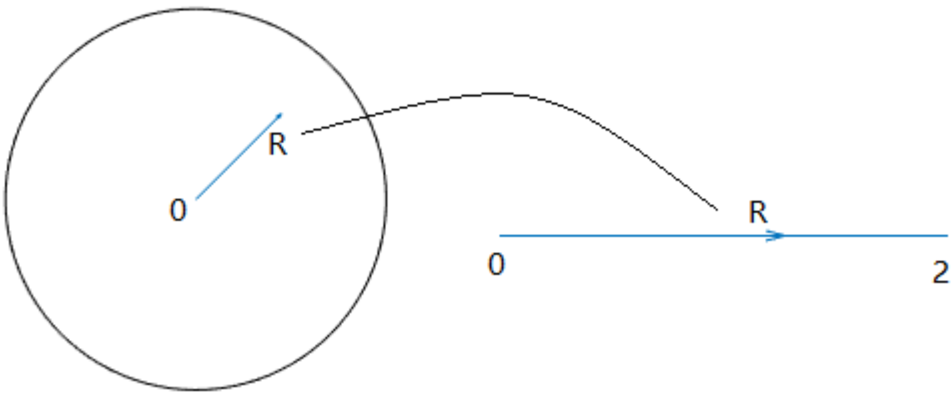
ECE 528: Introduction to Random Processes in ECE Fall 2020

Solutions to Homework Set No. 5

4.6

(a) $S=\left\{(x,y): x^2+y^2\le 4\right\}, S_R=\{R: 0\le R\le 2\}, R=\sqrt{x^2+y^2};$



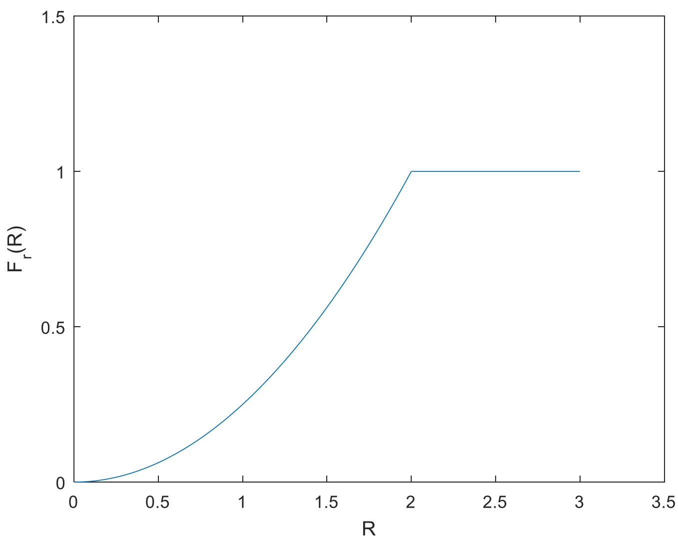

(b)

(c) $A_{S_R}=\{R: 0\le R\le 0.25\}, A_S=\left\{(x,y): \sqrt{x^2+y^2}\le 0.25\right\}$

$$P(A)=\frac{\pi * 0.25^2}{\pi * 2^2}=1/64.$$

(d) $F_r(R)=P(r\le R)=\dfrac{\pi * R^2}{\pi * 2^2}=(R/2)^2 \quad for \quad 0\le R\le 2,$

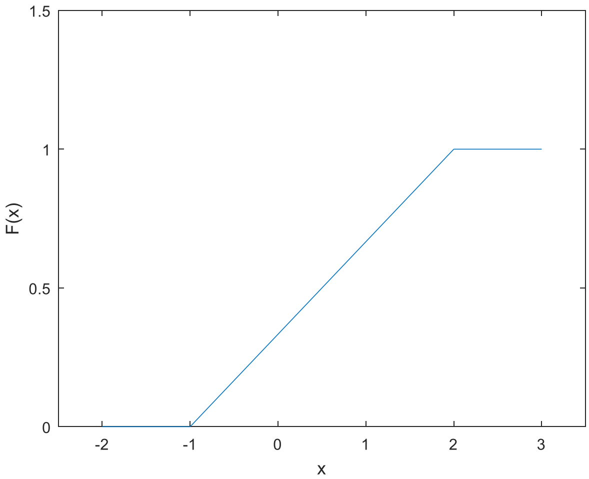
$F_r(R)=0 \quad for \quad R<0, F_r(R)=1 \quad for \quad R>2.$




4.11

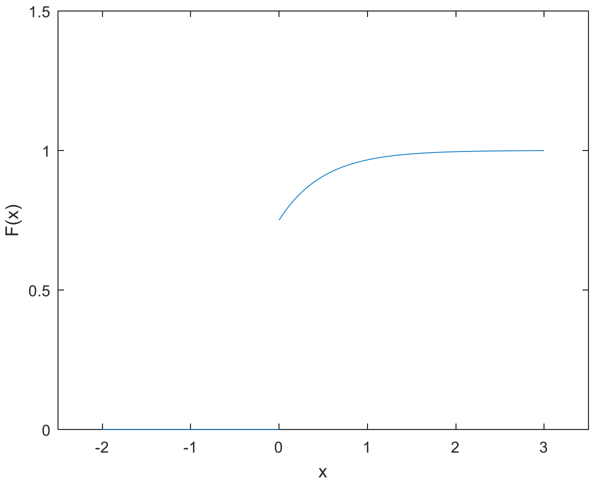
(a) $F(x)=(x+1)/3 \quad for -1\le x\le 2$

$F(x)=0 \quad for \quad x<-1, F(x)=1 \quad for \quad x>2.$




(b) $P(X<=0) = F(0) = 1/3$; $P(|X-0.5|<=1) = F(1.5)-F(0.5) = 2/3$; $P(X>=-1/2) = 1-F(-1/2)=5/6$.

4.13




(a)

This is a mixed type random variable.

(b) $P(X<=2) = F(2) = 0.9954$;

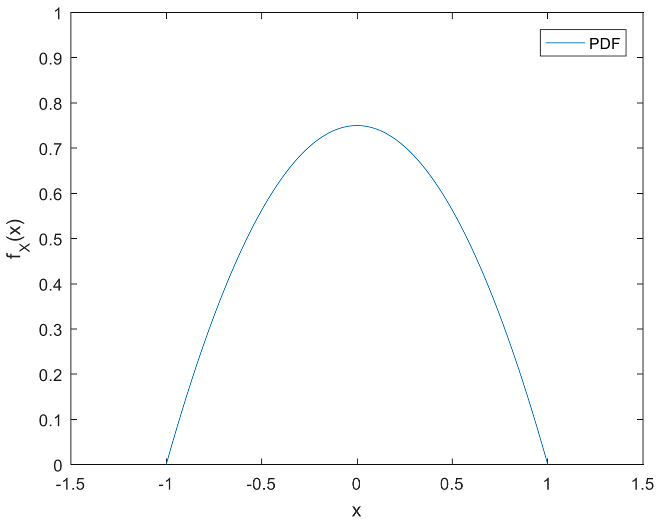
$P(X=0) = F(0) - 0 = 0.75$;

$P(2<X<6) = F(6)-F(2) = 0.0046$;

$P(X>10) = 1 - F(10) = 5.15e-10$.

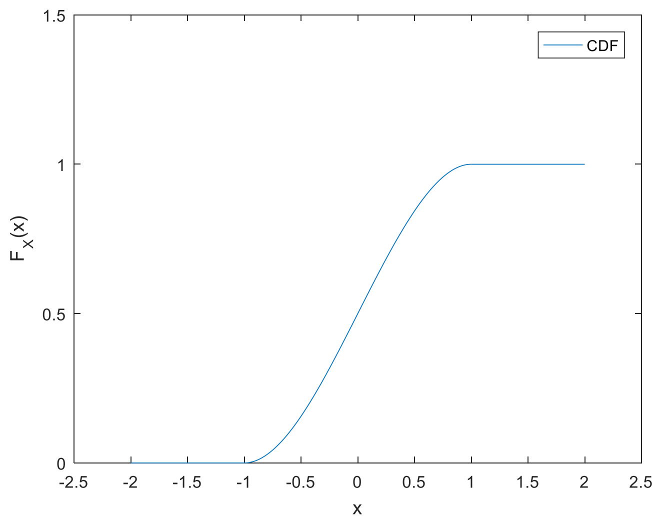
4.17

(a) $\int_{-1}^{1} c(1-x^2) = (cx - cx^3/3)|_{-1}^{1} = 1, c = 3/4$




(b) $F_X(x) = 3x/4 - x^3/4 + 1/2 \quad for -1 \le x \le 1$

$F_X(x) = 0 \quad for \quad x < -1, F_X(x) = 1 \quad for \quad x > 1.$



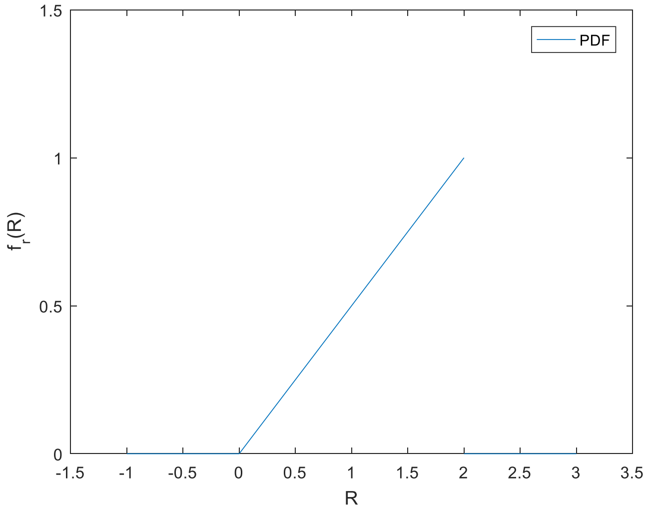

(c) P(X=0) = 0;

P(0<X<0.5) = F(0.5)-F(0) = 11/32;

P(|X-0.5|<0.25) = F(0.75)-F(0.25) = 0.2734.

4.19

(a) $f_r(R) = \frac{dF_r(R)}{dR} = R/2 \quad for \quad 0 \le R \le 2, f_r(R) = 0 \quad otherwise$

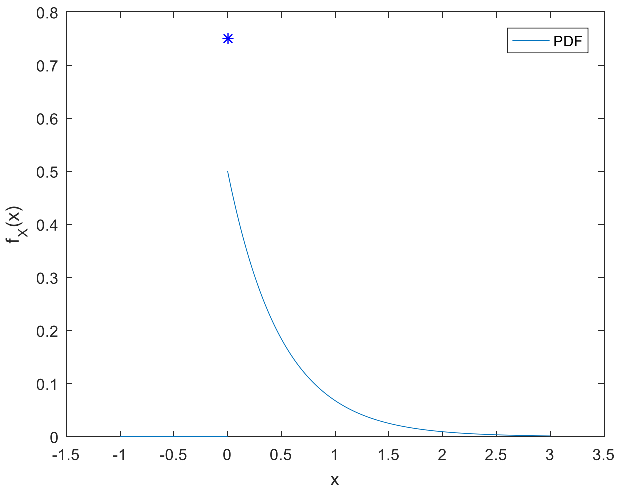



(b) P(R>1/4) = 1 – P(R<=1/4) = 1 – 1/2*1/4*1/4 = 63/64.

4.23

(a) $$f_X(x) = \frac{dF_X(x)}{dx} = e^{-2x}/2 \quad for \quad x > 0,$$

$$f_X(x) = 0.75\delta(x) \quad for \quad x = 0, f_X(x) = 0 \quad otherwise$$




(b) $P(x = 0) = 0.75;$

$$P(x > 8) = \int_8^\infty 0.5e^{-2x}dx = -0.25e^{-2x}|_8^\infty = 0.25e^{-16}$$

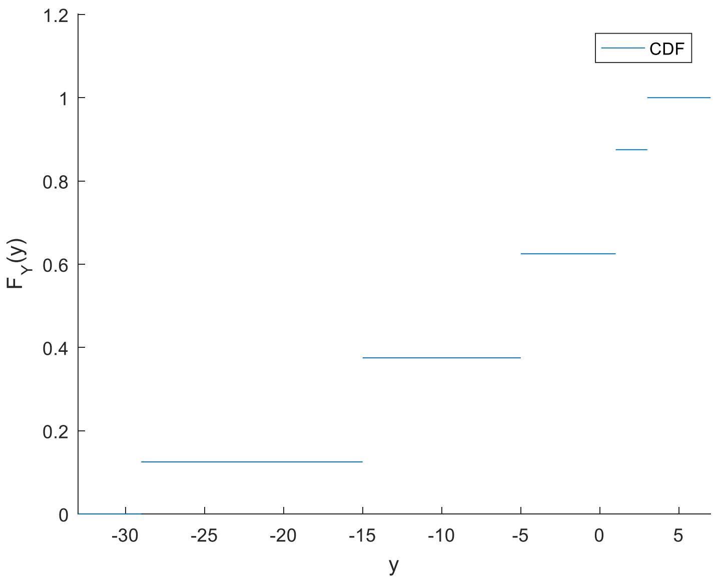
4.27

(a) $$f_X(x) = \sum_{j=-3}^{4} \frac{1}{8}\delta(x - j)$$

$$F_X(x) = \frac{j+3}{8}, \quad for \quad \frac{j+3}{7} \le x < \frac{j+4}{7};$$
$$F_X(x) = 1, x \ge 4; F_X(x) = 0, x < 3.$$

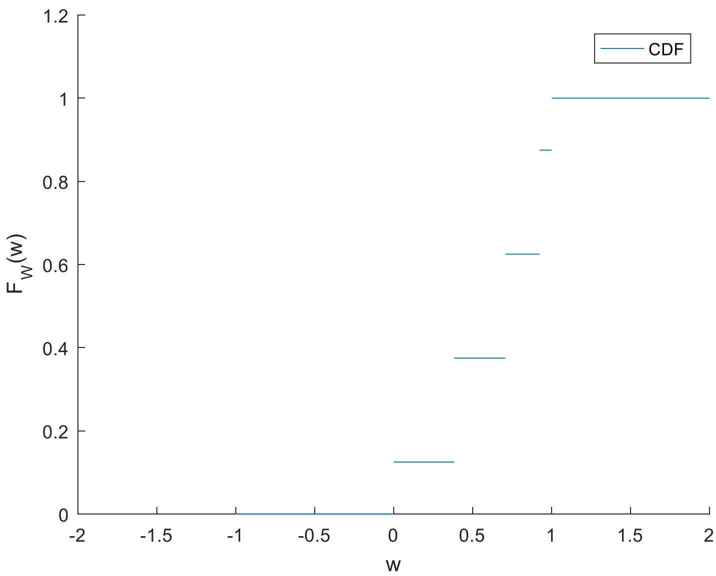
(b)

$$f_Y(y) = \frac{1}{8}\delta(y+29) + \frac{1}{4}\delta(y+15) + \frac{1}{4}\delta(y+5) + \frac{1}{4}\delta(y-1) + \frac{1}{8}\delta(y-3);$$



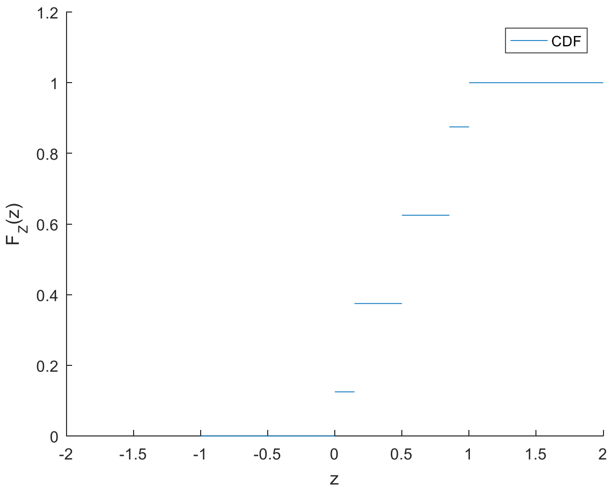

(c)

$$f_W(w) = \frac{1}{8}\delta(w) + \frac{1}{4}\delta(w - cos(3/8\pi)) + \frac{1}{4}\delta(w - 1/\sqrt{2}) + \frac{1}{4}\delta(w - cos(1/8\pi)) + \frac{1}{8}\delta(w-1)$$




(d)

$$f_Z(z) = \frac{1}{8}\delta(z) + \frac{1}{4}\delta(z - cos^2(3/8\pi)) + \frac{1}{4}\delta(z - 1/2) + \frac{1}{4}\delta(z - cos^2(1/8\pi)) + \frac{1}{8}\delta(z-1)$$




4.39

$$E[X] = \int_{-1}^{1} cx(1-x^2)dx = c(x^2/2 - x^4/4)|_{-1}^{1} = 0;$$

$$E[X^2] = \int_{-1}^{1} cx^2(1-x^2)dx = c(x^3/3 - x^5/5)|_{-1}^{1} = 1/5;$$

$$Var[X] = E[X^2] - E^2[X] = 1/5.$$

4.45

$$E[X] = \int_{0}^{\infty} (1 - F_X(x))dx = \int_{0}^{\infty} 0.25e^{-2x}dx = 1/8;$$

$$E[X^2] = \int_{0}^{\infty} \frac{1}{2}x^2e^{-2x}dx = 1/8;$$

$$Var[X] = E[X^2] - E^2[X] = 7/64.$$

4.46

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} dx = \int_{-\infty}^{\infty} (y+m) \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy = 0 + \frac{m}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} dy = m;$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} \frac{(x-m)^2}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} dx = \int_{-\infty}^{\infty} \frac{y^2}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy = \frac{1}{\sqrt{2\pi}\sigma} \left\{ 0 + \sigma^2 \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} dy \right\} = \sigma^2$$

4.50

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx = \frac{1}{2\pi} ln(1+x^2)|_{-\infty}^{\infty}$$

As $ln(1+x^2)$ goes to infinity when x tends to infinity or minus infinity, the integral does not exist. Thus Cauchy random variable does not have a mean value.

4.54

(a)

$$\begin{aligned}
\mathcal{E}[Y] &= \int_{-\infty}^{\infty} g(x) f_X(x) dx \quad \text{write integral into three parts} \\
&= -a \int_{-\infty}^{-a} f_X(x) dx + \int_{-a}^{a} x f_X(x) dx + a \int_{a}^{\infty} f_X(x) dx \\
&= -a F_X(-a) + \int_{-a}^{a} x f_X(x) dx + a(1 - F_X(a^-)) \\
\mathcal{E}[Y^2] &= a^2 F_X(-a) + \int_{-a}^{a} x^2 f_X(x) dx + a^2(1 - F_X(a^-)) \\
VAR[Y] &= \mathcal{E}[Y^2] - \mathcal{E}[Y]^2
\end{aligned}$$

(b)

$$E[Y] = -P[Y \leq -1] + P[Y \geq 1] + \int_{-1}^{1} 0.5 x e^{-|x|} dx = 0$$

$$Var[Y] = E[Y^2] = P[Y \leq -1] + P[Y \geq 1] + \int_{-1}^{1} 0.5 x^2 e^{-|x|} dx = 6e^{-1} - 2$$

(c)

$$E[Y] = -0.5P[Y \leq -1] + 0.5P[Y \geq 1] + \int_{-1/2}^{1/2} cx(1-x^2) dx = 0$$

$$Var[Y] = E[Y^2] = 0.25P[Y \leq -1] + 0.25P[Y \geq 1] + \int_{-1/2}^{1/2} cx^2(1-x^2)dx = 5/64 + 17/320 = 0.13125$$

(d)

$$E[Y] = -0.5P[Y \leq -1/2] + 0.5P[Y \geq 1/2] + \int_{-1/2}^{1/2} u^3 du/2 = 0$$

$$Var[Y] = E[Y^2] = 0.5P[Y \leq -1/2] + 0.25P[Y \geq 1/2] + \int_{-1/2}^{1/2} u^6 du/2 = \frac{1}{4}(1 - 0.5^{1/3}) + \frac{2}{7}0.5^7 = 0.0538$$