ECE 528: Introduction to Random Processes in ECE

Fall 2020

Solutions to Homework Set No. 2

2.1 a) The sample space consists of the twelve hours:

$$S = \{1,2,3,4,5,6,7,8,9,10,11,12\}$$

b) $A = \{1,2,3,4\} \quad B = \{2,3,4,5,6,7,8\} \quad D = \{1,3,5,7,9,11\}$

c)
$$A \cap B \cap D = \{2,3,4\} \cap \{1,3,5,7,9,11\} = \{3\}$$
$$A^c \cap B = \{5,6,7,8,9,10,11,12\} \cap \{2,3,4,5,6,7,8\} = \{5,6,7,8\}$$
$$A \cup (B \cap D^c) = \{1,2,3,4\} \cup (\{2,3,4,5,6,7,8\} \cap \{2,4,6,8,10,12\})$$
$$= \{1,2,3,4\} \cup \{2,4,6,8\} = \{1,2,3,4,6,8\}$$
$$(A \cup B) \cap D^c = \{1,2,3,4,5,6,7,8\} \cap \{2,4,6,8,10,12\} = \{2,4,6,8\}$$

2.2 The outcome of this experiment consists of a pair of numbers (x, y) where x = number of dots in first toss and y = number of dots in second toss. Therefore, S = set of ordered pairs (x, y) where $x, y \epsilon \{1,2,3,4,5,6\}$ which are listed in the table below:

a)

x \ y	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Checkmarks indicate elements of events below.

b)

x \ y	1	2	3	4	5	6
1	√					
2	√	√				
3	√	√	√			
4	√	√	√	√		
5	√	√	√	√	√	
6	√	√	√	√	√	√

$A = \{N_1 < N_2\}^c = \{N_1 \geq N_2\}$

c)

x \ y	1	2	3	4	5	6
1						
2						
3						
4						
5						
6	√	√	√	√	√	√

$B = \{N_1 = 6\}$

d) B is a subset of A so when B occurs then A also occurs, thus B implies A

e) $A \cap B^c = \{N_2 \le N_1 < 6\}$

x	1	2	3	4	5	6
1	√					
2	√	√				
3	√	√	√			
4	√	√	√	√		
5	√	√	√	√	√	
6						

f) C = "number of dots differ by 2"

	1	2	3	4	5	6
1			√			
2				√		
3	√				√	
4		√				√
5			√			
6				√		

Comparing the tables for A and C we see that $A \cap C = \{(3,1),(4,2),(5,3),(6,4)\}$.

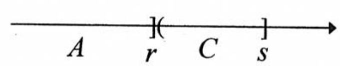
2.5 a) Each testing of a pen has two possible outcomes: "pen good" (g) or "pen bad" (b). The experiment consists of testing pens until a good pen is found. Therefore each outcome of the experiment consists of a string of "b's" ended by a "g". We assume that each pen is not put back in the drawer after being tested. Thus S = {*g, bg, bbg, bbbg, bbbbg*}.

b) We now simply record the number of pens tested, so $S = \{1,2,3,4,5\}$.

c) The outcome now consists of a substring of "b's" and one "g" in any order followed by a final "g". S = {*gg, bgg, gbg, gbbg, bbgg, gbbbg, bgbbg, bbgbg, bbbgg, gbbbg, bgbbbg, bbgbbg, bbbgbg, bbbbgg*}.

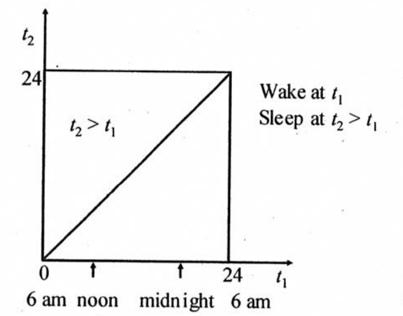
d) $S = \{2, 3, 4, 5, 6\}$

2.9 If we sketch the events A and B we see that $B = A \cup C$. We also see that the intervals corresponding to A and C have no points in common so $A \cap C = \varnothing$.



We also see that $(r,s] = (r,\infty) \cap (-\infty,s] = (-\infty,r]^c \cap (-\infty,s]$ that is $C = A^c \cap B$.

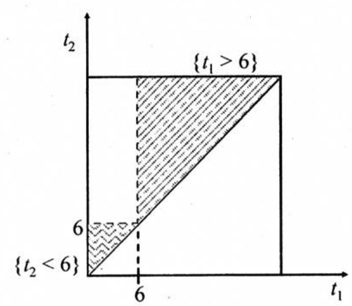
2.17 a)

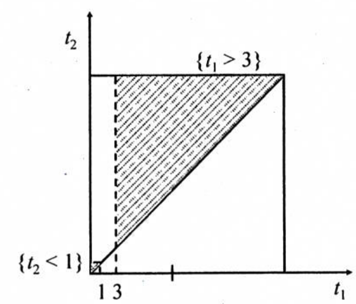


Note that the problem specifies that the student wakes up before returning to sleep in the specified time period. This condition constrains the sample space to the upper portion of the square region in the graph above.

b) "Asleep at noon" = wake up after noon or wake up and go to sleep before noon

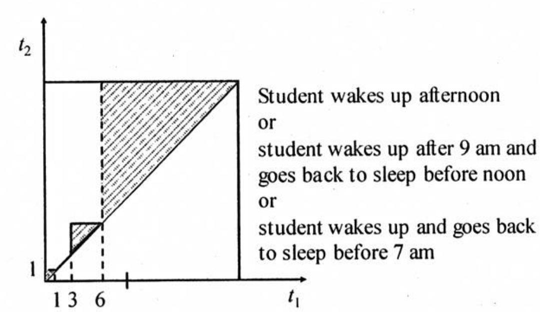
$= \{t_1 > 6\} \cup \{t_2 < 6\}$ where we label time from 0 to 24.





c) "Student sleeps through breakfast" $= \{t_1 > 3\} \cup \{t_2 < 1\}$

d) $A \cap B$ is found by taking the intersection of the regions in parts b) and c). We obtain the three triangular regions shown below, which are interpreted below.



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2.21 a) The sample space in tossing a die is $S = \{1,2,3,4,5,6\}$. Let $p_i = P[\{i\}] = p$ since all faces are equally likely. By Axiom 1

$$\begin{aligned} 1 &= P[S] \\ &= P[\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\}] \end{aligned}$$

The elementary events $\{i\}$ are mutually exclusive so by Corollary 4:

$$1 = p_1 + p_2 + \ldots + p_6 = 6p \Rightarrow p_i = p = \tfrac{1}{6} \qquad \text{for } i = 1, \ldots, 6$$

b) We express each event as the union of elementary events and then apply Axiom III′:

$$P[A] = P[> 3 \text{ dots}] = P[\{4,5,6\}] = P[\{4\}] + P[\{5\}] + P[\{6\}] = \tfrac{3}{6}$$

$$P[B] = P[\text{odd } \#] = P[\{1,3,5\}] = P[\{1\}] + P[\{3\}] + P[\{5\}] = \tfrac{3}{6}$$

c) We first find the elements in each event of interest and then apply Axiom III′:

$$P[A \cup B] = P[\{1,3,4,5,6\}] = P[\{1\}] + P[\{3\}] + P[\{4\}] + P[\{5\}] + P[\{6\}] = \tfrac{5}{6}$$

$$P[A \cap B] = P[\{5\}] = \tfrac{1}{6}$$

$P[A^c] = 1 - P[A] = \tfrac{3}{6}$ where we used Corollary 1.

2.26 Identities of this type are shown by application of the axioms. We begin by treating $(A \cup B)$ as a single event, then

$$\begin{aligned}
& P[A \cup B \cup C] \\
&= P[(A \cup B) \cup C] \\
&= P[A \cup B] + P[C] - P[(A \cup B) \cap C] && \text{by Cor. 5} \\
&= P[A] + P[B] - P[A \cap B] + P[C] && \text{by Cor. 5 on } A \cup B \\
&\quad - P[(A \cap C) \cup (B \cap C)] && \text{and by distributive property} \\
&= P[A] + P[B] + P[C] - P[A \cap B] \\
&\quad - P[A \cap C] - P[B \cap C] && \text{by Cor. 5 on} \\
&\quad + P[(A \cap B) \cap (B \cap C)] && (A \cap C) \cup (B \cap C) \\
&= P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] && \text{since} \\
&\quad - P[B \cap C] + P[A \cap B \cap C]. && (A \cap B) \cap (B \cap C) = A \cap B \cap C
\end{aligned}$$

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2.34 Assume that the probability of any subinterval I of $[-1, 2]$ is proportional to its length,

$$P[I] = k \text{ length } (I).$$

If we let $I = [-1, 2]$ then we must have that

$$1 = P[S] = [[-1,2]] = k \text{ length } ([-1,2]) = 3k \Rightarrow k = \tfrac{1}{3}.$$

a) $P[A] = \frac{1}{3}$ length $([-1,0)) = \frac{1}{3}(1) = \frac{1}{3}$

$P[B] = \frac{1}{3}$ length $((0,1)) = \frac{1}{3}1 = \frac{1}{3}$

$P[C] = \frac{1}{3}$ length $((\frac{3}{4},2]) = \frac{1}{3}\frac{5}{4} = \frac{5}{12}$

$P[A \cap B] = P[\varnothing] = 0$

$P[A \cap C] = P[\varnothing] = 0$

b) $P[A \cup B] = P[[-1,0) \cup (0,1)] = P[[-1,0)] + P[(0,1)] = \frac{2}{3}$

$P[A \cup C] = P[[-1,0) \cup (\frac{3}{4},2]] = P[[-1,0)] + P[(\frac{3}{4},2]] = \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$

$P[A \cup B \cup C] = P[[-1,0) \cup (0,2]] = P[S - \{0\}] = P[S] = 1$

Now use axioms and corollaries:

$P[A \cup B] = P[A] + P[B] - P[A \cap B] = \frac{1}{3} + \frac{1}{3} - 0 = \frac{2}{3}$ by Cor. 5

$P[A \cup C] = P[A] + P[C] - P[\underbrace{A \cap C}_{\varnothing}] = \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$ by Cor. 5

$$\begin{aligned} P[A \cup B \cup C] = &\ P[A] + P[B] + P[C] \\ & - P[A \cap B] - P[A \cap C] - P[B \cap C] \\ & + P[A \cap B \cap C] \\ = &\ \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{5}{12} - 0 - 0 - \tfrac{1}{12} + 0 = 1 \end{aligned}$$ by Eq. (2.7)

2.46 The order in which the 4 toppings are selected does not matter so we have sampling without ordering.

If toppings may not be repeated, Eq. (2.25) gives

$$\binom{15}{4} = 1365 \text{ possible deluxe pizzas}.$$

If toppings may be repeated, we have sampling with replacement and without ordering. The number of such arrangements is

$$\binom{14+4}{4} = 3060 \text{ possible deluxe pizzas}$$