ECE 528: Solutions to HW 1

1. (a) To calculate the probability of error, we note that we have two cases of (i) 0 being sent but received as 1 and (ii) 1 being sent but received as 0. Since we have a binary symmetric channel, the probability of these two events are equal and given by p. That is, $P(1|0) = P(0|1) = p$. The probability of error is given by

$$P_e = P(0)P(1|0) + P(1)P(0|1) \tag{1}$$

Since 0 or 1 are sent with equal probabilities, that is, $P(0) = P(1) = 0.5$.

$$P_e = 0.5p + 0.5p = p \tag{2}$$

(b) The decoding rule makes a decision by taking a majority vote. So, the probability of error now is given by:

$$\begin{aligned}
P_e &= 2P(0)\sum_{k=2}^{3}\binom{3}{k}P(1|0)^k P(0|0)^{3-k} \\
&= \sum_{k=2}^{3}\binom{3}{k}p^k(1-p)^{3-k} \\
&= 3p^2(1-p) + p^3 \\
&= 3p^2 - 2p^3 = p(3p - 2p^2)
\end{aligned}$$

The improvement of the new decoding rule in terms of the probability of error is given by:

$$P_e = \frac{P_e(\text{new})}{P_e(\text{old})} = 3p - 2p^2 \approx 3p. \tag{3}$$

Since typically p is small that means the improvement is approximately $3p$. In other words, the encoding will provide a lower probability of error (by approximately proportional to the number of replications).

2. (a) In the first draw the outcome can be black (b) or white (w). If the first draw is black, then the second outcome can be b or w. However if the first draw is white, then the run only contains black balls so the second outcome must be b. Therefore, $S = \{bb, bw, wb\}$.

(b) In this case all outcomes can be b or w. Therefore, $S = \{bb, bw, wb, ww\}$.

(c) In part (a) the outcome ww cannot occur so $f_{ww} = 0$. In part (b) let N be a large number of repetitions of the experiment. The number of times the first outcome is w is approximately $N/3$ since the run has one white ball and two black balls. Of these $N/3$ outcomes approximately $1/2$ are also white in the second draw. Thus $N/9$ if the outcome result in ww, and thus $f_{ww} = \frac{1}{9}$.

(d) In the first experiment, the outcome of the first draw affects the probability of the outcomes in the second draw. In the second experiment, the outcome of the first draw does not affect the probability of the outcomes in the second draw.

3. When the experiment is performed, either A occurs or it doesn't (i.e. B occurs); thus $N_A(n) + N_B(n) = n$ in n repetitions of the experiment, and

$$f_A(n) + f_B(n) = \frac{N_A(n)}{n} + \frac{N_B(n)}{n} = 1 \tag{4}$$

Thus $f_B(n) = 1 - f_A(n)$.

4. If A, B, or C occurs, then D occurs. Furthermore since A, B, or C cannot occur simultaneously, in n repetitions of the experiment we have

$$N_D(n) = N_A(n) + N_B(n) + N_C(n) \tag{5}$$

and dividing both sides by n

$$f_D(n) = f_A(n) + f_B(n) + f_C(n) \tag{6}$$

5.

$$\begin{aligned} < X >_n &= \frac{1}{n}\sum_{j=1}^{n} X(j) \\ &= \frac{n-1}{n}\frac{1}{n-1}\left\{\sum_{j=1}^{n-1} X(j) + X(n)\right\} \\ &= \left(1 - \frac{1}{n}\right) < X >_{n-1} + \frac{1}{n}X(n) \\ &= < X >_{n-1} + \frac{X(n) - < X >_{n-1}}{n} \end{aligned}$$