One-Dimensional Probability Distribution for Random Walker Position

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Our model is as follows: say the probability of finding the walker at position x_0 at a time $t_0+\tau$ $P(x_0, t_0+\tau)$. There are three possibilities for the walker's position at $t = t_0$: the walker was positioned some small distance δ to the left (then moved to the right), the walker was positioned some small distance δ to the right (then moved to the left), or the walker did not move. With the probability that the walker moves to the right being p_r and the probability the walker moves to the left p_ℓ, one can set up the equation

$$P(x_0, t_0 + \tau) = p_\ell \cdot P(x_0 + \delta, t_0) + (1 - p_r - p_\ell)P(x_0, t_0) + p_r \cdot P(x_0 - \delta, t_0)$$

We will use a multivariable Taylor expansion about x_0, t_0. On the left hand side, we will naturally evaluate the expansion at $(x_0, t_0 + \tau)$; all derivatives with respect to x evaluate to zero for they are in powers of $(x_0 - x_0)$. The left hand side is then

$$\begin{aligned} P(x_0, t_0) + \left.\frac{\partial P}{\partial t}\right|_{t=t_0} (t_0 - t_0 + \tau) + \frac{1}{2}\left.\frac{\partial^2 P}{\partial t^2}\right|_{t=t_0} (t_0 - t_0 + \tau)^2 + \cdots \\ = P(x_0, t_0) + \left.\frac{\partial P}{\partial t}\right|_{t=t_0} \tau + \frac{1}{2}\left.\frac{\partial^2 P}{\partial t^2}\right|_{t=t_0} \tau^2 + \cdots \end{aligned}$$

The right hand side, evaluated similarly, is

$$\begin{aligned} & p_\ell \left[P(x_0, t_0) + \left.\frac{\partial P}{\partial x}\right|_{x=x_0} \delta + \frac{1}{2}\left.\frac{\partial^2 P}{\partial x^2}\right|_{x_0} \delta^2 + \cdots \right] \\ & + (1 - p_r - p_\ell) \left[P(x_0, t_0) + \left.\frac{\partial P}{\partial x}\right|_{x=x_0} (x_0 - x_0) + \left.\frac{\partial^2 P}{\partial x^2}\right|_{x=x_0} (x_0 - x_0)^2 + \cdots \right] \\ & + p_r \left[P(x_0, t_0) - \left.\frac{\partial P}{\partial x}\right|_{x=x_0} \delta + \frac{1}{2}\left.\frac{\partial^2 P}{\partial x^2}\right|_{x_0} \delta^2 + \cdots \right] \end{aligned}$$

Every other term of the right hand side evidently disappears and we are left with

$$P(x_0, t_0) + \frac{1}{4}(p_\ell + p_r)\left.\frac{\partial^2 P}{\partial x^2}\right|_{x=x_0} \delta^2 + \cdots$$

With δ and τ small, we return to the equation and drop terms above the second order.

$$P(x_0, t_0) + \left.\frac{\partial P}{\partial t}\right|_{t=t_0} \tau + \frac{1}{2}\left.\frac{\partial^2 P}{\partial t^2}\right|_{t=t_0} \tau^2 = P(x_0, t_0) + \frac{1}{2}(p_\ell + p_r)\left.\frac{\partial^2 P}{\partial x^2}\right|_{x=x_0} \delta^2$$

With x_0 and t_0 arbitrary, we are free to drop the evaluation notation:

$$\begin{aligned} P(x_0, t_0) + \frac{\partial P}{\partial t}\tau + \frac{1}{4}\frac{\partial^2 P}{\partial t^2}\tau^2 &= P(x_0, t_0) + \frac{1}{2}(p_\ell + p_r)\frac{\partial^2 P}{\partial x^2}\delta^2 \\ \frac{\partial P}{\partial t}\tau + \frac{1}{2}\frac{\partial^2 P}{\partial t^2}\tau^2 &= \frac{1}{4}(p_\ell + p_r)\frac{\partial^2 P}{\partial x^2}\delta^2 \end{aligned}$$

where we've additionally cancelled the constant term. With τ and δ being arbitrary small quantities, we are free to set $\tau = \delta^2$ such that

$$\frac{\partial P}{\partial t}\delta^2 + \frac{1}{2}\frac{\partial^2 P}{\partial t^2}\delta^4 = \frac{1}{2}(p_\ell + p_r)\frac{\partial^2 P}{\partial x^2}\delta^2$$

We previously argued that δ^4 terms are too small to be considered, so we find

$$\frac{\partial P}{\partial t}\delta^2 = \frac{1}{2}(p_\ell + p_r)\frac{\partial^2 P}{\partial x^2}\delta^2$$

and, dividing both sides by δ^2,

$$\boxed{\frac{\partial P}{\partial t} = \frac{1}{2}(p_\ell + p_r)\frac{\partial^2 P}{\partial x^2}}$$

The one-dimensional heat equation!