MATH 4980 FINAL PROJECT

SUBMANIFOLD

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1 Introduction

This project is about to introduce smooth submanifold based on concept of smooth Manifold, and this project is break into 5 different part.

In first section we will review the definition of the Manifold and how to define a smooth structure on a manifold and the difference between smooth manifold with and smooth manifold with boundary.

Than in second section we will define smooth function on manifolds discuss the smooth function between smooth manifolds and definition and properties of diffeomorphism. After that we will introduce tangent space to a manifold at a point in order to make sense of calculus on a manifold and properties of tangent vector on Manifold and differential of smooth map and the properties of the smooth map.

In section 5 we will discuss the way in which geometric properties of smooth maps can be detected from their differentials. The maps for which differentials give good local models turn out to be the one whose differential have constant rank, the three categories of such maps are smooth submersions, smooth immersions and smooth embeddings.

And the focus of this project is the last section in first part of this section we will define the most important type of submanifolds, called embedded submanifolds. Which have the subspace topology inherited from their containing manifolds and turns out to be exactly the images of smooth embeddings, and we will also discuss how they presented as level sets of a smooth map.

In second part we discussed more general type of smooth submanifolds, called immersed submanifolds, which turn out to be the image of injective immersions. An immersed submanifolds may locally look like a embedded one, but globally it may have a topology that is different from the subspace topology.

In the last part of this section we will discuss when we are able to restrict the domain and the codomain of a smooth map to a smooth submanifold and still retain smoothness? Also how can we identify the tangent space to a smooth submanifold as a subspace of the tangent space of its ambient manifold? And generalized to the case of submanifold with boundary.

2 Manifold

Definition 2.1.1 (topological manifold of dimension n) or **topological n-manifold** if it has the following properties:

1.M is a **Hausdorff space**: for every pair of distinct points p,q $\in$ M, there are disjoint open subsets U, $V \subseteq$ M such that p $\in U$ and q $\in V$.

2.M is **second-countable**: there exists a countable basis for the topology of M.

3.M is **locally Euclidean of dimension n**: each point of M has a neighborhood that is homeomorphic to an open sets of $\mathbb{R}^n$.

Definition 2.1.2 (Coordinate Charts):Let M be a topological n-manifold. A coordinate chart on M is a pair (U,φ), where U is a open subset of M and φ :U $\rightarrow \hat{U}$ is a homeomorphism from U to an open subset $\hat{U}$=φ(U)$\subseteq \mathbb{R}^n$. For the chart(U,φ) we define U as the coordinate domain or coordinate neighborhood we picked . And φ we defined as (local) coordinate map. And the component functions (x^1,....,x^n) of φ, defined by φ(p)=(x^1(p),....,x^n(p)),are called local coordinates on U.

Definition 2.1.3 (Smooth map) Suppose U and V are subspace of $\mathbb{R}^n$ and $\mathbb{R}^m$, and we define a map φ: U $\rightarrow$ V we say the map φ is smooth when every component of φ has continuous partial derivatives.

Definition 2.1.4 (Atlas)

And for a n-dimensional Manifold M we define the **Atlas for M** denote by $\mathscr{A}$ for M as a collection of charts whose domains cover the entire M. And we define a Atlas $\mathscr{A}$ is a **smooth Atlas** when any two chart in $\mathscr{A}$ are smooth compatible with each other. which means pick any two charts (U,φ) and (V,ψ) exist in Atlas we defined the transition map we defined as $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are diffeomorphism.

One example of Atlas we can given is consider a unit sphere $\mathbb{S}^2$ a atlas $\mathscr{A}$ for this sphere could be.

$\mathscr{A}$={ (U_{upper},φ_1),(U_{lower},φ_2),(U_{left},φ_3),(U_{right},φ_4),(U_{front},φ_5),(U_{back},φ_6)}

Definition 2.1.5

(a) A atlas $\mathscr{A}$on M is maximal if it is not contained in another atlas.

(b) A smooth structure on M is a maximal smooth atlas, we define a smooth manifold as a pair (M,$\mathscr{A}$) where M is a topological manifold and $\mathscr{A}$ is smooth structure on M.

Proposition 2.1.6 Let M be a topological Manifold

(a) Every smooth atlas $\mathscr{A}$ for M is contained in a unique maximum smooth atlas, called the smooth structure determined by $\mathscr{A}$.

(b) Two smooth atlas for M determine the same smooth structure if and only if their union is a smooth atlas.

Definition 2.1.7. We say M is a **n-dimensional topological manifold with boundary** if it has the following properties:

1.M is a **Hausdorff space**: for every pair of distinct points p,q $\in$ M, there are disjoint open subsets U, $V \subseteq$ M such that p $\in U$ and q $\in V$.

2.M is **second-countable**: there exists a countable basis for the topology of M.

3.For a arbitrary point p $\in$ M, there exist a open subset U of M such that U contain p and there exist a homeomorphic φ : U $\rightarrow \hat{U}$, and $\hat{U} \subset \mathbb{H}^n$

$\mathbb{H}^n$ = {(x_1,...x_n $\in \mathbb{R}^n$: $x^n > 0$ } and $\mathbb{H}^n$ is called closed n-dimensional upper half-space.

So first and second condition of manifold with boundary is identical with first two condition of a topological manifold, but in the third condition $\hat{U}$ is a subset of $\mathbb{H}^n$ in order to have a boundary.

Theorem 2.1.8 (Topological Invariance of the Boundary). If M is a topological manifold with boundary, then each point of M is either a boundary point or a interior point, but not both. Both ∂M and Int M are disjoint sets whose union is M.

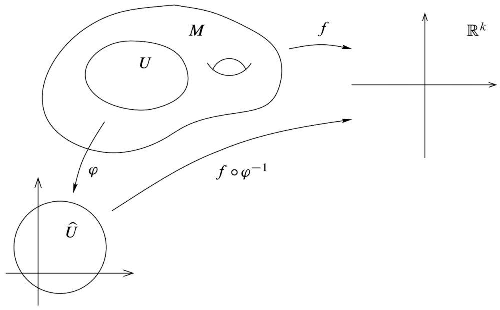
3 Smooth Map

3.1 Smooth maps

The main reason we talk about the smooth structure to force a manifold to be a smooth manifold was to enable us to define smooth function and smooth maps between manifolds.

3.1.1Definition(smooth function.) Suppose M is a smooth n-manifold, k is a nonnegative integer, and f : M $\to \mathbb{R}^k$ in any function. We say that f is a **smooth function** if for every p $\in$ M, there exists a smooth chart (U,φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\hat{U} = \varphi$(U) $\subseteq \mathbb{R}^n$.

When M is a smooth manifold with boundary, φ(U) is now an open subset of either $\mathbb{R}^n$ or $\mathbb{H}^n$, due to the fact the point p we pick is an interior point or a boundary point. And when p is a boundary point φ(U) is an open subset of $\mathbb{H}^n$ In this case since since $f \circ \varphi^{-1}$ is defined as a smooth function each point in φ(U) has a neighbourhood in $\mathbb{R}^n$.



The most important special case is that of smooth real-value function f : M $\to \mathbb{R}$; the set of all such functions is denoted by C^∞(M). Because sums and constant multiples of smooth functions are smooth, C^∞(M) is a vector space over R.

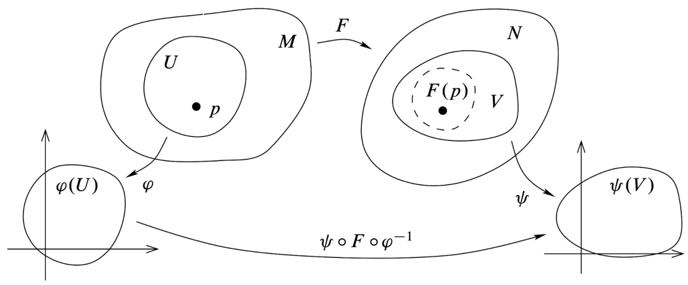
Exercise 3.1.2. Let M be a smooth manifold with or without boundary, and suppose f:M $\to \mathbb{R}^k$ is a smooth function. Such that $f \circ \varphi^{-1} : \varphi$(U) $\to \mathbb{R}^k$ is smooth for every smooth chart (U,φ) for M.

Proof: Let p $\in$ U by Def 3.1 there is a smooth chart (V,ψ) for M such that $f \circ \psi^{-1}$ is smooth and p $\in$V. But on φ^{-1} (U $\cap$V)we have

$$f \circ \varphi^{-1} = f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}$$

is smooth because $\psi \circ \varphi^{-1}$ is smooth by def of a smooth manifold and $f \circ \psi^{-1}$ is also smooth by def 3.1 and composition of 2 smooth maps still a smooth map. This shows that $f \circ \varphi^{-1}$ is smooth in a neighborhood of every point in φ(U), so $f \circ \varphi^{-1}$ is smooth.

Definition 3.1.3. Given a function f : M $\to \mathbb{R}^k$ and a chart (U,φ) for M, the function $\hat{f} : \varphi$(U) $\to \mathbb{R}^k$ defined by $\hat{f}$ (x) = $f \circ \varphi^{-1}$ is called the coordinate representation of f. Buy definition , f is smooth if and only if its coordinate representation is smooth in some smooth chart around each point. By exercise 3.2 , we have proved smooth function have smooth representation in every smooth chart.



Definition 3.1.4(Smooth Maps Between Manifolds). Let M,N be smooth manifolds, and let F:M $\to$ N be any map.F is a smooth map if for every p $\in$ M, there exist smooth charts (U,φ) containing p and (V,ψ) containing F(p) such that F(U) $\subseteq$ V and the composite map $\psi \circ$ F $\circ \varphi^{-1}$ is smooth from φ(U) to ψ(V).When M and N are smooth manifold with boundary, the only difference here is when we consider a map whose domain is a subset of $\mathbb{H}^n$ if it admits and extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ if it is smooth as a amp into $\mathbb{R}^n$.

proposition 3.1.5. Every smooth map is continuous.

Proof:Suppose M and N are smooth manifolds with or without boundary, and F: M $\to$ N is smooth. Given p $\in$ M; smoothness of F means there are smooth charts .(U,φ) containing p and (V,ψ) containing F (p) , such that F (U) $\subseteq$ and $\psi \circ$ F $\circ \varphi^{-1}\varphi$U$\to$ ψ(V) is smooth,hence continuous.Sinceφ: U $\to$ φUand ψ: V $\to$ ψV are homeomorphisms, this implies in turn that

F$|_U$ = ψ^{-1} $\circ$(ψ $\circ$ F $\circ$ φ^{-1})$\circ$ φ : U $\to$ V which is a composition of continuous maps. Since F is continuous in neighbourhood of each point, it is continuous on M.

3.1.6 Proposition Let M and N be smooth manifolds with or without boundary, and let F : M $\to$ N be a map.

(a) If every point p $\in$ M has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

3.1.7 Examples of Smooth map. Let M,N, and P be smooth manifolds with or without boundary.

(1).Every constant map c: M $\to$ N is smooth.

(2).The identity map of M is smooth .

(3).If U$\subseteq$ M is an open submanifold with or without boundary, then the inclusion map U $\hookrightarrow$ M is smooth.

(4). If F:M $\to$ N and G: N $\to$ P are smooth, then so is G $\circ$ F: M $\to$ P.

Proof: (we will proved 1 and 2 only since we haven't introduce submanifold and proof of 4 is introduced in the book)

(1).Let y $\in$ N and x $\in$ M and (U,φ) and (V,ψ) be smooth charts containing x and y. When y is defined as a constant the map $\psi \circ c \circ \varphi^{-1}$ is still a constant map so it's smooth by the fact we talked about in the beginning of this chapter.

(2).Define 2 charts (U,φ) and (V,ψ) be smooth charts in M and any since they are smooth compatible with each other(by the fact that M is a smooth Manifold), it should still be a smooth map.

3.2 Diffeomorphisms

Definition 3.2.1 If M and N are smooth manifolds with or without boundary, a diffeomorphism from M to N is a smooth bijective map F: M $\to$ N that has a smooth inverse. We say that M and N are diffeomorphic if there exists a diffeomorphism between them. In other word if F : M $\to$ N is diffeomorphism, we require the map F from M to N be smooth and with smooth inverse.

Proposition 3.2.2(properties of Diffeomorphisms)

(a).Every composition of diffeomorphisms is a diffeomorphism.

(b).Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.

(c).Every diffeomorphism is a homeomorphism and an open map.

(d) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.

Proof(a) Let f be diffeomorphism map M to N, so f : M → N. Let g be diffeomorphism map N to H, so f : N → H. Then $g \circ f$ as composition of 2 smooth map is smooth, and with smooth inverse $f^{-1} \circ g^{-1}$.

(b) Let f_i :$M_i \to N_i$ be diffeomorphisms for each i than $f_1 \times f_2 \times ... \times f_n$ is composition of smooth map with smooth inverse $f_1^{-1} \times f_2^{-1} \times ... \times f_n$

(c)By definition of diffeomorphism and prop 3.1.5 that every smooth map is continuous we can see that if F is smooth with smooth inverse then F is continuous with continuous inverse and by def of continuous it is indeed open map.

(d)By prop 3.16 (b) we know that every restrict function $F|_U$ is diffeomorphism and continuous by (c) we can say it homeomopic to it's image so it's clearly onto.

4 Tangent Vector

4.1 Geometric tangent vector

Definition 4.1.1. Geometric tangent space to $\mathbb{R}^n$ at a, denoted by $\mathbb{R}^n_a$, to be the set {a} × $\mathbb{R}^n$ = $\{(a,v) : v \in \mathbb{R}^n\}$. A geometric tangent vector in $\mathbb{R}^n$ is a element of $\mathbb{R}^n_a$for some a ∈ $\mathbb{R}^n$. And we denote (a,v) as v_a or $v|_a$, we think v_a as the vector v with it initial point a, and the set $\mathbb{R}^n_a$ is real vector space under the nature operations

v_a+ w_a = $(v+w)_a$ and $c(v_a) = (cv)_a$

The vectors $e_i|_a$ for i= 1,2,...,n, are basis for $\mathbb{R}^n_a$, and $\mathbb{R}^n_a$ is same as $\mathbb{R}^n$ itself; the reason to add index a is because that in this way $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ for distinct points a and b will be disjoint set

Definition 4.1.2. If a is a point of $\mathbb{R}^n$,a map w: C^∞ ($\mathbb{R}^n$) → $\mathbb{R}$ is called a **derivation** at a if it is linear over R and satisfies the following product rule:

$w(fg) = f(a)wg + g(a)wf$

Let T_a $\mathbb{R}^n$ denote the set of all derivations of C^∞ ($\mathbb{R}^n$) at a. Clearly, T_a $\mathbb{R}^n$ is a vector space under the operations

($(w_1 + w_2)f = w_1f + w_2f$, $(cw)f = c(wf)$

Lemma 4.1.3.(Properties of Derivations)Suppose a ∈ $\mathbb{R}^n$, w ∈ $T_a\mathbb{R}^n$, and f,g ∈ C^∞ ($\mathbb{R}^n$).

(a)If f is a constant function, then wf =0.

(b)If $f(a) = g(a) = 0$, then $w(fg) = 0$.

Definition 4.1.4. Let M be a Let M be a smooth manifold with or without boundary, and let p be a point of M. A linear map v :C^∞ ($\mathbb{R}^n$) → $\mathbb{R}^n$ is called a derivation at p if it satisfies;

v(fg)=f(p)vg + g(p)vf The set of all derivations ofC^∞ (M) at p, denoted by T_p M; is a vector space called the tangent space to M at p. An element of T_p M is called a tangent vector at p.

Definition 4.1.5.(Properties of Tangent Vectors on Manifolds). Suppose M is a smooth manifold with or without boundary, p ∈ M, v ∈ T_p M, and f,g ∈ C^∞ (M)

(a)If f is a constant function, then vf =0.

(b)If f(a) = g(a) = 0, then $v(fg)$=0.

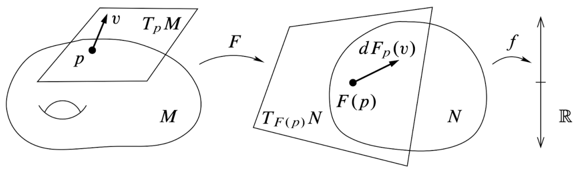
Theorem 4.1.6. Suppose M is a smooth manifold with or without boundary and $p \in M$. Every $v \in T_pM$ is the velocity of some smooth curve in M.

Definition 4.1.7.(The differential of smooth map): If M and N are smooth manifolds with or without boundary and F : M → N is a smooth map, for each p ∈ M we define a map

$$dF_p : T_p \text{ M} \to T_{F(p)} \text{ N}$$

called differential of F at p, as follows. Given $v \in T_pM$, we let dF_pv be the derivation at F(p) that acts on $f \in C^\infty(M)$, so $v(f \circ F)$ makes sense. the operator dF_pv: $C^\infty(M) \to \mathbb{R}$ is linear because v is , and is a derivation at F(p) because for any f,g ∈ $C^\infty(N)$ we have

$$\begin{aligned} dF_p(v)(fg) = v((fg)\circ F) = v((f\circ F)(g\circ F)) = f\circ F(p)v(g\circ F) + g\circ F(p)v(f\circ F) \\ = f(F(p))dF_p(c)g + g(F(p))dF_p(v)(f) \end{aligned}$$



Proposition 4.1.8.(Properties of Differentials). Let M, N, and P be smooth manifolds with or without boundary, let F: M → N and G: N → P be smooth maps, and let p ∈ M

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear

(b) $d(G\circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$

(c) $d(Id_M)_p = Id_{T_pM} : T_pM \to T_pM$.

(d)If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}$ N is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$

Proof (a) Let $v \in T_pM$ and $f \in C^\infty(N)$ we have $dF_p(v)(f) = (v)(f\circ F)$ by def 3.1.6 and for some constant c

$dF_p(cv)(f) = (cv)(f\circ F) = c(vf\circ F) = cdF_p(v)(f)$ which is linear

(b)Let $v \in T_pM$ and $d(G\circ F)_p(v)(f) = v(f\circ G\circ F) = dF_p(v)(f\circ G) = dG_{F(p)}\circ dF(p)$ such that $d(G\circ F)_p = dG_{F(p)}$

(c) Let $v \in T_pM$ $d(Id_M)_p(v)(f) = v(f\circ Id_M)$ but $f\circ Id_M$ is just f itself so we have $d(Id_M)_p(v)(f) = v(f\circ Id_M) = v(f)$ thus $d(Id_M)_p = Id_{T_pM}$

(d) In order to prove that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$, since $(dF_p)^{-1}$ is inverse of (dF_p) their composition must be the identity map, we can prove that (dF_p) composite with $d(F^{-1})_{F(p)}$ is still identity map, so we have

$d(F^{-1})_{F(p)} \circ dF_d = d(F^{-1})_p = d(Id_M)_p = Id_{T_pM}$

5 Submersions, Immersions, and Embeddings

5.1 Prerequisites for section 5

Definition 5.1.1. If $T : V \to W$ is a linear map between finite dimensional spaces, the dimension of ImT is called the **rank of T** , and the dimension of Ker$T$ is called its **nullity**.

Theorem 5.1.2(Canonical Form for a Linear Map). Suppose V and W are finite- dimensional vector spaces, and $T : V \to W$ is a linear map of rank r. Then there are bases for V and W with respect to which T has the following matrix representation:

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Theorem 5.1.3. Suppose V,W,X are finite-dimensional vector spaces, and $SV \to W$ and $T : W \to X$ are linear maps, Then the following holds

(a)rank S ≤ dim V, with equality if and only if S is injective.

(b)rank S ≤ dim W, with equality if and only if S is surjective.

(c)If dim V = dim W and S is either injective or surjective, then it is an isomorphism.

5.2 Submersions, Immersions, and Embeddings

Theorem 5.1.2 implies that rank is the only property that distinguishes different linear maps if we are free to bases independently for the domain and codomain.

Definition 5.2.1. Suppose M and N are smooth manifolds with or without boundary. Given a smooth map F: M → N and a point p ∈ M, we define the **rank of F at p** to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$;it is the rank of the Jacobian matrix of F in any smooth chart, or the dimension of Im $dF_p T_{F(p)}N$. If F has the same rank r at every point, we say that it has **constant rank**, and we write rank F =r.

By Theorem 5.1.3 we can see the fact that the rank of a linear map is never higher than the dimension of either its domain or its codomain, the rank of F is bounded above by the minimum of dim M, dim N. If the rank of dF_p is equal to this upper bound, we say that **F has full rank at p**, and if F has full rank everywhere, we say **F has full rank**.

Definition 5.2.2. A smooth map F: M → N is called a smooth submersion if its differential is surjective at each point. It is called a smooth immersion if its differential is injective at each point.

(Another word to say is F is smooth submersion if (rank F = dim N), and F is smooth immersion if (rank F = dim M)

Proposition 5.2.3. Suppose $F : M \to N$ is a smooth map and p∈ M.If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is injective, then p has a neighborhood U such that $F|_U$ is an immersion.

Example of Submersions and Immersions

(a) Suppose $M_1, ..., M_k$ are smooth manifolds. then each of the projection maps $\pi_i : M_1 \times .. \times M_k \to M_i$ is a smooth submersion.

(b) If $\psi : J \to M$ is a smooth curve in a smooth manifold M with or without boundary, then ψ is a smooth immersion if and only if ψ'(t) ≠ 0 for all $t \in J$

Proof (a). By Def5.2.2 in order to show this is a smooth submersion we need to show that d$\pi_i|_p$: $T_pM \to T_{p_i}M_i$ is a surjective map.

pick (p) ∈ $M_1 \times .. \times M_k$ then (p)= $p_1, ..., p_k$ and let $v_p \in T_pM$ $v_{p_i} \in T_{p_i}M$ and we have $d\pi_i|_p(v)f = v_p(f \circ \pi_i) = v_p f|_{M_i} = v_{p_i} f_i$ for some $f \in C^{\infty)}(M) and f_i \in C^{\infty)}(M_i)$ which is surjective.

(b).Suppose ψ'(t) = 0 for all t by Def 5.2.1 rank ψ is 0 for all t, and ψ is smooth immersion if and only if rank ψ = dim M, but dim(M) is can not be 0 thus this is contradict to out assumption.

Definition 5.2.4. Let M and N be smooth manifold with or without boundary, a map $F : M \to N$ is called a **local diffeomorphism** if every point p ∈M has a neighbourhood U such that F(U) is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 5.2.5 (Inverse Function Theorem for Manifold). Suppose M and N are smooth manifolds, and $F : M \to N$is a smooth map. If p ∈M is a point such that dF_p is invertible, then there are connected neighbourhoods U_0 of p and V_0 of F(p) such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

For Inverse Function Theorem for Manifold it is important to notice that this Theorem only apply for manifold without boundary. And it would fail for a map whose domain has nonempty boundary. Since if this is true we need to have a inclusion map $\iota : \mathbb{H}^n \to \mathbb{R}^n$ by Thm 5.2.5 for some $U \in \mathbb{H}^n$ and $V \in \mathbb{R}^n$ we need to construct a diffeomorphism map between them which is impossible.

But when we dealing with codomain with empty boundary, we can say that it provided the map takes its values in the interior of the codomain, the same conclusion holds because the interior is a smooth manifold without boundary which means if M is smooth manifold without boundary and N is a smooth manifold with boundary for some $p \in M$ $F(p) \in intN$.

Proposition 5.2.6(Elementary Properties of Local Diffeomorphism).

(a) Every composition of local diffeomorphisms is a local diffeomorphism.

(b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.

(c) Every local diffeomorphism is a local homeomorphism and an open map.

(d) The restriction of a local diffeomorphism to an open submanifold with or without boundary is a local diffeomorphism.

(e) Every diffeomorphism is a local diffeomorphism.

(f) Every bijective local diffeomorphism is a diffeomorphism.

(g) A map between smooth manifolds with or without boundary is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.

Proof.

(a)Suppose that $F : M \to N$ and $G : N \to H$ are both local diffeomorphism. Let x ∈ M and by Def 5.1.8 we can say there is a neighbourhood U ⊆ M of x, and V of F(x) such that $F|_U : U \to V$ is diffeomorphism, and there are neighbourhoods V' of F(x) and W ∈H of G(F(x)) = G ∘ F(x) such that $G|_{V'} : V' \to W$ is a diffeomorphism. Such that $(G \circ F)|_{F^{-1}(V \cap V')}$ should still be diffeomorphism.

(b)Suppose $F_i : M_i \to N_i$ is local diffeomorphism for each i, i is less than n. Such than by Def 5.1.8 we can say there is a neighbourhood $U_i \subseteq$ M of each x_i, and V_i ∈N of F(x_i) such that $F|_{U_i} : U_i \to V_i$ for each i is diffeomorphism, so F_i is a bijection smooth map with smooth inverse and we know the composition of bijection map is still a bijection map and a composition of smooth maps is still a smooth map. Thus it still diffeomorphism.

(c)Since every diffeomorphism map is a bijection and smooth map and every smooth map is a continuous map this is easily proved.

(d)By Prop 3.1.6(b) we know the restriction of a smooth map is still a smooth map so d is true.

(e)By def 5.1.8 and def3.2.2 that a diffeomorphism is just a bijective local diffeomorphism.

(f)Suppose $F : M \to N$ is a bijective local diffeomorphism, since it's a bijective map for some U ∈ M $F|_U$ is a diffeomorphism so $F|_U$ is a smooth map with smooth inverse by Prop 3.1.6 F is also a smooth map with smooth inverse.

Proposition 5.2.7. Suppose M and N are smooth manifolds (without boundary), and $F : M \to N$ is a map. (a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion. (b) If dim M = dim N and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Theorem 5.2.8(The Rank Theorem). Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F : M \to N$ is a smooth map with constant rank r. For each p ∈ M there exist smooth charts (U,φ) for M centered at p and (V, ψ) for N centered at F(p) such that F(U) ⊆ V , in which F has a coordinate representation of the form

$$\hat{F}(x^1,...,x^r,x^{r+1},...x^m) = (x^1,...,x^r,0,...,0).$$

In particular, if F is a smooth submersion, this becomes

$$\hat{F}(x^1,...,x^n,x^{n+1},...x^m) = (x^1,...,x^n).$$

and if F is a smooth immersion, it is

$$\hat{F}(x^1,...,x^m) = (x^1,...,x^m,0,...,0).$$

Definition 5.2.9(Embedding). If M and N are smooth manifolds with or without boundary, a **smooth embedding** of M into N is a smooth immersion $F : M \to N$ that is also a topological embedding, i.e., a homeomorphism onto its image F(M) ⊆N in the subspace topology. A smooth embedding is a map that is both a topological embedding and a smooth immersion, not just a topological embedding that happens to be smooth.

Example 5.2.10.

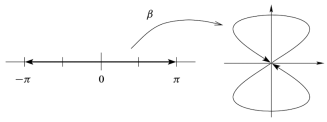
(a). If M is a smooth manifold with or without boundary and U ⊆ M is an open submanifold, the inclusion map U →M is a smooth embedding.

(b).(smooth topological embedding)The map ψ: $\mathbb{R} \to \mathbb{R}$ given by ψ(t): $(t^3,0)$ is a smooth map and a topological embedding, but it is not a smooth embedding because ψ'(0) =0 so consider example 5.1.7 (b) we know that it is not a smooth immersion.

(c)(A smooth Topological embedding).The map $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^3,0)$ is a smooth map and a topological embedding, but it is not a smooth embedding because $\gamma'(0) = 0$.

Example 5.2.11.(The Figure-Eight Curve) Consider the curve$\beta : (-\pi,\pi) \to \mathbb{R}^2$ defined by

$$\beta(t) = (sin2t, sint)$$



The image is a set looks like a figure-eight in the plane, sometimes called a lemniscate. It is easy to see that β is an injective smooth immersion because $\beta(t)'$ never vanishes (consider $D\beta(t) = (2cos(2t), cos(t))$ which is one to one); but it is not a topological embedding, because its image is compact in the subspace topology(Since its a closed under Hausdorff space), while its domain is not.

Proposition 5.2.12. Suppose M and N are smooth manifolds with or without boundary, and F : M →N is an injective smooth immersion. If any of the following holds, then F is a smooth embedding.

(a) F is an open or closed map.

(b) F is a proper map.

(c) M is compact.

(d) M has empty boundary and dim M = dim N .

Theorem 5.2.13(Local embedding Theorem). Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Then F is a smooth immersion if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U : U \to N$ is a smooth embedding.

Definition 5.2.14. Let $\pi : M \to N$be any continuous map, a section of π is a continuous right inverse of π ie. a continuous map $\psi : N \to M$ such that $\psi \circ \pi = Id_N$

A local section of π is a continuous map $\psi : U \to M$ defined on some open subset $U \subseteq N$ and $\psi \circ \pi = Id_U$

Theorem 5.2.15.(Local immersion Theorem of Manifold with Boundary) Suppose M is a smooth m-manifold with boundary, N is a smooth n-manifold, and $F : M \to N$ is a smooth immersion. For any $p \in \delta M$; there exist a smooth boundary chart .(U, φ) for M centered at p and a smooth coordinate chart .(V, ψ) for N centered at F (p) with $F(U) \subseteq V$, in which F has the coordinate representation

$$\hat{F}(x^1, .., x^m) = (x^1, .., x^m, 0, .., 0)$$

Definition 5.2.16. If $\pi : M \to N$ is any continuous map, a section of π is a continuous right inverse for π, i.e., a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = Id_N$:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \\ N. \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

A local section of π is a continuous map $\sigma : U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = Id_U$

Theorem 5.2.17.(Local Section Theorem)

Suppose M and N are smooth manifolds and $\pi : M \to N$ is a smooth map. Then π is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.

Proposition 5.2.18(Properties of Smooth Submersions). Let M and N be smooth manifolds, and suppose $\pi : M \to N$ is a smooth submersion. Then π is an open map, and if it is surjective it is a quotient map.

Quotient map For $p : X \to Y$ the map p is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$is open in X.

6 Submanifolds

6.1 Embedded submanifolds

Definition 6.1.1. Suppose M is a smooth manifold with or without boundary. An **embedded submanifold of M** is a subset S ⊆ M that is a manifold (without boundary) in the subspace topology, endowed with a smooth structure with respect to which the inclusion map$S \hookrightarrow M$ is smooth embedding. (Embedded submanifolds are also called regular submanifolds)
(Recall that a smooth embedding of M into N is a smooth immersion $F : M \to N$ also topological embedding so M is homeomorphic to its image $F(M) \subseteq N$)

If S is an embedded submanifold of M; the difference dim M - dim S is called the **codimension of S in M**, and the containing manifold M is called the **ambient manifold** for S. An embedded hypersurface is an embedded submanifold of codimension 1. The empty set is an embedded submanifold of any dimension.

Example(Open submanifolds). Let U be any open subset of $\mathbb{R}^n$. Then U is a topological n-manifold, and. the single chart (U, Id_U) defines a smooth structure on U.

More generally, let M be a smooth n-manifold and let U ⊆ M be any open subset. Define an atlas on U by

$$\mathscr{A}_U = \text{smooth charts}(V,\varphi) \text{ for M such that V} \subseteq U$$

Every point p ∈ U is contained in the domain of some chart (W,φ) for M, if we set V = W ∩ U, then $(V,\varphi|_V)$ is a chart in $\mathscr{A}_U$ whose domain contains p. Therefore, U is covered by the domains of charts in $\mathscr{A}_U$, and it is a smooth atlas for U. Thus any open subset of M is itself a smooth n-manifold , with it's own smooth structure. (And this example is the with codimension 0)

Proposition 6.1.2. (Open Submanifolds) Suppose M is a smooth manifold. The embedded submanifolds of codimension 0 in M are exactly the open submanifold.

Proof. Suppose $U \subseteq M$ is an open submanifold, and let $\iota : U \hookrightarrow M$ be the inclusion map. Previous example showed that U is a smooth manifold of the same dimension as M, and dimM -dim U, so it has codimension 0. In terms of the smooth charts for U constructed on previous question, ι is represented in coordinates by an identity map, so it is a smooth immersion; and because U has the subspace topology, which is topological embedding and ι is a smooth immersion, thus by definition U is an embedded submanifold.

Conversely, suppose U is any codimension-0 embedded submanifold of M. Then inclusion $\iota : U \hookrightarrow M$ is a smooth embedding by definition, and therefore it is a local diffeomorphism by proposition 5.2.7, and open map by 5.2.6(g). Thus U is an open subset of M.

Proposition 6.1.3. (Images of Embeddings as Submanifolds.)Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and $F : N \to M$ is a smooth embedding. Let S =F(N). With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.

Proof. If we give S the subspace topology that it inherits from M; then the assumption that F is an smooth embedding so F is both topological embedding and smooth immersion by definition we had for smooth embedding, thus we can say that F can be considered as a homeomorphism from N onto S, and thus S is a topological manifold.We give S a smooth structure by taking the smooth charts to be those of the form $(F(U),\varphi \circ F^{-1})$, where (U,φ) is any smooth chart for N and U is a subset of N; smooth compatibility of these charts follows immediately from the smooth compatibility of the corresponding charts for N. With this smooth structure on S, the map F is a diffeomorphism onto its image since F is smooth with smooth inverse by Def 3.1.4, and by uniqueness of the smooth structure this is the only smooth structure with this property. The inclusion map S ↪ M is equal to the composition of a diffeomorphism followed by a smooth embedding:

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M$$

therefor it is a smooth embedding. Which implies that embedded submanifolds are exactly the images of smooth embeddings.

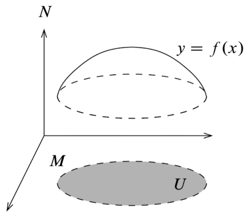
Proposition 6.1.4(Slice of product Manifolds). Suppose M and N are smooth Manifolds. For each p ∈ N, the subset M × p (called a slice of the product manifold) is an embedded submanifold of $M \times N$ diffeomorphic to M.

Proof. Define $F : M \to M \times N$ by $F(x) = (x,p)$ by proposition 6.1.2 F has codomain 0, The set $M \times p$ is the smooth embedding $x \to (x,p)$ and by proposition 5.2.7 F is a local diffeomorphism since F is clearly bijective so F is a diffeomorphism.

Proposition 6.1.5(Graph as Submanifolds). Suppose M is a smooth m-manifold(without boundary), N is a smooth n-manifold with or without boundary, U⊆ M is open, and $f : U \to N$ is a smooth map. Let

$$\Gamma(f) = (x, y) \in M \times N : x \in U, y = f(x)$$

Then $\Gamma(f)$ is an embedded m-dimensional submanifold of M × N.



Proof. Define a map $\Gamma(f) : U \to M \times N$ by $\Gamma(f)(x) = (x, f(x))$ It is a smooth map whose image is $\Gamma(f)$. Because the projection $\pi_M : M \times N \to M$ satisfies $\pi_M \circ \gamma_f(x) = x$ for $x \in U$, the composition $d(\pi_M)_{(x,f(x))} \circ d(\gamma_f)_x$ is the identity in T_xM for each x ∈ U. So $d(\gamma_f)_x$ is a injective map, by Def 5.2.2 γ_f is a smooth immersion. Since $\pi_M|_{\Gamma(f)}$ is a restriction of continuous a map so itself should also be a continuous map and is a continuous inverse of $\Gamma(f)$ which represent that it is homeomorphism, by Prop 5.2.6 this implies that $\Gamma(f)$ is an embedded submanifold diffeomorphic to U.

(A continuous map $f : X \to Y$ between topological spaces is called **proper** if for every compact space $K \subseteq Y$, $f^{-1}(K)$ is compact in X.)

Theorem 6.1.6(Sufficient Condition for properness). suppose X and Y are topological spaces, and $F : X \to Y$ is a continuous map.

(a) If X is compact and Y is Hausdorff, then F is proper.

(b) If F is a closed map with compact fibers, then F is proper.

(c) If F is a topological embedding with closed image, then F is proper.

(d) If Y is Hausdorff and F has a continuous left inverse ,then F is proper.

(e) If F is proper and A ⊆ X is a subset that is saturated with respect to F , then $F|_A : A \to F(A)$ is proper.

Theorem 6.1.7. Suppose X is a topological space and Y is a locally compact Hausdorff space. Then every proper continuous map $F : X \to Y$ is closed.

Definition 6.1.8 An embedded submanifold $S \subseteq M$ is said to be properly embedded if the inclusion $S \hookrightarrow M$ is a proper map.

Proposition 6.1.9 Suppose M is a smooth manifold with or without boundary and S ⊆ M is an embedded submanifold. Then S is properly embedded if and only if it is a closed subset of M .

proof. Part I: suppose S is properly embedded let $F : M \to S$ then F is a proper map, and by Prop 5.2.10 S is compact and S is also locally compact, then by Thm 6.1.7 it is closed.

Part II: Suppose S is a closed subset of M then by Thm 6.1.6 f is a proper map.

Corollary 6.1.10.Every compact embedded submanifold is properly embedded.

Proof. Since every compact subsets of Hausdorff space are closed by previous Proposition every compact embedded submanifold is properly embedded.

Proposition 6.1.11 (Global Graph Are Properly Embedded). Suppose M is a smooth manifold, N is a smooth manifold with or without boundary, and $f : M \to N$ is a smooth map. With the smooth manifold structure of the Prop 6.1.5, $\Gamma(f)$ is properly embedded in $M \times N$.

Proof. Consider the map $\pi_M : M \times N \to N$ which is the smooth left inverse for the embedding $\gamma_f : M \to M \times N$. By Thm 6.1.6(g) γ_d is a proper map.

In theorem 6.1.13 we will show, embedded submanifolds are modeled locally on the standard embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$, identifying $\mathbb{R}^k$ with the subspace

$$(x^1,...,x^k,x^{k+1},..x):x^{k+1}=...=x^n=0\subseteq\mathbb{R}^n$$

Definition 6.1.12 (k-slice). Let U be an open set of $\mathbb{R}^n$, and k $\in$ 0,...,n a **k-dimensional slice of U** (or simply a k-slice) is any subset of the form

$$S=\{(x^1,...,x^k,x^{k+1},..x)\in U:x^{k+1}=c^{k+1}...x^n=c^n\}$$

for some constant $c^{k+1},...,c^n$(When k=n, this just means $S=U$) and every k-slice is homeomorphic to an open subset of $\mathbb{R}^k$.

For smooth chart (U,φ) on a smooth manifold M, we also say S $\subseteq$ U is a k-slice of U if $\varphi(S)$ is a k-slice of $\varphi(U)$.

Given a subset S $\subseteq$ M and a nonnegative integer k, we say that S satisfies the **local k-slice condition** if each point of S is contained in the domain of a smooth chart (U,φ) for M such that $S\cap U$ is a single k-slice in U. Any such chart is called a **slice chart for S in M**, and the corresponding coordinates $(x^1,...,x^n)$ are called **slice coordinates**.

Theorem 6.1.13(Local Slice Criterion for Embedded Submanifolds).

Let M be a smooth n-manifold. If $S\subseteq M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition. Conversely, if $S\subseteq M$ is a subset that satisfies the local k-slice condition, then with the subspace topology, S is a topological manifold of dimension k, and it has a smooth structure making it into a k-dimensional embedded submanifold of M.

Proof. First suppose that $S\subseteq M$ is an embedded k-dimensional submanifold. Since the inclusion map $S\hookrightarrow M$ is an immersion, the rank theorem shows that for any $p\in S$ there are smooth charts (U,φ) for S (in its given smooth manifold structure) and (V,ψ) for M; both centered at p, in which the inclusion map $\iota|_U:U\to V$ has the coordinate representation

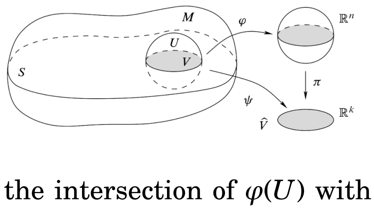
$$(x^1,...,x^k)\to(x^1,...,x^k,0,...,0)$$

Choose $\delta>0$ small enough that both U and V contain coordinate balls of radius δcentered at p, and denote these coordinate balls by $U_0\subseteq U$ and $V_0\subseteq V$. It follows that $(U_0=\iota(U_0))$ is exactly a single slice in V_0. Because S has the subspace topology, the fact that V_0 is open in S means that there is an open subset $W\subseteq M$ such that$U_0=W\cap S$. Setting $V_1=V_0\cap W$, we obtain a smooth chart$(V_1,\psi|_{V_1})$ for M containing p such that $V_1\cap S=U_0$, which is a single slice of V_1.

Conversely, suppose S satisfies the local k-slice condition. With the subspace topology, S is Hausdorff and second-countable, because both properties are inherited by subspaces. To see that S is locally Euclidean, we construct an atlas. The basic idea of the construction is that if $(x^1,...,x^k)$ are slice coordinates for S in M; we can use $(x^1,...,x^k)$ as local coordinates for S.

For this proof let $\pi:\mathbb{R}^n\to\mathbb{R}^k$ denote the projection onto the first k coordinates. Let (U,φ) be any slice chart for S in M and define

$$V=U\cap S,\hat{V}=\pi\circ\varphi(V),\psi=\pi\circ\varphi|_V:V\to\hat{V}$$

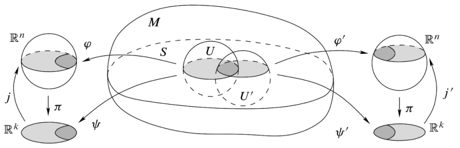


By definition of slice charts, $\varphi(V)$ is the intersection of $\varphi(U)$ with a certain k-slice $A\subseteq\mathbb{R}^n$ defined by setting $x^{k+1}=c^{k+1},...,x^n=c^n$, and therefore $\varphi(V)$ is open in A. Since $\pi|_A$ is a diffeomorphism from A to $\mathbb{R}^k$, it follows that $\hat{V}$ is open in $\mathbb{R}^k$. Moreover, is a homeomorphism because it has a continuous inverse given by $\varphi^{-1}\circ j|_{\hat{V}}$,where$j:\mathbb{R}^k\to\mathbb{R}^n$ is the map

$$j(x^1,...,x^k) = (x^1,...,x^k,c^{k+1},...,c^n)$$

Thus S is a topological k-manifold, and the inclusion map $\iota : S \hookrightarrow M$ is a topological embedding.

(This is because for any p $\in S$ we can define a chart (ψ,V), where V is open in S contain p, $\psi : V \to \hat{V} \subseteq \mathbb{R}^k$ and $\iota : S \hookrightarrow M$ is a topological embedding since S is a subspace topology so ι is homormophic to its image)



To put a smooth structure on S, we need to verify that the charts constructed above are smoothly compatible. Suppose (U,φ) and (U',φ') are two slice charts for S in M; and let (V,ψ), (V',ψ') be the corresponding charts for S. The transition map is given by $\psi' \circ \psi^{-1} = \pi \circ \varphi' \circ \varphi^{-1} \circ$▮, which is a composition of four smooth maps. Thus the atlas we have constructed is in fact a smooth atlas, and it defines a smooth structure on S. In terms of a slice chart (U,φ) for M and the corresponding chart (V,ψ) for S, the inclusion map $S \hookrightarrow M$ has a coordinate representation of the form

$$(x^1,...,x^k) \to (x^1,...,x^k,c^{k+1},...,c^n)$$

which is a smooth immersion. Since the inclusion is. a smooth immersion and a topological embedding, S is an embedded submanifold

Theorem 6.1.14. If M is a smooth n-manifold with boundary, then with the subspace topology, δM (boundary of M) is a topological (n-1)dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M .

Proof. Let $x \in \delta M$ let (U,φ) be the slice chart containing x, then $\varphi\ (\delta M \cap U) = (x^1,..,x^n \in \varphi(U) : x^n = 0)$ since the slice coordinate has x^n as δM satisfies the (n-1) slice result. And by Thm6.1.13 it is a topological manifold with dimension (k-1) and it has a smooth structure making it into a k-dimensional embedded submanifold of M. And by Def of boundary ($\delta A = \bar{A} \cap \bar{A^c}$) it is a closed subset of M so Prop6.1.9 it is also proper embedded.

Example 6.1.15(Sphere as Submanifolds). Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{R}^n$, and define functions $F^{\pm}(u) = \sqrt{1-|u|^2}$ for any $i \in (1,..,n)$ the intersection of ^{n} with the open set where $x^i > 0$ is the graph of the smooth function

$$x^i = F^+(x^1,..,x^{i-1},x^{i+1},...x^{n+1})$$

and similarly the intersection pf $\mathbb{S}^n$ with $x : x^i < 0$ is the graph for F^+. Since every point is $\mathbb{S}^n$ is one of these sets, $\mathbb{S}^n$ satisfies the local n-slice condition and is a embedded submanifold of $\mathbb{R}^{n+1}$ by Thm 6.1.13

Example 6.1.16 (Level set). Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $\Phi : U \to \mathbb{R}$ is a smooth function. For any $c \in \mathbb{R}$, the set $\Phi^{-1}(c)$ is called a level set of Φ. Choose some $c \in \mathbb{R}$, let $M = \Phi^{-1}(c)$, and suppose that the total derivative $D\Phi(a)$ is a row matrix whose entries are the partial derivatives $((\delta\Phi/\delta x^1(a)),...,(\delta\Phi/\delta x^n(a))$, for each a $\in$ M there is some i, such that $(\delta\Phi/\delta x^i(a)) \neq 0$. By implicit function theorem that there is a neighbourhood U_0 of a such that $M \cap U_0$ can be expressed as a graph of an equation of the form

$$x^i = f(x^1,...,\hat{x^i},...x^n)$$

for some smooth real-valued function f defined on an open subset of $\mathbb{R}^{n-1}$. Therefore, arguing just as in the case of the n-sphere, we see that M is a topological manifold of dimension (n-1), and has a smooth structure such that each of the graph coordinate charts associated with a choice of f as above is a smooth chart.

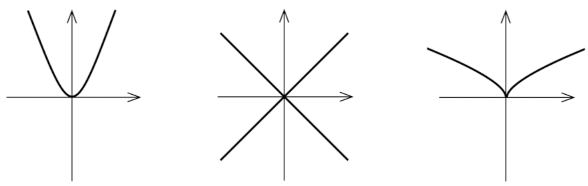
Definition 6.1.17. If $\Phi : M \to N$ is any point of N, we call the set $\Phi^{-1}(c)$ a **level set of** Φ (In the special case $N = \mathbb{R}^k$ and c=0, the level set $\Phi^{-1}(0)$ is usually called **zero set of** Φ.

Not all level sets of smooth functions are smooth submanifolds. For instance, take the three smooth functions Θ, Φ, Ψ defined by

$$\Theta(x,y) = x^2 - y, \Phi(x,y) = x^2 - y^2, \Psi(x,y) = x^2 - y^3$$

Even the zero set Θ is an embedded submanifold of $\mathbb{R}^2$ because it is the graph of the smooth function $f(x) = x^2$ but neither the zero set of Φ nor that of Ψ is an embedded submanifold.

This is because if we let $x^2 - y^2 = 0$ we have $(x+y)(x-y) = 0$ so we have union of two line $x = y$ and $x = -y$ which is not locally Euclidean at the point(0,0), and if we have $x^2 - y^3 = 0$ we have $y = x^{3/2}$ which is not smooth at point 0.



The previous example we discussed that by using implicit function thm we can show that certain level sets in Rn are smooth manifolds can be adapted to show that those level sets are in fact embedded submanifolds of $\mathbb{R}^n$. But using the rank theorem, we can prove something much stronger.

Theorem 6.1.18(Constant-Rank Level Set Theorem). Let M and N be smooth manifolds, and let $\Phi : M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M .

Proof. Let m = dim M, n = dim N, and k= m-r. Let $c \in N$ be arbitrary, and let S denote the level set $\Phi^{-1}(c) \subseteq M$. By rank thm for each p $\in$ S there exist smooth charts (U,φ) for M centered at p and (V, ψ) centered at $c(p)$ in which Φhas a coordinate representation of the form

$$\hat{F}(x^1, ..., x^r, x^{r+1}, ...x^m) = (x^1, ..., x^r, 0, ..., 0).$$

and therefor $S \cap U$ is the slice

$$\{(x^1, ..., x^r, x^{r+1}, ..., x^m) \in U : x^1 = ... = x^r = 0\}$$

Thus S satisfies the k -slice condition, so it is an embedded submanifold of dimension k. It is closed in M by continuity, by Prop 6.1.9 it is a compact properly embedded .

Corollary 6.1.19(Submersion Level Set Theorem). If M and N are smooth manifolds and $\Phi : M \to N$ is a smooth submersion, then each level set of Φ is a properly embedded submanifold whose codimension is equal to the dimension of N .

Proof. Since $\Phi : M \to N$ is a smooth submersion by Def 5.2.2 we know that you Rank Φ = Rank N, so it has constant rank equal to the dimension of its codomain, then by previous Thm each level set of Φ is a proper embedded submanifold.

Definition 6.1.20. If $\Phi : M \to N$ is a smooth map, a point $p \in M$ is said to be **regular point** of Φ if $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is surjective;it is a **critical point** of Φ otherwise. This means, in particular, that every point of M is critical if dim M < dim N , and every point is regular if and only if F is a submersion. (Note that the set of regular points of is always an open subset of M by Def 5.2.1)

A point $c \in N$ is said to be a **regular value** of Φ if every point of the level set $\Phi^{-1}(c)$ is a regular point, and a **critical value** otherwise. In particular, if $\Phi^{-1}(c) = \emptyset$, then c is a regular value.

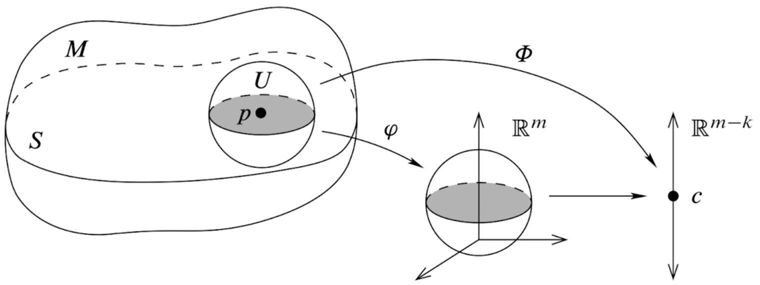
A level set $\Phi^{-1}(c)$ is called a **regular level set** if c is a regular value of Φ in other words, a regular level set is a level set consisting entirely of regular points of Φ.

Corollary 6.1.21. Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.

Proof. Let $\Phi M \to N$ be a smooth map and let $c \in N$ be a regular value. so $d\Phi_c : T_cM \to T_{\Phi(c)}N$ is surjective , and the set U of points $p \in M$ where rank $d\Phi_p = dimN$ by definition of smooth submersion is open in M by Prop 5.2.3 and since c is a regular poing $\Phi^{-1}(c)$ is contained in U. And by Prop 5.2.3 we can say that Φ is still a smooth submersion and by Corollary 6.1.19 $\Phi^{-1}(c)$ is still a n embedded manifold pf U. Since the composition on smooth embeddings $\Phi^{-1}(c) \hookrightarrow U \hookrightarrow M$ is again a somooth embedding, which implies that $\Phi^{-1}(c)$ is an embedded submanifold of M, and it is close by continuity.

Example 6.1.22.(Sphere). Since sphere is a regular level set of the smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x) = |x|^2$, and $df_x(v) = 2\sum_i x^i v^i$, which is surjective without origin. Thus by previous $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$.

Proposition 6.1.23. Let S be a subset of a smooth m-manifold M. Then S is an embedded k-submanifold of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi : U \to \mathbb{R}^{m-k}$.



Proof. Part I: Suppose S is an embedded k-submanifold there is map $f : S \to M$ is a smooth submersion. If $(x^1,..,x^m)$ are slice coordinates for S on an open subset $U \subseteq M$, the map $\Phi U \to \mathbb{R}^{m-k}$ given in coordinates by $\Phi(x) = (x^{k+1},..,x^m)$ as a restriction of f is smooth submersion, one of whose level set is $S \cap U$ as we indicated in the figure.

Part II: Suppose that around every point $p \in S$ there is a neighbourhood U and a smooth submersion $\Phi : U \to \mathbb{R}^{m-k}$ such that $S \cap U$ is a level set of Φ. By the submersion level set theorem, $S \cap U$ is an embedded submanifold of U, so it satisfies the local slice condition; Which means that S is itself an embedded submanifold of M by Thm 6.1.13.

Definition 6.1.24. If $S \subseteq M$ is an embedded submanifold, a smooth map $\Phi : M \to N$ such that S is a regular level set of Φ is called a defining map for S . In the special case $N = \mathbb{R}^{m-k}$ (so that Φ is a real-valued or vector-valued function), it is usually called a defining function.(Example 6.1.22 $f(x) = |x|^2$ is a example of a defining function for the sphere)

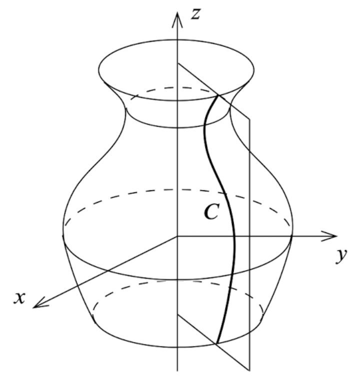
if U is an open subset of M and $\Phi : U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ , then Φ is called a local defining map (or local defining function) for S .(every embedded submanifold admits a local defining function in a neighborhood of each of its points)

Example 6.1.25.(Surface of Revolution). Let H be the half-plane $\{(r,z) : r > 0\}$, and suppose $C \subseteq H$ is an embedded 1-dimensional submanifold. The surface of revolution determined by C is the subset $S_C \subseteq \mathbb{R}^3$ given by

$$S_C = \{(x,y,z) : (\sqrt{x^2+y^2}, z) \in C\}$$

the set C is called its generating curve. If $\varphi : U \to \mathbb{R}$ is any local defining function for C in H, we get a local defining function Φ for S_C by

$$\Phi(x,y,z) = \varphi(\sqrt{x^2+y^2}, z)$$



defined on the open subset

$$\hat{U} = \{(x,y,z) : (\sqrt{x^2+y^2}, z) \in U\} \subseteq \mathbb{R}^3$$

A computation shows that the Jacobian matrix of Φ is

$$D\Phi(x,y,z) = (\tfrac{x}{r}\tfrac{\delta\varphi}{\delta r}(r,z)\tfrac{y}{r}\tfrac{\delta\varphi}{\delta r}(r,z)\tfrac{\delta\varphi}{\delta z}(r,z))$$

Where we have written $r = \sqrt{x^2+y^2}$. At any point $(x,y,z) \in S_C$, at least one of the components of $D\Phi(x,y,z)$ is nonzero, so S_C is a regular level set of Φ and is thus an embedded 2-dimensional submanifold of $\mathbb{R}^2$.

6.2 Immersed Submanifolds

Definition 6.2.1. An immersed submanifold of M is a subset S ⊆ M endowed with a topology (not necessarily the subspace topology) with respect to which it is a topological manifold (without boundary), and a smooth structure with respect to which the inclusion map S ↪ M is a smooth immersion. As for embedded submanifolds, we define the codimension of S in M to be dimM -dimS.(Every embedded submanifold is also an immersed submanifold, and we define smooth submanifold same as immersed one and smooth hpersurface represent an immersed submanifold of codimension 1.)

Proposition 6.2.2.Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and $F : N \to M$ is an injective smooth immersion. Let S =F(N) Then S has a unique topology and smooth structure such that it is a smooth submanifold of M and such that $F : N \to M$ is a diffeomorphism onto its image.

Proof. Proof of this similar to Prop6.1.3. But now we define the topology on S. We give S a topology by declaring a set $U \subseteq S$ to be open if and only if $F^{-1}(U) \subseteq N$ is open, and then give it a smooth structure by taking the smooth charts to be those of the form $(F(U), \varphi \circ F^{-1})$, where (U,φ) is any smooth chart for N . As in the proof of Prop6.1.3, the smooth compatibility condition follows from that for N . With this topology and smooth structure on S , the map F is a diffeomorphism onto its image, and these are the only topology and smooth structure on S with this property. As in the embedding case, the inclusion $S \hookrightarrow M$ can be writtern as the composition

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M$$

in this case, the first map is a diffeomorphism and the second s a smooth immersion, so the composition is a smooth immersion.

Exercise 6.2.3(The Figure-Eight Curve). Consider the Example 5.2.10 the Figure-Eight Curve we discussed which is a injective smooth immersion but not topological embedding it is a immersed submanifolds when given appropriate topologies and smooth structures. As smooth manifolds, they are diffeomorphic to $\mathbb{R}$. They are not embedded submanifolds.

Proof. For Figure-Eight Curve we have $\beta : (-\pi, \pi) \to \mathbb{R}^2$ defined by $\beta(t) = (sin2t, sint)$,we need to show this is not a embedded submanifold of $\mathbb{R}^2$ but the image of β is not even a smooth submanifold, since there is no neighbourhood of the center point which is homeomorphic to $\mathbb{R}$

Exercise 6.2.3.Suppose M is a smooth manifold and S ⊆ M is an immersed submanifold. Show that every subset of S that is open in the subspace topology is also open in its given submanifold topology; and the converse is true if and only if S is embedded.

Proof. Part I:Suppose that M is a smooth manifold and S ⊆ M is an immersed submanifold then by definition of immersed submanifold nap $F : S \to M$ should be injective smooth immersion which means so it's also open in submanifold topology.

Part II: and converse hold when F is a surjective map, and F is a surjective map if and only if that S is a embedded submanifold.

Proposition 6.2.4. Suppose M is a smooth manifold with or without boundary, and S ⊆ M is an immersed submanifold. If any of the following holds, then S is embedded.

(a) S has codimension 0 in M .

(b) The inclusion map S ⊆ M is proper.

(c) S is compact.

Proof.(a)Suppose that S has codimension 0 in M such that by definition of codimension we introduced in the beginning of this section we can say dimM = dimS, by Prop 5.2.10 we can say there is a map $F : S \to M$ is a injective smooth immersion.

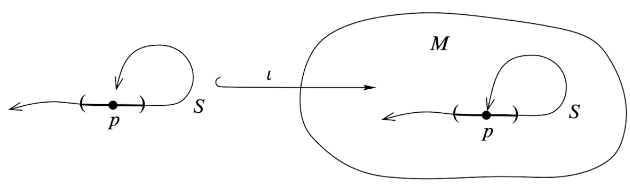
(b)Consider 5.2.10(b)

(c)Consider 5.2.10(c)

Proposition 6.2.5 (Immersed Submanifolds are locally Embedded) If M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold, then for each p ∈ S there exists a neighborhood U of p in S that is an embedded submanifold of M .

Proof. Suppose that $S \subseteq M$ is an immersed submanifold such that $F : S \to M$ is a injective smooth immersion by Thm 5.2.11 every point in S has a neighborhood U ⊆ S such that $F|_U : U \to M$ is a smooth embedding.

This Proposition is saying that given an immersed submanifold $S \subseteq M$ and a point $p \in S$, we are able to find a neighbourhood U of p in S such that U is embedded, but it may not be possible to find a neighbourhood V of p in M such that $V \cap S$ is embedded.



Definition 6.2.6 Suppose $S \subseteq M$ is an immersed k-dimensional submanifold. A **local parametrization of S** is a continuous map $X : U \to M$ whose domain is an open subset $U \subseteq \mathbb{R}^k$, whose image is an open subset of S , and which, considered as a map into S, is a homeomorphism onto its image. It is called a **smooth local parametrization** if it is a diffeomorphism onto its image (with respect to S 's smooth manifold structure). If the image of X is all of S , it is called a **global parametrization**.

Proposition 6.2.7 Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed k-submanifold, $\iota : S \hookrightarrow M$ is the inclusion map, and U is an open subset of $\mathbb{R}^k$. A map $X : U \to M$ is a smooth local parametrization of S if and only if there is a smooth coordinate chart (U,φ) for S such that $X = \iota \circ \varphi^{-1}$. Therefore, every point of S is in the image of some local parametrization.

Proof. Part I: assume that $X : U \to M$ is a smooth local parametrization of S then by Def 6.2.6 it is a diffeomorphism, and it's inverse $X^{-1} : X(U) \to U$ should also be a diffeomorphism, then we can construct a chart $(X(U), X^{-1})$ such that $X = \iota \circ (X^{-1-1})$

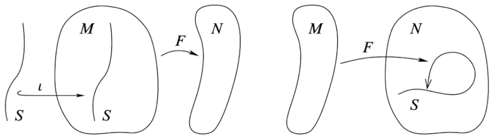
Part II: Suppose there is a smooth coordinate chart (U,φ) for S such that $X = \iota \circ \varphi^{-1}$ and $X(\varphi(U)) = \iota \circ \varphi^{-1} \circ \varphi(U) = U$ which is also open in S, so X is smooth and we can prove X^{-1} is also smooth thus we can conclude that X is a diffeomorphism onto V.

Example 6.2.8(Graph Parametrizations) Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is a smooth function. The map $\gamma_f : U \to \mathbb{R}^n \times \mathbb{R}^k$ given by $\gamma_f(u) = (u, f(u))$ is a smooth global parametrization of $\Gamma(f)$, called a graph parametrization.

And consider $S \subseteq \mathbb{R}^n$ and let S be the figure-eight curve in Exercise 6.2.3 considered as an immersed submanifold of $\mathbb{R}^2$. The map $\beta : (-\pi,\pi) \to \mathbb{R}^2$ we defined in Example 5.1.10 is a smooth global parametrization of S.

Restricting the Domain of a smooth Map. For a smooth map $F : M \to N$ the following Theorems allowed us to check whether F is still smooth when its domain or codomain is restricted to a submanifold.

Theorem 6.2.9 If M and N are smooth manifolds with or without boundary, $F : M \to N$ is a smooth map, and $S \subseteq M$ is an immersed or embedded submanifold, then $F|_S : S \to N$ is smooth.



Proof.The inclusion map $\iota : S \hookrightarrow M$ is smooth by definition of an immersed sub- manifold. Since $F|_S = F \circ \iota$, as the composition of 2 smooth maps $F|_S$ is smooth .

Example 6.2.10 Let $S \subseteq \mathbb{R}^2$ be the figure-eight submanifold, with the topology and smooth structure induced by the immersion β of Example 5.2.10. Define a smooth map $G : \mathbb{R} \to \mathbb{R}^2$ by

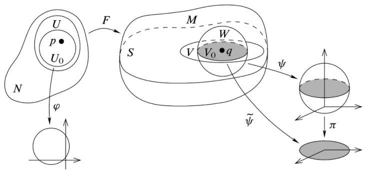
$$G(t) = (sin(2t), sin(t))$$

(This is the same formula that we used to define β, but now the domain is extended to the whole real line instead of being just a subinterval.) It is clear that the image of G lies in S. But in this case G as a map form $\mathbb{R} \to S$, it is not even continuous, since $\beta^{-1} \circ G$ is not continuous at $t = \pi$.(Since this function jump from some point near π in to 0)

Remark: This following hold only when the ambient manifold M is a smooth manifold (without boundary), because it is only in that case that we have constructed slice charts for embedded submanifolds of M.

Theorem 6.2.11(Restricting the Codomain of a smooth map) Suppose M is a smooth manifold (without boundary), $S \subseteq M$ is an immersed submanifold, and $F : N \to M$ is a smooth map whose image is contained in S. If F is continuous as a map from N to S, the $F : N \to S$ is smooth.(notice that we this theorem hold only when M is a smooth manifold without boundary, since it is only in that case that we have constructed slice charts for embedded submanifolds of M .)

Proof.Let p be a arbitrary point of N, let $q = F(p) \in S$ by Prop6.2.5 we can say there is a neighbourhood V of q in such that $\iota|_V : V \hookrightarrow M$ is a smooth embedding. by Thm6.1.13 there exist a smooth chart (W, ψ) for M that is a slice chart for V in M centered at q. The fact that (W, ψ) is a slice chart means that $(V_0, \hat{\psi})$ is a smooth chart for V, where $V_0 = W \cap V$ and $\hat{\psi} = \pi \circ \psi$, with $\pi : \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first k = dimS coordinates. Since $V_0 = (\iota|_V)^{-1}(W)$ is open in V, it is open in S in its given topology, by definition of subspace topology V_0 is also open in S, and $(V_0, \hat{\psi})$ is also a smooth chart for S. Since the smooth structure on V is induced by smooth structure of S.



Let $U = F^{-1}(V_0)$ since F is continuous it is a open subset of N containing p, choose a smooth chart (U_0, φ) for N such that $p \in U_0 \subseteq U$. Then the coordinate representation of $F : N \to S$ with respect to the charts (U_0, φ) and $(V_0, \hat{\psi})$ is

$$\hat{\psi} \circ \varphi^{-1} = \pi \circ (\psi \circ \varphi^{-1})$$

which is smooth because $F : N \to M$ is smooth.

Proposition 6.2.12 Let X be a topological space and let S be a subspace of X.If Y is a topological space, a map $F : Y \to S$ is continuous if and only is the the composition $\iota_S \circ F : Y \to X$ is continuous, where $\iota_S : S \hookrightarrow X$ is the inclusion map.

Corollary 6.2.13 Let M be a smooth manifold and $S \subseteq M$ be an embedded submanifold. Then every smooth map $F : N \to M$ whose image is contained in S is also smooth as a map from N to S.

Proof. Since $S \subseteq M$ has the subspace topology, a continuous map $F : N \to M$ whose image is contained in S is automatically continuous into S, by the previous Prop.

Definition 6.2.14 If M is a smooth manifold and $S \subseteq M$ is an immersed submanifold, then S is said to be **weakly embedded in M (or initial submanifolds)**if every smooth map $F : N \to M$ whose image lies in S is smooth as a map from N to S.

(Uniqueness of Smooth Structures on Submanifolds)

Theorem 6.2.15 Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. The subspace topology on S and the smooth structure described in Theorem 6.1.13 are the only topology and smooth structure with respect to which S is an embedded or immersed submanifold.

Proof. Suppose $S \subseteq M$ is an embedded k-dimensional submanifold. Theorem 6.1.13 shows that it satisfies the local k-slice condition, so it is an embedded submanifold with the subspace topology and the smooth structure of Theorem 6.1.13. Suppose there were some other topology and smooth structure on S making it into an immersed submanifold of some dimension. Let $\hat{S}$ denote the same set S, considered as a smooth manifold with the non-standard topology and smooth structure, and let $\hat{\iota} : \hat{S} \hookrightarrow M$ denote the inclusion map, which by assumption is an injective immersion (but not necessarily an embedding).

Because $\hat{\iota}(\hat{S}) = S$, Corollary 6.2.13 implies that $\hat{\iota}$ is also smooth when considered as a map from $\hat{S}$ to S. For $p \in \hat{S}$, the differential $d\hat{\iota}_p : T_p\hat{S} \to T_pM$ is equal to the composition

$$T_p\hat{S} \xrightarrow{d\hat{\iota}_p} T_pS \xrightarrow{d\iota_p} T_pM$$

where $\iota : S \hookrightarrow M$ is also inclusion. Because this composition is injective (since $\hat{S}$ is assumed to be a smooth submanifold of M), d Qp must be injective. In particular, this means that $\hat{\iota} : \hat{S} \to S$ is an immersion. Because it is bijective, it follows from the global rank theorem that it is a diffeomorphism. In other words, the topology and smooth manifold structure of $\hat{S}$ are the same as those of S .Which means it has uniques smooth structure.

Due to this uniqueness result, we can see that a subset $S \subseteq M$ is an embedded submanifold if and only if it satisfies the local slice condition, and and if so, its topology and smooth structure are uniquely determined. Because the local slice condition is a local condition, if every point $P \in S$ has a neighborhood $U \subseteq M$ such that $U \cap S$ is an embedded k-submanifold of U , then S is an embedded k-submanifold of M .

Theorem 6.2.16 Suppose M is a smooth manifold and $S \subseteq M$ is an immersed submanifold. For the given topology on S , there is only one smooth structure making S into an immersed submanifold.

Proof. Let $\iota : S \to M$ be the inclusion map, which is a smooth immersion. Let S another smooth structure such that $\iota' : S' \to M$ is also a smooth immersion. because S and S have the same topology, both ι, ι' are continuous, and Theorem 5.29 shows ι, ι' are also smooth and ι is clear bijective so this implies that S and S' are diffeomorpic with each other.

Theorem 6.2.17 If M is a smooth manifold and $S \subseteq M$ is a weakly embedded submanifold, then S has only one topology and smooth structure with respect to which it is an immersed submanifold.

Proof. Consider proof for previous Theorem, we can use the same method if there exists 2 smooth structure they must be diffeomorphic with each other.

Extending Functions from Submanifolds. Let M be a smooth manifold with or without boundary, and let $S \subseteq M$ be a smooth submanifold. If $f : S \to \mathbb{R}$ is a function, there are two ways we might interpret the statement "f is smooth": it might mean that f is smooth as a function on the smooth manifold S (i.e., each coordinate representation is smooth), or it might mean that it is smooth as a function on the subset $S \subseteq M$ (i.e., it admits a smooth extension to a neighborhood of each point). We adopt the convention that the notation $f \in C^\infty(S)$ always means that f is smooth in the former sense (as a function on the manifold S).

Lemma 6.2.18(Extension Lemma for Functions on Submanifolds). Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^\infty(S)$.

(a)If S is embedded, then there exist a neighbourhood U of S in M and a smooth function $\hat{f} \in C^\infty(U)$ such that $\hat{f}|_S = f$.

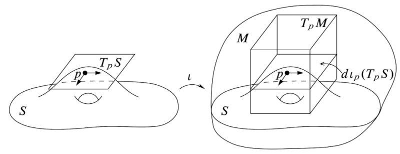
(b)If S is properly embedded, then the neighbourhood U in part(a) can be taken to be all of M

6.3 The Tangent Space to a Submanifold

Definition 6.3.1. Let M be a smooth manifold with or without boundary, and let $S \subseteq M$ be an immersed or embedded submanifold. Since the inclusion map $\iota : S \hookrightarrow M$ is a smooth immersion, at each point $p \in S$ we have an injective linear map $d_{\iota_P} : T_pS \to T_pM$. In terms of derivations, this injection works in the following way: for any vector $v \in T_pS$,the image vector$\hat{v} = d_{\iota_p}(v) \in T_pM$ acts on smooth functions on M by

$$\hat{v}f = d_{\iota_P}(v)f = v(f \circ \iota) = v(f|_S)$$

We adopt the convention of identifying T_pS with its image under this map, thereby thinking of T_pS as a certain linear subspace of T_pM.



Proposition 6.3.2. Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J, \gamma(0) = p$ and $\gamma'(0) = v$

Proof. Part I:By Thm4.1.6 we can say that there exist a smooth curve $\gamma : J \to S$ with v as it velocity. And let $\iota : S \hookrightarrow M$ be a inclusion map and $\iota \circ \gamma$ is the curve

Part II: If we have a curve γ it trivial to see that $\gamma'(0) = v$ and v is in T_pS

Proposition 6.3.3. Suppose M is a smooth manifold, $S \subseteq M$ is an embedded submanifold, and $p \in S$. As a subspace of T_pM the tangent space T_p is characterized by

$$T_pS = \{v \in T_pM : vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_S = 0\}$$

Proof. Part I: Suppose $v \in T_pS \subseteq T_pM$, and $v = d\iota_p(w)$ for some $w \in T_pS$, where $\iota : S \to M$ is inclusion. If f is any smooth real-valued function on M that vanishes on S, the $f \circ \iota \equiv 0$ so

$$vf = d_{\iota_p}(w)f = w(f \circ \iota) = 0$$

Part II: If $v \in T_pM$ satisfies $vf = 0$ whenever f vanishes. on S, we need to show that there is a vector $w \in T_pS$ such that $v = d\iota_p(w)$. Let $(x^1,..,x^n)$ be slice coordinates for S in some neighbourhood U of p, so that $U \cap S$ is the subset of U where $x^{k+1} = \ldots = x^n = 0$, and $(x^1,..,x^k)$ are coordinates for $U \cap S$. Because the inclusion map $\iota : S \cap U \hookrightarrow M$ has the coordinate representation

$$\iota(x^1,...,x^k) = (x^1,...,x^k,0,...,0)$$

in these coordinates, it follows that T_pS (that is, $d\iota_p(T_pS)$) is exactly the subspace of T_pM spanned by $\delta/\delta x^1|_p,...,\delta/\delta x^k|_p$. If we write the coordinate representation of v as

$$v = \textstyle\sum_n^{i=1} v^i \frac{\delta}{\delta x^i}|_p$$

we see that $v \in T_pS$ if and only if $v^i = 0$ for i>k.

Let φ be a smooth bump function supported in U that is equal to 1 in a neighbourhood of p. Choose an index j > k, and consider the function $f(x) = \varphi(x)x^j$, extends to be zero on $M\setminus\ supp\varphi$. Then f vanishes identically on S, so

$$0 = vf = \textstyle\sum_n^{i=1} v^i \frac{\delta(x)x^i}{\delta x^i}(p) = v^j$$

Thus $v \in T_pS$ as desired.

Note the following proposition is used to characterize T_pS in the embedded case only.

Proposition 6.3.4. Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi : U \to N$ is any local defining map for S, then $T_pS = Kerd\Phi_p : T_pM \to T_{\Phi_p}N$ for each $p \in S \cap U$.

Proof. Since that we identify T_pS with the subspace $d\iota_p(T_pS) \subseteq T_pM$, where $\iota : S \hookrightarrow M$ is the inclusion map. Because $\Phi \circ \iota$ is constant on $S \cap U$, it follows that $d\Phi_p \circ d\iota_p$ is the zero map from T_pS to $T_{\Phi(p)N}$, and therefore $Imd_{\iota_p} \subseteq Kerd\Phi_p$. On the other hand, $d\Phi_p$ is surjective by the definition of the defining map, so the rank-nullity law implies that

$$dimKerd\Phi_p = dimT_pM - dimT_{\Phi_{(p)}} = dimT_p(S) = dimImd\iota_p$$

which implies that $Imd\iota_p = Kerd\Phi_p$

Corollary 6.3.5. Suppose $S \subseteq M$ is a level set of a smooth submersion $\Phi = (\Phi^1, ..., \Phi^n) : M \to \mathbb{R}^k$ a vector $v \in T_pM$ is tangent to S if and only if $v\Phi^1 = ... = v\Phi^k = 0$

Proof. For $v \in T_pM$ then we have $d\Phi^i(v) = v\Phi^i = v\Phi^1, v\Phi^2, .., v\Phi^k$ and v is tangent to S if $d\Phi^i(v) = 0$ and and $d\Phi^i(v) = v\Phi^i = v\Phi^1, v\Phi^2, .., v\Phi^k = 0$ and $v \in Kerd\Phi$ but $v \in Kerd\Phi$ if and only if $v\Phi^1, v\Phi^2, .., v\Phi^k = 0$

Exercise 6.3.6. Suppose $S \subseteq M$ is a level set of a smooth map $\Phi : M \to N$ with constant rank. Show that $T_pS = Kerd\Phi_p$ for each $p \in S$.

Proof. By Constant-Rank Level Set Theorem when we have $S \subseteq M$ is a level set of a smooth map $\Phi : M \to N$ with constant rank. for each $p \in S$ we have smooth chart (U, φ) centered at p, and each $S \cap U$ is a local slice chart for S, thus it is a embedded submanifold of S, then by Prop 6.3.4 we have $T_pS = Kerd\Phi_p$ for each p.

Definition 6.3.7. If $p \in$; a vector $v \in T_pM \diagdown T_p\delta M$ is said to be inward- pointing if for some $\varepsilon > 0$ there exists a smooth curve $\gamma : [0, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and it is outward-pointing if there exists such a curve whose domain is $(-\varepsilon, 0]$.

Proposition 6.3.8. Suppose M is a smooth n-dimensional manifold with boundary, $p \in M$; and (x^i) are any smooth boundary coordinates defined on a neighborhood of p. The inward-pointing vectors in T_pM are precisely those with positive x^n-component, the outward-pointing ones are those with negative x^n-component, and the ones tangent to δM are those with zero x^n-component. Thus, T_pM is the disjoint union of $T_p\delta M$; the set of inward-pointing vectors, and the set of outward- pointing vectors, and $v \in T_pM$ is inward-pointing if and only if $-v$ is outward- pointing.

Proof. By the definition 6.3.7. a vector $v \in T_pM \diagdown T_p\delta M$ is said to be inward- pointing if for some $\varepsilon > 0$ there exists a smooth curve $\gamma : [0, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and it is outward-pointing if there exists such a curve whose domain is $(-\varepsilon, 0]$ and $[0, \varepsilon) \cap \{0\} \cap (-\varepsilon, 0] = (-\varepsilon, \varepsilon)$ and each $v \in T_pM \diagdown T_p\delta M$ such that T_pM is the disjoint union of $T_p\delta M$. And if $v \in T_pM$ is inward-pointing we have a smooth curve $\gamma : [0, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, which means there is also exist another smooth curve $\gamma : (-\varepsilon, \varepsilon)/[0, \varepsilon) = (-\varepsilon, 0] \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = -v$

Definition 6.3.9.f M is a smooth manifold with boundary, a **boundary defining function** for M is a smooth function $f : M \to [0, \infty)$ such that $f^{-1}(0) = \delta M$ and $df_p \neq 0$ for all $p \in \delta M$. For example, $f(x) = 1 - |x|^2$ is a boundary defining function for the closed unit ball $\bar{\mathbb{B}}^n$.

Proposition 6.3.10. Every smooth manifold with boundary admits a boundary defining function.

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a collection of smooth charts whose domains cover M.For each α, define a smooth function $f_\alpha : U_\alpha \to [0, \infty)$ as follows: if U_α is an interior chart, let $f_\alpha \equiv 1$; while if U_α is a boundary chart, let $f_\alpha(x^1, .., x^n) = x^n$ (the nth coordinate function in that chart). Thus, $f_\alpha(p)$ is positive if $p \in IntM$ and zero if $p \in \delta M$. Let Ψ_α be a partition of unity subordinate to this cover, and let $f = \sum_\alpha \Psi_\alpha f_\alpha$. Then f is smooth, identically zero on δM; and strictly positive in IntM. To see that df does not vanish on δM; suppose $p \in \delta M$ and v is an inward-pointing vector at p. For each α such that $p \in U_\alpha$, we have $f_\alpha(p) = 0$ and $df_\alpha|_p(v) = dx^n|_p(v) > 0$ by Prop 6.3.8.

$$df_p(v) = \sum_\alpha (f_\alpha(p) d\psi_\alpha|p(v) + \psi_\alpha(p) df_\alpha|_p(v))$$

For each α, the first term in parenthese is zero and the second is nonegative, and there is at least one α for which the second term is positive. Thus $df_p(v) > 0$, which implies that $df_p \neq 0$

Exercise 6.3.11. Suppose M is a smooth manifold with boundary, f is a boundary defining function, and $p \in \delta M$. Show that a vector $v \in T_pM$ is inward-pointing if and only if $vf > 0$, outward-pointing if and only if $vf < 0$, and tangent to δM if and only if $vf = 0$.

Proof. Suppose that f is a boundary defining function, so $f : M \to [0,\infty)$ which mean that $f > 0$ and by prop 6.3.8 we see that if The inward-pointing vectors in T_pM are precisely those with positive x^n-component, the outward-pointing ones are those with negative x^n-component and and the ones tangent to δM are those with zero x^n-component, thus since f is always positive we can see that $v \in T_pM$ is inward-pointing if and only if $vf > 0$, outward-pointing if and only if $vf < 0$, and tangent to δM if and only if $vf = 0$.

6.4 Submanifolds with Boundary

Definition 6.4.1. If M is a smooth manifold with or without boundary, a **smooth submanifold with boundary in M** is a subset $S \subseteq M$ endowed with a topology and smooth structure making it into a smooth manifold with boundary such that the inclusion map is a smooth immersion.

If the inclusion map is an embedding, then it is called an **embedded submanifold with boundary**; in the general case, it is an **immersed submanifold with boundary**. The terms **codimension** and **properly embedded** are defined just as in the submanifold case.

If M is a smooth manifold with or without boundary, a **regular domain in M** is a properly embedded codimension-0 submanifold with boundary.

Proposition 6.4.2. Suppose M is a smooth manifold without boundary and $D \subseteq$ is a regular domain. The topological interior and boundary of D are equal to its manifold interior and boundary, respectively.

Proof. Suppose $p \in D$ is arbitrary. If p is in the manifold boundary of D, by Thm 5.2.15 we can say there exist a smooth boundary chart (U,φ) for D centered at p and a smooth chart (V,ψ) for M centered at p in which F has the coordinate representation $F(x^1,..,x^m) = (x^1,..,x^m,0,..,0)$, where n = dimM = dimD. Since D has the subspace topology, $U = D \cap W$ for some open subset $W \subseteq M$, so $V_0 = V \cap W$ is a neighborhood of p in M such that $V_0 \cap D$ consists of all the points in V0 whose x^m coordinate is nonnegative. Thus every neighborhood of p intersects both D and $M\ D$, so p is in the topological boundary of D.

On the other hand, suppose p is in the manifold interior of D. The manifold interior is a smooth embedded codimension-0 submanifold without boundary in M; so it is an open subset by Proposition 6.1.2 Thus p is in the topological interior of D.

Conversely, if p is in the topological interior of D, then it is not in the topological boundary, so the preceding argument shows that it is not in the manifold boundary and hence must be in the manifold interior. Similarly, if p is in the topological boundary, it is also in the manifold boundary.

Proposition 6.4.3. Suppose M is a smooth manifold and $f \in C^\infty(M)$.

(a).For each regular value b of f, the sublevel set $f^{-1}((-\infty,b])$ is a regular domain in M

(b)If a and b are two regular values of f with a<b, then $f^{-1}([a,b])$ is a regular domain in M

Proof.(a)Let k be dimension of M, then we can say that $f^{-1}((-\infty,b))$ satisfies the local k-slice condition, with boundary slice chart around each point of $f^{-1}(\{b\})$

Definition 6.4.4. A set of the form $f^{-1}((-\infty, b])$ for b a regular value of f is called a regular sublevel set of f . Part previous part (a) of the preceding theorem shows that every regular sublevel set of a smooth real-valued function is a regular domain. If $D \subseteq M$ is a regular domain and $f \in C^{\infty}(M)$ is a smooth function such that D is a regular sublevel set of f , then f is called a defining function for D.

Proposition 6.4.6(Properties of Submanifolds with Boundary). Suppose M is a smooth manifold with or without boundary.

(a) Every open subset of M is an embedded codimension-0 submanifold with (possibly empty) boundary.

(b) If N is a smooth manifold with boundary and $F : N \to M$ is a smooth embedding, then with the subspace topology F(N) is a topological manifold with boundary, and it has a smooth structure making it into an embedded submanifold with boundary in M .

(c) An embedded submanifold with boundary in M is properly embedded if and only if it is closed.

(d) If $S \subseteq M$ is an immersed submanifold with boundary, then for each $p \in S$ there exists a neighborhood U of p in S that is embedded in M .

Proof. (a) By Open submanifolds example we talked about in Prop 6.1.2 it's clear that every open subset of M is a embedded submanifold of codimension 0, and this submanifold may or may not have boundary depend on it intersect the boundary of M or not.

(b) In Prop 6.1.3 we have discussed about that the images of F(N) is a topological manifold with boundary, and it has a smooth structure making it into an embedded submanifold in M, and it has same smooth structure as M it is a smooth manifold with boundary in M.

(c) In Def 6.1.9 we know an embedded submanifold in M is properly embedded if and only if it is closed and due to F is a proper embedded map it preserves the boundary structure of M.

(d) By Prop 6.2.5

Definition 6.4.7(Exhaustion function).M is a topological space, an exhaustion function for M is a continuous function $f : M \to \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$ is compact for each $c \in \mathbb{R}$ The name comes from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}((-\infty, c])$ form an exhaustion of M by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets.

Definition 6.4.8. If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function for D. If D is compact, then f can be taken to be a smooth exhaustion function for M

Definition 6.4.7(Local slice chart for manifold with boundary). Suppose M is a smooth manifold (without boundary). If $(U, (x^i))$ is a chart for M; a k-dimensional half-slice of U is a subset of the following form for some constants $c^{k+1}, ..., c^n$:

$$\{(x^1, .., x^n)) \in U : c^{k+1}, ..., x^n = c^n, and x^k \geq\}$$

We say that a subset $S \subseteq M$ satisfies the local k-slice condition for submanifolds with boundary if each point of S is contained in the domain of a smooth chart $(U, (x^i))$ such that $S \cap U$ is either an ordinary k-dimensional slice or a k-dimensional half-slice. In the former case, the chart is called an interior slice chart for S in M, and in the latter, it is a boundary slice chart for S in M.

Theorem 6.4.8. Let M be a smooth n-manifold without boundary. If $S \subseteq M$ is an embedded k-dimensional submanifold with boundary, then S satisfies the local k-slice condition for submanifolds with boundary. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition for submanifolds with boundary, then with the subspace topology, S is a topological k-manifold with boundary, and it has a smooth structure making it into an embedded submanifold with boundary in M.

Proof. Consider Thm 6.1.13.

Theorem 6.4.9.(Restricting Maps to Submanifolds with Boundary). Suppose M and N are smooth manifolds with boundary and $S \subseteq M$ is an embedded submanifold with boundary.

(a) RESTRICTING THE DOMAIN:If $F : M \to N$ is a smooth map,then $F|_S : S \to N$ is smooth.

(b) RESTRICTING THE CODOMAIN: If $\partial M = \emptyset$ and $F : N \to M$ is a smooth map whose image is contained in S, then F is smooth as a map from N to S

Proof.

Same as Thm 6.2.9 and 6.2.11 since the definition of smooth embedding is same for bot smooth manifold with or without boundary .

Reference

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