CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE
Lecture 1: Probability and Basic Concepts

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Note

- These slides cover material partially covered in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-set derivation and detailed discussions were need.

Deterministic vs. Random Processes

- In deterministic processes, the outcome can be predicted exactly in advance
  - Eg. Force = mass x acceleration. If we are given values for mass and acceleration, we exactly know the value of force

- In random processes, the outcome is not known exactly, but we can still describe the *probability distribution* of possible outcomes
  - Eg. 10 coin tosses: we don't know exactly how many heads we will get, but we can calculate the probability of getting a certain number of heads

Events

- An **event** is an outcome or a set of outcomes of a random process

Example: Tossing a coin three times

Event A = getting exactly two heads = {HTH, HHT, THH}

Example: Picking real number X between 1 and 20

Event A = chosen number is at most 8.23 = $\{X \leq 8.23\}$

Example: Tossing a fair dice

Event A = result is an even number = {2, 4, 6}

- Notation: P(A) = Probability of event A
- **Probability Rule 1:**

$$0 \leq P(A) \leq 1 \text{ for any event } A$$

Sample Space

- The **sample space** S of a random process is the set of all possible outcomes

  Example: one coin toss

  S = {H,T}

  Example: three coin tosses

  S = {HHH, HTH, HHT, TTT, HTT, THT, TTH, THH}

  Example: roll a six-sided dice

  S = {1, 2, 3, 4, 5, 6}

  Example: Pick a real number X between 1 and 20

  S = all real numbers between 1 and 20

- **Probability Rule 2: The probability of the whole sample space is 1**

$$P(S) = 1$$

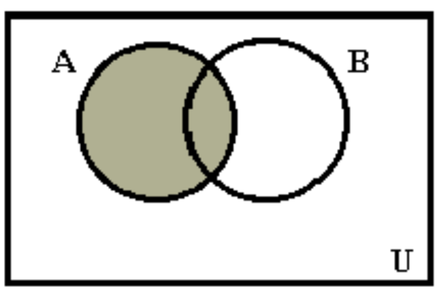
Combinations of Events

- The **complement** A^c of an event A is the event that A does not occur
- **Probability Rule 3:**

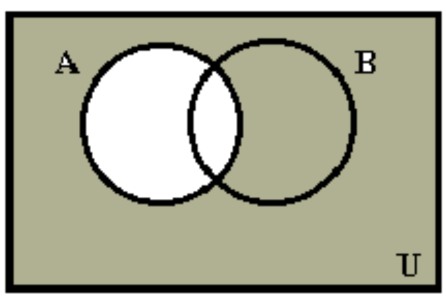
$$P(A^c) = 1 - P(A)$$

- The **union** of two events A and B is the event that either A or B or both occurs
- The **intersection** of two events A and B is the event that both A and B occur

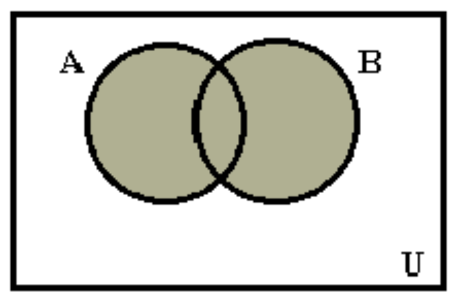
Event A



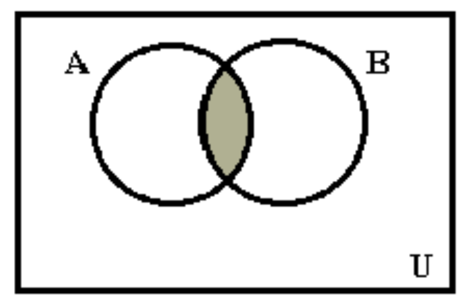
Complement of A



Union of A and B



Intersection of A and B



Disjoint Events

- Two events are called **disjoint** if they can not happen at the same time
  - Events A and B are disjoint means that the intersection of A and B is zero
- Example: coin is tossed twice
  - S = {HH,TH,HT,TT}
  - Events A={HH} and B={TT} are disjoint
  - Events A={HH,HT} and B = {HH} are not disjoint
- **Probability Rule 4: If A and B are disjoint events then**

$$P(A \text{ or } B) = P(A) + P(B)$$

Independent events

- Events A and B are **independent** if knowing that A occurs does not affect the probability that B occurs

- Example: tossing two coins

  Event A = first coin is a head

  Event B = second coin is a head

  Independent

- Disjoint events cannot be independent!
  - If A and B can not occur together (disjoint), then knowing that A occurs does change probability that B occurs

- **Probability Rule 5: If A and B are independent**

P(A and B) = P(A) x P(B)

multiplication rule for independent events

Equally Likely Outcomes Rule

- If all possible outcomes from a random process have the same probability, then
- P(A) = (# of outcomes in A)/(# of outcomes in S)
- Example: One Dice Tossed

P(even number) = |2,4,6| / |1,2,3,4,5,6|

- Note: equal outcomes rule only works if the number of outcomes is "countable"
  - Eg. of an uncountable process is sampling any fraction between 0 and 1. Impossible to count all possible fractions !

Combining Probability Rules Together

- Initial screening for HIV in the blood first uses an enzyme immunoassay test (EIA)
- Even if an individual is HIV-negative, EIA has probability of 0.006 of giving a positive result
- Suppose 100 people are tested who are all HIV-negative. What is probability that at least one will show positive on the test?
- First, use complement rule:

P(at least one positive) = 1 - P(all negative)

Combining Probability Rules Together

- Now, we assume that each individual is independent and use the multiplication rule for independent events:

$$P(\text{all negative}) = P(\text{test 1 negative}) \times \ldots \times P(\text{test 100 negative})$$

- P(test negative) = 1 - P(test positive) = 0.994

$$P(\text{all negative}) = 0.994 \times \ldots \times 0.994 = (0.994)^{100}$$

- So, we finally we have

$$P(\text{at least one positive}) = 1 - (0.994)^{100} = 1 - 0.548 = 0.452$$

Set Functions

- Define **Ω** as the set of all possible outcomes
- Define **A** as set of events
- Define A as an event – subset of the set of all experiments outcomes
- Set operations:
  - Complementation A^c: is the event that event A does not occur
  - Intersection $A \cap B$: is the event that event A and event B occur
  - Union $A \cup B$: is the event that event A or event B occurs
  - Inclusion $A \subseteq B$: an event A occurring implying event B occurs

Set Functions

- Note:
  - Set of events **A** is closed under set operations
  - Φ – empty set
  - $A \cap B = \Phi$ ➔ are mutually exclusive or disjoint

Axioms of Probability

- Let P(A) denote probability of event A:
  1. For any event A belongs **A**, $P(A) \geq 0$;
  2. For set of all possible outcomes **Ω**, $P(\mathbf{\Omega}) = 1$;
  3. If A and B are disjoint events, $P(A \cup B) = P(A) + P(B)$
  4. For countably infinite sets, $A_1, A_2, \ldots$ such that $A_i \cap A_j = \Phi$ for $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Additional Properties

- For any event, $P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A) \leq P(B)$ for $A \subseteq B$

Randomness in ECE Systems

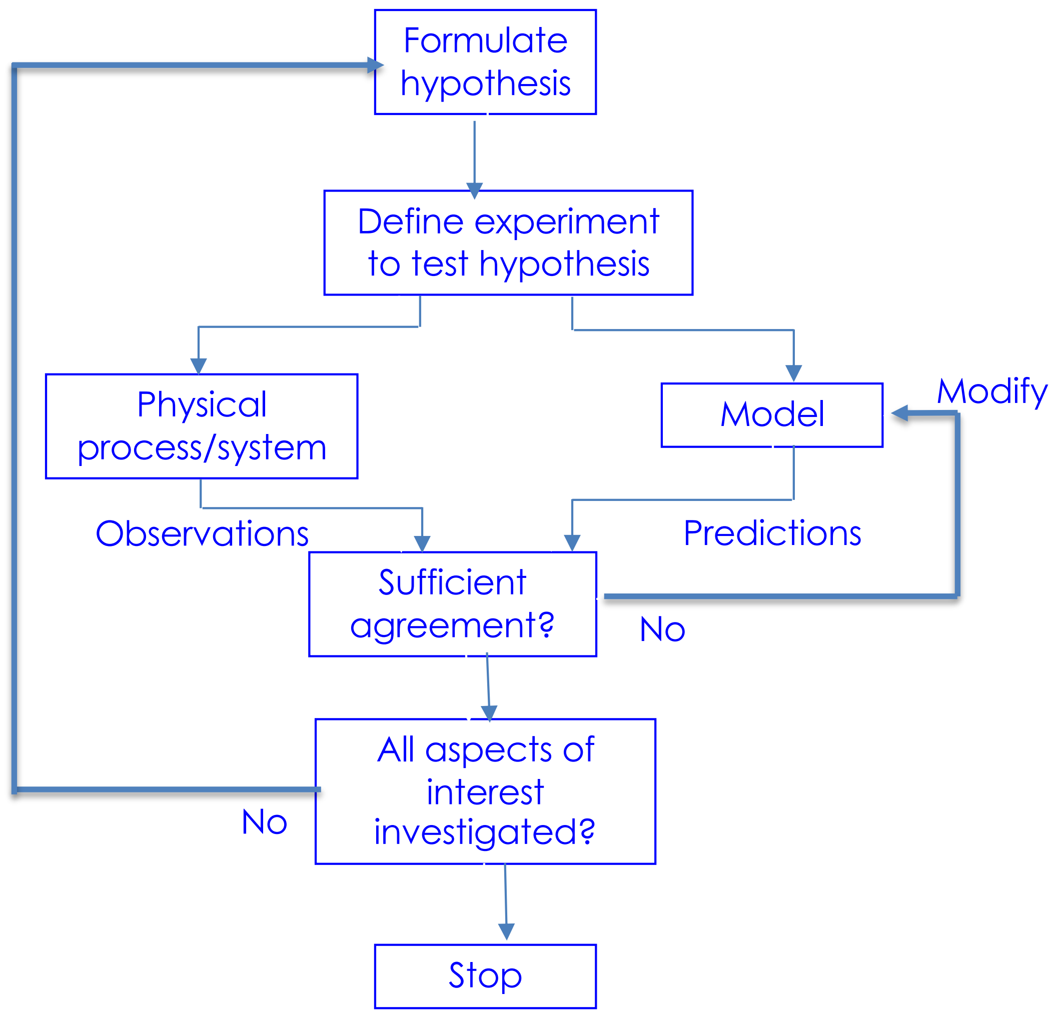
- Variability in environment
  - Noise & interference in communications
  - Variability in Internet traffic
- Incomplete control in system parameters
  - Wavelength of light produced by a laser
  - Fabrication of fault-free device
  - Variability in a speech utterance
- Insufficient measurement precision
  - Analog-to-digital conversion of audio signal

Designing Systems for Randomness

- Engineers design systems that:
  - Perform in predictable fashion
  - Provide reliable operation
  - Are efficient and cost-effective
- How can engineers accomplish this?
  - *Probability models*!
  - Exploit *statistical regularity*

Models

- Model
  - Approximate representation of a situation
  - Predict outcome of an experiment
- Modeling Process
  - Experimentation
  - What are relevant system parameters?
  - How do outcomes depend on these parameters?
- Mathematical Models
  - Mathematical relationships
  - Deterministic
  - Random



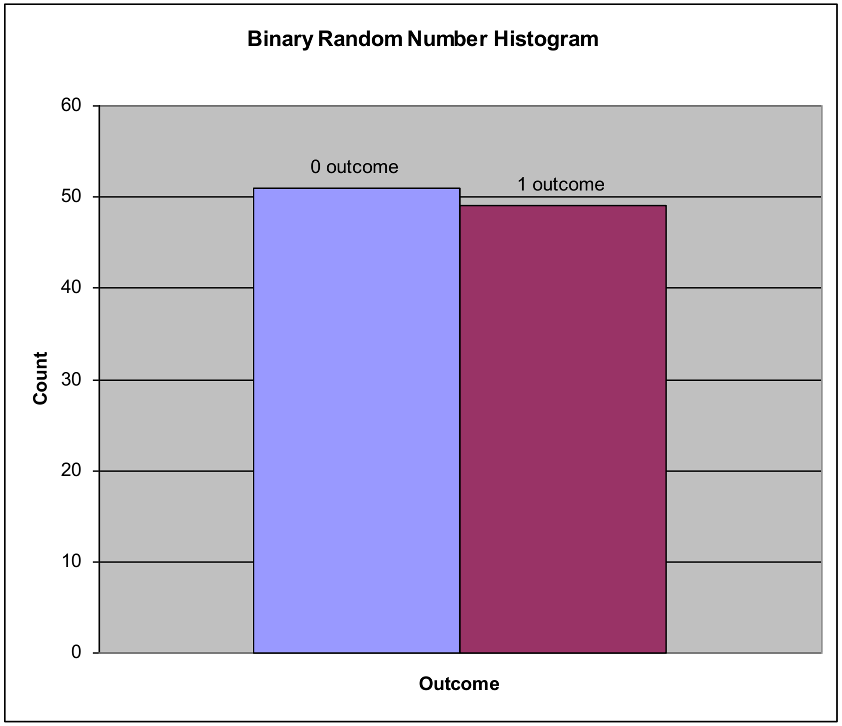
Probability Models

- Random Experiment
  - Outcome varies in unpredictable fashion
- Sample Space
  - Set of possible outcomes
- Probabilities
  - ***Conditions of experiment determine a "law" that determines probability of outcomes***

- Flip a fair coin once
  - Impossible to predict outcome consistently
- Two possible outcomes
  - Heads ("0")
  - Tails ("1")
- Fair Coin
  - Equally likely outcomes

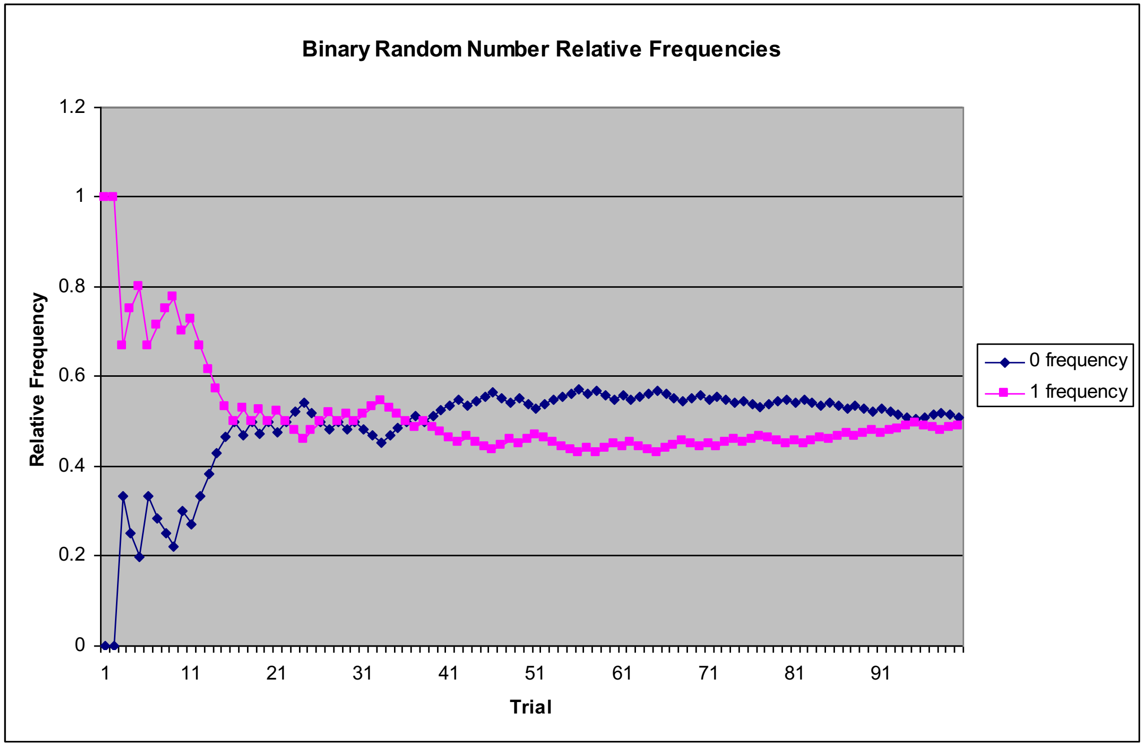
What is Probability?

- Repeat the experiment n independent times under identical conditions
- Histograms
  - N_0 = # heads
  - N_1 = # tails
- Relative frequency of an outcome
  - $f_0 = N_0/n$
  - $f_1 = N_1/n$



Statistical Regularity

- Averages over many repetitions of a random experiment yield approximately the same value.
- The relative frequency of an outcome tends toward the *probability* of the outcome.



Properties of Relative Frequency

- Let $S = \{1, 2, \ldots, K\}$
- Repeat random experiment n times
- Let $f_k = N_k/n$ be relative frequency of kth outcome

Then the following properties hold:

1. $0 \leq f_k$

2. $f_k \leq 1$

3. $f_1 + f_2 + \ldots + f_K = 1$

The Counting Principle

The Counting Principle

Consider a process that consists of r stages. Suppose that:

(a) There are n_1 possible results at the first stage.

(b) For every possible result at the first stage, there are n_2 possible results at the second stage.

(c) More generally, for any sequence of possible results at the first $i-1$ stages, there are n_i possible results at the ith stage.

Then, the total number of possible results of the r-stage process is

$$n_1 n_2 \cdots n_r.$$

Summary of Counting Results

Summary of Counting Results

- **Permutations** of n objects: $n!$.
- **k-permutations** of n objects: $n!/(n-k)!$.
- **Combinations** of k out of n objects: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.
- **Partitions** of n objects into r groups, with the ith group having n_i objects:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

Summary

- Probabilities and averages for a random experiment can by found by computing relative frequencies and sample averages in a large number of repetitions of a random experiment.
- Performance measures of many systems involve relative frequencies and long-term averages. Probability models are used in the design of such systems.