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Abstract Algebra

Groups In Topology

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February 8, 2022

1 Introduction

In this paper, we want to explore the elementary ideas of simplificial complexes, boundary and homology theory that all together lead to homology groups. All of this also belongs to the area of Mathematics which is called Topology. Topology is the aspect of Mathematics that kind of extends Geometry. We are allowed to use deformations, twistings, and stretchings of objects. Note that tearing, however, is not allowed. The most popular examples of topology ideas are a circle which topologically equivalent to an ellipse (into which it can be deformed by stretching) together with a sphere which is equivalent to an ellipsoid. Taken with analysis and algebra, those areas of Mathematics are considered the main areas. It has a huge amount of applications both in pure and applied mathematics.

Speaking about homology, it is a part of algebraic topology. This just means that we use the Abtract Algebra to study topological spaces. The basic goal of algebraic topology is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence. In other words, it is the way of connecting sequence of algebraic objects(modules, abelian groups, etc) to other mathematical objects. In algebraic topology, again, it is mostly topological space.

The Homology Theory is said to have its beginning with Euler's polyhedron formula (Recall: $V + F - E = 2$) and later followed by Riemann's definition of genus (simply saying, number of "holes" of a surface) and n-fold connectedness in the middle of 19 century which were followed by Betti's proof in 1871 of the independence of "homology numbers" from the choice of basis. Homology itself was developed as a way to analyse and classify manifolds according to their cycles – closed loops (or more generally submanifolds) that can be drawn on a given n dimensional manifold but not continuously deformed into each other. These cycles are also sometimes thought of as cuts which can be glued back together, or as zippers which can be fastened and unfastened. Cycles are classified by dimension. We will come back to this idea(s) later in the paper.

2 Simplicial Complexes and Homology Groups

2.1 Simplificial Simplexes and Complexes

The term *simplificial complex* can refer to two different at first glance concepts. One of them is *abstract simplificial complex* which describes a family of sets that is closed under taking subsets. It is purely combinatorics part of something we call just a *simplificial complex* which is usually called *geometric simplificial complex.* The second concept describes a geometric object in Euclidean space R^n consisting of simplexes[1] of various dimensions kind of glued together according to certain rules. In this paper we will be mostly dealing with geometric concept of *simplificial complex.*

How would we approach this concept though? Let's recall what simplexes (or simplices) are in small dimensional spaces:

- in 0-dimensional space - **0-simplex** - a point
- in 1-dimensional space - **1-simplex** - a line segment
- in 2-dimensional space - **2-simplex** - a triangle
- in 3-dimensional space - **3-simplex** - a tetrahedron

and so on. If those simplexes are given orientation(they are being directed) they are called **oriented** k-simplexes for the k-dimensional space. This can remind one the concept of vectors. As far as the simplex describes the same path, the description is the same. If the path is exactly opposite direction of the initial one, it can be written as the one but with the negative sign. For example, for the oriented 3-simplex $P_1P_2P_3 = P_2P_3P_1 = P_3P_1P_2 = -P_1P_3P_2 = -P_2P_1P_3 = -P_3P_2P_1$. Note that for k-simplex $P_1P_2P_3...P_k = P_{i_1}P_{i_2}P_{i_3}...P_{i_k}$ if $\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & k \\ i_1 & i_2 & i_3 & \cdots & i_k \end{smallmatrix}\right)$ is an even permutation, and it is equal to $-P_{i_1}P_{i_2}P_{i_3}...P_{i_k}$ in another case.

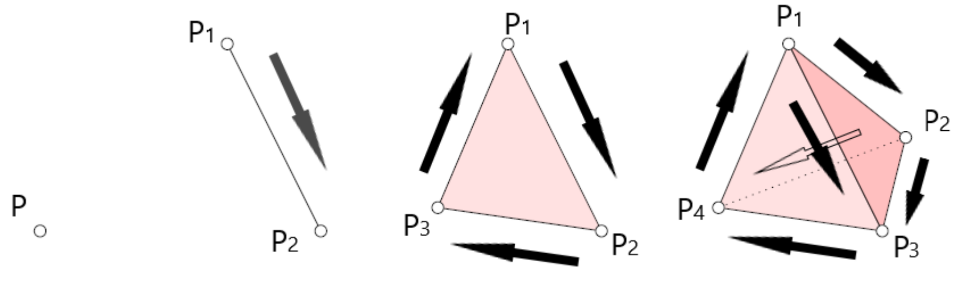


Figure 2.1: From left to right: 0-,1-,2-,3-simplexes, all oriented

[1]Briefly, a simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.

Starting from now on, the adjective *oriented* will be omitted as we will be dealing with only this type of simplexes. We will also be looking mostly at examples of k-simplexes for $k = 0, 1, 2, 3$ as for these cases the simplexes can be easily visualised.

The **boundary** of a k-simplex $\partial_k(P_1 P_2 ... P_k)$ consists of a union of simplexes of lower dimensions, which have been glued by their boundaries. The boundary of the 0-simplex is defined to be the *empty simplex* which is denoted by 0. Also, taking a 1-simplex $P_1 P_2$ and a 2-simplex $P_1 P_2 P_3$ the boundaries are defined as $\partial_1(P_1 P_2) = P_2 - P_1$ and $\partial_2(P_1 P_2 P_3) = P_2 P_3 - P_1 P_3 + P_1 P_2$.

$$\cancel{P_1}P_2 \qquad P_1\cancel{P_2} \qquad\qquad \cancel{P_1}P_2P_3 \qquad P_1\cancel{P_2}P_3 \qquad P_1P_2\cancel{P_3}$$

$$+ \qquad\quad - \qquad\qquad\quad + \qquad\quad - \qquad\quad +$$

The same way one can calculate other boundaries ∂_k for $k > 2$: list k-simplex k times, every i'th time without i'th vertex, the signs(+ or -) alternate starting with + before the first simplex of lower dimension. Each individual *summand* (together with + or - before it) of the boundary of a simplex is called a **face** of a simplex. Note that the boundary of an k-simplex contains 0-faces (0-dimensional faces), 1-faces (1-dimensional faces), . . . , and all the way up to (k-1)-faces.

A **simplicial complex** is an object that results from the union of several simplices(or vice versa - a space divided into simplexes), such that:

- Each point of the space belongs to finite number(> 0) of simplexes;
- The intersection of any two simplexes of the simplicial complex is a face of each of the simplices

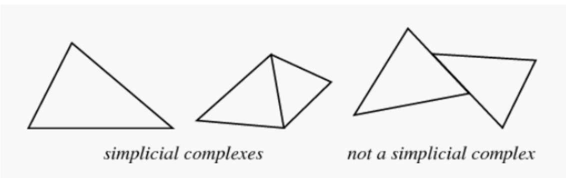


Figure 2.2: Examples

On picture 2.2 we can see examples of configurations only some of which are simplificial complexes. The example of not a simplificial complex breaks on the second condition: the intersection of the 2 simplexes is not the face of each of them, but only part of the face.

What happens to the boundary when we glue two simplexes together? The proper definition will be given shortly but let's take a particular example on the Fig.2.3 where the simplexes are kind of "chained" together. The boundary of the union of this chain would consist of the points A_1 and A_6. The points that have been used for gluing are no longer part of the boundary, as they sort of cancel each other out.

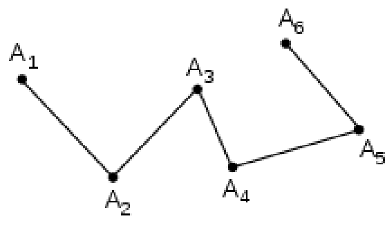


Figure 2.3: 5 1-simplexes

2.2 Chains, Cycles and Boundaries

A **free abelian group with a basis** X is an abelian group having a generating set X. X is a **generating set** of a group G if each nonzero element a in G can be expressed uniquely(up to order of summands) in the form $a = n_1x_1 + n_2x_2 + ... + n_rx_r$, for $n_i \neq 0$ in $\mathbb{Z}$ and distinct $x_i \in X$.
First, take all the k-simplexes σ_i of a simplicial complex S and give each one of them an orientation. The group $C_k(S)$ of (oriented) k–chain complex of S is the free abelian group generated by all the k-simplexes of S. In other words, C_k is formed by all the possible linear combinations of the k-simplexes of S:

$$\sum_{i=1} m_i\sigma_i, m_i \in \mathbb{Z}$$

Note: Associativity and communicativity follows from the same properties of addition on $\mathbb{Z}$. The identity element is $0 = 0\sum(\sigma_i)$. The inverse of m is $-m$ since $m + (-m) = 0$.

A negative sign of one of the m_i can be interpreted as giving the opposite orientation to the k-simplex accompanied by the m_i. For example, if surface S is taken in 3-dimensional space, an element of $C_2(S)$ has a form $m_1P_2P_3P_4 + m_2P_1P_3P_4 + m_3P_1P_2P_4 + m_4P_1P_2P_3$, an element of $C_1(S)$ has a form $m_1P_1P_2 + m_2P_1P_3 + m_3P_1P_4 + m_4P_2P_3 + m_5P_2P_4 + m_6P_3p_4$, and an element of $C_0(S)$ has a form $m_1P_1 + m_2P_2 + m_3P_3 + m_4P_4$. One can see that if the surface S is in n-dimensional space, the number of different simplexes as an element of $C_k(S)$ for $k < n$ is $\binom{n+1}{k+1}$.
Addition of chains can be approached as taking the algebraic sum of the coefficients of each occurrence in the chains in the fixed simplex.

Example 1. $(2P_2P_3P_4 - 6P_1P_3P_4 + 3P_1P_2P_4) + (-7P_2P_3P_4 + P_1P_3P_4 + 2P_1P_2P_3) = -5P_2P_3P_4 - 5P_1P_3P_4 + 3P_1P_2P_4 + 2P_1P_2P_3$

Let's also define the **boundary operator** on the chain complexes: $\partial_k : C_k \to C_{k-1}$, thus, the boundary operator takes each k-chain to a (k-1)-chain. For this to work, we also need to define $C_{-1}(S)$ as $\{0\}$, the trivial group of one element, so $\partial_0 \in C_{-1}(S)$. Now, as a k-chain is a linear combination of k-simplexes, the boundary of a k-chain is defined as the linear combination of the boundaries of its k-simplexes(which is a property of a free abelian group):

$$\partial_k(\sum_{i=1} m_i\sigma_i) = \sum_{i=1} m_i\partial_k(\sigma_i)$$

Example 2. $\partial_1(3P_1P_2 - 4P_1P_3 + 5P_2P_4) = 3\partial_1(P_1P_2) - 4\partial P_1P_3 + 5\partial P_2P_4 = 3(P_2 - P_1) - 4(P_3 - P_1) + 5(P_4 - P_2) = P_1 - 2P_2 - 4P_3 + 5P_4$

This means that ∂_k is actually a group homomorphism which is called a **boundary homomorphism**. Recall that the kernel of a homomorphism is generally the inverse image of the identity element and the image is the set of elements in C_{k-1} that have an inverse map.

A **k-cycle** is a k-chain with empty boundary, $\partial_k = 0$. Since ∂ commutes with addition, we have a group of k-cycles, denoted as $Z_k \leq C_k$ which is a subgroup of the group of k-chains. In other words, the group of k-cycles is the kernel of the k-th boundary homomorphism, $Z_k = ker(\partial_k)$. Since the chain groups are abelian, their cycle subgroups are abelian as well. Take $k = 0$ as an example: the boundary of every vertex S is zero, and 0 is indeed the only element in $C_{-1}(S)$. Hence, $Z_0 = \ker(\partial_0) = C_0$.

A **k-boundary** c is a k-chain that is the boundary of a (k+1)-chain d. Since ∂ commutes with addition, we have a group of k-boundaries, denotes as $B_k \leq C_k$. In other words, the group of k-boundaries is the image of the (k+1)-st boundary homomorphism, $B_k = \mathrm{Im}(\partial_{k+1})$. Since the chain groups are abelian, their boundary subgroups are too. Let's look back at the case $k = 0$: every 1-chain consists of some number of edges with twice as many endpoints. Taking the boundary cancels duplicate endpoints in pairs leaving an even number. Hence, every 0-chain with an even number of vertices in each component is a 0-boundary.

Example 3. *Let's go back to the 3-simplex(triangle) S in the figure 2.1. If we take a chain $z = P_1P_2 + P_2P_3 + P_3P_1$, we can see that the boundary $\partial_1(z) = \partial_1(P_1P_2) + \partial_1(P_2P_3) + \partial_1(P_3P_1) = P_2 - P_1 + P_3 - P_2 + P_1 - P_3 = 0$. Thus, z describes a circuit(a 1-cycle) around the triangle. On the other hand, a chain $q = 2P_1P_2 + P_2P_3 + P_3P_1$ is not a cycle as $\partial_1(q) = 2\partial_1(P_1P_2) + \partial_1(P_2P_3) + \partial_1(P_3P_1) = 2P_1 - 2P_2 + P_3 - P_2 + P_1 - P_3 = P_1 - P_2 \neq 0$*

Example 4. *Let's go back to the 3-simplex (tetrahedron) S in the figure 2.1 and compute the groups $Z_k(S)$ and $B_k(S)$. For any $k > 2$, $C_k = Z_k = B_k = 0$ since the largest possible dimension for a chain in this example is 2 (triangle). Also as $B_k = \partial(C_{k+1})$, we can conclude that $B_2 = 0$. Recall that $C_0 = Z_0 = \ker(\partial_0)$, $C_1(S) = \langle P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4 \rangle$ and $C_2(S) = \langle P_1P_2P_3, P_1P_2P_4, P_1P_3P_4, P_2P_3P_4 \rangle$. $B_0(S)$ is generated by $P_2 - P_1, P_3 - P_1, P_4 - P_1, P_3 - P_2, P_4 - P_2, P_4 - P_3$, although, it is free abelian only on $P_2 - P_1, P_3 - P_1, P_4 - P_1$ (for ex. since $P_3 - P_2 = (P_3 - P_1) - (P_2 - P_1)$ while each element should have a unique representation). We can describe an element of $Z_1(S)$ as a chain of edges P_iP_j where a beginning point is an end point too. For the tetrahedron all 1-cycles are $\{P_2P_3 + P_3P_4 + P_4P_2, P_1P_4 + P_4P_3 + P_3P_1, P_1P_2 + P_2P_4 + P_4P_1, P_1P_3 + P_3P_2 + P_2P_1\}$ which do indeed generate $Z_1(S)$. On the way, those cycles describe the boundaries of all 2-simplexes which means that $B_1(S) = Z_1(S)$. Let's show that $Z_2(S)$ is infinite cyclic group generated by $P_2P_3P_4 + P_3P_1P_4 + P_1P_2P_4 + P_2P_1P_3$. Let $c \in C_2(S)$ with $r_1P_1P_2P_3$ and $r_2P_1P_2P_4$. The common* *edge P_1P_2 has $(r_1 - r_2)$ as a coefficient which has to be 0 for c to be a cycle. Therefore, all coefficients before each of 2-simplexes have to be equal.*

2.3 Homology Groups

Theorem 2.1 (Fundamental Lemma of Homology)**.** $\forall k \in \mathbb{Z}^+, \forall$ *k-chains* $\tau, \partial_{k-1} \circ \partial_k(\tau) = 0$

Proof I. Let τ be a k-chain, thus, it is a k-dimensional figure. The boundary, $\partial_{k-1}(\tau)$, consists of all k-1)-faces of τ . Every (k-1)-face of τ belongs to exactly two k-faces which are opposite to each other, so $\partial_{k-1}(\partial_k(\tau)) = 0$. □

Proof II. Note that because $\partial_k : C_k \to C_{k-1}$ is a homomorphism for any $k \geq 0$, the composition $\partial_{k-1} \circ \partial_k : C_k \to C_{k-2}$ is a homomorphism $\forall k \geq 1$ as well. Using the Theorem 39.12 from the book[2], it is enough to show that for a k-simplex τ, $\partial_{k-1} \circ \partial_k(\tau) = 0$. Let's take $\tau = P_1P_2P_3...P_k$, then the boundary $\partial_k(\tau) = \sum_{i=1}^{k}(-1)^{i+1}P_1...P_{i-1}P_{i+1}...P_k$.

We know it is a (k-1)-chain, so let's figure out the coefficient of a simplex where P_i and P_j are missing $(i < j)$: we could get such a simplex 2 ways(and for sure both of them happened): when i-th P got "cancelled" during the first step of finding the boundary and j-th P had to get deleted during the second step, and vice versa. Coefficient before the first one after the first step of the sum the coefficient would be $(-1)^{i+1}$ and after the second step it would also get multiplied by $(-1)^{j+1-1}$. Together it would give us $(-1)^{i+j-1}P_1P_2...P_{i-1}P_{i+1}...P_{j-1}Pj+1...P_k$. Coefficient before the second such a simplex can be found the same way: after the first step it would get multiplied by $(-1)^{j+1}$, after the second one by $(-1)^{i+1}$. This gives $(-1)^{i+j+2}P_1P_2...P_{i-1}P_{i+1}...P_{j-1}Pj+1...P_k$, and together in summation it would result in the coefficient $(-1)^{i+j-1} + (-1)^{i+j+2} = (-1)^{i+j-1}(1 + (-1)^3) = 0$. □

It follows that every k-boundary is also a k-cycle or, equivalently, that $B_p \leqslant Z_k$. Figure 2.4 illustrates the subgroup relations among the three types of groups and their connection across dimensions through boundary homomorphisms.

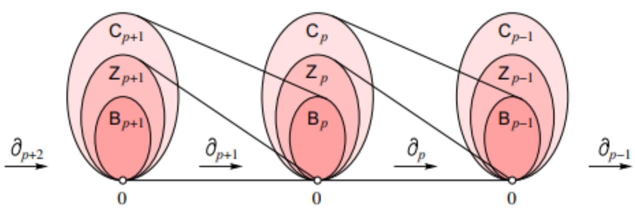


Figure 2.4:

The factor group $H_k(S) = Z_k(S)/B_k(S) = \frac{\ker(\partial)}{\mathrm{Im}(\partial)}$ is called **k-dimensional homology group** of S. We also say that an n-cycle x of a k-complex K is **homologous to zero** if it is the boundary of an (n + 1)-chain of K, $n < k$. A boundary is then any cycle that is homologous to zero. This relation

[2]Fraleigh, J. B., A First Course in Abstract Algebra, 8th Edition

is written $x \sim 0$. It is an equivalence relation: for two chains x, y, $(x - y) \sim 0 \Longleftrightarrow x \sim y$, and we call x and y **homologous**.

Example 4 (Continued)**.** *Let's compute the $H_k(S), \forall k \geq 0$.*

Since $\forall k \geq 3$ we have $C_k(S) = Z_k(S) = B_k(S) = 0$, $H_k = 0$ respectively. $Z_2(S)$ is an infinite cyclic group while $B_2(S) = 0$. Thus, $H_2(S)$ is also an infinite cyclic group and $H_2(S) \simeq \mathbb{Z}$. Also, since $Z_1(S) = B_1(S)$ the factor group is the trivial group of one element(identity) and, therefore, $H_1(S) = 0$. Recall that $Z_0(S)$ is a free abelian group (with all 4 vertices as generators) and $B_0(S) = \langle P_2 - P_1, P_3 - P_1, P_4 - P_1, P_3 - P_2, P_4 - P_2, P_4 - P_3 \rangle$. Let $z \in Z_0(S)$ such that z_i is a coefficient before P_i. Then

$$z - (z_2(P_2 - P_1) + z_3(P_3 - P_1) + z_4(P_4 - P_1)) = rP_1$$

for some r. Therefore, $z \in B_0(S) + rP_1$ which means that every coset contains an element rP_1 for some r. If a coset contains an element $r'P_1$ then $(r' - r)P_1 \in B_0(S)$. The only multiple of P_1 that is the boundary is 0, so $r = r'$. Thus, the coset contains exactly one element of the form rP_1. If we choose rP_1 as representatives of the cosets, we can see that $H_0(S)$ is infinite cyclic and, therefore, $H_0(S) \simeq \mathbb{Z}$.

Example 5. *The same way we defined the groups C_k, B_k, Z_k and the boundary operator ∂_k, we can define:*

- *group $C^{(k)}$ of k-cochain complex of S which describes the same group as $C_k(S)$*
- *coboundary operator $\delta^{(k)} : C^{(k)} \to C^{(k+1)}$ that maps the group of k-cochains to the group of (k+1)-cochains. It is a dual homomorphism as well so*

$$\delta^{(k)}(\sum_{i=1} m_i \sigma_i) = \sum_{i=1} m_i \delta^{(k)}(\sigma_i)$$

- *group of k-cocycles $Z^{(k)}(S)$ of S is the kernel of the coboundary homomorphism $\delta^{(k)}$. It describes a cyclic cochain that has a zero boundary*
- *group of k-coboundaries $B^{(k)}$ of $C^{(k)}(S)$ to be the image of $\delta^{(k-1)}$ or, more specifically, $\delta^{(k-1)}[C^{(k-1)}(S)]$. It describes the space of k-cochains which are boundaries of (k-1)-cochains. Because $\delta^{(k)}[B^{(k)}] = \delta^{(k)}(\delta^{(k-1)}[C^{(k-1)}(S)]) = 0$, we see that $B^{(k)}(S) \leq Z^{(k)}(S)$*
- *k-dimensional cohomology group $H^{(k)}(S)$ of S is a factor group $\frac{Z^{(k)}(S)}{B^{(k)}(S)}$*

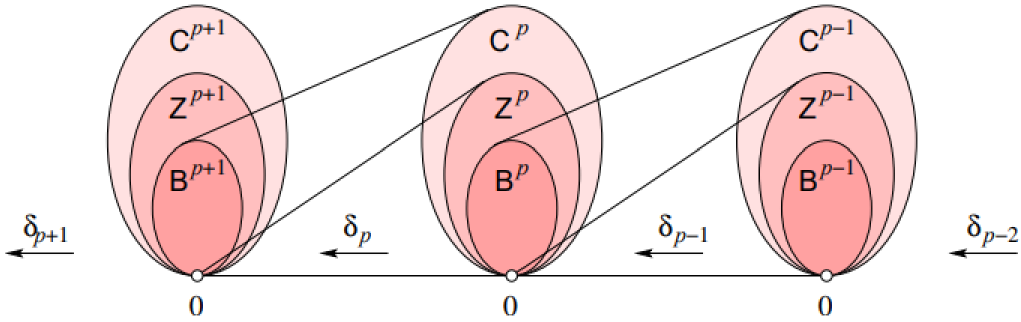


Figure 2.5: Relations between $C^{(k)}$, $B^{(k)}$, $Z^{(k)}$

For example, $\delta^{(0)(P_1)} = P_1P_2 + P_1P_3 + P_1P_4$ *as* P_1 *is a summand of each of the pairs described in the sum. The same way,* $\delta^{(1)(P_1P_2)} = P_1P_2P_3 + P_1P_2P_4$ *and* $\delta^{(2)(P_1P_2P_3)} = P_1P_2P_3P_4$*.*
Recall the Fundamental Lemma of Homology which says that $\partial(\partial(\sigma)) : C_{k+1} \to C_{k-1}$ *is the zero homomorphism. We therefore have* $\langle \delta \circ \delta(\varphi), \sigma \rangle = \langle \delta(\varphi), \partial(\sigma) \rangle = \langle \varphi, \partial \circ \partial(\sigma) \rangle = 0$*, where* φ *is a (k-1)-cochain. In other words,* $\delta(\delta) : C^{(k-1)} \to C^{(k+1)}$ *is also the zero homomorphism. Hence the coboundary groups are subgroups of the cocycle groups and we have the familiar picture, except that the maps now go from right to left, as in figure 2.5.*

3 Computations of Homology Groups

This section will be devoted to further computations of homology groups of certain important spaces. Consider two spaces, sphere and tetrahedron. For a painter, it is impossible to draw a perfect surface of a sphere by tetrahedrons because a sphere has a curved surface while the face of a tetrahedron are all plane. However, image you have a piece plasticine, it is easy to knead it from a solid sphere into a solid tetrahedron without being torn or cut, from a tetrahedron to a sphere. In topology, we will say they are homeomorphic with each other and also this is what *triangulation* of a space meaning. In general, triangulation refers a division into n-simplexes for any $n \geqslant 0$. Based on this fact, we say a **triangulation of the space** is the original division of the space which is divided up into pieces in the way we mentioned before, and the space can be deformed to look like a part of some Euclidean space $\mathbb{R}^n$ and the pieces into which the space was divided appear after this deformation as part of a simplical complex near each point.

Now, before we actually start to talk about the concept, there are two important invariance properties of homology groups we need to know:

- The homology groups of a space are defined in terms of a triangulation, and the group will be same no matter how the space is triangulated.
- If one triangulated space is homeomorphic to another, the homology groups of two spaces are the same in each dimension n.

Two important types of spaces in topology are sphere and cells

- The n-sphere $\mathbb{S}^n$ is a set of points a distance of 1 unit from the origin in (n+1 dimension) Euclidean space $\mathbb{R}^{n+1}$. (The 2-sphere $\mathbb{S}^2$ is usually called the surface of a sphere in $\mathbb{R}^3$, $\mathbb{S}^1$ is the rim of circle, and $\mathbb{S}^0$ is the disjoint union of two points)
- The n-cell or n-ball $\mathbb{E}^n$ is the set of all points in $\mathbb{R}^n$ a distance ≤ 1 from the origin. ($\mathbb{E}^3$ usually think as a solid sphere, $\mathbb{E}^2$ is a region of circle, and $\mathbb{E}^0$ is a line segment.)

3.1 Connected and Contractible Spaces

Definition 3.1.1. Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$.

Example 1. By second invariance properties and our computation of $H_2(S^2)$ and $H_0(S^2)$ implies that they both isomorphic to $\mathbb{Z}$ and $H_1(S^2) = 0$.

Definition 3.1.2. A space is **connected** if any two point can be joined by a path lying totally in the space. And a space is **not connected** if there exist a separation of the space.

(A separation of a space is a pair U, V of disjoint nonempty subsets of the space whose union is the entire space)

Definition(Betti number). The Betti number is any of a sequence of numbers, denoted b_n, which characterise a given topological space K by giving, for each dimension, the number of holes in K of said dimension; the rank of the nth homology group, H_n, of K.

Theorem 3.1.3. If a space X is trangulated into a finite number of simplexes, then $H_0(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, and the Betti number m of factors $\mathbb{Z}$ is the number of connected components of X.

Proof. Since we know that $C_0(X)$ is the free abelian group generated by the finite number of vertices P_i in the triangulation of X. Also, $B_0(X)$ is generated by expressions of the form $P_{i_2} - P_{i_1}$, where $P_{i_1}P_{i_2}$ is an edge in the triangulation. If we fixed P_{i_1}, any vertex P_{i_r} in the same connected component of X as P_{i_1} can be joined to P_{i_1} by a finite sequence

$$P_{i_1}P_{i_2}, P_{i_2}P_{i_3}, ..., P_{i_{r-1}}P_{i-r}$$

of edges. Then we have,

$$P_{i_r} = P_{i_1} + (P_{i_2} - P_{i_1}) + (P_{i_3} - P_{i_2}) + ... + (P_{i_r} - P_{i_{r-1}}).$$

This equation showing that $P_{i_r} \in [P_{i_r} + B_0(X)]$. Clearly, if P_{i_s} is not in the same connected component with P_{i_1}, then $P_{i_s} \notin [P_{i_r} + B_0(X)]$, since no edge joins the two different components. Thus, if we select one vertex from each connected component, each coset of $H_0(X)$ contains exactly one representative that is an integral multiple of one of the selected vertices. Then this theorem follows at once. □

Example 2. We have proved that $H_0(S^n) \simeq \mathbb{Z}$ for $n > 0$, Since S^n is connected for $n > 0$.

But since S^0 is the disjoint union of two points P and P' we say (we will prove this in next example)

$$H_0(S^0) \simeq \mathbb{Z} \times \mathbb{Z}$$
$$H_0(E^0) \simeq \mathbb{Z}$$

for $n \geqslant 1$.

Exercise 3. Describe $C_i(X), Z_i(X), B_i(X), H_i(X)$ for the space X consisting of two distinct 0-simplexes, P and P'.

Proof. By Definition $C_0 = \{m_1 P + m_2 P' | m_1, m_2 \in \mathbb{Z}\}$ and due to the fact we had in previous section we know

$$Z_0(X) = C_0(X) = \{m_1 P + m_2 P' | m_1, m_2 \in \mathbb{Z}\}$$

Since $C_0(X) = 0$ we can say$Z_0(X) = 0$ and by definition of homology group

$$H_0(X) = Z_0(X)/B_0(X) = \{m_1 P + m_2 P' | m_1, m_2 \in \mathbb{Z}\} \ / \ 0 = \{m_1 P + m_2 P' | m_1, m_2 \in \mathbb{Z}\}$$

$\square$

Definition 3.1.4. A space is **contractible** if it can be compressed to a point without being torn or cut, but always kept within the space it originally occupied.

Theorem 3.1.5. If X is a contractible space triangulated into a finite number of simplexes, then $H_n(X) = 0$ for $n \geqslant 1$.

Proof. If X is a contractible space triangulated into a finite number of simplexes, then X is homeomorphic to a single point P. Which implies that $H_n(X) = 0$ for $n \geqslant 1$. Note that it is also true for all $n > 0$. $\square$

Example 4. Consider about the sphere S^2, it is not contractible. You can think about that you cannot compressed the "surface of a sphere" to a single point without tearing it, and we cannot compress it all to the "center of the sphere". Note that from Example 1 we know $H_2(S^2) \neq 0$ but is isomorphic to $\mathbb{Z}$.

Now we consider $H_2(E^3)$, where E^3 is a solid sphere and we can regard it as solid tetrahedron of figure 2.1. And E^3 is contractible. Note that the surface S of this tetrahedron is homomorphic to S^2. The simplexes for E^3 and for S (or S^2) are the same, except for the whole 3-simplex that now we need. Remember that a generator of $Z_2(S)$, and hence of $Z_2(E^3)$, was $P_2P_3P_4+P_3P_1P_4+P_1P_2P_4+P_2P_1P_3$. Viewed in E^3, we get this is an element of $B_2(E^3)$, since it is $\partial_3(\sigma)$. Hence, $Z_2(E^3) = B_2(E^3)$ and $H_2(E^3) = 0$.

In general, E^n is contractible for $n \leq 1$, so by Theorem 3.1.5, we have $H_i(E^n) = 0$ for $i > 0$.

3.2 Futher Computations

Remark. 1-cycles in a triangulated space are generated by closed curve of the space formed by edges of the triangulation. 2-cycles can be thought of as generated by 2-spheres or other 2-dimensional surface in the space. Forming the factor group $H_1(X) = Z_1(X)/B_1(X)$

amounts that roughly computing the closed curves appear as boundary of a 2-dimensional pieces(collection of 2-simplexes) of the space. Similarly forming $H_1(X) = Z_1(X)/B_1(X)$ amounts roughly counting the closed 2-dimensional surface in the space that cannot be "filled in solid" whithin the space.

Example 5. Consider the S^1. We have three vertices $\{P_1, P_2, P_3\}$ in order, and edges between each pair of these vertices, P_1P_2, P_2P_3, and P_3P_1. Based on this, we have three 0-simplex and three 1-simplex, without any other simplex. To be specific, there is no 2-simplex and hence there is no 2-dimensional space in S^1.

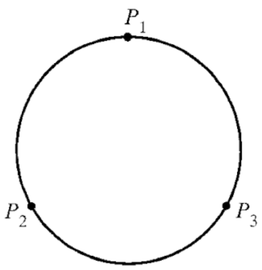


Figure 3.1

As the triangulation shown in figure above, $C_1(S^1)$ is generated by P_1P_2, P_2P_3, and P_3P_1. Since the highest dimensional simplex for the surface is a 1-simplex, we have $C_2(S^1) = 0$, so $B_1(S^1) = \partial_2[C_2(S^1)] = 0$. In order to find $Z_1(S^1)$, we need to find $ker(\partial_1)$. Compute the boundary map $\partial_1 : C_1 \to C_0$. The chain group C_1 has basis $\{P_1P_2, P_2P_3, P_3P_1\}$ and the chain group C_0 has basis $\{P_1, P_2, P_3\}$. We have $\partial_1(P_1P_2) = P_2 - P_1$, $\partial_1(P_2P_3) = P_3 - P_2$, and $\partial(P_1P_3) = P_3 - P_1$. Then for $a, b, c \in \mathbb{Z}$,

$$\partial_1(aP_1P_2+bP_2P_3+cP_1P_3) = a(P_2-P_1)+b(P_3-P_2)+c(P_3-P_1) = -(a+c)P_1+(a-b)P_2+(b+c)P_3.$$

Thus $aP_1P_2 + bP_2P_3 + cP_1P_3 \in ker(\partial_1)$ if and only if $a = b = -c$. That is, $ker(\partial_1)$ is isomorphic to $\mathbb{Z}$, generated by $P_1P_2 + P_2P_3 - P_1P_3$. So, $H_1(S^1) \simeq \mathbb{Z}/0 \simeq \mathbb{Z}$. In general,

$$H_n(S^n), H_0(S^n) \simeq \mathbb{Z}, n > 0$$

$$H_i(S^n) = 0, 0 < i < n$$

Definition 3.2.1. We call an element of $H_n(X)$, that is, a coset of $B_n(X)$ in $Z_n(X)$, a homology class. Cycles in the same homology class are **homologous**.

Example 6. Now let's compute the homology groups of a plane annular region X between two concentric circles. A triangulation is indicated as follow figure.

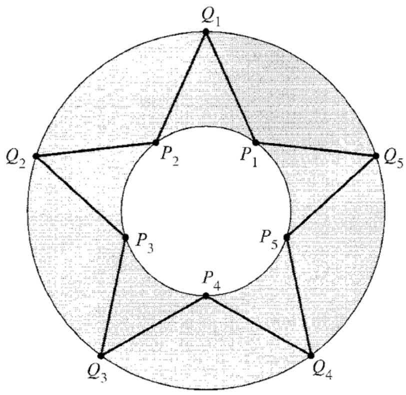


Figure 3.2

Since X is connected, and by Theorem 3.1.3 we know that $H_0(X) \simeq \mathbb{Z}$.

When z is an 1-cycle, and if P_1P_2 has coefficient r in z, then $z - r\partial_2(P_1P_2Q_1)$ should be a cycle without P_1P_2 homologous to z consider what we had in example 4 and the fact that $Z_1 = B_1$. Keeping apply this $P_1P_5, P_5P - 4, P_4P_3, P_3P_2$ we will obtain a 1-cycle homologous to z containing no edge on the inner circle of the annulus. Using the "outside" triangle, apply same logic to multiples of $\partial_2(Q_iP_iQ_j)$, therefore we construct a new z' containing no edge Q_iP_i either. But then if Q_5P_1 appears in z' with nonzero coefficient, P_1, then z' is not a free abelian group which contradict with z' is a cycle. Also keep applying same logic we have above no edges Q_iP_{i+1} can occur for $i = 1, 2, 3, 4$. Thus z is homologous to a cycle made up of edges on the outer cycle. So due to the fact that if 1-chain is a cycle so that its boundary is zero, then it must contain Q_1 Q_2, Q_2Q_3, Q_3Q_4, Q_4Q_5 and Q_5Q_1 same number of times so this cycle must be of the form

$$\text{n}(Q_1Q_2 + Q_2Q_3 + Q_3Q_4 + Q_4Q_5 + Q_5Q_1)$$

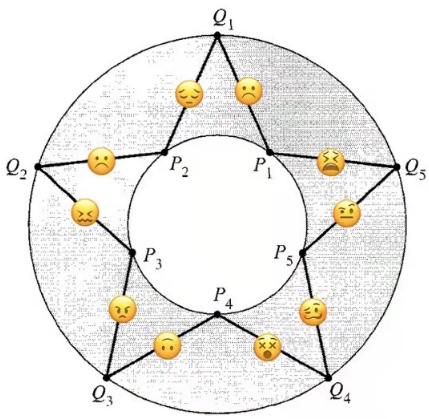


Figure 3.3

Since we have coefficient n, we could conclude that $H_1(X) \simeq \mathbb{Z}$. So we "push" any 1-cycle to the outside circle, and we could use same method we could also push the outside circle to the inside circle.

Since $Z_2(X) = 0$ due to the fact that every 2-simplex has their unique boundary compare to other simplexes. Then the boundary of any nonzero 2-chain must then contain some nonzero multiples of these edges its impossible to find such chain in Z. Due to that fact we could conclude that $H_2(X) = 0$.

Example 7. Consider about torus surface X which looks like the surface of a doughnut. Imagine that you cut torus according to steps 1 and 2 in the figure.

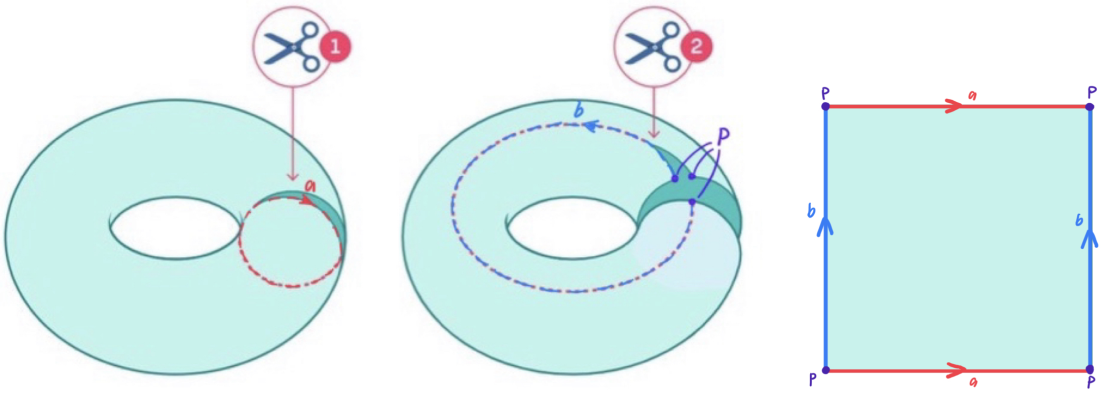


Figure 3.4

Clear that, to recover the torus from the flatten plane we create before by join edge a with edge a and edge b with edge b in the arrow direction, four corners will actually concentrate at a same point on the surface of torus. Meaning that boundary of all edges, a and b, are 0, and this could correspond the fact that torus is connected. By Theorem 3.1 we have $H_0(X) \simeq \mathbb{Z}$.

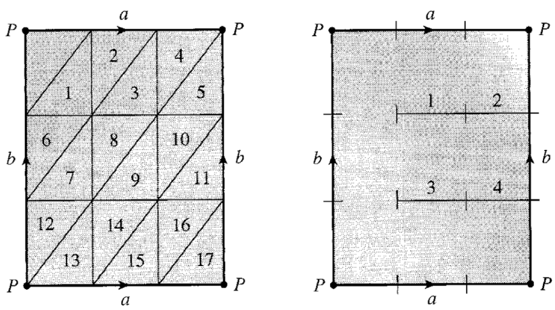


Figure 3.5

For $H_1(X)$, let z be a 1-cycle. And we use same method we used in the previous example changing z by a multiple of the boundary of the triangle numbered 1 in the above figure on the left, and we can get a homology cycle not containing side / of triangle1. Then by changing this new 1-cycle by suitable multiple of the boundary of triangle 2, we can further eliminate side | of 2. Continuing, we can then eliminate one side of each triangle on the figure which is / of 3, | of 4, /of 5, — of 6, / 0f 7, | of 8, / of 9, | of 10, / of 11, — of 12, / of 13, | of 14, / of 15, | of 16, and / of 17.

After eliminate 17 edges from 17 triangles, the result circle we have left homologous to z, can only contain edges shown in right above figure. But this resulting circle we have will not have boundary 0 if it contain nonzero coefficient for any of the edges we labeled in the upper right figure. So z is homologous to a 1 -cycle having edges only on the circle a or the circle b; Any every edges on circle a must appear the same number of times, and same is also true for edges on circle b in order to get a 0 boundary. But this does not require an edge on circle b appear same number of times as an edge appear on a. Furthermore, if a 2-chain is to have a boundary just containing a and b, all the triangles oriented counterclockwise must appear with the same coefficient so that the inner edges will cancel out. The boundary of such 2-chain is 0. Thus every homology class(coset) contain one and only one element

$$ra + sb \text{ where } r \text{ and } s \text{ are integers.}$$

Hence $H_1(X)$ is free abelian on two generators, represent by two circle a and b. Therefore $H_1(X) \simeq \mathbb{Z} \times \mathbb{Z}$.

Finally, for $H_2(X)$, a 2-cycle must contain the triangle numbered 2 in the above figure with counterclockwise orientation the same number of times as it contains the triangle numbered 3, also with counterclockwise orientation, in order for cancel the common edge / of these triangle so it will not be the boundary. These triangle are illustrated in the following figure.

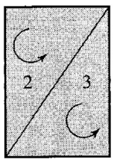


Figure 3.6

Same logic apply to other two adjacent triangles in order to obatain 0 boundary, and thus every triangle with the counterclockwise orientation must appear same number of times in a 2-cycle, we have only one coefficient left. Thus $Z_2(X)$ is infinite cyclic, isomorphic to $\mathbb{Z}$. Also, $B_2(X) = 0$, because there being no 3-simplexes, so

$$H_2(X) \simeq \mathbb{Z}$$

3.3 Exercise

Exercise 1. Compute the homology groups of the space consisting of two tangent 1-spheres, i.e., a figure eight.

Proof. Let X be a space consisting of two tangent 1-spheres since X is connected

$$H_0(X) \simeq \mathbb{Z}$$

Each tangent 1-sphere is a 1-cycle so $Z_1(X) \simeq \mathbb{Z} \times \mathbb{Z}$ and we have showed that $B_1(S^1) = 0$ in example 5 section 3.2, thus we can conclude that

$$H_1(X) = Z_1(X)/B_1(S^1) = \mathbb{Z} \times \mathbb{Z}.$$

Since there is no higher dimension, we can say that $H_i(X) = 0$ for all $i > 0$. $\square$

Exercise 2. Compute the homology groups of the space consisting of a tours tangent to 2-sphere at all points of a great circle of the 2-sphere, i.e., a balloon wearing an inner tube.

Proof. The space X is connected, by Theorem 3.1.3 we know that $H_0(X) \simeq \mathbb{Z}$. Let b be the 1-cycle that is the intersection of the torus with the 2-sphere. Now a every 1-cycle that is a circle going the long way around the torus, like the circle b in figure 3.7, is homologous to this 1-cycle that is the circle of intersection, as you can "push" it into this circle of intersection. Imaging to cut the figure 3.7 along the circle of intersection, we can actually get two hemispheres of S^2 while the circle of intersection is the 1-boundary of it. This means that every 1-cycle of type b is homologous to 0. Hence, the elements of $H_1(X)$ are elements of the cosets $ma + B_1(X)$ where $m \in \mathbb{Z}$ and a is another 1-cycle going the short way around the torus. Thus, $H_1(X) \simeq \mathbb{Z}$.

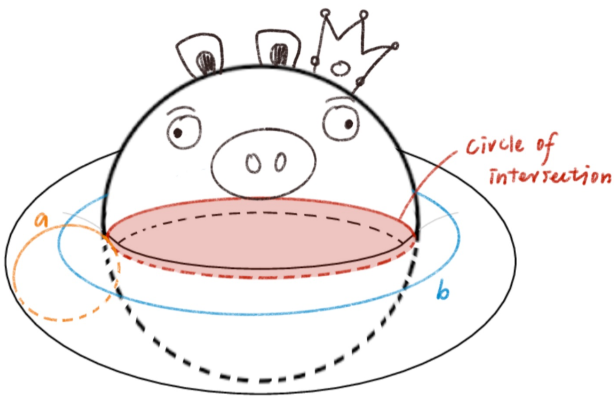


Figure 3.7: Piggy with a Hula-Hoop

Since a torus is a 2-cycle while 2-sphere is also a 2-cycle, and there are no 2-boundaries, then $H_2(X) \simeq \mathbb{Z} \times \mathbb{Z}$.

For the dimension, they are not even exist, hence we can conclude that $H_i(X) = 0$ for $i > 2$. $\square$

4 More Homology Computations And Applications

4.1 One-Sided Surfaces

Example 1. (Klein bottle) We have discussed some homology groups which are free abelian groups, so there were no nonzero elements of finite order. This can be shown always to be the case for the homology groups of a closed surface that has two sides. Now we want to investigate some one side closed surface (Klein Bottle). Figure 4.1 represent the Klein bottle cut apart, the only difference compare to torus here is the arrow on the top and bottle edge a of the rectangle are in opposite directions.

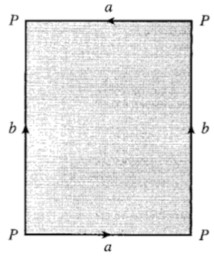


Figure 4.1: Klein bottle cut apart

In order to recover a Klein bottle we by bend the rectangle joining the edges labeled b so that the directions of the arrow match up. This gives a cylinder that is shown somewhat deformed, with the bottom end pushed a little way up inside the cylinder, which is called legitimate in topology. Now the circles a must be joined so that the arrow go around same way which is impossible to be done in $\mathbb{R}^3$, so you must image that you are in $\mathbb{R}^4$, so that you can bend the neck of the bottle around and "through" the side without intersecting the side. So this is a one side surface, if you start at any place and begin to paint "one side", you will wind up painting the whole thing.

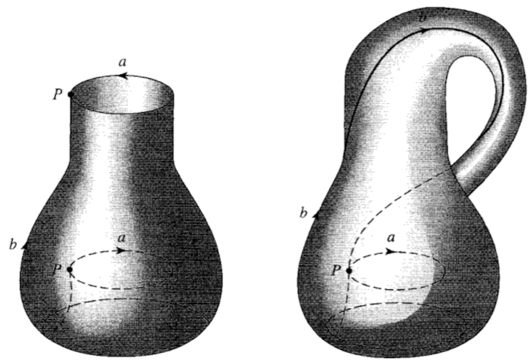


Figure 4.2

Like previous example since S is connected $H_0(X) \simeq \mathbb{Z}$.

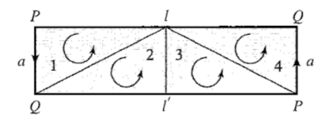
Now let we triangulate the Klein bottle by dividing figure 4.1 into triangles, like what we did for the torus, then every 1-cycle is homologous to a cycle of the form $ra + sb$ where r and s are integers. We follows the same technique as in the case of the tours, we can cancel each inner edegs. However, the difference from torus is that the boundary of such a 2-chain is $k(2a)$, where k is the number of times each triangle appears, since $\partial_2 = -b + a + b + a = 2a$. Therefore,

$$H_1(X) = \frac{ker(\partial_1)}{Im(\partial_2)} = \frac{\mathbb{Z} \times \mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}_2.$$

And by what we proved above, there are no 2-cycles this time, so $ker(\partial_2) = 0$, hence

$$H_2(X) = 0.$$

Example 2. (M*ö*bius strip) Let X be the M*ö*bius strip, which we can form by taking a rectangle of paper and joining the two ends marked a with half twist so that the arrows match up, as indicated in the following figure.



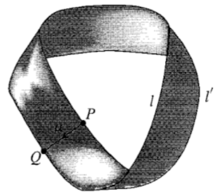


Figure 4.3

Note that the M*ö*bius strip is a surface with boundary, and the boundary is just one closed curve (homomorophic to a circle) made up of l and l', and M*ö*bius strip has just one side like the Klein bottle. Since S is connected, one can conclude that

$$H_0(X) \simeq \mathbb{Z}.$$

Let z be any 1-cycle, like previous examples we wish to subtracting every edges until we only have boundary left. By subtracting in succession suitable multiples of the triangles number 2, 3 and 4 we have above in the figure 4.3. We can eliminate edges / of triangle 2, | of triangle 3, and \ of triangle 4. Thus z is homologus to a cycle z' have the edges on only l, l' and a and both edges on l' must appear the same number of times. But if c is a 2-chain consisting of the formal sum of the triangles oriented as shown in above figure. We see that $\delta_2(c)$ consists of the edges on l and l' plus $2a$. Since both edges on l' must appear in z' the same number of times, so when we subtract a suitable multiple of $\delta_2(c)$, we see that z is homologous to a cycle with edges just lying on l and a. Which means that all these edges properly oriented must appear same number of times in this new cycle, then the homology class containing their sum of finitely many elements from the basis set multiplied by integer coefficients is a generator for $H_1(X)$, and hence

$$H_1(X) \simeq \mathbb{Z}.$$

This generating cycle starts at Q and goes around the strip, then cuts across it at P via a, and arrives at its starting point.

If z'' were a 2-cycle, it would it would contain the triangles 1, 2, 3 ,4 of figure above an equal number r of times with the indicated orientation. But the $\delta(z'')$ would be $r(2a+l+l') \neq 0$. Thus $Z_2(X) = 0$ so

$$H_2(X) = 0$$

4.2 The Euler Characteristic

Let X be a finite simplicial complex (or triangulated space) consisting of simplexes no more than dimension 3. Given some new notations for these n-simplexes we known:

- n_0 denotes the total number of vertices in the triangulation,
- n_1 be the numbers of edges,
- n_2 be the numbers of 2-simplexes,
- n_3 be the numbers of 3-simplexes.

Then the number

$$n_0 - n_1 + n_2 - n_3 = \sum_{i=0}^{3} (-1)^i n_i$$

will be the same no matter how the space X is triangulated. This is number is called **Euler characteristic** $\chi(X)$ of the sapce.

Theorem 4.2.1. Let X be a finite simplicial complex of dimension ≤ 3. Let $\chi(X)$ be the Euler characteristic of the space X, and let β_i be the Betti number of $H_i(X)$. Then

$$\chi(X) = \sum_{i=0}^{3} (-1)^i \beta_i.$$

This theorem holds also for X of dimension greater greater than 3.

Example 3. Consider the solid tetrahedron E^3. It is clear that $n_0 = 4, n_1 = 6, n_2 = 4, n_3 = 1$ and hence by what the previous theorem said,

$$\chi(E^3) = n_0 - n_1 + n_2 - n_3 = 4 - 6 + 4 - 1 = 1.$$

Since $H_3(E^3) = H_2(E^3) = H_1(E^3) = 0$ and $H_0(E^3) \simeq \mathbb{Z}$ as what we proved before, we have $\beta_3 = \beta_2 = \beta_1 = 0, \beta_0 = 1$ and hence

$$\chi(E^3) = \sum_{i=0}^{3}(-1)^i \beta_i = 1$$

Example 4. Consider the surface tetrahedron S^2. It is clear that $n_0 = 4, n_1 = 6, n_2 = 4, n_3 = 0$ and hence

$$\chi(S^2) = n_0 - n_1 + n_2 - n_3 = n_0 - n_1 + n_2 = 4 - 6 + 4 = 2.$$

Since $H_3(S^2) = H_1(S^2) = 0$ and $H_2(S^2), H_0(S^2) \simeq \mathbb{Z}$ as what we proved before, we have $\beta_3 = \beta_1 = 0, \beta_2 = \beta_0 = 1$ and hence

$$\chi(S^2) = \sum_{i=0}^{3}(-1)^i \beta_i = 2$$

Example 5. Consider the surface tetrahedron S^1. It is clear that $n_0 = 3, n_1 = 3, n_2 = n_3 = 0$ and hence

$$\chi(S^1) = n_0 - n_1 + n_2 - n_3 = 3 - 3 = 0.$$

Since $H_3(S^1) = H_2(S^1) = 0$ and $H_1(S^1), H_0(S^1) \simeq \mathbb{Z}$, then $\beta_3 = \beta_2 = 0, \beta_1 = \beta_0 = 1$ and hence

$$\chi(S^1) = \sum_{i=0}^{3}(-1)^i \beta_i = 0$$

4.3 Mapping of Spaces

A continuous function f mapping a space X into a space Y gives rise to a homomorphism f_{*n} mapping $H_n(X)$ into $H_n(Y)$ for $n \geq 0$

Theorem 4.3.1. If $z \in Z_n(X)$, and if $f(z)$, regarded as the result of picking up z and setting it down in Y in the naively obvious way, we should be an n-cycle in Y, then

$$f_{*n}(z + B_n(X)) = f(z) + B_n(Y)$$

that is, if z represents a homology class in $H_n(X)$ and $f(z)$ is an n-cycle in Y, then $f(z)$ represents the image homology class under f_{*n} of the homology class containing z.

Example 6. Consider the unit circle $S^1 = \{(x, y)|x^2 + y^2 = 1\}$ in $\mathbb{R}^2$. Any point in S^1 has coordinates $(cos\theta, sin\theta)$, like what we indicated in the following diagram.

Let $f : S^1 \to S^1$ be defined as

$$f((cos\theta, sin\theta)) = ((cos3\theta, sin3\theta))$$

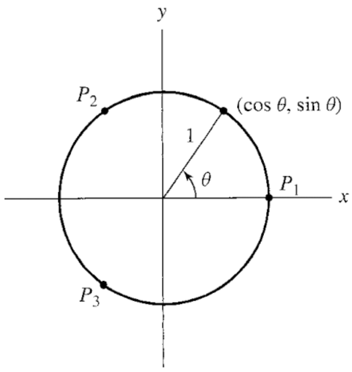


Figure 4.4

it's clear to see that function f is continuous. Now f could induce

$$f_{*1} : H_1(S^1) \to H_1(S^1).$$

We know that $H_1(S^1) \simeq \mathbb{Z}$ with generator $z = P_1P_2 + P_2P_3 + P_3P_1$ and a circle could be evenly spaced by three vertices, say P_1, P_2, P_3, by what we already proved in example 5 in section 3.2. By the definition of the continuous map f, each arcs P_1P_2, P_2P_3 and P_3P_1 onto the perimeter of the entire circle (thinking about how the angle θ changes between two vertices), which we can say that

$$f(P_1P_2) = f(P_2P_3)f(P_3P_1) = P_1P_2 + P_2P_3 + P_3P_1.$$

As what we mentioned before, $z = P_1P_2 + P_2P_3 + P_3P_1$ is the generator of a homology class. By the concept of cosets, we get

$$f_{*1}(z + B_1(S^1)) = 3z + B_1(S^1)$$

which corresponding the fact that the continuous function f will send the original S^1 into three times of itself. This example demonstrates the homomorphism of homology groups associated with a continuous mapping f may mirror important properties of the mapping.

Now we will introduce Brouwer Fixed-Point Theorem next this theorem state that for any such function f there is at least one point x such that $f(x) = x$, so this function f map x to itself.

Definition (Fixed Point) For a function $f : X \to X$, a fixed point $x \in X$ is a point where $f(x) = x$.

Example 7. Brouwer Fixed-Point Theorem works for any $n > 1$. When $n = 1$, suppose there is a function $f : E^1 \to E^1$ the theorem simply state that any continuous path joining the left and right side of the square must cross the diagonal somewhere.

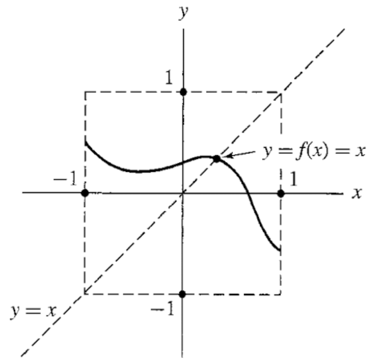


Figure 4.5

This theorem in 1-dimensional case is similar to the Intermediate Value Theorem from calculus: for a continuous function f on interval $[-1, 1]$: if $f(-1) < 0$ and $f(1) > 0$, then $f(x) = 0$ for at least one number x between -1 and 1.

Theorem 4.3.2. (Brouwer Fixed-Point Theorem). A continuous map f of E^n into itself has a fixed point for $n \geq 1$.

Proof. We already talked about the case when $n = 1$. Now we consider about when $n > 1$, let $f : E^n \to E^n$ for $n > 1$ be a map. Assume that f has no fixed point and we will prove there is a contradiction.

Since f has no fixed point, $f(x) \neq x$ for all $x \in E^n$. Consider a line segment from $f(x)$ to x, and extend this line segment in the direction from $f(x)$ to x until the boundary S^{n-1} of E^n at point y. Then we get a new function $g : E^n \to S^{n-1}$ with $g(x) = y$. Note that if point y on the boundary, then we will have $g(y) = y$. Since f is continuous, then we have g is also continuous. By the definition of mappings of spaces, we get a homomorphism map

$$g_{*(n-1)} : H_{n-1}(E^n) \to H_{n-1}(S^{n-1}).$$

By the Theorem 3.1.5, since E^n is contractible, we can say that $H_{n-1}(E^n) = 0$ for $n > 1$. Also, since $g_{*(n-1)}$ is a homomorphism, then $g_{*(n-1)}(0) = 0$. Meanwhile, since (n-1)-cycle actually represents that the homology class 0 of $H_{n-1}(E^n)$ is the whole complex S^{n-1} while all simplexes are in proper order, then $g(S^{n-1}) = S^{n-1}$ as $g(y) = y$ for any $y \in S^{n-1}$. By definition of the homology class,

$$g(S^{n-1}) + B_{n-1}(S^{n-1}) = S^{n-1} + B_{n-1}(S^{n-1}) = 0$$

which implies that

$$g_{*(n-1)}(0) = 0 = S^{n-1} + B_{n-1}(S^{n-1}).$$

Note, this is a nonzero generator of $H_{n-1}(S^{n-1})$. Therefore, contradiction. $\square$

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