ECE 528 – Introduction to Random Processes in ECE

Lecture 13: Sum Process and Binomial Counting Process

Bijan Jabbari, PhD
Dept. of Electrical and Computer Eng.
George Mason University
Fairfax, VA 22030-4444, USA
bjabbari@gmu.edu
http://cnl.gmu.edu/bjabbari

November 18, 2020

Multiple Random Processes

- Very frequently we deal with multiple interrelated random processes.
- The joint behavior of $X(t)$ and $Y(t)$ is specified by all possible joint density functions of $X(t_1),\ldots, X(t_k), Y(t'_1),\ldots, Y(t'_j)$ for all k, j and all choices of $t_1,\ldots, t_k$ and $t'_1,\ldots, t'_j$.

$$f_{X(t_1),\ldots,X(t_k),Y(t_1'),\ldots,Y(t_j')}(x_1, x_2, \ldots, x_k, y_1, \ldots, y_j)$$

- $X(t)$ and $Y(t)$ are **independent** if the vector random variables $(X(t_1),\ldots, X(t_k))$ and $(Y(t'_1),\ldots, Y(t'_j))$ are independent for all k, j and all choices of $t_1,\ldots, t_k$ and $t'_1,\ldots, t'_j$.

$$f_{X(t_1),\ldots,X(t_k),Y(t_1'),\ldots,Y(t_{k'}')}(x_1, x_2, \ldots, x_k, y_1, \ldots, y_{k'}) = f_{\mathbf{X}}(\mathbf{x}) f_{\mathbf{Y}}(\mathbf{y})$$

Cross Moments of Random Processes

- **Cross-correlation $R_X(t_1, t_2)$** of X(t) and Y(t):

$$R_{X,Y}(t_1,t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1),Y(t_2)}(x,y)\, dx\, dy$$

- X(t) and Y(t) are **orthogonal** if

$$R_{X,Y}(t_1,t_2) = 0 \quad \text{for all } t_1 \text{ and } t_2$$

- **Cross-covariance $C_{X,Y}(t_1, t_2)$** is

$$\begin{aligned} C_{X,Y}(t_1,t_2) &= E\left[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}\right] \\ &= R_{X,Y}(t_1,t_2) - m_X(t_1)m_Y(t_2) \end{aligned}$$

- X(t) and Y(t) are **uncorrelated** if

$$C_{X,Y}(t_1,t_2) = 0 \quad \text{for all } t_1 \text{ and } t_2$$

Example: Sinusoids w Random Phase

- Let $X(t) = \cos(\omega t + \Theta)$ and $Y(t) = \sin(\omega t + \Theta)$, where Θ is a random variable uniformly distributed in $[-\pi, \pi]$.

- Find the cross-covariance of X(t) and Y(t).

Example: Signal Plus Noise

- Let Y(t) consists of a desired signal X(t) plus noise N(t):

$$Y(t) = X(t) + N(t)$$

- Find the cross-correlation between the observed signal and the desired signal assuming that X(t) and N(t) are independent random processes.

Random Processes with Special Properties

- Many important random processes are obtained through modeling process that builds complex models from simple components.
- Three important properties that occur frequently are:
  - Independent Identically Distributed Sequences of RVs
  - Independent Increments
  - Markov Dependence
- We will develop several important examples of random processes by building on IID sequences.

Independent Increments

- $X(t)$ is said to have **independent increments** if for any k and any choice of sampling instants $t_1 < t_2 < \ldots < t_k$, the random variables that represent the increments in an interval

$$X(t_2) - X(t_1),\ X(t_3) - X(t_2), \ldots,\ X(t_k) - X(t_{k-1})$$

are independent random variables.

- The joint probabilities and pdfs can then be expressed in terms of the probabilities and pdfs of the increments.

Markov Random Processes

- $X(t)$ is said to be **Markov** if the future of the process given the present is independent of the past; that is, for any k and any choice of sampling instants $t_1 < t_2 < \ldots < t_k$, and for any $x_1, x_2, \ldots, x_k$,

$$f_{X(t_k)}\left(x_k \mid X(t_{k-1}) = x_{k-1}, \ldots, X(t_1) = x_1\right) = f_{X(t_k)}\left(x_k \mid X(t_{k-1}) = x_{k-1}\right)$$

Continuous valued

$$P[X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}, \ldots, X(t_1) = x_1] = P[X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}]$$

Discrete valued

- Only the most recent value is relevant.

iid Random Process

- Let X_n be a discrete-time random process consisting of a sequence of iid RVs with common cdf $F_X(x)$, mean m, and variance σ^2. X_n is an **iid random process**.

$$F_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) = P[X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k]$$
$$= F_X(x_1)F_X(x_2)\ldots F_X(x_k)$$

- The mean of X_n is:

$$m_X(n) = E[X_n] = m \quad \text{for all } m$$

Properties of iid Random Process

- Autocovariance of iid process if $n_1 \neq n_2$:

$$\begin{aligned} C_X(n_1,n_2) &= E[(X_{n_1}-m)(X_{n_2}-m)] \\ &= E[(X_{n_1}-m)]E[(X_{n_2}-m)] = 0 \end{aligned}$$

- Autocovariance of iid process if $n_1 = n_2$:

$$C_X(n_1,n_2) = E[(X_n - m)^2] = \sigma^2$$ or $$C_X(n_1,n_2) = \sigma^2 \delta_{n_1,n_2}$$

- Autocorrelation of iid process:

$$R_X(n_1,n_2) = C_X(n_1,n_2) + m^2$$

Sum Process

- Many interesting random processes are obtained as the sum of a sequence of iid random variables, X_1, X_2,… (where $S_0 = 0$):

$$S_n = X_1 + X_2 + \ldots + X_n = S_{n-1} + X_n \qquad n = 1, 2, \ldots$$

- S_n is the **sum process**.

- S_n is dependent on the "past," S_1, S_2,…, S_{n-2}, only through S_{n-1}.
- S_n is a Markov process.

Increments in Sum Process

- S_n has **independent increments**.
- Consider intervals: $n_0 < n \le n_1$ and $n_2 < n \le n_3$, where $n_1 \le n_2$.

$$S_{n_1} - S_{n_0} = X_{n_0+1} + \cdots + X_{n_1}$$
$$S_{n_3} - S_{n_2} = X_{n_2+1} + \cdots + X_{n_3}.$$

- Note that $P[S_{n'} - S_n = y] = P[S_{n'-n} = y]$
- S_n has **stationary increments** that depend only on n' – n, not on the absolute time instants.

Joint pmf of Sum Process

- The joint pmf of S_n at times n_1, n_2, and n_3:

$$P[S_{n_1} = y_1, S_{n_2} = y_2, S_{n_3} = y_3]$$

Find the joint pmf for the binomial counting process at times n_1 and n_2. Find the probability that $P[S_{n_1} = 0, S_{n_2} = n_2 - n_1]$, that is, the first n_1 trials are failures and the remaining trials are all successes.

Following the above approach we have

$$\begin{aligned} P[S_{n_1} = y_1, S_{n_2} = y_2] &= P[S_{n_1} = y_1]P[S_{n_2} - S_{n_1} = y_2 - y_1] \\ &= \binom{n_2 - n_1}{y_2 - y_1} p^{y_2 - y_1}(1 - p)^{n_2 - n_1 - y_2 + y_1} \binom{n_1}{y_1} p^{y_1}(1 - p)^{n_1 - y_1} \\ &= \binom{n_2 - n_1}{y_2 - y_1}\binom{n_1}{y_1} p^{y_2}(1 - p)^{n_2 - y_2}. \end{aligned}$$

The requested probability is then:

$$P[S_{n_1} = 0, S_{n_2} = n_2 - n_1] = \binom{n_2 - n_1}{n_2 - n_1}\binom{n_1}{0} p^{n_2 - n_1}(1 - p)^{n_1} = p^{n_2 - n_1}(1 - p)^{n_1}$$

which is what we would obtain from a direct calculation for Bernoulli trials.

$$\begin{aligned} &P[S_{n_1} = y_1, S_{n_2} = y_2, \ldots, S_{n_k} = y_k] = \\ &\qquad P[S_{n_1} = y_1]P[S_{n_2 - n_1} = y_2 - y_1] \cdots P[S_{n_k - n_{k-1}} = y_k - y_{k-1}] \end{aligned}$$

Mean & Autocovariance of Sum Process

- S_n is the sum of n iid RVs, so:

$$m_S(n) = E[S_n] = nE[X] = nm$$
$$\mathrm{VAR}[S_n] = n\mathrm{VAR}[X] = n\sigma^2$$

- Autocovariance of S_n is:

$$C_S(n,k) =$$

$$\begin{aligned} C_S(n,k) &= E[(S_n - nm)^2] + E[(S_n - nm)]E[(S_k - S_n - (k-n)m)] \\ &= E[(S_n - nm)^2] \\ &= \mathrm{VAR}[S_n] = n\sigma^2, \end{aligned}$$

Lecture Summary

- Multiple random processes are specified by the joint probabilities of samples at arbitrary times and by cross-moment functions.
- The independent increments and Markov properties simplify the specification of joint probabilities of random processes.
- The sum process of an iid sequence of random variables has independent increments.
- The Binomial counting process is an important example of a sum process.