CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE

Lecture 14: Poisson Process; Stationary Random Processes

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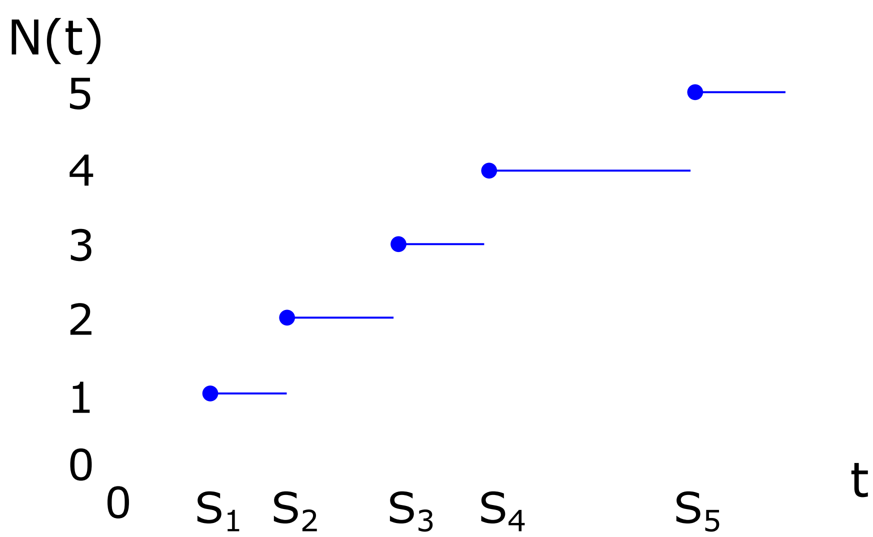
December 2, 2020

Continuous-Time Counting Process

- Consider a situation in which events occur at random instants of time at an average rate of λ events per second.

- Let N(t) be the number of event occurrences in the time interval [0, t].

- N(t) is then a non-decreasing, integer-valued, continuous-time random process.



Uniform "Random" Arrivals

- Suppose that the interval [0, t] is divided into n subintervals of very short duration $\delta = t/n$.
- Assume:
    1. Probability of > 1 event occurrence in a subinterval is negligible compared to the probability of observing 1 or 0 events.
    2. Whether an event occurs in a subinterval is independent of the outcomes in other subintervals.

Poisson Process

- The number of event occurrences N(t) in the interval [0, t] has a Poisson distribution with mean λt:

$$P[N(t)] = (\lambda t)^k / k!\, e^{-\lambda t}$$

- For this reason N(t) is called the **Poisson process**.
- N(t) inherits the property of independent and stationary increments from the underlying binomial process.

Joint Distribution and Covariance Function

$$P[N(t_1)=i, N(t_2)=j\ ]=$$

$$\begin{aligned} P[N(t_1) = i, N(t_2) = j] &= P[N(t_1) = i]P[N(t_2) - N(t_1) = j - i] \\ &= P[N(t_1) = i]P[N(t_2 - t_1) = j - i] \\ &= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \frac{(\lambda(t_2 - t_1))^j e^{-\lambda(t_2-t_1)}}{(j-i)!}. \end{aligned}$$

$$C_N(t_1, t_2)=E[(N(t_1) - \lambda\ t_1)\ (N(t_2) - \lambda\ t_2)]=$$

$$\begin{aligned} C_N(t_1, t_2) &= E[(N(t_1) - \lambda t_1)(N(t_2) - \lambda t_2)] \\ &= E[(N(t_1) - \lambda t_1)\{N(t_2) - N(t_1) - \lambda t_2 + \lambda t_1 + (N(t_1) - \lambda t_1)\}] \\ &= E[(N(t_1) - \lambda t_1)]E[(N(t_2) - N(t_1) - \lambda(t_2 - t_1)] + \text{VAR}[N(t_1)] \\ &= \text{VAR}[N(t_1)] = \lambda t_1. \end{aligned}$$

Interarrival Times

- Consider the time T between event occurrences in a Poisson process. Suppose the interval [0, t] is divided into n subintervals of length $\delta = t/n$.

$$
\begin{aligned}
P[T > t] &= P[\text{no events in } t \text{ seconds}] \\
&= (1-p)^n \\
&= \left(1 - \frac{\lambda t}{n}\right)^n \\
&\to e^{-\lambda t} \quad \text{as } n \to \infty
\end{aligned}
$$

- The inter-event times in a Poisson process form an iid sequence of exponential random variables with mean $1/\lambda$.

Arrival Times are Uniformly Distributed

- Suppose only one arrival occurred in an [0, *t*].
- Let X be the arrival time of the single customer.
- For 0 < x < t, let N(x) be the number of events up to time x, and let N(t) – N(x) be the increment in the interval (x, t], then $P[X \leq x]$ =

$$
\begin{aligned}
&= P[N(x) = 1 \mid N(t) = 1] \\
&= \frac{P[N(x) = 1 \text{ and } N(t) = 1]}{P[N(t) = 1]} \\
&= \frac{P[N(x) = 1 \text{ and } N(t) - N(x) = 0]}{P[N(t) = 1]} \\
&= \frac{P[N(x) = 1]P[N(t) - N(x) = 0]}{P[N(t) = 1]} \\
&= \frac{\lambda x e^{-\lambda x} e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}} \\
&= \frac{x}{t}.
\end{aligned}
$$

- If there are *k* arrivals in the interval [0, *t*], then the individual arrival times are distributed independently and uniformly in the interval.

Stationary Random Processes

- For many random processes: the nature of the randomness in the process does not change with time.
- An observation of the process in the time interval (t_0, t_1) exhibits the same type of random behavior as an observation in some other time interval $(t_0 + \tau, t_1 + \tau)$.

Mean & Variance of Stationary RP

- First-order CDF of a stationary random process must be independent of time, thus, mean and variance are constant and independent of time:

$$m_X(t) = E[X(t)] = m \quad \text{for all } t$$

$$\mathrm{VAR}[X(t)] = E[(X(t) - m)^2] = \sigma^2 \quad \text{for all } t$$

Correlation/Covariance of Stationary RP

- Second-order CDF of a stationary random process can depend only on the time difference between the samples and not on the particular time of the samples.

$$F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(0),X(t_2-t_1)}(x_1,x_2) \quad \text{for all } t_1,t_2$$

$$R_X(t_1,t_2) = R_X(t_2-t_1) \quad \text{for all } t_1,t_2$$

$$C_X(t_1,t_2) = C_X(t_2-t_1) \quad \text{for all } t_1,t_2$$

Stationary Random Processes (Cont'd)

- A (discrete-time/continuous-time) random process X(t) is **stationary** if the joint distribution of any set of samples does not depend on the placement of the time origin.
- Joint cdf of $X(t_1)$, $X(t_2)$,…, $X(t_k)$ is the same as that of $X(t_1 + \tau)$, $X(t_2 + \tau)$,…, $X(t_k + \tau)$:

$$F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k)$$

- Two processes X(t) and Y(t) are said to be **jointly stationary** if the joint CDFs of $X(t_1)$,…, $X(t_k)$ and $Y(t'_1)$,…, $Y(t'_j)$ do not depend on the placement of the time origin for all k and j and all choices of sampling times $t_1,\ldots,t_k$ and $t'_1,\ldots,t'_j$.

Example: iid Random Process

- Is the iid random process stationary?

Example 9.31 iid Random Process

Show that the iid random process is stationary.

The joint cdf for the samples at any k time instants, $t_1, \ldots, t_k$, is

$$F_{X(t_1),\ldots,X(t_k)}(x_1, x_2, \ldots, x_k) = F_X(x_1)F_X(x_2)\ldots F_X(x_k)$$
$$= F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1, \ldots, x_k),$$

for all $k, t_1, \ldots, t_k$. Thus Eq. (9.55) is satisfied, and so the iid random process is stationary.

Example: Sum Random Process

- Is the sum process a stationary process?

Example 9.32

Is the sum process a discrete-time stationary process?

The sum process is defined by $S_n = X_1 + X_2 + \cdots + X_n$, where the X_i are an iid sequence. The process has mean and variance

$$m_S(n) = nm \qquad \text{VAR}[S_n] = n\sigma^2,$$

where m and σ^2 are the mean and variance of the X_n. It can be seen that the mean and variance are not constant but grow linearly with the time index n. Therefore the sum process cannot be a stationary process.

Example: Random Telegraph Signal

- Show that the random telegraph signal is a stationary random process when $P[X(0) = \pm 1] = ½$.
- Show that X(t) settles into stationary behavior as $t \to \infty$ even if $P[X(0) = \pm 1] \neq ½$.
- Need to show that

$$P[X(t_1) = a_1, \ldots, X(t_k) = a_k] = P[X(t_1 + \tau) = a_1, \ldots, X(t_k + \tau) = a_k]$$

$$P[X(t_{j+1}) = a_{j+1} \mid X(t_j) = a_j] = \begin{cases} \frac{1}{2}\{1 + e^{-2\alpha(t_{j+1}-t_j)}\} & \text{if } a_j = a_{j+1} \\ \frac{1}{2}\{1 - e^{-2\alpha(t_{j+1}-t_j)}\} & \text{if } a_j \neq a_{j+1} \end{cases}$$

$$= P[X(t_{j+1} + \tau) = a_{j+1} \mid X(t_j + \tau) = a_j].$$

Example: Random Telegraph (Cont'd)

- If $P[X(0) = \underline{+}1] \neq ½$ then the two joint pmfs are not equal because $P[X(t_1) = a_1] \neq P[X(t_1 + \tau) = a_1]$.

$$
\begin{aligned}
P[X(t) = a] &= P[X(t) = a \mid X(0) = 1]1 \\
&= \begin{cases} \frac{1}{2}\{1 + e^{-2\alpha t}\} & \text{if } a = 1 \\ \frac{1}{2}\{1 - e^{-2\alpha t}\} & \text{if } a = -1. \end{cases}
\end{aligned}
$$

Lecture Summary

- Number of arrivals in a Poisson process has a Poisson distribution with mean λt.
- Poisson process has iid exponential interarrival times with mean λ.
- An arrival in the interval $[0,t]$ in a Poisson process is uniformly distributed in $[0,t]$.
- Poisson process has independent and stationary increments.

Lecture Summary (Cont'd)

- One or more random processes are strict-sense stationary if their joint distribution is independent of the choice of time origin.
- A strict-sense stationary process has a mean function that is constant and an autocovariance that depends only on the time difference.