CNL
Communications & Networking
Laboratory

ECE 528 – Introduction to Random Processes in ECE

Lecture 10: More on Multiple Random Variables

No Slides - This Lecture Was Given Using My iPAD – Summary Class Notes

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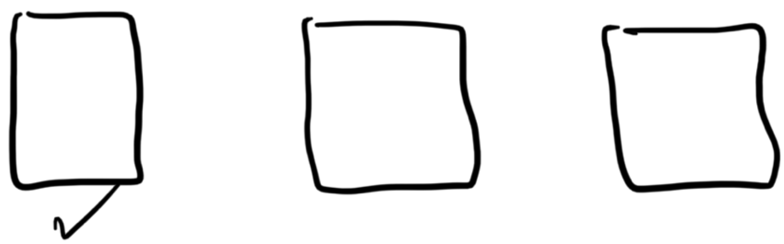
November 11, 2020

Lecture 10, Nov 11, 2020

Monty Hall Problem



2 Goats (G)
1 Ferrari (F)

Do Not Switch	Switch
$P(W) = 1/3$	$P(W) = 2/3$
$P(L) = 2/3$	$P(L) = 1/$

You Pick	Prize Door	No Sw	Sw
1	1	W	L
1	2	L	W
1	3	L	W
2	1	L	W
2	2	W	L
2	3	L	W
3	1	L	W
3	2		W

3 3 $\overset{L}{W}$ L

$\frac{}{\text{3 wins}}$ 6 wins

$1/3$ $2/3$

Pick door #1

Monty shows G behind #2

event A : ~~Ferrari~~ Car behind #1

event B : open door #2 showing G

$$Pr(A|B) = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)}$$

$$= \frac{1/2 \times 1/3}{1/3 \times 1/2 + 1/3 \times 0 + 1/3 \times 1}$$

$$= 1/3$$

More on Conditional Exp.

$$\underbrace{E[X \mid Y=y]}_{\text{a R.V.}}$$ exists for every y in Y.

$$E[E[X \mid Y]] = E[X]$$

Law of Iterated Expectati

Discrete RVs

$$E[E[X \mid Y]] = \sum_{y \in R_Y} \overbrace{E[X \mid Y=y]}\, p_Y(y)$$

Cond. EXP.

$$= \sum_{y \in R_Y} \underbrace{\sum_{x \in R_X} x\, p_{X \mid Y=y}(x)}\, p_Y(y)$$

$$\vdots$$

$$= E[X]$$

For Cont. RVs

$$E[E[X \mid Y]] = \int^{\infty} E[X \mid Y=y]\, f_Y(y)\, d$$

$$= \int_{-\infty}^{\infty} \overbrace{\int_{-\infty}^{\infty} x f_{X|Y}(x)\,dx}^{\text{Cond. Exp.}} \; f_Y(y)\,dy$$

$$= \;\vdots$$

$$= E[X]$$

Example

roll a dice, whatever number comes up toss a coin that many times.

Q: What is the expected number of Heads?

Please do it at home!

Conditional Variance

If X is a cont. RV

$$E[X^2 \mid A] = \int_A x^2 f(x \mid A)\,dx$$

$$Var[X \mid A] = E[X^2 \mid A] - [E[X \mid A$$

If X is a discr. RV

$$E[X^2 \mid A] = \sum_A x^2 p(x \mid A)$$

$$Var[X \mid A] = E[X^2 \mid A] - [E[X \mid$$

- Once X is observed,

  $E[Y \mid X]$ is a numb

- Before X is observed

  $E[Y \mid X]$ is a RV

- If X & Y are indep RVs,

then

$$E[Y|X] = E[Y]$$

and $\mathrm{Var}[Y|X] = \mathrm{Var}[Y]$

- If X & Y RVs, a & b cons

then

$$E[a + bY | X] = a + bE[Y|X$$

Proof

$$E[\underbrace{a + bY}_{g(y)} | X = x] = \int (a + by) \cdot f_{Y|X}(y|x)\,dy$$

$$= a \underbrace{\int f_{Y|X}(y|x)\,dy}_{=1} + b \underbrace{\int y f_{Y|X}(y|$$

$E[Y|X=$

$$= a + b\,E[Y|X]$$

Conditional Expectation

For X & Y disc. RV,

$$E[Y \mid X = x] = \sum_{y} y \cdot P_{Y|X}(y|x$$

cond. mean

Conditional variance

$$V(Y \mid X = x) = \sum (y - E[Y|X=x]$$

$$p_{Y|X}(y|x)$$

$$= E[Y^2 \mid X = x] - (E(Y|X=x$$

In general

$$E[h(Y) \mid X = x] = \sum_{y} h(y) p_{Y|X}(y$$

For X & Y continuous RV,

$$E[Y \mid X = x] = \int^{\infty} y \cdot f_{Y|X}(y|x) d$$

$$\mathrm{Var}[Y \mid X=x] = \int_{-\infty}^{\infty} [y - E(Y \mid X=x)]^2 f_{Y|X}(y \mid x)\,dy$$

$$= E[Y^2 \mid X=x] - (E[Y \mid X=$$

In General

$$E[h(Y) \mid X=x] = \int_y h(y)\, f_{Y|X}(y \mid$$

Example: X & Y cont. RVs
with joint pdf

$$f_{XY}(x,y) = \begin{cases} e^{-y} & y>0,\ 0 \leq x \leq 1 \\ 0 & \text{ow} \end{cases}$$

Q: Find $E($

$E(X \mid Y = 2) = ?$

Conditional Expectation

For RVs X & Y, we have

$$V(X) = V[E(X|Y)] + E[V(X|Y$$

Proof

(1) $Var[X] = E[X^2] - E[X]^2$

(2) $Var[X|Y=y] = E[X^2|Y=y] - E[X|Y=y]^2$

$\rightarrow Var[X]$

$= E[X^2] - E[X]^2$ use iterative Expect.

$= E[\underbrace{E[X^2|Y=y]}] - E[E[X|Y=$

$= E[\ Var[X|Y=y] \ \ldots$

$$= E\left[Var[X|Y=y] + E[X|Y=y] \right.$$

$$- E[E[X|Y=y]]^2$$

$$= E[Var[X|Y=y]] + E[E[X|Y=y$$

$$- E[E[X|Y=y]]^2$$

$$= \boxed{E[Var[X|Y=y]] + Var[E[X|Y=}$$

$$= Var[X]$$

Covariance & Correlation

RV X & Y

μ_X & μ_Y means

σ_X^2 & σ_Y^2 variances

$$Cov(X,Y) = \begin{cases} \sum_i \sum_j (x_i - \mu_X) \cdot (y_j - \mu_Y) \cdot \end{cases}$$

$f(x_i, y_j$

Continuous

discrete

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{(x-\mu_x)(y-\mu_x)f}(x,y)\,dx\,dy$$

If X & Y are indep.

$$Cov(X,Y) = E[X-\mu_x]E[Y-\mu_y]$$

$$= (\mu_x - \mu_x)(\mu_y - \mu_y) = 0$$

$$(X-\mu_x)(Y-\mu_y) = XY - X\mu_y - \mu_x Y + \mu_x \mu_y$$

$$Cov(X,Y) = E[XY] - \mu_x \mu_y$$

Ex rolling 2 fair dice

$\downarrow$

$Y \quad 1 \quad 0 \qquad \qquad | f_Y(y)$

x		1	
0	21/36	4/36	25/36
1	8/36	2/36	10/36
2	1/36	0	1/36
$f_Y(y)$	30/36	6/36	1

$$E[XY] = \sum_i \sum_j x_i y_j f(x_i, y_j)$$

$$= 1 \cdot 1 \cdot \frac{2}{36} = \frac{1}{18}$$

$$\mu_X = E[X] = \sum_i x_i f_X(x_i)$$

$$= 0 \cdot \frac{25}{36} + 1 \, \frac{10}{36} + 2 \cdot \frac{1}{3}$$

$$= \frac{1}{3}$$

$$\mu_Y = E[Y] = \sum_j y_j f_Y(y_j)$$

$= 0 \cdot 30/ \quad 6 \quad 1$

$= 0 \cdot \frac{30}{36} + 1 \cdot \frac{}{36} = \frac{1}{6}$

$$cov(X, Y) = \frac{1}{18} - \frac{1}{3} \cdot \frac{1}{6} = 0$$

X & Y are NOT independently

Cov = 0

$$f(2,1) = 0 \neq \frac{1}{36}\left(\frac{6}{36}\right) = f_X(2) f_Y(1)$$

Example

$$f_{XY}(x, y) = \begin{cases} x e^{-x(y+1)} & 0 < x < \infty \\ & 0 < y < \infty \\ 0 & \text{ow} \end{cases}$$

$\mu_X = 1$

μ_Y : doesn't exist

$\Rightarrow$ cov(X, Y) as undefined

$$E[XY] = \int_0^\infty \int_0^\infty xy\, e^{-x(y+1)}\, dx\, dy = 1$$

Example

$$f_{XY}(x, y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty \\ & 0 < y < \infty \\ 0 & \text{ow} \end{cases}$$

$$\mu_x = \mu_y = 1$$

Covariance ?

$$\underbrace{E[XY]}_{} - \mu_x \mu_y$$

$$E[XY] \overset{?}{=} \int_0^\infty \int_0^\infty xy\, e^{-(x+y)}\, dx\, dy =$$

$$E[XY] - \mu_x \mu_y = 1 - 1 = 0$$

$$X^* = \frac{X - \mu_x}{\sigma_x}$$

$$Y^* = \frac{Y - \mu_y}{\sigma_y}$$

$$\rho = \mathrm{Cov}(X^*, Y^*) = \frac{\mathrm{Cov}(X, Y)}{\sigma_x \sigma_y}$$

Correlation coeff.

1. $-1 \le \rho \le +1$
2. if X & Y indep then $\rho = 0$
3. X & Y satisfy linear identity $Y = aX + b$

iff $\rho = \pm 1$

X & Y RV

$S = X + Y$ Variance?

$$E[S] = \mu_x + \mu_y$$

$$Var[S] = Var[X] + Var[Y] \text{ if}$$

X & Y ind

In general

$$Var[S] = E\left[(S - E(S))^2\right]$$

$$= E\left[((x - \mu_x) + (y - \mu_y))^2\right]$$

$$= E\left[(x - \mu_x)^2\right] + E\left[(y - \mu_y)^2\right]$$

$$+ 2E\left[(x - \mu_x)(y - \mu_y)\right]$$

$$\boxed{Var[s] = Var[x] + Var[y] + 2\,Cov[x, y]}$$

Move on jointly Gaussian RUs

Change of coordinates from Cartesian to polar.

obtaining Rayleigh distribution

Rayleigh $\longrightarrow$ Exponential distribution