CALIFORNIA STATE UNIVERSITY,NORTHRIDGE

$E(s^2)$-Optimal Supersaturated Designs for $N = 14$

A thesis submitted in partial fulfillment of the requirements
For the Degree of Master of Science in Mathematics
By
Liyuan Zhu

May 2022

The thesis of Liyuan Zhu is approved:

Professor Neubauer

Date

Professor Mojirsheibani

Date

Professor Domagni, Chair

Date

California State University Northridge

Table of Content

Abstract

$E(s^2)$-Optimal Supersaturated Designs for $N = 14$

By

Liyuan Zhu

Master of Science in Mathematics

We provide a list of $E(s^2)$-optimal supersaturated designs for $N = 14$ and $14 \leq m \leq 26$.In the process we revisit and provide simpler proofs for some known result and point out a combinatorial restriction that has led to an improved lower bound.

1 Introduction

A supersaturated design is a design that accommodate more factors than runs an experimenter can afford. It is a useful choice for screening experiments. When the number of factors exceeds its the run size, a supersaturated design can be very cost-effective in the preliminary stage of scientific investigations. Compared to traditional fractional factorial design, a supersaturated design can effectively reduce the number of runs without compromising the validity of the experiment. Throughout this article, we consider a supersaturated X with m factors and an even number of runs N. We assume that each factor has two levels, coded +1 and 1 for high level and low level respectively. Obviously, $m > N - 1$ and each column x of X is balanced in the sense that they have exactly $\frac{N}{2}$ entries of 1 and $\frac{N}{2}$ entries of -1.In addition any two columns of X cannot be fully aliased,that is for two different columns x and y of X,we have $x \neq y$ and $x \neq -y$. Under these assumptions, for an even number N it's not difficult to draw conclusion that the largest supersaturated design one can construct has $m = \frac{1}{2}\binom{N}{N/2}$ factors. Given our design matrix X as defined above, the information matrix $X^T X$ is not invertible. We only have N degrees of freedom and it is impossible to obtain the unbiased estimation of all the m factors when m is larger than N. Under the *sparsity of effect principle* only a small portion of the m factors is active and the goal of the experimenter is to conduct a preliminary supersaturated design to identify the factors that are important. Since traditional techniques such as OLS cannot be used to obtain the unbiased estimation of the parameters of the m factors. Various existing and novel techniques have been used to analyse a supersaturated design. One of the method one can use is the the well-known stepwise forward selection. In section we give a real world example of supersaturated designs where the stepwise forward selection technique was used to identify the important factors out of 24 predictor variables.

Since the aforementioned criteria cannot be used to select the optimal supersaturated design matrix, Booth and Cox(1962) proposed two critera namely the minmax and the

$E(s^2)$-optimal criteria for selecting a supersaturated design. In this article we will mainly focus on the $E(s^2)$-optimal criterion. For an $N \times m$ supersaturated design matrix X, denote $X^T X = (s_{ij})$. Then the $E(s^2)$ value of X is given by $s^2(X) = \sum_{i<j} s_{ij}^2 / \binom{m}{2}$. Note that if all the columns of X are orthogonal, then $s^2(X)$ is equal to zero. Thus, the $s^2(X)$ is a measure of the departure from orthogonality of the pairwise correlation of the factors average through all the pair of factors.

The $E(s^2)$-optimal criterion find design matrix that minimize the $s^2(X)$ value for given N and m. Concretely, let $ssd(N, m)$ be the set of all supersaturated design matrices for given N and m. Then we have

$$E(s^2) = \min\left\{ s^2(X) \quad | \quad X \in ssd(N, m) \right\} \tag{1}$$

For given N and m, Nguyen [9] and Tang and Wu [11] independently derived a global lower bound for $E(s^2)$ Which is given by:

$$E(s^2) \geq \frac{m - N + 1}{(m-1)(N-1)} N^2 \tag{2}$$

The lower bound in Equation (2) is achievable only in two cases. The first case is when $N \equiv 0 \pmod 4$, m is a multiple of $N-1$ and a BIBD$(N-1, q(N-1), N/2-1)$ exists, where $N-1$, $m = q(N-1)$ and $N/2-1$ represent respectively the number of treatments, the number of blocks, and the block size. The second case is when $N \equiv 0 \pmod 4$, m is an even multiple of $N-1$ and a BIBD$(N-1, q(N-1), N/2-1)$ exists, where $N-1$, $m = q(N-1)$ and $N/2-1$ represent respectively the number of treatments, the number of blocks, and the block size.

Bulutoglu and Cheng [2] proved general lower bounds for $E(s^2)$ that outperformed Nguyen [9] and Tang and Wu [11] bound given in Equation 2. For $N = 10$, Bulutoglu and Cheng [2] bounds appear to be tight for vast majority of the given m. In Domagni and Bauer [?]. All the $E(s^2)$-optimal design were given for $N = 10$ and any value of m.

Roughly 90% of those designs have an $E(s^2)$ that attain the lower bound in Bulutoglu and Cheng [2].

In this article, we focus on finding $E(s^2)$-optimal supersaturated designs for $N = 14$ and and given m for $14 \leq m \leq 26$. Our strategy is to design a computer algorithm to search for supersaturated designs that attain the lower bounds of Bulutoglu and Cheng [2]. In many cases we are able to find $E(s^2)$ designs that reach the lower bounds and in other cases we showed why the lower bound is not achievable. In Section we constraint on $s^2(X)$ and a new bound.

2 A real world application of Supersaturated Designs

Table 1: Half Fractions of William"s(1968) Data

	Factor																								
Run	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	Response
1	+	+	+	−	−	−	+	+	+	+	+	−	+	−	−	+	+	−	−	+	−	−	−	+	133
2	+	−	−	−	−	−	+	+	+	−	−	−	+	+	+	+	−	+	−	−	+	+	−	−	62
3	+	+	−	+	+	−	−	−	−	+	−	+	+	+	+	+	+	−	−	−	−	+	+	−	45
4	+	+	−	+	−	+	−	−	−	+	+	−	+	−	+	+	−	+	+	+	−	−	−	−	52
5	−	−	+	+	+	+	−	+	+	−	−	−	+	−	+	+	+	−	−	+	−	+	+	+	56
6	−	−	+	+	+	+	+	−	+	+	+	−	−	+	+	−	+	+	+	+	+	+	−	−	47
7	−	−	−	−	+	−	−	+	−	+	−	+	+	+	−	+	+	+	+	+	+	−	−	+	88
8	−	+	+	−	−	+	−	+	−	+	−	−	−	−	−	−	−	−	+	−	+	+	+	−	193
9	−	−	−	−	−	+	+	−	−	−	+	+	−	−	+	−	+	+	−	−	−	−	+	+	32
10	+	+	+	+	−	+	+	+	−	−	−	+	−	+	+	−	+	−	+	−	+	−	−	+	53
11	−	+	−	+	+	−	−	+	+	−	+	−	−	+	−	−	−	+	+	−	−	−	+	+	276
12	+	−	−	−	+	+	+	−	+	+	+	+	+	−	−	+	−	−	+	−	+	+	+	+	145
13	+	+	+	+	+	−	+	−	+	−	−	+	−	−	−	−	−	+	−	+	+	−	+	−	130
14	−	−	+	−	−	−	−	−	−	−	+	+	−	+	−	−	−	−	−	+	−	+	−	−	127

In table we give an example of a supersaturated design with $N = 14$ runs and $m = 24$ factors. The corresponding response values were collected for each run. This example was first created by Williams [12] and was reused in Lin [8]. The original experiment involved 24 predictor variables with 28 runs.According to the relative size of mean squared terms,we can get a conclusion that factors 15, 20 and 17 were significantly important. Factors 4, 22, 14 and 8 were moderate. But when it comes to other experimental results,this conclusion doesn't hold.Combining all kind of results and data,they found that factors 15,10,20 and 4 are most significant.To clarify this further,Lin used the half fraction method:choosing and removing a brunching column whose signs equal $+1$ or -1,for

Table 2: Step-wise(forward) Anova

Step	Df	Deviance	Resid. Df	Resid. Dev	AIC
			13	62,754.360	119.711
+ X15	-1	39,644.640	12	23,109.710	107.725
+ X12	-1	6,802.149	11	16,307.570	104.845
+ X20	-1	8,184.008	10	8,123.557	97.089
+ X4	-1	5,284.556	9	2,839.001	84.370
+ X10	-1	1,146.452	8	1,692.549	79.129
+ X11	-1	857.641	7	834.907	71.236
+ X7	-1	724.332	6	110.575	44.933
+ X13	-1	76.521	5	34.054	30.444
+ X14	-1	19.515	4	14.538	20.528
+ X1	-1	11.939	3	2.600	-1.571
+ X17	-1	2.154	2	0.446	-24.264
+ X2	-1	0.396	1	0.050	-52.979
+ X5	-1	0.050	0	0	-*Inf*

Table 3: Step-wise (forward) Selection

	Entering variables						
Step	***15***	***12***	***20***	***4***	***10***	σ	R^2
1	**−53.2**					**43.9**	**63%**
	(−4.54)						
2	**−56.4**	**−22.3**				**38.5**	**74%**
	(−5.42)	**(−2.14)**					
3	**60.5**	**−26.4**	**24.8**			**28.5**	**87%**
	(−7.75)	**(−3.38)**	**(−3.17)**				
4	**−70.5**	**−25.3**	**−29.2**	**22.1**		**17.8**	**95%**
	(−12.96)	**(−5.19)**	**(−5.86)**	**(4.09)**			
5	**−71.3**	**−26.8**	**−28.0**	**20.7**	**−9.4**	**14.5**	**97%**
	(−15.96)	**(−6.63)**	**(−6.80)**	**(4.64)**	**(−2.33)**		

NOTE: The numbers given in the table are estimated effects and their *t* ratios. The constant term is 102.8 at all steps.

example,$(+-++-+-+++--+---+----++++--+)$,since $(24 < 28-2)$ factors are involved,so we can remove a certain column from the complete design.Then the $(28-2)$ columns other than the branching column will form a supersaturated design for 24 factors in 14 runs.See table 1 for the supersaturated design and corresponding observations. Next,Lin used the step-wise selection method to identify the important effects.B y AIC criterion The most important factors turned out to be the factor 15 with AIC 107.725,and it followed by the factor 12 with AIC 104.845,factor 20 with AIC 97.089, factor 4 with AIC84.370 and factor 10with AIC 79.129,see Table 2.The five important factors are given in table 3.see(Lin table 4) with $R^2 = 0.973$ which included all important effects mentioned before. And with this method, only 14 observations were involved.

3 $E(s^2)$-Optimal Supersaturated Designs for $N = 14$ and $m \leq 28$

In this section our goal is to find the $E(s^2)$-Supersaturated Designs for $N = 14$ runs and each m for $m \leq 28$. Our method of search consists of designing an efficient computer algorithm that searches for the $E(s^2)$-optimal supersaturated designs. Key to our algorithm is the $E(s^2)$ lower bounds found in Theorem 3.1 of Bulutoglu [2]. The idea is to have our computer algorithm search for a supersaturated design that has an $E(s^2)$ value equals to the lower bound in Theorem 3.1. It turns out that in the majority of the cases our algorithm was able to find the $E(s^2)$-optimal supersaturated designs for $N = 14$. However for some cases the lower bounds in Theorem 3.1 is not achievable. In those cases we prove why the lower bounds in Theorem 3.1 were not achievable and we give the correct value of the $E(s^2)$-optimal values. For $N = 14$, Our algorithm has allowed us to find $E(s^2)$-optimal supersaturated designs for $m \in \{14, 15, 17, 18, 19, 20, 23, 24, 25, 26\}$. The $E(s^2)$-values for these values of m correspond exactly to the lower bounds in Theorem 3.1. Before we discuss about the cases for $m = 16, 21$ and 22 we give the following the following lemma and theorems that are very useful.

Lemma 3.1. *For $S = (s_{ij}) = X^T X$ we have $s_{ij} \in \Big\{ \pm (N - 4k) : k = 1, 2, \cdots, \lfloor \frac{N}{4} \rfloor \Big\}$. ie, $s_{ij} \equiv N$ (mod 4) for $1 \leq i, j \leq m$.*

This constraint on the modulus of the off-diagonal entries of $X^T X$ limits the values that $s^2(X)$ takes on.

Theorem 3.2. *Let $N = 4t + 2 = 2k$. Let $X \in ssd(m, N)$ and k_i be the number of off-diagonal entries of $X^T X$ of absolute value $2 + 4i$. Then*

$$s^2(X) = 4 + \frac{64}{m(m-1)} \sum_{i=1}^{t} \frac{i(i+1)}{2} \frac{k_i}{2}. \tag{3}$$

From Theorem 3.2, it is clear the for given N and m and $X \in ssd(m, N)$ the $s^2(X)$ is necessarily on the form $4 + \frac{64k}{m(m-1)}$. For $N = 14$ and $m = 21$, the lower bound in Theorem 3.1 is 6.6666 which cannot be written on the aforementioned form. The smallest value that is larger than 6.6666 and can be written in the form of Thereom3.2 is 6.7429. Our computer search for $N = 14$ and $m = 21$ found a supersaturated design with $E(s^2) = 6.7429$ which is optimal.

Table 4: N=14,m=14,E(s^2) = 4

1	-1	1	1	1	1	1	1	1	1	1	1	1	1
-1	-1	1	1	-1	-1	-1	-1	1	1	-1	1	1	-1
-1	1	-1	1	1	-1	1	1	1	1	-1	-1	1	1
1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	-1	1
-1	1	1	-1	1	1	1	1	1	-1	-1	1	-1	-1
-1	-1	-1	-1	-1	1	1	-1	1	1	1	-1	-1	-1
1	1	1	1	1	-1	-1	-1	1	-1	1	-1	1	-1
1	1	-1	-1	-1	1	-1	-1	-1	1	-1	1	1	-1
-1	1	-1	1	1	1	-1	-1	-1	1	1	1	-1	1
1	-1	-1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
1	1	1	1	-1	1	1	-1	-1	-1	-1	-1	-1	1
-1	-1	-1	-1	1	-1	1	-1	-1	-1	-1	1	1	1
-1	-1	1	-1	-1	1	-1	1	-1	-1	1	-1	1	1
1	-1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	-1

Table 5: N=14,m=15,E(s^2) = 4

1	-1	1	1	1	-1	-1	1	-1	-1	-1	1	-1	1	-1
-1	1	-1	1	-1	-1	-1	1	-1	1	-1	-1	1	-1	1
-1	1	1	1	1	1	-1	1	1	-1	1	1	-1	-1	1
-1	-1	1	1	1	-1	1	-1	1	1	1	-1	-1	1	1
1	-1	-1	1	1	1	-1	-1	1	1	-1	1	1	-1	-1
1	1	1	1	-1	-1	1	-1	-1	-1	1	-1	1	-1	-1
1	1	-1	1	-1	1	1	-1	1	-1	-1	-1	-1	1	1
1	1	-1	-1	1	1	1	1	-1	1	1	1	1	1	1
-1	-1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	-1
1	-1	1	-1	-1	1	-1	1	1	1	1	-1	1	1	-1
-1	1	1	-1	-1	-1	1	1	1	1	-1	1	-1	-1	-1
-1	1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	1	1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	1
-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1

Table 6: N=14,m=16,E(s^2) = 4.53

1	1	-1	1	1	1	1	1	-1	-1	1	-1	1	1	-1	-1
-1	1	1	1	-1	1	1	-1	1	1	1	1	1	1	1	1
-1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	-1	1	1
-1	-1	-1	-1	1	1	-1	1	-1	1	-1	1	1	1	1	-1
1	1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	1	-1
-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	-1	1
1	-1	-1	-1	-1	-1	1	-1	-1	1	1	1	1	-1	-1	1
1	1	1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	1
1	-1	1	1	1	1	-1	1	1	1	1	-1	-1	-1	-1	1
1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1	1	-1
-1	-1	1	1	1	-1	1	1	-1	-1	1	1	-1	-1	1	-1
-1	1	-1	-1	1	1	1	-1	1	-1	-1	1	-1	-1	-1	1
1	1	-1	1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	1	-1
-1	-1	-1	1	-1	-1	1	1	1	1	-1	-1	-1	1	-1	-1

Table 7: x

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
-1	1	-1	-1	1	1	1	1	1	-1	-1	-1	1	-1	1	-1
1	-1	-1	-1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1	1
1	-1	1	-1	-1	1	-1	-1	-1	-1	1	1	-1	1	-1	-1
1	-1	-1	-1	1	-1	-1	1	-1	1	1	1	1	1	1	-1
-1	-1	1	1	-1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1
-1	-1	-1	1	-1	1	1	-1	1	1	1	1	-1	-1	1	1
1	1	1	1	1	-1	1	-1	1	-1	-1	1	1	-1	1	-1
-1	-1	1	1	1	1	-1	1	1	-1	-1	-1	-1	1	-1	1
-1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	1	1	-1	1
1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1	-1
-1	1	1	1	-1	-1	1	-1	1	1	1	1	-1	-1	1	1
-1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1	-1
1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

Table 8: $X^T X$

14	-2	2	-2	2	-2	-2	-2	-2	2	-2	2	2	6	-2	-6
-2	14	2	-2	2	-2	6	2	2	2	-2	2	6	-2	6	-2
2	2	14	6	-2	-2	-2	2	2	-2	2	2	2	2	-2	2
-2	-2	6	14	2	2	2	-2	6	-2	2	2	-2	-6	2	6
2	2	-2	2	14	6	2	6	2	-6	-2	2	2	-2	6	-6
-2	-2	-2	2	6	14	2	2	2	-6	2	2	-6	-2	2	-2
-2	6	-2	2	2	2	14	-2	10	2	2	6	-2	-6	10	2
-2	2	2	-2	6	2	-2	14	-2	-2	2	-2	6	2	2	-2
-2	2	2	6	2	2	10	-2	14	2	-2	2	-2	-2	6	6
2	2	-2	-2	-6	-6	2	-2	2	14	2	2	2	6	2	6
-2	-2	2	2	-2	2	2	2	-2	2	14	10	-2	-2	6	2
2	2	2	2	2	2	6	-2	2	2	10	14	-2	-2	10	-2
2	6	2	-2	2	-6	-2	6	-2	2	-2	-2	14	2	2	-2
6	-2	2	-6	-2	-2	-6	2	-2	6	-2	-2	2	14	-6	2
-2	6	-2	2	6	2	10	2	6	2	6	10	2	-6	14	-2
-6	-2	2	6	-6	-2	2	-2	6	6	2	-2	-2	2	-2	14

Table 9: N=14,m=17,$E(s^2)$ = 4.9412

-1	1	1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1
1	1	-1	1	1	1	-1	1	1	1	-1	-1	-1	1	1	1	1
1	1	-1	-1	1	-1	1	-1	-1	1	-1	1	1	-1	1	1	1
-1	-1	1	1	1	1	1	1	1	-1	-1	1	1	1	1	1	-1
1	-1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	1	-1	-1
-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	1	1	-1	-1	1
-1	1	1	1	1	-1	-1	1	-1	1	1	-1	1	-1	1	-1	-1
1	-1	1	1	-1	1	1	-1	1	1	1	-1	1	-1	-1	-1	1
1	1	-1	1	-1	-1	1	-1	-1	-1	1	-1	1	1	-1	1	-1
-1	-1	-1	-1	-1	1	-1	1	-1	1	1	1	1	1	-1	1	1
1	-1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	1	1
-1	1	-1	1	-1	-1	1	1	1	1	-1	1	-1	-1	-1	-1	-1
-1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1
1	-1	1	-1	-1	-1	-1	-1	1	1	1	1	-1	1	1	-1	-1

Table 10: N=14,m=18,E(s^2) = 5.6732

1	1	1	1	1	-1	1	1	1	1	1	1	1	-1	1	1	1	1
1	1	-1	1	-1	-1	1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1
-1	-1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1
-1	1	1	-1	1	1	1	-1	1	1	-1	1	-1	1	1	1	-1	-1
-1	-1	-1	-1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	1
1	-1	-1	-1	-1	1	-1	1	1	1	-1	-1	1	1	-1	1	1	-1
1	-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	-1	1	-1	-1
-1	-1	-1	1	-1	1	1	1	-1	-1	1	1	1	1	1	-1	1	-1
1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	-1	-1	1	-1	1	-1
-1	1	-1	-1	1	-1	1	1	-1	-1	-1	-1	1	-1	-1	-1	1	-1
1	1	1	1	1	1	-1	1	-1	-1	-1	-1	1	1	1	-1	-1	1
1	1	-1	-1	-1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	1
-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	1	-1	-1	-1	-1	1
-1	1	1	1	-1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	-1

Table 11: N=14,m=19,E(s^2) = 6.0585

-1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	1	1	1	-1	1	-1	1
-1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	-1	1	1	-1	1	-1
1	1	1	1	1	-1	-1	-1	1	1	-1	-1	1	-1	1	-1	1	1	1
1	-1	1	-1	-1	1	1	-1	1	1	-1	1	1	1	1	1	-1	1	-1
-1	1	1	-1	-1	1	-1	-1	-1	1	1	1	-1	-1	1	1	1	-1	1
1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	1	1	-1	-1	1	1
1	-1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	1	1	-1	1	1	-1
-1	-1	-1	1	1	1	1	-1	1	-1	1	-1	1	1	-1	1	1	1	1
1	-1	1	1	1	1	-1	1	-1	-1	-1	1	1	-1	-1	1	-1	-1	1
1	1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	-1	-1	-1	-1
-1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	-1	-1	-1	-1	-1
-1	1	-1	1	1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	1
-1	1	1	1	-1	1	-1	1	1	-1	-1	-1	-1	1	-1	-1	1	-1	-1

Table 12: N=14,m=20,E(s^2) = 6.3579

-1	-1	1	1	-1	1	1	-1	1	-1	-1	1	1	1	-1	-1	1	-1	-1	1
1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	1	-1	-1	-1	-1	1	-1	1
1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	-1	1	1	-1	1	-1	1	1
-1	-1	1	1	1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	1	1
-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1
1	-1	-1	-1	1	-1	1	1	1	1	-1	1	-1	1	-1	1	1	1	-1	1
-1	1	1	1	1	-1	-1	1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1
-1	1	1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	-1	1	1	-1
1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	1	-1	1	-1	-1	1	-1
1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	1	1	1	-1	1	-1
1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1
-1	-1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1
1	-1	-1	1	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	-1	-1	-1	-1
-1	1	-1	1	1	1	-1	-1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	-1

Table 13: N=14,m=21,E(s^2) = 6.7429

1	1	-1	1	1	1	-1	-1	-1	1	-1	1	1	-1	1	1	1	1	1	1	-1
1	-1	1	-1	-1	1	1	1	-1	-1	1	-1	1	-1	1	-1	-1	1	1	1	-1
-1	1	1	1	-1	1	1	1	1	1	-1	-1	-1	1	-1	1	-1	1	1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	1	1	-1	-1	1	-1	-1
-1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	1	-1	1
1	1	-1	-1	-1	-1	1	-1	1	1	-1	-1	1	-1	-1	-1	1	-1	1	-1	1
-1	-1	1	-1	1	-1	-1	1	1	-1	-1	-1	1	1	1	1	1	-1	1	1	-1
-1	-1	1	-1	1	1	1	-1	1	1	1	1	1	1	1	-1	1	1	-1	-1	1
1	1	1	1	1	-1	1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1	1
1	-1	-1	1	-1	1	-1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1
-1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1
-1	1	-1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	-1
1	-1	-1	-1	1	1	1	1	-1	-1	-1	1	-1	1	-1	1	1	-1	-1	-1	1
1	1	1	1	-1	-1	-1	1	1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1

Table 14: N=14,m=22,E(s^2) = 7.0476

1	1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	1	-1
1	-1	1	-1	1	1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	1	1
1	-1	1	1	1	1	1	1	1	-1	1	-1	1	1	1	-1	1	-1	-1	-1	-1	1
1	-1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	1	-1	-1	1	-1	1	1	1
1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	1	-1	1	1	-1	-1	1
1	-1	-1	-1	1	-1	-1	-1	-1	1	1	-1	1	1	-1	1	1	1	1	-1	-1	-1
1	1	1	-1	-1	1	1	1	-1	-1	-1	1	1	1	1	-1	1	1	1	1	-1	-1
-1	1	1	1	-1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	1	1	-1	-1	1	-1	-1
-1	1	1	-1	1	-1	1	1	-1	-1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1	1
-1	1	-1	1	1	-1	1	1	1	1	1	1	-1	1	1	1	1	1	1	1	1	1
-1	1	-1	-1	1	1	-1	1	1	-1	1	1	-1	-1	-1	-1	1	-1	1	-1	1	-1
-1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	1
-1	-1	-1	1	-1	1	1	1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	1	-1	-1
-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1	1	1	1	1	-1

Table 15: N=14,m=23,E(s^2) = 7.4150

-1	-1	1	1	-1	1	-1	-1	-1	1	1	1	1	1	1	-1	1	1	1	1	1	1	1	1
-1	1	1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	-1	-1	-1	-1
1	-1	1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	-1	1	1	1	1	1
1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	-1	-1	1	-1	-1	1	-1
-1	1	-1	1	1	-1	1	-1	1	-1	1	1	1	1	1	1	-1	1	-1	-1	-1	1	-1	-1
1	1	-1	-1	1	1	-1	1	1	1	-1	-1	-1	1	1	1	-1	1	1	1	-1	-1	1	1
1	-1	1	-1	-1	1	1	1	-1	1	-1	-1	1	-1	1	1	1	1	1	-1	-1	1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	1
1	-1	1	1	1	1	1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	-1	-1	1
-1	1	1	-1	-1	-1	1	-1	1	1	-1	1	1	1	-1	1	-1	-1	1	-1	1	-1	1	1
1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	-1	-1	-1	1	1	-1	1	1	-1	1	-1	-1
1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	1	1	1	-1	-1	1	1	-1	-1	1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	-1	1	1	-1	1
-1	1	-1	1	1	1	1	1	1	1	1	-1	1	-1	-1	-1	1	-1	-1	1	1	1	1	-1

Table 16: N=14,m=24,$E(s^2) = 7.8261$

1	1	1	-1	-1	1	1	-1	1	1	1	-1	1	1	1	1	1	-1	-1	-1	-1	1	-1	1
1	-1	-1	1	1	-1	1	1	1	1	1	1	1	1	1	-1	1	1	1	1	1	1	1	1
-1	-1	1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	-1	1	-1	-1	1	1
-1	-1	1	1	1	1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	1	-1	-1	1	1	1	1
1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	-1	1	-1	1	-1	1	1	-1	1
1	-1	1	-1	1	1	1	1	-1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	-1	1
-1	1	-1	-1	-1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1
1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	-1	1	-1
-1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	1	-1	-1	1	1	-1
-1	1	1	-1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	1	-1	-1	1	1	1	1	-1
-1	-1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	-1	1	1	1	1	-1	-1	1	-1	-1	-1
1	1	1	-1	1	-1	-1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	1	-1	1	1	-1
1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	-1
-1	1	1	1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	-1

Table 17: N=14,m=25,$E(S^2) = 7.8400$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	-1	-1
-1	-1	1	-1	1	1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	-1	1	-1
-1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	1	1
-1	1	-1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1
-1	-1	-1	-1	-1	1	-1	-1	1	1	-1	1	1	-1	1	-1	1	1	-1	-1	1	1	1	-1	-1
-1	1	1	1	1	-1	-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	-1
-1	1	-1	-1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	-1	1	1
-1	-1	1	1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1	1
1	1	1	-1	-1	1	1	-1	-1	-1	1	1	1	1	-1	1	-1	-1	-1	-1	1	1	-1	-1	-1
1	-1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	-1	-1	1	-1	-1	1	1	-1	1	1	-1	1
1	-1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	-1
1	1	-1	1	-1	1	1	-1	1	1	-1	-1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	1	-1
1	1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	1	1	1	1	1

Table 18: N=14,m=26,$E(S^2)$ = 7.8400

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	-1	-1	1
-1	-1	1	-1	1	1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	-1	1	-1	1
-1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1
-1	1	-1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
-1	-1	-1	-1	-1	1	-1	-1	1	1	-1	1	1	-1	1	-1	1	1	-1	-1	1	1	1	-1	-1	1
-1	1	1	1	1	-1	-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	-1	-1
-1	1	-1	-1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	-1	1	1	1
-1	-1	1	1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1	1	1
1	1	1	-1	-1	1	1	-1	-1	-1	1	1	1	1	-1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1
1	-1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	-1	-1	1	-1	-1	1	1	-1	1	1	-1	1	-1
1	-1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	-1	-1
1	1	-1	1	-1	1	1	-1	1	1	-1	-1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	1	-1	-1
1	1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	1	1	1	1	1	1

4 The proofs

References

[1] BOOTH, K. H. V. and COX, D. R. (1962). Some systematic supersaturated designs. *Technometrics 4* 489–495.

[2] BULUTOGLU, D.A. and CHENG, C.-S. (2004). Construction of $E(s^2)$-optimal supersaturated designs. *The Annals of Statistics* 4 1662-1678.

[3] Butler, N.A., Mead, R., Eskridge, K.M., Gilmour, S.G., A general method of constructing $E(s^2($-optimal supersaturated designs, 63, 2001 621-632.

[4] Das, A., Dey A., Chan, L-Y., Chatterjee, K. (2008). *Journal of Statistical Planning and Inference* 13 3749-3757.

[5] CHENG, C.-S. (1997). $E(s^2)$-Optimal Supersaturated designs. *Statistica Sinica* 7 929-939.

[6] Eskridge, K.M., Gilmour, S.G., Meads, R., Butler, N.A., Travnicek, D.A. Large supersaturated designs, *J. Statistical Computation and Simulation*—, 74 (7) 2004 525-542

[7] DOMAGNI, K. F. and HEDAYAT, A.S. and SINHA, B. K. On the nature of saturated 2^k-factorial designs for unbiased estimation of non-negligible parameters. *Journal of Statistical Planning and Inference* 211 33-40.

[8] Lin, Dennis K. J. A New Class of Supersaturated Designs *Technometrics* Feb., 1993, Vol. 35, No. 1 (Feb., 1993), pp. 28-31

[9] NGUYEN, N.-K. (1996). An algorithmic approach to constructing supersaturated designs. *Technometrics* 38 69–73.

[10] Ryan, K.J., Bulutoglu, D.A., $E(s^2)$-optimal supersaturated designs with good minimax properties, *J. Statistical Planning and Inference*, 137 2007 2250-2562.

[11] TANG, B. and WU, C. F. J. (1997). A method for constructing supersaturated designs and its E(s2)-optimality. *Canad. J. Statist.* 25 191–201.

[12] Williams,K.R.(1968). Designed Experiments. *Rubber Age*,100,65-71.