CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE

Lecture 9: Multiple Random Variables

(Pairs & Vector Random Variables)

Bijan Jabbari, PhD
Dept. of Electrical and Computer Eng.
George Mason University
Fairfax, VA 22030-4444, USA
bjabbari@gmu.edu
http://cnl.gmu.edu/bjabbari

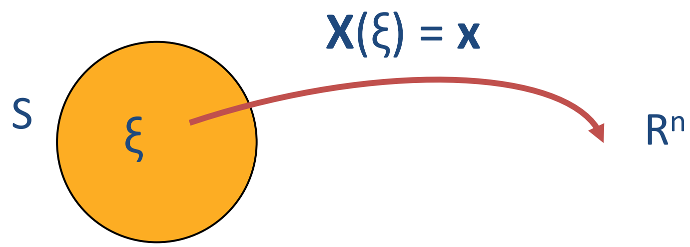
October 28 & November 4, 2020

Note

- These slides cover material partially presented in class. They are provided to help students to follow the textbook. The material here are partly taken from the book by A. Leon-Garcia, Probability and Random Processes for Electrical Engineering, 3rd edition, whom I am thankful.
- There are many other topics which have been covered in class using the blackboard as step-by-step derivation and detailed discussions were needed.

Vector Random Variables

- A **vector random variable** is a function that assigns a vector of real numbers to each outcome ξ in S.



- An event involving $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ has a corresponding region in n-dimensional real space.

- For $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ consider **product-form event:**

$$A = \{X_1 \text{ in } A_1\} \cap \{X_2 \text{ in } A_2\} \cap \cdots \cap \{X_n \text{ in } A_n\}$$

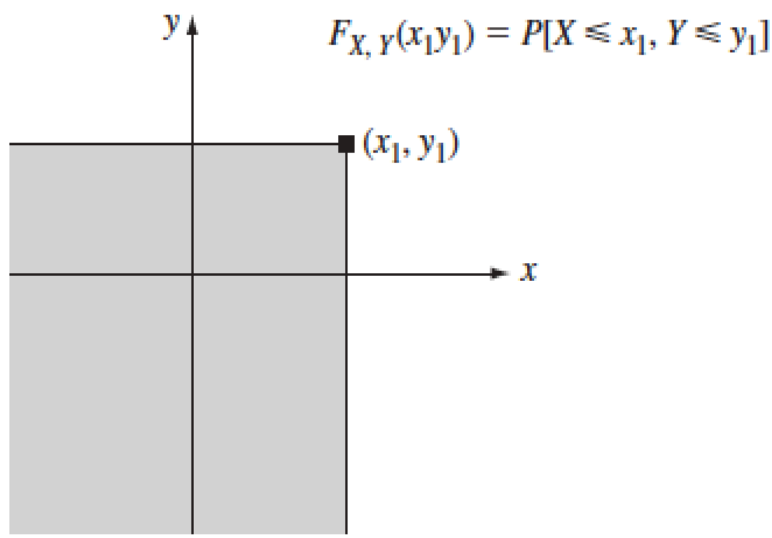
Joint cdf of *X* and *Y*

- A basic building block:

$$\{(x_1, y_1) : x < x_1, y < y_1\}$$

- The **joint cumulative distribution function** of **X** and **Y** is defined as

$$F_{X,Y}(x_1, y_1) = P[X \le x_1, Y \le y_1]$$



Joint cdf of *Random Vector*

- A basic building block:

- The **joint cumulative distribution function** is defined as:

$$F_{\mathbf{X}}(\mathbf{x}) \triangleq F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n].$$

Properties of the Joint cdf

i. $F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$
   if $x_1 \leq x_2$ and $y_1 \leq y_2$

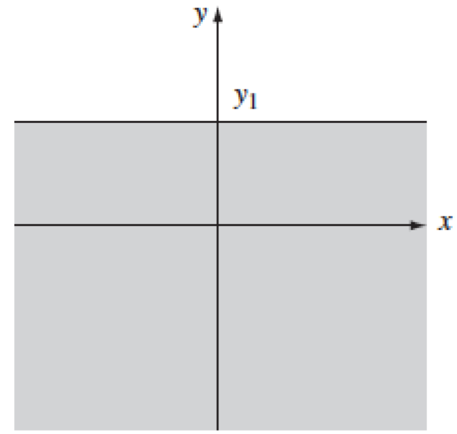
ii. $F_{X,Y}(-\infty, y_1) = F_{X,Y}(x_1, -\infty) = 0$

iii. $F_{X,Y}(\infty, \infty) = 1$

Properties of the Joint cdf (Cont'd)

iv. $F_X(x) = F_{X,Y}(x,\infty) = P[X \le x \ , Y \le \infty]$

$F_Y(y) = F_{X,Y}(\infty,y) = P[Y \le y]$



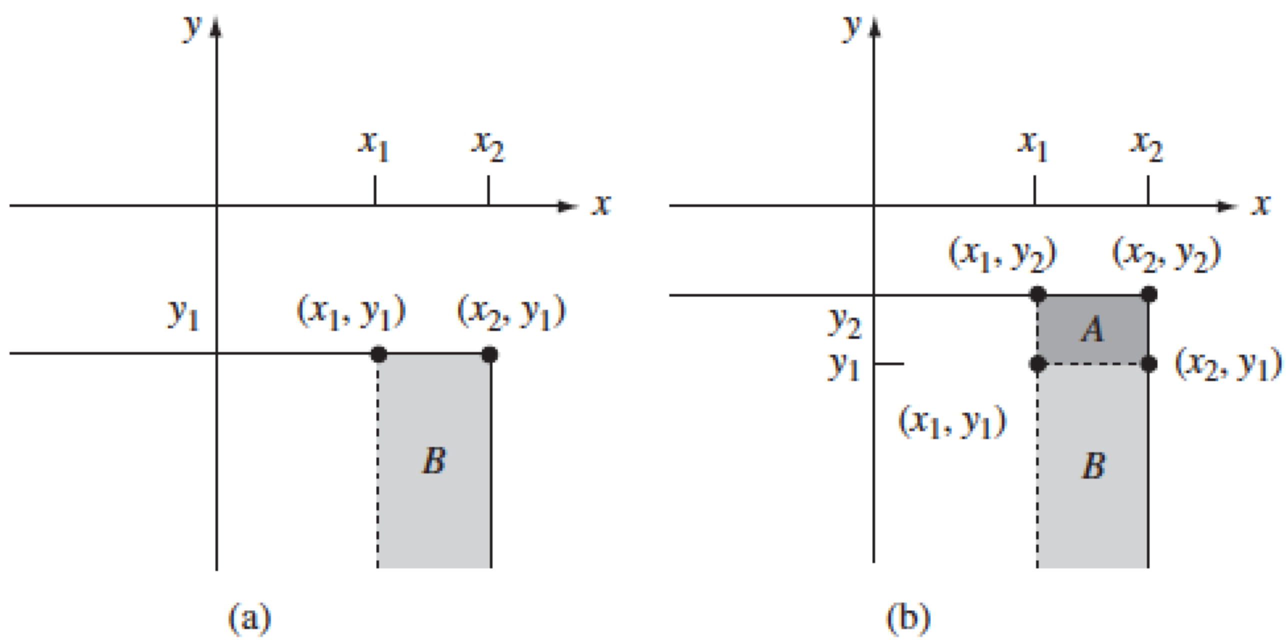
$F_Y(y_1) = P[X < \infty, Y \le y_1]$

v. $\lim_{x \to a^+} F_{X,Y}(x,y) = F_{X,Y}(a,y)$

$\lim_{y \to b^+} F_{X,Y}(x,y) = F_{X,Y}(x,b)$

Properties of the Joint cdf (Cont'd)

vi. $$P\left[x_1 < X \le x_2, y_1 < Y \le y_2\right]$$
$$= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$



General Events

- The probability of a non-product-form event B is approximated by:

$$P[B] = P\left[\bigcup_k B_k\right] = \sum_k P[B_k]$$

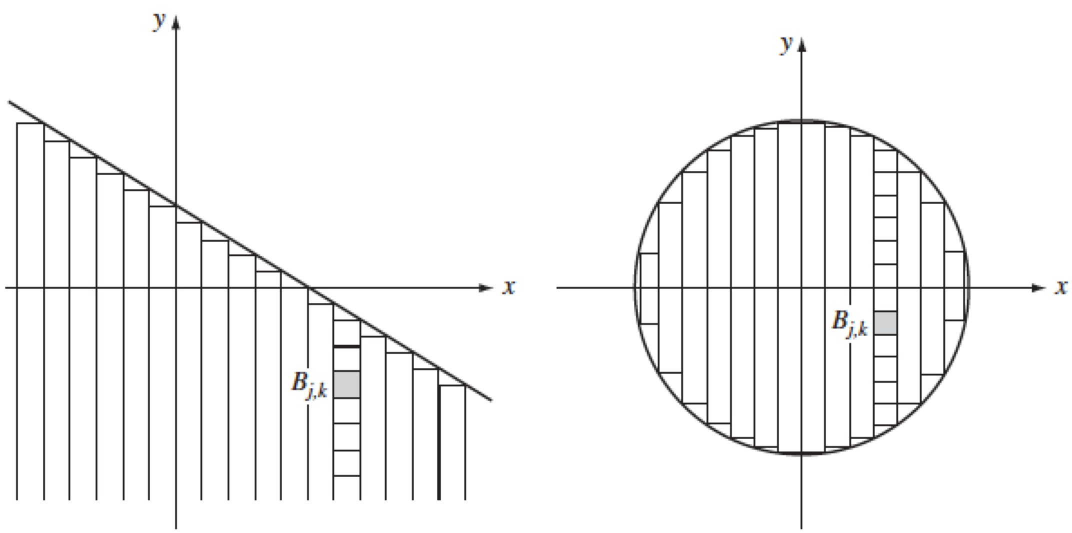


FIGURE 5.12
Some two-dimensional non-product form events.

Jointly Continuous Random Variables

- X and Y are **jointly continuous** if the probabilities of events involving (X, Y) can be expressed as an integral of a **joint probability density function:**

$$P\left[x < X \le x + dx, y < Y \le y + dy\right]$$
$$= \int_x^{x+dx} \int_y^{y+dy} f_{X,Y}(x', y')\, dx', dy'$$
$$\simeq f_{X,Y}(x, y)\, dx, dy$$

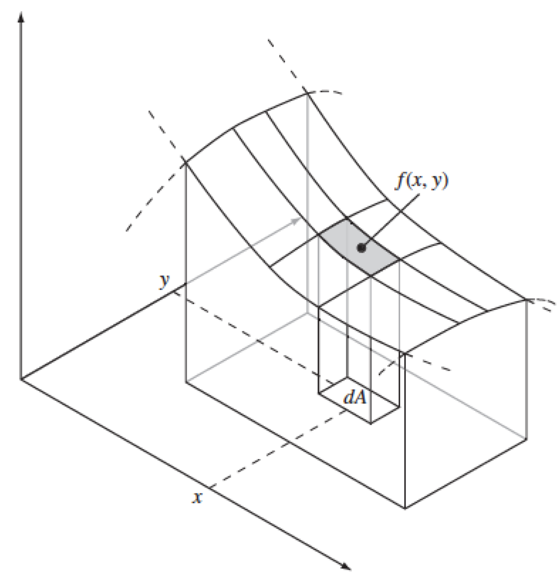


FIGURE 5.13
The probability of A is the integral of $f_{X,Y}(x, y)$ over the region defined by A.

$$P[\mathbf{X} \text{ in } A] = \int_A \int f_{X,Y}(x', y')\, dx', dy'$$

$$P\left[a_1 < X \le b_1, a_2 < Y \le b_2\right] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_{X,Y}(x', y')\, dx', dy'$$

Joint cdf from Joint pdf

- The **joint pdf** is a nonnegative function that satisfies:

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x',y')dx',dy'$$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x',y')dx',dy'$$

If X and Y are jointly continuous then the pdf can be obtained from the cdf by differentiation.

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Joint pdf of Vector Random Variables

- $X_1, X_2, \ldots, X_n$ are jointly continuous if the probability of any n-dimensional event A is given by an n-dimensional integral of a pdf:

$$P\left[(X_1,\ldots,X_n) \text{ in } A\right] = \int_{\mathbf{x} \text{ in } A} \cdots \int f_{X_1,\ldots,X_n}(x_1',\ldots,x_n')\,dx_1',\ldots,dx_n'$$

$$\triangleq \int_{\mathbf{x} \text{ in } A} \cdots \int f_{\mathbf{X}}(\mathbf{x}')\,d\mathbf{x}'$$

Where

$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) \geq 0$ is the **joint probability density function**.

$$F_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(x_1',\ldots,x_n')\,dx_1',\ldots,dx_n'$$

$$f_{\mathbf{X}}(\mathbf{x}) \triangleq f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = \frac{\partial^n}{\partial x_1,\ldots,\partial x_n} F_{X_1,\ldots,X_n}(x_1',\ldots,x_n')$$

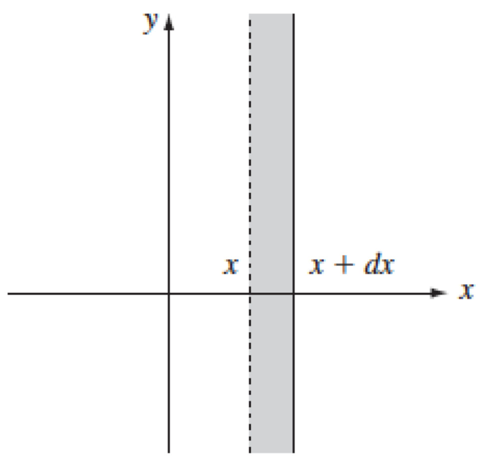
Marginal pdf

$$f_X(x) = \frac{d}{dx}\int_{-\infty}^{x}\left\{\int_{-\infty}^{\infty} f_{X,Y}(x',y')dy'\right\}dx'$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x,y')dy'$$

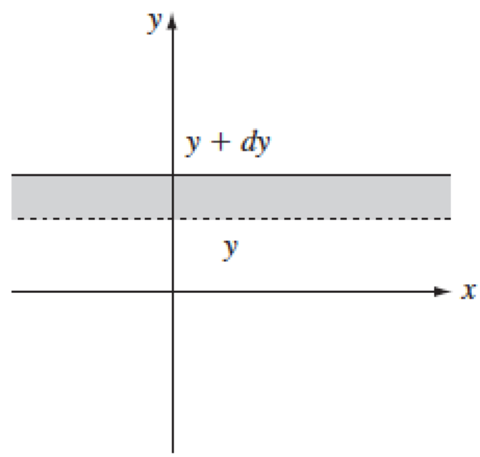
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x',y)dx'$$

$$f_X(x)dx \simeq P\left[x < X \leq x + dx, Y \leq \infty\right]$$



$f_X(x)dx \cong P[x < X \leqslant x + dx, Y < \infty]$

(a)

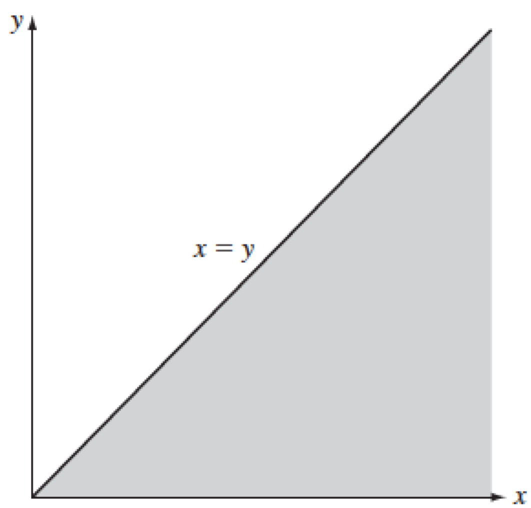


$f_Y(y)dy \cong P[X < \infty, y < Y \leqslant y + dy]$

(b)

Example:

Let (X, Y) be selected at random from the lower part of the unit square (i.e., $x > y$). Find the joint pdf and marginal pdf



Independence of Two Random Variables

- X and Y are **independent** if **any** event A_1 that only involves X is independent of **any** event A_2 that only involves Y.

$$P[X \in A_1, Y \in A_2] = P[X \in A_1]P[Y \in A_2]$$

- X and Y are independent if and only if the joint cdf equals the product of the marginal cdfs.

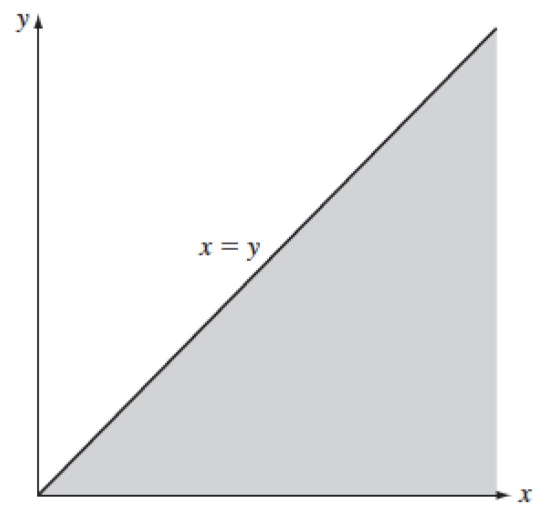
$$F_{X,Y}(x,y) = F_X(x)F_Y(y) \quad \text{for all } x \text{ and } y$$

- If X and Y are jointly continuous , then X and Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \quad \text{for all } x \text{ and } y$$

Example: Joint pdf

- Are (X, Y) in previous example independent RVs?



Example: Jointly Gaussian RVs

- Determine whether *X* and *Y* are independent if they have joint pdf:

$$f_{X,Y}(x,y) = \frac{e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \quad \text{all } x,\ y$$

Independent Variables (Cont'd)

- If X and Y are independent random variables, then the $g(X)$ and $h(Y)$ are also independent.

$$\begin{aligned} P[g(X) \text{ in } A, h(Y) \text{ in } B] &= P[X \text{ in } A', Y \text{ in } B'] \\ &= P[X \text{ in } A']P[Y \text{ in } B'] \\ &= P[g(X) \text{ in } A]P[h(Y) \text{ in } B] \end{aligned}$$

Independence of Vector RVs

- $X_1, \ldots, X_n$ are independent if:

$$P[X_1 \text{ in } A_1, \ldots, X_n \text{ in } A_n] = P[X_1 \text{ in } A_1] \ldots P[X_n \text{ in } A_n]$$

for any one-dimensional events $A_1, \ldots, A_n$.

- $X_1, \ldots, X_n$ are independent if and only if for all $x_1, \ldots, x_n$:

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = F_{X_1}(x_1)\ldots F_{X_n}(x_n)$$

- If they are discrete then equivalently:

$$p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = p_{X_1}(x_1)\ldots p_{X_n}(x_n) \quad \text{for all } x_1,\ldots,x_n$$

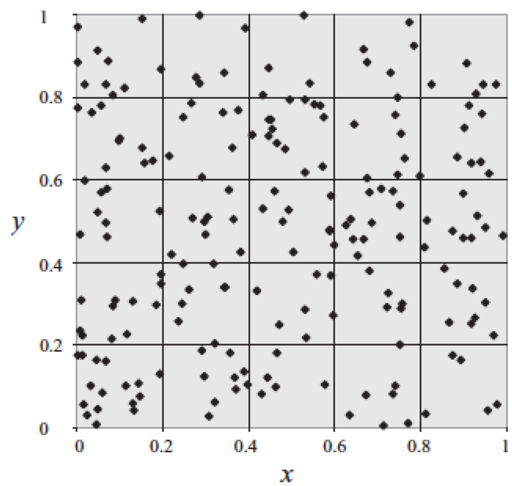
- If they are continuous then equivalently:

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = f_{X_1}(x_1)\ldots f_{X_n}(x_n) \quad \text{for all } x_1,\ldots,x_n$$

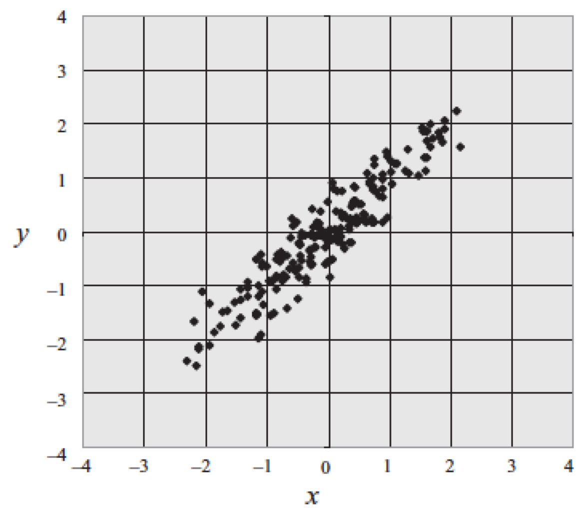
Summary from Last Lecture

- The joint statistical behavior of a pair of random variables X and Y is specified by the joint cumulative distribution function, the joint probability mass function, or the joint probability density function.
- The probability of any event involving the joint behavior of these random variables can be computed from these functions.
- The statistical behavior of individual random variables from $\mathbf{X}$ is specified by the marginal CDF, marginal pdf, or marginal pmf that can be obtained from the joint CDF, joint pdf, or joint pmf of $\mathbf{X}$.
- Two random variables are independent if the probability of a product-form event is equal to the product of the probabilities of the component events. Equivalent conditions for the independence of a set of random variables are that the joint CDF, joint pdf, or joint pmf factors into the product of the corresponding marginal functions.

Covariance of Two Random Variables



(a)



Correlation & Covariance

- The **correlation of X and Y** is defined by:

$$E[XY] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy & X,Y \text{ jointly continuous} \\ \sum_i \sum_n x_i y_n p_{X,Y}(x_i, y_n) & X,Y \text{ discrete} \end{cases}$$

- **X and Y are orthogonal** if E[XY] = 0.

- The **covariance of X and Y** is defined as:

$$\mathrm{COV}(X,Y) = E\left[(X - E[X])(Y - E[Y])\right]$$

- **X and Y are uncorrelated** if COV(X,Y) = 0.

Joint Moments of Two RVs

- The *jk*th **joint moment of X and Y** is defined by:

$$E\left[X^j Y^k\right] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{X,Y}(x,y)dxdy & X,Y \text{ jointly continuous} \\ \sum_i \sum_n x_i^j y_n^k p_{X,Y}(x_i, y_n) & X,Y \text{ discrete} \end{cases}$$

- The jkth **central moment of X and Y** is defined by:

$$E\left[(X - E[X])^j (Y - E[Y])^k\right]$$

Correlation Coefficient of Two Random Variables

- The **correlation coefficient of X and Y** is:

$$\rho_{X,Y} = \frac{COV(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

- X and Y are uncorrelated if $\rho_{X,Y} = 0$.
- What does the correlation coefficient measure?
  - The covariance and the correlation coefficient of two random variables are measures of the linear dependence between the random variables.

$$COV(X,Y) = E\left[\left(\frac{X - E[X]}{\sigma_X}\right)\left(\frac{Y - E[Y]}{\sigma_Y}\right) \right]$$

- Correlation (alignment) of X – E[X] with Y – E[Y].

Correlation Coefficient

- Correlation coefficient has a value: $-1 \leq \rho \leq 1$.

$$0 \leq E\left\{\left(\frac{X-E[X]}{\sigma_X} \pm \frac{Y-E[Y]}{\sigma_Y}\right)^2\right\} = 1 \pm 2\rho + 1 = 2(1 \pm \rho)$$

  - If Y = aX + b, then $\rho = \pm 1$.
  - If X and Y are independent, then $\rho = 0$
  - If X and Y are uncorrelated, they are not necessarily independent
  - IF X and Y are jointly Gaussian and $\rho = 0$, then they are independent.

Conditional cdf: Discrete RV

- If X is discrete, then the conditional cdf of Y given X = x_k:

$$F_Y[y \mid x_k] = \frac{P[Y \le y, X = x_k]}{P[X = x_k]} \quad \text{for } P[X = x_k] > 0.$$

- Conditional pdf of Y given X = x_k:

$$f_Y[y \mid x_k] = \frac{d}{dy} F_Y(y \mid x_k)$$

- Probability of A given X = x_k:

$$P[Y \text{ in } A \mid X = x_k] = \int_{y \text{ in } A} f_Y(y \mid x_k) dy$$

Conditional CDF: *X* Continuous

$$P[Y \text{ in } A \mid X = x] = \frac{P[Y \text{ in } A, X = x]}{P[X = x]}$$

- If X is continuous then P[X = x] = 0
- **Conditional CDF** of Y given X is defined by the limit:

$$F_Y(y \mid x) = \lim_{h \to 0} F_Y(y \mid x < X \leq x + h)$$

Conditional cdf: *X* Continuous (Cont'd)

Suppose $f_X(x) > 0$.

$$F_Y(y \mid x < X \le x+h) = \frac{P(Y \le y, x < X \le x+h)}{P(x < X \le x+h)}$$

$$= \frac{\int_{-\infty}^{y} \int_{x}^{x+h} f_{X,Y}(x', y')dx', dy'}{\int_{x}^{x+h} f_X(x')dx'} = \frac{\int_{-\infty}^{y} f_{X,Y}(x, y')dy'h}{f_X(x)h}$$

As h approaches 0, $F_Y(y \mid x) = \frac{\int_{-\infty}^{y} f_{X,Y}(x, y')dy'}{f_X(x)}$

- Conditional pdf of Y given *X* = *x*:

$$f_Y(y \mid x) = \frac{d}{dy} F_Y(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

Conditional pdfs of Vector RVs

- A family of marginal and conditional pdfs is obtained from the joint pdf:

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1, X_2, \ldots, X_n}(x_1, x_2', \ldots, x_n') \, dx_2', \ldots, dx_n'$$

$$f_{X_1, \ldots, X_{n-1}}(x_1, \ldots, x_{n-1}) = \int_{-\infty}^{\infty} f_{X_1, \ldots, X_n}(x_1, \ldots, x_{n-1}, x_n') \, dx_n'$$

$$f_{X_n}(x_n \mid x_1, \ldots, x_{n-1}) = \frac{f_{X_1, \ldots, X_n}(x_1, \ldots, x_n)}{f_{X_1, \ldots, X_{n-1}}(x_1, \ldots, x_{n-1})}$$

$$f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = f_{X_n}(x_n \mid x_1, \ldots, x_{n-1}) \times f_{X_{n-1}}(x_{n-1} \mid x_1, \ldots, x_{n-2}) \cdots f_{X_2}(x_2 \mid x_1) f_{X_1}(x_1)$$

Independent *X* and *Y*

- If *X* and *Y* are independent, then:
  - $f_{X,Y}(\mathrm{x},y) = f_X(\mathrm{x})\, f_Y(y)$
  - $f_Y(\ y\ | x\ ) = f_Y(y)$
  - $F_Y(\ y\ | x\ ) = F_Y(y)$

Total Probability

- Joint pdf can be expressed as follows:

$$f_{X,Y}(x,y) = f_Y(y \mid x) f_X(x)$$

- then

$$P[Y \text{ in } A] = \int_{-\infty}^{\infty} \int_{y \text{ in A}} f_{X,Y}(x,y) dx dy$$

$$= \int_{-\infty}^{\infty} \int_{y \text{ in A}} f_Y(y \mid x) dy) f_X(x) dx$$

$$= \int_{-\infty}^{\infty} P[Y \text{ in } A \mid X = x] f_X(x) dx$$

- Bayes' rule:

$$f_X(x \mid y) = \frac{f_{X,Y}(x,y)}{f_y(y)}$$

Example: Jointly Gaussian RVs

- The joint pdf of a pair of jointly Gaussian random variables is determined by the means, variances, and covariance:

$$f_{X,Y}(x,y) = \frac{e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \quad \text{all } x,\ y$$

- Find $f_Y(y|x)$
- All **marginal** pdf's and **conditional** pdf's are also Gaussian pdf's.

Example: Splitting of Poisson Counts

- Events in a Poisson process are classified as type 1 or type 2 according to independent Bernoulli trials.
- Find the joint pmf of N_1 and N_2.

Conditional Expectation

- Conditional expectation of Y given X is:

$$E[Y \mid x] = \int_{-\infty}^{\infty} y f_Y(y \mid x) dy$$

- Let g(x) = E[Y|x], then g(X) is a function of a RV, and

$$E[g(X)] = E[E[Y \mid X]] = \int_{-\infty}^{\infty} E[Y \mid x] f_X(x) dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_Y(y \mid x) dy f_X(x) dx = \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx\, dy$$

$$= \int_{-\infty}^{\infty} y f_Y(y) dy = E[Y]$$

$$E[Y] = E[E[Y \mid X]]$$

Example: $E[E[Y|X]]$

- Find mean of type 1 events in Poisson splitting.

Joint Characteristic Function

- The joint characteristic function of two RVs is:

$$\Phi_{X,Y}(\omega_1,\omega_2) = E[e^{j(\omega_1 X+\omega_2 Y)}]$$

- If X and Y are continuous, then

$$\Phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)e^{j(\omega_1 x+\omega_2 y)}dxdy$$

$$f_{X,Y}(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{X,Y}(\omega_1,\omega_2)e^{-j(\omega_1 x+\omega_2 y)}d\omega_1 d\omega_2$$

Joint Characteristic Function (Cont'd)

- The marginal characteristic function is obtained from the joint characteristic function:

$$\Phi_X(\omega) = \Phi_{X,Y}(\omega, 0)$$

$$\Phi_Y(\omega) = \Phi_{X,Y}(\overset{0}{\cancel{\omega}}, \overset{\omega}{0})$$

- The **joint characteristic function** of n RVs is defined as:

$$\Phi_{X_1, X_2, \dots, X_n}(\omega_1, \omega_2, \dots, \omega_n) = E[e^{j(\omega_1 X_1 + \omega_2 X_2 + \dots + \omega_n X_n)}]$$

Joint Characteristic Function of Independent RVs

- Let X and Y be independent RVs:

$$\Phi_{X,Y}(\omega_1,\omega_2) = E[e^{j(\omega_1 X+\omega_2 Y)}] = E[e^{j\omega_1 X}e^{j\omega_2 Y}]$$
$$= E[e^{j\omega_1 X}]E[e^{j\omega_2 Y}] = \Phi_X(\omega_1)\Phi_Y(\omega_2)$$

- The characteristic function of sum Z = aX + bY is

$$\Phi_Z(\omega) = E[e^{j\omega(aX+bY)}] = E[e^{j(\omega aX+\omega bY)}] = \Phi_{X,Y}(a\omega,b\omega)$$

- If Z = aX + bY and X and Y are independent:

$$\Phi_Z(\omega) = \Phi_{X,Y}(a\omega,b\omega) = \Phi_X(a\omega)\Phi_Y(b\omega)$$

Joint Moments of *X* and *Y*

- Joint moments of *X* and *Y* obtained from derivatives of joint characteristic function.

$$\Phi_{X,Y}(\omega_1,\omega_2) = E[e^{j\omega_1 X} e^{j\omega_2 Y}]$$

$$= E\left[\sum_{i=0}^{\infty} \frac{(j\omega_1 X)^i}{i!} \sum_{j=0}^{\infty} \frac{(j\omega_2 Y)^k}{k!}\right]$$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} E[X^i Y^k] \frac{(j\omega_1 \cancel{X})^i}{i!} \frac{(j\omega_2 \cancel{Y})^k}{k!}$$

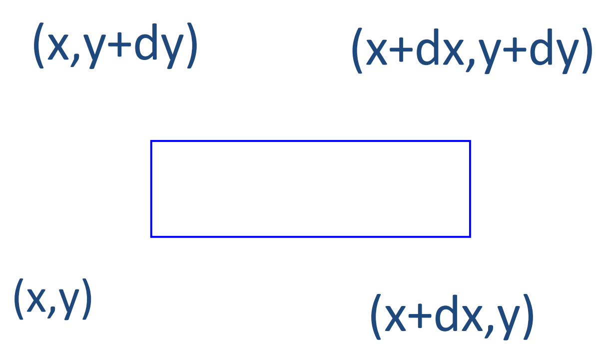
$$E[X^i Y^k] = \frac{1}{j^{i+k}} \frac{\partial^i \partial^k}{\partial \omega_1^i \partial \omega_2^k} \Phi_{X,Y}(\omega_1,\omega_2)\Big|_{\omega_1=0,\omega_2=0}$$

Linear Transformations

- Consider **linear transformation** of two RVs:

$$V = aX + bY$$
$$W = cX + eY$$

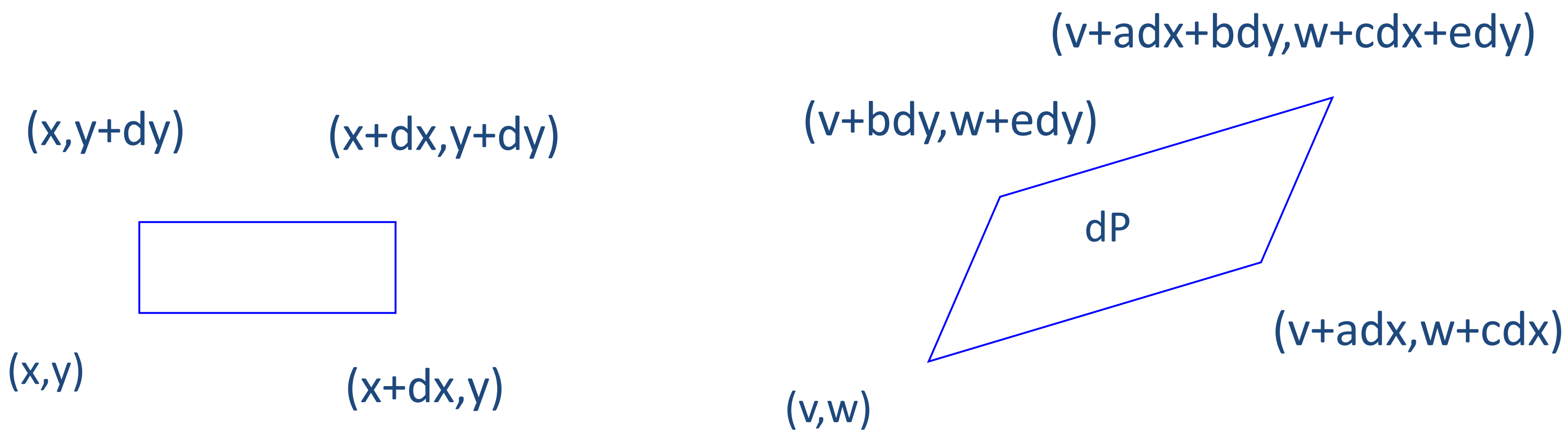
- Assume A has inverse: determinant $|ae - bc| \neq 0$.
- Consider how infinitesimal rectangle in (x,y) is mapped into (v,w)

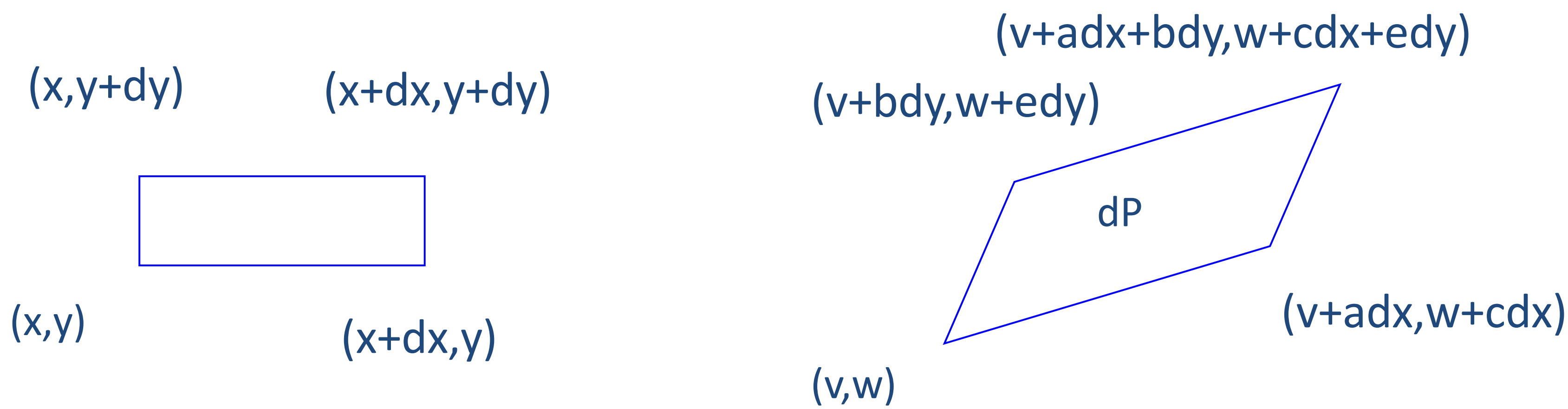


Linear Transformations (Cont'd)

- Note the "stretch factor"

$$\left|\frac{dP}{dx\,dy}\right| = \frac{|ae-bc|(dx\,dy)}{(dx\,dy)} = |ae-bc| = |A|$$





$$f_{V,W}(v,w)dP = f_{X,Y}(x,y)\left|dxdy\right|$$

$$f_{V,W}(v,w) = \frac{f_{X,Y}(x,y)}{\left|\dfrac{dP}{dx\,dy}\right|} = \frac{f_{X,Y}(x,y)}{\left|A\right|}$$

Example: Rotation of Jointly Gaussian RVs

$$Z_1 = \frac{1}{\sqrt{2}} X + \frac{1}{\sqrt{2}} Y$$

$$Z_1 = -\frac{1}{\sqrt{2}} X + \frac{1}{\sqrt{2}} Y$$

$$f_{X,Y}(x,y) = \frac{e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \quad \text{all } x, y$$

Example 5.48 Rotation of Jointly Gaussian Random Variables

The ellipse corresponding to an arbitrary two-dimensional Gaussian vector forms an angle

$$\theta = \frac{1}{2}\arctan\left(\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}\right)$$

relative to the x-axis. Suppose we define a new coordinate system whose axes are aligned with those of the ellipse as shown in Fig. 5.27. This is accomplished by using the following rotation matrix:

$$\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

To show that the new random variables are independent it suffices to show that they have covariance zero:

$$\begin{aligned}
\text{COV}(V, W) &= E[(V - E[V])(W - E[W])] \\
&= E[\{(X - m_1)\cos\theta + (Y - m_2)\sin\theta\} \\
&\quad \times \{-(X - m_1)\sin\theta + (Y - m_2)\cos\theta\}] \\
&= -\sigma_1^2 \sin\theta\cos\theta + \text{COV}(X, Y)\cos^2\theta \\
&\quad -\text{COV}(X, Y)\sin^2\theta + \sigma_2^2 \sin\theta\cos\theta \\
&= \frac{(\sigma_2^2 - \sigma_1^2)\sin 2\theta + 2\,\text{COV}(X, Y)\cos 2\theta}{2} \\
&= \frac{\cos 2\theta[(\sigma_2^2 - \sigma_1^2)\tan 2\theta + 2\,\text{COV}(X, Y)]}{2}.
\end{aligned}$$

Example: Rotation of Jointly Gaussian RVs (Cont'd)

If we let the angle of rotation θ be such that

$$\tan 2\theta = \frac{2\,\text{COV}(X, Y)}{\sigma_1^2 - \sigma_2^2},$$

then the covariance of V and W is zero as required.

General Linear Transformation

- Let the n-dimensional vector **Z** be:

$$\mathbf{Z} = A\mathbf{X}$$

where A is an n x n invertible matrix.

- The joint pdf of **Z** is:

$$f_{\mathbf{Z}}(\mathbf{z}) \triangleq f_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n) = \left.\frac{f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)}{|A|}\right|_{\mathbf{x}=A^{-1}\mathbf{z}} = \frac{f_{\mathbf{X}}(A^{-1}\mathbf{z})}{|A|}$$

Gaussian Vector Random Variables

We now present a method for generating unit-variance, uncorrelated (and hence independent) jointly Gaussian random variables. Suppose that X and Y are two independent zero-mean, unit-variance jointly Gaussian random variables with pdf:

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

In Example 5.44 we saw that the transformation

$$R = \sqrt{X^2 + Y^2} \qquad \text{and} \qquad \Theta = \tan^{-1} Y/X$$

leads to the pair of independent random variables

$$f_{R,\Theta}(r, \theta) = \frac{1}{2\pi} r e^{-r^2/2} = f_R(r) f_\Theta(\theta),$$

where R is a Rayleigh random variable and Θ is a uniform random variable. The above transformation is invertible. Therefore we can also start with independent Rayleigh and uniform random variables and produce zero-mean, unit-variance independent Gaussian random variables through the transformation:

$$X = R \cos \Theta \quad \text{and} \quad Y = R \sin \Theta. \tag{5.65}$$

Consider $W = R^2$ where R is a Rayleigh random variable. From Example 5.41 we then have that: W has pdf

$$f_W(w) = \frac{f_R(\sqrt{w})}{2\sqrt{w}} = \frac{\sqrt{w} e^{-\sqrt{w}^2/2}}{2\sqrt{w}} = \frac{1}{2} e^{-w/2}.$$

$W = R^2$ has an exponential distribution with $\lambda = 1/2$.

Example: Gaussian Random Vectors

- Find the joint pdf of **X** if X_1, X_2,…, X_n are independent Gaussian random variables.

Mean Vector

- Mean Vector = vector of means

$$\mathbf{m_x} = E[\mathbf{X}] = E\begin{bmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{bmatrix} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \dots \\ E[X_n] \end{bmatrix}.$$

Correlation Matrix

- Correlation Matrix= matrix of correlations

$$\mathbf{R}_{\mathbf{X}} = \begin{bmatrix} E[X_1^2] & E[X_1X_2] & \dots & E[X_1X_n] \\ E[X_2X_1] & E[X_2^2] & \dots & E[X_2X_n] \\ . & . & \dots & . \\ E[X_nX_1] & E[X_nX_2] & \dots & E[X_n^2] \end{bmatrix}.$$

- Correlation Matrix is symmetric
- Diagonal elements are second moments

Covariance Matrix

- Covariance Matrix= matrix of covariances

$$K_{\mathbf{X}} = \begin{bmatrix} E[(X_1 - m_1)^2] & E[(X_1 - m_1)(X_2 - m_2)] & \dots & E[(X_1 - m_1)(X_n - m_n)] \\ E[(X_2 - m_2)(X_1 - m_1)] & E[(X_2 - m_2)^2] & \dots & E[(X_2 - m_2)(X_n - m_n)] \\ . & . & \dots & . \\ E[(X_n - m_n)(X_1 - m_1)] & E[(X_n - m_n)(X_2 - m_2)] & \dots & E[(X_n - m_n)^2] \end{bmatrix}.$$

- Covariance Matrix is symmetric
- Diagonal elements are variances
- **m**=**0** implies $K_{\mathbf{X}}=R_{\mathbf{X}}$
- X_i and X_j uncorrelated all i,j implies $K_{\mathbf{X}}$ is diagonal
- X_i and X_j independent all i,j implies $K_{\mathbf{X}}$ is diagonal

Matrix Notation is Compact

$$\mathbf{X}\mathbf{X}^{\mathrm{T}} = \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{bmatrix} [X_1, X_2, \ldots, X_n] = \begin{bmatrix} X_1^2 & X_1 X_2 & \ldots & X_1 X_n \\ X_2 X_1 & X_2^2 & \ldots & X_2 X_n \\ . & . & \ldots & . \\ X_n X_1 & X_n X_2 & \ldots & X_n^2 \end{bmatrix}.$$

$$\mathbf{R}_{\mathbf{X}} = E[\mathbf{X}\mathbf{X}^{\mathrm{T}}].$$

$$\begin{aligned} \mathbf{K}_{\mathbf{X}} &= E[(\mathbf{X} - \mathbf{m}_{\mathbf{X}})(\mathbf{X} - \mathbf{m}_{\mathbf{X}})^{\mathrm{T}}] \\ &= E[\mathbf{X}\mathbf{X}^{\mathrm{T}}] - \mathbf{m}_{\mathbf{X}} E[\mathbf{X}^{\mathrm{T}}] - E[\mathbf{X}]\mathbf{m}'_{\mathbf{X}} + \mathbf{m}_{\mathbf{X}}\mathbf{m}'_{\mathbf{X}} \\ &= \mathbf{R}_{\mathbf{X}} - \mathbf{m}_{\mathbf{X}}\mathbf{m}'_{\mathbf{X}} \end{aligned}$$

Linear Transformations of X

- Input X and Output Y

$$\mathbf{Y} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{bmatrix} = \mathbf{AX}.$$

$$E[Y_k] = E\left[\sum_{j=1}^{n} a_{kj} X_j\right] = \sum_{j=1}^{n} a_{kj} E[X_j]$$

$$E[\mathbf{Y}] = \mathbf{A}E[\mathbf{X}] = \mathbf{A}\mathbf{m}_{\mathbf{X}}.$$

Covariance and Cross-Covariance

$$\mathbf{K_Y} = E[(\mathbf{Y}-\mathbf{m_Y})(\mathbf{Y}-\mathbf{m_Y})^{\mathrm{T}}] = (\mathbf{AX}-\mathbf{Am_X})(\mathbf{AX}-\mathbf{Am_X})^{\mathrm{T}}]$$
$$= E[\mathbf{A}(\mathbf{X}-\mathbf{m_X})(\mathbf{X}-\mathbf{m_X})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}] = \mathbf{A}E[(\mathbf{X}-\mathbf{m_X})(\mathbf{X}-\mathbf{m_X})^{\mathrm{T}}]\mathbf{A}^{\mathrm{T}}$$
$$= \mathbf{AK_X A}^{\mathrm{T}}.$$

$$\mathbf{K_{XY}} = E[(\mathbf{X}-\mathbf{m_X})(\mathbf{Y}-\mathbf{m_Y})^{\mathrm{T}}] = E[(\mathbf{X}-\mathbf{m_X})(\mathbf{X}-\mathbf{m_X})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}]$$
$$= \mathbf{K_X A}^{\mathrm{T}}.$$

Example: nD Gaussian RV

- **X** is determined by **m** and $K_{\mathbf{X}}$ is a covariance matrix

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{e^{-\{\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{K}^{-1}(\mathbf{x}-\mathbf{m})\}}}{(2\pi)^{n/2} \, |K|^{1/2}}$$

- Argument in the exponent is a quadratic form (ellipsoid in R^n)