MATH1120 - Partial Differential Equations

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Contents

Introduction

> In order to solve this differential equation you look at it until a solution occurs to you.
>
> George Polya

title

In PDEs you will use one of two things: integration by parts or the maximum principle.

- Jungang Li

Chapter 1

Where PDEs Come From

1.1 Types of Partial Differential Equations

Partial differential equations are equations involving partial derivatives and several variables. We typically denote partial derivatives of u

$$u_x \equiv \frac{\partial u}{\partial x}$$

A function u is said to be a solution of the PDE if it satisfies the equation for every x, y in the domain.

Specifically, a solution that satisfies the equation for every x, y is a classical solution, whereas one that does not is called a weak solution. We will not cover these in the course.

1.1.1 Order of a PDE

The order of a PDE is the order of the highest-order derivative (of *any* variable) that appears in the equation. Formally:

$$\begin{aligned}\text{1st order PDE} &\equiv F(x, y, u(x, y), u_x, u_y)\\ \text{2nd order PDE} &\equiv F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy})\end{aligned}$$

Example $u_x + u_y = 0$ is a first order PDE, $u_{xy} + u$ is a second order PDE.

1.1.2 Linearity of a PDE

We conventionally consider the left side of the equation an **operator** applied to the function in question. For example,

$$u_x + 2u_y = 0 \Rightarrow (\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y})u = 0$$

The operator (and the equation in question) is called **linear** if it satisfies

$$\begin{cases}\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v\\ \mathcal{L}(cu) = c\mathcal{L}u\end{cases}$$

We can check whether an operator is linear by substituting $u+v$ and cu into u and checking whether the result is linear.

EXAMPLE $u_t - u_{xxt} + uu_x = 0$ is not a linear PDE:

$$\begin{aligned}
\mathcal{L} &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2 \partial t} + u\frac{\partial}{\partial x}\right) \\
\mathcal{L}(u+v) &= (u+v)_t - (u+v)_{xxt} + (u+v)(u+v)_x \\
&= u_t + v_t - u_{xxt} - v_{xxt} + (u+v)(u_x + v_x) \\
&= u_t + v_t - u_{xxt} - v_{xxt} + uu_x + uv_x + vu_x + vv_x \\
&\neq \mathcal{L}(u) + \mathcal{L}(v)
\end{aligned}$$

1.2 Homogeneity

If the sum of all instances of u and its derivative is zero, the equation is considered homogeneous. If this is not the case, the equation is inhomogeneous, and the leftover term is called the inhomogeneity.

1.3 Inhomogeneous Equations and Linearity

Suppose an inhomogeneous equation with a linear differential operator $\mathcal{L}$

$$\mathcal{L}u = h$$

Suppose we know one solution of it, f. Then, by linearity, the general solution is some

$$u = f + g$$

Where g is the general solution of the homogeneous equation $\mathcal{L}u = 0$.

1.4 1st Order Linear Constant Coefficient Equation

First order linear partial differential equations of constant coefficients are of the form

$$\boxed{au_x + bu_y = 0}$$

1.4.1 Geometric Method

We can separate the differential operator:

$$\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)u = 0$$

Recall that the directional derivative of a function is $\nabla u \cdot \hat{\mathbf{n}}$. Then the above implies that the directional derivative of u along the vector (a, b) is zero, and that the function is constant in that direction. Specifically, it will be constant along any line parallel to (a, b). These are the lines

$$y = \frac{b}{a}x + c \tag{1.1}$$

for an arbitrary c. These are called **characteristic lines**. Then u depends solely on C, and we can write

$$u = u(c) = f(y - (b/a)x)$$

where we just solved for c using 1.1, and where f is an arbitrary function. Then, if we are given a condition for, say, $x = 0$, we can just set that equal to f, and plug in $y - \frac{b}{a}x$.

EXAMPLE Solve $2u_x - 3u_y = 0$. The operator is

$$(2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y})u = 0$$

The direction is $(2, -3)$, and the characteristic line is $y = -\frac{3}{2}x + c$. Then u is some

$$u = f(c) = f(y + \frac{3}{2}x)$$

We are given $u(0, y) = y^2$. Then plug in $y + \frac{3}{2}x$, and we get

$$u = f(y + \frac{3}{2}x) = \left(y + \frac{3}{2}x\right)^2$$

1.4.2 Change of Variables/Coordinate Method

We are still trying to solve

$$au_x + bu_y = 0$$

The idea is to rotate the coordinate system so that solution only depends on some x'. We perform a linear transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Giving us

$$\begin{cases} x' = ax + by \\ x' = bx - ay \end{cases}$$

Then we use these to substitute u_x, u_y via the chain rule:

$$\begin{aligned} u_x = \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} \\ &= u_{x'} \cdot a + u_{y'} \cdot b \\ u_y = u_{x'} \cdot \frac{\partial x'}{\partial y} + u_{y'} \cdot \frac{\partial y'}{\partial y} &= u_{x'} \cdot b + u_{y'} \cdot (-a) \end{aligned}$$

And we plug into the original equation

$$
\begin{aligned}
a(u_{x'} \cdot a + u_{y'} \cdot b) + b(u_{x'} \cdot b + u_{y'} \cdot (-a)) &= 0 \\
a^2 u_{x'} + ab u_{y'} + b^2 u_{x'} - ab u_{y'} &= 0
\end{aligned}
$$

And we will, without fail, get cancellation that leads to an ODE:

$$(a^2 + b^2) u_{x'} = 0$$

Then $u = f(y')$, where f is arbitrary, and, substituting,

$$u = f(ax - by)$$

The advantage of the change of variables method is that it works for inhomogeneous equations as well, whereas it is much more difficult to use the geometric method in that case. Solve inhomogeneous equations by setting the left side of the equation in the proper form.

Example Solve $u_x + u_y = 2u$.

$$
\begin{cases}
x' = x + y \\
y' = x - y
\end{cases}
$$

$$
\begin{aligned}
u_x &= u_{x'} \frac{\partial x'}{\partial x} + u_{y'} \frac{\partial y'}{\partial x} = u_{x'} + u_{y'} \\
u_y &= u_{x'} \frac{\partial x'}{\partial y} + u_{y'} \frac{\partial y'}{\partial y} = u_{x'} - u_{y'}
\end{aligned}
$$

Substituting, this simplifies to

$$
\begin{aligned}
2u_{x'} &= 2u \\
\frac{\partial u}{\partial x'} &= u \\
\frac{1}{u}\, \mathrm{d}u &= \mathrm{d}x' \\
\ln|u| &= x' + f(y') \\
u &= e^{f(y')} e^{x'}
\end{aligned}
$$

f is arbitrary so denote $e^{f(y')} = f(y')$, and

$$
\begin{aligned}
u &= f(y') e^{x'} \\
u &= f(x - y) e^{x+y}
\end{aligned}
$$

Chapter 2

Wave and Diffusion Equations

2.1 Initial and Boundary Conditions

Initial conditions are conditions imposed when $t = 0$. Boundary conditions are imposed when x is some specific value; usually, we are considering values between 0 and some length l, in which case boundary conditions are values at $x = 0, l$.

In higher dimensions, boundary value conditions occur at some boundary surface denoted ∂D, where $D \subseteq \mathbb{R}^n$, and where $\hat{\mathbf{n}}$ is the outer normal vector.

Initial conditions for the heat equation look like

$$\begin{cases} u_t = k\nabla u \\ u(x, t_0) = \phi(x) \end{cases}$$

and for the wave equation, we need *two* initial conditions because the highest time derivative is of order two:

$$\begin{cases} u_{tt} = k\nabla u \\ u(x, t_0) = \phi(x) \\ u_t(x, t_0) = \psi(x) \end{cases}$$

2.1.1 Types of Conditions

1. **Dirichlet condition:** u is specified on the boundary:
$$u(x, t) = f(x, t) \text{ for } x \in \partial D$$
2. **Neumann condition:** directional derivative $\frac{\partial u}{\partial \hat{\mathbf{n}}}$ is specified, where $\hat{\mathbf{n}}$ is the outer normal vector:
$$\frac{\partial u}{\partial \hat{\mathbf{n}}} = g(x, t) \text{ for } x \in \partial D$$
3. **Robin condition:** $\frac{\partial u}{\partial \hat{\mathbf{n}}} + au$ is specified (both u and its directional derivative.)

For example, with a fluid bounded by a container, the flux of substance in that fluid through the boundary of the container is zero, so, by Fick's law ($\mathbf{J}$ is the diffusion flux vector),

$$\begin{aligned} &\mathbf{J} = -k\nabla u \\ &0 = -k\nabla u \text{ for } x, y, z, \in \partial D \end{aligned}$$

2.2 The Wave Equation

Consider a flexible, elastic, homogeneous string or thread of length l, which undergoes relatively small transverse vibration. Let $u(x,t)$ be its displacement from equilibrium position at time t and position x. Because the string is flexible, the tension T points tangentially along the string. Let $T(x,t)$ be the magnitude of this tension vector. Let ρ be the density of the string. We know it to be constant.

Write down Newton's law for the part of the string between any two points at $x = x_0$ and $x = x_1$. Slope of string at x_1 is $u_x(x_1,t)$. Newton's law in the longitudinal and transverse directions are

$$\begin{aligned} \left.\frac{T}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} &= 0 \\ \left.\frac{Tu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} &= \int_{x_0}^{x_1} \rho u_{tt}\,\mathrm{d}x \end{aligned}$$

Where the left side we obtained by constructing an, infinitesimal triangle for length of the string, with sides 1, u_x, and $\sqrt{1+u_x^2}$, then using $\cos\theta = \text{Adj/Hyp}$ and $\sin\theta = \text{Opp/Hyp}$ to solve for the component of T along the transverse and longitudinal directions. We assume no motion along longitudinal direction, so the right side of the first equation is zero. Then, we use Newton's law, for the right side of the bottom equation, with u_{tt} being acceleration.

Assume the motion is small, and that $|u_x|$ is small. Then replace $\sqrt{1+u_x^2}$ with one. (We can justify this by using the Taylor expansion $\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \ldots$). Then

$$\begin{aligned} \left.\frac{T}{u_x}\right|_{x_0}^{x_1} &= \int_{x_0}^{x_1} \rho u_{tt}\,\mathrm{d}x \\ T(u_x)_x &= \rho u_{tt} \end{aligned}$$

Assuming T is independent of x and t,

$$\boxed{u_{tt} = c^2 u_{xx}}$$

Where $x = \sqrt{T/\rho}$.

2.3 Solving the Wave Equation

2.3.1 Change of Variables Method

Our equation is

$$u_{tt} = c^2 u_{xx} \Rightarrow u_{tt} - c^2 u_{xx} = 0$$

We start by "factoring" the differential operator:

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u &= 0 \\ \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u &= 0 \end{aligned}$$

We use a similar procedure as with linear equations, performing a change of variables. We choose variables that are "perpendicular" to the directional derivatives given by the linear operators:

$$\begin{cases} \xi = ct + x \\ \eta = ct - x \end{cases}$$

Then, by the chain rule:

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} \end{aligned}$$

Plugging in to the factored operators above:

$$\begin{aligned} \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} &= \left(c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right) - c\left(\frac{\partial}{\partial}\right)\xi - \frac{\partial}{\partial \eta}) \\ &= 2c\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} &= \left(c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right) + c\left(\frac{\partial}{\partial}\right)\xi - \frac{\partial}{\partial \eta}) \\ &= 2c\frac{\partial}{\partial \xi} \end{aligned}$$

Substituting into the original equation, we have

$$\begin{aligned} \left(2c\frac{\partial}{\partial \eta}\right)\left(2c\frac{\partial}{\partial \xi}\right)u &= 0 \\ \frac{\partial^2}{\partial \eta \partial \xi}u &= 0 \end{aligned}$$

Which we can then integrate one variable at a time:

$$\begin{aligned} (u_\xi)_\eta &= 0 \\ u_\xi &= f(\eta) \\ u &= \int f(\xi)\,\mathrm{d}\xi + g(\eta) \end{aligned}$$

$\int f(\xi)\,\mathrm{d}\xi$ is also some arbitrary function of ξ, so we abuse notation to

$$u = f(\xi) + g(\eta)$$

And substituting our original variables:

$$u(x,t) = f(ct + x) + g(ct - x)$$

2.3.2 General and Particular Solution Method

We once again return to

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$$

We choose to denote

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u$$

and thus

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)v = 0$$

We're left with a system of equations

$$\begin{cases} (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})v = 0 \\ (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u = v \end{cases}$$

We know that $v(x,t) = h(x+ct)$, where h is arbitrary, as the upper equation is a homogeneous linear equation. Then we're left with the bottom equation:

$$u_t + cu_x = h(x+ct)$$

Which is an inhomogeneous equation. By section 1.3, we know that some solution to

$$u_t + cu_x = 0$$

will be a component of the general solution. We will call this $g(x-ct)$.

Now we just need to find the particular solution, which should be of the form $f(x+ct)$. We let $s = x+ct$ for simplicity, and apply this linear operator to f in order to solve the equation:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)f &= h(x+ct) \\
\frac{\partial}{\partial t}(f(x+ct)) + c\frac{\partial}{\partial x}(f(x+ct)) &= h(s) \\
cf'(x+ct) + cf'(x+ct) &= h(s) \\
2cf'(x) &= h(s) \\
f'(s) &= \frac{h(s)}{2c} \\
f(s) &= \frac{1}{2c}\int h(s)\,\mathrm{d}s
\end{aligned}$$

And so

$$u(x,t) = g(x-ct) + f(x+ct)$$

To summarize, we solved the topmost homogeneous equation for v, leading to some $h(x+ct)$. Then we set that solution h equal to the bottom-most inhomogeneous equation.

To solve that inhomogeneous equation, we solve for the general solution $\mathcal{L}u = 0$, which gave us some $g(x-ct)$, as per linear homogeneous equations.

Then we guessed a particular solution of form $f(x+ct)$ in order to set $\mathcal{L}f = h$ collapse the equation to an ordinary differential equation.

2.3.3 Generalized Change of Variables

So, why did we make the change of variables

$$\begin{cases} \xi = ct + x \\ \eta = ct - x \end{cases}$$

Consider the general case

$$\left(a_{11}\frac{\partial}{\partial t} - a_{12}\frac{\partial}{\partial x}\right)\left(a_{21}\frac{\partial}{\partial t} + a_{22}\frac{\partial}{\partial x}\right)$$

Then we assume the change of variables will be of a form

$$\begin{cases} \xi = b_{11}t + b_{12}x \\ \eta = b_{21}t + b_{22}x \end{cases}$$

And now we are left with the task of finding b_{ij}. We effectively have a linear transformation of the form

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$$

We abbreviate the matrix with entries b_{ij} B. Then notice that

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial \xi}\frac{\partial xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = b_{11}\frac{\partial}{\partial \xi} + b_{21}\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} - b_{12}\frac{\partial}{\partial \xi} + b_{22}\frac{\partial}{\partial \eta} \end{aligned}$$

And thus, in matrix form,

$$\begin{aligned} \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{bmatrix} &= \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \\ &= B^T \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \end{aligned}$$

And this implies that

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \left(B^T\right)^{-1} \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{bmatrix}$$

We want the new variables ξ, η to be parallel to the directional derivatives defined by a_{ij}. In symbols,

$$\frac{\partial}{\partial \xi} \parallel a_{11}\frac{\partial}{\partial t} + a_{12}\frac{\partial}{\partial x}$$
$$\frac{\partial}{\partial \eta} \parallel a_{21}\frac{\partial}{\partial t} + a_{22}\frac{\partial}{\partial x}$$

Then, in matrix form

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Defining the matrix with entries $a_{ij} \equiv A$, and returning to B,

$$\begin{aligned} \left(B^T\right)^{-1} &= A \\ B^T &= A^{-1} \\ B &= \left(A^{-1}\right)^T \end{aligned}$$

Now we know the change of variables matrix is the transpose of the inverse of the matrix A. In symbols:

$$\boxed{\text{Change of variables matrix} = \left(A^{-1}\right)^T}$$

For differential operator $\mathcal{L} = \left(a_{11}\frac{\partial}{\partial t} - a_{12}\frac{\partial}{\partial x}\right)\left(a_{21}\frac{\partial}{\partial t} + a_{22}\frac{\partial}{\partial x}\right)$.

2.3.4 Generalized Particular Solution Method

We once again have the differential operation

$$\left(a\frac{\partial}{\partial t} - b\frac{\partial}{\partial x}\right)\left(c\frac{\partial}{\partial t} + d\frac{\partial}{\partial x}\right)u = 0$$

We denote

$$v = \left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\right)u$$

Then v satisfies

$$\left(c\frac{\partial}{\partial t} - d\frac{\partial}{\partial x}\right)v = 0$$

And our system of equations is

$$\begin{cases} \left(c\frac{\partial}{\partial t} - d\frac{\partial}{\partial x}\right)v = 0 \\ \left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\right)u = v \end{cases}$$

$v(x,t) = h(dx + ct)$, where h is arbitrary. Then second equation becomes

$$au_t + bu_x = h(dt + cx)$$

The general solution will be of form

$$g(bt - ax)$$

And we choose the particular solution to be some $f(dt + cx)$ such that

$$\begin{aligned}
a\frac{\partial}{\partial t}f(dt+cx) + b\frac{\partial}{\partial x}f(dt+cx) &= h(s) \\
ad \cdot f'(dt+cx) + bc \cdot f'(dt+cx) &= h(s) \\
(ad+bc)f'(s) &= h(s) \\
f'(s) &= \frac{1}{ad+bc}h(s) \\
f(s) &= \frac{1}{ad+bc}\int h(s)\,\mathrm{d}s
\end{aligned}$$

Then the solution is of the form

$$u(x,t) = g(bt - ax) + f(dt + cx)$$

2.4 Wave Equation - Initial Value Problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

We know the solution is of the form $u = f(x+ct) + g(x-ct)$. Let us plug in $t = 0$,
$u = f(x) + g(x) = \phi(x)$
And let us find u_t then plug in $t = 0$:

$$\begin{aligned}
cf'(x+ct) - cg'(x-ct) &= \psi(x) \\
cf'(x) - cg'(x) &= \psi(x)
\end{aligned}$$

Now we leverage the fact that f and g just became single variable functions:

$$\begin{cases} f' + g' = \phi' \\ f' - g' = \frac{1}{c}\psi \end{cases}$$

Then, solving for f' and g':

$$\begin{cases} f' = \frac{1}{2}\left(\phi' + \frac{1}{c}\psi\right) \\ g' = \frac{1}{2}\left(\phi' - \frac{1}{c}\psi\right) \end{cases}$$

Then, recognizing that f and g are functions of some $x \pm ct$, and integrating,

$$\begin{cases} f(r) = \frac{1}{2}\psi(r) + \frac{1}{2c}\int \psi\,\mathrm{d}r + A \\ g(s) = \frac{1}{3}\psi(s) - \frac{1}{2c}\int \psi\,\mathrm{d}s + B \end{cases}$$

We know that $u(x,0) = f + g = \phi(x)$, so $A + B = 0$, and then

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(s)\,\mathrm{d}s + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(s)\,\mathrm{d}s \end{aligned}$$

By $\int_a^b f\,\mathrm{d}s = -\int_b^a f\,\mathrm{d}s$, we can combine the integrals and

$$\boxed{u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s}$$

What we get is actually quite intuitive: ϕ is a function that depends on x, but gets transposed to the right and to the left as time increases due to the $\pm ct$ term in the input. Then, we get a "propagation" of ϕ at a speed c.

The ψ term, since it's under an integral, effectively *expands* over x as t increases. We see that the limits of integration become larger as t goes increases, and so the area covered gets larger and we see an expanding graph.

2.5 Energy of Vibrating String

$$\frac{\mathrm{d}}{\mathrm{d}t}KE = \rho\int_{-\infty}^{\infty} u_t u_{tt}\,\mathrm{d}x$$

Then, by $\rho u_{xx} = T u_{xx}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}KE &= T\int_{\infty}^{\infty} u_t u_{xx}\,\mathrm{d}x \\ &= T\left(\int_{-\infty}^{\infty} (u_t u_x)_x\,\mathrm{d}x - \int_{-\infty}^{\infty} u_{tx}u_x\,\mathrm{d}x\right) \\ &= T\left((u_t u_x)\Big|_{-\infty}^{\infty} - \frac{1}{2}\int_{\infty}^{\infty} \frac{\partial}{\partial t}(u_x)^2\,\mathrm{d}x\right) \\ &= -\frac{T}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}(u_x)^2\,\mathrm{d}x \end{aligned}$$

We can take the derivative out of the integral because the integral is not with respect to t,

$$\frac{\mathrm{d}}{\mathrm{d}t}KE = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T}{2}\int_{-\infty}^{\infty} (u_x)^2\right)\mathrm{d}x$$

The right side of the equation is the potential energy, and so we get the familiar result

$$\frac{\mathrm{d}}{\mathrm{d}t}KE = -\frac{\mathrm{d}}{\mathrm{d}t}PE$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(KE + PE) = 0$$

When wave traveling, kinetic energy and potential energy experience exchange, and there is no net loss.

We can also generalize PE in three dimensions to

$$PE = \frac{T}{2}\int_{-\infty}^{\infty} |\nabla u|^2 \,\mathrm{d}\hat{\mathbf{x}}$$

2.6 The Diffusion Equation

Let us imagine a motionless liquid filling a straight tube or pipe and a chemical substance, say a dye, which is diffusing through the liquid . Simple diffusion is characterized by the following law: The dye moves from regions of higher concentration to regions of lower concentration. The rate of motion is proportional to the concentration gradient. (Fick's law). Let $u(x,t)$ be the concentration of the dye at position x of the pipe at time t.

In the section of the pipe from x_0 to x_1, the mass of the dye is

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,\mathrm{d}x\,,\ \text{so}\ \frac{\mathrm{d}M}{\mathrm{d}t} = \int_{x_0}^{x_1} u_t(x,t)\,\mathrm{d}x$$

The mass in this section of pipe cannot change except by flowing in and our of its ends. By Fick's law,

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \text{flow in} - \text{flow out} = -ku_x(x_1,t) - ku_x(x_0,t)$$

Then

$$\int_{x_0}^{x_1} u_t(x,t)\,\mathrm{d}x = ku_x(x_1,t) - ku_x(x_0,t)$$

Differentiate with respect to x,

$$\boxed{u_t = ku_{xx}}$$

In three dimensions we have

$$\iiint_D u_t \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\mathrm{bdy}D} k(\hat{\mathbf{n}}\cdot\nabla u)\,\mathrm{d}S$$

By the divergence theorem, we get the three-dimensional diffusion equation

$$\boxed{u_t = k(u_{xx}+u_{yy}+u_{zz}) = k\delta u}$$

If there is an external source f, and if the rate k of diffusion is variable, we get the more general inhomogeneous

$$u_t = \nabla\cdot(k\nabla u) + f(x,t)$$

2.7 The Maximum Principle

With the diffusion equation defined

$$\begin{cases} u_t = ku_{xx} \\ 0 \leq x \leq l \\ 0 \leq t \leq T \end{cases}$$

The solution will attain its maximum and minimum value on the boundary $t = 0$, $x = 0$, $x = l$

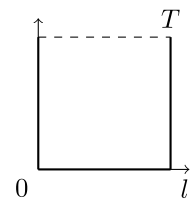

Figure 2.1: The solution will be at a maximum or a minimum along the bolded lines.

Proof: We will perturb the solution $u(x,t)$ with a small εx^2:

$$v(x,t) = u(x,t) + \varepsilon x^2$$

$$\begin{cases} v_t = u_t \\ v_{xx} = u_x x + 2\varepsilon \end{cases}$$

Then plug into the heat equation:

$$\begin{aligned} u_t - kv_{xx} &= u_t - k(u_{xx} + 2\varepsilon) \\ &= u_t - ku_{xx} - 2k\varepsilon \end{aligned}$$

The $u_t - ku_{xx}$ term should be zero, which means

$$v_t - kv_{xx} = -2k\varepsilon < 0 \tag{2.1}$$

We know this is negative because ε and k are positive. Suppose v attains its maximum value *inside* the boundaries. Then

$$\begin{cases} v_t = 0 \\ v_{xx} \leq 0 \end{cases}$$

Plugging these into (2.1):

$$0 \leq -kv_{xx} = -2k\varepsilon < 0$$

Where we know this is greater than or equal to zero because $v_x x$ is negative. The above is a contradiction: a number cannot be greater than or equal to zero and also less than zero. This means v only attains max along the boundary; it remains to show that this is also the case for u.

Recall $v = u + \varepsilon x^2$, so, by the definition of the maximum

$$v(x,t) \leq \max v(x,t) \text{ for } (x,t) \in \text{boundary}$$

And

$$\max v(x,t) \leq \max u + \varepsilon l^2 \text{ for } (x,t) \in \text{boundary}$$

,

so,

$$\begin{aligned} v(x,t) &\leq \max u + \varepsilon l^2 \text{ for } (x,t) \in \text{boundary} \\ u + \varepsilon x^2 &\leq \max u + \varepsilon l^2 \text{ for } (x,t) \in \text{boundary} \\ u &\leq \max u + \varepsilon(l^2 - x^2) \text{ for } (x,t) \in \text{boundary} \end{aligned}$$

Then let $\varepsilon \Rightarrow 0$, and

$$u \leq \max u \text{ for } (x,t) \in \text{boundary}$$

Which is precisely the definition of a maximum. WE can show the minimum is also along the boundary by proving this for $-u$ (the maximum of $-u$ is the minimum of u). We get

$$\min u \text{ for } (x,t) \in \text{boundary} \leq u \leq \max u \text{ for } (x,t) \in \text{boundary}$$

2.7.1 Applications

We can use the maximum principle to show **uniqueness**. For

$$\begin{cases} u_t - ku_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(l,t) = h(t) \end{cases}$$

If u_1 and u_2 are two solutions of above problem then $u_1 = u_2$. I.e., the above equation has a unique solution.

Proof: Consider $w = u_1 - u_2$. Then

$$\begin{cases} w_t - kw_{xx} = 0 \\ w(x,0) = 0 \\ w(0,t) = w(l,t) = 0 \end{cases}$$

The topmost equation is equal to zero because u_1 and u_2 both must satisfy the inhomogeneity, so, by linearity, the inhomogeneities are equal and their difference is zero. The boundary and initial conditions are equal to zero because of the same reason: they must satisfy the same equation.

Then, if the function is zero at the boundaries, and the maximum and minimum at the boundaries:

$$0 = \min w \leq w \leq \max w = 0$$

And thus $w = 0$ and $u_1 = u_2$.

Now we move on to **stability**. Our two equations are

$$\begin{cases} u_t = ku_xx \\ u(x,0) = \phi_1(x) \\ u(0,t) = u(l,t) = 0 \end{cases}$$

$$\begin{cases} u_t = ku_xx \\ u(x,0) = \phi_2(x) \\ u(0,t) = u(l,t) = 0 \end{cases}$$

Given these different initial conditions, how different will u_1, u_2 be?

Let $w = u_1 - u_2$, then

$$\begin{cases} w_t = kw_{xx} \\ w(x,0) = \phi_1 - \phi 2 \\ w(0,t) = w(l,t) = 0 \end{cases}$$

Then, because the boundaries are equal to zero,

$$w(x,t) \leq \max w \leq \max |\phi_1 - \phi_2| \text{ for } x \in [0,l]$$

And also

$$\min(\phi_1 - \phi_2) \leq w \leq \max(\phi_1 - \phi_2)$$

So that

$$|w| \leq \max |\phi_1 - \phi_2| \text{ for } x \in [0,l]$$

And thus

$$|u_1 - u_2| \leq \max |\phi_1 - \phi_2| \text{ for } x \in [0,l]$$

]

If we enforce small ε in $|\phi_1 - \phi_2| < \varepsilon$, u_1 and u_2 are then quite similar.

2.8 Solving the Heat/Diffusion Equation (Initial Condition)

Our equation is

$$\begin{cases} u_t = ku_{xx} \\ u(x,0) = \phi(x) \end{cases}$$

We proceed with the observations made in section 2.8.1.

Suppose $Q(x,t)$ is a solution. We want a particular solution. Use dilation invariance:

$$\frac{\sqrt{c}x}{\sqrt{ct}} \Rightarrow \frac{x}{\sqrt{t}}$$

Let $Q(x,t) = g(\frac{x}{\sqrt{t}})$, where g is arbitrary. For convenience, let $p = \frac{x}{\sqrt{t}}$. Then

$$Q_t = g'\left(\frac{x}{\sqrt{t}}\right) \cdot -\frac{1}{2}\frac{x}{t^{3/2}}$$
$$Q_x = g'\left(\frac{x}{\sqrt{t}}\right) \cdot \frac{1}{\sqrt{t}}$$
$$Q_{xx} = g''\left(\frac{x}{\sqrt{x}}\right)\frac{1}{t}$$

We know $Q_t - kQ_{xx} = 0$, so

$$-\frac{1}{2}\frac{x}{t^{3/2}}g' - k\frac{1}{t}g'' = 0$$
$$g'' + \frac{x}{k2\sqrt{t}}g' = 0$$
$$g'' + \frac{p}{2k}g' = 0$$

Letting $y' = g''$, this is just a first order linear differential equation. Solving with an integrating factor, we eventually obtain

$$Q = g = C_1 \int_0^{x/\sqrt{t}} e^{-\frac{p^2}{4k}}\,\mathrm{d}p + C_2$$

Set Q to satisfy some initial condition, so that we may solve for its constants. For simplicity, we choose:

$$Q(x,0) = \begin{cases} 1 \text{ when } x > 0 \\ 0 \text{ when } x \leq 0 \end{cases}$$

Recall $t > 0$ by definition, and also by $x/\sqrt{t}$. Then, when $x > 0$, $x/\sqrt{t}$ approaches ∞ as $t \to 0$ (we can't plug in $t = 0$ directly):

$$C_1 \int_0^{\infty} e^{-\frac{p^2}{4k}}\,\mathrm{d}p + C_2 = 1$$

,

But when $x \leq 0$,

$$-C_1 \int_{-\infty}^{0} e^{-\frac{p^2}{4k}}\,\mathrm{d}p + C_2 = 0$$

Now we solve for C_1 and C_2. Adding the two equations above (the integrals cancel because the integrand is even):

$$2C_2 = 1 \Rightarrow C_2 = \frac{1}{2}$$

Now solve for C_1:

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$$
$$\int_0^{\infty} e^{-\frac{p^2}{4k}}\,\mathrm{d}p = \int_0^{\infty} e^{-u^2}\sqrt{4k}\,\mathrm{d}u$$
$$= \sqrt{4k}\frac{\sqrt{\pi}}{2}$$

And once again adding the two equations:

$$\sqrt{4k\pi}C_1 = 1 \Rightarrow C_1 = \frac{1}{\sqrt{4k\pi}}$$

And thus the particular solution is

$$Q(x,t) = \frac{1}{\sqrt{4k\pi}}\int_0^{x/\sqrt{t}} e^{\frac{p^2}{4k}}\,\mathrm{d}p + \frac{1}{2}$$

In section 2.8.1, we showed that the derivative of a solution is also a solution, as well as that we can shift the solution along the y axis, and convolve it, and the resulting function is still a solution. Now we put this to use. Let $Q_x = K$, and:

$$u(x,t) = \int_{\infty}^{\infty} K(x-y,t)\phi(y)\,\mathrm{d}y$$

Which is the convolution of Q and some initial condition ϕ. It can be shown that this u satisfies the initial condition but I omit the steps here.

Let us determine K:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{4k\pi}}\int_0^{x/\sqrt{t}} e^{\frac{-p^2}{4k}}\,\mathrm{d}p + \frac{1}{2}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-u^2}\,\mathrm{d}u + \frac{1}{2}\right)$$
$$= \frac{1}{\sqrt{4k\pi t}}e^{-(\sqrt{x}/2kt)^2}$$

And we end up with

$$\boxed{K = \frac{1}{4k\pi t}\exp\left[-\left(\frac{x}{\sqrt{4kt}}\right)^2\right]}$$

Which is called kernel—it is effectively a Gaussian curve that spreads out and decreases as t increases. Then, the solution is

$$\boxed{u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_{\infty}^{\infty} \exp\left[-\left(\frac{|x-y|^2}{4kt}\right)\right]\phi(y)\,\mathrm{d}y}$$

Which is the heat kernel convolved with the initial condition.

Note: The heat kernel solution is only valid when the initial condition coincides with its Fourier series (see chapter 3). Thus, you might be tempted to use the heat kernel to show certain properties of the solution or of the heat equation in general. This is not recommended; instead, use the maximum principle of integration by parts for better generality.

2.8.1 Observations on the Solution

1. **Shift invariance:** If $u(x,t)$ is a solution, $u(x-y,t)$ is still a solution.
2. **Divergence free:** If $u(x,t)$ is a solution, u_t, u_x, u_{xx} are still solutions. Note that this this means

$$(u_t)_t = k(u_{xx})_t$$
$$(u_t)_t = k(u_t)_{xx}$$

   because we are free to change the order of derivatives.
3. If $u_1, u_2, \ldots, u_m$ are solutions, then their linear combinations are also a solution.
4. If $k(x,t)$ is a solution, then

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)g(y)\,\mathrm{d}y$$

   is also a solution, with arbitrary g. This is because

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} k(x-y,t)g(y)\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}k(x-y,t)g(y)\,\mathrm{d}y\end{aligned}$$

   The same is true for the partial derivative with respect to x, allowing us to plug in to the heat equation and verify the equality. Note that $\int k(x-y,t)g(y)\,\mathrm{d}t$ is a **convolution** integral. In effect, this is the weighted sum of the values of k. g gives the weight.
5. **Dilation invariance:** If $u(x,t)$ is a solution, then $u(\sqrt{c}x, ct)$ is also a solution. By the chain rule:

$$cu(\sqrt{c}x, ct) = \sqrt{c}^2 u(\sqrt{c}x, ct)$$

2.9 Boundary Condition Solutions

2.9.1 Wave Equation with Dirichlet Condition

Our equation is

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(l,t) = 0 \end{cases}$$

We assume we are capable of separation of variables; that is, that the solution has the form

$$u(x,t) = X(x)T(t)$$

Now we plug back into the wave equation

$$\begin{aligned}
(X(x)T(t))_{tt} &= c^2(X(x)T(t))_{xx} \\
X(x)T''(t) &= x^2X''(t)T(t) \\
\frac{X''(x)}{X'(x)} &= \frac{T''(t)}{x^2T(t)} \\
-\frac{X''(x)}{X'(x)} &= -\frac{T''(t)}{x^2T(t)}
\end{aligned}$$

Where in the last line we added negative signs for convenience.

These will only be equal to each other if they are both equal to a constant:

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \lambda$$

Now we have two ordinary differential equations:

$$\begin{cases} -X''(x) = \lambda X(x) - T''(t) = \lambda c^2 T(t) \end{cases}$$

Solve for X:

$$X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

Note λ is positive, as we will prove later. For simplicity, let $\lambda = \beta^2$. Then

$$X = A\cos\beta x + B\sin\beta x$$

Now let us use the boundary condition, $u(0,t) = u(l,t) = 0$, to solve for some constants. We can set solely $X(x) = 0$, ignoring $T(t)$, because the whole expression is separable and we can divide both sides by $T(t)$:

$$\begin{cases} A = 0 \\ A\cos\beta l + B\sin\beta l = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B\sin\beta l = 0 \end{cases}$$

We don't want $B = 0$, because then the whole solution is zero, and this is a trivial solution. Then

$$\begin{aligned}
\sin\beta l &= 0 \\
\beta l &= n\pi, n \in \mathbb{Z} \\
\beta &= \frac{n\pi}{l}
\end{aligned}$$

Infinitely many β values to choose, hence

$$X(x) = B\sin\frac{n\pi}{l}x, n \in \mathbb{N}$$

Because $\sin -\theta = -\sin\theta$, we can restrict n to $1, 2, 3$, because you can put the negative sign in the constant.

Now let us solve the other ODE for T. We already know λ:

$$
\begin{aligned}
-\frac{T''(t)}{c^2T(t)} &= \left(\frac{n\pi}{l}\right)^2 \\
-T''(t) &= \left(\frac{cn\pi}{l}\right)^2 T(t) \\
T(t) &= C\cos\frac{cn\pi}{l}t + D\frac{cn\pi}{l}t
\end{aligned}
$$

Because there is n in the trigonometric functions, each constant is unique to the particular n:

$$T_n(t) = C_n\cos\frac{cn\pi}{l}t + D_n\sin\frac{cn\pi}{l}t$$

$u(x,t)$ is then

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} X_nT_n \\
&= \sum_{n=1}^{\infty}\left(C_n\cos\frac{cn\pi}{l}t + D_n\sin\frac{cn\pi}{l}t\right)\sin\frac{n\pi}{l}x
\end{aligned}
$$

Then our solution for the boundary conditions is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{cn\pi}{l}t + D_n\sin\frac{cn\pi}{l}t\right)\sin\frac{n\pi}{l}x}$$

Now we show this solution works with **initial conditions**:

$$
\begin{cases}
u_{tt} = c^2u_{xx} \\
u(0,t) = u(l,t) = 0 \\
u(x,0) = \phi(x) \\
u_t(x,0) = \phi(x)
\end{cases}
$$

We notice

$$\phi(x) = \int_{n=1}^{\infty} C_n\sin\frac{n\pi}{l}x$$

And

$$
\begin{aligned}
u_t(x,t) &= \sum_{n=1}^{\infty}\left(-C_n\frac{cn\pi}{l}\sin\frac{cn\pi}{l}t + D_n\frac{cn\pi}{l}\cos\frac{cn\pi}{l}t\right)\cdot\sin\frac{\pi t}{l}x \\
\psi(x) &= \sum_{n=1}^{\infty} D_n\frac{cn\pi}{l}\sin\frac{n\pi}{l}x
\end{aligned}
$$

Then separation of variables only works for functions that satisfy their sine Fourier series:

$$
\begin{cases}
\phi(x) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi}{l}x \\
\psi(x) = \sum_{n=1}^{\infty} D_n\frac{cn\pi}{l}\sin\frac{n\pi}{l}x
\end{cases}
$$

It can be shown that practically any odd function on the interval $(0,l)$ can be expressed in such a series.

2.9.2 Diffusion Equation with Dirichlet Condition

$$\begin{cases} u_t = ku_{xx}, 0 \le x \le l \\ u(0,t) = u(l,t) = 0 \end{cases}$$

We use the separation of variables method once again: assume

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ X(x)T'(t) &= kX''(x)T(t) \\ -\frac{X''(x)}{X(x)} &= -\frac{T'(t)}{kT(t)} - \lambda \end{aligned}$$

X'' will be the same, but T' is a single derivative, so we must solve it differently (recall $\lambda = (n\pi/l)^2$):

$$\begin{aligned} X(x) &= B_n \sin\frac{n\pi}{l}x \\ -\frac{T'(t)}{kT(t)} &= \left(\frac{n\pi}{l}\right)^2 \\ T'(t) &= -k(\frac{n\pi}{l})^2 T(t) \\ T(t) &= C_1 \exp\left[-k\left(\frac{n\pi}{l}\right)^2 t\right], n = 1,2,3 \end{aligned}$$

And thus our solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n \exp\left[-k\left(\frac{n\pi}{l}\right)^2 t\right] \sin\frac{n\pi}{l}x}$$

And the initial value condition must satisfy

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}x$$

2.9.3 Why Lambda Must be Positive

For the Dirichlet eigenvalue problem,

$$\begin{cases} -X'' = \lambda X \\ X(0) = X(l) = 0 \end{cases}$$

λ is always positive. **Proof:** If $\lambda = 0$,

$$\begin{aligned} X'' &= 0 \\ X(x) &= Cx + D \end{aligned}$$

Then, solving for constants:

$$\begin{cases} X(0) = D \\ X(l) = Cl + D \end{cases} \Rightarrow \begin{cases} D = 0 \\ C = 0 \end{cases}$$

Then $X(x) = 0$, and since we do not consider the zero vector an eigenvector, $\lambda = 0$ is not an eigenvalue. Now consider the case $\lambda < 0$. Taking λ negative such that $-\lambda$ is positive:

$$\begin{aligned} X'' &= (-\lambda)X \\ X'' &= \beta^2 X \\ X(x) &= Ce^{\beta x} + De^{-\beta x} \end{aligned}$$

Then, solving for the constants:

$$\begin{cases} X(0) = C + D \\ X(l) = Ce^{l\beta} = De^{-l\beta} \end{cases} \Rightarrow \begin{cases} C + D = 0 \\ Ce^{\beta l} + De^{-\beta l} = 0 \end{cases}$$

By the top equation, $C = -D$. We enforce $C \neq 0$ for the same reasons as above (it would result in the zero vector). Then:

$$\begin{aligned} Ce^{\beta l} - Ce^{\beta l} &= 0 \\ e^{\beta l} - e^{-\beta l} &= 0 \\ e^{\beta l} &= e^{-\beta l} \\ e^{2\beta l} &= 1 \\ \beta &= 0 \end{aligned}$$

But we enforced $\lambda < 0$! By contradiction, we have proved λ must be positive.

2.9.4 Wave Equation with Neumann Condition

$$\begin{cases} u_{tt} = c^2 u_{xx}, 0 \leq x \leq l \\ u_x(0,t) = u_x(l,t) = 0 \end{cases}$$

Consider

$$-X''(x) = \lambda X(x)$$

Because the Neumann condition has to do with the x derivative of u, we cannot make the same claims about λ. Then we consider three different cases. **Assume** $\lambda > 0$: Again, let $\lambda = \beta^2$

$$\begin{aligned} X(x) &= A\cos\beta x + B\sin\beta x \\ X'(x) &= -A\beta\sin\beta x + B\beta\cos\beta x \end{aligned}$$

Then, solving for the constants:

$$\begin{cases} X'(0) = B\beta \\ X'(l) = -A\beta\sin\beta l + B\beta\cos\beta l \end{cases} \Rightarrow \begin{cases} B\beta = 0 \Rightarrow B = 0 \\ -A\beta\sin\beta l = 0 \end{cases}$$

Then $\sin \beta l = 0$ and

$$\beta l = n\pi \Rightarrow \frac{n\pi}{l}, n = 1, 2, 3, \ldots$$

Then we have the exact same case as before, and we get the case $\lambda > 0$ implies

$$\begin{aligned} X_n(x) &= A_n \cos \frac{n\pi}{l} x \\ T_n(t) &= C_1 \cos \frac{cn\pi}{l} t + D_n \sin \frac{cn\pi}{l} t \end{aligned}$$

Then

$$u_n(x,t) = \left(C_n \cos \frac{2n\pi}{l} t + D_n \sin \frac{cn\pi}{l} \right) \cos \frac{n\pi}{l}$$

Now **assume** $\lambda = 0$, then:

$$\begin{aligned} X''(x) &= 0 \\ X(x) &= Cx + D \\ X'(x) &= C \end{aligned}$$

If we want

$$\begin{cases} X(0) = 0 \\ X(l) = 0 \end{cases}$$

we only need $c = 0$. I.e., $X(x) = D$, some constant. For T,

$$\begin{aligned} T''(t) &= 0 \\ T(t) &= At + B \end{aligned}$$

Then

$$\begin{aligned} u(x,t) = X(x)T(t) &= D(At + B) \\ &= At + B \end{aligned}$$

This is the particular solution for $\lambda = 0$, so denote

$$u_0(x,t) = A_0 t + B_0$$

At last **assume** $\lambda < 0$,

$$\begin{aligned} -X''(x) &= \lambda X(x) \\ X''(x) &= (-\lambda) X(x) \\ X''(x) &= \beta^2 X(x) \\ X(x) &= Ce^{\beta x} + De^{-\beta x} \end{aligned}$$

Then

$$X'(x) = C\beta e^{\beta x} - D\beta e^{-\beta x}$$

We attempt to solve for β and the constants:

$$\begin{cases} C\beta - D\beta = 0 \\ C\beta e^{\beta l} - D\beta e^{-\beta l} = 0 \end{cases} \Rightarrow \begin{cases} C = D \\ C\beta e^{\beta l} - C\beta e^{-\beta l} = 0 \end{cases}$$

Then, solving for β:

$$\begin{aligned} e^{\beta l} - e^{-\beta l} &= 0 \\ e^{2\beta l} &= 1 \\ \beta &= 0 \end{aligned}$$

WE have a contradiction again. We enforced $\lambda < 0 \Rightarrow \beta^2 > 0$. So, we may have $\lambda \geq 0$, but **not** $\lambda < 0$.

Then, the **general solution** to the Neumann condition is

$$\boxed{u(x,t) = A_0 t + B_0 + \sum_{n=1}^{\infty} \left(C_n \cos \frac{cn\pi}{l} t + D_n \sin \frac{cn\pi}{l} t \right) \cos \frac{n\pi}{l} x}$$

2.9.5 Diffusion Equation with Neumann Condition

$$\begin{cases} u_t = ku_{xx} \\ u_x(0,t) = u_x(l,t) = 0 \\ x(x,0) = \phi(x) \end{cases}$$

We take the same results as with the wave equation, except the time derivative is of order one. Starting with $\lambda = 0$:

$$\begin{aligned} X''(x) &= 0 \\ X(x) &= Cx + D \end{aligned}$$

But for the boundary condition to be satisfied, $C = 0$, and $X(x) = D$. And for T:

$$\begin{aligned} \frac{T'(t)}{kT(t)} &= 0 \\ T'(t) &= 0 \\ T_0(t) &= A_0 \end{aligned}$$

We combine these solutions (the constants combine into one) with the solutions for $\lambda > 0$:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} C_n \exp \left[-k \left(\frac{n\pi}{l} \right)^2 t \right] \sin \frac{n\pi}{l} x$$

Chapter 3

Fourier Series

3.1 The Coefficients

Several solutions to equations in chapter 2 involved Fourier series of functions—i.e., series involving sines and cosines. We now explore the coefficients of such series so that we may completely describe an arbitrary function with a periodic series.

3.1.1 Fourier Sine Series

The Fourier sine series is of form

$$\boxed{\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}}$$

in the interval $(0, l)$.

We use the fact that

$$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} \, \mathrm{d}x = 0 \text{ if } m \neq n, \frac{l}{2} \text{ if } m = n \tag{3.1}$$

Which we can show by using the trig identity $\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$:

$$\int_0^l -\frac{1}{2} \cos\Big(\frac{n\pi x}{l} + \frac{m\pi x}{l}\Big) + \frac{1}{2} \cos\Big(\frac{n\pi x}{l} - \frac{m\pi x}{l}\Big) \, \mathrm{d}x$$
$$\int_0^l -\frac{1}{2} \cos\left(\frac{(n+m)\pi x}{l}\right) + \frac{1}{2} \cos\left(\frac{(n-m)\pi x}{l}\right) \mathrm{d}x$$

Consider the case $m \neq n$:

$$-\frac{1}{2} \frac{l}{(m+n)\pi} \sin \frac{(n+m)\pi x}{l} \bigg|_0^l + \frac{1}{2} \frac{l}{(n-m)\pi} \sin \frac{(n-m)\pi x}{l} \bigg|_0^l$$

$n \pm m$ is an integer and thus the entire expression goes to zero. Considering $m = n$, the integral becomes

$$\begin{aligned}\int_0^l \sin^2 \frac{n\pi x}{l}\,\mathrm{d}x &= \int_0^l \frac{1}{2} - \frac{1}{2}\cos 2\frac{n\pi x}{l}\,\mathrm{d}x \\ &= \frac{l}{2} - \frac{1}{2}\frac{l}{2n\pi}\sin\frac{2n\pi x}{l}\bigg|_0^l \\ &= \frac{l}{2}\end{aligned}$$

Where once again n is an integer so $\sin(2n\pi x)/l$ goes to zero for $x = l$ and $x = 0$.

Now we use this fact. Multiply both sides of (3.1) by $\sin(m\pi x/l)$ and integrate:

$$\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^l A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,\mathrm{d}x$$

Then all terms in the sum vanish but the one with $m = n$. In that case, this becomes

$$\int_0^l \phi(x)\sin\frac{m\pi x}{l} = A_m\frac{l}{2}$$

Which yields

$$\boxed{A_m = \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,\mathrm{d}x}$$

3.1.2 Fourier Cosine Series

The fourier cosine series is of form

$$\boxed{\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{l}}$$

Once again,

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,\mathrm{d}x = 0 \text{ if } n \neq m,\ \frac{l}{2} \text{ if } m = n$$

Which we can verify by using a similar trig identity:

$$\int_0^l \cos\left(\frac{n\pi x}{l} + \frac{m\pi x}{l}\right) + \frac{1}{2}\cos\left(\frac{n\pi x}{l} - \frac{m\pi x}{l}\right)\mathrm{d}x$$
$$\int_0^l \frac{1}{2}\cos\left(\frac{(n+m)\pi x}{l}\right) + \frac{1}{2}\cos\left(\frac{(n-m)\pi x}{l}\right)$$

Take the case $m \neq n$. Then

$$\frac{1}{2}\frac{l}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\bigg|_0^l + \frac{1}{2}\frac{l}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l}\bigg|_0^l$$

Which again is zero because $m \pm n$ is an integer. Considering $m = n$:

$$\int_0^l \cos^2\frac{n\pi x}{l}\,\mathrm{d}x = \int_0^l \frac{1}{2} + \frac{1}{2}\cos\frac{2\pi nx}{l}\,\mathrm{d}x = \frac{l}{2} + \int_0^l \cos\frac{2\pi nx}{l}\,\mathrm{d}x$$

Where once again the second term becomes zero.

Multiplying both sides by $\cos(m\pi x/l)$ and integrating:

$$\int_0^l \phi(x)\cos\frac{m\pi x}{l}\,\mathrm{d}x = A_m\int_0^l \cos^2\frac{m\pi x}{l}\,\mathrm{d}x$$

Which leads to

$$\boxed{A_m = \frac{2}{l}\int_0^l \phi(x)\cos\frac{m\pi x}{l}\,\mathrm{d}x}$$

There is a nonzero term for $n = 0$ in this cosine series, so we consider $m = 0$ as well:

$$\int_0^l \phi(x)\cdot 1\,\mathrm{d}x = \frac{1}{2}A_0\int_0^l 1^2\,\mathrm{d}x = \frac{l}{2}A_0$$

Which yields

$$\boxed{A_0 = \frac{2}{l}\int_0^l \phi(x)\,\mathrm{d}x}$$

3.1.3 Full Fourier Series

We now define the full Fourier series on the interval (-l,l)

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right)$$

And we are given

$$\int_{-l}^l \cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,\mathrm{d}x = 0 \text{ for all } n, m$$
$$\int_{-l}^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,\mathrm{d}x = 0 \text{ for } n \neq m$$
$$\int_{-l}^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,\mathrm{d}x = 0 \text{ for } n \neq m$$
$$\int_{-l}^l 1\cdot\cos\frac{n\pi x}{l}\,\mathrm{d}x = 0 = \int_{-l}^l 1\cdot\sin\frac{m\pi x}{l}\,\mathrm{d}x$$

Which is a property of mutually orthogonal basis, as we will investigate later. Finally,

$$\int_{-l}^l \cos^2\frac{n\pi x}{l}\,\mathrm{d}x = \int_{-l}^l \sin^2\frac{n\pi x}{l}\,\mathrm{d}x = l$$
$$\int_{-l}^l 1^2\,\mathrm{d}x = 2l$$

This yields

$$\boxed{A_n = \frac{1}{l}\int_{-l}^l \phi(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x}$$

$$\boxed{B_n = \frac{1}{l}\int_{-l}^l \phi(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x}$$

3.2 Complex Form of Full Fourier Series

Recall that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \text{ and } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Putting the $e^{i\theta}$ and $e^{-i\theta}$ terms together, this means we can use a sum of complex numbers as a Fourier series, with $e^{in\pi\theta}$ where n can be a positive or negative integer:

$$\boxed{\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}}$$

We use the fact that

$$\begin{aligned}
\int_{-l}^{l} e^{in\pi x/l} e^{-im\pi x/l}\,\mathrm{d}x &= \int_{-l}^{l} e^{i(n-m)\pi x/l}\,\mathrm{d}x \\
&= \frac{l}{i\pi(n-m)}\left[e^{i(n-m)\pi} - e^{i(n-m)\pi}\right] \\
&= \frac{l}{i\pi(n-m)}\left[(-1)^{n-m} - (-1)^{m-n}\right] \\
&= \frac{l}{i\pi(n-m)}\left[(-1)^{-(m-n)} - (-1)^{m-n}\right] \\
&= 0
\end{aligned}$$

If $n \neq m$. When $n = m$

$$\int_{-l}^{l} e^{i(n-m)\pi x/l}\,\mathrm{d}x = \int_{-l}^{l} 1\,\mathrm{d}x = 2l$$

And the coefficients are given by

$$\boxed{c_n = \frac{1}{2l}\int_{-l}^{l} \phi(x) e^{-in\pi x/l}\,\mathrm{d}x}$$

3.3 General Fourier Series

We call back to linear algebra and define a vector space of functions of x and an inner product

$$(f, g) \equiv \int_a^b f(x)g(x)\,\mathrm{d}x$$

$f(x)$ and $g(x)$ are orthogonal if $(f, g) = 0$. $(f, f) = 0$ only if $f = 0$, and, most importantly, every eigenfunction is orthogonal to every other eigenfunction. Let us show this.

We are studying the operator $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with some boundary conditions. Let $X_1(x)$ and $X_2(x)$ be two different eigenfunctions. Then

$$-X_1'' = -\frac{\mathrm{d}^2 X_1}{\mathrm{d}x^2} = \lambda X_1 \tag{3.2}$$

$$-X_2'' = -\frac{\mathrm{d}^2 X_2}{\mathrm{d}x^2} = \lambda X_2 V \tag{3.3}$$

Assume $\lambda_1 = \lambda_2$. We use the identity

$$-X_1'' X_2 + X_1 X_2'' = (-X_1' X_2 + X_1 X_2')'$$

Integrate to obtain

$$\int_a^b (-X_1''X_2 + X_1X_2'')\,\mathrm{d}x = (-X_1'X_2 + X_1X_2')\int_a^b$$

(This is Green's second identity. We may think of this as two integrations by parts.)

On the left side use the eigenvalue relations (3.2) and (3.3). On the right side, we may use different boundary conditions:

- **Dirichlet:** Both functions vanish at both ends: $X_1(a) = X_1(b) = X_2(a) = X_2(b) = 0$. The right side is zero.
- **Neumann:** The first derivatives vanish at both ends, and we again end with zero.
- **Periodic:** $X_j(a) = X_j(b), X_j'(a) = X_j'(b)$ for $j = 1, 2$. *again* we end with zero.
- **Robin:** Likewise zero.

Then in all cases, we have

$$(\lambda_1 - \lambda_2)\int_a^b X_1X_2\,\mathrm{d}x = 0$$

Thus eigenfunctions of the second derivative operator X_1 and X_2 are orthogonal. This explains why Fourier's method works for *all eigenvalue problems*. We will have a set of λ_n and thus a set of different X_n which are all orthogonal to each other, which in turn allows us to obtain Fourier coefficients.

3.4 Symmetric Boundary Conditions

Consider any pair of boundary conditions

$$\alpha_1X(a) + \beta_1X(b) + \gamma_1X'(a) + \delta_1X'(b) = 0\alpha_2X(a) + \beta_2X(b) + \gamma_2X'(a) + \delta_2X'(b) = 0$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i, i = 1, 2$ are real constants. Such a set of boundary conditions is called **symmetric** if

$$\boxed{f'(x)g(x) - f(x)g'(x)\Big|_{x=a}^{x=b} = 0}$$

With eigenfunction meaning a solution to $-X'' = \lambda X$, we have the theorem

> With symmetric boundary conditions, any two eigenfunctions that correspond to distinct eigenvalues are orthogonal. Then if any function is expanded in a series of these eigenfunctions, the coefficients are determined.

Proof: Take two different eigenfunctions $X_1(x)$ and $X_2(x)$ with $\lambda_1 \neq \lambda_2$. Write Green's second identity

$$\int_a^b (-X_1''X_2 + X_1X_2'')\,\mathrm{d}x = (-X_1'X_2 + X_1X_2')\int_a^b$$

Because the boundary conditions are symmetric, the right hand side vanishes, and we obtain that $(X_1, X_2) = 0$.

If $X_n(x)$ denotes eigenfunction with eigenvalue λ_n and if

$$\boxed{\phi(x) = \sum_n A_nX_n(x)}$$

is a convergent series with constants A_n, then

$$(\phi, X_m) = \left(\sum_n A_n X_n, X_m\right) = \sum_n A_n(X_n, X_m) = A_m(X_m, X_m) \text{ by orthogonality}$$

Denote $c_m = (X_m, X_m)$, and we have

$$\boxed{A_m = \frac{(\phi, X_m)}{c_m}}$$

Two important details:

- Later we will investigate the convergence of such series. So far we have assumed convergence.
- If there are two functions with $\lambda_1 = \lambda_2$, they are not necessarily orthogonal, but they can be made so with the Gram-Schmidt procedure.

3.4.1 Complex Eigenvalues

In complex function space, we define

$$\boxed{(f, g) = \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x}$$

Where the bar denotes the complex conjugate. It remains that orthogonal g and f have $(f, g) = 0$. In the complex case, we have symmetric (or hermitian) boundary conditions if

$$f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\Big|_a^b = 0$$

And we likewise have the theorem

> With symmetric boundary conditions, any two eigenfunctions that correspond to distinct eigenvalues are orthogonal. Then if any function is expanded in a series of these eigenfunctions, the coefficients are determined. Additionally, these eigenvalues are *real*, and the eigenfunctions can be chosen real valued.

3.4.2 Negative Eigenvalues

Most the eigenvalues we have seen so far have been positive. When is this not the case?

With symmetric boundary conditions, there are *no negative eigenvalues* if

$$f(x)f'(x)\Big|_{x=a}^{x=b} \leq 0$$

for all real valued $f(x)$ satisfying the boundary conditions.

Note that all of these theorems are simply theorems concerning real symmetric matrices.

3.5 Even, Odd, Periodic Functions

A function is called **periodic** if there is a number $p > 0$ (the period) such that

$$\phi(x+p) = \phi(x) \text{ for all } x$$

If a function is only defined on an interval $-l < x < l$ of length p, we may perform a **periodic extension** of that function:

$$\phi_{\text{per}}(x) = \phi(x - 2lm) \text{ for } -l + 2lm < x < +l + 2lm$$

for all integers m. This effectively creates a periodic function out of $\phi(x)$ with period p. This extension will have jumps at the endpoints $x = l + 2lm$ unless $\lim_{x \to -l^-} \phi(x) = \lim_{x \to +^+} \phi(x)$.

The full Fourier series has period $2l$, and we may consider the Fourier series of an arbitrary function to be either the expansion of that function on the interval $(-l, l)$, or an expansion of a periodic function of period $2l$ defined on $-\infty < x < \infty$.

A function $\phi(x)$ is called **even** if

$$\phi(-x) = \phi(x)$$

I.e., te graph $y = \phi(x)$ is symmetric about the y axis. An **odd** function $\phi(x)$ satisfies

$$\phi(-x) = -\phi(x)$$

for an odd function to be continuous, we require

$$\phi(0) = 0$$

Since this yields $\phi(0) = -\phi(0)$, which is only true for $\phi(0) = 0$.

Some properties of the parity (even/oddness) of a function:

- A monomial x^n is even if n is even and odd if n is odd.
- The products of functions follow the usual rules:

$$\begin{aligned} \text{even} \times \text{even} &= \text{even} \\ \text{odd} \times \text{odd} &= \text{even} \\ \text{odd} \times \text{even} &= \text{odd} \end{aligned}$$

- The sum of even functions is even, and the sum of odd functions is odd.
- The sum of an even and odd function can be anything. Let $f(x)$ be any function defined on $(-l, l)$. Let

$$\begin{aligned} \phi(x) &= \frac{1}{2}[f(x) + f(-x)] \\ \psi(x) &= \frac{1}{2}[f(x) - f(-x)] \end{aligned}$$

  Then $f(x) = \phi(x) + \psi(x)$, $\phi(x)$ is even, and $\psi(x)$ is odd. These are the even and odd parts of f, respectively. E.g.,

$$e^x = \cosh x + \sinh x$$

  Where cosh is even and sinh is odd.

- Integration and differentiation change the parity of a function. If $\phi(x)$ is even, then $\mathrm{d}\phi/\mathrm{d}x$ and $\int \phi \, \mathrm{d}x$ ara odd.

3.6 Fourier Series and Boundary Conditions

The Fourier sine series is a sum of odd functions. Then the convergence of its sum is also odd. Likewise, each of its terms has period $2l$, and, as such, its sum likewise is periodic with period $2l$. Then the Fourier sine series is an expansion of an arbitrary function that is odd and has period $2l$ defined on $-\infty < x < \infty$.

Likewise since all cosine functions are even with period $2l$, the Fourier cosine series is an expansion of an arbitrary function that is even and has period $2l$ defined on $-\infty < x < \infty$.

- $u(0,t) = u(l,t) = 0$: Dirichlet boundary conditions correspond to the odd extension.
- $u_x(0,t) = u_x(l,t)$: Neumann boundary conditions correspond to the even extension.
- $u(l,t) = u(-l,t), u_x(l,t) = u_x(-l,t)$: Periodic boundary conditions correspond to the periodic extension.

3.7 Completeness

We will explore different manners in which functions may converge. Theorems 2, 3, and 4 state sufficient conditions on a function $f(x)$ such that its Fourier series converge to it in these three senses.

Consider the eigenvalue problem

$$X'' + \lambda X = 0 \text{ in } (a,b) \text{ with any symmetric } BC$$

Theorem 1: There are an infinite number of eigenvalues forming a sequence $\lambda_n \to \infty$.

Now we turn to proving some theorems regarding convergence. This is important because not all integrable $f(x)$ have converging Fourier series.

3.7.1 Three Notions of Convergence

1. **Pointwise Convergence:** An infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f(x)$ in (a,b) if for each $a < x < b$
$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \text{ as } N \to \infty$$
2. **Mean-square convergence:** A series converges in the mean square sense (or L^2 sense) if
$$\int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(X) \right|^2 \mathrm{d}x \to 0 \text{ as } N \to \infty$$
3. **Uniform Convergence:** A series converges uniformly to $f(x)$ in $[a,b]$ if
$$\max_{a \le x \le b} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \text{ as } N \to \infty$$

These three forms of convergence are listed in order of strength. Uniform convergence is stronger than mean-square convergence and pointwise convergence.

3.7.2 Convergence Theorems

Let $f(x)$ be any function defined on $a \leq x \leq b$. Consider the Fourier series with any given symmetric boundary conditions.

Theorem 2, Uniform Convergence: The Fourier series $\sum A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

1. $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous on $[a, b]$.
2. $f(x)$ satisfies the given boundary conditions.

Theorem 3, L^2 Convergence: The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is such that

$$\int_a^b |f(x)|^2 \, \mathrm{d}x \ \text{ is finite.}$$

Two more definitions are in order. A function $f(x)$ has a **jump discontinuity** at a point $x = c$ if $\lim_{x\to c+} f(x) \neq \lim_{x\to c-}$. The value of the jump discontinuity is $f(c+) - f(c-)$.

A function is called **piecewise continuous** on $[a, b]$ if it is continuous at all but a finite number of points and has jump discontinuities at these points.

Theorem 4, Pointwise Convergence of a Classical Fourier Series:

1. The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b) provided that $f(x)$ is continuous on $[a, b]$ and $f'(x)$ is piecewise continuous on $[a, b]$.

2a. More generally if $f(x)$ itself is only piecewise continuous on $[a, b]$ and $f'(x)$ is also piecewise continuous on $[a, b]$, then the classical Fourier series converges at every $x \in (-\infty, \infty)$. The sum is

$$\sum_n A_n X_n(x) = \frac{1}{2}[f(x+) + f(x-)] \text{ for all } a < x < b$$

This includes the points at which the function is discontinuous.

Alternatively:

3b. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical Fourier series converges to

$$\frac{1}{2}[f(x+) + f(x-)] \text{ for } x \in (-\infty, \infty)$$

Chapter 4

Harmonic Functions

4.1 Laplace's Equation

If a diffusion or wave process is time independent, $u_t = 0, u_{tt} = 0$ and we have the **Laplace equation**:

$$\boxed{\Delta u = 0 \text{ or } \nabla^2 u = 0}$$

Expanding, this gives us

$$\begin{aligned} u_{xx} &= 0 \text{ in one dimension} \\ \nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} &= 0 \text{ in two dimensions} \\ \nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} + u_{zz} &= 0 \text{ in three dimensions} \end{aligned}$$

A solution of the Laplace equation is called a **harmonic function**. The inhomogeneous Laplace equation is **Poisson's equation**

$$\boxed{\Delta u = f}$$

Some applications of Laplace's and Poisson's equations are

1. **Electrostatics:** From Maxwell's equations, $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = 4\pi\rho$ with $\rho =$ charge density. The curl of the electric field being zero implies $\mathbf{E} = -\nabla\phi$ for some scalar electric potential ϕ. Then

$$\Delta\phi = -\nabla \cdot \nabla\phi = -\nabla \cdot \mathbf{E} = -4\pi\rho$$

2. **Steady Fluid Flow:** Assume that the flow is irrotational so that $\nabla \times \mathbf{v} = 0$, where $\mathbf{v} = \mathbf{v}(x, y, z)$ is the velocity at the position (x, y, z). Assume the fluid is incompressible, and assume no sources or sinks. Then $\nabla \cdot \mathbf{v} = 0$. Thus $\mathbf{v} = -\nabla\phi$ for some potential ϕ, and $\Delta\phi = -\nabla \cdot \mathbf{v} = 0$.

3. **Analytic Functions of a Complex Variable**

4. **Brownian Motion:** Consider Brownian motion in a container D. Particles inside D move randomly until they hit the boundary, at which point they stop. Divide the boundary arbitrarily into two pieces C_1 and C_2. Let $u(x, y, z)$ be the probability that a particle that begins at the point (x, y, z) stops at some point of C_1. Then a particle that starts in C_1 is guaranteed to stay in C_1, and guaranteed to never end in C_2. We then obtain

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = 1 \text{ on } C_1 \\ u = 2 \text{ on } C_2 \end{cases}$$

4.2 The Maximum Principle

An open set is a set that contains none of its boundary points.

> Let D be a connected bounded open set in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let u be a harmonic function D that is continuous on $\overline{D} = D \cup (\text{bdy } D)$. Then the maximum and minimum values of u are attained on bdy D and nowhere inside.

Use the shorthand $\mathbf{x} = (x, y)$ in $\mathbb{R}^2$ and $\mathbf{x} = (x, y, z)$ in$\mathbb{R}^3$. The maximum principle then asserts that there are points $\mathbf{x}_M$ and $\mathbf{x}_m$ on bdy D such that

$$u(\mathbf{x}_m) \le u(\mathbf{x}) \le u(\mathbf{x}_M)$$

for all $\mathbf{x} \in D$.

Proof: If there were a maximum point inside D, at that point $u_{xx} \le 0$ and $u_{yy} \le 0$, so $u_{xx} + u_{yy} \le 0$. At most maximum points, $u_{xx} < 0$ and $u_{yy} < 0$ such that we contradict Laplace's equation. But it is possible $u_{yy} = u_{xx} = 0$. Let us perturb the system by a $\varepsilon > 0$, such that

$$v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$$
$$\Delta v = \Delta u + \varepsilon \Delta |\mathbf{x}|^2 = 0 + 4\epsilon > 0 \text{ in } D$$

By the second derivative test, $\Delta v = v_{xx} + v_{yy} \le 0$ at a maximum (in the two dimensional case, for example). Then we have a contradiction and $v(\mathbf{x})$ has no interior maximum in D.

Since $v(\mathbf{x})$ is continuous, it must have some maximum in the closure $\overline{D} = D \cup \text{bdy } D$. Let's call that a point $\mathbf{x}_0 \in \text{bdy } D$. Then, for all $\mathbf{x} \in D$,

$$u(\mathbf{x}) \le v(\mathbf{x}) \le v(\mathbf{x}_0) = u(\mathbf{x}_0) + \varepsilon|\mathbf{x}_0|^2 \le \max_{\text{bdy } D} u + \varepsilon l^2$$

where l is the greatest distance from bdy D to the origin (coming from $|\mathbf{x}|^2$). Take the limit as $\varepsilon \to 0$ and we have

$$u(\mathbf{x}) \le \max_{\text{bdy } D} u \text{ for all } \mathbf{x} \in D$$

We may set $w = -u$ and this proof still works, so the minimum also occurs on the boundary.

4.2.1 Uniqueness of the Dirichlet Problem

Suppose

$$\begin{cases} \Delta u = f \text{ in } D \\ u = h \text{ on bdy } D \end{cases} \quad \text{and} \quad \begin{cases} \Delta v = f \text{ in } D \\ v = h \text{ on bdy } D \end{cases}$$

Show $w = u - v = 1$ in D. Then $\Delta w = 0$ in D and $w = 0$ on bdy D. By the maximum principle

$$0 = w(\mathbf{w}_m) \le w(\mathbf{x}) \le w(\mathbf{x}_M) = 0 \text{ for all } \mathbf{x} \in D$$

Thus the maximum and minimum of $w(\mathbf{x})$ are zero and $w = 0 \Rightarrow v = u$.

4.3 Invariance

4.3.1 Invariance in Two Dimensions

The Laplace equation is invariant under rotations and translations. A **translation** is a change of variables

$$\begin{cases} x' = x + a \\ y' = y + b \end{cases}$$

Invariance under translations means that $u_{xx} + u_{yy} = u_{xx} + u_{yy}$. By the chain rule:

$$\begin{cases} \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} = \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}\frac{\partial y}{\partial y'} = \frac{\partial}{\partial y} \end{cases} \Rightarrow \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

A **rotation** by an angle α is a change of variables

$$\begin{cases} x' = x\cos\alpha + y\sin\alpha \\ y' = -x\sin\alpha + y\cos\alpha \end{cases}$$

By the chain rule, we have

$$\begin{aligned} u_x &= u_{x'}\cos\alpha - u_{y'}\sin\alpha \\ u_y &= u_{x'}\sin\alpha + u_{y'}\cos\alpha \\ u_{xx} &= (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{x'}\cos\alpha - (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{y'}\sin\alpha \\ u_{yy} &= (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{x'}\sin\alpha + (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{y'}\cos\alpha \end{aligned}$$

Adding,

$$\begin{aligned} u_{xx} + u_{yy} &= (u_{x'x'} + u_{y'y'})(\cos^2\alpha + \sin^2\alpha) + u_{x'y'} \cdot 0 \\ &= u_{x'x'} + u_{y'y'} \end{aligned}$$

4.3.2 Invariance in Three Dimensions

The three-dimensional Laplacian is invariant under all translations and rotations. Any **rotation** in three dimensions is given by the matrix multiplication is given by

$$\mathbf{x}' = B\mathbf{x}$$

where B is an *orthogonal* matrix; i.e., it does not alter the length of vectors it transforms. The Laplacian is given by $\Delta u = \sum_{i=1}^{3} u_{ii} = \sum_{i,j=1}^{3} \delta_{ij} u_{ij}$ where δ is the Kronecker delta. Therefore

$$\begin{aligned} \Delta u &= \sum_{k,l}\left(\sum_{i,k} b_{ky}\delta_{ij}b_{lj}\right)u_{k'l'}i \\ &= \sum_{k,l}\delta_{kl}u_{k'l'} \\ &= \sum_{k} u_{k'k'} \end{aligned}$$

since the new coefficient matrix is

$$\sum_{i,j} b_{ki}\delta_{ij}b_{lj} = \sum_{i} b_{ki}b_{li} = (B^T B)_{kl} = \delta_{kl}$$

So, in the primed coordinates,

$$\Delta u = u_{x'x'} + u_{y'y'} + u_{z'z'}$$

4.4 Radially Symmetric Harmonic Functions

4.4.1 In Two Dimensions

Rotation invariance suggests the usage of polar coordinates and a radially symmetric (or rotationally invariant) solution $u(r)$.

The two-dimensional Laplacian in polar coordinates is

$$\boxed{\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}}$$

With $u = u(r)$, we have Laplace's equation

$$0 = u_{rr} + \frac{1}{r}u_r$$

This is an ODE, which we may solve by

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r &= 0 \\ ru_{rr} + u_r &= 0 \\ (ru)_r &= 0 \\ u_r &= \frac{c_1}{r} \\ u &= c_1 \log r + c_2 \end{aligned}$$

This will be an important result later on.

4.4.2 In Three Dimensions

The Laplacian operator in spherical coordinates (r, θ, ϕ) is

$$\boxed{\Delta_3 = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right]}$$

If $u = u(r)$, we have partial derivatives with respect to θ and ϕ are zero and

$$\begin{aligned} u_{rr} + \frac{2}{r}u_r &= 0 \\ r^2 u_{rr} + 2ru_r &= 0 \\ (r^2 u_r)_r &= 0 \\ r^2 u_r &= c_1 \\ u_r &= \frac{c_1}{r} \\ u &= -\frac{c_1}{r} + c_2 \end{aligned}$$

Which is also an important result.

4.5 Harmonic Functions in Rectangles and Cubes

More general solutions are available to us. The procedure is

1. Look for separated solutions of the PDE.
2. Apply in the homogeneous boundary conditions to get the eigenvalues.
3. Sum the series.
4. Put in the inhomogeneous initial or boundary conditions.

Example Take the equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,b) = g(x) \\ u(x,0)_y = 0 \\ u(0,y) = 0 \\ u_x(a,y) = 0 \end{cases}$$

Assume separability

$$u(x,y) = X(x)Y(y)$$

Plug into $\Delta u = 0$,

$$\begin{aligned} X''(x)Y(y) + X(x)Y''(y) &= 0 \\ \frac{X''}{X} + \frac{Y''}{Y} &= 0 \end{aligned}$$

As usual, we only have $X''/X = -Y''/Y$ if this equation is equal to some constant λ. Then we have the eigenvalue problem

$$\begin{cases} X'' = -\lambda X \\ Y'' = \lambda Y \end{cases}$$

With initial conditions, we have

$$\begin{cases} X(0)Y(y) = 0 \\ X'(a)Y(y) = 0 \end{cases} \Rightarrow \begin{cases} X(0) = 0 \\ X'(a) = 0 \end{cases}$$

We turn to showing the eigenvalue is positive.

$$-X'' = \lambda X$$

Multiply both sides by X and integrate along the boundary:

$$\begin{aligned} \int_0^a -X'' \cdot X \,\mathrm{d}x &= \int_0^a \lambda X^2 \,\mathrm{d}x \\ \int_0^a -(X'X)' + (X')^2 \,\mathrm{d}x &= \lambda \int_0^a X^2 \,\mathrm{d}x \\ -(X'X)\Big|_0^a + \int_0^a (X')^2 \,\mathrm{d}x &= \lambda \int_0^a X^2 \,\mathrm{d}x \end{aligned}$$

By the boundary conditions, $(X'X)\Big|_0^a$ is zero.

$$\int_0^a (X')^2 \,\mathrm{d}x = \lambda \int_0^a X^2 \,\mathrm{d}x$$

Both the expressions in the integrals are squared, so λ must be positive. Then we may set $\lambda = \beta^2$ and

$$X = A\cos\beta x + B\sin\beta x$$

Applying the boundary conditions, we have

$$\begin{cases} A = 0 \\ -A\beta\sin\beta a + B\beta\cos\beta a = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ \cos\beta a = 0 \end{cases}$$

Thus

$$\beta = \frac{1}{a}\left(\frac{\pi}{2} + n\pi\right) = \left(\frac{1}{2} + n\right)\frac{\pi}{a}$$

and

$$X_n = \sin\left[(n + \frac{1}{2})\frac{\pi}{a}x\right] \text{ for } n = 0, 1, 2, \ldots$$

Turning to Y,

$$\begin{cases} Y'' = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{a^2} Y \\ Y'(0) = 0 \end{cases}$$

We have a solution of the form

$$Ae^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}y} + Be^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}y}$$

Applying the boundary conditions, we have

$$\begin{aligned} Y' = A\left(n + \frac{1}{2}\right)\frac{\pi}{a}e^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}y} - B\left(n + \frac{1}{2}\right)\frac{\pi}{a}e^{-\left(\frac{1}{2}+n\right)\frac{\pi}{a}} = 0 \\ A - B = 0 \\ A = B \end{aligned}$$

Then

$$U(x,y) = \sum_{n=1}^{\infty} X_n Y_n = \sum_{n=0}^{\infty} A_n \sin\left[\frac{\pi}{a}\left(n + \frac{1}{2}\right)x\right]\left(e^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}y} + e^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}y}\right)$$

With the boundary condition $g(x)$,

$$g(x) = \sum_{n=0}^{\infty} A_n \sin\left[\frac{\pi}{a}\left(n + \frac{1}{2}\right)x\right]\left(e^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}b} + e^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}b}\right)$$

We find A_n by taking the inner product:

$$\begin{aligned} \int_0^a g(x)\sin\left[\frac{\pi}{a}\left(n + \frac{1}{2}\right)x\right]\mathrm{d}x = A_n \int_0^a \left(\sin\left[\frac{\pi}{a}\left(n + \frac{1}{2}\right)\right]\right)^2 \cdot \left(e^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}b} + e^{\left(\frac{1}{2}+n\right)\frac{\pi}{a}b}\right)\mathrm{d}x \\ A_n = \frac{2}{a}\left(e^{(n+\frac{1}{2})\frac{\pi b}{a}} + e^{-(n+\frac{1}{2})\frac{\pi b}{a}}\right)\int_0^a g(x)\sin\left[\frac{\pi}{a}\left(n + \frac{1}{2}\right)\right]x\,\mathrm{d}x \end{aligned}$$

EXAMPLE We take an example in $\mathbb{R}^3$:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0 \\ u(a,y,z) = g(y,z) \\ u(0,y,z) = u(x,0,z) = u(x,a,z) = u(x,y,0) = u(x,y,a) \end{cases}$$

Assume separability once again

$$u = X(x)Y(y)Z(z)$$

Plug this into the equation:

$$X''YZ + XY''Z + XYZ'' = 0 \tag{4.1}$$

With conditions

$$X(0) = Y(0) = Z(0) = Y(a) = Z(a) = 0$$

This yields the eigenvalue problem (we choose λ_1 positive for reasons we will soon see):

$$\begin{cases} X'' = \lambda_1 X \\ Y'' = -\lambda_2 Y \\ Z'' = -\lambda_3 Z \end{cases}$$

One can show $\lambda_1, \lambda_2 > 0$, but we omit this process here. We start with Y, which has solution

$$Y = A\cos\beta_2 y + B\sin\beta_2 y$$

Applying the boundary conditions,

$$\begin{aligned} Y(0) = 0 &= A \\ Y &= B\sin\beta_2 t \\ \beta_2 a &= n\pi \\ \beta_2 &= \frac{n\pi}{a} \end{aligned}$$

Notice that, although Z can be solved identically, it has a different eigenvalue, and we in turn use a different integer when solving for it:

$$\begin{aligned} Z = A\cos\beta_3 z &+ B\sin\beta_3 z \\ \beta_3 &= \frac{m\pi}{a} \end{aligned}$$

Equation (4.1) requires:

$$\lambda_1 = \lambda_2 + \lambda_3 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a}$$

With that, we turn to X:

$$\begin{aligned} X'' &= \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a}\right)X \\ X(x) &= A\sinh\left(\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a}}x\right) \end{aligned}$$

Then the solution is

$$u(x,y,z) = \sum_{m,n}^{\infty} A_{mn}\sinh\left(\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a}}x\right)\sin\frac{m\pi}{a}y\sin\frac{n\pi}{a}z$$

For simplicity, say $a = \pi$. Let us substitute the inhomogeneous condition at $x = a = \pi$.

$$g(y,z) = \sum_{m,n}^{\infty} A_{mn}\sinh\left(\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a}}\pi\right)\sin ny\sin mz$$

Section 3.3 showed that any such problem has orthogonal basis functions. Then we may take this as a *double* Fourier series with

$$A_{mn} = \frac{(g,X)}{(X,X)} = \frac{4}{\pi^2 \sinh\sqrt{m^2+n^2}\pi}\int_0^\pi\int_0^\pi g(y,z)\sin my \sin nz \,\mathrm{d}y\,\mathrm{d}z$$

Where we used the fact that

$$\int_0^\pi\int_0^\pi (\sin my \sin nz)^2 \,\mathrm{d}y\,\mathrm{d}z = \frac{\pi^2}{4}$$

4.6 Harmonic Functions in Polar Coordinates (Poisson's Formula)

We consider polar coordinates and a circular domain:

$$\begin{cases} u_{xx} + u_{yy} = 0 \text{ for } x^2+y^2 < a^2 \\ u = h(\theta) \text{ for } x^2+y^2 = a^2 \end{cases}$$

with radius a and any boundary data $h(\theta)$

Assume separability

$$u = R(r)\Theta(\theta)$$

Use the polar Laplacian and substitute in the :

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Divide by $R\Theta$ and multiply by r^2:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

Then we have the eigenvalue problem

$$\begin{cases} \Theta'' = -\lambda\Theta \\ r^2R'' + rR' = \lambda R \end{cases}$$

We've reduced this to two ordinary differential equations. For $\Theta(\theta)$ we should enforce a periodic boundary condition:

$$\Theta(\theta + 2\pi) = \Theta(\theta) \text{ for } -\infty\theta\infty$$

Thus, assuming positive eigenvalues $\lambda = n^2$, we have

$$\Theta(\theta) = A\cos n\theta + B\sin n\theta \text{ for } n = 1,2,3,\ldots$$

There is also a solution $\lambda = 0$ with $\Theta(\theta) = A$.

We turn to the equation for R. This is an Euler type equation with solutions of the form $R(r) = r^\alpha$. Since $\lambda = n^2$, for $n \neq 0$, we have

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0$$

where $\alpha = \pm n$. Then we have

$$R(r) = Cr^n + Dr^{-n}$$

And the solution is

$$u = \left(Cr^n + \frac{D}{r^n}\right)(A\cos n\theta + B\sin n\theta)$$

If $n = 0$, we have the solution $R = \log r$, and thus

$$u = C + D \log r$$

At $r = 0$, solutions $\log r$ and $1/r^n$ are undefined. We *reject* these solutions and keep only the "nice" solutions:

$$\boxed{u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)}$$

Use the inhomogeneous boundary condition at $r = a$, we have

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

Which is a Fourier series the coefficients of which are

$$\boxed{A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi \, \mathrm{d}\phi}$$

$$\boxed{B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi \, \mathrm{d}\phi}$$

4.7 Poisson's Formula

The just-obtained result can actually be summed *explicitly.* We plug the Fourier coefficients into the formula for u,

$$\begin{aligned} u(r,\theta) &= \int_0^{2\pi} h(\phi) \frac{\mathrm{d}\phi}{2\pi} + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} h(\phi) [\cos n\phi \cos n\theta + \sin n\phi \sin n\theta] \\ &= \int_0^{2\pi} h(\phi) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right] \frac{\mathrm{d}\phi}{2\pi} \end{aligned}$$

The term in brackets can actually be expressed as a series of complex numbers:

$$1 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} = 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} = \frac{a^2 - r^2}{a^2 2ar \cos(\theta - \phi) + r^2}$$

Then u becomes **Poisson's Formula**:

$$\boxed{u(r,\theta) = (a^2 - r^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{\mathrm{d}\phi}{2\pi}}$$

Which is, as with the solution to the heat equation, a kernel convoluted with the initial condition. This may replace our pervious solution for the harmonic function inside a circle.

We may write Poisson's formula. We write the position vector $\mathbf{x} = (r, \theta)$:

$$\begin{aligned} \mathbf{x} &= \text{polar coordinates } (r, \theta) \\ \mathbf{x}' &= \text{polar coordinates } (a, \theta) \end{aligned}$$

The origin and the points $\mathbf{x}$ and $\mathbf{x}'$ form a triangle with sides $r = |\mathbf{x}|, a = |\mathbf{x}'|, |\mathbf{x} - \mathbf{x}'|$. Then Poisson's formula takes the form

$$\boxed{u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{x}'|} \frac{u(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} \, \mathrm{d}s'}$$

for $\mathbf{x} \in D$ where $u(\mathbf{x}') = h(\phi)$, and $ds' = a \, \mathrm{d}\phi$. This is a line integral along a circle of radius a!

Here's a careful statement of Poisson's formula:

Theorem 1: Let $h(\phi) = u(\mathbf{x}')$ be any continuous function on the circle $C = \operatorname{bdy} D$. Then the Poisson formula provides the *only* harmonic function in D for which

$$\lim \mathbf{x} \to \mathbf{x}_0 u(\mathbf{x}) = h(\mathbf{x}_0) \text{ for all } \mathbf{x}_0 \in C$$

This means that $u(\mathbf{x})$ is a continuous function on $\overline{D} = D \cup C$. It is all differentiable to all orders inside D. This is proved in 4.7.4

Some remarks:

1. Notice that this means that the solution to the Laplace equation is completely determined by the condition $h(\theta)$!
2. What if the domain is not a circle? Turns out that, if the domain is smooth enough,

   $$u = \int_{\operatorname{bdy}\Omega} P(x, y) g(y) \, \mathrm{d}Sy \ \text{ for } y \in \operatorname{bdy} Omega$$

   where this is a line integral along the boundary.
3. $u(\mathbf{x}) \to g(\mathbf{x}')$ only when $\mathbf{x}$ approaches $\mathbf{x}'$ non-tangentially.

4.7.1 Mean Value Property

Let u be a harmonic function in a disk D, continuous in its closure $\overline{D}$. Then the value of u at the center of D equals the average of u on its circumference.

Proof: Choose coordinates with the origin 0 at the center of the circle. Put $\mathbf{x} = 0$ in Poisson's formula. Then

$$u(0) = \frac{a^2}{2\pi a} \int_{|\mathbf{x}'|} \frac{u(\mathbf{x}')}{a^2} \, \mathrm{d}s'$$

Which is the average of u on the circumference $|\mathbf{x}'| = a$.

4.7.2 Maximum Principle

This was stated and partially proved in section 4.2. We now provide a complete proof of the **strong** form.

Let $u(\mathbf{x})$ be harmonic in D. The maximum is attained somewhere (by the continuity of u on $\overline{D}$), say, at $\mathbf{x}_M \in \overline{D}$. We have to show that $\mathbf{x}_M \notin D$ unless $u \equiv$ constant. We know that, by definition,

$$u(\mathbf{x}) \leq u(\mathbf{u}_M) = M \text{ for all } \mathbf{x} \in D$$

(Chain of balls method.) Draw a small circle centered around $\mathbf{x}_M$ entirely contained in D. By the mean value property $u(\mathbf{x}_M)$ is equal to its average around the circumference. Notice that the average is not greater than the maximum, so we have the string of inequalities

$$M = u(\mathbf{x}_M) = \text{average on circle } \leq M$$

Therefore $u(\mathbf{x}) = M$ for all $\mathbf{x}$ on the circumference. We can repeat this process for several circles around the domain with different centers. If these centers lie on the circumference of the previous circle, we have that every center of those circumferences are also $\mathbf{x}_M$. Then $u(\mathbf{x}) = M$ throughout D, and so $u =$ constant.

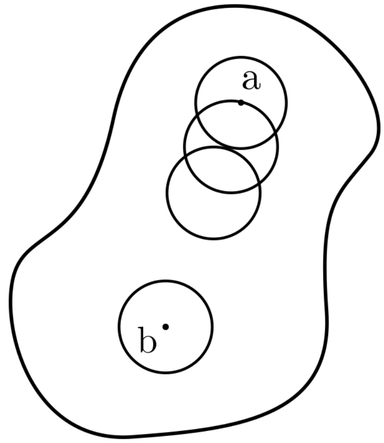

Figure 4.1: The chain of balls method.

4.7.3 Differentiability

Let u be a harmonic function in an open set D of the plane. Then $u(\mathbf{x}) = u(x, y)$ possesses all partial derivatives of all orders in D.

Show this for the case D is a disk with center at the origin. Observe the second form of Poisson's formula. The integrand is differentiable to all orders for $\mathbf{x} \in D$. Note that $\mathbf{x}' \in \text{bdy}\, D$ so that $\mathbf{x} \neq \mathbf{x}'$. By the theorem about differentiating integrals, we can differentiate under the integral sign. So $u(\mathbf{x})$ is differentiable to any order in D.

Second, let D be any domain at all, and let $\mathbf{x}_0 \in D$. Let B be a disk contained in D with center at $\mathbf{x}_0$. We just showed that $u(\mathbf{x})$ is differentiable inside B, and hence at $\mathbf{x}_0$. But $\mathbf{x}_0$ is an arbitrary point in D. So u is differentiable to all orders at all points of D.

4.7.4 Poisson Kernel, Proof of the Limit

Begin by writing Poisson's formula in the form

$$u(r, \theta) = \int_0^{2\pi} P(r, \theta - \phi) h(\phi) \frac{\mathrm{d}\phi}{2\pi}$$

for $r < a$, where

$$P(r, \theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n\theta$$

is called the **Poisson kernel**.

The Poisson kernel has the following properties

1. $P(r, \theta) > 0$ for $r < a$. This property follows from the observation that $a^2 - 2ar\cos\theta + r^2 \geq a^2 - 2ar + r^2 = (a - 2) > 0$. That is, the Poisson kernel is positive inside the circle of radius a.

2. $$\int_0^{2\pi} P(r, \theta) \frac{\mathrm{d}\theta}{2\pi} = 1$$

   This property follows from integrating the Poisson kernel, since $\int_0^{2\pi} \cos n\theta \, \mathrm{d}\theta = 0$ for any integer n.

3. $P(r,\theta)$ is a harmonic function inside the circle. This follows from the fact that each term $(r/a)^n \cos n\theta$ is harmonic and thus the entire sum is as well.

4. The kernel is not well-defined on $r = a, \theta = 0$. Take the limit

$$\lim_{r\to a} \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = \frac{0}{2a^2(1-\cos\theta)} = 0$$

So at an arbitrary θ, the kernel vanishes at the boundary. However, when $\theta = 0$, the

$$P(r,0) = \frac{a^2 - r^2}{a^2 - 2ar + r^2} = \frac{a^2 - r^2}{(a-r)^2}$$

Taking the limit,

$$\lim_{r\to a} \frac{a^2 - r^2}{(a-r)^2} = \lim_{r\to a} \frac{-2r}{2(a-r)(-1)} = \frac{-2r}{-0} = \infty$$

Then, given that we have the integral with respect to θ is unity, and all points other than $\theta = 0$ have $P(r,\theta) = 0$, we see that the Poisson kernel at the boundary is akin to the Dirac delta function.

Turn to proving the limit. Differentiate the Poisson formula under the integral sign to get

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= \int_0^{2\pi} \left(P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\theta\theta}\right)(r,\theta-\phi)h(\phi)\frac{\mathrm{d}\phi}{2\pi} \\ &= \int_0^{2\pi} 0 \cdot h(\phi)\,\mathrm{d}\phi \\ &= 0 \end{aligned}$$

Since the kernel is harmonic. Then u is harmonic in D. Now prove that the limit is true. Fix an angle θ_0 and consider a radius r near a. Then we estimate the difference

$$u(r,\theta_0) - h(\theta_0) = \int_0^{2\pi} P(r,\theta_0 - \phi)[h(\phi) - h(\theta_0)]\frac{\mathrm{d}\phi}{2\pi} \tag{4.2}$$

By property 2. $P(r,\theta)$ is concentrated near $\theta = 0$. That is, for $\delta \le \theta \le 2\pi - \delta$,

$$|P(r,\theta)| = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = \frac{a^2 - r^2}{(a-r)^2 + 4ar\sin^2(\theta/2)} < \varepsilon$$

For r sufficiently close to a. This means that for each small $\delta > 0$ and each small $\varepsilon > 0$ the above equation is true for r sufficiently close to a. Plugging this into (4.2) and using property 1,

$$|u(r,\theta_0) - h(\theta_0)| \le \int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta_0-\phi)\varepsilon\frac{\mathrm{d}\phi}{2\pi} + \varepsilon\int_{|\phi-\theta_0|>\delta} |h(\phi) - h(\phi_0)|\frac{\mathrm{d}\theta}{2\pi}$$

for r sufficiently close to a. The ε in the first integral came from the continuity of h. In fact, there is some $\delta > 0$ such that $|h(\phi) - h(\theta_0)| < \varepsilon$ for $|\phi - \theta_0| < \delta$. Since the function $|h| \le H$ for some constant H and in view of property 2, we deduce from the above equation that

$$|u(r,\theta_0) - h(\theta_0)| \le (1 + 2H)\varepsilon$$

provided r is sufficiently close to a. This is exactly the limit.

Chapter 5

Green's Identities and Green's Functions

5.1 Green's First Identity

We will use $\mathbb{R}^3$ as our space, but all of the following holds for $\mathbb{R}^n$. Recall the divergence theorem

$$\iiint_D \nabla \cdot \mathbf{F} \, \mathrm{d}\mathbf{x} = \iint_{\mathrm{bdy}\, D} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$$

Start from the product rule:

$$\begin{cases} (vu_x)_x = v_x u_x + v u_{xx} \\ (vu_y)_y = v_y u_y + v u_{yy} \\ (vu_z)_z = v_z u_z + v u_{zz} \end{cases}$$

Summing, we obtain

$$\nabla \cdot (v \nabla u) = \nabla v \cdot \nabla u + v \nabla u$$

Integrate on both sides

$$\iiint_D \nabla \cdot (v \nabla u) \, \mathrm{d}\mathbf{x} = \iiint_D \nabla v \cdot \nabla u \, \mathrm{d}\mathbf{x} + \iiint_D v \nabla u \, \mathrm{d}\mathbf{x}$$

Apply the divergence theorem on the left side to obtain

$$\boxed{\iint_{\mathrm{bdy}\, D} v \frac{\partial u}{\partial n} \, \mathrm{d}S = \iiint_D \nabla v \cdot \nabla u \, \mathrm{d}\mathbf{x} + \iiint_D v \nabla u \, \mathrm{d}\mathbf{x}} \tag{5.1}$$

where we used $\partial u / \partial n \equiv \nabla u \cdot \mathbf{n}$. This is effectively high-dimensional integration by parts. Reducing the above to one dimension should make this clear; simply recall that the boundary integral in one dimension is simply the value at the endpoints.

5.2 Mean Value Property

In three dimensions, we have that the average value of a harmonic function over any sphee equals its value at the center. Let's prove this using Green's first identity. Let D be a ball of radius a. Then $\mathrm{bdy}\, D$ is the surface of this ball. Let $\Delta u = 0$ in any region that contains D and bdy D. For a sphere, $\mathbf{n}$ points directly away from the origin in the direction of r such that

$$\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u = \frac{\mathbf{x}}{r} \cdot \nabla u = \frac{x}{r} u_x + \frac{y}{r} u_y + \frac{z}{r} u_z = \frac{\partial u}{\partial r}$$

We use Green's first identity with $v = 1$ to obtain

$$\iint_{\text{bdy } D} \frac{\partial u}{\partial n} \, \mathrm{d}S = \iiint_D \Delta u \, \mathrm{d}\mathbf{x} \Rightarrow \iint_{\text{bdy } D} \frac{\partial u}{\partial r} \, \mathrm{d}S = 0$$

Write this integral explicitly in spherical coordinates

$$\int_0^{2\pi} \int_0^{\pi} u_r(a, \theta, \phi) a^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = 0$$

with $r = a$ on bdy D. Divide by $4\pi a^2$, the area of the boundary. Since this is all valid for any a, we're going to rename $a = r$. Then we pull the partial derivative operator out of the integral to obtain

$$\frac{\partial}{\partial r} \left[\frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} u(r, \theta, \phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \right]$$

Then the expression in the brackets is independent of r. This is exactly the average value of u on the sphere. As we take the limit $r \to 0$, the value of this integral approaches exactly $u(0)$:

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} u(0) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = u(0)$$

But if the integral is independent of r,

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} u(r, \theta, \phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = u(0)$$

5.2.1 Maximum Principle

Using the ball and chain method, we can likewise

If D is any solid region, a nonconstant harmonic function in D cannot take its maximum value inside D, only on bdy D.

The **Hopf maximum principle** also shows that the outward normal derivative $\partial u/\partial n$ is strictly positive at a maximum point.

5.2.2 Uniqueness of Dirichlet Problem

Let's use a different method to prove 4.2.1's theorem. Use the energy method. Suppose two harmonic functions u_1, u_2 with the same boundary data. Define $u = u_1 - u_2$. Use Green's identity with both functions set to u. Then, since u is harmonic (Δ is linear),

$$\iint_{\text{bdy } D} u \frac{\partial u}{\partial n} \, \mathrm{d}S = \iiint_D |\nabla u|^2 \, \mathrm{d}\mathbf{x}$$

Since $u = 0$ on bdy D, the left side vanishes. Then

$$\iiint_D |\nabla u|^2 \, \mathrm{d}\mathbf{x} = 0$$

One can prove that this shows $|\nabla u|^2 = 0$. A function with a vanishing gradient must be constant. But this function is 0 on the boundary. Thus $u = 0$.

One can also prove that the Neumann problem is unique: $\Delta u = 0$ in D and $\partial u/\partial n = 0$ on bdy D.

5.2.3 Dirichlet's Principle

Among all the functions $w(\mathbf{x})$ in D satisfying

$$w = h(\mathbf{x}) \text{ on } \operatorname{bdy} D$$

The *lowest energy* occurs for the *harmonic* function that satisfies the above, where energy is defined

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 \,\mathrm{d}\mathbf{x}$$

In precise mathematical language: Let $u(x)$ be the unique harmonic function in D satisfying the boundary condition. Let $w(x)$ be any function in D that satisfies the boundary condition. Then

$$E[w] \geq E[u]$$

Proof: Let $v = u - w$ and expand the square in the integral.

$$\begin{aligned} E[w] &= \frac{1}{2}\iiint_D |\nabla(u-v)|^2 \,\mathrm{d}\mathbf{x} \\ &= E[u] - \iiint_D \nabla u \cdot \nabla v \,\mathrm{d}\mathbf{x} + E[v] \end{aligned}$$

Apply Green's first identity to the pair of functions u and v. Two of the terms are zero because $v = 0$ on $\operatorname{bdy} D$ and $\Delta u = 0$ in D. Then the middle term in the above equation is also zero. Thus

$$E[w] = E[u] + E[v]$$

Since v is squared in $E[v]$, then $E[v] \geq 0$ and $E[w] \geq E[u]$.

5.3 Green's Second Identity

Green's first identity does not change if we switch the order of u and v. So we write

$$\begin{aligned} \iint_{\operatorname{bdy} D} v\frac{\partial u}{\partial n}\,\mathrm{d}S &= \iiint_D \nabla v \cdot \nabla u \,\mathrm{d}\mathbf{x} + \iiint_D v\nabla u \,\mathrm{d}\mathbf{x} \\ \iint_{\operatorname{bdy} D} u\frac{\partial v}{\partial n}\,\mathrm{d}S &= \iiint_D \nabla u \cdot \nabla v \,\mathrm{d}\mathbf{x} + \iiint_D u\nabla v \,\mathrm{d}\mathbf{x} \end{aligned}$$

Then subtract these two equations to obtain

$$\boxed{\iiint_D (u\Delta v - v\Delta u)\,\mathrm{d}\mathbf{x} = \iint_{\operatorname{bdy} D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S}$$

which is **Green's second identity**.

A boundary condition is called **symmetric** for the operator Δ if the right side of Green's second identity vanishes for all pairs of functions that satisfy the boundary conditions. Each of the three classical boundary conditions (Dirichlet, Neumann, Robin) is symmetric. This is analogous to section 3.4.

5.3.1 Representation Formula

Consider $\Delta u = 0$ on $D \subseteq \mathbb{R}^3$. If $u = u(r)$, then recall we have the fundamental solution (from reducing Δ to a one dimensional operator)

$$u = -\frac{1}{4\pi}\frac{1}{r}$$

Let us shift the fundamental solution:

$$v = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}$$

v is not well-defined at $\mathbf{x}_0$, so we restrict ourselves to the domain excluding a small ball around $\mathbf{x}_0$: $D - B_\varepsilon$.

Let u be some harmonic function. We know v is harmonic. Applying Green's second identity yields

$$0 = \int_{\text{bdy } D} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S$$

Because there are two boundaries, the outer boundary and the boundary of the small ball, we split the integral of the two boundaries and flip the sign for the small ball (since it is an inner boundary):

$$0 = \int_{\text{bdy } D} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S - \int_{\text{bdy } B_\varepsilon} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S \tag{5.2}$$

Note that, for the small ball, $\partial/\partial n = \partial/\partial r$. Evaluate the second term:

$$\int_{\text{bdy } B_\varepsilon} u\frac{\partial}{\partial n}\left(-\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) - \left(-\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right)\frac{\partial u}{\partial n}\,\mathrm{d}S$$

On the boundary of the ball, $|\mathbf{x}-\mathbf{x}_0| = \varepsilon$ such that

$$\int_{\text{bdy } B_\varepsilon} u\frac{\partial}{\partial r}\left(-\frac{1}{4\pi}\frac{1}{\varepsilon}\right) + \int_{\text{bdy } B_\varepsilon} \frac{1}{4\pi}\frac{1}{\varepsilon}\frac{\partial u}{\partial n}\,\mathrm{d}S = \int_{\text{bdy } B_\varepsilon} u\cdot\frac{1}{4\pi}\frac{1}{\varepsilon^2}\,\mathrm{d}S + \frac{1}{4\pi\varepsilon}\int_{\text{bdy } B_\varepsilon}\frac{\partial u}{\partial n}\,\mathrm{d}S$$

Take the limit $\varepsilon \to 0$ to obtain, by the mean value property, ($\mathrm{d}S$ has an ε^2 term so that the first term has $\varepsilon^2/\varepsilon^2$ and the second has $\varepsilon^2/\varepsilon$; thus, the second goes to zero as $\varepsilon \to 0$):

$$= u(\mathbf{x}_0) + 0$$

Returning to (5.2), we have

$$\boxed{u(\mathbf{x}_0) = \iint_{\text{bdy } D}\left[-u(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) + \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial n}\right]\mathrm{d}S}$$

5.4 Green's Functions

In the preceding section, we obtained a representation formula. This formula is fine, although we want to simplify it by having its second term disappear. That is, we want some G such that

$$\boxed{u(\mathbf{x_0}) = \iint_{\text{bdy } D} u(\mathbf{x})\frac{\partial G}{\partial n}\,\mathrm{d}S} \tag{5.3}$$

There is a theorem that states that for Green's function G, the above is a solution of the Dirichlet problem.

So we want to define **Green's function** G such that, if the operator is $-\Delta$, we have a domain D and a fixed point $\mathbf{x}_0 \in D$,

1. $G(\mathbf{x})$ possesses continuous second derivatives and $\Delta G = 0$ in D except at $\mathbf{x} = \mathbf{x}_0$.
2. $G(\mathbf{x}) = 0$ for $x \in \operatorname{bdy} D$
3. The function $G(\mathbf{x}) + 1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ is finite at $\mathbf{x}_0$ and had continuous second derivatives everywhere and is harmonic at $\mathbf{x}_0$.

It is an unique function that exists. The only one of these properties that $-1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ does not satisfy is property 2.

For now, we suppose G is some

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + H(\mathbf{x})$$

5.4.1 Symmetry of Green's Function

For any region D we have Green's function $G(\mathbf{x}, \mathbf{x}_0)$ such that

$$G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x}) \text{ for } \mathbf{x} \neq \mathbf{x}_0$$

Proof: Let $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{a})$ and$v(\mathbf{x}) = G(\mathbf{x}, \mathbf{b})$, let D_ε be the domain D with two small spheres of radii ε around the points $\mathbf{a}$ and $\mathbf{b}$.

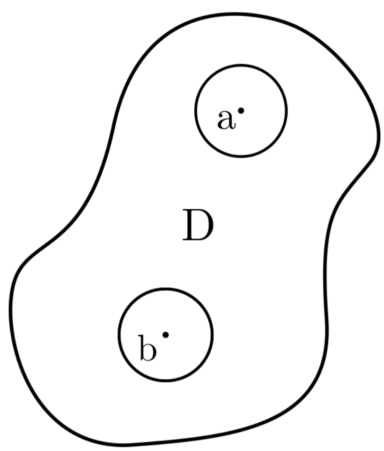

Figure 5.1: The domain D_ε.

Apply Green's second identity:

$$\int_{D_\varepsilon} u\Delta v - v\Delta u \,\mathrm{d}\mathbf{x} = \int_{\operatorname{bdy} D} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S - \int_{B_\varepsilon(\mathbf{a})} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S - \int_{B_\varepsilon(\mathbf{b})} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S$$

By definition, u and v vanish both when Δ is applied and when on the boundary D. Then the left hand side vanishes, and the first term of the right hand side vanishes as well:

$$0 = -\int_{B_\varepsilon(\mathbf{a})} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S - \int_{B_\varepsilon(\mathbf{b})} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S$$

Now write out v and u explicitly

$$\begin{aligned} 0 = &-\int_{B_\varepsilon(\mathbf{a})} u\frac{\partial}{\partial n}\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|}+H(\mathbf{x},\mathbf{a})\right)-\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|}+H(\mathbf{x},\mathbf{a})\right)\frac{\partial u}{\partial n}\,\mathrm{d}S \\ &-\int_{B_\varepsilon(\mathbf{b})}\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{b}|}+H(\mathbf{x},\mathbf{b})\right)\frac{\partial v}{\partial n}-v\frac{\partial}{\partial n}\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{b}|}+H(\mathbf{x},\mathbf{b})\right)\mathrm{d}S \end{aligned}$$

Once again use the fact that $\partial/\partial n$ is just $\partial/\partial r$ and $1/|\mathbf{x}-\mathbf{a}|$ is just r from the center of a or b.

$$\begin{aligned} 0 = &-\int_{B_\varepsilon(\mathbf{a})} u\left(-\frac{1}{4\pi\varepsilon^2}+\frac{\partial H}{\partial n}\right)-\left(-\frac{1}{4\pi\varepsilon}+H(\mathbf{x},\mathbf{a})\right)\frac{\partial u}{\partial n}\,\mathrm{d}S \\ &-\int_{B_\varepsilon(\mathbf{b})}\left(-\frac{1}{4\pi\varepsilon}+H(\mathbf{x},\mathbf{b})\right)\frac{\partial v}{\partial n}-v\left(-\frac{1}{4\pi\varepsilon^2}+\frac{\partial H}{\partial n}\right)\mathrm{d}S \end{aligned}$$

Take the limit as $\varepsilon \to 0$. We see that, since the surface area goes to zero,

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon(\mathbf{a})} u\cdot\frac{\partial H}{\partial n}\approx 4\pi\varepsilon^2\cdot u(\mathbf{a})\frac{\partial H}{\partial n}\bigg|_{\mathbf{x}=\mathbf{a}}=0$$

The only nonzero terms are, with the ε cancelling,

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon(\mathbf{a})} v\cdot\frac{1}{4\pi\varepsilon^2}\varepsilon^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=v(\mathbf{a})$$

A similar calculation yields $-u(\mathbf{b})$ (the sign is flipped because of Green's second identity), such that

$$0=v(\mathbf{a})-u(\mathbf{b})=G(\mathbf{a},\mathbf{b})-G(\mathbf{b},\mathbf{a})$$

5.5 Method of Reflection for Green's Function

We saw in section 5.4 that the function $(4\pi|\mathbf{x}-\mathbf{x}_0|)^{-1}$ satisfies two properties of Green's function. The only additional piece we need is that the function be zero on the boundary. We use the **method of reflection** to yield this property. We simply add a term that cancels out $(4\pi|\mathbf{x}-\mathbf{x}_0|)^{-1}$ at the boundary. The same process can be yielded for any n dimensional fundamental solution; for example, in two dimensions this function is analogous to

$$\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x_0}|$$

5.5.1 Half-Space

The half space is the region lying on one side of the plane. It is an infinite domain, but things will work out ig we impose the boundary condition at infinity: the limit of functions and their derivatives tend to zero as $\mathbf{x}\to\infty$.

As mentioned above, we want $G(\mathbf{x},\mathbf{x_0})=0$ for $\mathbf{x}\in\operatorname{bdy}D$. In this case the boundary is the xy plane. We can simply add a term that is the negative of $(4\pi|\mathbf{x}-\mathbf{x}_0|)^{-1}$ at the boundary. We produce a point that is a reflection of $\mathbf{x}$ over the boundary:

$$\mathbf{x}^*=(x,y,-z)$$

Such that this is exactly $\mathbf{x}$ on the boundary. Then

$$\boxed{G(\mathbf{x},\mathbf{x_0})=-\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}+\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0^*|}}$$

Verify: Let's make sure this is indeed Green's function for this domain.

1. It is finite and differentiable everywhere but $\mathbf{x}_0$, and $\Delta G = 0$.
2. Let $\mathbf{x} \in \text{bdy}\, D$ such that $z = 0$. Then $|\mathbf{x} - \mathbf{x}_0| = |\mathbf{x} - \mathbf{x}_0^*|$ and $G(\mathbf{x}, \mathbf{x}_0) = 0$.
3. Because $\mathbf{x}_0^*$ is outside the domain, the second term has no singularity inside the domain, and the only singularity is indeed $\mathbf{x}_0$.

Example Solve the Dirichlet problem

$$\begin{cases} \Delta u = 0, \ z > 0 \\ u(x, y, 0) = h(x, y) \end{cases}$$

We use formula (5.3). Notice $\partial G/\partial n = -\,\partial G/\partial z\,|_{z=0}$. Then

$$-\frac{\partial G}{\partial z} = \frac{1}{4\pi}\left(\frac{z + z_0}{|\mathbf{x} - \mathbf{x}_0|^3} - \frac{z - z_0}{|\mathbf{x} - \mathbf{x}_0|^3}\right) = \frac{1}{2\pi}\frac{z_0}{|\mathbf{x} - \mathbf{x}_0|^3}$$

on $z = 0$. Then the solution is

$$u(\mathbf{x_0}) = \frac{z_0}{2\pi}\iint_{\text{bdy}\, D} \frac{h(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|^3}\, \mathrm{d}S$$

5.5.2 Sphere

We again employ the reflection method. The reflection is over the surface of the sphere in this case. Fix a nonzero point $\mathbf{x}_0$ in the sphere. To reflect, we use two properties.

1. $\mathbf{x}_0^*$ is collinear with the origin and the point $\mathbf{x}_0$.
2. It has length

$$|\mathbf{x_0}| = \frac{R^2}{|\mathbf{x}_0|}$$

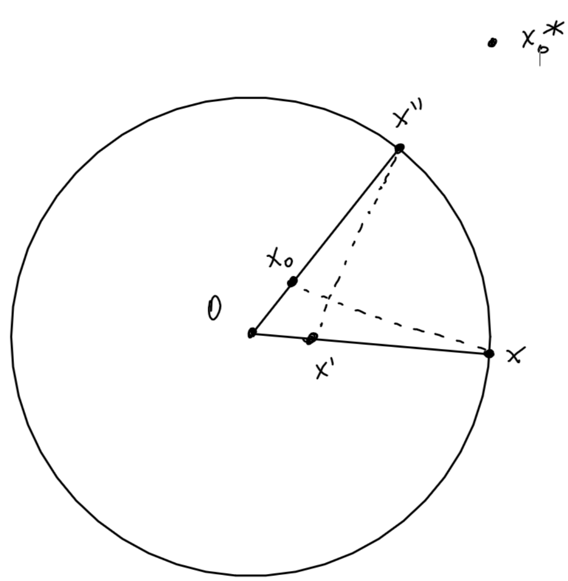

Figure 5.2: The sphere has radius R.

Observe figure 5.2 to see why property 2 is the case. We have that $\mathbf{x}'' = R \cdot \mathbf{x}_0/|\mathbf{x}_0|$, and $\mathbf{x}' = \mathbf{x} \cdot |\mathbf{x}_0|/R$ (since $|\mathbf{x}| = R$ and $\mathbf{x}'$ has the same length as $\mathbf{x}_0$). This means

$$\begin{aligned} |\mathbf{x} - \mathbf{x}_0| &= |\mathbf{x}' - \mathbf{x}''| \\ &= \left| \frac{|\mathbf{x}_0|}{R}\mathbf{x} - \frac{R}{|\mathbf{x}_0|}\mathbf{x}_0 \right| \\ &= \frac{|\mathbf{x}_0|}{R}\left| \mathbf{x} - \frac{R^2}{|\mathbf{x}_0|^2}\mathbf{x}_0 \right| \end{aligned} \tag{5.4}$$

We want $|\mathbf{x} - \mathbf{x}_0| = |\mathbf{x} - \mathbf{x}_0^*|$ for $|\mathbf{x}| = R$. Evidently we have that property with the above equation. Additionally, the second term in the absolute value sign clearly is collinear with $\mathbf{x_0}$. Then we have found our

$$\mathbf{x}_0^* = \frac{R^2}{|\mathbf{x}_0|^2}\mathbf{x}_0$$

And we have

$$\boxed{G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{R}{|\mathbf{x}|_0}\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0^*|}}$$

or

$$\boxed{G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi||\mathbf{x_0}|\mathbf{x}/R - R\mathbf{x}_0/|\mathbf{x}_0||}}$$

both of which are for the case $\mathbf{x_0} \neq 0$. In the case $\mathbf{x}_0 = 0$, we have

$$G(\mathbf{x}, 0) = -\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a}$$

Verify:

- G has no singularities except at $\mathbf{x} = \mathbf{x}_0$, since the reflected point $\mathbf{x}_0^*$ is outside the domain.
- We've already shown that

  $$\begin{aligned} \left| \frac{|\mathbf{x}_0|}{R}\mathbf{x} - \frac{R}{|\mathbf{x}_0|}\mathbf{x}_0 \right| &= |\mathbf{x} - \mathbf{x}_0| \\ \frac{|\mathbf{x}_0|}{R}\left| \mathbf{x} - \frac{R^2}{|\mathbf{x}_0|^2}\mathbf{x}_0 \right| &= |\mathbf{x} - \mathbf{x}_0| \end{aligned}$$

  Substituting into G makes the entire function vanish.
- The two terms are harmonic—they're simply translations of $1/r$, and the coefficient on the second term makes no difference.

In two dimensions, this is

$$\boxed{G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}\log|\mathbf{x} - \mathbf{x}_0| - \frac{1}{2\pi}\log\left(\frac{|\mathbf{x}_0|}{a}|\mathbf{x} - \mathbf{x}_0^*|\right)}$$

Example Let us solve the three-dimensional Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } |\mathbf{x}| < R \\ u = h \text{ on } |\mathbf{x}| = R \end{cases}$$

using (5.3). We know $u(0)$ is the average of $h(\mathbf{x})$ on the sphere. Then consider $\mathbf{x}_0 \neq 0$. Note that

$$\frac{\partial G}{\partial n} = \nabla G \cdot \mathbf{n} = \nabla \cdot \frac{\mathbf{x}}{|\mathbf{x}|}$$

Now, component wise

$$\frac{\partial G}{\partial x_j} = -\frac{1}{4\pi}\frac{\partial}{\partial x_j}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) + \frac{R}{4\pi|\mathbf{x}_0|}\frac{\partial}{\partial x_j}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0^*|}\right)$$
$$= \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2}\frac{\partial}{\partial x_j}|\mathbf{x}-\mathbf{x}_0| - \frac{R}{4\pi|\mathbf{x}_0|}\frac{1}{|\mathbf{x}-\mathbf{x}_0^*|}\frac{\partial}{\partial x_j}(|\mathbf{x}-\mathbf{x}_0^*|)$$

Use the fact that

$$\frac{\partial}{\partial x_j}|\mathbf{x}-\mathbf{x}_0| = \frac{x_j - x_{0j}}{|\mathbf{x}-\mathbf{x}_0|}$$

To obtain

$$= \frac{1}{4\pi}\frac{x_j - x_{0j}}{|\mathbf{x}-\mathbf{x}_0|^3} - \frac{R}{4\pi|\mathbf{x}_0|}\frac{x_j - x_{0j}^*}{|\mathbf{x}-\mathbf{x}_0^*|^3}$$

Note that, when $|\mathbf{x}| = R$, with (5.4),

$$\frac{|\mathbf{x}_0|}{R}|\mathbf{x}-\mathbf{x}^*| = |\mathbf{x}-\mathbf{x}_0|$$
$$|\mathbf{x}-\mathbf{x}_0^*|^3 = \left(\frac{R}{|\mathbf{x}_0|}|\mathbf{x}-\mathbf{x}_0|\right)^3$$

Since we are integrating over the boundary, this holds. Further,

$$x_j - x_{0j}^* = x - \frac{R^2}{|\mathbf{x}_0|^2}x_{0j}$$

So that

$$= \frac{1}{4\pi}\frac{x_j - (|\mathbf{x}_0|^2/R^2)x_j}{|\mathbf{x}-\mathbf{x}_0|^3} = \frac{1}{4\pi}\frac{R^2 - |\mathbf{x}_0|^2}{R^2}\frac{\mathbf{x}_j}{|\mathbf{x}-\mathbf{x}_0|^3}$$

Summing, we have

$$\nabla G = \frac{1}{4\pi}\frac{R^2 - |\mathbf{x}_0|^2}{R^2}\frac{\mathbf{x}}{|\mathbf{x}-\mathbf{x}_0|^3}$$
$$\nabla G \cdot \mathbf{n} = \frac{1}{4\pi}\frac{R^2 - |\mathbf{x}_0|^2}{R^2}\frac{1}{|\mathbf{x}-\mathbf{x}_0|^3}\mathbf{x} \cdot \frac{\mathbf{x}}{|\mathbf{x}|}$$
$$= \frac{1}{4\pi}\frac{R^2 - |\mathbf{x}_0|^2}{R}\frac{1}{|\mathbf{x}-\mathbf{x}_0|^3}$$

where we used $|\mathbf{x}| = R$ on the surface.
Then we obtain

$$u(\mathbf{x}_0) = \frac{R^2 - |\mathbf{x}_0^2|}{4\pi R}\int_{\text{bdy } D}\frac{h(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^3}\,\mathrm{d}S$$

Which is exactly the Poisson formula!

Chapter 6

General Eigenvalue Problems

6.1 Eigenvalues and Minima

The eignvalue problem with Dirichlet boundary conditions is

$$\begin{cases} -\Delta u = \lambda u \text{ in } \operatorname{bdy} D \\ u = 0 \text{ on } \operatorname{bdy} D \end{cases} \tag{6.1}$$

where D is an *arbitrary* domain in $\mathbb{R}^3$ with a piecewise smooth boundary. For possible functions u_i, we have eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots$$

We define

$$(f, g) = \iiint_D f(\mathbf{x})\overline{g(\mathbf{x})}$$

We saw in section 5.2.3 that the function that minimizes the energy and satisfies a homogeneous boundary condition is the harmonic function.

Consider the minimum of the **Rayleigh quotient**:

$$m = \min \left\{ \frac{|\nabla w|^2}{|w|^2} : \; w = 0 \text{ on } \operatorname{bdy} D, w \neq 0 \right\}$$

this means we consider the minimum value of the quotient for all functions that vanish on the boundary and are not zero. The "solution" of the above problem is the function u that yields such a minimum value.

6.1.1 Minimum Principle for the First Eigenvalue

Assume $u(\mathbf{x})$ is a solution to the minimum problem above. Then the value of the minimum *equals* the first eigenvalue of 6.1 and $u(\mathbf{x})$ is its eigenfunction:

$$\boxed{\lambda_1 m = \min \left\{ \frac{|\nabla w|^2}{|w|^2} \right\} \text{ and } -\Delta u = \lambda_1 u \text{ in } D}$$

"The first eigenvalue is the minimum of the energy."

Proof: We use calculus of variations. We use trial functions $w(\mathbf{x})$ with continuous second derivatives, $w = 0$ on bdy D and $w \neq 0$. Let $u(\mathbf{x})$ be a solution of the minimum problem. By assumption,

$$m = \frac{\iiint |\nabla u|^2 \,\mathrm{d}\mathbf{x}}{\iiint |u|^2 \,\mathrm{d}\mathbf{x}} \leq \frac{\iiint |\nabla w|^2 \,\mathrm{d}\mathbf{x}}{\iiint |w|^2 \,\mathrm{d}\mathbf{x}}$$

For all trial functions. Let $v(\mathbf{x})$ be any other trial function and let $w(\mathbf{x}) = u(\mathbf{x}) + \varepsilon v(\mathbf{x})$ where ε is any constant. Then define the functional

$$f(\varepsilon) = \frac{\int |\nabla(u + \varepsilon v)|^2}{\int |u + \varepsilon v|^2}$$

which has a minimum at $\varepsilon = 0$. By single-variable calculus, we know that $f'(0) = 0$. Expanding both squares yields

$$f(\varepsilon) = \frac{\int |\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2}{\int u^2 + 2\varepsilon uv + \varepsilon^2 v^2}$$

Then the derivative with respect to ε at $\varepsilon = 0$ is

$$0 = f'(0) = \frac{(\int u^2)(2\int \nabla u \cdot \nabla v) - \left(\int |\nabla u|^2\right)(2\int uv)}{(\int u^2)^2}$$

Rearranging,

$$\int \nabla u \cdot \nabla v = \frac{\int |\nabla u|^2}{\int u^2} \int uv = m \int uv$$

By Green's first identity, equation (5.1), and by the boundary condition $v = 0$, we have

$$\iiint (\Delta u + mu)(v) \,\mathrm{d}\mathbf{x} = 0$$

Since $v(\mathbf{x})$ is arbitrary, we see that

$$\Delta u + mu = 0 \Rightarrow -\Delta u = mu$$

Now show m is the *smallest* eigenvalue. Let $-\Delta v_j = \lambda_j v_j$ where λ_j is any eigenvalue. By the definition of m as the minimum of the Rayleigh quotient, and by Green's first identity (5.1) (the left hand side is zero by the boundary condition),

$$\begin{aligned} m \leq \frac{\int |\nabla v_j|^2}{\int v_j^2} &= \frac{\int (-\Delta v_j)(v_j)}{\int v_j^2} \\ &= \frac{\int (\lambda_j v_j) v_j}{\int v_j^2} \\ &= \lambda_j \frac{\int v_j^2}{\int v_j^2} = \lambda_j \end{aligned}$$

6.1.2 For the Neumann Condition

We have

$$\begin{cases} -\Delta u = \lambda u \text{ on } D \\ \frac{\partial u}{\partial n} = 0 \text{ on bdy } D \end{cases}$$

Consider

$$\boxed{m = min\left\{\frac{\int_D |\nabla w|^2}{\int_D |w|^2} : \frac{\partial w}{\partial n} = 0 \text{ on } \operatorname{bdy} D\right\}}$$

We proceed identically as before, and eventually get

$$\int_{\operatorname{bdy} D} \frac{\partial u}{\partial n} v \, \mathrm{d}S - \int_D \Delta u \cdot v \, \mathrm{d}\mathbf{x} = m \int_D uv \, \mathrm{d}\mathbf{x}$$

Which leads us to $m = \lambda_1$ as well.

Notice that we then don't need to require $v = 0$ on $\operatorname{bdy} D$ since by the Neumann condition $\partial u/\partial n = 0$ on the boundary anyway. So it turns out the requirement we enforced above with $\partial w/\partial n = 0$ is not necessary since $w = u+\varepsilon v$. Then we are allowed more candidate functions and it turns out $\lambda_N \leq \lambda_D$ and the Neumann eigenvalues are smaller than the Dirichlet eigenvalues.

6.1.3 The Other Eigenvalues

The other eigenvalues of (6.1) are also minima, simply with additional constraints.

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are already known, with eigenfunctions $v_1, v_2, \ldots, v_{n-1}$ respectively. Then

$$\boxed{\lambda_n = \min\left\{\frac{|\Delta w|^2}{|w|^2} : w \neq 0, w = 0 \text{ on } \operatorname{bdy} D, w \in C^2, 0 = (w, v_1) = \cdots = (w, v_{n-1})\right\}}$$

Assuming such a minimum exists. In words, we have that the eigenvalue is the minimum of the set of nonzero functions that vanisih on the boundary, have continuous second derivatives, and are orthogonal to all previous eigenfunctions.

With more constraints as the integer n increases (the eigenfunction must be orthogonal to more functions), there are less candidates for eigenfunctions and the minimum value is greater than or equal to the previous minima. This is a general fact: with more constraints, there are less candidates and thus a $\geq$ eigenvalue.

Proof: Let

$$m^* = \min\left\{\frac{|\Delta w|^2}{|w|^2} : \cdots\right\}$$

attained by some u such that $u = 0$ on $\operatorname{bdy} D$. Consider some $w = u + \varepsilon v$ with the same constraints as the previous section. Then the functional is

$$f(\varepsilon) = \frac{\int_D |\nabla u + \varepsilon \nabla u|^2}{|u + \varepsilon v|^2}$$

which attains its minimum at $\varepsilon = 0$ such that $f'(0) = 0$. Using the same procedure as before, we have

$$\int_D (\Delta u + m^* u) v = 0 \tag{6.2}$$

We now consider perturbations $w = u + \varepsilon v_j$, and with the same logic

$$\begin{aligned} \iiint_D (\Delta u + m^* u) v_j \, \mathrm{d}\mathbf{x} &= \iiint_D u(\Delta v_j + m^* v_j) \, \mathrm{d}\mathbf{x} && (6.3) \\ &= (m^* - \lambda_j) \iiint uv_j \, \mathrm{d}\mathbf{x} = 0 && (6.4) \end{aligned}$$

Since u orthogonal to v_j for $j = 1, \ldots, n-1$. Now let $h(\mathbf{x})$ be an arbitrary trial function with $h = 0$ on bdy D and $h \neq 0$. Let

$$v(\mathbf{x}) = h(\mathbf{x}) - \sum_{k=1}^{n-1} c_k v_k(\mathbf{x}), \text{ where } c_k = \frac{(h, v_k)}{v_k, v_k}$$

Then we've just created v as the part of h that is orthogonal to the $v_1, v_2, \ldots, v_{n-1}$. That is, we've orthogonalized h from the subspace formed by v_j, then we set it equal to v such that v and thus w satisfy all the constraints we've required for this section. Thus (6.2) is valid for this v. A linear combination of (6.2) and (6.3) (such that we sum the component of h that is in the subspace formed by the v_j to the component of h orthogonal to this subspace, yieldig h again) gives

$$\iiint_D (\Delta u + m^* u) h \, \mathrm{d}\mathbf{x} = 0$$

Which holds for any arbitrary h that disappear on the boundary. Then, again, we must hav e

$$\Delta u + m^* u = 0 \Rightarrow -\Delta u = m^* u$$

Which means m^* is an eigenvalue.

Now it remains to show that $m^* = \lambda_n$:

1. Show $m^* \geq \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$. Compared to λ_{n-1}, m^* requires $(w, v_{n-1}) = 0$, which means there are less choices of w, and thus $m^* \geq \lambda_{n-1}$.

2. Show $m^* \leq \lambda_{n+1}, \lambda_{n+1}, \ldots$. v_j corresponds to λ_j for $j \geq n+1$ with

$$(v_j, n) = (v_j, v_{n-1}), = \cdots = (v_j, v_1) = 0$$

Then

$$m^* \leq \frac{\int |\nabla v_j|^2}{\int |v_j|^2} = \lambda_j$$