ECE 528: Introduction to Random Processes in ECE Fall 2020

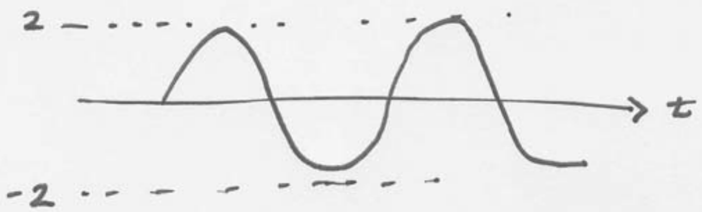
Problem Solving Session: Chapter 1: 1.10; Chapter 2: 2.38, 2.56, 2.102; Chapter 3: 3.17, 3.31 and Chapter 4: 4.8, 4.34

1.10. Suppose that the signal $2\cos 2\pi t$ is sampled at random instants of time.

(a) Find the long-term sample mean.

(b) Find the long-term relative frequency of the events "voltage is positive"; "voltage is less than -2."

(c) Do the answers to parts a and b change if the sampling times are periodic and taken every τ seconds?



(a) Sample values equally likely to be in positive and negative amplitude regions

Symmetry of function in positive & negative range

$\Rightarrow$ longterm average of samples $= 0$

(b) "Voltage positive" $\Leftrightarrow$ half samples $\Leftrightarrow P[>0] = \frac{1}{2}$

"Voltage < -2" does not occur $\Rightarrow P[<-2] = 0$

(c) The observed frequencies can change if τ is a rational number times 2π

for example: $\tau = 2\pi$ gives only one observed value

$1 = 2\cos 2\pi = 2\cos 4\pi = 2\cos 6\pi = \ldots$

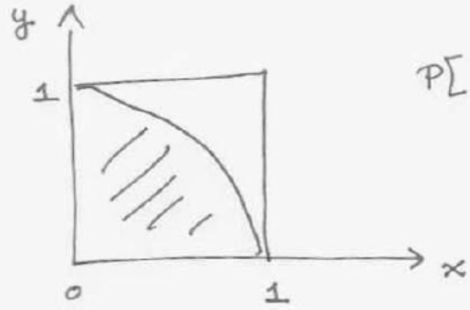
2.38. Two numbers (x, y) are selected at random from the interval $[0, 1]$.

(a) Find the probability that the pair of numbers are inside the unit circle.

(b) Find the probability that $y > 2x$.

2.38

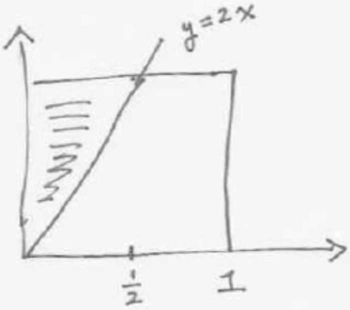
(a)



$P[x^2+y^2<1] = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$

Area inside circle

(b)



$P[y>2x] = \frac{1}{4}$

Area in right triangle

2.56. A lot of 50 items has 40 good items and 10 bad items.

(a) Suppose we test five samples from the lot, with replacement. Let X be the number of defective items in the sample. Find $P[X = k]$.

(b) Suppose we test five samples from the lot, without replacement. Let Y be the number of defective items in the sample. Find $P[Y = k]$.

2.56

(b) $P[X=k] = \frac{\binom{10}{k}\binom{40}{5-k}}{\binom{50}{5}}$ $\quad k=0,1,\ldots,5$ without replacement

Hypergeometric probabilities

(a) With replacement:

pick $\underbrace{k \text{ defective balls}}_{10^k}$ then pick $\underbrace{5-k \text{ nondefective balls}}_{40^{5-k}}$

There are $\binom{5}{k}$ arrangements of this composition

$$\#\text{ ways of obtaining } k \text{ defective in 5 tested} = \frac{\binom{5}{k} 10^k 40^{5-k}}{50^5}$$

$$= \binom{5}{k}\left(\frac{10}{50}\right)^k\left(\frac{40}{50}\right)^{5-k} \quad k=0,1,\ldots,5$$

Binomial probabilities.

2.102. A machine makes errors in a certain operation with probability p. There are two types of errors. The fraction of errors that are type 1 is α, and type 2 is $1 - \alpha$.

(a) What is the probability of k errors in n operations?

(b) What is the probability of k_1 type 1 errors in n operations?

(c) What is the probability of k_2 type 2 errors in n operations?

(d) What is the joint probability of k_1 and k_2 type 1 and 2 errors, respectively, in n operations?

2.102

a) $P[k \text{ errors}] = \binom{n}{k} p^k (1-p)^{n-k}$

b) Type 1 errors occur with problem $p\alpha$ and do not occur with problem $1 - p\alpha$

$$P[k_1 \text{ type 1 errors}] = \binom{n}{k_1} (p\alpha)^{k_1} (1-p\alpha)^{n-k_1}$$

c) $P[k_2 \text{ type 2 errors}] = \binom{n}{k_2} (p(1-\alpha))^{k_2} (1 - p(1-\alpha))^{n-k_2}$

d) Three outcomes: type 1 error, type 2 error, no error

$$P[k_1, k_2, n - k_1 - k_2] = \frac{n!}{k_1! k_2! (n - k_1 - k_2)!} (p\alpha)^{k_1} (p(1-\alpha))^{k_2} (1-p)^{n-k_1-k_2}$$

3.17. A modem transmits a +2 voltage signal into a channel. The channel adds to this signal a noise term that is drawn from the set $\{0, -1, -2, -3\}$ with respective probabilities $\{4/10, 3/10, 2/10, 1/10\}$.

(a) Find the pmf of the output Y of the channel.

(b) What is the probability that the output of the channel is equal to the input of the channel?

(c) What is the probability that the output of the channel is positive?

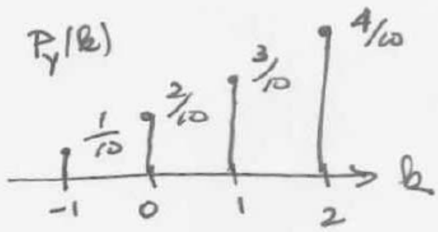
3.17

(a) $Y = 0 + 2 = 2$ with prob. $4/10$

$Y = -1 + 2 = 1$ " $3/10$

$Y = -2 + 2 = 0$ " $2/10$

$Y = -3 + 2 = -1$ " $1/10$



(b) $P[Y = 2] = 4/10$

(c) $P[Y > 0] = P[Y = 2] + P[Y = 1] = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$

3.31. **(a)** Suppose a fair coin is tossed n times. Each coin toss costs d dollars and the reward in obtaining X heads is $aX^2 + bX$. Find the expected value of the net reward.

(b) Suppose that the reward in obtaining X heads is a^X, where $a > 0$. Find the expected value of the reward.

3.31

$$P[X=k] = \binom{n}{k}\left(\tfrac{1}{2}\right)^n$$

$$E[aX^2+bX] = aE[X^2] + bE[X]$$

$$E[X] = \sum_{j=0}^{n} j\binom{n}{j}\left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^n \sum_{j=0}^{n} j \frac{n!}{j!\,(n-j)!}$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{j=1}^{n} \frac{n!}{(j-1)!\,(n-j)!} \qquad \text{let } j' = j-1$$

$$= \left(\tfrac{1}{2}\right)^n n \sum_{j'=0}^{n-1} \frac{(n-1)!}{j'!\,(n-1-j')!} = n\left(\tfrac{1}{2}\right)^n \sum_{j'=0}^{n-1} \binom{n-1}{j'}$$

$$= n\left(\tfrac{1}{2}\right)^n 2^{n-1} = \frac{n}{2}.$$

$$E[X^2] = \sum_{j=0}^{n} j^2 \binom{n}{j}\left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^n n \sum_{j=1}^{n} j \frac{(n-1)!}{(j-1)!\,(n-j)!}$$

$$= n\left(\tfrac{1}{2}\right)^n \sum_{j'=0}^{n-1} (j'+1)\binom{n-1}{j'}$$

$$= n\left(\tfrac{1}{2}\right)\Bigg[\underbrace{\sum_{j'=0}^{n-1} j'\binom{n-1}{j'}\left(\tfrac{1}{2}\right)^{n-1}}_{\substack{(n-1)\frac{1}{2} \\ \text{expected value of} \\ \text{binomial}}} + \underbrace{\sum_{j'=0}^{n-1}\binom{n-1}{j'}\left(\tfrac{1}{2}\right)^{n-1}}_{\substack{1 \\ \text{binomial probs}}}\Bigg]$$

$$= \frac{n}{2}\left[\frac{n}{2}+1\right]$$

$$\therefore E[aX^2+bX] = a\frac{n}{2}\left(\frac{n}{2}+1\right) + b\frac{n}{2}$$ ✓ average reward.

3.31b

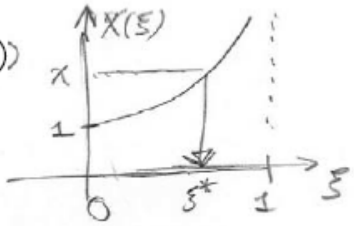
$$E[a^X] = \sum_{j=0}^{n} a^j \binom{n}{j} \left(\frac{1}{2}\right)^j = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{a}{2}\right)^j$$

$$= \left(1 + \frac{a}{2}\right)^n$$

4.8. Let ζ be a point selected at random from the unit interval. Consider the random variable $X = (1 - \zeta)^{-1/2}$.

- **(a)** Sketch X as a function of ζ.
- **(b)** Find and plot the cdf of X.
- **(c)** Find the probability of the events $\{X > 1\}$, $\{5 < X < 7\}$, $\{X \leq 20\}$.

4.8

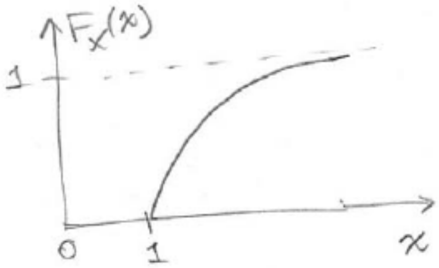


$$X(\xi) = \frac{1}{\sqrt{1-\xi}}$$

$$S_X = \{x : 1 \leq x < \infty\}$$

$$P[X(\xi) \leq x] = P\left[\frac{1}{\sqrt{1-\xi}} \leq x\right] = P\left[\frac{1}{1-\xi} \leq x^2\right]$$

$$= P\left[\frac{1}{x^2} \leq 1 - \xi\right] = P\left[\xi \leq 1 - \frac{1}{x^2}\right] = 1 - \frac{1}{x^2}$$



$$P[X > 1] = 1 - F_X(1)$$

$$= 1 - \left(1 - \frac{1}{1^2}\right) = 1$$

$$P[5 < X < 7] = F_X(7) - F_X(5) = \left(1 - \frac{1}{49}\right) - \left(1 - \frac{1}{25}\right)$$

$$= \frac{1}{25} - \frac{1}{49} = 0.01959$$

$$P[X \leq 20] = 1 - \frac{1}{400} = 0.9975$$

4.34. The Pareto random variable X has cdf:

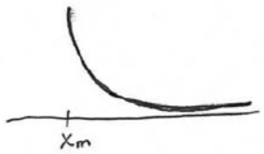
$$F_X(x) = \begin{cases} 0 & x < x_m \\ 1 - \dfrac{x_m^\alpha}{x^\alpha} & x \geq x_m. \end{cases}$$

(a) Find and plot the pdf of X.

(b) Repeat Problem 4.33 parts a and b for the Pareto random variable.

(c) What happens to $P[X > t + x \mid X > t]$ as t becomes large? Interpret this result.

a) $f_X(x) = \begin{cases} 0 & x < x_m \\ \dfrac{\alpha x_m^\alpha}{x^{\alpha-1}} & x \geq x_m \end{cases}$



b) $F_X(x \mid X>t) = \dfrac{P[\{X \leq x\} \cap \{X > t\}]}{P[X>t]} = \dfrac{P[t < X \leq x]}{P[X>t]}$

$$= \begin{cases} 0 & x \leq t \\ \dfrac{F_X(x) - F_X(t)}{1 - F_X(t)} & x \geq t \end{cases}$$

if $t \geq x_m$ $\quad F_X(x \mid X>t) = \dfrac{1 - \frac{x_m^\alpha}{x^\alpha} - 1 + \frac{x_m^\alpha}{t^\alpha}}{1 - \left(1 - \frac{x_m^\alpha}{t^\alpha}\right)} = t^\alpha\left(\dfrac{1}{t^\alpha} - \dfrac{1}{x^\alpha}\right) = 1 - \left(\dfrac{t}{x}\right)^\alpha \quad x \geq t$

if $t < x_m$ $\quad F_X(x \mid X>t) = 1 - \left(\dfrac{x_m}{x}\right)^\alpha \quad x \geq x_m$

$f_X(x \mid X>t) = \dfrac{f_X(x)}{1 - F_X(t)} \quad x \geq t$

if $t \geq x_m$ $\quad f_X(x \mid X>t) = \dfrac{\alpha \frac{x_m^\alpha}{x^{\alpha-1}}}{\frac{x_m^\alpha}{t^\alpha}} = \alpha t \left(\dfrac{t}{x}\right)^{\alpha-1} \quad x \geq t$

if $t < x_m$ $\quad f_X(x \mid X>t) = \alpha \dfrac{x_m^\alpha}{x^{\alpha-1}} \quad x \geq x_m$

c) $\dfrac{P[\{X > t+x\} \cap \{X>t\}]}{P[X>t]} \xrightarrow[t \to \infty]{} 1$

The longer you wait the longer you are likely to wait more!