The Dynamics of Cosmological Perturbation via Quantum Reference Frames Algorithm

Zhen Li* and Chun-Yen Lin†
Department of Physics, Beijing Normal University, Beijing 100875, China

Quantum reference frames is an algorithm of extracting Schrödinger theories under all viable physical time from the Einstein-Hilbert path integral, formulated as the timeless transition amplitudes $\hat{\mathbb{P}}: H \to H^*$ between the boundary states in a kinematic Hilbert space H. The only input is this transition amplitude generating from constraint operator of the gravitation system. The outputs are the physical Hamiltonian up to any wanted order of $\hbar$, which enable us to investigate the quantum effect of the spacetime. In this paper, we exhibit an concrete application of this method, and apply this algorithm to the cosmological perturbations in the scalar field frame. The zeroth order physical Hamiltonian agree with that of in dressed metric approach which is one of the common strategy dealing with the cosmological perturbations in loop quantum cosmology. What's more, we also obtained the first order $\hbar^1$ quantum corrections of the physical Hamiltonian, as well as the effective Hamiltonian and the equation of motion for perturbations. All these proved the effectiveness and correctness of the quantum reference frame algorithm.

I. INTRODUCTION

As a non-perturbative and background-independent canonical theory of quantum gravity, loop quantum gravity (LQG) has made extensive progresses in the past three decades[1][2][3][4]. One crucial perspective of the development lies in the application of the full LQG theory to the symmetry-reduced models, known as loop quantum cosmology (LQC)[5][6][7][8].

Through the simplified cosmological setting, LQC has shown the remarkable potential of LQGs characteristic spatial quantum geometric effects, in resolving important difficulties faced by the conventional quantum cosmology. While reproducing the usual FRW dynamics in the large scale classical regime, a wide range of LQC models demonstrate the robust behavior of singularity resolution in the Plankian region, in which the Big Bang singularity is replaced by a Big Bounce regular evolution[5][6][7][8]; more recently, it is also shown that the Big Bounce evolution in LQC in the may even supply favourable initial conditions for the onset required for the proper inflation era [9][10]. Nevertheless, it is through the latest advances of LQC in the arena of cosmic perturbations that the quantum geometric effects in LQG may finally become testable by observations, hopefully manifesting as signature corrections to CMB power spectrum. To achieve that, one faces the challenge of deriving the precise quantum evolution of the cosmic perturbations from a proper LQC model, in which the physical quantum states have more intricated structure with inhomogeneous field degrees of freedom, and represent the dynamical spacetimes without any pre-given background.

Loop quantum cosmology, just as the full LQG theory, follows the familiar canonical Wheeler-DeWitt framework[5][6][7][8] in the definition of physical quantum states. That is, due to the back-ground independence, the physical quantum states are not defined as the solutions to the Schrödinger equation in a fixed spacetime, but rather as the solutions to the quantized ADM constraints (which may be partially reduced in cosmological models), as the dynamical quantum spacetimes themselves. The novel technical ingredients of LQG and LQC lie in the new canonical quantization[1][2][3][4] of the kinematic phase space, with the metric and curvature variables respectively described by the smeared quantities of the holonomy and flux variables. As a major achievement, the kinematic Hilbert space describing the quantum geometry of space, arising from the holonomy and flux excitations, has been rigorously constructed and explored in the various models. In all cases, the true physical results would be obtained in the physical state space that is anhilated by the quantum constraints constructed involving the holonomy and flux operators. Just as for any Wheeler-DeWitt theory, it has been a crucial challenge for LQG to extract the physical Schrödinger evolution of the physical quantum states starting from the quantum constraint equations[11]. Among the many approaches to this problem currently being studied, the Dirac approach[12][13][14]is widely regarded as genuinely fundamental. In the Dirac approach, the well-constructed kernel projectors of the quantum constraints, known as the rigging map[14], has been introduced to define its image as a physical Hilbert space of the quantum spacetimes, with the natural physical inner product given by the rigging map matrix elements. In the context of LQG, it is also commonly believed that true dynamics of LQG should arise from the relational Dirac quantum observables[14] in the physical Hilbert space, constructed to be localized in a relative description amongst the degrees of freedom in the system.

Aiming for results in direct contact with observational data such as the CMB power spectrum, the exploration of cosmic perturbation in LQC concerns all the above constructions and challenges. Recently, a method

* zhen_li@mail.bnu.edu.cn
† cynlin@ucdavis.edu

called the hybrid LQC[15][16][17] has been extensively studied, in which the (partially reduced) kinematic Hilbert space– typically for the system of gravity and a scalar matter field– has been constructed via applying the loop qauntization to the homogeneous degrees of freedom, and the conventional Fock quantization to the inhomogeneous perturbations. The (partially reduced) quantum constraints for the model are correspondingly constructed with the hybrid set of loop and Fock operators, and they are expected to capture the interactions amongst the quantum geometry of the homogeneous sector and the usual QFT quantum perturbations. To obtain the Schrödinger evolutions for computing the quantum n-point functions, a prevailing approach called the "dressed metric" approach[18][19][20] has been applied to many models. In applying this method, one treats the homogeneous sector as a background upon which the perturbations propagates as test-fields with the back-reactions ignored. Under such approximation, one may consider an ansatz for the physical wave-finctions given by the direct product between that of the homogeneous background sector and that of the perturbation sector, with the background sector wavefunction specified to be a solution to the purely FRW part of the quantum constraints. Under certain proper forms for the original quantum constraints, together with the small-value condition for the scalar field potential, it is showned[18][19][20] that the perturbation sector of the direct product ansatz satisfies an effective Schrödinger equation, with the homogeneous scalar field being the time parameter. Specifically, the quantum perturbation fields appear to propagate in an effective spacetime metric given by the background state's expectation values for a set of quantum geometric operators. Within the merits of the approximations and assuptions, the physical picture is clear and persuasive, allowing the study the effects of the quantum geometry on the test fields, such as those coming form the big-bounce[20]. The efforts to go beyond the linear perturbation in the context of hybrid LQC have taken places[19], with interesting results suggesting the non-Gaussian deformation of the CMB spectrum.

In these latest attempts to explore the higher-order LQG predictions from a model governed by given quantum constraints, it becomes essential to derive the precise Schrödinger evolutions from the constraints in a more universal and fundamental manner–with minimal dependence on the model specific schemes, assumptions and truncations– that allows the capturing of the full quantum interactions to each order of perturbation truncation. Indeed, while the test-field assumption is consistent in the linear perturbation level, it becomes questionable for computing the higher-order ceffects that can be comparable to the back reactions in magnitudes. The prevailing way of extracting Schrödinger equations from the constraint equations relies on the specific form of the constraint equations, that may not be preserved when higher order terms are included, or when the scalar potential can no-longer be assume to be small. Further, one would also need a universal principle to eliminate the vast quantum ambiguities, coming from the "square-root" manipulation of the Wheeler-DeWitt into the Schrödinger equation.

We are thus motivated to derive the Schrödinger dynamics of the theories through the more fundamental Dirac approach. Just as described, the given quantum constraints in this approach are taken seriously as being fundamental. Instead of finding a way to approximate the constraint equations with a Schrödinger equation, we would construct the physical Hilbert space for the model with the rigging map of the given quantum constraints. The crucial task in this approach is then to constructing the relarional local Dirac observables in the physical Hilbert space, in a universal manner for all orders of perturbations, without scheme dependence or interaction truncation, such that the quantum observables can describe the Schrödinger evolutions. For this goal we propose to apply the quantum reference frame approach, developed in a previous series of works[21][22], to quamtum cosmic purturbation. The only input of quantum reference frame is indeed the rigging map of the quantum ADM constraints acting on a kinematic Hilbert space, which defines the physical Hilbert space. As it was realized, a special type of relational Dirac observables[21], refering to any choice of proper reference quantum fields in the system, can be fully described in terms of the rigging map's relevant matrix elements between the eigenspaces of the reference quantum fields. Enjoying the associated elementary kinematic algebra, these local observables provide the desired instantaneous complete set for for the Schrödinger wavefunctions. Furthermore, the explicit form of the unitary Schrödinger propagator for such relational quantum dynamics has been expressed as a universal function of the relevant rigging map elements.

Based upon a fundamental construction for a generic Dirac theory, the functions of the relevant rigging map elements describing the relational observables and their corresponding Schrödinger propagators are universal– independent of the perturbation truncations and approximation assumptions to quantum constraints; thus the dynamics are always derived in a "first principle" manner as in the full theory. That is, once the quantum constraints are constructed, the full evolutions are directly determined by the given quantum constraints with no further truncations, assumptions, or quantum ambiguity. In particular, all the interactions encoded in the quantum constraints– including the back-reactions from the reference fields– are always captured in the relational Schrödinger evolutions. From the perspectives of perturbative computation, our approach draws its strength from a variety of well-developed computation tools. It is known that the the rigging map is the operator realization of the canonical Fadeev-Popov path integral for GR[23][24][25], and thus it can be computed with many familiar methods, ranging from the familiar Feymann-diagrams to the spin-foam expansion for LQG[26][27][28]. By construction, each of these methods leading to a systematic expansion for the rigging map– in the powers of the cosmic perturbations and $\hbar$– automatically induces the corresponding systematic expansion for the relational Schrödinger propagator via its form as a function of the rigging map.

In Sec.II of this work, we will first introduce an existing hybrid LQC model[19] of cosmic perturbation, briefly reviewing the classical setting and hybrid quantization that lead to the quantum constraint governing the model. In Sec.III, we provide a short review on the QRF algorithm in the context of a general Dirac theory of quantum cosmic perturbation, and then we demonstrate the concrete application of the algorithm to the introduced hybrid LQC model, obtianing the relational Schrödinger propagator with the reference field chosen to be the scalar field, and subsequently the physical Hamiltonian operator. We will also introduce a specific power expansion for the rigging map via the expression of right-ordering the momentum operators, such that the resulted power expansion for the Hamiltonian operator takes a rather transparent form for physical interpretations. In Sec.IV, we will explicitly carry out the computation of the physical Hamiltonian operator, to the first order in quantum correction in our expansion. Finally, for a comparison with the results from the dressed metric approach, in Sec.V we will apply the back-reaction truncation to our full Hiessenberg equations, and show that they may also reduce to the equations of motion for the perturbation quamtum fields propagating on an effective and homogeneous quantum spacetime background. This test field dynamics is similar to the result of the original work[19] obtained using the dressed metric approach; however, even with the back reactions ignored, we will show that specific new quantum corrections are present in our effective metric, which in our regards should be taken seriously together with the computable back-reactions in the future works toward the higher-order and deeper-quantum computations.

II. A HYBRID LQC MOODEL OF COSMIC PERTURBATION

To demonstrate our approach in comparison to the dressed metric approach, in this section we introduce an existing perturbation model studied in [19] using the dressed metric approach. The model is based on the ADM formalism of general relativity with a minimally coupled massless scalar field, treated with partial-gauge reduction such the theory is governed by only one remaining global scalar constraint truncated to second order of perturbations. In the quantization of the partially-reduced phase space, the quantum geometry of LQG is encoded in the homogeneous gravitational degrees of freedom through the loop quantization, while the inhomogeneous perturbations are quantized through the conventional Fock quantization. The goal of the model is to explore the possible effects of the introduced quantum geometry upon the evolution of the perturbations.

In this section, we briefly introduce the basic construction leading to the quantized constraint operator governing the model, which serves as the common starting point for both the dressed metric method and our approach.

A. Classical setting

Let us consider general relativity minimally coupled to a scalar field Φ on a spacetime manifold $M = R \times \Sigma$. In this paper, we consider Σ having the R^3 topology. We will also follow the notions in the refrence[19]. In the Arnowitt-Deser-Misner(ADM), or Hamiltonian formulation the phase space Γ is made of quadruples of fields defined on the space manifold Σ, i.e., $\left(\Phi\left(\vec{x}\right), P_\Phi\left(\vec{x}\right), q_{ij}\left(\vec{x}\right), \pi^{ij}\left(\vec{x}\right)\right)$. (Latin indices i, j run from 1 to 3.) The only non-zero Poisson brackets between these canonical variables are

$$\left\{\Phi\left(\vec{x}\right), P_\Phi\left(\vec{x}'\right)\right\} = \delta^{(3)}\left(\vec{x}-\vec{x}'\right) \tag{1a}$$

$$\left\{q_{ij}\left(\vec{x}\right), \pi^{kl}\left(\vec{x}'\right)\right\} = \delta^k_{(i}\delta^l_{j)}\delta^{(3)}\left(\vec{x}-\vec{x}'\right) \tag{1b}$$

where $\delta^k_{(i}\delta^l_{j)} \equiv \frac{1}{2}\left(\delta^k_i\delta^l_j + \delta^k_j\delta^l_i\right)$ is the symmetrized Kronecker delta. The phase space carries the four constraints of general relativity, the scalar $S\left(\vec{x}\right)$ and vector constraints $V_i\left(\vec{x}\right)$.

$$\begin{aligned} S\left(\vec{x}\right) = &\frac{2\kappa}{\sqrt{q}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2) - \frac{\sqrt{q}}{2\kappa}{}^{(3)}R + \frac{1}{2\sqrt{q}}P_\Phi^2 \\ &+ \sqrt{q}V\left(\Phi\right) + \frac{\sqrt{q}}{2}D_i\Phi D^i\Phi \approx 0 \end{aligned} \tag{2}$$

$$V_i\left(\vec{x}\right) = -2\sqrt{q}q_{ij}D_k(q^{-1/2}\pi^{kj}) + P_\Phi D_i\Phi \approx 0 \tag{3}$$

In terms of the ordinary derivative associated with a reference frame, the components of vector constraint read

$$V_i\left(\vec{x}\right) = -2\partial_k(q_{ij}\pi^{jk}) + \pi^{jk}\partial_i q_{jk} + P_\Phi\partial_i\Phi \approx 0 \tag{4}$$

The Hamiltonian that generates time evolution in Γ is a combination of constraints

$$\mathbb{H} = \int d^3x \left[N\left(\vec{x}\right)S\left(\vec{x}\right) + N^i\left(\vec{x}\right)V_i(\vec{x})\right] \tag{5}$$

where the Lagrange multipliers $N\left(\vec{x}\right)$ and $N^i\left(\vec{x}\right)$ are the so-called lapse function and shift vector field.

1. perturbation expansion to the basic variables

Since Σ has the non-compact R^3 topology, the spatial integrals diverge. Following the regular method in LQC, we will constrain all the integrals of the fields within in a compact fiducial cubic coordinate volume V_0. Then, if we do fourier transformation to the fiels within the fiducial cell V_0, we will get discrete modes $\vec{k}$. $\vec{k} = 0$ modes of these fields represents the purely homogenous background, and $\vec{k} \neq 0$ modes represents the small purely inhomogeneous perturbation fields around the homogenous background.

Now we can do fourier transformation to these fields. For the gravitational variables,

$$q_{ij}(\vec{x}) = \frac{1}{V_0}(\widetilde{q}_{ij}(0) + \sum_{\vec{k}\neq 0} \widetilde{q}_{ij}(\vec{k} \neq 0)e^{i\vec{k}\cdot\vec{x}}) \tag{6a}$$

$$\pi^{ij}(\vec{x}) = \frac{1}{V_0}(\widetilde{\pi}^{ij}(0) + \sum_{\vec{k}\neq 0} \widetilde{\pi}^{ij}(\vec{k} \neq 0)e^{i\vec{k}\cdot\vec{x}}) \tag{6b}$$

Then, the non-zero Poisson brackets (1b) now translate to

$$\left\{\widetilde{q}_{ij}(\vec{k}), \widetilde{\pi}^{kl}(\vec{k}')\right\} = V_0 \delta^k_{(i}\delta^l_{j)}\delta_{\vec{k},-\vec{k}'} \tag{7}$$

If we define $\widetilde{q}_{ij}(\vec{k} \neq 0) \equiv \widetilde{\delta q}_{ij}(\vec{k})$ and $\widetilde{\pi}^{ij}(\vec{k} \neq 0) \equiv \widetilde{\delta\pi}^{ij}(\vec{k})$, that means

$$\left\{\widetilde{q}_{ij}(0), \widetilde{\pi}^{kl}(0)\right\} = V_0 \delta^k_{(i}\delta^l_{j)} \tag{8a}$$

$$\{\widetilde{\delta q}_{ij}(\vec{k}), \widetilde{\delta\pi}^{kl}(\vec{k}')\} = V_0 \delta^k_{(i}\delta^l_{j)}\delta_{\vec{k},-\vec{k}'} \tag{8b}$$

For the zero $\vec{k}$ mode, we can decompose the background metric and its conjugate momentum in a basis, namely $A^{(n)}_{ij}(0)$, for example we can choose a set of basis like

$$A^{(1)}_{ij}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A^{(2)}_{ij}(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^{(3)}_{ij}(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A^{(4)}_{ij}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^{(5)}_{ij}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A^{(6)}_{ij}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We use δ_{ij} and its inverse to raise or low indices. These six matrices form an orthonormal basis, with respect to the inner product $A^{*(n)}_{ij}(0)\, A^{ij}_{(m)}(0) = \delta_{nm}$. Now we can expand $\vec{k} = 0$ modes in this basis

$$\widetilde{q}_{ij}(0) = \sum_{n=1}^{6} \widetilde{\gamma_n}(0)\, A^{(n)}_{ij}(0), \quad \widetilde{\pi}^{ij}(0) = \sum_{n=1}^{6} \widetilde{\pi}^n(0)\, A^{ij}_{(n)}(0) \tag{9}$$

where $\widetilde{\gamma}_n(0) \equiv A^{ij}_{(n)}(0)\, \widetilde{q}_{ij}(0)$ and $\widetilde{\pi}^n(0) \equiv A^{(n)}_{ij}(0)\, \widetilde{\pi}^{ij}(0)$. Then the non-zero poisson brackets (8a) let

$$\begin{aligned}\{\widetilde{\gamma}_n(0), \widetilde{\pi}^m(0)\} &= A^{ij}_{(n)}(0)\, A^{(m)}_{kl}(0) \times \{\widetilde{q}_{ij}(0), \widetilde{\pi}^{kl}(0)\} \\ &= V_0 \delta_{nm}\end{aligned} \tag{10}$$

Here and after, we choose a gauge in which the background metric variables on Σ take the manifestly homogenous and isotropic form, i.e.the flat FLRW metric, when there is no perturbations

$$\overset{0}{q}_{ij} = a^2 \delta_{ij} \qquad \overset{0}{\pi}{}^{ij} = \frac{\pi_a}{a}\delta^{ij} \tag{11}$$

where δ_{ij} is the Euclidean metric on Σ and δ^{ij} its inverse, and $\{a, \pi_a\} = \frac{1}{V_0}$. Under this consideration, we can get

$$\widetilde{q}_{ij}(0) = \sum_{n=1}^{3} \widetilde{\gamma_n}(0)\, A^{(n)}_{ij}(0) = \widetilde{\overset{0}{q}_{ij}}(0) \tag{12a}$$

$$\widetilde{\pi}^{ij}(0) = \sum_{n=1}^{3} \widetilde{\pi}^n(0)\, A^{ij}_{(n)}(0) = \widetilde{\overset{0}{\pi}{}^{ij}}(0) \tag{12b}$$

where $\widetilde{\overset{0}{q}_{ij}}(0)$ and $\widetilde{\overset{0}{\pi}{}^{ij}}(0)$ are the fourier transformation of the background FLRW metric $\overset{0}{q}_{ij}$ and its conjugate momentum $\overset{0}{\pi}{}^{ij}$.

For the non-zero $\vec{k}$ modes of the fields or the perturbation parts, the matrices $\widetilde{\delta q}_{ij}(\vec{k})$ belongs to the vector space of 3*3 symmetric matrices. We can write $\widetilde{\delta q}_{ij}(\vec{k})$ in a convenient basis in this space, namely

$$A^{(1)}_{ij} = \frac{\overset{0}{q}_{ij}}{\sqrt{3}} \qquad A^{(2)}_{ij} = \sqrt{\frac{3}{2}}\left(\widehat{k}_i\widehat{k}_j - \frac{\overset{0}{q}_{ij}}{3}\right)$$

$$A^{(3)}_{ij} = \sqrt{\frac{1}{2}}\left(\widehat{k}_i\widehat{x}_j + \widehat{k}_j\widehat{x}_i\right) \quad A^{(4)}_{ij} = \sqrt{\frac{1}{2}}\left(\widehat{k}_i\widehat{y}_j + \widehat{k}_j\widehat{y}_i\right)$$

$$A^{(5)}_{ij} = \sqrt{\frac{1}{2}}\left(\widehat{x}_i\widehat{y}_j + \widehat{x}_j\widehat{y}_i\right) \qquad A^{(6)}_{ij} = \sqrt{\frac{1}{2}}\left(\widehat{x}_i\widehat{x}_j - \widehat{y}_i\widehat{y}_j\right)$$

where $\widehat{k}$ is the unit vector in the direction $\vec{k}$, and $\widehat{k}$, $\widehat{x}$, $\widehat{y}$ form an orthonormal set of unit vector (with respect to $\overset{0}{q}_{ij}$). We use $\overset{0}{q}_{ij}$ and its inverse to raise or low indices. These six matrices form an orthonormal basis, with respect to the inner product $A^{*(n)}_{ij} A^{ij}_{(m)} = \delta_{nm}$. Now we expand the perturbation fields in this basis

$$\widetilde{\delta q}_{ij}(\vec{k}) = \sum_{n=1}^{6} \widetilde{\gamma}_n(\vec{k}) A^{(n)}_{ij}(\vec{k}) \tag{13a}$$

$$\widetilde{\delta\pi}^{ij}(\vec{k}) = \sum_{n=1}^{6} \widetilde{\pi}^n(\vec{k}) A^{ij}_{(n)}(\vec{k}) \tag{13b}$$

where $\widetilde{\gamma}_n(\vec{k}) \equiv A^{ij}_{(n)}\widetilde{\delta q}_{ij}(\vec{k})$ and $\widetilde{\pi}^n(\vec{k}) \equiv A^{(n)}_{ij}\widetilde{\delta\pi}^{ij}(\vec{k})$. $\widetilde{\gamma}_n(\vec{k})$ and $\widetilde{\pi}^n(\vec{k})$ are called scalar modes for n=1,2, vector modes for n=3,4, tensor modes for n=5,6. Then the non-zero poisson brackets (8b) let

$$\begin{aligned}\{\widetilde{\gamma}_n(\vec{k}), \widetilde{\pi}^m(\vec{k}')\} &= A^{ij}_{(n)}(\vec{k})A^{(m)}_{kl}(\vec{k}) \times \{\widetilde{\delta q}_{ij}(\vec{k}), \widetilde{\delta\pi}^{kl}(\vec{k}')\} \\ &= V_0 \delta_{nm}\delta_{\vec{k},-\vec{k}'}\end{aligned} \tag{14}$$

For the scalar field variables, in the same way we can know

$$\Phi(\vec{x}) = \frac{1}{V_0}(\widetilde{\Phi}(0) + \sum_{\vec{k}\neq 0} \widetilde{\Phi}(\vec{k} \neq 0)) \quad (15a)$$

$$P_\Phi(\vec{x}) = \frac{1}{V_0}(\widetilde{P}_\Phi(0) + \sum_{\vec{k}\neq 0} \widetilde{P}_\Phi(\vec{k} \neq 0)) \quad (15b)$$

The non-zero poisson brackets (1a) are

$$\{\widetilde{\Phi}(\vec{k}), \widetilde{P}_\Phi(\vec{k}')\} = V_0 \delta_{\vec{k},-\vec{k}'} \quad (16)$$

If we define that $\widetilde{\Phi}(0) \equiv \widetilde{\phi}$,$\widetilde{\Phi}(\vec{k} \neq 0) \equiv \widetilde{\delta\phi}(\vec{k})$, and $\widetilde{P}_\Phi(0) \equiv \widetilde{p}_\phi$, $\widetilde{P}_\Phi(\vec{k} \neq 0) \equiv \widetilde{\delta}p_\phi(\vec{k})$, then the non-zero poisson brackets take forms as

$$\{\widetilde{\phi}, \widetilde{p}_\phi\} = V_0 \qquad \{\widetilde{\delta\phi}(\vec{k}), \widetilde{\delta}p_\phi(\vec{k}')\} = V_0 \delta_{\vec{k},-\vec{k}'} \quad (17)$$

where $\widetilde{\phi}$ and $\widetilde{p}_\phi$ are the fourier transformation of the background scalar field ϕ and its conjugate momentum p_ϕ.

Now, we back to the position space, the canonical variables in the phase space can be written as

$$\begin{aligned} \Phi(\vec{x}) &= \phi + \delta\phi(\vec{x}) \\ P_\Phi(\vec{x}) &= p_\phi + \delta p_\phi(\vec{x}) \\ q_{ij}(\vec{x}) &= \overset{0}{q}_{ij} + \delta q_{ij}(\vec{x}) \\ \pi^{ij}(\vec{x}) &= \overset{0}{\pi}{}^{ij} + \delta\pi^{ij}(\vec{x}) \end{aligned}$$

where ϕ, p_ϕ are the inverse fourier transformation of $\widetilde{\phi}, \widetilde{p}_\phi$ and together with $\overset{0}{q}_{ij}, \overset{0}{\pi}{}^{ij}$ belong to Γ_{FLRW} which describe the homogenous and isotropic background, $\delta\phi(\vec{x}), \delta p_\phi(\vec{x}), \delta q_{ij}(\vec{x}), \delta\pi^{ij}(\vec{x})$ are the inverse fourier transformation of $\widetilde{\delta\phi}(\vec{k}), \widetilde{\delta}p_\phi(\vec{k}), \widetilde{\delta}q_{ij}(\vec{k}), \widetilde{\delta\pi}^{ij}(\vec{k})$ without $\vec{k} = 0$ modes

$$\delta\phi(\vec{x}) = \frac{1}{V_0}\sum_{\vec{k}\neq 0} \widetilde{\delta\phi}(\vec{k}) \qquad \delta p_\phi(\vec{x}) = \frac{1}{V_0}\sum_{\vec{k}\neq 0} \widetilde{\delta}p_\phi(\vec{k})$$

$$\delta q_{ij}(\vec{x}) = \frac{1}{V_0}\sum_{\vec{k}\neq 0} \widetilde{\delta}q_{ij}(\vec{k}) \qquad \delta\pi^{ij}(\vec{x}) = \frac{1}{V_0}\sum_{\vec{k}\neq 0} \widetilde{\delta\pi}^{ij}(\vec{k})$$

$\delta\phi(\vec{x}), \delta p_\phi(\vec{x}), \delta q_{ij}(\vec{x}), \delta\pi^{ij}(\vec{x})$ belong to Γ_{Pert} which describe the small "purely" inhomogeneous perturbation fields around the homogenous and isotropic background. The phase space are thus can be written as Γ=$\Gamma_{FLRW} \times \Gamma_{Pert}$.

Now, the basic poisson brackets between background variables are

$$\{\phi, p_\phi\} = \frac{1}{V_0} \qquad \{\overset{0}{q}_{ij}, \overset{0}{\pi}{}^{kl}\} = \frac{1}{V_0}\delta^k_{(i}\delta^l_{j)} \quad (18)$$

The basic poisson brackets for perturbation fields are

$$\{\delta\phi(\vec{x}), \delta p_\phi(\vec{x}')\} = \delta^{(3)}(\vec{x} - \vec{x}') - \frac{1}{V_0} \quad (19a)$$

$$\{\delta q_{ij}(\vec{x}), \delta\pi^{ij}(\vec{x}')\} = \delta^k_{(i}\delta^l_{j)}(\delta^{(3)}(\vec{x} - \vec{x}') - \frac{1}{V_0}) \quad (19b)$$

2. Partial gauge reduction into system with one global scalar constraint

Regarding to the degrees of freedom in classical phase space, we have a total 9 degrees of freedom in configuration variables: for the background or $\vec{k} = 0$ mode, we have 1 in FLRW metric $\widetilde{\overset{0}{q}_{ij}}$, and 1 in scalar field $\widetilde{\phi}$; For the perturbations or $\vec{k} \neq 0$ modes(since there are infinity $\vec{k} \neq 0$ modes, for simplicity, we only focus on one representative $\vec{k} \neq 0$ mode in analysis), we have 6 in metric perturbations $\widetilde{\gamma}_n(\vec{k})$, n=1-6; and 1 in scalar field perturbation $\widetilde{\delta\phi}$. 9 more in the conjugate momenta.

Next, we need analyze the constraints in the phase space to see how many physical degrees of freedom are there. Before that, we should first truncate the scalar and vector constraints to the second order

$$S(\vec{x}) = S^0 + S^1(\vec{x}) + S^2(\vec{x}) \approx 0 \quad (20a)$$

$$V_i(\vec{x}) = V_i{}^0 + V_i{}^1(\vec{x}) + V_i{}^2(\vec{x}) \approx 0 \quad (20b)$$

What's more, we need also expand the lapse and shift function into homogenous and purely inhomogeneous part

$$N(\vec{x}) = \overset{0}{N} + \delta N(\vec{x}) \quad (21a)$$

$$N^i(\vec{x}) = \overset{0}{N}{}^i + \delta N^i(\vec{x}) \quad (21b)$$

where $\overset{0}{N}, \overset{0}{N}{}^i$ are homogenous background and $\delta N(\vec{x}), \delta N^i(\vec{x})$ are the small purely inhomogeneous perturbation. For consistency with the flat FLRW gauge fixing (Eqn.11), the $\overset{0}{N}{}^i$ must be vanished. Because only when $\overset{0}{N}{}^i = 0$ can we get $ds^2 = -\overset{0}{N}{}^2 dt^2 + \overset{0}{q}_{ij} dx^i dx^j$ from the zeroth order of general 3+1 decomposition of spacetime

$$ds^2 = -N^2 dt^2 + q_{ij}(N^i dt + dx^i)(N^j dt + dx^j) \quad (22)$$

Then, we are ready to truncate the action to the second order regrading to the perturbations

$$\mathbb{S} = \mathbb{S}^0 + \mathbb{S}^1 + \mathbb{S}^2 \quad (23)$$

where

$$\mathbb{S}^0 = \int dt \int_{V_0} d^3\vec{x}\, \overset{0}{\pi}{}^{ij}\, \dot{\overset{0}{q}}_{ij} + p_\phi\, \dot{\phi} - \int dt \mathbb{H}^0 \quad (24a)$$

$$\mathbb{S}^1 = \int dt \int_{V_0} d^3\vec{x} \delta\pi^{ij}\, \dot{\overset{0}{q}}_{ij} + \overset{0}{\pi}{}^{ij}\, \delta\dot{q}_{ij} + \delta p_\phi\, \dot{\phi} + p_\phi\, \delta\dot{\phi} - \int dt \mathbb{H}^1 \quad (24b)$$

$$\mathbb{S}^2 = \int dt \int_{V_0} d^3\vec{x} \delta\pi^{ij} \delta\dot{q}_{ij} + \delta p_\phi\, \delta\dot{\phi} - \int dt \mathbb{H}^2 \quad (24c)$$

and the Hamiltonian $\mathbb{H} = \mathbb{H}^0 + \mathbb{H}^1 + \mathbb{H}^2$ given by

$$\mathbb{H}^0 = \int_{V_0} d^3\vec{x} \overset{0}{N} S^0 = V_0 \overset{0}{N} S^0 \tag{25a}$$

$$\mathbb{H}^1 = \int_{V_0} d^3\vec{x}(\overset{0}{N} S^1(\vec{x}) + \delta N(\vec{x}) S^0) \tag{25b}$$

$$\mathbb{H}^2 = \int_{V_0} d^3\vec{x}(\overset{0}{N} S^2(\vec{x}) + \delta N(\vec{x}) S^1(\vec{x}) + \delta N^i(\vec{x}) V_i{}^1(\vec{x})) \tag{25c}$$

Since each term of $\mathbb{H}^1$ is an integral of homogenous ($k = 0$) and purely inhomogeneous fields ($k \neq 0$), it is vanished identically. The constraints in the cosmological perturbation system can be given by

$$\frac{\delta \mathbb{H}}{\delta \overset{0}{N}} = \frac{\delta\left(\mathbb{H}^0 + \mathbb{H}^2\right)}{\delta \overset{0}{N}} = V_0 S^0 + \int_{V_0} d^3\vec{x}\, S^2(\vec{x}) = 0 \tag{26a}$$

$$\frac{\delta \mathbb{H}}{\delta(\delta N)} = \frac{\delta \mathbb{H}^2}{\delta(\delta N)} = S^1(\vec{x}) = 0 \tag{26b}$$

$$\frac{\delta \mathbb{H}}{\delta(\delta N^i)} = \frac{\delta \mathbb{H}^2}{\delta(\delta N^i)} = V_i{}^1(\vec{x}) = 0 \tag{26c}$$

Form above we can see that, we have 5 constraints in the classical phase space, they are: $V_0 S^0 + \int_{V_0} d^3\vec{x}\, S^2(\vec{x})$, $S^1(\vec{x})$ and $V_i{}^1(\vec{x})$. Hence leaving a total of 4 physical degrees of freedom in configuration variables. In order to compare the results with the previous works, we will use the same gauge fixing choices[19] for perturbations as named "spatially flat gauge", i.e, $\widetilde{\gamma}_i(\vec{k}) = 0$ for $i = 1, 2, 3, 4$. The physical degrees of freedom for perturbations are therefore encoded in the scalar perturbations $\widetilde{\delta\phi}(\vec{k})$ and the tensor modes $\widetilde{\gamma}_5(\vec{k})$ and $\widetilde{\gamma}_6(\vec{k})$. Then, we leave one physical degree of freedom and one gauge degree of freedom which will be fixed in the quantum level by solving the only left quantum constraint operator in section IV.

What's more, we only interest in the scalar perturbations, therefore, we will follow [19] not to write terms containing vector and tensor modes. When we say "perturbations" in the following, we refer to the scalar modes of the full perturbations. Then the explicit forms of the scalar and vector constraints up to the second order are as follow[19]:

$$S^0 = -\frac{\kappa \pi_a^2}{12a} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) \tag{27a}$$

$$V_i{}^0 = 0 \tag{27b}$$

$$S^1(\vec{x}) = \frac{p_\phi}{a^3}\delta p_\phi(\vec{x}) - \frac{\kappa \pi_a}{\sqrt{3}a^2}\pi_1(\vec{x}) + a^3 V_\phi \delta\phi(\vec{x}) \tag{27c}$$

$$V_i{}^1(\vec{x}) = p_\phi \partial_i \delta\phi(\vec{x}) - \frac{2}{\sqrt{3}}\partial_i \pi_1(\vec{x}) - 2\sqrt{\frac{2}{3}}\partial_i \pi_2(\vec{x}) \tag{27d}$$

$$\begin{aligned} S^2(\vec{x}) = {} & \frac{1}{2a^3}\delta p_\phi^2(\vec{x}) - \frac{\kappa}{a^3}\pi_1^2(\vec{x}) - \frac{\kappa}{a^3}\pi_2^2(\vec{x}) \\ & + \frac{1}{2}a^3 \partial_i \delta\phi(\vec{x})\partial^i \delta\phi(\vec{x}) + \frac{a^3 V_{\phi\phi}}{2}\delta\phi^2(\vec{x}) \\ & + \frac{3\kappa \partial_i \partial_j \partial^{-2}\pi_2(\vec{x})\partial^i \partial^j \partial^{-2}\pi_2(\vec{x})}{a^3} \end{aligned} \tag{27e}$$

$$V_i{}^2(\vec{x}) = \delta p_\phi(\vec{x})\partial_i \delta\phi(\vec{x}) \tag{27f}$$

where $\pi_i(\vec{x})$ is the inverse fourier transformation of $\widetilde{\pi_1}(\vec{k})$, same for other variables. V_ϕ means the derivative of $V(\phi)$ with respect to ϕ.

For simplicity of the quantization, we can solve 4 constraints $S^1(\vec{x})$ and $V_i{}^1(\vec{x})$ in the classical phase space. We can get

$$\pi_1(\vec{x}) = \frac{\sqrt{3}a^5 V_\phi}{\kappa \pi_a}\delta\phi(\vec{x}) + \frac{\sqrt{3}p_\phi}{\kappa a \pi_a}\delta p_\phi(\vec{x}) \tag{28a}$$

$$\pi_2(\vec{x}) = \sqrt{\frac{3}{2}}\left[\left(\frac{p_\phi}{2} - \frac{a^5 V_\phi}{\kappa \pi_a}\right)\delta\phi(\vec{x}) - \frac{p_\phi}{\kappa a \pi_a}\delta p_\phi(\vec{x})\right] \tag{28b}$$

Carrying the results to the $S^{(2)}(\vec{x})$ expression(27e), it turns out

$$S^2(\vec{x}) = \frac{1}{2a^3}\delta \overline{p}_\phi^2(\vec{x}) + \frac{a^3}{2}(\partial\delta\phi(\vec{x}))^2 + \frac{a^3}{2}\mathfrak{U}\delta\phi^2(\vec{x}) \tag{29}$$

where $\partial\delta\phi(\vec{x})$ means the spatial derivative to $\delta\phi(\vec{x})$, and the effective potential $\mathfrak{U}$ is composed of background variables, and given by

$$\mathfrak{U} = -18\frac{p_\phi^4}{a^8 \pi_a^2} + 3\frac{\kappa p_\phi^2}{a^6} - 12\frac{p_\phi}{a\pi_a}V_\phi + V_{\phi\phi} \tag{30}$$

Here we have performed a canonical transformation $(\delta\phi(\vec{x}), \delta p_\phi(\vec{x})) \to (\delta\phi(\vec{x}), \delta\overline{p}_\phi(\vec{x}) = \delta p_\phi(\vec{x}) - \frac{3p_\phi^2}{a\pi_a}\delta\phi(\vec{x}))$ as [19] to simplify the expressions in the $S^2(\vec{x})$. From now on we will work with $\delta\overline{p}_\phi(\vec{x})$, but drop the bar to simplify the notation.

Since we have solved $S^1(\vec{x})$ and $V_i{}^1(\vec{x})$, we only left one constraint $C \equiv (V_0 S^0 + \int_{V_0} d^3\vec{x}\, S^2(\vec{x})) = 0$, and the dynamics can be generated by imposing the constraint C in the classical phase space.

B. Quantization and the constraint operator

Since we already know the explicit forms of the constraint in the classical phase space, the next step is to write all the quantities by operators in a Hilbert space.

In analog of the classical phase space $\Gamma_{FLRW} \times \Gamma_{\text{Pert}}$, the total Kinematical Hilbert space can be written as $H = H_{FLRW} \otimes H_{Pert}$, where H_{FLRW} is the usual LQC Hilbert space: $H_{FLRW} = H^{gr}_{kin} \otimes H^{sc}_{kin}$, with $H^{gr}_{kin} \equiv L^2(R_{Bohr}, d\mu_H), H^{sc}_{kin} \equiv L^2(R, d\mu)$. H_{Pert} is the perturbation Hilbert space which belongs to the usual Fock space in the quantum field theory.

From LQC FLRW classical gravitational phase space, we know that $|p| = {V_0}^{2/3} a^2$ and $c = \text{sgn}\,(p)\,\gamma {V_0}^{1/3} \frac{\dot{a}}{N}$ are the new coordinates in the gravitational phase space rather than metric and its conjugate momenta. In analogy, p plays a role of the configuration while c as its conjugate momenta. Each pair of (c, p) gives a fix point in phase space. The poisson brackets between these two variables is $\{c, p\} = \frac{\kappa\gamma}{3}$. V_0 is the fiducial volume, and $V{=}|p|^{3/2} = a^3 V_0$ is the physical volume. We can rewrite the constraint S^0 as

$$S^{(0)} = -\frac{3}{\kappa\gamma^2} c^2 \sqrt{|p|} + \frac{{p_\phi}^2}{2|p|^{3/2}} + |p|^{3/2} V\,(\phi) \tag{31}$$

Since there is no operator corresponds to c, we can use the loop representation technic in LQG to regularize cp as

$$(cp)_{reg} = 2\pi\gamma G\hbar \sqrt{v} \sin\left(\frac{b}{\hbar}\right) \sqrt{v} \equiv \Omega \tag{32}$$

with the definitions

$$b \equiv \hbar\overline{\mu}\,(p)\,c \quad \overline{\mu}\,(p) \equiv \sqrt{\frac{\Delta}{|p|}} \quad \Delta \equiv 2\sqrt{3}\pi\gamma G\hbar$$
$$v \equiv \left(2\pi\gamma G\hbar\sqrt{\Delta}\right)^{-1} \text{sgn}\,(p)\,|p|^{3/2} \tag{33}$$

where (b, v) is also the canonical pair in the phase space with Poisson brackets $\{b, v\} = 2$. Then we can promote

$$S^{(0)} \Rightarrow \left[\frac{1}{V}\right]^{\frac{1}{2}} \frac{1}{2\kappa} \left(-\frac{6}{\gamma^2}\Omega^2 + \kappa {p_\phi}^2 + 2\kappa |p|^3 V\,(\phi)\right) \left[\frac{1}{V}\right]^{\frac{1}{2}} \tag{34}$$

where $V = |p(t)|^{3/2}$. Now the regularized Hamiltonian in LQC FLRW classical phase space takes forms

$$\mathbb{H}^0_{reg} = V_0 N S^{(0)} \tag{35}$$

The evolution of V along the physical time t (which depends on the lapse function N), is given by $\partial_t V = \{V, \mathbb{H}^0_{reg}\} = N V_0 \frac{3}{4\gamma} \Omega(t) \cos\left(\frac{b(t)}{\hbar}\right) \propto v \sin\left(\frac{2b(t)}{\hbar}\right)$.

In addition, we can rewrite the effective potential by the LQC variables as

$$\begin{aligned}\mathfrak{U} = &-18\kappa^2\gamma^2 V_0^{8/3} V^{-2} \Omega^{-2} p_\phi^4 + 3\kappa V_0^2 V^{-2} p_\phi^2 \\ &- 12\kappa\gamma V_0^{1/3} \Omega^{-1} p_\phi V_\phi + V_{\phi\phi}\end{aligned} \tag{36}$$

Now, we are ready move to the operator forms. In order to compare the results with dressed metric approach, we will choose the similar operators forms based on the works of [8]. We consider the quantum background scalar constraint operator as

$$\begin{aligned}\widehat{S}^{(0)} &= \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \frac{1}{2\kappa} \left(-\frac{6}{\gamma^2}\hat{\Omega}^2 + \kappa \widehat{p_\phi}^2 + 2\kappa \widehat{V}^2 \widehat{V}(\phi)\right) \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \\ &\equiv \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \widehat{\mathcal{S}}^{(0)} \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}}\end{aligned} \tag{37}$$

where $\widehat{\mathcal{S}}^{(0)}$ called densitized scalar constraint operator, and the operators defined as

$$\begin{aligned}\widehat{\Omega} \equiv &\frac{-i}{2\sqrt{\Delta}} |\widehat{p}|^{3/4} \left[\left(\widehat{\mathcal{N}}_{2\overline{\mu}} - \widehat{\mathcal{N}}^\dagger_{2\overline{\mu}}\right) \widehat{sign\,(p)} \right.\\ &\left. + \widehat{sign\,(p)} \left(\widehat{\mathcal{N}}_{2\overline{\mu}} - \widehat{\mathcal{N}}^\dagger_{2\overline{\mu}}\right)\right] |\widehat{p}|^{3/4} \\ = &\, 2\pi\gamma G\hbar \sqrt{\widehat{v}} \widehat{\sin\left(\frac{b}{\hbar}\right)} \sqrt{\widehat{v}}\end{aligned} \tag{38}$$

where the operators defined as

$$\widehat{\mathcal{N}}_{2\overline{\mu}} \equiv \widehat{e^{i\overline{\mu}c}}, \qquad \left[\widehat{\frac{1}{V}}\right] \equiv \left[\widehat{\frac{1}{\sqrt{|p|}}}\right]^3 \tag{39}$$

$\widehat{S}^{(0)}$ acting on a state of the kinematical Hilbert space, it gives a quantum constraint equation which is know as a Klein-Gordan kind difference equation[7]. Correspondingly, we can also transforms S^2 to operator as

$$\begin{aligned}\widehat{S}^{(2)}(\vec{x}) &\equiv \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \widehat{\mathcal{S}}^{(2)} \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \\ &= \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \frac{V_0}{2} (\delta\hat{p}^2_\phi(\vec{x}) + \hat{a}^6 (\partial\widehat{\delta\phi}(\vec{x}))^2 \\ &\quad + \hat{a}^6 \widehat{\mathfrak{U}} \widehat{\delta\phi}^2(\vec{x})) \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}}\end{aligned} \tag{40}$$

where $\widehat{\mathcal{S}}^{(2)}$ called densitized second order scalar constraint operator, and $\widehat{a}^3 V_0 = \widehat{V}$. In consistent with the literatures [18], We will use the similar strategy of [8] that working with the densitized constraint

$$\widehat{\mathcal{C}} \equiv (V_0 \widehat{\mathcal{S}}^{(0)} + \int_{V_0} d^3\vec{x}\, \widehat{\mathcal{S}}^{(2)}(\vec{x})) \tag{41}$$

rather than $\widehat{C} \equiv (V_0 \widehat{S}^{(0)} + \int_{V_0} d^3\vec{x}\, \widehat{S}^{(2)}(\vec{x}))$, where $\widehat{\mathcal{C}}$ called densitized constraint operator. Its relation with the full constraint operator is

$$\widehat{C} = \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \widehat{\mathcal{C}} \left[\widehat{\frac{1}{V}}\right]^{\frac{1}{2}} \tag{42}$$

This densitized constraint operator has several advantages[8]: under its action, the zero-volume state decouples, and states with different orientation of the triad are not mixed. The big bang singularity is kinematically resolved inasmuch as its quantum analog, namely the zero-volume state, is removed from the kinematical Hilbert space. Once we have removed the kernel of the inverse volume operator, then it is well defined. These features enable us to establish a bijection between any element $(\,\Phi|$ annihilated by the (adjoint of the) operator $\widehat{C}$ and any other element $(\,\Psi| = (\,\Phi|\,(\widehat{1/V})^{1/2}$ annihilated by the (adjoint of the) densitized version of the constraint $\widehat{\mathcal{C}}$. If we denote the subspace of nonzero-volume states as $\widetilde{H}$, both the state $(\,\Phi|$ and $(\,\Psi|$ belong to the dual of $\widetilde{H}$. In conclusion, this equivalent form of the densitized constraint is easier to impose and allow us to solve it explicitly.

III. QUANTUM REFERENCE FRAME FOR DIRAC THEORY OF COSMIC PERTURBATION

Before going in to the specific calculations, we first illustrate the principle and main steps of QRF algorithm[21]. Here we give an updated review of the approach of quantum reference frame, in the context of a wide class of cosmic perturbation theories to arbitrary order.

A. algorithm under general setting

We assume a Dirac theory of canonical quantum gravity based on either ADM formulation of general relativity or its partially gauge-reduced theories and cosmological models. Those classical theories or models are thus governed by a set of (reduced) ADM constraints $\{C_\mu(X_i, \delta X_i, P_i, \delta P_i)\}$, where we use (X_i, P_i) and $(\delta X_i, \delta P_i)$ to denote repectively the homogeneous and inhomogeneous modes of the (reduced) ADM phase space coordinates. We also assume sufficient analyticity in the constraints, such that the nth order truncated constraints $\{{}^nC_\mu(X_i, \delta X_i, P_i, \delta P_i)\}$, containing only the terms to power up to n-th order in $(\delta X_i, \delta P_i)$, may be defined for arbitrary n with the correct exact limit of $C_\mu = {}^\infty C_\mu$. For the cases with more than one constraints, we also introduce one master constraint ${}^nM \equiv \sum_\mu {}^nC_\mu^2$ which is equivalent in giving the same constraint surface.

We assume the existence of the exact quantum theory obtained by applying the Dirac canonical quantization to the exact classical theory. That is, there is a kinematic Hilbert space $H = \mathrm{Span}\{|X_i, \delta X_i\rangle\}$, in which the constraint is quantized into a self-adjoint constraint operator $\hat{M} = \hat{M}^\dagger$. This then allow us to define the rigging map operator $\hat{\mathbb{P}} \equiv \hat{M} : H \to H^*$ and it serves as a generalized kernel projector for the quantum constraint $\hat{M}$. The rigging map serves double functions– its image $\mathbb{D} \subset H^*$ gives the physical state space, while its elements naturally define the physical inner product in $\mathbb{D}$, making it a physical Hilbert space. Explicitly, the inner product between two physical states $\{\,|\Psi_1) \equiv \hat{\mathbb{P}}\,|\psi_1\rangle\,, |\Psi_2) \equiv \hat{\mathbb{P}}\,|\psi_2\rangle\,\} \subset \mathbb{H}$ is defined by

$$(\Psi_1|\Psi_2) \equiv \mathbb{P}\big(\,|\psi_1\rangle\,, |\psi_2\rangle\,\big). \tag{43}$$

Note that the dynamics has to emerge from these physical states which are constructed without any notion of time. In our approach, we implicitly construct a complete set of local observables, called the elementary relational observables, identified using a quantum reference frame in which a specific set of reference quantum fields $\hat{T}_\mu$ takes a set of given values $\bar{T}_\mu(t)$ at any moment t of time. The calculation of the dynamics for these observables is given by an algorithm consisting of the following steps.

1. To specify a quantum reference frame, we first choose from H a reference sector, with the rest being the dynamic sector. Concretely, we introduce a splitting to the total set of the conjugate pairs as $\{(X_i, P_i), (\delta X_i, \delta P_i)\} = \{(X_A, P_A), (\delta X_B, \delta P_B)\} \cup \{(X_C, P_C), (\delta X_D, \delta P_D)\}$, where the two disjoint sets $\{(X_A, P_A), (\delta X_B, \delta P_B)\} \equiv \{(\mathbf{X}_\mu, \mathbf{P}_\mu)\}$ and $\{(X_C, P_C), (\delta X_D, \delta P_D)\} \equiv \{(\mathbf{X}_I, \mathbf{P}_I)\}$ represent respectively the reference and dynamic sectors. The specific reference quantum fields are constructed as $\{\hat{T}_\mu = \hat{T}_\mu(\hat{\mathbf{X}}_\nu, \hat{\mathbf{P}}_\nu)\}$, and the choice of a reference frame can be completed by specifying the set of functions $\{\bar{T}_\mu(t)\}$ of the moments t, taking values from the eigenspectrum of $\hat{T}_\mu$. Therefore, each t is now associated with an eigenspace of $\{\hat{T}_\mu\}$, given by $\mathbb{S}_t \equiv span\{|\bar{T}_\mu(t), \mathbf{P}_I\rangle\} \subset H$ (or equivalently $\mathbb{S}_t \equiv span\{|\bar{T}_\mu(t), \mathbf{X}_I\rangle\} \subset H$) which under $\hat{\mathbb{P}}$ is projected to the image $\mathbb{D}_t$.

2. Calculate the relevant transition amplitude elements

$$\hat{\mathbb{P}}_{t't}[\,\mathbf{P}'_I\,,\,\mathbf{P}_I\,] \equiv \hat{\mathbb{P}}(\,|\bar{T}_\mu(t'), \mathbf{P}'_I\rangle\,, |\bar{T}_\mu(t), \mathbf{P}_I\rangle\,) \tag{44}$$

   among arbitrary t' and t. Carrying the meaning of the inner products for the space $\mathbb{D}_t$, the square matrix $\mathbb{P}_{tt}$ in the dynamical sector must be diagonalizable into a diagonal matrix with non-negative real elements. Upon this given condition, the matrix $\mathbb{P}_{tt}$ for a valid quantum reference frame must further be densely positive-definite in $\mathbb{S}_t$, implying a bijection between $\mathbb{D}_t$ and $\mathbb{S}_t$ through the rigging map.

3. Suppose the $\mathbb{P}_{tt}$ is indeed invertible, then we may introduce the correction factor operator $\hat{\Lambda}_t \equiv \hat{\mathbb{P}}_{tt}^{-\frac{1}{2}} : \mathbb{S}_t \to \mathbb{S}_t$ which would be also (densely) well-defined and positive-definite. In matrix notation, the relational propagator $\hat{U}_{t't} : \mathbb{S}_t \to \mathbb{S}_{t'}$ is then defined as

$$\hat{U}_{t't}[\hat{\mathbb{P}}_{t't}] \equiv \hat{\Lambda}_{t'}\,\hat{\mathbb{P}}_{t't}\,\hat{\Lambda}_t\,. \tag{45}$$

Note that, when $t' = t$ by construction we have the desired condition $\hat{U}_{tt} = \hat{I}$. Upon this given condition, the matrix $U_{t't}$ for a valid quantum reference frame must further satisfy $U^{\dagger}_{t't} = U^{-1}_{t't}$, implying $\mathbb{D}_t = \mathbb{D}_{t'} \equiv \mathbb{D}$ and the time independent Heissenberg physical state space $\mathbb{D}$.

4. When the above holds, we derive a Schrödinger dynamics emerging from the timeless Dirac theory. The wave functions take the form $\Psi_{\mathbb{D}}[\mathbf{X}_I](t)$ and is evolved by $\hat{U}_{t't}$. The theory in its Heissenberg picture is given by the physical states in the domain $\mathbb{D}$, and the complete sets of elementary relational observables $\{\hat{\mathbf{P}}_J(t) : \mathbb{D} \to \mathbb{D}\}$ and $\{\hat{\mathbf{X}}_I(t) : \mathbb{D} \to \mathbb{D}\}$, which by construction enjoy their correspondent kinematic algebra $[\hat{\mathbf{P}}_J(t), \hat{\mathbf{X}}_I(t)] = [\hat{\mathbf{P}}_J, \hat{\mathbf{X}}_I](t)$.

Having introduced the formulation for the exact theory, we now look into its application under perturbation truncations mentioned above. For that, we assume the truncated classical constraints $\{{}^nC_\mu\}$ can also be properly quantized such that we are given $\{{}^n\hat{C}_\mu\}$ with $\hat{C}_\mu = {}^\infty\hat{C}_\mu$. In a straight forward manner, we may introduce the truncated rigging map as

$$ {}^n\hat{\mathbb{P}} \equiv \delta(\,{}^n\hat{M}\,) \equiv \delta(\sum_\mu {}^n\hat{C}^2_\mu\,). \tag{46} $$

We may then apply the algorithm above to the truncated theory governed by ${}^n\hat{\mathbb{P}}$, with the relevant transition elements being instead the ${}^n\hat{\mathbb{P}}_{t',t}[\,\mathbf{P}'_I\,,\,\mathbf{P}_I\,]$. Recall that the only input for quantum reference frame algorithm is the values of the elements ${}^n\hat{\mathbb{P}}_{t',t}[\,\mathbf{P}'_I\,,\,\mathbf{P}_I\,]$ which should approach the exact values $\hat{\mathbb{P}}_{t',t}[\,\mathbf{P}'_I\,,\,\mathbf{P}_I\,]$ when the perturbation components of $(\mathbf{P}_J\,,\,\mathbf{X}_I)$ are small enough. Clearly, in obtaining the truncated propagator ${}^n\hat{U}_{t't}$ through the algorithm, we should consistently keep the results only to the proper orders in a manner consistent to the truncation upon the quantum constraints.

B. algorithm application to the hybrid LQC model

The quantum model governed by $\widehat{\mathcal{C}}$ is a special case of the theories referred to in above subsection. Specifically, in this case we have the second-order truncated quantum constraint system of $\{{}^2\hat{C}_\mu\} = \{\widehat{\mathcal{C}}\}$ with $\widehat{\mathcal{C}}$ as given in (41). The truncated rigging map is then given by

$$ {}^2\hat{\mathbb{P}} \;\equiv \delta(\widehat{\mathcal{C}}) = \int_{-\infty}^{+\infty} d\alpha e^{i\alpha\widehat{\mathcal{C}}} \equiv \widehat{P} \tag{47} $$

where we introduce the abbreviation $\widehat{P} \equiv {}^2\hat{\mathbb{P}}$ in our special case just for notation simplicity. We will also use the form $\delta(\widehat{\mathcal{C}}) = \delta(\widehat{\Omega}, \widehat{V}, \widehat{\phi}, \widehat{p_\phi}, \widehat{\delta\phi}, \widehat{\delta p_\phi})$, which means the constraint operator is a function of the operators in the bracket.

Before going in to the specific calculations, more concretely, we will show the main steps of QRF algorithm in the cosmological perturbation scenario.

Step 1: To Specify the Quantum Reference Frame. We have discussed in the previous section about the degrees of freedom in classical phase space. Since we only focus on the scalar perturbations, and we already made gauge fixing and solved the constraints for perturbations (there is only one physical physical degrees of freedom in our consideration–$\delta\phi(\vec{x})$ for perturbation variables), so we only left one constraint $\widehat{\mathcal{C}}$ in classical phase space. The number of fields in reference sector must consist with the number of constraints. Then, we need choose one of the background variables as the reference sector, in this paper, we will choose the scalar field as the reference frame, that is

$$ \{\mathbf{X}_\mu, \mathbf{P}_\mu\} \equiv \{(\phi, p_\phi)\} \tag{48a} $$
$$ and $$
$$ \begin{aligned} \{\mathbf{X}_I, \mathbf{P}_I\} &= \{X_i, P_i\} - \{X_\mu, P_\mu\} \\ &\equiv \{(V, \Omega), (\delta\phi(\vec{x}), \delta p_\phi(\vec{x}))\} \end{aligned} \tag{48b} $$

In the LQC FLRW classical phase space, each given set of values to $(V, \Omega, \delta\phi, \delta p_\phi)(t)$ and $\phi(t)$ corresponds to two points on the constraint surface given by the two constraints solutions $\pm p_\phi > 0$. Therefore we are instructed to use only one of this solution $p_\phi > 0$, corresponds to $\phi^+ \equiv \Theta(p_\phi)\phi$ where $\Theta(p_\phi)$ project ϕ to the positive solution. In the quantum theory, we then represent ϕ^+ with the operator $\widehat{\phi}^+$ constructed in the previous work [21], such that

$$ \left|\phi^+(t)\right\rangle \equiv \Theta(\widehat{p_\phi})\left|\phi(t)\right\rangle. $$

Then, for this work we set $\{\hat{T}_\mu\} = \{\widehat{\phi}^+\}$ with the assigned $\{\bar{T}_\mu(t)\} = \phi^+(t)$, which specifies a one-parameter family of kinematical eigenspaces where the transition happens

$$ \{\mathbb{S}_t\} \equiv span\left\{\left|\phi^+(t), \Omega, \delta p_\phi(\vec{x})\right\rangle\right\} \subset \widetilde{H} \tag{49} $$

where $|\phi^+(t)\rangle \equiv \Theta(\widehat{p_\phi})\,|\phi(t)\rangle$, $|\Omega\rangle$, $|\delta p_\phi(\vec{x})\rangle$ are the eigenstate of operators $\widehat{\phi}, \widehat{\Omega}, \widehat{\delta p}_\phi(\vec{x})$ respectively.

Step 2: Calculate the Relevant Transition Amplitude, Correction Factor and Propagator. Transition amplitudes can be obtained from the truncated rigging map operator

$$ \begin{aligned} \widehat{P}_{t't}[\mathbf{P}'_I, \mathbf{P}_I] &= \langle \mathbf{P}'_I|\hat{P}_{t't}\,|\mathbf{P}_I\rangle \\ &= \langle \mathbf{P}'_I, \phi^+(t')|\hat{P}\left|\mathbf{P}_I, \phi^+(t)\right\rangle \end{aligned} \tag{50} $$

we will choose a subspace of the small perturbations $\mathbb{S}^\varepsilon_t \subset \mathbb{S}_t$, or, equally to say, the wave function will vanished when the perturbations get large enough for consisting with the cosmological observations

$$ \begin{aligned} \{\mathbb{S}^\varepsilon_t\} \equiv span\,\{\,|\Psi, \phi(t)\rangle|\,\Psi(\Omega, \delta p_\phi) = \Psi(V, \delta\phi) = 0, \\ when\,|\delta p_\phi| > \varepsilon, |\delta\phi| > \varepsilon\} \end{aligned} \tag{51} $$

ε is a small positive quantity.

As mentioned earlier, we should now supply the rigging map with a systematic power expansion, which will then define the associated expansion for the physical Hamiltonian operator. Specifically, let us introduce the phase space function associated to the rigging map as

$$\hat{P}[\,\mathbf{P}_j\,,\,\mathbf{X}_i\,] \equiv \langle\,\mathbf{P}_j\,|\;\hat{P}\;|\,\mathbf{X}_i\,\rangle\,. \tag{52}$$

Assumed with the proper analyticity, the function $\hat{P}[\,\mathbf{P}_j\,,\,\mathbf{X}_i\,]$ would have a unique power expansion with respect to the perturbations and $\hbar$. In this work we will make use of the Taylor expansion of $\hat{P}[\,\mathbf{P}_j\,,\,\mathbf{X}_i\,]$ in $\hbar$ given by

$$\hat{P}[\,\mathbf{P}_j\,,\,\mathbf{X}_i\,] \equiv \sum_{m=0}^{\infty} \hat{P}^{(m)}[\,\mathbf{P}_j\,,\,\mathbf{X}_i\,] \tag{53}$$

Here $\hat{P}^{(m)}[\,\mathbf{P}_j\,,\,\mathbf{X}_i\,]$ represents the term of $\hbar^m$ in the Taylor expansion of $\hat{P}[\,\mathbf{P}_j\,,\,\mathbf{X}_i\,]$ over $\hbar$. Then, the rigging map operator can be correspondingly expanded as

$$\hat{P} \equiv \sum_{m=0}^{\infty} \varepsilon^m \hat{P}^{(m)}\,|_{\varepsilon=1}. \tag{54}$$

where the $\hbar^m$ order rigging map operator $\hat{P}^{(m)}$ is defined by

$$\hat{P}^{(m)}[\,\mathbf{P}_j\,,\,\mathbf{X}_i\,] \equiv \langle\,\mathbf{P}_j\,|\;\hat{P}^{(m)}\;|\,\mathbf{X}_i\,\rangle\,. \tag{55}$$

Note that in (54) we have introduced a dummy parameter $\varepsilon = 1$ purely to track the order of expansion here and later on.

Accordingly, we define the expansions for the relevant transition amplitude operator $\widehat{P}_{t't}$ as

$$\widehat{P}_{t't} \equiv \sum_{m=0} \varepsilon^m \widehat{P}^{(m)}_{t't}\,|_{\varepsilon=1} \tag{56}$$

where we introduce

$$\widehat{P}^{(m)}_{t't}[\mathbf{P}'_I, \mathbf{P}_I] = \langle \mathbf{P}'_I, \phi^+(t')|\,\hat{P}^{(m)}\,|\mathbf{P}_I, \phi^+(t)\rangle \tag{57}$$

The expansion (56) for the relevant transition amplitude operator is the foundation of our systematic power expansion for the Schrödinger dynamics, which would be expanded in the powers of ε. Indeed, the opeator $\hat{\Lambda}_t$ and hence the propagator $\hat{U}_{t't}$ are all given functions of $\widehat{P}_{t't}$, therefore we may expand both of them in the powers of ε as

$$\hat{\Lambda}_t\left(\hat{P}_{t't}\right) \equiv \sum_{m=0} \varepsilon^m \hat{\Lambda}^{(m)}_t(\hat{P}^{(0)}_{t't}, \cdots, \hat{P}^{(m)}_{t't}) \tag{58}$$

$$\hat{U}_{t't}\left(\hat{P}_{t't}\right) \equiv \sum_{m=0} \varepsilon^m \hat{U}^{(m)}_{t't}(\hat{P}^{(0)}_{t't}, \cdots, \hat{P}^{(m)}_{t't}) \tag{59}$$

where $\hat{\Lambda}^{(m)}_t$ and $\hat{U}^{(m)}_{t't}$ are simply the terms of the order of ε^m arising through the expression (56), and thus they are functions of only the components $(\hat{P}^{(0)}_{t't}, \cdots, \hat{P}^{(m)}_{t't})$ of the orders up to m. In the special cases of the ε^0 order, we have

$$\hat{\Lambda}^{(0)}_t(\hat{P}^{(0)}_{t't}) = \hat{\Lambda}_t(\hat{P}_{t't}) + o(\varepsilon^1) = \left(\hat{P}^{(0)}_{tt}\right)^{-\frac{1}{2}} \tag{60}$$

$$\hat{U}^{(0)}_{t't}(\hat{P}^{(0)}_{t't}) = \hat{U}_{t't}(\hat{P}_{t't}) + o(\varepsilon^1) = \hat{\Lambda}^{(0)}_{t'}\hat{P}^{(0)}_{t't}\hat{\Lambda}^{(0)}_t \tag{61}$$

Finally, since the mth order term $\hat{P}^{(m)}$ in the rigging map is of $\hbar^m$ by definition, the terms with higher m in $\hat{\Lambda}_t$ or $\hat{U}_{t't}$ would be more suppressed by higher powers of $\hbar$. However, we also note that due to the additional factors of $\hbar$ appearing in the above function expressions of the rigging map, the mth order term $\hat{U}^{(m)}_{t't}$ in $\hat{U}_{t't}$ would nolonger simply be of $\hbar^m$. Rather, as we will see soon (in IV B), it becomes a combination of the terms of $\hbar^m$ and $\hbar^{m-1}$ when t' approaches t.

Step 3: Verifying the Unitary Conditions. If we denote $\{\mathbb{D}_t\}$ as a family of eigenspaces in the physical Hilbert space, the unitary conditions are: 1, $\hat{U}_{tt} = \mathbb{I}$ 2, $\mathbb{D}_t = \mathbb{D}_{t'>t} \equiv \mathbb{D}$, or $U^{\dagger}_{t't}U_{t't} = \mathbb{I}$. Then the relation between propagator and physical Hamiltonian operator satisfies

$$\widehat{U}_{t't} = e^{-\frac{i}{\hbar}\int_t^{t'}\widehat{H}(t)dt} \tag{62}$$

If we assume P_{tt} is non-degenerate, the solution given by the above subsection $\hat{\Lambda}_t \equiv \hat{P}^{-1/2}_{tt}$ will make $\hat{U}_{t't}\left(\hat{P}_{t't}\right)$ satisfies the unitary conditions 1, so does to each ε^m order term. We will assume this solution as well as its every ε^m order term also satisfy the unitary conditions 2, because in the FLRW situation, it has been proved[21]. Since the perturbations is quite small, we can expect that it will be the same with the FLRW situation.

Step 4: Calculating the Physical Hamiltonian. We can obtain the physical Hamiltonian directly from the propagator, when $\phi(t') - \phi(t) \ll 1$, by definition,

$$\hat{H}(\phi(t)) \equiv i\hbar\frac{\partial}{\partial t'}\hat{U}_{t't}(\hat{P}_{t't})\,|_{t'=t} \tag{63}$$

especially, we can also apply the $\hbar^1$ truncation, this will give us the physical Hamiltonian arising from the zeroth order transition amplitude,

$$\hat{H}^{(0)}(\phi(t)) \equiv i\hbar\frac{\partial}{\partial t'}\hat{U}^{(0)}_{t't}(\hat{P}^{(0)}_{t't})\,|_{t'=t} \tag{64}$$

At the end, we derive a Schrödinger dynamics emerging for the cosmological perturbation from the timeless Dirac theory. The wave functions take the form $\Psi_{\mathbb{D}}[\Pi](t)$ and is evolved by these physical Hamiltonian.

Now we are ready to calculate the above mentioned quantities for cosmological perturbations based on the Quantum Reference Frames algorithm.

IV. EXPLICIT COMPUTATION OF PHYSICAL HAMILTONIAN OPERATOR

A. the expansion to the relevant transition amplitude

Referring to (54), one can see clearly that the unique expansion for $\hat{P}$ can also be generated by the unique operator power expansion with each term of $\hbar^m$ having all the position operators ordered to the right. To extract these terms order-by-order through the successive reordering operations, we introduce an operation $[\,]^{\widehat{x}}_{R,ord}$ which move all the position operators $\widehat{x}$ in the bracket to the right and all the corresponding momentum operators $\widehat{p_x}$ to the left without changing the rest of the ordering. For example, we would have

$$[\widehat{p_x}\widehat{x}\widehat{y}\widehat{p_y}\widehat{p_x}\widehat{x}]^{\widehat{x}}_{R,ord} \equiv \widehat{p_x}^2\widehat{y}\widehat{p_y}\widehat{x}^2.$$

Starting from the $m=0$ order, we first reorder the rigging map operator which will give us the quantum corrections with $m\geq 1$,

$$\widehat{P} = \delta(\widehat{\mathcal{C}}) = \varepsilon^0\left[\delta(\widehat{\mathcal{C}})\right]^{\widehat{\phi}}_{R,ord} + \sum_{m=1}^{\infty}\varepsilon^{(m)}\widehat{\alpha}^{(m)} \tag{65}$$

where $\widehat{\alpha}^{(m)}\sim\hbar^m$ represents the $\hbar^m$ order quantum corrections of reordering $(\widehat{\phi},\widehat{p_\phi})$. Then, the transition amplitude operator is that

$$\begin{aligned}\widehat{P}_{t't} &= \left\langle\phi^+(t')\right|\left[\delta(\hat{\Omega},\hat{V},\widehat{\phi},\widehat{p_\phi},\widehat{\delta\phi},\widehat{\delta p_\phi})\right]^{\widehat{\phi}}_{R,ord}\left|\phi^+(t)\right\rangle\\ &+\sum_{m=1}^{\infty}\varepsilon^m\left\langle\phi^+(t')\right|\widehat{\alpha}^{(m)}(\hat{\Omega},\hat{V},\widehat{\phi},\widehat{p_\phi},\widehat{\delta\phi},\widehat{\delta p_\phi})\left|\phi^+(t)\right\rangle\\ &=\int_0^{+\infty}dp_\phi\delta(\hat{\Omega},\hat{V},\phi^+(t),p_\phi,\widehat{\delta\phi},\widehat{\delta p_\phi})\cdot\\ &\qquad e^{\frac{i}{\hbar}p_\phi(\phi^+(t')-\phi^+(t))}\\ &+\sum_{m=1}^{\infty}\varepsilon^m\int_0^{+\infty}dp_\phi\widehat{\alpha}^{(m)}(\hat{\Omega},\hat{V},\phi^+(t),p_\phi,\widehat{\delta\phi},\widehat{\delta p_\phi})\cdot\\ &\qquad e^{\frac{i}{\hbar}p_\phi(\phi^+(t')-\phi^+(t))}\end{aligned} \tag{66}$$

where we have used the completeness relations $\mathbb{I}=\int dp_\phi\left|p_\phi\right\rangle\left\langle p_\phi\right|$ and $\left\langle\phi\mid p_\phi\right\rangle = e^{\frac{i}{\hbar}\phi p_\phi}$. $\alpha^m\sim\hbar^m$ is the eigenvalues of operator $\widehat{\alpha}^m$. For simplicity, we will not to write the "+" sign above each of eigenvalue $\phi(t)$ after, and it doesn't loose any information. $\widehat{P}_{t't}$ is the operator act on the eigenstate of $|\Omega,\delta p_\phi\rangle$, and we also need to reorder this operator as

$$\begin{aligned}\widehat{P}_{t't} &= \left[\int_0^{+\infty}dp_\phi\delta(\hat{\Omega},\hat{V},\phi(t),p_\phi,\widehat{\delta\phi},\widehat{\delta p_\phi})e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))}\right]^{\widehat{V},\widehat{\delta\phi}}_{R,ord}\\ &+\sum_{m=1}^{\infty}\varepsilon^m\widehat{\sigma}^{(m)}_{t't}\end{aligned} \tag{67}$$

where $\widehat{\sigma}^{(m)}_{t't} = \widehat{\beta}^{(m)}_{t't} + \langle\phi^+(t')|\,\widehat{\alpha}^{(m)}\,|\phi^+(t)\rangle$, and $\widehat{\beta}^{(m)}_{t't}$ represents the quantum corrections of reordering $(\hat{\Omega},\hat{V},)$ and $(\widehat{\delta\phi},\widehat{\delta p_\phi})$.

Then we can calculate the transition amplitude elements between $\mathbb{S}_{t'}$ and $\mathbb{S}_t$ by using $\widehat{P}_{t't}$

$$\begin{aligned}P_{t't} &= \langle\Omega',\delta p_\phi{}'|\,\widehat{P}_{t't}\,|\Omega,\delta p_\phi\rangle\\ &= \langle\Omega',\delta p_\phi{}'|\\ &\left[\int_0^{+\infty}dp_\phi\delta(\hat{\Omega},\hat{V},\phi(t),p_\phi,\widehat{\delta\phi},\widehat{\delta p_\phi})e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))}\right]^{\widehat{V},\widehat{\delta\phi}}_{R,ord}\\ &\quad|\Omega,\delta p_\phi\rangle\\ &\quad+\sum_{m=1}^{\infty}\varepsilon^m\langle\Omega',\delta p_\phi{}'|\,\widehat{\sigma}^{(m)}_{t't}\,|\Omega,\delta p_\phi\rangle\\ &=\sum_{V''}\int d\delta\phi'' B e^{\frac{i}{\hbar}\underline{p^+_\phi}(\phi(t')-\phi(t))}\\ &\quad\langle\Omega',\delta p_\phi{}'\mid V'',\delta\phi''\rangle\,\langle V'',\delta\phi''\mid\Omega,\delta p_\phi\rangle\\ &\quad+\sum_{m=1}^{\infty}\varepsilon^m\langle\Omega',\delta p_\phi{}'|\,\widehat{\sigma}^{(m)}_{t't}\,|\Omega,\delta p_\phi\rangle\\ &\equiv\varepsilon^0P^{(0)}_{t't}+\sum_{m=1}^{\infty}\varepsilon^mP^{(m)}_{t't}\end{aligned} \tag{68}$$

where we have used the completeness relations $\mathbb{I}=\sum_{V''}|V''\rangle\langle V''| = \int d\delta\phi''\,|\delta\phi''\rangle\,\langle\delta\phi''|$, and B is the Fadeev-Popov path integral determinant which only paly the role of normalization here, $\underline{p^+_\phi}$ is the positive solution to the constraint $\mathcal{C}$ given by

$$B=\int_0^{+\infty}dp_\phi\delta(\Omega',V'',\phi(t),p_\phi,\delta\phi'',\delta p_\phi{}')\approx\frac{1}{\underline{p^+_\phi}} \tag{69}$$

$$\begin{aligned}\underline{p^+_\phi} &= \Bigg(\frac{6}{\kappa\gamma^2}\Omega'^2-2V''^2V(\phi)\\ &\quad-\int_{V_0}d^3\vec{x}\left(\delta p'^2_\phi(\vec{x})+a''^6(\partial\delta\phi''(\vec{x}))^2+\mathbf{U}a''^6\delta\phi''^2(\vec{x})\right)\Bigg)^{\frac{1}{2}}\end{aligned} \tag{70}$$

where $a''^3V_0=V''$. Here we want to remark that this is an approximation rather than an exact solution to p_ϕ since there are "p_ϕ" high order terms (see equation (36)) in the effective potential $\mathfrak{U}$ which is a function of background quantities $(\Omega,V,p_\phi,\phi(t))$. However, since it appears in the perturbations part, we can always use the positive solution $p_{\phi+}$ of the regularized background densitized constraint $\mathcal{S}^{(0)}=\frac{1}{2\kappa}\left(-\frac{6}{\gamma^2}\Omega^2+\kappa p_\phi{}^2+2\kappa V^2V\left(\phi\right)\right)=0$ to replace those "p_ϕ"s in $\mathfrak{U}$. Then $\mathfrak{U}$ become a function of background quantities $\mathbf{U}(\Omega',V'',\phi(t))$ here which takes form as

$$\mathbf{U} \approx (-648V_0^{8/3} + 18V_0^2)\frac{\Omega'^2 V''^{-2}}{\gamma^2} - 72\kappa^2\gamma^2 V_0^{8/3}\Omega'^{-2}V''^2V^2(\phi) + (72V_0^{8/3} - V_0^2)6\kappa V(\phi) - 12\sqrt{6\kappa}V_0^{1/3}(1 - \frac{\kappa\gamma^2}{6}\Omega'^{-2}V''^2V(\phi))V_\phi + V_{\phi\phi} \quad (71)$$

B. the physical Hamiltonian arising from the ε^0 order transition amplitude $\widehat{P}_{t't}^{(0)}$

Let us first focus on the zeroth order or the first term in $P_{t't}$, from the rigging map matrix elements (68) and $P_{t't} = \langle\Omega', \delta p_\phi'|\,\widehat{P}_{t't}\,|\Omega, \delta p_\phi\rangle$, we can recognize that

$$\widehat{P}_{t't}^{(0)} = \left[\widehat{B}(\phi(t)) \cdot e^{\frac{i}{\hbar}(\phi(t')-\phi(t))\widehat{\underline{p_\phi^+}}(\hat{\Omega},\hat{V},\phi(t),\widehat{\delta\phi},\widehat{\delta p_\phi})}\right]_{R,ord}^{\widehat{V},\widehat{\delta\phi}} \quad (72)$$

where $\widehat{B}$ and $\widehat{p_\phi^+}$ represent the operator form of (69) and (70) respectively, which just replace all the quantities by corresponding operators. Following the requirements of unitary evolution, the only meaningful solution for the zeroth order correction factor is

$$\widehat{\Lambda}_t^{(0)} = \left(\widehat{P}_{tt}^{(0)}\right)^{-\frac{1}{2}} = (\left[\widehat{B}(\phi(t))\right]_{R,ord}^{\widehat{V},\widehat{\delta\phi}})^{-\frac{1}{2}} \approx (\left[\widehat{\underline{p_\phi^+}}^{-1}\right]_{R,ord}^{\widehat{V},\widehat{\delta\phi}})^{-\frac{1}{2}} \quad (73)$$

Then, we can get the propagator operator in terms of the zeroth order transition amplitude

$$\hat{U}_{t't}^{(0)}(\hat{P}_{t't}^{(0)}) = \widehat{\Lambda}_{t'}^{(0)\dagger}\widehat{P}_{t't}^{(0)}\widehat{\Lambda}_t^{(0)} = (\left[\widehat{B}(\phi(t'))\right]_{R,ord}^{\widehat{V},\widehat{\delta\phi}})^{-\frac{1}{2}}\widehat{P}_{t't}^{(0)}(\left[\widehat{B}(\phi(t))\right]_{R,ord}^{\widehat{V},\widehat{\delta\phi}})^{-\frac{1}{2}} \quad (74)$$

According the definition (63), the physical Hamiltonian arising from the zeroth order transition amplitude take form of

$$\begin{aligned}\hat{H}^{(0)}(\phi(t)) &\equiv i\hbar\frac{\partial}{\partial t'}\hat{U}_{t't}^{(0)}(\hat{P}_{t't}^{(0)})\,|_{t'=t} \\ &= -\partial_t\phi(t)(\left[\widehat{B}(\phi(t))\right]_{R,ord}^{\widehat{V},\widehat{\delta\phi}})^{-\frac{1}{2}}(\left[\widehat{B}(\phi(t))\cdot\widehat{\underline{p_\phi^+}}(\phi(t))\right]_{R,ord}^{\widehat{V},\widehat{\delta\phi}})(\left[\widehat{B}(\phi(t))\right]_{R,ord}^{\widehat{V},\widehat{\delta\phi}})^{-\frac{1}{2}} \\ &\quad - \frac{i\hbar}{2}\partial_t\phi(t)(\partial_\phi\left[\widehat{B}(\phi(t))\right]_{R,ord}^{\hat{V},\widehat{\delta\phi}})(\left[\widehat{B}(\phi(t))\right]_{R,ord}^{\hat{V},\widehat{\delta\phi}})^{-1} \\ &\approx \hat{H}^{01}(\phi(t)) + \hat{H}^{02}(\phi(t))\end{aligned} \quad (75)$$

where we have use the definition (69) to get the approximation result in the last line. $\hat{H}^{01}(\phi(t))$ is the zeroth order $\hbar^0$ physical Hamiltonian, and the second term $\hat{H}^{02}(\phi(t))$ is the first order $\hbar^1$ quantum correction to $\hat{H}^{01}(\phi(t))$. They are

$$\hat{H}^{01}(\phi(t)) = -\partial_t\phi(t)(\left[\widehat{\underline{p_\phi^+}}^{-1}\right]_{R,ord}^{\hat{V},\widehat{\delta\phi}})^{-1} \quad (76)$$

$$\hat{H}^{02}(\phi(t)) = -\frac{i\hbar}{4}\partial_t\phi(t)\cdot(\left[\widehat{\underline{p_\phi^+}}^{-1}\right]_{R,ord}^{\hat{V},\widehat{\delta\phi}})^2(-2\widehat{V}^2\widehat{V}_\phi - \int_{V_0}dx^3\widehat{\mathbf{U}}_\phi\hat{a}^6\widehat{\delta\phi}^2) \quad (77)$$

$\widehat{\mathbf{U}}_\phi$ means the derivative of $\widehat{\mathbf{U}}$ with respect to ϕ.

Remark: *From equation (66), we can see that $\hat{P}_{t't}^{(m)} \propto \varepsilon^m e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))}$. When considering the condition $\phi(t') - \phi(t) \ll 1$, we have $e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))} \approx 1 + \frac{i}{\hbar}p_\phi(\phi(t') - \phi(t))$, Then it is clear that $\hat{P}_{t't}^{(m)} \propto \varepsilon^m(1 + \frac{i}{\hbar}p_\phi(\phi(t') - \phi(t))$. That means $\hat{P}_{t't}^{(m)}$ is not only containing the $\hbar^m$ order terms but also the $\hbar^{m-1}$ order terms. So does to the $\widehat{\Lambda}_t^{(m)}$ and $U_{t't}^{(m)}$, they all contain $\hbar^m$ order terms and the $\hbar^{m-1}$ order terms. Then, when we calculate the physical Hamiltonian according to (63), it will give us $\hat{H}^{(m)}(\phi(t)) \propto a\hbar^{m+1} + b\hbar^m$ (a,b are some $\hbar^0$ quantities), which means the physical Hamiltonian arising from ε^m order transition amplitude will obtain $\hbar^{m+1}$ order terms and the $\hbar^m$ order terms. In a special case, $\hat{H}^{(0)}(\phi(t))$ will contain $\hbar^1$ order terms and the $\hbar^0$ order terms, and $\hat{H}^{(1)}(\phi(t))$ will contain $\hbar^2$ order terms and the $\hbar^1$ order terms.*

In order to make sure this Hamiltonian bounded from below, we can let $\partial_t\phi(t) = -1$ here and after, and it is allowed since t is only a parameter monotonic with respect to ϕ.

We first focus on the zeroth order physical Hamiltonian $\hat{H}^{01}(\phi(t))$. Since $\widehat{\underline{p_\phi^+}}$ is a function involving square root of the operators $\widehat{\Omega}, \hat{V}, \widehat{\delta p_\phi}, \widehat{\delta\phi}$, it is quite complicate. However, The potential and perturbation terms in the expression is quite small relative to the first term in the deep-planck regime[19], so we can do the Taylor expansion and keep the terms up to the second order of perturbations

$$(\left[\widehat{\underline{p_\phi^+}}^{-1}\right]_{R,ord}^{\hat{V},\widehat{\delta\phi}})^{-1} \approx \widehat{H}^{01} - \widehat{H}^{02} + \sum_{m=1}^{\infty}\hat{u}^{(m)} \quad (78)$$

$$\widehat{H}^{011} \equiv \sqrt{\frac{6}{\kappa\gamma^2}}\left|\widehat{\Omega}\right| - \sqrt{\frac{\kappa\gamma^2}{6}}V(\phi)\left|\widehat{\Omega}\right|^{-\frac{1}{2}}\widehat{V}^2\left|\widehat{\Omega}\right|^{-\frac{1}{2}} \quad (79)$$

$$\widehat{H}^{012} \equiv \frac{1}{2}\sqrt{\frac{\kappa\gamma^2}{6}}\left|\widehat{\Omega}\right|^{-\frac{1}{2}}\left(\int_{V_0}d^3\vec{x}(\widehat{\delta p}_\phi^2 + \hat{a}^6(\partial\widehat{\delta\phi})^2 + \hat{a}^3\widehat{\mathbf{U}}\hat{a}^3\widehat{\delta\phi}^2)\right)\left|\widehat{\Omega}\right|^{-\frac{1}{2}} \quad (80)$$

where the absolute sign of $\left|\widehat{\Omega}\right|$ means that we only consider the positive solution, for simplicity we will not write the absolute value sign for operators after. $\widehat{u}^{(m)} \sim \hbar^m$ represents the m-th order quantum corrections of reordering $(\hat{\Omega}, \hat{V})$ into symmetrized order just for comparing with the dressed metric approach later. Keep to the first order of $\hbar^1$, the explicit forms for $\widehat{u}^{(1)}$ can be seen from Appendix B.

In addition, we also have the $\hbar^1$ quantum corrections $\hat{H}^{02}(\phi(t))$. If we also keep to the second order of perturbations, we get

$$\hat{H}^{02}(\phi(t)) \approx \hat{H}^{021} - \hat{H}^{022} \tag{81}$$

where $\hat{H}^{021}$ and $\hat{H}^{022}$ represents the quantum corrections to the zeroth order physical Hamiltonian of background and perturbation respectively

$$\hat{H}^{021} = -\frac{i\hbar}{2}\Big(\frac{\kappa\gamma^2}{6}\widehat{\Omega}^{-2}\widehat{V}^2 V_\phi + \frac{\kappa^2\gamma^4}{18}\widehat{\Omega}^{-4}\widehat{V}^4 V(\phi)V_\phi\Big) \tag{82}$$

$$\begin{aligned}\hat{H}^{022} = & \frac{i\hbar}{4}\Big(\frac{\kappa\gamma^2}{6}\widehat{\Omega}^{-2} + \frac{\kappa^2\gamma^4}{18}\widehat{\Omega}^{-4}\widehat{V}^2 V(\phi)\Big)\widehat{\mathbf{U}}_\phi \widehat{a}^6 \int_{V_0} dx^3 \widehat{\delta\phi}^2 \\ & + \frac{i\hbar}{2}\Big(\frac{\kappa^2\gamma^4}{36}\widehat{\Omega}^{-4}\widehat{V}^2 V_\phi\Big)\cdot \\ & \int_{V_0} d^3\vec{x}\big(\widehat{\delta p_\phi}^2(\vec{x}) + \widehat{a}^6(\partial\widehat{\delta\phi}(\vec{x}))^2 + \widehat{\mathbf{U}}\widehat{a}^6\widehat{\delta\phi}^2(\vec{x})\big)\end{aligned} \tag{83}$$

In the Heisenberg picture, the dynamics of background and perturbations can be obtained by

$$\partial_\phi \widehat{V} = \frac{1}{i\hbar}[\widehat{V}, \hat{H}^{(0)}(\phi(t))] \tag{84a}$$

$$\partial_\phi \widehat{\Omega} = \frac{1}{i\hbar}[\widehat{\Omega}, \hat{H}^{(0)}(\phi(t))] \tag{84b}$$

$$\partial_\phi \widehat{\delta\phi} = \frac{1}{i\hbar}[\widehat{\delta\phi}, \hat{H}^{(0)}(\phi(t))] \tag{84c}$$

$$\partial_\phi \widehat{\delta p_\phi} = \frac{1}{i\hbar}[\widehat{\delta p_\phi}, \hat{H}^{(0)}(\phi(t))] \tag{84d}$$

C. the physical Hamiltonian arising from the ε^1 order transition amplitude $\hat{P}^{(1)}_{t't}$

According to the definition (68), the ε^1 order correction to the transition amplitude operator is (we take $\varepsilon = 1$ here and after)

$$\begin{aligned}\hat{P}^{(1)}_{t't} &= \hat{\sigma}^{(1)}_{t't} = \hat{\beta}^{(1)}_{t't} + \langle\phi^+(t')|\hat{\alpha}^{(1)}|\phi^+(t)\rangle \\ &= \hbar\int_0^{+\infty} dp_\phi \int_{-\infty}^{+\infty} d\alpha \widehat{F}\,(\phi(t))\, e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))}\end{aligned} \tag{85}$$

the quantity $\widehat{F}\,(\phi(t))$ is composed of quantum corrections of recording operators, given by

$$\begin{aligned}\widehat{F}\,(\phi(t)) \equiv & -\left(4\pi\gamma G\hbar\sqrt{\Delta}\right)\left(\alpha e^{i\alpha\widehat{C_I}(p_\phi)}\right)\hat{V}^1_{crrec} \\ & -\frac{i}{2}\left(\alpha^2 e^{i\alpha\widehat{C_I}(p_\phi)}\right)\Big(4\pi\gamma G\hbar\sqrt{\Delta}\hat{V}^2_{crrec} \\ & +\widehat{\delta\phi}_{crrec} + \hat{\phi}_{crrec}\Big)\end{aligned} \tag{86}$$

where the definition for $\widehat{C}_I(p_\phi)$ could be found in Appendix A. $\hat{V}^1_{crrec}$, $\hat{V}^2_{crrec}$ is the $\hbar^1$ order quantum corrections of reordering $(\hat{V}, \hat{\Omega})$. $\widehat{\delta\phi}_{crrec}$, $\hat{\phi}_{crrec}$ are the $\hbar^1$ order quantum corrections of reordering $(\widehat{\phi}, \hat{p}_\phi)$, $(\widehat{\delta\phi}(x), \widehat{\delta p}_\phi(x))$ respectively. The exact expressions for these quantities are shown in the Appendix A. Then, up to $\hbar^1$ order, the total transition amplitude operator is

$$\hat{P}_{t't} \approx \hat{P}^{(0)}_{t't} + \hat{P}^{(1)}_{t't} \tag{87}$$

one should be noted that $\hat{P}^{(1)}_{t't}$ is much smaller than $\hat{P}^{(0)}_{t't}$. Then the only meaningful $\mathbb{S}_t$-preserving operator $\widehat{\Lambda}_t$ can be given by

$$\begin{aligned}\hat{\Lambda}_t(\hat{P}_{t't}) &\approx \left(\hat{P}^{(0)}_{tt} + \hat{P}^{(1)}_{tt}\right)^{-\frac{1}{2}} \\ &\approx \left(\hat{P}^{(0)}_{tt}\right)^{-\frac{1}{2}} - \frac{1}{2}\left(\hat{P}^{(0)}_{tt}\right)^{-\frac{3}{2}}\hat{P}^{(1)}_{tt}\end{aligned} \tag{88}$$

we have used Taylor expansion and keep to the first order regarding to $\hbar$. From this equation, we can see the $\hat{\Lambda}^{(1)}_t$ is

$$\hat{\Lambda}^{(1)}_t(\hat{P}^{(0)}_{t't}, \hat{P}^{(1)}_{t't}) \equiv -\frac{1}{2}\left(\hat{P}^{(0)}_{tt}\right)^{-\frac{3}{2}}\hat{P}^{(1)}_{tt} \tag{89}$$

We can now calculate the propagator operator arising from the $\hbar^1$ order rigging map $\hat{P}^1$, we can get

$$
\begin{aligned}
\hat{U}^{(1)}_{t't}(\hat{P}^{(0)}_{t't},\hat{P}^{(1)}_{t't}) \equiv{}& \hat{\Lambda}^{(0)\dagger}_{t'}\hat{P}^{(0)}_{t't}\hat{\Lambda}^{(1)}_{t} + \hat{\Lambda}^{(0)\dagger}_{t'}\hat{P}^{(1)}_{t't}\hat{\Lambda}^{(0)}_{t} + \hat{\Lambda}^{(1)\dagger}_{t'}\hat{P}^{(0)}_{t't}\hat{\Lambda}^{(0)}_{t} \\
={}& -\frac{\hbar}{2}\Big(\hat{B}(\phi(t'))\Big)^{-\frac{1}{2}}\Big(\hat{B}(\phi(t))\Big)^{-\frac{1}{2}}\left(\int_0^{+\infty}dp_\phi\int_{-\infty}^{+\infty}d\alpha\widehat{F}\left(\phi(t)\right)\right)e^{\frac{i}{\hbar}\underline{\widehat{p^+_\phi}}(\phi(t))(\phi(t')-\phi(t))} \\
&+\hbar\Big(\hat{B}(\phi(t'))\Big)^{-\frac{1}{2}}\Big(\hat{B}(\phi(t))\Big)^{-\frac{1}{2}}\int_0^{+\infty}dp_\phi\int_{-\infty}^{+\infty}d\alpha\widehat{F}\left(\phi(t)\right)e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))} \\
&-\frac{\hbar}{2}\Big(\hat{B}(\phi(t'))\Big)^{-\frac{3}{2}}\Big(\hat{B}(\phi(t))\Big)^{\frac{1}{2}}\left(\int_0^{+\infty}dp_\phi\int_{-\infty}^{+\infty}d\alpha\widehat{F}\left(\phi(t')\right)\right)e^{\frac{i}{\hbar}\underline{\widehat{p^+_\phi}}(\phi(t))(\phi(t')-\phi(t))}
\end{aligned}
\tag{90}
$$

the operators ordering in the $\hat{U}^{(1)}_{t't}$ doesn't matter, because every term is at least proportional to the $\hbar$, any reordering will give to at least $\hbar^2$ which can be safely neglected. So we are free to take any convenience order in the $\hat{U}^{(1)}_{t't}$. According to the definition, keep to the $\hbar^1$ order, we can get another part of $\hbar^1$ quantum correction to the physical Hamiltonian (75), given by

$$
\begin{aligned}
\hat{H}^{(1)}(\phi(t)) &= i\hbar\frac{\partial}{\partial t'}\hat{U}_{t't}(\hat{P}^{(0)}_{t't},\hat{P}^{(1)}_{t't})\,|_{t'=t} \\
&= \hbar\Big(\hat{B}(\phi(t))\Big)^{-1}\int_0^{+\infty}dp_\phi \\
&\quad\int_{-\infty}^{+\infty}d\alpha\widehat{F}\left(\phi(t)\right)\Big(p_\phi-\underline{\widehat{p^+_\phi}}(\phi(t))\Big)
\end{aligned}
\tag{91}
$$

If we keep to the second order of perturbations, the above equation can be simplified to

$$
\hat{H}^{(1)}(\phi(t)) \approx i\hbar\hat{H}^1_a\int_{V_0}d^3\vec{x}\widehat{\delta\phi}^2 \tag{92}
$$

where

$$
\hat{H}^1_a = \widehat{H}^{011}\int_0^{+\infty}dp_\phi\int_{-\infty}^{+\infty}d\alpha\hat{F}_a(p_\phi-\widehat{H}^{011}) \tag{93a}
$$

$$
\begin{aligned}
\hat{F}_a &= i\left(4\pi\gamma G\hbar\sqrt{\Delta}\right)\left(\alpha e^{i\alpha\widehat{C_I}(p_\phi)}\right)\hat{V}^1_a \\
&\quad-\frac{i}{2}\left(\alpha^2e^{i\alpha\widehat{C_I}(p_\phi)}\right)\left(4\pi\gamma G\hbar\sqrt{\Delta}\hat{V}^2_a+\widehat{\phi}_a\right)
\end{aligned}
\tag{93b}
$$

Then, when considering the quantum corrections, the equations (84a,84b,84c,84d) become

$$
\partial_\phi\widehat{V} = \frac{1}{i\hbar}[\widehat{V},\hat{H}^{(0)}(\phi(t))+\hat{H}^{(1)}(\phi(t))] \tag{94a}
$$

$$
\partial_\phi\widehat{\Omega} = \frac{1}{i\hbar}[\widehat{\Omega},\hat{H}^{(0)}(\phi(t))+\hat{H}^{(1)}(\phi(t))] \tag{94b}
$$

$$
\partial_\phi\widehat{\delta\phi} = \frac{1}{i\hbar}[\widehat{\delta\phi},\hat{H}^{(0)}(\phi(t))+\hat{H}^{(1)}(\phi(t))] \tag{94c}
$$

$$
\partial_\phi\widehat{\delta p_\phi} = \frac{1}{i\hbar}[\widehat{\delta p_\phi},\hat{H}^{(0)}(\phi(t))+\hat{H}^{(1)}(\phi(t))] \tag{94d}
$$

V. THE EFFECTIVE DYNAMICS OF COSMOLOGICAL PERTURBATIONS

By using the physical Hamiltonians obtained in the above section, we can analyze the effective and semiclassical dynamics of this theory by taking the expectation values on a state.

A typical state in $\{\mathbb{S}^\varepsilon_t\}$ is given by

$$
|\,\Psi) \equiv \int d\Omega d\delta p_\phi\Psi(\Omega(t),\delta p_\phi(t))\,|\,\Omega(t),\delta p_\phi(t)) \tag{95}
$$

Since there are interactions between the background and perturbation, the state (95) can not directly be written as a direct product of background part and perturbations part. Because even if the initial state could be written in this way, the time evolution will cause the states entangled between the two parts.

However, in order to compare the QRF algorithm results with that of in dressed metric approach, we can use the same test field approximation proposed in[18], that we ignore both the back reactions from perturbation and the quantum corrections to the background spacetime. That means we use the LQC equations to determine the evolution of background spacetime without considering perturbations and $\hbar^1$ quantum corrections, or to put it more clearly, we will replace the Hamiltonian $\hat{H}^0(\phi(t))+\hat{H}^1(\phi(t))$ in (94a,94b) by the LQC Hamiltonian $\widehat{H}^{011}$ in (79). In this way, the state can be written as a direct product of background and perturbation $\Psi(\Omega(t),\delta p_\phi(t)) = \Psi_0(\Omega(t))\otimes\delta\Psi(\Omega(t),\delta p_\phi(t))$.

We are interest in how the perturbation fields evolve on a given background spacetime. Thus we consider a background state $|\,\Psi_0)$,

$$
|\,\Psi_0) \equiv \int d\Omega\Psi_0(\Omega(t))\,|\,\Omega(t)) \tag{96}
$$

Then we can calculate the expectation value of the physical Hamiltonian under this state, and it will give us the effective Hamiltonian.

A. the zeroth order effective dynamics

We first focus on the zeroth order physical Hamiltonian $\hat{H}^{01}(\phi(t))$. The zeroth order effective Hamiltonian can be obtained as

$$\hat{H}^{01}_{eff} = (\Psi_0 \,|\, \hat{H}^{01}(\phi(t)) \,|\, \Psi_0) = (\Psi_0 \,|\, (\left[\underline{\widehat{p_\phi^+}}^{-1} \right]^{\hat{V}, \widehat{\delta\phi}}_{R,ord})^{-1} \,|\, \Psi_0) \tag{97}$$

where the hat means $\hat{H}^{01}_{eff}$ is still an operator acting on the state of perturbation physical Hilbert space. If we consider the $\hbar^0$ order term of (78) only, after we carry the (79) and (80) into the (97), we can get

$$\hat{H}^{01}_{eff} \approx H^{011}_{eff} - \hat{H}^{012}_{eff} \tag{98}$$

where $H^{011}_{eff}(\phi(t))$ and $\hat{H}^{012}_{eff}(\phi(t))$ represent the background and perturbation effective Hamiltonian, given by

$$H^{011}_{eff} = \sqrt{\frac{6}{\kappa\gamma^2}} \left\langle \hat{\Omega} \right\rangle - \sqrt{\frac{\kappa\gamma^2}{6}} V(\phi) \left\langle \hat{\Omega}^{-\frac{1}{2}} \hat{V}^2 \hat{\Omega}^{-\frac{1}{2}} \right\rangle \tag{99}$$

$$\widehat{H}^{012}_{eff} = \frac{1}{2}\sqrt{\frac{\kappa\gamma^2}{6}} \int_{V_0} d^3\vec{x} \, \Big(\left\langle \hat{\Omega}^{-1} \right\rangle \widehat{\delta p_\phi}^2 + \left\langle \hat{\Omega}^{-\frac{1}{2}} \hat{a}^6 \hat{\Omega}^{-\frac{1}{2}} \right\rangle (\partial\widehat{\delta\phi})^2 + \left\langle \hat{\Omega}^{-\frac{1}{2}} \hat{a}^3 \hat{\mathbf{U}} \hat{a}^3 \hat{\Omega}^{-\frac{1}{2}} \right\rangle \widehat{\delta\phi}^2 \Big) \tag{100}$$

where the "$\langle\rangle$" represents the expectation value on the background state $|\,\Psi_0)$. What the above equation tells us is that if we choose the scalar field $\phi(t)$ as the relational time, then, the quantum effective dynamics of the background is generated by the first term H^{011}_{eff}, while the effective dynamics of perturbation fields are generated by the second term $\hat{H}^{012}_{eff}$.

For the background, this theory truly has the semiclassical limits of the Hamiltonian dynamics in the reduced phase space (V, Ω). For a semiclassical state, the dynamics governed by the effective Hamiltonian

$$H^{011}_{eff} = \sqrt{\frac{6}{\kappa\gamma^2}} \, |\Omega| - \sqrt{\frac{\kappa\gamma^2}{6}} V(\phi) \left\langle \hat{\Omega}^{-\frac{1}{2}} \hat{V}^2 \hat{\Omega}^{-\frac{1}{2}} \right\rangle \tag{101}$$

For the perturbation fields, one will immediately notice that it is quite closely to the dressed metric results. Actually, we will show below that it coincident each other as long as we recognize

$$\frac{\widetilde{N}_\phi}{\widetilde{a}^3(\phi)} \equiv \sqrt{\frac{\kappa\gamma^2}{6}} \left\langle \hat{\Omega}^{-1} \right\rangle \tag{102a}$$

$$\widetilde{N}_\phi \widetilde{a}^3(\phi) \equiv \sqrt{\frac{\kappa\gamma^2}{6}} \left\langle \hat{\Omega}^{-\frac{1}{2}} \hat{a}^6 \hat{\Omega}^{-\frac{1}{2}} \right\rangle \tag{102b}$$

$$\widetilde{N}_\phi \widetilde{a}^3(\phi) \widetilde{\mathbf{U}}(\phi) \equiv \sqrt{\frac{\kappa\gamma^2}{6}} \left\langle \hat{\Omega}^{-\frac{1}{2}} \hat{a}^3 \hat{\mathbf{U}} \hat{a}^3 \hat{\Omega}^{-\frac{1}{2}} \right\rangle \tag{102c}$$

here the states $(\,\Psi_0|$ we use have a relation with the states $(\,\Phi_0|$ used in dressed metric appraoch, that is $(\,\Psi_0| = (\,\Phi_0|\,(\widehat{1/V})^{1/2}$ (see the discussions above equation (47)).

If we choose the lapse function as $N = \widetilde{N}_\phi$, then

$$\hat{H}^{012}_{eff} = \frac{\widetilde{N}_\phi}{2} \int_{V_0} d^3\vec{x} \, (\frac{\widehat{\delta p_\phi}^2}{\widetilde{a}^3(\phi)} + \widetilde{a}^3(\phi) (\partial\widehat{\delta\phi})^2 + \widetilde{a}^3(\phi) \widetilde{\mathbf{U}}(\phi) \widehat{\delta\phi}^2) \tag{103}$$

Now the second term becomes an quantum field propagate on an effective spacetime which described by the effective scale factor $\widetilde{a}(\phi)$ and the time lapse function $N = \widetilde{N}_\phi$ determined by the first term in the effective Hamiltonian (99), or, to put it in other way, it is the Hamiltonian of the quantum perturbation fields on an effective dressed metric $\widetilde{g}_{ab} dx^a dx^b = -\widetilde{N}^2_\phi d\phi^2 + \widetilde{a}^2(\phi)(dx^2 + dy^2 + dz^2)$ which satisfy the LQC FLRW equations rather than Einstein's general relativity. We set $\phi(t) = \phi$, in Heisenberg picture, the equations (94c,94d) will reduce to

$$\partial_\phi \widehat{\delta\phi} = \frac{1}{i\hbar} [\widehat{\delta\phi}, \hat{H}^{01}_{eff}] = \frac{1}{i\hbar} [\widehat{\delta\phi}, \hat{H}^{012}_{eff}] \tag{104a}$$

$$\partial_\phi \widehat{\delta p_\phi} = \frac{1}{i\hbar} [\widehat{\delta p_\phi}, \hat{H}^{01}_{eff}] = \frac{1}{i\hbar} [\widehat{\delta p_\phi}, \hat{H}^{012}_{eff}] \tag{104b}$$

Now by simple algebraic manipulations, these equations in the conformal time η where $\widetilde{N}_\phi = \widetilde{a}(\phi)$ can be written as the second-order differential equation

$$(\widetilde{\Box} - \widetilde{\mathbf{U}}(\eta)) \widehat{\delta\phi}(\vec{x}, \eta) = 0 \tag{105}$$

Where $\widetilde{\Box}$ is the d'Alembertian of the effective background dressed metric. This results is exactly consist with that of in [19]. Here we recover the dressed metric results through the Quantum Reference Frame Algorithm. It is the results of the zeroth order in term of $\hbar$.

B. the effective dynamics with the $\hbar^1$ order quantum corrections

Regarding to the $\hbar^1$ quantum corrections of perturbations dynamics, there are three contribution parts: the first part comes from $\hat{H}^{022}$ in (75), the second part comes from the $\hbar^1$ order Hamiltonian (91), the third part comes from the last term in (78) due to the symmetrized ordering. We will discuss one by one.

We first focus on the quantum correction $\hat{H}^{022}$, The corresponding effective Hamiltonian for the perturbations can be given by

$$\hat{H}^{022}_{eff} = (\Psi_0 \,|\, \hat{H}^{022} \,|\, \Psi_0) \tag{106}$$

Then the effective dynamics (104a,104b) will become

$$\partial_\phi \widehat{\delta\phi} = \frac{1}{i\hbar} [\widehat{\delta\phi}, \hat{H}^{012}_{eff} + \hat{H}^{022}_{eff}] \tag{107a}$$

$$\partial_\phi \widehat{\delta p_\phi} = \frac{1}{i\hbar} [\widehat{\delta p_\phi}, \hat{H}^{012}_{eff} + \hat{H}^{022}_{eff}] \tag{107b}$$

After simple calculations, the EOM (105) is updated as

$$(\widetilde{\Box} - \widetilde{\mathbf{U}} + i\hbar\widetilde{\zeta}_1(\eta)\partial_\phi + i\hbar\widetilde{\zeta}_2(\eta)\partial^2 - i\hbar\widetilde{\zeta}_3(\eta)) \widehat{\delta\phi}(\vec{x}, \eta) = 0 \tag{108}$$

where the $\widetilde{\zeta}_i$ (i from 1 to 3) are defined as the expectation value of the Hermitian symmetrized operators $\widetilde{\zeta}_i = \left\langle \widehat{\zeta}_i + \widehat{\zeta}_i^\dagger \right\rangle$, these operators given by

$$\widehat{\zeta}_1 = \frac{\kappa^2\gamma^4}{36}\widehat{\Omega}^{-4}\widehat{V}^2(V_{\phi\phi} + 6\hat{a}'\hat{a}^{-1}V_\phi + 2\hat{a}'\hat{a}^{-3}V_\phi) \quad (109a)$$

$$\widehat{\zeta}_2 = \frac{\kappa^2\gamma^4}{18}\widehat{\Omega}^{-4}\widehat{V}^2\widehat{a}^2V_\phi \quad (109b)$$

$$\widehat{\zeta}_3 = \frac{\kappa^2\gamma^4}{36}\widehat{\Omega}^{-4}\widehat{V}^2(\frac{3}{\kappa\gamma^2}\widehat{\Omega}^2\widehat{V}^{-2} + V(\phi) + 2V_\phi\widehat{a}^2\widehat{\mathbf{U}})\widehat{\mathbf{U}}_\phi\widehat{a}^2 \quad (109c)$$

where $\widehat{a}'$ means the derivative of $\widehat{a}$ with respect to conformal time η.

For the second and third part of $\hbar^1$ quantum corrections, the effective Hamiltonian for the perturbations, together, can be written as

$$\hat{H}^1_{crec}(\hbar^1) = H^1_{crec}(back) + \hat{H}^1_{crec}(pert) \quad (110)$$

where $H^1_{crec}(back)$ is the corrections to the background and we will ignore in this paper. $\hat{H}^1_{crec}(pert)$ is the corrections to the perturbations. They are

$$H^1_{crec}(back) = \frac{i\hbar}{2}\left\langle \widehat{u}^1_a - (\hat{u}^1_a)^\dagger \right\rangle \quad (111a)$$

$$\hat{H}^1_{crec}(pert) = \frac{i\hbar}{2}\int_{V_0} d^3\vec{x}\left(\left\langle \widehat{u}^1_b - (\hat{u}^1_b)^\dagger \right\rangle (\partial\widehat{\delta\phi})^2 + \left\langle \widehat{u}^1_c - (\hat{u}^1_c)^\dagger + \hat{H}^1_a - (\hat{H}^1_a)^\dagger \right\rangle \widehat{\delta\phi}^2\right) \quad (111b)$$

we have also applied Hermitian symmetrization to the operators because of the conditions of unitary evolution.

For simplicity, we can denote $\left\langle \hat{u}^1_b - (\hat{u}^1_b)^\dagger \right\rangle \equiv \widetilde{u^1_b}$, $\left\langle \hat{u}^1_c - (\hat{u}^1_c)^\dagger + \hat{H}^1_a - (\hat{H}^1_a)^\dagger \right\rangle \equiv \widetilde{u^1_c} + \widetilde{H^1_a}$. After taking into account the zeroth order effective Hamiltonian and all the $\hbar^1$ quantum corrections, the equations (107a,107b) end up with

$$\partial_\phi\widehat{\delta\phi} = \frac{1}{i\hbar}[\widehat{\delta\phi}, \hat{H}^{012}_{eff} + \hat{H}^{022}_{eff} + \hat{H}^1_{crec}(pert)] \quad (112a)$$

$$\partial_\phi\widehat{\delta p_\phi} = \frac{1}{i\hbar}[\widehat{\delta p_\phi}, \hat{H}^{012}_{eff} + \hat{H}^{022}_{eff} + \hat{H}^1_{crec}(pert)] \quad (112b)$$

then the effective equation of motion for perturbations (108) becomes

$$\left(\widetilde{\Box} - \widetilde{\mathbf{U}}(\eta) + i\hbar\widetilde{\zeta}_1(\eta)\partial_\phi + i\hbar\widetilde{\zeta}_2(\eta)\partial^2 - i\hbar\widetilde{\zeta}_3(\eta) + \frac{i\hbar}{\widetilde{a}^4(\eta)}\widetilde{u^1_b}(\eta)\partial^2 - \frac{i\hbar}{\widetilde{a}^4(\eta)}(\widetilde{u^1_c}(\eta) + \widetilde{H^1_a}(\eta))\right)\widehat{\delta\phi}(\vec{x},\eta) = 0 \quad (113)$$

we can then expand the perturbation field operator into creation and annihilation operators

$$\widehat{\delta\phi}(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^3}\widehat{\delta\phi}_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} = \int \frac{d^3k}{(2\pi)^3}(\widehat{A}_{\vec{k}}\varphi_k(\eta) + \widehat{A}^\dagger_{-\vec{k}}\varphi^*_k(\eta))e^{i\vec{k}\cdot\vec{x}} \quad (114)$$

with the only non-vanishing commutator $\left[\widehat{A}_{\vec{k}}, \widehat{A}^\dagger_{-\vec{k}'}\right] = \hbar(2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')$, and the set of mode functions $\varphi_k(\eta)$ forms a basis of the solutions to the equation

$$\varphi_k'' + 2\frac{a'}{a}\varphi_k' + (k^2 + a^2\widetilde{\mathbf{U}})\varphi_k - i\hbar a^2\widetilde{\zeta}_1\varphi_k' + \frac{i\hbar}{a^2}(a^4\widetilde{\zeta}_2k^2 + a^4\widetilde{\zeta}_3 + \widetilde{u}^1_bk^2 + \widetilde{u}^1_c + \widetilde{H}^1_a)\varphi_k = 0 \quad (115)$$

we have omitted the tilde above a for simplicity. In order to better understand this equation, it is convenient to re-scale mode functions by using $v_k = a\varphi_k$, then the above function takes form of

$$v_k''(\eta) - t(\eta)v_k'(\eta) + (k^2 + f(\eta) + i\hbar g(\eta)k^2 + i\hbar h(\eta))v_k(\eta) = 0 \quad (116)$$

where

$$t(\eta) = i\hbar a^2(\eta)\widetilde{\zeta}_1$$

$$f(\eta) = a^2(\eta)\mathbf{U}(\eta) - \frac{a''(\eta)}{a(\eta)} = a^2(\mathbf{U} - \frac{R}{6})$$

$$g(\eta) = a^2(\eta)\widetilde{\zeta}_2(\eta) + \frac{\widetilde{u}^1_b(\eta)}{a^2(\eta)}$$

$$h(\eta) = a(\eta)a'(\eta)\widetilde{\zeta}_1 + a^2(\eta)\widetilde{\zeta}_3(\eta) + \frac{\widetilde{u}^1_c(\eta) + \widetilde{H}^1_a(\eta)}{a^2(\eta)}$$

R is the Ricci scalar. From this equation, we can see that v_k', g and h terms are the quantum corrections, it will add effects on the dynamics of the perturbations. When we ignored these quantum correction terms, this equation will reduced to the result of dressed metric approach. Although one can see the quantum corrections effects are quite small, they maybe have larger impact on the power spectrum. In addition, when we take the classical limits, it will reduce to the general relativity results.

In the case ${v_k}'' \sim t{v_k}'$ or ${v_k}'' \leq t{v_k}'$, the solutions will be exponentially suppressed and it is also not physical. Then, it is also obvious that whenever ${v_k}'' \gg t{v_k}'$ and $k^2 \gg \left|f + i\hbar g k^2 + i\hbar h\right|$, the solutions to this equation are simple oscillatory functions with time independent frequency of k. In the other hand, when ${v_k}'' \gg t{v_k}'$ and $k^2 \ll \left|f + i\hbar g k^2 + i\hbar h\right|$, the solutions will behave more complicatedly. This will contribute to the power spectrum of perturbations which is exactly the quantum effects of spacetime. Especially, we can get a unique scale

$$\begin{aligned} k_{LQC\hbar} &\equiv \sqrt{\frac{f_B + i\hbar h_B}{1 + i\hbar g_B}} \\ &\approx a(\eta_B)\sqrt{\frac{R_B}{6}} - \frac{i\hbar a(\eta_B)}{2}\sqrt{\frac{R_B}{6}}(g_B - \frac{6h_B}{a^2(\eta_B)R_B}) \\ &\approx a(\eta_B)\sqrt{\frac{R_B}{6}} + o(\hbar) \end{aligned} \tag{118}$$

where the subscript B means taking the values at the bounce. In the first approximation, We have used the condition that $f(\eta)$ remains approximately a constant and is proportional to the scalar curvature R during inflation era[19], and we also did Taylor expansion in terms of $\hbar$ and keep to the first order. The zeroth order is the same dressed metric results, and $\sqrt{\frac{R_B}{6}} \approx 3.21$ approximately[29]. We expect the power spectrum will be significantly affected by the quantum spacetime or Big Bounce for the modes $k \leq k_{LQC\hbar}$, and the bounce has a little effect on $k \geq k_{LQC\hbar}$ since it is too ultraviolet.

VI. SUMMARY AND DISCUSSION

In this paper, we have demonstrated the application of quantum reference frames algorithm to cosmological perturbations in LQC. We started from the classical phase space, made perturbation expansions to the constraints. Then, we applied the loop representation technic and arrive at the rigging map operator. From the transition amplitude or the matrix elements of $\widehat{P}$ in the eigenspaces $\{\mathbb{S}_t\}$, we derived the interesting Schrödinger theory in the scalar field reference frame. At the end, The physical and effective Hamiltonian were presented, and the equations of motion for perturbation was also obtained in this paper. It shows that for the scalar field frame, the zeroth order effective dynamics of cosmological perturbations exactly agrees with the results obtained by the dressed metric approach[19], and we also presented the $\hbar^1$ order quantum corrections for the dynamics of perturbations. These also proved the effectiveness and advantages of the quantum reference frames algorithm.

The most important thing of this paper is that we provide an alternative approach to the quantum theory of perturbations in LQC. It relies on the property of the quantum reference frames algorithm which enable us to explore the dynamics in the deep-planck regime. However, this algorithm can be applied in a more broad area. It can fulfill the demand to exact the dynamics in a universally manner, describing them using the elementary relational Dirac oberservables. Since we only derived the dynamics of the scalar mode of the cosmological perturbations in the scalar field frame, there are some works to be done in the future: 1, with the flexility of this algorithm, we can change the choice of (48a) and choose the gravitation field as the reference frame which allow us to study the cross Big Bounce dynamics of perturbations. 2, for the continuity of this work, we can investigate the power spectrums derived from the equations of motion of perturbations in this theory, and comparing them with the observations to show how the quantum corrections impact on the power spectrums. Especially for the gravitation frame, we can investigate the impact of Big Bounce on the evolution of the perturbations; 3, for the other research objects, such as the tensor modes which is the primordial gravitational waves, primordial non-Gaussianity, and so on. In these new objects, we may take an different route – spin foam model which discussed in [30][31][32][33] to get the elements of the rigging map. However, the vertex expansion for LQC illuminate certain conceptual issues[34]. Anyway, it is a very interesting topic in LQC. We will leave this discussions in the future researches.

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Appendix A: First order quantum corrections in scalar field frame

We will calculate the $\hbar^1$ order quantum corrections of reordering operators in the scalar field frame.

1. The $\hat{\alpha}^{(1)}$

Let us first focus on $\hat{\alpha}^1$. We know that

$$\delta\left(\widehat{\mathcal{C}}\right) = \int_{-\infty}^{+\infty} d\alpha e^{i\alpha\widehat{\mathcal{C}}} = \int_{-\infty}^{+\infty} d\alpha(1 + i\alpha\widehat{\mathcal{C}} + \frac{(i\alpha)^2}{2!}\widehat{\mathcal{C}}^2 + \frac{(i\alpha)^3}{3!}\widehat{\mathcal{C}}^3 + \cdots) \tag{A1}$$

Thus, the reordering of $\left(\hat{\phi}, \widehat{p}_\phi\right)$ will give us

$$\left[\delta(\widehat{\mathcal{C}})\right]_{R,ord}^{\hat{\phi}} = \int_{-\infty}^{+\infty} d\alpha \left[e^{i\alpha\widehat{\mathcal{C}}}\right]_{R,ord}^{\hat{\phi}} = \int_{-\infty}^{+\infty} d\alpha(1+i\alpha \left[\widehat{\mathcal{C}}\right]_{R,ord}^{\hat{\phi}} + \frac{(i\alpha)^2}{2!}\left[\widehat{\mathcal{C}}^2\right]_{R,ord}^{\hat{\phi}} + \frac{(i\alpha)^3}{3!}\left[\widehat{\mathcal{C}}^3\right]_{R,ord}^{\hat{\phi}} + \cdots) \tag{A2}$$

Keep to the $\hbar^1$ order, we know that

$$\hat{\alpha}^{(1)} = \delta(\widehat{\mathcal{C}}) - \left[\delta(\widehat{\mathcal{C}})\right]_{R,ord}^{\hat{\phi}} \tag{A3}$$

so, the important quantity for us to calculate $\hat{\alpha}^{(1)}$ is $\widehat{\mathcal{C}}^n - \left[\widehat{\mathcal{C}}^n\right]_{R,ord}^{\hat{\phi}}$. Since $\widehat{\mathcal{C}}^n$ is highly non-liner which is hard to deal with. However, we can use Taylor expansion regarding to: the perturbations since they are very small to the background, and the scalar field potential terms since it is also quite small compare to the kinetic term. We can write

$$\widehat{\mathcal{C}}^n = \left(\widehat{\mathcal{C}}_I + \widehat{\mathcal{C}}_{II}\right)^n \tag{A4}$$

where $\widehat{\mathcal{C}}_I$ represents the main part and $\widehat{\mathcal{C}}_{II}$ represents the relatively small part. The definition for $\widehat{\mathcal{C}}_I$ and $\widehat{\mathcal{C}}_{II}$ given by

$$\widehat{\mathcal{C}}_I = -\frac{3V_0}{\kappa\gamma^2}\widehat{\Omega}^2 + \frac{V_0\widehat{p}_\phi^{\,2}}{2} \tag{A5}$$

$$\widehat{C}_{II} = V_0\hat{V}^2\hat{V}(\phi) + \frac{V_0}{2}\int_{V_0} d^3\vec{x}(\widehat{\delta p}_\phi^{\,2}(\vec{x}) + \frac{\hat{V}^2}{V_0^2}(\partial\widehat{\delta\phi}(\vec{x}))^2 + \frac{\hat{V}^2}{V_0^2}\widehat{\mathfrak{U}}\widehat{\delta\phi}^{\,2}(\vec{x})) \tag{A6}$$

Then we can do expansion for $\widehat{\mathcal{C}}^n$, keep terms to the first three terms of the series, we can get

$$\begin{aligned}\widehat{\mathcal{C}}^n &= \left(\widehat{\mathcal{C}}_I + \widehat{\mathcal{C}}_{II}\right)^n = \left(\widehat{\mathcal{C}}_I\left(1 + \widehat{\mathcal{C}}_I^{-1}\widehat{\mathcal{C}}_{II}\right)\right)^n \\ &\approx \widehat{\mathcal{C}}_I^n\left(1 + \widehat{\mathcal{C}}_I^{-1}\widehat{\mathcal{C}}_{II}\right)^n + i\hbar\widehat{\mathcal{C}}_{crrec1} \\ &\approx \widehat{\mathcal{C}}_I^n + n\widehat{\mathcal{C}}_I^{n-1}\widehat{\mathcal{C}}_{II} + \frac{n(n-1)}{2}\widehat{\mathcal{C}}_I^{n-2}\widehat{\mathcal{C}}_{II}^2 \\ &\quad + i\hbar\widehat{\mathcal{C}}_{crrec2} + i\hbar\widehat{\mathcal{C}}_{\mathrm{crrec1}}\end{aligned} \tag{A7}$$

and also for

$$\begin{aligned}\left[\widehat{\mathcal{C}}^n\right]_{R,ord}^{\hat{\phi}} \approx{}& \left[\widehat{\mathcal{C}}_I^n\right]_{R,ord}^{\hat{\phi}} + n\left[\widehat{\mathcal{C}}_I^{n-1}\widehat{\mathcal{C}}_{II}\right]_{R,ord}^{\hat{\phi}} \\ &+ \frac{n(n-1)}{2}\left[\widehat{\mathcal{C}}_I^{n-2}\widehat{\mathcal{C}}_{II}^2\right]_{R,ord}^{\hat{\phi}} \\ &+ i\hbar\widehat{\mathcal{C}}_{crrec1} + i\hbar\widehat{\mathcal{C}}_{\mathrm{crrec2}}\end{aligned} \tag{A8}$$

where $\widehat{\mathcal{C}}_{crrec1}$ represents the $\hbar^1$ quantum corrections of reordering $(\hat{\phi}, \widehat{p}_\phi)$ in the third "equal" sign. $\widehat{\mathcal{C}}_{crrec2}$ represents the $\hbar^1$ quantum corrections of reordering $(\hat{\phi}, \widehat{p}_\phi)$ in the last "equal" sign. These quantities are not important, since we only keep to the $\hbar^1$ order, all the quantum correction of operators reordering in the terms proportional to $\hbar$, can be ignored, because it will give us the $\hbar^2$ terms. In other words, operators can be in any order in the terms proportional to $\hbar$. Thus, $\widehat{\mathcal{C}}_{crrec1}$ and $\widehat{\mathcal{C}}_{crrec2}$ will be the same in $\widehat{\mathcal{C}}^n$ and $\left[\widehat{\mathcal{C}}^n\right]_{R,ord}^{\hat{\phi}}$, and will be cancelled out in the final results of $\hat{\alpha}^1$.

One can also noticed that

$$\widehat{\mathcal{C}}_I^n = \left[\widehat{\mathcal{C}}_I^n\right]_{R,ord}^{\hat{\phi}} \tag{A9}$$

$$\widehat{\mathcal{C}}_I^{n-1}\widehat{\mathcal{C}}_{II} = \left[\widehat{\mathcal{C}}_I^{n-1}\widehat{\mathcal{C}}_{II}\right]_{R,ord}^{\hat{\phi}} \tag{A10}$$

So the first two terms will have no quantum corrections, that's why we keep the first three terms in the Taylor expansion. We must move to the third term, and

$$\left[\widehat{\mathcal{C}}_I^{n-2}\widehat{\mathcal{C}}_{II}^2\right]_{R,ord}^{\hat{\phi}} = \widehat{\mathcal{C}}_I^{n-2}\left[\widehat{\mathcal{C}}_{II}^2\right]_{R,ord}^{\hat{\phi}} \tag{A11}$$

this means the only difference of reordering lie on $\widehat{\mathcal{C}}_{II}^2$. By using the commutation relation $[p_\phi, V(\phi)] = -i\hbar V_\phi$, we can get

$$\widehat{\mathcal{C}}_{II}^2 - \left[\widehat{\mathcal{C}}_{II}^2\right]_{R,ord}^{\hat{\phi}} = i\hbar\widehat{\phi}_{crrec} \tag{A12}$$

where $\widehat{\phi}_{crrec}$ is the quantum corrections we are looking for, and given by

$$\widehat{\phi}_{crrec} = \widehat{\phi}_a\int_{V_0} d^3\vec{x}\widehat{\delta\phi}^{\,2}(\vec{x}) + \widehat{\phi}_b\int_{V_0} d^3\vec{x}\widehat{\delta\phi}^{\,2}(\vec{x})\int_{V_0} d^3\vec{y}\widehat{\delta\phi}^{\,2}(\vec{y}) \tag{A13}$$

where

$$\widehat{\phi}_a = -\left(36\kappa^2\gamma^2V_0^{8/3}\widehat{V}^2\widehat{\Omega}^{-2}\right)p_\phi^3V_\phi + \left(3\kappa V_0^2\widehat{V}^2\right)p_\phi V_\phi - \left(6\kappa\gamma V_0^{1/3}\widehat{V}^4\Omega^{-1}\right)V_\phi^2 \tag{A14}$$

$$\begin{aligned}\widehat{\phi}_b ={}& \left(36\kappa^2\gamma^2V_0^{-4/3}\widehat{V}^2\Omega^{-1}\widehat{V}^2\Omega^{-1}\right)p_\phi V_{\phi\phi}V_\phi \\ &+ \left(216\kappa^3\gamma^3V_0\widehat{V}^2\Omega^{-3}\right)p_\phi^4V_{\phi\phi} - \left(18\kappa^2\gamma V_0^{1/3}\widehat{V}^2\Omega^{-1}\right)p_\phi^2V_{\phi\phi} \\ &- \left(18\kappa^2\gamma^2V_0^{2/3}\widehat{V}^2\widehat{\Omega}^{-2}\right)p_\phi^3V_{\phi\phi\phi} + \left(\frac{3\kappa}{2}\widehat{V}^2\right)p_\phi V_{\phi\phi\phi} \\ &- \left(3\kappa\gamma V_0^{-5/3}\widehat{V}^4\Omega^{-1}\right)V_{\phi\phi\phi}V_\phi\end{aligned} \tag{A15}$$

then we can get

$$\widehat{\mathcal{C}}^n - \left[\widehat{\mathcal{C}}^n\right]^{\hat{\phi}}_{R,ord} = i\hbar\frac{n(n-1)}{2}\widehat{\mathcal{C}}_I^{n-2}\widehat{\phi}_{crrec} \tag{A16}$$

Then, we can get the final results, which can be written in a very concise way

$$\begin{aligned}\hat{\alpha}^{(1)} &= \delta(\widehat{\mathcal{C}}) - \left[\delta(\widehat{\mathcal{C}})\right]^{\hat{\phi}}_{R,ord} \\ &= \int_{-\infty}^{+\infty} d\alpha \left(\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\left(\widehat{\mathcal{C}}^n - \left[\widehat{\mathcal{C}}^n\right]^{\hat{\phi}}_{R,ord}\right)\right) \\ &= \int_{-\infty}^{+\infty} d\alpha \left(\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\left(\frac{n(n-1)}{2}i\hbar\widehat{\mathcal{C}}_I^{n-2}\widehat{\phi}_{crrec}\right)\right) \\ &= \int_{-\infty}^{+\infty} d\alpha \left(\sum_{n=0}^{\infty}\frac{(i\alpha)^{n-2}}{(n-2)!}\left(\widehat{\mathcal{C}}_I^{n-2}\right)\frac{-i\hbar\alpha^2}{2}\widehat{\phi}_{crrec}\right) \\ &= -\frac{i\hbar}{2}\left[\int_{-\infty}^{+\infty} d\alpha\left(\alpha^2 e^{i\alpha\widehat{\mathcal{C}}_I}\right)\right]\widehat{\phi}_{crrec}\end{aligned} \tag{A17}$$

2. The $\widehat{\beta}^{(1)}_{t't}$

From definition, we can know

$$\widehat{\beta}^{(1)}_{t't} = \hat{P}_{t't} - \hat{P}^{(0)}_{t't} \tag{A18}$$

so the important quantities for us to calculate $\widehat{\beta}^{(1)}_{t't}$ is again

$$\delta(\widehat{\mathcal{C}}) - \left[\delta(\widehat{\mathcal{C}})\right]^{\hat{V},\widehat{\delta\phi}}_{R,ord} \tag{A19}$$

or equally to say the $\widehat{\mathcal{C}}^n - \left[\widehat{\mathcal{C}}^n\right]^{\hat{V},\widehat{\delta\phi}}_{R,ord}$. We will write

$$\widehat{\beta}^{(1)}_{t't} = \widehat{\beta}^{1\widehat{V}}_{t't} + \widehat{\beta}^{1\widehat{\delta\phi}}_{t't} \tag{A20}$$

where $\widehat{\beta}^{1\widehat{V}}_{t't}$ and $\widehat{\beta}^{1\widehat{\delta\phi}}_{t't}$ represent the contributions of reordering $(\hat{V},\hat{\Omega})$ and $(\widehat{\delta\phi},\delta\hat{p}_\phi)$ respectively.

Let us first focus on the reordering of $(\hat{V},\hat{\Omega})$. By using the same Taylor expansion strategy of calculating $\hat{\alpha}(1)$, and also the commutation relation $\left[\widehat{V},\widehat{\Omega}^{-1}\right] = -2i\hbar\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)\widehat{\Omega}_b^{-1}$, where $\widehat{\Omega}_b^{-1} \equiv \frac{\partial}{\partial b}\widehat{\Omega}^{-1}$. We can get

$$\begin{aligned}\widehat{\mathcal{C}}^n - \left[\widehat{\mathcal{C}}^n\right]^{\widehat{V}}_{R,ord} &= n\widehat{\mathcal{C}}_I^{n-1}\left(\widehat{\mathcal{C}}_{II} - \left[\widehat{\mathcal{C}}_{II}\right]^{\widehat{V}}_{R,ord}\right) \\ &+ \frac{n(n-1)}{2}\widehat{\mathcal{C}}_I^{n-2}\left(\widehat{\mathcal{C}}_{II}^2 - \left[\widehat{\mathcal{C}}_{II}^2\right]^{\widehat{V},\widehat{\delta\phi}}_{R,ord}\right) \\ &= n\widehat{\mathcal{C}}_I^{n-1}i\hbar\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)\hat{V}^1_{crrec} \\ &+ i\hbar\frac{n(n-1)}{2}\widehat{\mathcal{C}}_I^{n-2}\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)\hat{V}^2_{crrec}\end{aligned} \tag{A21}$$

The difference with $\hat{\alpha}^{(1)}$ is that the second term of Taylor expansion has contribution to the corrections. V^1_{crrec} and V^2_{crrec} represent the second term and third term quantum corrections of recording $(\hat{V},\hat{\Omega})$, given by

$$\begin{aligned}\hat{V}^1_{crrec} &= 24\kappa\gamma V_0^{-2/3}\hat{\Omega}_b^{-1}\hat{V}p_\phi V_\phi\int_{V_0} d^3\vec{x}\,\widehat{\delta\phi}^2(\vec{x}) \\ &\equiv \hat{V}^1_a\int_{V_0} d^3\vec{x}\widehat{\delta\phi}^2\end{aligned} \tag{A22}$$

$$\begin{aligned}\hat{V}^2_{crrec} &= \hat{V}^2_a\int_{V_0} d^3\vec{x}\widehat{\delta\phi}^2(\vec{x}) + \hat{V}^2_b\int_{V_0} d^3\vec{x}\widehat{\delta\phi}^2(\vec{x})\int_{V_0} d^3\vec{y}\widehat{\delta\phi}^2(\vec{y}) \\ &+ \hat{V}^2_c\int_{V_0} d^3\vec{x}(\partial\widehat{\delta\phi}(\vec{x}))^2(\vec{x})\int_{V_0} d^3\vec{y}\widehat{\delta\phi}^2(\vec{y})\end{aligned} \tag{A23}$$

where

$$\begin{aligned}\hat{V}^2_a &= 72\kappa^2\gamma^2 V_0^{8/3}\widehat{\Omega}^{-1}\hat{\Omega}_b^{-1}\widehat{V}p_\phi^4 V(\phi) \\ &+ 96\kappa\gamma V_0^{1/3}\hat{\Omega}_b^{-1}\widehat{V}^3 p_\phi V(\phi)V_\phi \\ &+ 24\kappa\gamma V_0^{-2/3}\hat{\Omega}_b^{-1}\widehat{V}^3 p_\phi V_\phi V(\phi)\end{aligned} \tag{A24}$$

$$\begin{aligned}\hat{V}^2_b &= -720\kappa^2\gamma^2 V_0^{-4/3}\widehat{\Omega}^{-1}\hat{\Omega}_b^{-1}\widehat{V}^3 p_\phi^2 V_\phi^2 \\ &- 216\kappa^3\gamma^3 V_0\widehat{\Omega}^{-2}\hat{\Omega}_b^{-1}\widehat{V}p_\phi^5 V_\phi + 12\kappa\gamma V_0^{-5/3}\hat{\Omega}_b^{-1}\widehat{V}^3 p_\phi V_\phi V_{\phi\phi} \\ &+ 48\kappa\gamma V_0^{-5/3}\hat{\Omega}_b^{-1}\widehat{V}^3 p_\phi V_{\phi\phi}V_\phi + 36\kappa^2\gamma^2 V_0^{2/3}\widehat{\Omega}^{-1}\hat{\Omega}_b^{-1}\widehat{V}p_\phi^4 V_{\phi\phi} \\ &- 864\kappa^3\gamma^3 V_0\widehat{\Omega}^{-2}\hat{\Omega}_b^{-1}\widehat{V}p_\phi^5 V_\phi\end{aligned}$$

$$\begin{aligned}\hat{V}^2_c &= 36\kappa^2\gamma^2 V_0^{2/3}\widehat{\Omega}^{-1}\hat{\Omega}_b^{-1}\widehat{V}p_\phi^4 + 48\kappa\gamma V_0^{-5/3}\hat{\Omega}_b^{-1}\widehat{V}^3 p_\phi V_\phi \\ &+ 12\kappa\gamma V_0^{-5/3}\hat{\Omega}_b^{-1}\widehat{V}^3 p_\phi V_\phi\end{aligned}$$

Then carry the results to (A19) and (A18), we can get

$$\begin{aligned}\widehat{\beta}^{1\widehat{V}}_{t't} &= -\hbar\left(4\pi\gamma G\hbar\sqrt{\Delta}\right)\int_0^{+\infty} dp_\phi \\ &\left[\int_{-\infty}^{+\infty} d\alpha\left(\alpha e^{i\alpha\widehat{\mathcal{C}}_I}\right)\widehat{V}^1_{crrec}\right]e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))}\end{aligned} \tag{A25}$$

$$\begin{aligned}&-\frac{i\hbar}{2}\left(4\pi\gamma G\hbar\sqrt{\Delta}\right)\int_0^{+\infty} dp_\phi \\ &\left[\int_{-\infty}^{+\infty} d\alpha\left(\alpha^2 e^{i\alpha\widehat{\mathcal{C}}_I}\right)\widehat{V}^2_{crrec}\right]e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))}\end{aligned} \tag{A26}$$

Next, we consider the quantum corrections of reordering $(\widehat{\delta\phi},\widehat{\delta p}_\phi)$. By using the same Taylor expansion strategy of calculating $\hat{\alpha}^1$, and also the commutation relation $\left[\widehat{\delta\phi}(\vec{x}),\widehat{\delta p}_\phi(\vec{y})\right] = i\hbar\left(\delta^{(3)}(\vec{x}-\vec{y}) - \frac{1}{V_0}\right)$, we can get

$$\widehat{\mathcal{C}}^n - \left[\widehat{\mathcal{C}}^n\right]^{\widehat{\delta\phi}}_{R,ord} = i\hbar\frac{n(n-1)}{2}\widehat{\mathcal{C}}_I^{n-2}\widehat{\delta\phi}_{crrec} \tag{A27}$$

where $\widehat{\delta\phi}_{crrec}$ given by

$$\begin{aligned}\widehat{\delta\phi}_{crrec} &= \widehat{V}^2\int_{V_0} d^3\vec{x}\int_{V_0} d^3\vec{y}\left(\delta^{(3)}(\vec{x}-\vec{y}) - \frac{1}{V_0}\right) \\ &\left[-\widehat{\delta p}_\phi(\vec{x})\partial^2\widehat{\delta\phi}(\vec{y}) + \widehat{\mathfrak{U}}\widehat{\delta p}_\phi(\vec{x})\widehat{\delta\phi}^2(\vec{y})\right]\end{aligned} \tag{A28}$$

where $\widehat{\mathfrak{U}}$ is that

$$\widehat{\mathfrak{U}} = -18\kappa^2\gamma^2 V_0^{8/3}\widehat{V}^{-2}\widehat{\Omega}^{-2}p_\phi^4 + 3\kappa V_0^2\widehat{V}^{-2}p_\phi^2 - 12\kappa\gamma V_0^{1/3}\widehat{\Omega}^{-1}p_\phi V_\phi + V_{\phi\phi} \tag{A29}$$

Then,

$$\widehat{\beta}_{t't}^{1\widehat{\delta\phi}} = -\frac{i\hbar}{2}\int_0^{+\infty} dp_\phi \left[\int_{-\infty}^{+\infty} d\alpha \left(\alpha^2 e^{i\alpha\widehat{C_I}}\right)\widehat{\delta\phi}_{crrec}\right] e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))} \tag{A30}$$

Collecting the corrections together, we can get

$$\begin{aligned}\widehat{\beta}_{t't}^{(1)} &= \widehat{\beta}_{t't}^{1\widehat{V}} + \widehat{\beta}_{t't}^{1\widehat{\delta\phi}} \\ &= -\hbar\left(4\pi\gamma G\hbar\sqrt{\Delta}\right)\int_0^{+\infty} dp_\phi \left[\int_{-\infty}^{+\infty} d\alpha\left(\alpha e^{i\alpha\widehat{C_I}}\right)\widehat{V}_{crrec}^1\right] e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))} \\ &\quad - \frac{i\hbar}{2}\int_0^{+\infty} dp_\phi \left[\int_{-\infty}^{+\infty} d\alpha\left(\alpha^2 e^{i\alpha\widehat{C_I}}\right)\left(4\pi\gamma G\hbar\sqrt{\Delta}\widehat{V}_{crrec}^2 + \widehat{\delta\phi}_{crrec}\right)\right] e^{\frac{i}{\hbar}p_\phi(\phi(t')-\phi(t))}\end{aligned} \tag{A31}$$

Appendix B: The quantum corrections of $\widehat{u}^1$ in the scalar field frame

$$\widehat{u}^1 = i\hbar\widehat{u}_a^1 + i\hbar\widehat{u}_b^1\int_{V_0} d^3\vec{x}(\partial\widehat{\delta\phi})^2 + i\hbar\widehat{u}_c^1\int_{V_0} d^3\vec{x}\widehat{\delta\phi}^2 \tag{B1}$$

where $\widehat{u}_a^1, \widehat{u}_b^1, \widehat{u}_c^1$ are defined as

$$\begin{aligned}\widehat{u}_a^1 &\equiv \sqrt{\frac{\kappa\gamma^2}{6}}\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)^2\hat{\Omega}^{-\frac{1}{2}}\widehat{V}\hat{\Omega}_b^{-\frac{3}{2}}V(\phi) \\ \widehat{u}_b^1 &\equiv \sqrt{\frac{\kappa\gamma^2}{6}}\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)^2 V_0^{-2}\hat{\Omega}^{-\frac{1}{2}}\widehat{V}\hat{\Omega}_b^{-\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}\widehat{u}_c^1 \equiv{}& \sqrt{\frac{\kappa\gamma^2}{6}}\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)^2(-144\kappa^2\gamma^2V_0^{2/3}\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)^2\hat{\Omega}^{-\frac{5}{2}}\widehat{V}^2\hat{\Omega}_b^{-\frac{3}{2}}V^2(\phi) + \left(72V_0^{2/3}-1\right)6\kappa V(\phi)\hat{\Omega}^{-\frac{1}{2}}\widehat{V}\hat{\Omega}_b^{-\frac{3}{2}} \\ &- 12\sqrt{6\kappa}V_0^{-5/3}(\hat{\Omega}^{-\frac{1}{2}}\widehat{V}\hat{\Omega}_b^{-\frac{3}{2}} - \frac{\kappa\gamma^2}{3}\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)^2\hat{\Omega}^{-\frac{5}{2}}\widehat{V}^2\hat{\Omega}_b^{-\frac{3}{2}}V(\phi))V_\phi + V_0^{-2}\hat{\Omega}^{-\frac{1}{2}}\widehat{V}\hat{\Omega}_b^{-\frac{3}{2}}V_{\phi\phi})) \\ &- 2\sqrt{\frac{\kappa\gamma^2}{6}}\left(2\pi\gamma G\hbar\sqrt{\Delta}\right)(((-648V_0^{5/3}+18V_0)\hat{\Omega}^{\frac{1}{2}}\hat{\Omega}_b\frac{\widehat{V}^{-2}}{\gamma^2} - 72\kappa^2\gamma^2V_0^{5/3}\hat{\Omega}^{\frac{1}{2}}\hat{\Omega}_b^{-2}\widehat{V}^2V^2(\phi) \\ &+ \frac{\sqrt{6\kappa^3}\gamma^2}{2}V_0^{-2/3}\hat{\Omega}^{\frac{1}{2}}\hat{\Omega}_b^{-2}\widehat{V}^2V(\phi)V_\phi)\frac{\widehat{V}}{V_0}\hat{\Omega}^{-\frac{1}{2}})\end{aligned}$$

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