George Mason University

Department of Electrical and Computer Engineering

ECE 528: Introduction to Random Processes in ECE — Fall Semester

Homework Set 7 Solutions

1. Consider the random variable N taking values $n \geq 0$ with probabilities $p_1, p_2, p_3, ...$ and the indicator function $u(x) = 1, x \geq 1$ and 0 otherwise.

$$u(x) = \begin{cases} 1, & x \geq 1 \\ 0, & otherwise \end{cases}$$

   (a) Find $E[u(n)]$

   (b) Find $E[nu[n]]$

   Solutions:

   (a)

$$E[u(n)] = \sum_{n=0}^{+\infty} u[n]p_n = u[0]p_0 + \sum_{n=1}^{+\infty} u[n]p_n = 0 + \sum_{n=1}^{+\infty} 1 * p_n = \sum_{n=1}^{+\infty} p_n = 1 - p_0.$$

   (b)

$$\begin{aligned} E[nu(n)] &= \sum_{n=0}^{+\infty} nu[n]p_n = 0 * u[0] * p_0 + \sum_{n=1}^{+\infty} nu[n]p_n \\ &= 0 + \sum_{n=0}^{+\infty} 1 * n * p_n = \sum_{n=0}^{+\infty} np_n = E[n]. \end{aligned}$$

2. In the class, we derived the distribution of a linear function of a random variable when the scaling was positive. In other words, consider the random variable X which has Cumulative Distribution Function (CDF) $F_X(x)$, and suppose further that the random variable Y is a linear function of X, that is,

$$Y = aX + b$$

   where a and b are constants. We obtained the CDF $G_Y(y)$ and the PDF $g_Y(y)$ of Y in terms of $F_X(x)$ and $f_X(x)$ respectively when $a > 0$. Now do this for any general case of a (i.e., for both cases of $a > 0$ and $a < 0$) and derive a single formula for the PDF $g_Y(y)$.

   Solutions:

If $a > 0$, we have:

$$G_Y(y) = P(Y \le y) = P(aX + b \le y) = P(X \le \frac{y-b}{a}) = F_X(\frac{y-b}{a});$$
$$g_Y(y) = \frac{\partial G_Y(y)}{\partial y} = \frac{1}{a} f_X(\frac{y-b}{a}).$$

If $a < 0$, we have:

$$G_Y(y) = P(Y \le y) = P(aX + b \le y) = P(X \ge \frac{y-b}{a}) = 1 - P(X \le \frac{y-b}{a}) = 1 - F_X(\frac{y-b}{a}).$$
$$g_Y(y) = \frac{\partial G_Y(y)}{\partial y} = -\frac{1}{a} f_X(\frac{y-b}{a}).$$

Therefore, the PDF $g_Y(y)$ can be summarized as:

$$g_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$$

3. Apply the above problem to the case where the random variable X has a standard negative exponential distribution, ($f_X(x) = e^{-x}$ for $x \ge 0$, $f_X(x) = 0$ for $x < 0$, $E[X] = \mu = 1$) and Y = aX + b with a and b positive constants. What is the CDF of Y? Can you think of what a might represents?

   Solutions:

   The cdf of X is as follows:

   $$F_X(x) = \int_0^x e^{-x} dx = 1 - e^{-x}, x \ge 0;$$

   When $a, b > 0$, we know that $F_Y(y) = F_X(\frac{y-b}{a})$. Therefore, the CDF of y becomes:

   $$F_Y(y) = \int_0^x e^{-x} dx = 1 - e^{-\frac{y-b}{a}}, y \ge b.$$

   Parameter a denotes the mean value in exponential distribution.

4. Apply the above problem to the case where X is a standard normally distributed random variable, and Y = aX + b. What is the CDF of Y? What do a and b represent?

   Solutions:

   In standard Gaussian distribution, the CDF can be written as $F_X(x) = 1 - Q(x)$. The CDF of Y thus becomes $F_Y(y) = 1 - Q(\frac{y-b}{a})$. a is the standard deviation, and b is the mean in Gaussian distribution.

5. Consider the random variable X which has a uniform distribution over the interval [0, 1], that is, the pdf of X, $f_X(x)$ is equal to 1 for $0 \leq x \leq 1$, and is 0 otherwise and let $Y = e^X$. In this case, Y is not a linear function of X, but the method used in problem 2 can be used to obtain the distribution function and probability density function of Y.

Solutions:

The CDF of X is as follows:

$$
\begin{aligned}
F_X(x) &= \int_0^x 1dx = x, 0 \leq x \leq 1; \\
F_X(x) &= 1, x > 1.
\end{aligned}
$$

The CDF of Y becomes:

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq log(y)) = F_X(log(y));$$

Therefore,

$$F_Y(y) = \begin{cases} 0, & y < 1; \\ log(y), & 1 \leq y \leq e; \\ 1, & y > e. \end{cases}$$

The PDF of Y is derived by taking the derivative of $F_Y(y)$, i.e., $f_Y(y) = \frac{\partial F_Y(y)}{\partial y}$:

$$f_Y(y) = \begin{cases} 0, & y < 1; \\ 1/y, & 1 \leq y \leq e; \\ 0, & y > e. \end{cases}$$

6. Consider a non-negative Random Variable, X with CDF of $F_X(x)$ and show that:

$$E[X] = \int_0^{+\infty} [1 - F_X(x)]dx$$

Solutions:

$$
\begin{aligned}
\int_0^{+\infty} [1 - F_X(x)]dx &= x[1 - F_X(x)]\Big|_0^{+\infty} - \int_0^{+\infty} xd(1 - F_X(x)) \\
&= (0 - 0) - \int_0^{+\infty} xd(-F_X(x)) \\
&= - \int_0^{+\infty} x(-f_X(x))dx \\
&= \int_0^{+\infty} xf_X(x)dx = E[X].
\end{aligned}
$$