- The Borel field does not include all subsets of R.

(-∞,b]

$$\{x: -\infty < x \le b\}.$$

(a,∞) is in B

(a,b] is in B

$$(a, \infty) \cap (-\infty, b] = (a, b] \quad \text{for} \quad a < b.$$

(-∞,b) is in B

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}\{x: -\infty < x \leq b - 1/n\} = (-\infty, b).$$

Axioms of Probability

A **probability space** (S, F, P) consists of sample space S, a sigma field F of events from S, and a function that assigns to each event A a number $P[A]$, the probability of A, that satisfies the following axioms:

Axiom 1: $0 \leq P[A]$

Axiom 2: $P[S] = 1$

Axiom 3′: If $A_1, A_2, \ldots$ is a sequence of events such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P[A_k]$$

Increasing Sequence: Let A_1, A_2,… be a sequence of events from a sigma field, such that, $A_1 \subset A_2 \ldots \subset A_n \ldots$

$$\lim_{k\to\infty} A_k \triangleq \bigcup_{k=1}^{\infty} A_k$$

Ex: $[a, b - 1/n]$ with $a < b + 1$; $(-n, a]$

Decreasing Sequence: Let A_1, A_2,... be a sequence of events from a sigma field, such that, $A_1 \supset A_2 \ldots \supset A_n \ldots$

$$\lim_{k\to\infty} A_k \triangleq \bigcap_{k=1}^{\infty} A_k$$

Ex: $(a-1/n, a+1/n)$; $(-\infty, a+1/n]$.

Corollary 8: Continuity of Probability

Let A_1, A_2,… be an increasing and decreasing sequence of events in F, then:

$$\lim_{k\to\infty} P[A_k] = P[\lim_{k\to\infty} A_k]$$

Find the limiting probability for each sequences of events :

$[a, b - 1/n]$, $(-n, a]$, $(a - 1/n)$, $(-\infty, a + 1/n]$.

Example 2.51

Find an expression for the probabilities of the following sequences of events from the Borel field: $[a, b - 1/n], (-n, a], (a - 1/n, a + 1/n), (-\infty, a + 1/n]$.

$$\lim_{n\to\infty} P[\{x: a \le x \le b - 1/n\}] = P[\lim_{n\to\infty} \{x: a \le x \le b - 1/n\}] = P[\{x: a \le x < b\}].$$

$$\lim_{n\to\infty} P[\{x: -n < x \le a\}] = P[\lim_{n\to\infty} \{x: -n < x \le a\}] = P[\{x: -\infty < x \le a\}].$$

$$\lim_{n\to\infty} P[\{x: a - 1/n < x < a + 1/n\}] = P[\lim_{n\to\infty} \{x: a - 1/n < x < a + 1/n\}] = P[\{x = a\}].$$

$$\lim_{n\to\infty} P[\{x: -\infty < x \le a + 1/n\}] = P[\lim_{n\to\infty} \{x: -\infty < x \le a + 1/n\}]$$

$$= P[\{x: -\infty < x \le a\}].$$