Log 2023

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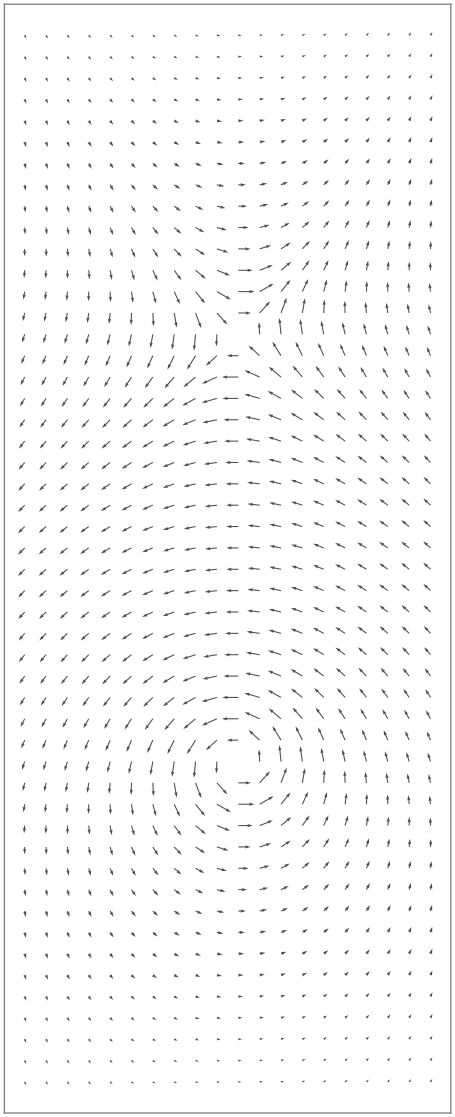

Contents

Chapter 1

March

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1.1 Metropolis-Hastings Monte Carlo

We want to generate a set of samples from a distribution $P(x)$. The goal is to use a Markov process that asymptotically reaches a distribution that coincides with $P(x)$.

We start with the detailed balanced condition: consider a system in thermal equilibrium, and two states x and x' of that system. The probability of being in state x and going from x to x' is the same as the probability of being at x' and going back to state x:

$$P(x \to x')P(x) = P(x' \to x)P(x'). \tag{1.1}$$

Here $P(x \to x')$ is occasionally written as $P(x' \mid x)$, as in a Markov process we can consider the probability of going to x' *given* a previous state x. This is fact would require us to factor $P(x' \mid x)$ into $P(x' \mid x) = g(x' \mid x)A(x', x)$, where $g(x' \mid x)$ captures the *conditional probability of considering x' as the next state* that might arise if this process is such that the probability distribution *changes* depending on the previous state, and $A(x', x)$ is the probability of actually making that transition. In statistical mechanics the equilibrium probability distribution does not change depending on which of the states of the ensemble you're in, so the above is not really a point of concern and we can unambiguously write $P(x \to x')$.

The idea is to write eq. (1.1) as

$$\frac{P(x \to x')}{P(x' \to x)} = \frac{P(x')}{P(x)} \tag{1.2}$$

and to choose $P(x \to x')$ and $P(x' \to x)$ such that this condition is always fulfilled. An intuitive choice is the so-called *Metropolis* choice, which is the most important bit to understand:

$$P(x \to x') = \min\left(1, \frac{P(x')}{P(x)}\right)$$

Make the move from x to x' no matter what if $P(x') > P(x)$ (i.e., if x' is a more likely state); make the move with probability $P(x')/P(x)$ if it is a less likely state.

Where in the algorithm we generate a random number between 0 and 1 and make the move if that number is less than $P(x \to x')$ (hence with certainty if $P(x') > P(x)$ and with chance to $P(x')/P(x)$ otherwise). Here implicitly the same choice is also made for $P(x' \to x)$, where we just switch $x \longleftrightarrow x'$ in the above definition (note that we only have to compute $P(x \to x')$, though, since we are only considering the move from x to x'). This fulfilled eq. (1.2) in the following sense:

- If $P(x') > P(x)$, we set $P(x \to x') = 1$ and $P(x' \to x) = P(x)/P(x')$. Then the ratio of $P(x \to x')$ and $P(x' \to x)$ is the right hand side of eq. (1.2):

$$\frac{P(x \to x')}{P(x' \to x)} = \frac{1}{P(x)/P(x')} = \frac{P(x')}{P(x)}$$

- If $P(x') < P(x)$, we set $P(x \to x') = P(x')/P(x)$ and $P(x' \to x) = 1$ and again the ratio eq. (1.2) is satisfied.

The power in its application to statistical mechanics lies in the fact that we only need to compute the relative change in energy $P(x')/P(x)$! E.g., in the Ising model with coordination number z, if we are in an all-up (or down) ferromagnetic state and flip spin k:

$$\frac{P(x')}{P(x)} = \frac{\exp\left(\beta J \sum S_i S_j\right)}{\exp\left(\beta J \sum S_i S_j\right)} = \frac{\cancel{\exp\left(\beta J \sum_{i,j\neq k} S_i S_j\right)} \cdot \exp\left(\beta J \sum_j S_j S_k\right)}{\cancel{\exp\left(\beta J \sum_{i,j\neq k} S_i S_j\right)} \cdot \exp\left(\beta J \sum_j S_j S_k\right)}$$

$$= \frac{\exp\left(\beta J \sum_j (+1)(-1)\right)}{\exp\left(\beta J \sum_j (+1)(+1)\right)} = \frac{\exp\left(-z\beta J\right)}{\exp\left(z\beta J\right)} = \exp(-2zJ\beta).$$

1.2 Symmetries and Degeneracy

We use here the example of rotational symmetry, although the argument is general unless otherwise specified. Our goal is essentially to derive the m (magnetic quantum number) degeneracy present in central potentials/rotationally invariant systems such as the hydrogen

atom. Consider the $(2\ell+1)$-dimensional irrep of L_z, the angular momentum operator. We know this generates the group of rotations about the z axis by $\exp(-i\theta L_z)$. Consider also a Hamiltonian with z-rotational symmetry. Then

$$e^{-i\theta L_z} H e^{i\theta L_z} = H \implies \left[e^{i\theta L_z}, H\right] = 0$$

We can ascertain from this two facts:

1. z-rotated states are also eigenstates of H with the same energy: $\left[e^{i\theta L_z}, H\right] = 0 \implies He^{i\theta L_z}|E_n\rangle = e^{i\theta L_z} H|E_n\rangle$, so $H(e^{i\theta L_z}|E_n\rangle) = E_n(e^{i\theta L_z}|E_n\rangle)$. In a sense this can be thought of as a degeneracy, but this is not a particularly useful line of thinking: applying the rotation operator $e^{i\theta L_z}$ does not alter the quantum numbers of E_n with respect to the quantization axis L_z, so there are no other eigenstates of L_z with the same energy, and this "degeneracy" is fictitious, so to speak. Besides, $e^{i\theta L_z}$ is not Hermitian and thus is not an observable.

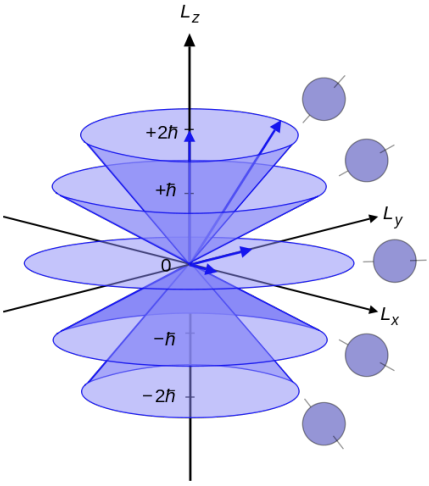

Figure 1.1

2. Eigenstates of H are also eigenstates of L_z: $\left[e^{i\theta L_z}, H\right] = 0 \implies [L_z, H] = 0$. Note this does not mean that the L_z-eigenvalues are the same as H eigenvalues! That is, we've not yet derived the degeneracy; in fact, if the eigenvalues were the same, we might've indeed derived a degenerate set of states. What this does mean is that we can use both E_n and m, ℓ to label the eigenstates.

In reference to the latter fact, I will only use m henceforth, dropping the quantum number ℓ. This amounts to implicitly picking the irreducible representation of dimension $2\ell+1$ of the rotation group.

So far we've only considered z-rotational symmetry. This does not give us the degeneracy in m. To do this, we need some more physical intuition: we know that the magnetic quantum number m roughly corresponds to the projection of the angular momentum along the z axis. Degeneracy in m corresponds to the Hamiltonian produce *the same energies independent of the direction of the angular momentum* (specifically, its projection along the z axis). To obtain this, the system needs to also be invariant with respect to rotations along the x- and y- axes, as such rotations will alter the projection along the z-axis. See fig. 1.1. This is the main insight:

Symmetries lead to Hamiltonians that have solutions with the same energy related by a certain transformation. If that transformation changes a certain quantum number, it should be the case that those new states (labelled by the new quantum numbers) are also solutions with the same energies, and thus those transformations generate the degeneracy in that quantum number.

Then the generator L_x should correspond to the transformations that lead to the degeneracy. This is seen by considering the transformation law $L_x = \frac{1}{2}(L_+ + L_-)$ (again, if m is the projection along the z axis, we can see that rotations $e^{i\theta L_x}$ increase, and, as a superposition, increase the projection of the angular momentum along the z axis, exactly as if we are rotating the angular momentum vector). Then, if the Hamiltonian is symmetric with respect to x-rotations,

$$\begin{aligned} L_x H\left|E_n, m\right\rangle &= H L_x \left|E_n, m\right\rangle \\ \frac{1}{2}E_n(L_+ + L_-)\left|E_n, m\right\rangle &= \frac{1}{2}H(L_+ + L_-)\left|E_n, m\right\rangle \\ \frac{1}{2}E_n L_\pm \left|E_n, m\right\rangle &= \frac{1}{2}H L_\pm \left|E_n, m\right\rangle \qquad (|m \pm 1\rangle \text{ are linearly indep.}) \\ E_n \left|E_n, m \pm 1\right\rangle &= H\left|E_n, m \pm 1\right\rangle . \end{aligned}$$

Thus altering the value of m leads to another eigenstate with the same energy.

Some other remarks: its possible that the symmetry group we are considering is Abelian. In this case, it turns out that the irreducible representation is trivial (see here). This means that for Abelian symmetries, the state is only altered by an overall phase factor and there is no degeneracy: if $\left[H, e^{it\theta}\right] = 0$

$$E_n e^{it\phi}\left|E_n\right\rangle = H e^{it\phi}\left|E_n\right\rangle \implies e^{it\phi}E_n\left|E_n\right\rangle = e^{it\phi}H\left|E_n\right\rangle \implies E_n\left|E_n\right\rangle = H\left|E_n\right\rangle$$

where ϕ is the one dimensional representation. The last equality, of course is something we already knew.

An example is that of a one-dimensional symmetric potential; the symmetry is $\mathbb{Z}_2$, which is Abelian. Such a system has no degeneracy.

LRL vector and ℓ degeneracy

Inverse-square potentials are known to have an additional conserved quantity, the Laplace-Runge-Lenz vector. This turns out to correspond to a 4-dimensional spherical symmetry, which

in turn leads to a degeneracy in the ℓ (azimuthal) quantum numbers, such as those present in the Hydrogen atom.

See also

- The examples section of this document.

1.3 Connections, Generally

There is a connection between connections (haha); that is, when we hear people refer to the potentials A_μ as "connections" in gauge theory (quantum field theory), this actually does correspond to the Riemannian connection $\nabla_X Y$ we see in differential geometry, and similarly for notions such as curvature, covariant derivatives, etc.

There is one quick terminological clarification to make. What we refer to as a connection in gauge theory (e.g., A_μ) really corresponds to the Christoffel symbols Γ^k_{ij}; the (gauge) covariant derivative is, under this terminology, really $\nabla_X Y$. The reason the two are referred to interchangeably is that the Γ^k_{ij} can be thought of as implicitly determined upon specifying $\nabla_X Y$, and vice-versa. It's not unlike how people will refer to the representation space as a "representation." Which infuriates me to this day.

One discrepancy that might be concerning at first is the fact that, in gauge theory (let's generalize to non-Abelian here), the connection appears to have one index: e.g., in Peskin and Schroeder §15.2 we see A^i_μ, where i indexes the generator of the gauge symmetry group. From Stack Exchange:

> Mathematicians would refer to a gauge theory setting as a fiber bundle. A_μ takes the role of a connection on the fibre bundle. Of course, out of the three indices the Cristoffel symbols Γ^i_{jk} have, two indices "live" in the fiber. You can appreciate this more easily if you look at non-Abelian gauge theories, in which $(A_\mu)_{ij} = A^a_\mu T^a_{ij}$

So here $(A_\mu)_{ij}$ corresponds to Γ^i_{jk}.

To understand the correspondence between curvatures, we need an expression of the curvature of Riemannian geometry $R(X,Y)Z$ in terms of the connection Γ^k_{ij}. The bottom equation of page 92 of Do Carmo's *Riemannian Geometry* gives us what we need:

$$R(X_i, X_j)X_k = \nabla_{X_j}(\Gamma^\ell_{ik} X_\ell) - \nabla_{X_i}(\Gamma^\ell_{jk} X_\ell). \tag{1.3}$$

(X_i are basis vectors, so it's not difficult to convert to index notation from this.)

- **Berry's phase:** The connection is the *Berry connection* $A^n(\mathbf{R}) = i\langle n(\mathbf{R})|\,\partial_{\mathbf{R}}\,|n(\mathbf{R})\rangle$ (what is the fiber bundle?), from which we obtain the *Berry curvature* $\omega_{\mu\nu}(\mathbf{R}) = \partial_{R_\mu} A^n_\nu(\mathbf{R}) - \partial_{R_\nu} A^n_\mu(\mathbf{R})$. Cf. eq. (1.3); the missing index relative to $R(X_i, X_j)X_k$ comes from the fact that we've written $\mathbf{R}$ in the vector form as opposed to index form.

See also

- Girvin and Yang's *Modern Condensed Matter Physics*, §13.1 has an excellent explanation of Berry's phase.
- These notes.

1.4 Legendre Transforms

This is one I've been meaning to sit down and understand more carefully for one or two years, actually. It's a remarkably simple concept that somehow gets mangled by thoughtless pedagogy in almost every single treatment. The big idea is: we'd like to turn a function $f(x)$ into a function of $p = \mathrm{d}f/\mathrm{d}x$. Put in other words, we can think of it as "a map from the graph of the function ot the tangents of the graph" (Wikipedia).

Say we have a convex function $f(x)$ (in the case the function isn't convex everywhere, we can restrict ourselves to an interval where it is convex), and a line $y = px + b$ which we'd like to make the tangent line of $f(x)$ at $x = x_0$ by tweaking p and b. For this to be the case,

$$y = px + b = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} x + b \Longrightarrow p = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}$$
$$\text{and} \qquad y(x = x_0) = px_0 + b = f(x_0)$$

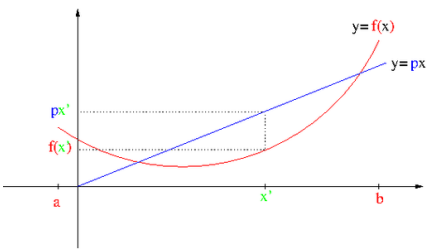

Figure 1.2

(Note this is *not* related to the Taylor series expansion, we're just looking for the tangent line at the point $f(x_0)$). Since $f(x)$ is convex $\mathrm{d}f/\mathrm{d}x = p$ can be inverted, and we can solve for $x_0 = (f')^{-1}(p)$. Now it remains to solve for b by using the last equality above:

$$b = f(x_0) - px_0$$

Since we know $x_0 = (f')^{-1}(p)$ and we're ultimately looking to convert f into a function of p:

$$b = f((f')^{-1}(p)) - p \cdot (f')^{-1}(p) \equiv f^*(p)$$

were we've defined the Legendre transform in the last equality. As it turns out, the value of x_0 is such that the distance between $f(x_0)$ and px_0 is maximized: we could have found x_0

as a function of p by maximizing $b = b(x)$:

$$b(x) = f(x) - px \Longrightarrow \left.\frac{\partial b}{\partial x}\right|_{x_0} = f'(x_0) - p = 0$$
$$\Longrightarrow x_0 = (f')^{-1}(p),$$

which is what commonly drawn figures such as fig. 1.2 are trying to tell you.

> The Legendre transform is defined by
>
> $$f^*(p) = f(x_0) - px_0, \qquad x_0 = (f')^{-1}(p),$$
>
> It states that for a convex function f we can determine the function's tangent lines' y-intercepts for given values p of their slopes. $f^*(p)$ is precisely the y-intercept of the function for a slope p.

Here's an interactive Desmos graph that demonstrates this. In this sense it is a very simple procedure—one a high school calculus student could discover by accident if, say, they are trying to make an animation of a tangent line coasting along the plot of a function.

This revisit was informed by an issue Elijah brought up. It was something like: one encounters a "path integral" with "Lagrangian" $\pi\phi - H[\pi, \phi]$ integrated over both the field *and* the canonical momentum π; this it *not* the usual path integral, as there is a mysterious integration over π. We were curious as to why. The answer is that the the Legendre transform exchanges

$$H(p, q) \longleftrightarrow L(\dot{q}, q), \qquad \text{i.e.,} \qquad p \longleftrightarrow \dot{q},$$

where $p \equiv \partial L/\partial \dot{q}$ *is evaluated at a point specified by* q, t, so properly it should be written $\partial L/\partial q\,|_{\dot{q}_0,q,t}$. This is why the path integral resulting from the proper Legendre transform only includes integration over the field ϕ, π is implicitly given by ϕ.

Chapter 2

April

Contents

2.1 Miscellaneous Bash Things

2.1.1 How to run a command without admin

E.g., using the BasicTeX distribution of LaTeX, I used to have to write `sudo tlmgr install` every time I wanted to install a package. You can fix this by (1) giving yourself (or any user) "no password" privilege to run that specific command, then (2) *alias* the command `sudo tlmgr` to just `tlmgr`.

Specifically, step (1) is to add the following line to the `/etc/sudoers` file, which controls what users have which `sudo` privilege to do what:

```
<username> ALL = (root) NOPASSWD: <command>
```

Note that `<command>` requires the *full* path to the executable. Also note that you should place this after the root specifications. See this.

Step (2) is to make the shell alias your command on startup by adding the following line to `~/.bashrc` (or `~/.zshrc`):

```
alias <command>='sudo <command>'
```

From Stack Exchange, `sudoers` formatting is:

> ```
> user_spec host_spec = (runas_spec) NOPASSWD:cmd_spec
> ```

- `user_spec`: which users can use the rule
- `host_spec`: which hosts the rule applies to. This is optional and defaults to all.
- `runas_spec`: which users the commands can be run as.
- `NOPASSWD` or `PASSWD`: whether a password is required. Optional and defaults to `PASSWD` unless the default has been changed in the sudeors configuration.
- `cmd_spec`: identifies which commands the rule can be run for.

2.1.2 Running a terminal script in the background

Say you have an application that is started on the terminal and runs in a GUI or something, and you want to close the terminal while still using the application; you can start it, then

1. Use Ctrl-Z to suspend it
2. Use `bg` to make it a background job
3. Run `disown` to remove the job from the current shell
4. Feel free to then close the terminal.

If you want to start the script already in the background in such a way that logout doesn't terminate the script, you can add a `&` to the end, which automatically backgrounds (i.e., runs `bg`) it, and run it in "no hangup" mode `nohup` such that it stays running and outputs everything to a file called `nohup.out`; e.g.,

```
nohup qitview &
```

2.2 Solving a ODE as a Matrix Problem

I find that the matrix method in, say, computational single-particle quantum mechanics, is rarely derived in a nice explicit way. Here's a quick calculation that makes it more

unambiguous. Do a backward and forward Taylor expansion:

$$\psi(x_i + \Delta x) \approx \psi(x_i) + \left.\frac{d\psi}{dx}\right|_{x_i} \Delta x + \frac{1}{2}\left.\frac{d^2\psi}{dx^2}\right|_{x_i} (\Delta x)^2$$
$$\psi(x_i - \Delta x) \approx \psi(x_i) - \left.\frac{d\psi}{dx}\right|_{x_i} \Delta x + \frac{1}{2}\left.\frac{d^2\psi}{dx^2}\right|_{x_i} (\Delta x)^2$$

Call $\psi(x_i \pm \Delta x) \equiv \psi_{i\pm1}$ and add these two equations:

$$\psi_{i+1} + \psi_{i-1} \approx 2\psi_i + \left.\frac{d^2\psi}{dx^2}\right|_{x_i} (\Delta x)^2 \Longrightarrow \left.\frac{d^2\psi}{dx^2}\right|_{x_i} \approx \frac{\psi_{i+1} + \psi_{i-2} - 2\psi_i}{(\Delta x)^2}$$

Now we can obtain an approximate matrix representation of the second derivative

$$\frac{1}{(\Delta x)} \sum_i D_{ij}\psi_i = \frac{1}{(\Delta x)^2}(\psi_{i+1} + \psi_{i-1} - 2\psi_i) = \frac{1}{(\Delta x)^2} \sum_i \big(\delta_{j(i+1)} + \delta_{j(i-1)} - 2\delta_{ij}\big)\psi_i$$

$$\Longrightarrow \quad D_{ij} = \delta_{j(i+1)} + \delta_{j(i-1)} - 2\delta_{ij}$$

which is a tridiagonal or band matrix.

Chapter 3

May

Contents

3.1 Carroll 3.6: Curvature Approach to Satellite Orbit

The metric outside the earth can be approximated by

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + (1-2\Phi)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\mathrm{d}\phi^2), \qquad \Phi = -\frac{GM}{r}$$

First, we can compute the flow of time at radii R_1 and R_2. This is done by picking a rest frame at a point with radius $R_{1,2}$ such that $\mathrm{d}r = \mathrm{d}\phi = \mathrm{d}\theta = 0$. So the expression for the metric reduces to

$$\mathrm{d}s^2 = -(1+2\Phi_{1,2})\,\mathrm{d}t^2\,.$$

We have just two degrees of freedom so that we can solve it as a differential equation. In fact, since we're working in the rest frame, the arclength s is exactly the proper time $\tau_{1,2}$. Taking the square root of both sides (properly, giving both sides some vector as described in Carroll §2.6)

$$\mathrm{d}\tau_{1,2} = \sqrt{-(1+2\Phi_{1,2})}\,\mathrm{d}t$$

We then integrate both sides to obtain

$$\tau_{1,2} = \sqrt{-1-2\Phi_{1,2}}t$$

So that we can find

$$\tau_1 = \left[\frac{R_2(2GM-R_1)}{R_1(2GM-R_2)}\right]^{1/2}\tau_2$$

If $R_2 > R_1$, the coefficient is greater than one and we find that time passes faster farther away from earth.

Now suppose that there's a particle orbiting the earth at a fixed radius and $\theta = \pi/2$. In this case $\mathrm{d}t \neq 0$ and $\mathrm{d}\phi = 0$; there are two nonvanishing differentials and we can't approach the problem like we did above. We will need to use the geodesic equation

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + \Gamma^r_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0$$

We can assume that in this orbit $\mathrm{d}\phi\, t$ is some constant ω. Then the only relevant geodesic equation is for $\mathrm{d}^2 r/\mathrm{d}\lambda^2$. Since r is fixed, $\mathrm{d}r/\mathrm{d}\lambda = \mathrm{d}^2 r/\mathrm{d}\lambda^2 = 0$, and similarly for θ. The only relevant Christoffel symbols are then $\Gamma^r_{\phi\phi}$ and Γ^r_{tt}; the remaining terms in the geodesic equation above vanish. The geodesic equation becomes

$$\Gamma^r_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 + \Gamma^r_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = 0$$

There is no time dependence in any of the elements of the metric and thus t is equal to λ times some constant of proportionality. Applying the chain rule to all terms above and cancelling the factors of $(\mathrm{d}\lambda/\mathrm{d}t)^2$ finds

$$\Gamma^r_{\phi\phi}\dot{\phi}^2 + \Gamma^r_{tt} = 0$$

Substituting $\mathrm{d}\phi/\mathrm{d}t = \omega$ and solving for it yields

$$\omega = \sqrt{\frac{\Gamma^r_{tt}}{\Gamma^r_{\phi\phi}}}$$

Lastly, we can determine the proper time elapsed per complete orbit for someone on the surface of the earth as compared to the orbiting particle itself. We will have to use the expression for the proper time:

$$\tau = \int \left(-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)^{1/2} \mathrm{d}\lambda\,.$$

In our case an observer on earth will find

$$\tau_{\mathrm{E}} = \int \left[(1+2\Phi) - r^2\omega^2\right]^{1/2} \mathrm{d}t = \sqrt{(1+2\Phi) - r^2\omega^2}\cdot t$$

(we've again used the fact that we can reparameterize from λ to t).

3.2 Comments on Killing Vectors

The Killing vectors of Minkowski spacetime are precisely the Lie algebra of the Poincaré group:

- Time and space translations ∂_μ.
- Rotation generators $-x_i\partial_j + x_j\partial_i$.
- Boost generators $x_i\partial_0 + x_0\partial_i$.

The boost generators (and, I think the rotation generators as well) can be seen to have the above form by considering a very small boost by a rapidity ϕ. For this purpose we can jettison the two other spatial coordinates and just focus on one spatial direction. Then the boost is

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\phi & -\sin\phi \\ \sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \approx \begin{bmatrix} 1 & -\phi \\ \phi & 1 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} t \\ x \end{bmatrix} + \phi \begin{bmatrix} x \\ t \end{bmatrix}$$

writing the second vector (without the ϕ prefactor) in the standard differential geometric notation recovers $x_i\partial_0 + x_0\partial x_i$. The procedure is analogous for the other elements of the Lie algebra, e.g., ∂_μ comes from

$$\begin{bmatrix} t' \\ x' \end{bmatrix} \approx \begin{bmatrix} t \\ x \end{bmatrix} + \delta x \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The precise connection to the exponential map in Lie theory is not crystal clear to me (especially when in Hilbert space); I assume the matrices multiplying our vectors above (e.g., the rapidity matrix) is the matrix exponential which for a small parameter turns approximately into a simple linear transformation as above, but according to that picture the generators should be matrices (operators), not vectors as we see here. Will investigate further when I have time.

3.2.1 Static and Stationary Spacetimes

A metric is called stationary if it has a timelike Killing vector, which essentially means that you can find observers such that the spacetime is unchanging. A static metric (or static spacetime) is that of a spacetime that is additionally irrotational. It turns out that an equivalent definition of a static metric is a stationary metric that is symmetric with respect to $\partial_t \to -\partial_t$, or, alternatively, a metric such that ∂_t is everywhere orthogonal to a spacelike hypersurface. We can show these two definitions are equivalent.

Some preliminaries: a vector v_μ orthogonal to a hypersurface of constant f can be written $v_\mu = h\nabla_\mu f = h\partial_\mu f$ for a function h (we can write it in terms of a regular partial derivative because a function is a rank-$(0,0)$ tensor). The Frobenius theorem then tells us we can write

$$v_{[\sigma}\nabla_\mu v_{\nu]} = 0 \tag{3.1}$$

The proof is simple; just write $v_\mu = h\nabla_\mu f$ and use the fact that the covariant derivative commutes when applied to functions. Lastly, note the Killing equation can be written as $\mathcal{L}_\xi g_{\mu\nu}$ for a Killing vector ξ (this just says that the metric tensor doesn't change along a Killing vector, the defining property of Killing vectors).

One last fact that we need: if ξ is timelike, we can write it ∂_t for some choice of coordinates. Expanding the Lie derivative above and substituting $\xi = \partial_t$:

$$\mathcal{L}_\xi g = \partial_\gamma g_{\alpha\beta} \cdot \xi^\gamma - g_{\mu\beta}\partial_\mu \xi^\alpha + g_{\alpha\mu}\partial_\beta \xi^\mu = 0 \quad \Longrightarrow \quad \partial_\gamma g_{\alpha\beta}\xi^\gamma = 0 \quad \Longrightarrow \quad \partial_t g_{\alpha\beta} = 0$$

This means that if ξ is timelike, the metric should be independent of time.

Now, to show that if the metric is symmetric with respect to $\partial_t \to -\partial_t$, ∂_t is everywhere orthogonal to spacelike hypersurfaces. Note that if the metric has this symmetry, there can be no terms in $\mathrm{d}s$ linear to $\mathrm{d}t$, since this transformation has $\mathrm{d}t \to -\mathrm{d}t$. This means that $g_{0i} = 0$ for all spatial components i. Then, it is a covariant fact of the matter that

$$g_{\mu\nu}\xi^\mu = g_{0\nu} = \delta^0_\nu$$

where in the last equation we used g_{0i}. This is true everywhere and thus we can write ξ as the covariant derivative of some function of constant t.

Now show hypersurface orthogonality implies reversal symmetry. We can write $\xi_\mu = \xi^2 \nabla_\mu f = \xi^2 \partial_\mu f$[1]. Since $\xi^2 = g_{\mu\nu}\xi^\mu\xi^\nu = g_{tt}$ and $\xi_\mu = g_{\mu\nu}\xi^\nu = g_{0\nu}$ in these coordinates, this equation is equivalent to

$$g_{0\mu} = \partial_\mu f \Longrightarrow g_{00} = g_{00}\partial_\mu f \Longrightarrow \partial_0 f = 1$$

Solving this equation, we find $f = t + g(\mathbf{x})$ for some arbitrary function $g(\mathbf{x})$ of the spatial coordinates. Now if we transform coordinates $t \to t' = t - g(\mathbf{x})$ and plug into our equation above again

$$g'_{0i} = g'_{00}\partial_i(t' - g(\mathbf{x}) + g(\mathbf{x})) = g'_{00}\partial_i t' = 0$$

which we showed above is equivalent to the metric being time-reversal symmetric. This transformation preserves the Killing vector, since

$$\xi^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\xi^\mu = \frac{\partial x^{\mu'}}{\partial x^0}\xi^0 = \frac{\partial t'}{\partial t}\delta^{\mu'}_0 = (1, 0, 0, 0).$$

so we can repeat the argument above to show the metric remains independent of t'

This transformation does change the metric, though (hence we wrote $g \to g'$), because $\mathrm{d}t' = \mathrm{d}t - \mathrm{d}g$. However, this does not ultimately matter; we are free to choose whatever coordinates make demonstrating a certain fact most convenient and the geometry should remain the same, as the changes in coordinates are appropriately accounted for by the Jacobian, etc.

[1] At least I'm pretty sure you can; I justify it by noting it's still the case that $\xi = \partial_t$ by hypothesis, which means we can choose f such that ∂_μ has unit length.

3.3 Canonical Density Matrix with Truncated Spectrum

This is something I ran into working with data from finite-temperature trials of SU(2) spin chains. The upshot is that if we are trying to compute the canonical density matrix $e^{-\beta H}$ such a chain, the computed density matrix is inaccurate when we don't have access to the full spectrum (such as with sparse diagonalizations), and the error depends on chain size and temperature.

Recall by the Lieb-Schultz-Mattis theorem the energy gap will vanish as we increase the number of sites (it goes as 1/N). Also recall that the Boltzmann weights go as $e^{-\beta E_i}$.

What this means is that as we crank up the number of sites while keeping β, the gap will start to close and we will tend to sample higher and higher energy eigenstates (i.e., states further up the spectrum). The issue is if we only include a few eigenstates (as we would need to in a sparse matrix diagonalization) the canonical density matrix would require more and more higher energy eigenstates that are missing in the numerics.

Note that if we keep our value of β at the same scale as the energy gap, then the temperature range shrinks with the energy gap and we don't need to worry about sampling states that are not included in our numerics (because the exponential would scale with the gap and those states would remain in the vanishing tail of the distribution). This is what Alex meant by "currently, my temperature scale is the width of the whole spectrum rather than the gap..."

Chapter 4

September

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4.1 Spontaneous Parametric Down-Conversion

I have been working in the instructional labs at Brown setting up a demonstration of Bell's inequality. We are using type II Spontaneous Parametric Down-Conversion (SPDC), a nonlinear optical phenomenon which produces pairs of photons entangled in the polarization basis. In our case we are using a barium borate (BBO) crystal.

The crystal produces two cones of photons bearing horizontal (ordinary) and vertical (extraordinary polarizations). One of these photons is referred to as the idler photon and the other the signal photon. The incoming photon is referred to as a pump photon. The effective Hamiltonian is

$$H = i\hbar\kappa \left(a_i a_s a_p^\dagger e^{i\Delta\mathbf{k}\cdot\mathbf{r} - i\Delta\omega\cdot t} + a_i^\dagger a_s^\dagger a_p e^{-i\Delta\mathbf{k}\cdot\mathbf{r} + i\Delta\omega\cdot t} \right)$$

where $\Delta\omega = \omega_p - \omega_i - \omega_s$, a_i creates an idler photon, so on. In the case of Type II SPDC, the nonlinear crystal produces the two photons in two cones corresponding to horizontal and vertical polarizations; the origin of these cones lies of course in conservation of momentum. The cone the photon is in is therefore perfectly correlated with the polarization, and if we know the cone, we know the polarization. The exception is the intersection of the two cones, where photons of either polarization could be found. Since the photons are in this case indistinguishable, the state of the system must

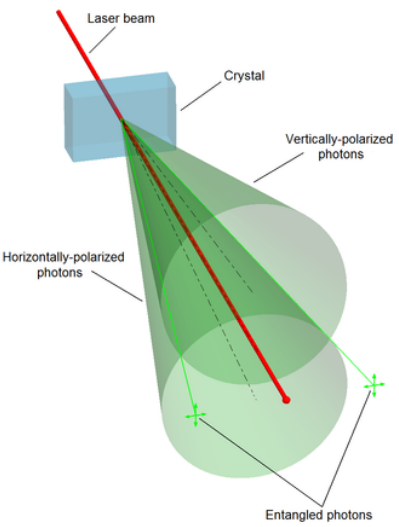

Figure 4.1:

be symmetrized and we have

$$|\psi\rangle = |H_i\rangle\,|V_s\rangle + |V_i\rangle\,|H_s\rangle\,.$$

We can then test Bell's inequality by looking for *coincident* (simultaneously arriving) photons hitting our detectors, which should be entangled. There's also an arbitrary phase in front of the second term above, which I've set to 1. Don't really understand why.

Sources

- C. Cocteau, Spontaneous Parametric Down-Conversion.

4.2 Some Combinatorics Intuitions

Binomial coefficient How might we construct the formula for n choose k? We start with the number of permutations of the objects, then divide by the objects in the collection we don't care about. This is $n!/(n-k)!$, so there are n ways to choose the first object, $n-1$ ways to choose the second, and so on until we've chosen k, so we truncate at $n-k$. But we also decide that the order doesn't matter, so we must divide by $k!$ to eliminate the counts from the different permutations of those $k!$ objects we picked.

$$n \text{ choose } k = \binom{n}{k} = \frac{n!}{k!\;(n-k)!}$$

permutation

accounts for order not mattering

eliminates objects we're not considering

. . . with ordering This suggests a simple way to get the number of possible choices *keeping track of order*: just get rid of the factor that eliminates the "ordering equivalence"

$$\frac{n!}{(n-k)!}.$$

. . . with replacement Now say we choose r objects from a collection of n *with replacement*. We keep the $r!$ in the numerator to ignore ordering. We are sampling r times from that collection of objects, so to sample with repetition we have to add that object back into the collection for every time we sample. Since the collection *already starts* with the object, we add $r-1$ (not r), so the numerator takes on $(n+r-1)!$. The numerator then goes to $(n-k)!$ to $((n+r-1)-r)!$ so as to serve the same function as before, getting rid of the

objects we're not sampling.

$$n \text{ choose } k = \binom{n}{k} = \frac{(n+r-1)!}{r!\ (n-1)!}$$

permuatation replacing $r-1$ times

accounts for order not mattering

from $(n+r-1-r)$

4.3 Noether's Theorem and Time-Dependent Symmetries

The upshot is that the conserved quantity J has nonvanishing partial derivative but its Poisson bracket compensates such that the total derivative vanishes

$$\frac{\partial J}{\partial t} = 0, \quad \text{but} \quad \frac{dJ}{dt} = -\{H, K\} + \frac{\partial J}{\partial t} = 0$$

Here is a differential geometric way of viewing the above equation: consider some space with coordinates (q, p, t), and some trajectory $q(t) = x(t)$, and some function J on this manifold. The differential/exterior derivative of J is

$$\begin{aligned} \mathrm{d}J &= \frac{\partial J}{\partial x}\,\mathrm{d}x + \frac{\partial J}{\partial p}\,\mathrm{d}p + \frac{\partial J}{\partial t}\,\mathrm{d}t \\ \implies \frac{dJ}{dt} &= \frac{\partial J}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial J}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial J}{\partial t} \end{aligned}$$

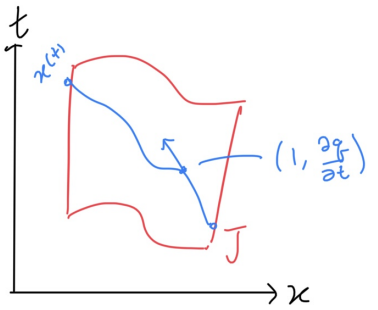

Figure 4.2

Using Hamilton's equations $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$, we find

$$\frac{dJ}{dt} = \frac{\partial J}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial J}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial J}{\partial t} = -\{H, K\} + \frac{\partial J}{\partial t} \tag{4.1}$$

This is represented in cartoon form by fig. 4.2.

Galilean Boost The Galilean boost is of the form $x \to x + vt$, which is a symmetry of Newtonian mechanics. The conserved quantity is

$$mx_{\text{COM}} - pt = \text{const.} \implies mx_{\text{COM}} = \text{const.} + pt$$

I.e., the center of mass moves according to the total momentum. The special relativistic analogue is that the conserved quantity of the Lorenz boost is the center of mass-energy. We

can work out eq. (4.1) for $J = mx_{\text{com}} - pt$, with $H = p^2/2m$:

$$\begin{aligned}\frac{dJ}{dt} &= \frac{\partial}{\partial x}(mx_{\text{COM}} - pt) \cdot \frac{\partial}{\partial p}\left(\frac{p^2}{2m}\right) - \frac{\partial}{\partial p}(mx_{\text{COM}} - pt) \cdot \cancel{\frac{\partial}{\partial x}\left(\frac{p^2}{2m}\right)} + \frac{\partial}{\partial p}(-pt) \\ &= m \cdot \frac{p}{m} - p = p - p = 0.\end{aligned}$$

4.4 Cosmological Constant

Consider some nonvanishing vacuum energy-momentum tensor $T^{\mu\nu}$. We postulate the vacuum is frame-independent (Lorentz-invariant), and also know that the only rank-two tensor that is Lorentz invariant is the metric. (This is "lore" but I'm not sure what the proof is.) (The invariance of the metric is what *defines* Lorentz transformations). $T^{\mu\nu}$ is then of the form

$$T^{\mu\nu} = Eg^{\mu\nu}$$

Then write a new general energy-momentum tensor capturing the vacuum energy:

$$T^{\mu\nu} \to T^{\mu\nu} + Eg^{\mu\nu}$$

and the Einstein equation becomes

$$G^{\mu\nu} = \kappa(T^{\mu\nu} + Eg^{\mu}\nu) \implies G^{\mu\nu} - \kappa E g^{\mu\nu} = \kappa T^{\mu\nu}$$

setting $-\kappa E = \Lambda$ recovers the usual form of the Einstein equation with a cosmological constant. The whole "cosmological constant" problem is that the vacuum energy of most quantum field theories yields a much larger cosmological constant than is actually observed.

4.5 Newton-Cartan Gravity

Chapter 5

October

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5.1 The Higgs Mechanism

Gauging a symmetry Consider the linear sigma model

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial_\mu \phi) + m^2 \phi \phi^* - \frac{\lambda}{4} \phi^2 \phi^{*2}$$

which has a $U(1)$ symmetry. We now "gauge" this symmetry by promoting to a local transformation: $\phi \to e^{i\alpha(x)}\phi$. To do this we need to

- make the kinetic terms covariant wrt the gauge field and change $\partial_\mu \to D_\mu$,
- and introduce the corresponding dynamical term for the gauge field:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} + D_\mu \phi^* D_\mu \phi + m^2 |\phi|^2 - \frac{\lambda}{4} |\phi|^4$$

For a non-Abelian theory we'd have

$$\mathcal{L} = -\frac{1}{4} F^{(a)\mu\nu} F^{(a)}_{\mu\nu} + (\partial_\mu \phi^* - ieA^{(a)}_\mu(x) T^{(a)})(\partial_\mu \phi + ieA^{(a)}_\mu(x) T^{(a)}) + \cdots$$

In differential geometric language, recall that the fields ϕ are sections of the (in this case, vector) bundle, $A_\mu(x)$ are the connections corresponding to the principal G bundle where G is the Lie group of this theory.

Higgs mechanism Recall that spontaneous symmetry breaking puts us in some classical minimum configuration which we can write as

$$\phi(x) = \left(\frac{v+\sigma(x)}{\sqrt{2}}\right) e^{i\pi(x)/F_\pi}$$

where $\pi(x)$ is the massless Goldstone boson (colloquially referred to as a pion) which will eventually be "eaten" by the gauge boson to gain mass, Now focus on the kinetic term only

$$-\frac{1}{4}F_{\mu\nu}^2 + \left(\frac{v+\sigma}{\sqrt{2}}\right)^2 \left[-i\frac{\partial_\mu \pi}{F_\pi} + \frac{\partial_\mu \sigma}{v+\sigma} + ieA_\mu\right]^2$$

This quadratic factor with three terms in the brackets will mix the dynamics of the three fields A_μ, π, and σ, but we single out the behavior of A_μ and π by considering the sector where σ becomes decoupled from the other two fields. This is the so-called **decoupling limit**, which is achieved by giving σ an infinite mass $m \to \infty$. Then σ is not at all dynamical and we can set it to zero. The potential term, had we included it, would have been

$$-m^2\sigma^2 - \frac{1}{2}\sqrt{\lambda}\sigma^3 - \frac{1}{16}\lambda\sigma^4$$

which shows that $m \to \infty$ is the right limit. The kinetic term then becomes

$$\frac{v^2}{2}\left[i\frac{\partial_\mu \pi}{F_\pi} + ieA_\mu\right]^2 = -\frac{v^2e^2}{2}\left[\frac{\partial_\mu \pi}{F_\pi e} + A_\mu\right]^2$$

The gauge invariance transforms the fields nonlinearly[1] as

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x), \qquad \pi(x) \to \pi(x) - F_\pi \alpha(x)$$

The argument is then that we can choose the **unitary gauge** where $\alpha(x) = \pi(x)/F_\pi$, and the kinetic term becomes

$$\frac{v^2e^2}{2}A_\mu^2 = \frac{1}{2}m_A A_\mu^2, \quad v^2e^2 = m_A$$

and our gauge boson now has a mass.

More complicated groups In the case of a bigger (more generators) group, Goldstone's theorem says we have a massless Goldstone boson for every broken generator. E.g., if we get the vev

$$\langle\phi\rangle = \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix}$$

[1] When people say this they just mean that the transformation induces the addition of a term as opposed to transforming the field according to some linear operator acting on it.

we break $SO(3)$ down to $SO(2)$, for the vacuum is still invariant under $SO(2)$ of the first two entries of the vector. $SO(2)$ has one generator, so there are two Goldstone bosons for the two broken generators of $SO(3)$. If we gauge this $SO(3)$ symmetry, those two Goldstone bosons are then Higgsed away to produce two massive gauge bosons.

More generally, for a group G, if we spontaneously break to a subgroup H, this means the vacuum is still invariant with respect to H but transforms nontrivially under the quotient set G/H (it is a set not a subgroup because it is possible that H is not a normal subgroup). Sometimes we understand the unbroken gauge coset as the "maximal torus" of the group (or something like that lol). The Goldstone mode is a fluctuation in the direction of the broken generators, the generators of G/H; intuitively, if we pick a VEV $\langle\phi\rangle = (0, 0, 1)$ for a theory with a global SO(3) symmetry picking a vacuum configuration pointing in the z-direction, the Goldstone mode is in the generators θ_x and θ_y. More about this in **??**

Condensed matter people will then say that the ground states of the system are labelled by G/H in the sense that the cosets of H are in one-to-one correspondence with the ground states. In particular, this means that elements H do not change the vacuum, but any element of a fixed coset g_1H will transform the vacuum, and the different vacua can be reached by the different cosets. E.g., if the cosets are g_1H, g_2H and g_3H there are three different vacua.

5.1.1 Elitzur's Theorem

Now, something is amiss. Symmetry breaking is introduced typically as a feature of *global symmetries*, which are bona-fide transformations to make to your system. On the other hand, gauge invariance is understood as a feature of the Hilbert space we are working with, i.e., it's understood as an "overlabelling" of the states (see, e.g., Wen's book on many-body theory).

This is not a physical feature but a mathematical one (which is why some people, myself included, use the term invariance as opposed to symmetry in context of gauging), meaning that it shouldn't really be possible to "break" such an invariance.

Indeed, Elitzur's theorem says

> The only operators with non-vanishing expectation values are those that are gauge invariant.

But note that the principal G-bundle has a section corresponding to that global transformation. E.g., in the case of a $U(1)$ symmetry, the transformation $e^{i\alpha(x)}\phi(x)$ can be of the form $e^{i\alpha_0}\phi(x)$ if we just choose $\alpha(x)$ to be a constant function $\alpha(x) = \alpha_0$. It is postulated that the gauge "symmetry" breaking just corresponds to the spontaneous breaking of that section. It's really tempting to associate these transformations with so-called **large gauge transformations**, but I am pretty sure that they're unrelated, despite a comment to that effect by Humberto. Large gauge transformations are gauge transformations that are not

related to the trivial transformation by a homeomorphism. They will come up in the gauge theory of Heisenberg models, §**??**.

5.2 Roots and Weights of a Lie Algebra

The short of it is

- **Weights** are the eigenvalues of the largest set of mutually commuting elements of your Lie Algebra (the Cartan subalgebra) *for a given representation*, meaning that they can be used to label your eigenstates. Think $\pm 1/2$ for $S = 1/2$, $\pm 1, 0$ for $S = 1$, etc.
- **Roots** are the weights of your *adjoint representation*; they turn out to correspond to the change in the quantum number that is induced by the raising and lowering operators and thus tell you how to get around your weight diagram.

SU(3) example The matter fields in one's gauge theory transform in the **fundamental representation** (or in the **defining representation**, which is the same as the fundamental representation for most groups of interest to physicists), meaning that the weights of the fundamental representation comprise the possible charges of the particles in your theory. For example, in $SU(3)$, the Cartan subalgebra is comprised of

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

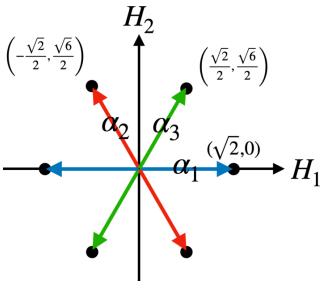

Figure 5.1: The colors represent the actual color charge.

Which are two of the Gell-Mann matrices. This Lie algebra has two raising and lowering operators, and two elements of the Cartan subalgebra; we thus need two numbers to distinguish the weights, one for each element of the subalgebra. The matrices above are already diagonalized, so the eigenvectors are just the unit vectors $(1, 0, 0), \ldots$; we then understand that the two numbers labelling the weights are just the eigenvalues of each member of the Cartan subalgebra, organized together (and divided by $1/2$ for technical reasons). In this case we have that they should be $(\lambda_{3,ii}, \lambda_{8,ii})$, where ii denotes the i-th diagonal element:

$$(\mu_1, \mu_2) = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right), \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right), \left(0, -\frac{1}{\sqrt{3}}\right)$$

These are related to each other by the roots (the weights of the adjoint representation) of which there are six (see fig. 5.1).

Constraint on the angles One can prove that generally for a weight $\boldsymbol{\mu}$ and a root $\boldsymbol{\alpha}$

$$\frac{\boldsymbol{\mu}\cdot\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}\in\frac{1}{2}\mathbb{Z}$$

(which also holds for two roots, obviously). This amounts to the statement that the angle between roots must be

- Angle of 90 degrees
- Angle of 60 or 120 degrees with a length ratio of 1
- Angle of 45 or 135 degrees with a length ratio of $\sqrt{2}$
- Angle of 30 or 150 degrees with a length ratio of $\sqrt{3}$.

In more complicated root systems, however, the roots might live in some higher-dimensional space (i.e., $\mathbb{R}^n$ where $n>2$), or the roots might not be unit vectors; nonetheless this condition will still hold. The group of reflections of the root system comprises the Weyl group (Not sure why it needs to be the reflections only; usually this does not comprise the whole symmetry group of the roots.).

5.2.1 Co-roots

The co-root of a root $\boldsymbol{\alpha}$ is

$$\boldsymbol{\alpha}^\wedge\equiv\frac{2\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}\quad\Longrightarrow\quad\boldsymbol{\alpha}^\wedge\cdot\boldsymbol{\mu}\in\mathbb{Z}$$

where the right side of the implication can be thought of as a defining property of the co-roots. The Dirac quantization condition

$$e^{i\mathbf{m}\cdot\boldsymbol{\mu}}=1$$

becomes the condition that $\mathbf{m}\in 2\pi\Lambda_{\text{co-root}}(\mathfrak{g})$, where $\Lambda_{\text{co-root}}(\mathfrak{g})$ is the lattice spanned by the co-roots (known as the Goddard-Nuyts-Olive quantization condition). Here lattice is meant in the usual sense of a set of vectors, not a condition on the ordering.

It's interesting to note that while for $\mathrm{SU}(N)$ gauge theory the 't Hooft lines are in $\Lambda_{\text{co-roots}}(\mathfrak{g})=\Lambda_{\text{roots}}(\mathfrak{g}^\wedge)$, but for $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory the representations are not labelled by the weight lattice $\Lambda_w(\mathfrak{g})$ but by $\Lambda_{\text{co-roots}}(\mathfrak{g})$, which can be understood as a consequence of

$$\Lambda_w(\mathfrak{g})/\Lambda_{\text{root}}(\mathfrak{g})=\mathbb{Z}_N$$

(lattices have a group structure so we can take their quotient). This means that the 't Hooft lines are labelled by $\Lambda_w(\mathfrak{g})$.

Actually, it's a general feature of Lie algebras that the center of the Lie group is given by the quotient $\Lambda_w/\Lambda_{\text{root}}$; see here for a technical proof. Though this is not too hard to see how this is the case by drawing some root lattices overlaid on weight lattices. For instance in SU(2) we'd have the set of integers for the roots, and the root lattice would be the integers and the integers displaced by 1/2; modding by the root lattice then just gives us $\{0, 1/2\} \simeq \mathbb{Z}_2$. The argument follows for SU(N): see for instance here. A crucial property is that the root lattice vectors have larger norms than the weight lattice vectors so that the quotient is not the trivial group.

5.3 Projective Representations, Central Extensions

Mathematically, the projective representation of a group G is defined as the group homomorphism

$$G \to \mathrm{GL}(V)/F^*$$

where $\mathrm{GL}(V)$ is the general linear group over some vector space V over a field F, and $F^* \lhd \mathrm{GL}(V)$ is the normal subgroup consisting of scalar multiples of the identity operator.

In quantum mechanics we map the group elements to unitary operators on Hilbert space, but quotient by operators $e^{i\phi} \cdot \mathbb{I} = e^{i\phi}$ because states are equivalent up to a phase. The allowable scalar multiples all have to be of magnitude one since we are working with unitary representations, hence they may be written as a phase; hence the representation of G we must work with in quantum mechanics is the projective unitary representation

$$G \to U(\mathcal{H})/e^{i\phi}, \quad \phi \in \mathbb{R}.$$

Central extension If we want to move from a projective representation to an ordinary representation (no modulo by F^*), we can promote G to its universal cover $\tilde{G}$:

$$G \curvearrowright \mathcal{H} \text{ by projective rep.} \longrightarrow \text{universal cover } \tilde{G} \curvearrowright \mathcal{H} \text{ by ordinary rep.}$$

The universal cover $\tilde{G}$ turns out to be the **central extension** of $\rho^{-1}(F^*)$ by G, where $\rho^{-1}(F^*)$ is the preimage of F^* under the representation ρ. That is, G is the quotient $\tilde{G}/\rho^{-1}(F^*)$, $\rho^{-1}(F^*)$ is in the center of $\tilde{G}$, and the following is a short exact sequence:

$$1 \to \rho^{-1}(F^*) \to \tilde{G} \to G \to 1$$

Generally, the central extension of a group N by G is some $\tilde{G}$ such that N is in the center of $\tilde{G}$ and $\tilde{G}/N \simeq G$. This leads to the statement that the quantum group corresponding to a symmetry is the central extension of that symmetry group, for that central extension gives you a group of which the projective representation is an ordinary representation.

There is an additional statement to the effect that the second cohomology group of a Lie group classifies the central extension, but I do not know algebraic topology yet so I

haven't a clue what this means. Regardless this does lead to a simple way to understand why semi-simple finite dimensional Lie algebras do not possess a central charge whereas the Virasoro algebra does. Will return to this later!

Example An example is SO(3), which becomes the central extension/universal cover SU(2). The quantum representation of SO(3) is a projective representation, but an ordinary representation of SU(2).

5.4 Short Exact Sequences

One might see short exact sequences thrown around a ton (see previous section); reading the Wikipedia/nLab pages doesn't do much to answer why a certain set of objects forming a short exact sequence is of note. The actual reason is that they essentially express, in the most general and unambiguous way, a quotient relationship between groups.

Namely, if $G/H = N$, then we cannot simply write $G = H \times N$; in fact, it could be the case that $G = H \ltimes N$ for a semidirect product. The short exact sequence, however, expresses this relationship exactly (no pun intended):

$$A \rtimes C \simeq B \quad \iff \quad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \quad \iff \quad C \simeq B/A$$

If the sequence *splits*, the semidirect product turns into the direct product: $A \rtimes C \to A \times C$.

We can see that in the case of groups this is in line with the textbook definition: the kernel of ... is the image of ....

5.5 Semi-Direct Product

A mnemonic for which of the groups does not possess an induced group composition is that the side of $\ltimes$ with the vertical line points toward the group with the induced composition. So the side that looks like the normal direct product is the opposite of what you would expect it to be.

We have that if a system enjoys two symmetries A and B, and A and B don't commute, $G \not\simeq A \times B$, or G cannot simply be the direct product of its two symmetries A and B. It's easiest to see this by proving the contrapositive:

$$G \simeq A \times B \implies A \text{ and } B \text{ commute.}$$

which follows easily from the definition of the direct product: if $a \in A$, $b \in B$,

$$(a, e) \cdot (e, b) = (a, b) = (e, b) \cdot (a, e).$$

We can see by construction that the semidirect product admits a noncommutative pair of symmetries like this. Let $a, e_A \in A$ and $b, e_B \in B$. Then for $A \ltimes B$ we have

$$(a, e_B) \cdot (e_A, b) = (a, e_B \cdot \varphi_{e_A}(b)) = (a, \varphi_{e_A}(b))$$
$$(e_A, b) \cdot (a, e_B) = (a, b \cdot \varphi_a(e_B)) = (a, b)$$

(Recall that automorphisms, being homomorphisms, must send the identity to itself.) In general we can choose φ_{e_A} such that $b \neq \varphi_{e_A}(b)$. This means that a and b don't commute provided we choose our automorphism appropriately, which is what we would like.

We can go a step further and make this construction for the symmetry enjoyed by the particle on a ring (section 6.4), $\mathbb{Z}_2 \ltimes \mathrm{U}(1)$

$$PT_\alpha P = (P, e) \cdot (e, T_\alpha) \cdot (P, e) = (P, e) \cdot (P, T_\alpha) = (P^2, \varphi_P(T_\alpha)) = (e, T_{-\alpha})$$

where we have to stipulate $\varphi_P(T_\alpha) = T_{-\alpha}$. Had we just used $\mathbb{Z}_2 \times \mathrm{U}(1)$ we would find

$$PT_\alpha P = (P, e) \cdot (e, T_\alpha) \cdot (P, e) = (P, e) \cdot (P, T_\alpha) = (e, T_\alpha).$$

5.6 Pontryagin Dual

The Pontryagin dual is occasionally referred to as the generalization of the Fourier transform, although this isn't quite right. The correct statement is that the Pontryagin dual generalizes the domain of the Fourier transform. Where the time domain—also $\mathbb{R}$—has a dual frequency domain $\mathbb{R}$ over which the Fourier transform is defined, or where the crystal lattice has a dual reciprocal lattice, any locally compact Abelian group G has a Pontryagin dual $\widehat{G}$ over which a generalization of the Fourier transform may be defined. In fact, we can calculate the Pontryagin duals for a selection of spaces to prove the following familiar Fourier analysis correspondences:

G	**"Time" domain**	$\widehat{G}$	**"Frequency" domain**
$\mathbb{R}$	(continuous, non-compact)	$\mathbb{R}$	(continuous, non-compact)
$\mathbb{Z}$	(discrete, non-compact)	S^1	(continuous, compact)
S^1	(continuous, compact)	$\mathbb{Z}$	(discrete, non-compact)
$\mathbb{Z}_n$	(discrete, compact)	$\mathbb{Z}_n$	(discrete, compact)

5.7 Axial Current and Anomaly

I recently did a problem involving an axial (or chiral) anomaly. In some Dirac fermion theory the Lagrangian has the $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ symmetry

$$\begin{aligned} \text{Vector current:} \quad \psi(x) \to e^{i\theta}\psi(x) \quad &\implies \quad J^\mu = \bar{\psi}(x)\gamma^\mu\psi(x) \\ \text{Axial current:} \quad \psi(x) \to e^{i\gamma^5\theta}\psi(x) \quad &\implies \quad J^\mu = \bar{\psi}(x)0\gamma^5\gamma^\mu\psi(x) \end{aligned}$$

The vector current is associated with charge conservation [relationship to gauge symmetry?]. Let's study this axial conserved current more explicitly. The in Weyl basis:

$$J^5_\mu = \psi^\dagger\gamma^0\gamma^5\gamma^\mu\psi$$

so that

$$J^5_k = \begin{bmatrix}\psi_L^\dagger & \psi_R^\dagger\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{bmatrix}\begin{bmatrix}\psi_L\\ \psi_R\end{bmatrix} = -\psi_L^\dagger\sigma^k\psi_L - \psi_R^\dagger\sigma^k\psi_R = -J_{\mu=k}$$

where J_μ is vector current.

$$J^5_0 = \begin{bmatrix}\psi_L^\dagger & \psi_R^\dagger\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_L\\ \psi_R\end{bmatrix} = \psi_L^\dagger\psi_L - \psi_R^\dagger\psi_R$$

In $(1+1)$d (the Schwinger model), we can express the magnitude of this non-conserved quantity as some scalar field which ends up being the bosonized field in the bosonization of the Schwinger model.

Consequences (chat with Humberto) I briefly spoke with Humberto about the physical consequences of this anomaly. One thing he said is that indeed you can think of this type of anomaly as a source term in your Lagrangian, so that the field theory is producing fermions of a certain chirality. As for how this shows up in experiment, the upshot was that in certain high energy processes you find that the incoming and outgoing states do not preserve chirality.

Chapter 6

November

Contents

6.1 Hubbard-Stratonovich, MFT, etc.

A few things are said about the relationship between the Hubbard-Stratonovich transformation, mean-field theory, and the saddle-point approximation:

- The saddle-point approximation of the Hubbard-Stratonovich transformation is mean-field theory.
- The saddle-point approximation provides corrections to the mean-field approximations.
- The Hubbard-Stratonovich transformation renders mean-field theory exact.

6.2 Heisenberg Model and QED_3

The Heisenberg model is not exactly solvable in $(2+1)$-d, but one can still gain insight into the structure of its ground state and excitations by pursuing a large-N approximation in an $SU(N)$ version of the theory. This is achieved through a menagerie of transforms and approximations which will eventually take us to a solvable $U(1)$ lattice gauge theory with a zoo of background gauge fields corresponding to different minima or phases of the theory.

One particular minimum corresponds to a field theory whose continuum limit is massless QED$_3$. We will also see that this theory will afford us insight into the ground of state of the actual SU(2) Heisenberg model by providing us variational wavefunction ansatze based on the exact large-N phases of the theory.

6.2.1 Gauge Fields via Hubbard-Stratonovich

Starting from the fact that the Heisenberg interaction can be written

$$\mathbf{S}_i \cdot \mathbf{S}_j = \frac{1}{2} c^\dagger_{\alpha i} c_{\beta i} c^\dagger_{\beta j} c_{\alpha j} - \frac{1}{4} n_i n_j$$

(the repeated greek indices sum over spin components) with half-filling the fermion number term becomes a constant so that the SU(2) Heisenberg Hamiltonian is simply

$$H = \frac{J}{2} \sum_{\langle i,j \rangle} c^\dagger_{\alpha i} c_{\beta i} c^\dagger_{\beta j} c_{\alpha i}$$

To perform a Hubbard-Stratonovich transformation it is necessary to work in the path integral representation and thus we Legendre transform to find the Lagrangian

$$\mathcal{L} = i \sum_i c^\dagger_{\alpha i}(t) \partial_t c_{\alpha i}(t) + \sum_i \varphi_i(t) (c^\dagger_{\alpha i}(t) c_{\alpha i}(t) - 1) - H$$

where we have introduced the background field $\varphi_i(t)$ as a Lagrange multiplier enforcing the half-filling. The presence of this field will turn out to be crucial to our understand of this as a relativistic gauge theory.

Aside

Fields without a kinetic term appearing in a path integral become Lagrange multipliers enforcing a constraint. This can be seen from the fact that in the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = 0$$

the second term in the LHS vanishes and so the field's equation of motion becomes an *algebraic* (contrast *differential*) equation for a constant ϕ. Note that this does mean the Lagrange multiplier does not experience quantum fluctuations.

Now proceeding with the Hubbard–Stratonovich transformation, we are able to convert a quartic interaction into an auxiliary field–fermion^2 interaction and an auxiliary field

quadratic interaction. The auxiliary field we introduce is a complex bosonic link variable $\chi_{ij} = \rho_{ij}(t)e^{iA_{ij}(t)} = \chi^*_{ji}$; where we anticipated its analogy with $U(1)$ lattice gauge theory by expressing it as an amplitude ρ and phase A. The Lagrangian is then

$$\begin{aligned}\mathcal{L} = i\sum_i c^\dagger_{\alpha i}(t)\partial_t c_{\alpha i}(t) + \sum_i \varphi_i(t)(c^\dagger_{\alpha i}(t)c_{\alpha i}(t) - 1) \\ - \frac{2}{J}\sum_{\langle i,j\rangle}|\chi_{ij}|^2 + \sum_{\langle ij\rangle}\Big(c^\dagger_{\alpha i}(t)\chi_{ij}(t)c_{\alpha j}(t) + c^\dagger_{\alpha j}(t)\chi^*_{ij}(t)c_{\alpha i}(t)\Big)\end{aligned} \tag{6.1}$$

where we've had to introduce χ and χ^* to account for the presence of two spin-indices in the interaction term. Now, even before the Hubbard–Stratonovich transformation the theory has a gauge symmetry in the form of $c_{\alpha i} \to e^{i\theta_i(t)}c_{\alpha i}$ corresponding to local fermion number conservation. Upon introducing gauge fields we see they transform as

$$\begin{gathered}\chi_{ij}(t) \to e^{-(\theta_j(t)-\theta_i(t))}\chi_{ij}(t), \quad \varphi(t)_i \to \varphi_i(t) + \partial_t\theta_i(t) \\ A_{ij}(t) \to A_{ij} + \Delta_{ij}\theta_i(t), \quad \Delta_{ij} \equiv \text{lattice derivative}\end{gathered}$$

It is not hard to see that these are the gauge transformations of a U(1) gauge theory; in particular, we find that χ_{ij} serves the role of a link variable or Wilson line with the Abelian linear gauge transformation

$$\chi_{ij}(t) = \rho_{ij}e^{iA_{ij}(t)} \quad \Longrightarrow \quad e^{A_{ij}(t)} \to G_i(t)e^{A_{ij}(t)}G_j(t)^{-1}$$

(see, e.g., Rothe, *Lattice Gauge Theories*), so that the fields (A, φ) together make $A_{\mu,i}(t)$ with timelike component φ. Note, however, that the Lagrangian is not fully gauge invariant under this transformation: the first two terms of eq. (6.1) transform as

$$\begin{aligned}&ic^\dagger_{i\alpha}e^{-i\theta_i}\partial_t(e^{i\theta_i}c_{i\alpha}) + [\varphi_i + \partial_t\theta_i](c^\dagger_{\alpha i}c_{\alpha i} - 1) \\ &\quad = ic^\dagger_{\alpha i}\partial_t c_{\alpha i} + \varphi_i(c^\dagger_{\alpha i}c_{\alpha i} - 1) - c^\dagger_{\alpha i}c_{\alpha i}\partial_t\theta_i + \partial_t\theta_i c^\dagger_{\alpha i}c_{\alpha i} + \partial_t\theta_i \\ &\quad = ic^\dagger_{\alpha i}\partial_t c_{\alpha i} + \varphi_i(c^\dagger_{\alpha i}c_{\alpha i} - 1) + \partial_t\theta_i\end{aligned}$$

And we find there is a hanging $\partial_t\theta_i$ term. The culprit is the $-\varphi_i$ term in the original Lagrangian; fortunately, it is not hard to find the proper constraint on ϕ such that gauge invariance is restored. We simply note that its amplitude is

$$\exp\left(i\int dt\sum_i \varphi_i(t)\right) = \prod_i \exp\left(i\int dt\ \varphi_i(t)\right) = \prod_i \exp\left(i\int\ dtA_{0i}(t)\right)$$

and that for periodic boundary conditions in time, we can construct a timelike Wilson loop along a path Γ_i closing around the spacetime manifold, which transforms under gauge transformations as

$$\prod_i \exp\left(i\int_{\Gamma_i} dx^\mu A_{\mu i}\right) \quad \to \quad \prod_i \left(i\exp\left(\int_{\Gamma_i} dx^\mu A_{\mu i}\right) + \exp\left(i\int_\Gamma dx^\mu\theta_{\mu i}\right)\right)$$

Using Stokes' theorem, we find that provided the form $d\theta$ is closed, gauge invariance is restored. Note, however, that the action itself is not invariant under large gauge transformations, which leave the amplitude invariant but wind around from 0 to $2\pi k$, $k \in \mathbb{Z}$ such that the action gains an additional factor $2\pi k$; this suggests the term acts as a sort of Chern-Simons term for this theory. For the effective action we will later derive to be consistent with the half-fililng constraint, one must keep track of this term, as the saddle point of the temporal component $\delta S_{\text{eff}}/\delta A_0$ should recover the constraint. In what follows, however, we only consider the saddle point of the spatial components of the gauge fields, and we ignore the term for the remainder of the discussion. This is justified by the fact that in the SU(N) theory with $S_{\text{eff}} = N \int \mathcal{L}$this term is of order $1/N$ and the half-filling constraint is irrelevant as $N \to \infty$.

6.2.2 Large N and Saddle-Points

Our Hubbard-Stratonovich transformation has then turned this Heisenberg model into a U(1) gauge theory with a Abelian gauge field without a dynamical term $F^{\mu\nu}$ and with a fluctuating magnitude ρ. In the saddle point approximation, we will find that the fluctuations in ϕ and χ are suppressed and they play the role of background fields by taking on classical minima of an effective action. This will produce a mean-field Hamiltonian that is nothing more than a tight-binding model with non-uniform hopping parameter, and which we will be able to solve for different classical gauge field configurations. This saddle point approximation is justified by making a large-N approximation which renders the mean field theory exact when $N \to \infty$; such an approximation provides $1/N$ as a parameter about which to organize perturbative corrections.

To employ this approximation we generalize to an SU(N) Heisenberg model and choose a group representation such that the theory describes N different fermion species or "flavors," and each site is populated by $N/2$ fermions (other representations are possible, most notably that of Read and Sachdev). The fermion operator algebra is then indexed by flavor a in addition to spin α—$c_{a\alpha}$. The Hubbard-Stratonovich transformation also supplies the gauge fields with these flavor indices, and the $N/2$-filling constraint is modified to $c^\dagger_{\alpha a} c_{\alpha b} = \delta_{ab} N/2$ such that the SU(N) Lagrangian is

$$\begin{aligned}\mathcal{L} = i \sum_i c^\dagger_{\alpha a,i}(t) \partial_t c_{\alpha a,i}(t) + \sum_i \varphi_{ab,i}(t)(c^\dagger_{\alpha a,i}(t) c_{\alpha b,i}(t) - \delta_{ab}\tfrac{N}{2}) \\ - \frac{N}{J} \sum_{\langle i,j\rangle} |\chi_{ab,ij}|^2 + \sum_{\langle ij\rangle} \left(c^\dagger_{\alpha a,i}(t) \chi_{ab,ij}(t) c_{\alpha b,j}(t) + c^\dagger_{\alpha b,j}(t) \chi^*_{ab,ij}(t) c_{\alpha a,i}(t) \right)\end{aligned} \tag{6.2}$$

where here we think of χ_{ab} as a complex matrix on each of the links of the lattice, signalling that this is now a non-Abelian gauge theory with Yang-Mills fields $A^{ab}_{ij}(t)$. To find the saddle point of χ and φ we must first integrate out the fermions; as the Hubbard-Stratonovich transformation renders the Lagrangian quadratic $c_{\alpha a}$, the integral is Gaussian and reduces

to a functional determinant (temporarily suppressing spin indices):

$$\det\left(-i\partial_t\delta_{ab}+\varphi_{ab,i}(t)+\chi_{ab,ij}(t)+\chi_{ij,ab}(t)^*\right)$$

The identity $\det A=\exp(\operatorname{tr}\ln A)$ lets us reinsert these terms into the Lagrangian to obtain an effective action:

$$S_{\text{eff}}=-i2N\operatorname{tr}\ln\left(i\partial_t\delta_{ab}+\varphi_{ab,i}(t)+\chi_{ab,ij}(t)+\chi_{ij,ab}(t)^*\right)-\frac{N}{J}\int dt\sum_{\langle i,j\rangle}|\chi_{ab,ij}|^2 \tag{6.3}$$

where the factor of $2N$ comes from the spin indices and the trace is understood to integrate over time and space. That this Lagrangian has an overall factor of N is crucial for the saddle point to become exact when $N\to\infty$, at least from a mathematical standpoint.

The physical grounds on which the large-N approximation is understood to render the saddle-point approximation exact as follows: A valence bond between two sites is annihilated (created) by the operator $c_{\alpha i}^{(\dagger)}c_{\alpha j}^{(\dagger)}$ (in the SU(N) case actually N such valence bonds between the different species are created); the Heisenberg Hamiltonian is then nothing more a sum over valence bond number operators:

$$H=-\frac{J}{N}\sum_{\langle i,j\rangle}\left(c_{\alpha i}^\dagger c_{\alpha j}^\dagger\right)\left(c_{\beta i}c_{\beta j}\right)$$

such that when one such local term acts on a valence bond, the bond is preserved and we obtain an eigenenergy $-J$: $H\left|\text{bond}\right\rangle=-J\left|\text{bond}\right\rangle$. When the Hamiltonian acts on a link without a bond, it creates one bond at that link, and annihilates two bonds terminating at the link—each of which is one among $N/2$ possible choices—and creates bonds between the two newly unbonded sites. The Hamiltonian produces an equal-weight superposition over all possible ways to do this, each of which possesses an amplitude $-J/N$. Indeed, there N possible ways of doing this, so that the probability of such a process is actually $O(1)$ in aggregate; nevertheless, only one bond is annihilated at a time, so that the fluctuations are $O(1/n)$ and the bond number operator takes on a classical value as $n\to\infty$.

6.2.3 Solving the Mean Field Hamiltonian

In fact, when the saddle-point approximation is exact, we can read off of eq. (6.1) a corresponding exact mean-field tight-binding-like Hamiltonian:

$$H_{\text{MF}}=\sum_{ij}c_{\alpha i}^\dagger\chi_{ij}c_{\alpha j}+c_{\alpha j}^\dagger\chi_{ij}^*c_{\alpha i} \tag{6.4}$$

Here we've suppressed the flavor index and dropped the constant term $|\chi_{ij}|^2$. In order to find the actual classical values of χ_{ij} we must extremize the action by finding the saddle point; we now turn to this task.

A saddle point corresponds to a configurations of χ, or, more precisely, ρ and A, such that the action in eq. (6.3) is stationary:

$$\frac{\delta S_{\text{eff}}}{\delta \rho_{ij}} = 0, \quad \frac{\delta S_{\text{eff}}}{\delta A_{ij}} = 0,$$

Two further constraints reduce the ansatze we must consider: (1) Elitzur's theorem informs us that only gauge invariant operators carry non-vanishing expectation values, meaning that different saddle points must correspond to different expectation values of some gauge invariant parameter. The natural operator to choose is the plaquette loop $W(\square) = e^{i\Phi} = \prod_{\square} e^{iA_{ij}}$ with flux $\Phi = \sum_{\square} A_{ij}(t)$. (2) By stipulating that $\chi_{ij}(t)$ should be in an equilibrium configuration, i.e., it is time-reversal invariant, we place additional constraints on the flux of the theory. In particular, since the time-reversal operator T has the effect

$$\mathsf{T} e^{iA} \mathsf{T}^{-1} = e^{-iA},$$

it must take Φ to $\sum_{\square} -A_{ij}(t) = -\Phi$, and the requirement that this describe an equilibrium distribution limits the ansatze to those with flux $\Phi = 0$ *or* $\Phi = \pi$ (since A_{ij} is defined up to integer multiples 2π, we have $-\Phi = -\pi \sim -\pi + 2\pi = \pi = \Phi$). As it turns out, we may choose saddle-point ansatze with constant ρ_{ij} and $A_{ij} = 0, \pi$, and those choices correspond to two different phases of this model. Shortly, we will specialize to $\Phi = \pi$, the so-called π-flux phase. However, we first return to eq. (6.4) to find its spectrum for

$$\chi_{ij} = \begin{cases} \chi_x \exp(-i\Phi/4) & \text{horizontal link} \\ \chi_y \exp(-i\Phi/4) & \text{vertical link} \end{cases}$$

this ensures that the flux through a plaquette is Φ and the bond variables are constant throughout the lattice. Equation (6.4) is then

$$H_{\text{MF}} = \sum_{\langle ij \rangle} c^{\dagger}_{\alpha i} \chi_{x,y} e^{i\Phi} c_j + c^{\dagger}_j \chi_{x,y} e^{-i\Phi} c_i$$

whhere $\chi_{x,y}$ denotes χ_x or χ_y depending on the link $\langle ij \rangle$. The usual technique is to Fourier transform to $c_{\alpha k}$ operators; before doing this, however, the trick is to introduce a local 2×2 Hamiltonian in the spirit of a Nambu spinor diagonalization. In what follows we specialize to bipartite lattices; in particular, we notice that the sum over links $\langle i, j \rangle$ will always contain one index in an odd sublattice and one index in an even sublattice. Then we write

$$H_{\text{MF}} = \sum_{\langle ij \rangle} \begin{bmatrix} c_i \\ c_j \end{bmatrix}^{\dagger} \begin{bmatrix} 0 & \chi_{x,y} e^{-i\Phi/4} \\ \chi_{x,y} e^{i\Phi/4} & 0 \end{bmatrix} \begin{bmatrix} c_i \\ c_j \end{bmatrix}$$

where i is always in an odd sublattice and j is always in an even sublattice. It is easy to diagonalize this matrix and find

$$H_{\text{MF}} = \sum_{\langle ij \rangle} \begin{bmatrix} c_i \\ c_j \end{bmatrix}^{\dagger} \begin{bmatrix} e^{i\Phi/4} & e^{i\Phi/4} \\ e^{-i\Phi/4} & -e^{-i\Phi/4} \end{bmatrix}^{\dagger} \begin{bmatrix} \chi_{x,y} & 0 \\ 0 & -\chi_{x,y} \end{bmatrix} \begin{bmatrix} e^{i\Phi/4} & e^{i\Phi/4} \\ e^{-i\Phi/4} & -e^{-i\Phi/4} \end{bmatrix} \begin{bmatrix} c_i \\ c_j \end{bmatrix} \tag{6.5}$$

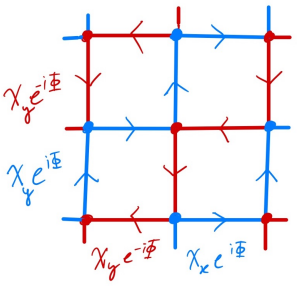

Figure 6.1

Now we are ready to Fourier transform. Consider first nearest neighbor interactions along a horizontal link: $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} + (1, 0)$, such that $\mathbf{k} \cdot \mathbf{x}' = \mathbf{k} \cdot \mathbf{x} + k_x$. Such a term Fourier transforms as

$$\sum_{\mathbf{x},\mathbf{x}'} c^\dagger_{\mathbf{x}} c^\dagger_{\mathbf{x}'} \to c^\dagger_{\mathbf{k}} c_{\mathbf{k}'} e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{x}} e^{ik_x}$$

since the sums over vertical and horizontal nearest neighbor sites is disjoint, considering the vertical links just induces a new term $\exp(ik_y)$:

$$\sum_{\langle \mathbf{x_1 x_2} \rangle} c^\dagger_{\mathbf{x}} c^\dagger_{\mathbf{x}'} \to \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} c_{\mathbf{k}'} e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{x}} (e^{ik_x} + e^{ik_y}).$$

where the $\exp(-i(\mathbf{k} - \mathbf{k}'))$ turns into $\delta(\mathbf{k} - \mathbf{k}')$ upon resolving the spatial sum.

Also note that in this bipartition of χ, every i index is associated with a horizontal χ_x bond but a vertical χ^*_y bond (see fig. 6.1). This means that when we combine the vertical and horizontal links in eq. (6.5) we find that they contribute terms $\chi^*_y \exp(ik_y)$ and links $\chi_x \exp(ik_x)$ respectively. Equation (6.5) then becomes

$$H_{\text{MF}} = \sum_{\langle k \rangle} (\cdots) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Bigg(\begin{bmatrix} e^{i\Phi/4} & e^{i\Phi/4} \\ e^{-i\Phi/4} & -e^{-i\Phi/4} \end{bmatrix} \begin{bmatrix} c_k \chi_x e^{ik_x} \\ c_k \chi_x e^{-ik_x} \end{bmatrix} + \begin{bmatrix} e^{-i\Phi/4} & e^{-i\Phi/4} \\ e^{i\Phi/4} & -e^{i\Phi/4} \end{bmatrix} \begin{bmatrix} c_k \chi_y e^{ik_y} \\ c_k \chi_y e^{-ik_y} \end{bmatrix} \Bigg)$$

By combining the latter two factors we find that the excitations diagonalizing this system are

$$\text{Odd sites:} \quad c_o = \left(e^{i\Phi/4} \chi_x \cos k_x + e^{-i\Phi/4} \chi_y \cos k_y \right) e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$\text{Even sites:} \quad c_e = \left(e^{-i\Phi/4} \chi_x \cos k_x + e^{i\Phi/4} \chi_y \cos k_y \right) e^{i\mathbf{k}\cdot\mathbf{x}}$$

where in the even excitations we have made the shift $\mathbf{k} \to \mathbf{k} + (\pi/2, \pi/2)$ which is permitted by the perfect nesting of the Fermi surface (thereby taking $\sin k_{xy} \to \cos k_{xy}$). By noticing that the factor in the even sites is the conjugate of the factor of the odd sites, we can pull out the even-site factor at the meaningless cost of an overall phase and find that

$$\text{Odd sites:} \qquad c_o = \frac{g}{\sqrt{gg^*}} e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \text{Even sites:} \qquad c_e e^{i\mathbf{k}\cdot\mathbf{x}} \tag{6.6}$$

with

$$g = e^{i\Phi/4}\chi_x \cos k_x + e^{-i\Phi/4}\chi_y \cos k_y$$

The presence of two types of excitations represents the doubling of the unit cell and is reflected in a two-branch spectrum corresponding to the ± 1 eigenvalues of the Nambu local Hamiltonian.

The π-flux phase in particular has $\Phi = \pi$ and the spectrum is

$$E(\mathbf{k}) = \pm 2|\chi| \left(\cos^2 k_x + \cos^2 k_y\right)^{1/2}$$

There are four Fermi points[1] $\mathbf{k} = (\pm\pi/2, \pm\pi/2)$, near which we can use $\cos(\pi/2 + \epsilon) \approx \epsilon$ to approximate

$$E(\mathbf{k}) \approx \left(k_x^2 + k_y^2\right)^{1/2}$$

meaning that the dispersion is linear near the gap, the first signature of a low-energy effective theory corresponiding to a massless relativistic QFT and a Dirac cone. We now explicitly demonstrate this fact.

We first gauge transform to $\chi_x = |\chi| e^{i\pi/2} = i|\chi|$ and $\chi_y = |\chi|$ (notice the plaquette flux is still π). The Hamiltonian of eq. (6.5) now reads

$$H_{\text{MF}} = |\chi| \sum_{\mathbf{r}\in\text{odd}} c_{\mathbf{r}}^\dagger [i(c_{\mathbf{r}-\hat{\mathbf{x}}} + c_{\mathbf{r}+\hat{\mathbf{x}}}) + (c_{\mathbf{r}-\hat{\mathbf{y}}} + c_{\mathbf{r}+\hat{\mathbf{y}}})] + \text{H.c.}$$

where we've explicitly combined all of the links associated with the odd bipartition of the lattice into the summand so that we run over $\mathbf{r}$ in the odd sublattice. Now Fourier transforming with eq. (6.6) and considering a large system size such that the spectrum becomes continuous, we have

$$2|\chi| \int \frac{d^2k}{(2\pi)^2} \left(i\cos k_x + \cos k_y\right) c_e^\dagger(\mathbf{k}) c_o(\mathbf{k}) + \text{H.c.}$$

where as usual the cosines come from relative displacements in the Fourier factor $e^{i(x+1)k_x}$. We can consider a continuum theory by limiting ourselves to small wavevectors such that the system does not see the Brillouin zone. To do so we shift to momenta $\mathbf{k} \to \mathbf{k} + (\pi/2, \pm\pi/2)$

[1] Note however that because of the perfect nesting of the Fermi surface only two of those Fermi points are inequivalent.

corresponding to two inequivalent Fermi points (producing what is sometimes referred to as Fermion doubling) and find

$$-2|\chi| \int \frac{d^2k}{(2\pi)^2} \left(ik_x + k_y\right) c^\dagger_{1e}(\mathbf{k}) c_{1o}(\mathbf{k}) + \text{H.c.} + \left(ik_x - k_y\right) c^\dagger_{1e}(\mathbf{k}) c_{2o}(\mathbf{k}) + \text{H.c.}$$

We turn our attention to the first Fermi point alone, the $c_{(o,e)1}$ excitations. We find that the terms can be written in matrix form

$$\begin{aligned}
&\left(ik_x + k_y\right) c^\dagger_{1e}(\mathbf{k}) c_{1o}(\mathbf{k}) + \left(-ik_x + k_y\right) c^\dagger_{1o}(\mathbf{k}) c_{1e}(\mathbf{k}) \\
&\quad = k_x \begin{bmatrix} c_{e1} \\ c_{o1} \end{bmatrix}^\dagger \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \begin{bmatrix} c_{e1} \\ c_{o1} \end{bmatrix} + k_y \begin{bmatrix} c_{e1} \\ c_{o1} \end{bmatrix}^\dagger \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_{e1} \\ c_{o1} \end{bmatrix} \\
&\quad = -k_x \psi_1^\dagger \sigma_2 \psi + k_y \psi_1^\dagger \sigma_1 \psi \\
&\quad = -\psi_1^\dagger \sigma_2 \partial_1 \psi + \psi_1^\dagger \sigma_1 \partial_2 \psi \equiv \mathcal{H}_1/2|\chi| \qquad \text{(Hamiltonian density)}
\end{aligned}$$

where we've introduced the Dirac spinors $\psi_1 = (c_{e1}, c_{o1})$, substituted the Pauli matricesm, and Fourier transformed in the final step. We've also anticipated the Lorentz covariant form of the Lagrangian density by using relativistic indices $x \to 1$, $y \to 2$. Legendre transforming these terms introduces a new term

$$\mathcal{L}_1 = i\psi_1 \partial_0 \psi_1 + 2|\chi| i\psi_1^\dagger (-\sigma_2 \partial_1 + \sigma_1 \partial_2) \psi_1$$

Using the fact that $-\sigma_2 = i\sigma_2\sigma_1$ and $\sigma_1 = i\sigma_3\sigma_1$, and $\sigma_3^2 = 1$

$$\mathcal{L}_1 = i\psi_1 \sigma_3\sigma_3 \partial_0 \psi_1 + 2|\chi| i\psi_1^\dagger (i\sigma_3\sigma_1 \partial_1 + i\sigma_3\sigma_2 \partial_2) \psi_1$$

Now we notice that the $(2+1)$d representation of the Clifford algebra is given by $\gamma_\mu = (i\sigma_3, \sigma_1, \sigma_2)$, so that idenfifying $2|\chi|$ as the speed of "light," we find

$$\mathcal{L}_1 = i\bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1, \qquad \bar{\psi} = \psi^\dagger \gamma^0.$$

The analysis is identical for the second Fermi point, so we find that the Lagrangian density is the sum over the two fermions (indexed by a):

$$\mathcal{L} = i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a.$$

It turns out that if one includes fluctuations about the saddle point of χ, we indeed obtain

$$\mathcal{L} = i\bar{\psi}_a \gamma^\mu D_\mu \psi_a, \qquad D^\mu = \partial^\mu + igA^\mu,$$

the Lagrangian of a $(2+1)$d massless U(1) gauge theory.

6.2.4 Concluding Remarks

By considering an unphysical SU(N) generalization of the Heisenberg model, we have derived a mean-field Hamiltonian eq. (6.5) containing a classical background gauge field manifesting as complex, non-uniform hopping parameters χ_{ij}, sometimes referred to as *parton construction*. To make contact with the actual SU(2) Heisenberg antiferromagnet, one must explicitly reintroduce the single-occupancy constaint by way of a Gutzwiller projection of the $c_{(o,e),\mathbf{k}}$ excitations. In particular, we construct a Slater determinant of such excitations, $|\psi_{\text{free}}\rangle$, and fully project out double occupancy:

$$|\psi\rangle = \Pi_i(1 - n_{i\uparrow}n_{i\downarrow})\,|\psi_{\text{free}}\rangle$$

We may then treat the π-flux phase wavefunction as a variational ansatz with which to explore the ground state of the SU(2) Heisenberg model.

References

- Rothe, *Lattice Gauge Theories*.
- Fradkin, *Field Theories of Condensed Matter*. See in particular §8.4.
- Marston, Affleck, *Large-n limit of the Hubbard-Heisenberg Model*, [PhysRevB.39.11538].
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6.3 Gauge Theories from SSB

6.4 't Hooft Anomaly: Particle on S^1

t' Hooft anomalies are often described as "obstructions to gauging a global symmetry." One illustration Humberto gave is a theory with a $\mathrm{U}(1)_{\mathrm{A}} \times \mathrm{U}(1)_{\mathrm{V}}$ symmetry wherein you cannot gauge $\mathrm{U}(1)_{\mathrm{A}}$ without rendering $\mathrm{U}(1)_{\mathrm{V}}$ anomalous and vice-versa. In quantum mechanics (in field theory the story is a bit different) the description of 't Hooft anomalies can be given in terms of what we learned in §5.3: if we cannot rescale the action of the (global) symmetry operator on the state so as to make it a linear representation, that is, if it is *strictly* a projective representation, we say that the symmetry has an 't Hooft anomaly. This is considered an "obstruction to gauging" because the gauge symmetry of quantum mechanics is the overlabelling of Hilbert space up to a phase; if we can't get rid of that phase, there is an obstruction. This obstruction manifests itself as a physical inequivalence between states that are related by such a phase. Shao claims that this is so general a phenomenon as to encompass the fact that qubits are in the projective representation of SO(3): qubits are 't Hooft anomalous!?

Particle on a ring Here we illustrate the phenomenon with a supposedly representative $(0+1)$-d example, a particle on a ring with a theta term:

$$\mathcal{L} = \frac{m}{2}\dot{x}(t)^2 + \frac{\theta}{2\pi}\dot{x}(t) \quad \xrightarrow{t=i\tau} \quad \mathcal{L} = \frac{m}{2}\dot{x}(\tau)^2 - \frac{i\theta}{2\pi}\dot{x}(\tau)$$

The theta term is indeed topological. Additionally, being a total derivative, it has no effect on the classical equations of motion (see §6.4.3). Periodic boundary conditions amount to $x \simeq x + 2\pi$, and additionally $\theta \simeq \theta + 2\pi$. This θ-periodicity is not postulated, but rather a consequence of the invariance of the path integral with respect to $\theta \to \theta + 2\pi$. Two quick ways of seeing this, from Tong:

- The Euclidean partition function will have a total derivative term which captures instanton effects by yielding a topological charge k corresponding to the winding number.
$$i\frac{\theta}{2\pi}\int_0^\beta d\tau\, \partial_\tau x = i\frac{\theta}{2\pi}2\pi k = i\theta k$$
(the i comes from $t = i\tau$). Put more crudely, this we have that the integral over $\partial_\tau x$ must be $x(\beta) - x(0) = 2\pi k$ due to the periodicity in $x(\tau)$. The weight is then $\exp(i\theta k)$, which is clearly invariant under $\theta \to \theta + 2\pi$.

- The Hamiltonian should be invariant with respect to conjugation by a unitary operator. If we choose the unitary operator $\exp(ix)$:
$$e^{ix}H_\theta e^{-ix} = H_{\theta+2\pi}$$
we can use Baker-Campbell-Hausdorff and $[ix, p] = -1$ to see this fact. Namely,
$$e^{ix}He^{-ix} = H - [H, ix] \tag{6.7}$$
where expanding $H = (p^2 - p\theta/2\pi + \theta^2/(2\pi)^2)/2m$ we find
$$[H, ix] = \frac{1}{2m}\left(2p + \frac{\theta}{\pi}\right)$$
where we used $[A, B^2] = [A, B]B + B[A, B]$ with $B = p$, $A = ix$. We now just have to recognize that
$$H_{2\pi+\theta} = \left(p - \frac{\theta + 2\pi}{2\pi}\right) = \left(p - 1\frac{\theta}{2\pi}\right) = p^2 - 2p + \frac{\theta}{2\pi}(p+1) - \frac{\theta^2}{2\pi}$$
which agrees with eq. (6.7) given what we've just found for the commutator.

This system has a $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ translational symmetry $\theta \to \theta + \alpha$ and, provided $\theta = 0, \pi$, a "conjugation" symmetry $x \to -x$. Here's why: The kinetic term is automatically

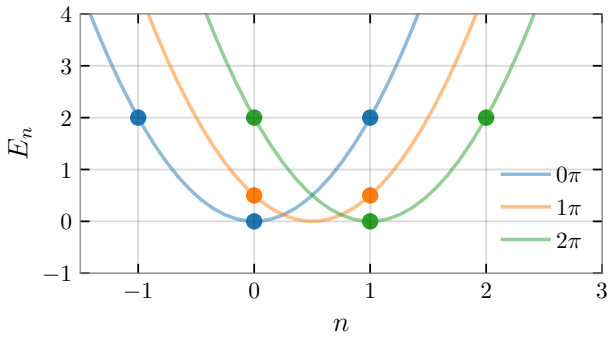

Figure 6.2

symmetric under $x \to -x$ regardless of the value of θ. If $\theta = 0$ the antisymmetric vanishes, so it is trivially symmetric. If $\theta = \pi$, the antisymmetric term goes to

$$\frac{i\pi}{2\pi}\dot{x}(t) \to -\frac{i\pi}{2}\dot{x}(t)$$

Here the periodicity of θ is pivotal: since $\pi \to -\pi \simeq -\pi + 2\pi \simeq \pi$, the θ-values π and $-\pi$ are identified, and so

$$\frac{i\pi}{2\pi}\dot{x}(t) \to -\frac{i\pi}{2}\dot{x}(t) \simeq \frac{i\pi}{2}\dot{x}(t)$$

and the sign is absorbed by $\theta = \pi$. Since $\mathrm{SO}(2) \ltimes \mathbb{Z}_2 \simeq \mathrm{O}(2)$ the model possesses a global $\mathrm{O}(2)$ symmetry at $\theta = 0, \pi$. (Think about it—$\mathrm{SO}(2)$ is just $\mathrm{O}(2)/\{-I\}$ where $-I$ is the negative of the identity matrix, i.e., a parity transformation. Thus we just need to include this reflection to take the special orthogonal group to the orthogonal group). At $\theta = \pi$, however, the $\mathrm{O}(2)$ symmetry is anomalous. In particular, we will now demonstrate that the $\mathrm{O}(2)$ acts on the quantized theory not as a linear representation but as a projective representation. We then show how this amounts to an obstruction to gauging (I think? we just show that the gauge fields reveal the anomaly) and discuss the dynamical consequences of the anomaly.

Before doing so, we quickly solve this system. The canonical momentum corresponding to the above Lagrangian is

$$p \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{\theta}{2\pi} \quad \Longrightarrow \quad \dot{x} = \frac{1}{m}\left(p - \frac{\theta}{2\pi}\right)$$

Which means that the Hamiltonian is

$$\begin{aligned} H = p\dot{x} - L &= p \cdot \frac{1}{m}\left(p - \frac{\theta}{2\pi}\right) - \frac{m}{2}\frac{1}{m^2}\left(p - \frac{\theta}{2\pi}\right)^2 - \frac{\theta}{2\pi}\frac{1}{m}\left(p - \frac{\theta}{2\pi}\right) \\ &= \frac{p^2}{m} - \frac{1}{2m}\left(p^2 - \frac{p\theta}{2\pi} + \frac{\theta^2}{(2\pi)^2}\right) + \frac{\theta^2}{(2\pi)^2 m} \\ &= \frac{p^2}{2m} - \frac{p\theta}{2\pi m} + \frac{\theta^2}{2(2\pi)^2 m} = \frac{1}{2m}\left(p - \frac{\theta}{2\pi}\right)^2 \end{aligned}$$

Promoting this to an operator:

$$\hat{H} = \frac{1}{2m}\left(\hat{p} - \frac{\theta}{2\pi}\right)^2$$

So the Schrödinger equation is

$$\frac{1}{2m}\left(-i\partial_x - \frac{\theta}{2\pi}\right)^2 \psi_n(x) = E_n \psi_n(x)$$

Since we're trying to be quick, we just Fourier transform:

$$\frac{1}{2m}\left(n - \frac{\theta}{2\pi}\right)^2 e^{inx} = E_n e^{inx}$$

which means that the wavefunctions and energies are

$$\psi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{2\pi}} e^{inx}, \qquad E_n = \frac{1}{2m}\left(n - \frac{\theta}{2\pi}\right)^2 \tag{6.8}$$

The spectrum is discrete but infinite, as is expected from Pontryagin dual of U(1) being $\mathbb{Z}$. The fact that the spectrum is infinite is necessary for it to be invariant with respect to $\theta \to \theta + 2\pi$ as this transformation takes $n \to n + 1$:

$$E_n = \frac{1}{2m}\left(n + \frac{\theta}{2\pi}\right)^2 \to \frac{1}{2m}\left(n + \frac{\theta + 2\pi}{2\pi}\right)^2 = \frac{1}{2m}\left(n + 1 + \frac{\theta}{2\pi}\right)^2 = E_{n+1}$$

From eq. (6.8) we can read off the action of the parity P and translation (by α) operators T_α:

$$P\,|n\rangle = |-n\rangle\,, \qquad T_\alpha\,|n\rangle = e^{in\alpha}\,|n\rangle$$

where the last line holds because we have $\psi_n(x) \to \psi_n(x + \alpha)$ which is $e^{in(x+\alpha)} = e^{in\alpha}e^{inx}$. This is just the statement that the state $|n\rangle$ carries translation charge n.

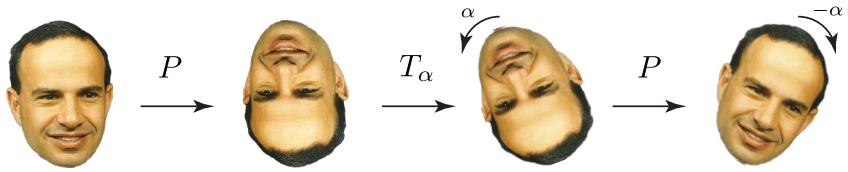

Figure 6.3: Nati Seiberg demonstrates the action of the parity and translation transformations.

The anomaly is the central extension Now return to the case of $\theta = \pi \sim -\pi$. This system has a doubly-degenerate ground state:

$$E_n = \frac{1}{2m}\left(n + \frac{1}{2}\right)^2 = \frac{1}{2m}\left(-n + 1 + \frac{1}{2}\right)^2 \implies E_0 = E_1 = \frac{1}{2m}\left(\frac{1}{2}\right)^2$$

(I think there's a subtlety here—if you decide to show this by leveraging $\theta = \pi \sim -\pi$, you can make the same argument to show that $E_n = E_{-n}$ which we will find below that is not the case. Thus I'm pretty sure you can't use the fact that $\pi \sim -\pi$ when discussing the spectrum.) When $\theta = 0$ this is not the case: there is a unique ground state $|n = 0\rangle$. In fact, generally, we expect that because O(2) is a symmetry of the Hamiltonian, the eigenstates eq. (6.8) furnish a representation of O(2), with each set of degenerate eigenstates transforming as an irreducible representation of $O(2)$. We see that when $\theta = 0$ this is indeed the case, for $|n\rangle$ and $|-n\rangle$ share energy $E_{\pm n} = n^2/2m$: the degenarate states furnish the two-dimensional irreps of O(2) for they transform into each other under parity: $|n\rangle \to |-n\rangle$. I'm not sure if the ground state uniqueness $\theta = 0$ has a deeper meaning—prima facie, this is a consequence of that eigenstate carrying no translation charge. Either way, by this reasoning, we have

$$|n\rangle \xrightarrow{P} |-n\rangle \xrightarrow{T_\alpha} e^{-i\alpha n}|-n\rangle \xrightarrow{P} e^{i(-\alpha)n}|n\rangle$$

so that when $\theta = 0$,

$$PT_\alpha P = T_{-\alpha}$$

which is demonstrated in fig. 6.3.

Aside

It's worth emphasizing that this is a very common (I do not know of any counterexamples) property of degeneracy, which I already discussed a bit in §1.2: eigenstates with the same energy are related to each other by symmetry transformations, so the degenerate blocks of the Hamiltonian are irreps* of the symmetry group. (*We will see that they can also be irreps of the central extension instead.)

When $\theta = \pi$, however, something special happens. We have argued above that for this value of θ the eigenstates $|n\rangle$ and $|-n+1\rangle$ are degenerate: thus, it is those states, not $|\pm n\rangle$, that transform under the irreps of the symmetry of the Hamiltonian. This has the funny consequence that

$$|n\rangle \xrightarrow{P} |-n+1\rangle \xrightarrow{T_\alpha} e^{-i\alpha n}e^{i\alpha}\,|-n+1\rangle \xrightarrow{P} e^{i\alpha}e^{i(-\alpha)n}\,|n\rangle \quad \Longrightarrow \quad PT_\alpha P = e^{i\alpha}T_{-\alpha}$$

which is rooted in the fact that $|-n+1\rangle$ has a different translation charge than $|n\rangle$. The states thus no longer transform under a faithful representation (TODO check defn faithful) of O(2); instead, they transform under a projective representation of O(2) and thus a representation of the central extension of O(2).

If we gauge the $U(1)$ symmetry by adding a background gauge field A_0 to this theory with $\dot{x} \to \dot{x} + A_0$, and subsequently include a Chern-Simons term kA_0, we see the sense in which 't Hooft anomalies are thought of as "obstructions to gauging." The action is

$$\int dt\, \frac{m}{2}(\dot{x}+A_0)^2 + \frac{\theta}{2\pi}(\dot{x}+A_0) + kA_0$$

where the U(1) is now $x \to x + \alpha(t)$ and the Lagrangian is gauge invariant if we choose $A \to A - \dot{\alpha}(t)$ for the gauge field and, as usual, enforce integer Chern level $k \in \mathbb{Z}$.

In the past we argued that the periodicity in θ is a consequence of the invariance of the partition function in shifts by 2π. This is almost the case here, except the action accrues a change in the Chern level:

$$\frac{\theta+2\pi}{2\pi}(\dot{x}+A_0) + kA_0 = \underbrace{\frac{\theta+2\pi}{2\pi}\dot{x}}_{\text{Still invariant}} + \frac{\theta}{2\pi}A_0 + A_0 + kA_0 = \frac{\theta}{2\pi}(\dot{x}+A_0) + (1+p)A_0$$

Now for the parity symmetry: we postulate this symmetry also takes $A_0 \to -A_0$ and

$$\frac{m}{2}(\dot{x}+A_0)^2 + \frac{-\theta}{2\pi}(\dot{x}+A_0) - kA_0$$

At $k = 0$, $\theta = 0$ this is still a good symmetry of the theory, although almost trivially so. At $\theta = \pi$

$$\frac{m}{2}(\dot{x} + A_0)^2 + \frac{-\pi}{2\pi}(\dot{x} + A_0) - kA_0$$

to restore the initial form of the Lagrangian and establish that it is invariant, we must shift $\theta \to \theta + 2\pi$ for this induces $-\pi \to -\pi + 2\pi = \pi$. However, we have just argued that this shift changes the Chern level to $-k \to -k - 1$; there is no k satisfying $-k - 1 = k$ and we conclude that there is no Chern level such that the action is invariant under parity. No matter what, a parity transformation which induce a change in the Chern level. Notice that this means that even if there is *no* Chern-Simons term (Chern level $k = 0$) the action is not invariant; i.e., the obstruction to gauging does not require a Chern-Simons term. Something like this comes up in §section 6.2. This is what is meant when people speak of a *mixed* 't Hooft anomaly: gauging translation symmetry breaks parity symmetry. (I have heard, but am not convinced, that this means that gauging parity would conversely break translation, and that this is a general feature of mixed anomalies.)

Dynamical consequences Consider a potential that explicitly breaks the SO(2) translation symmetry to $\mathbb{Z}_2$. One such potential is

$$L = \frac{m}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x} + \lambda \cos 2x$$

which breaks translations $\{T_\alpha\}$ to $\{T_0, T_\pi\}$. The symmetry group of $\theta = 0$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, but the symmetry group of $\theta = \pi$ obeys

$$PT_\pi P = -T_\pi$$

By choosing $r = P$ and $t = T_\pi P$ it is not hard to show that this makes up the group D_4, the dihedral group of the square. The irreducible representations of D_4 are of dimension 2:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R = -I \quad \text{(reflection)}, \qquad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (\pi/2 \text{ rotation}),$$

and all possible products of these matrices. Notably this is *not* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

6.4.1 Separating into topological sectors

To calculate the partition function of this system we can perform a trick where we separate the field configurations into different topological sectors; i.e., field configurations with different winding numbers. This is done by isolating the topological terms and rewriting the path integral as a sum over the topological charges Q

$$Z = \int \mathcal{D}\phi \, e^{iS[\phi] + iS_{\text{top}}[\phi]} = \sum_{Q=-\infty}^{\infty} e^{iAQ} \int \mathcal{D}\phi_Q \, e^{S[\phi]}$$

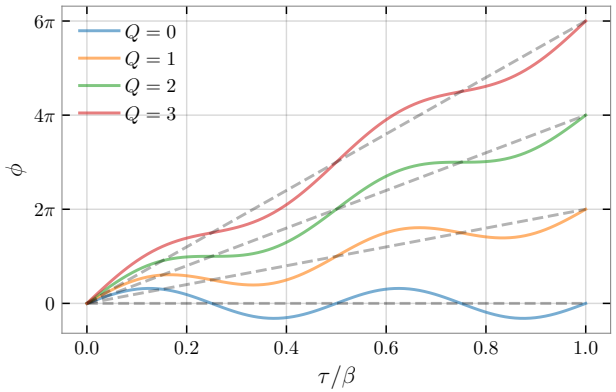

Figure 6.4: Field value $\phi(\tau)$ for a particle on a ring with various winding numbers Q and for a fixed mode $\ell = 4$ (see eq. (6.10)). The periodicity of general Fourier components (i.e., arbitrary ℓ) ensures that indeed $\phi(\beta) - \phi(0) = 2\pi Q$.

where $S_{\text{top}}[\phi]$ denotes the topological term, A is some factor associated with it, and $\mathcal{D}\phi_Q$ denotes field configurations with the particular winding number Q.

Particle on a ring For a particle on a ring with Wick-rotated path integral

$$Z = \int \mathcal{D}\phi \, \exp - \int_0^\beta d\tau \left[\frac{m}{2}\dot{\phi}^2 - \frac{i\theta}{2\pi}\dot{\phi} \right] \tag{6.9}$$

we can Fourier-expand as

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_\ell \phi_\ell e^{\frac{2\pi\ell}{\beta}\tau} \tag{6.10}$$

where again Q is the topological charge or winding number.That this is the winding number can be seen in fig. 6.4. Note that $\phi(\tau)$ is real and thus the reality condition is enforced: $\phi_\ell^* = \phi_{-\ell}$. The derivatives are

$$\dot{\phi} = \frac{2\pi}{\beta}\left(Q + \sum_{\ell\in\mathbb{Z}} \phi_\ell e^{\frac{i2\pi\ell\tau}{\beta}} \ell i \right)$$

$$\dot{\phi}^2 = \left(\frac{2\pi}{\beta}\right)^2 \left(Q^2 - \sum_{\ell\ell'} \phi_\ell \phi_{\ell'} e^{i\frac{2\pi}{\beta}(\ell+\ell')\tau} \ell\ell' + 2Q \sum_\ell i\ell e^{i\frac{2\pi}{\beta}\phi_\ell} \right)$$

and correspondingly eq. (6.9) reads

$$Z = \sum_{Q=-\infty}^{\infty} \int \prod_\ell d\phi_\ell \, e^{-S[\phi_\ell, Q]}$$

with

$$S = -\int_0^\beta d\tau \left[\frac{m}{2}\left(\frac{2\pi}{\beta}\right)^2 \left(Q^2 - \sum_{\ell\ell'} \phi_\ell \phi_{\ell'} e^{\frac{i2\pi}{\beta}(\ell+\ell')\tau} \ell\ell' + 2Q \sum_\ell i\ell e^{\frac{i2\pi}{\beta}\ell\tau}\phi_\ell \right) - i\frac{\theta}{2\pi}\frac{2\pi}{\beta}\left(Q + \sum_\ell \phi_\ell e^{i\frac{2\pi}{\beta}\ell\tau}\ell i \right) \right]$$

Now we carry out the β integral, which in this Fourier decomposition will induce a number of simplifications by the orthogonality of the modes. Namely:

$$\int_0^\beta \sum_{\ell\ell'} \phi_\ell \phi_{\ell'} e^{\frac{i2\pi}{\beta}(\ell+\ell')\tau}\ell\ell' = \beta \sum_\ell \phi_\ell \phi_{\ell'} \ell\ell' \delta(\ell+\ell') = -\beta \sum_\ell \ell^2 \phi_\ell \phi_{-\ell}$$

$$\int_0^\beta \sum_\ell \phi_\ell e^{i\frac{2\pi}{\beta}\ell\tau}\ell = \int_0^\beta \sum_\ell \phi_\ell \ell e^{i\frac{2\pi}{\beta}\ell\tau} e^{i\frac{2\pi}{\beta}(0)\tau} = \sum_\ell \phi_\ell \ell \delta(\ell - 0) = 0$$

We are thus left with

$$S = \frac{m}{2}\frac{(2\pi)^2}{\beta}\left(Q^2 - \sum_\ell \ell^2 \phi_\ell \phi_{-\ell} \right) + i\theta Q$$

and the partition function is

$$Z = \sum_Q e^{-\frac{m}{2}\frac{(2\pi)^2}{\beta}Q^2 + i\theta Q} \int \prod_\ell \phi_\ell \exp\left(-\frac{m}{2}\frac{(2\pi)^2}{\beta} \sum_\ell \ell^2 \phi_\ell \phi_{-\ell} \right) \tag{6.11}$$

The ϕ_ℓ integral is Gaussian with

$$\int \prod_\ell d\phi_\ell \exp\left(\vec{\phi}^* A \vec{\phi} \right), \qquad A_{\ell\ell'} = \delta_{\ell\ell'}\ell^2 \frac{m(2\pi)^2}{\beta}$$

which yields the functional determinant

$$\sqrt{\det\left(\frac{\beta\delta_{\ell\ell'}}{2\pi m \ell^2} \right)} = \sqrt{\prod_\ell \frac{\beta}{2\pi m \ell^2}} \equiv \mathcal{N}_\phi$$

(note to self: determinant of diagonal matrix is the product of the diagonals not the sum, dummy!) We treat this as an overall constant $\mathcal{N}_\phi$. For the topological sector, we use the sum formula (which can be derived with the Poisson summation formula)

$$\sum_{m=\infty}^{\infty} \exp\left[-\frac{1}{2}Am^2 + iBm \right] = \sqrt{\frac{2\pi}{A}} \sum_{n=\ell}^{\infty} \exp\left[-\frac{1}{2A}(B - 2\pi n)^2 \right]$$

To find that

$$\sum_Q \exp\left(-\frac{m}{2}\frac{(2\pi)^2}{\beta}Q^2 + i\theta Q\right) = \sqrt{\frac{\beta}{2\pi m}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\beta}{2m(2\pi)^2}(\theta - 2\pi n)^2\right)$$

and in aggregate the partition function becomes

$$Z = \mathcal{N}_\phi \frac{\beta}{2\pi m}\sum_n \exp\left[-\frac{\beta}{2m}\left(\frac{\theta}{2\pi} - n\right)^2\right]$$

(This β dependence outside the sum is a little funny, but I'm pretty sure it's right.) Since it should have the form $Z \propto \sum_n \exp(-\beta E_n)$, we can immediately read off the spectrum of this system:

$$E_n = \frac{1}{2m}\left(\frac{\theta}{2\pi} - n\right)^2.$$

6.4.2 Metric Independence of Topological Term

With that, the action of the particle on a ring is

$$S = \int dt_p \left(\frac{m\dot{\phi}^2}{2} - \frac{\theta}{2\pi}\dot{\phi}\right) \tag{6.12}$$

where we have written t as t_p to signal that the action takes this form for a proper time coordinate t_p. We can parametrize $t_p = f(t)$ so that $dt_p = f'dt$, $dt_p^2 = f'^2 dt^2$ and identify $f'^2 = g_0 0$, $g^0 0 = f'^{-2}$ as the metric.

We then find that

$$\frac{d\phi}{dt_p} = \frac{dt}{dt_p}\frac{d\phi}{dt} = \frac{1}{f'}\frac{d\phi}{dt}$$

and eq. (6.12) will become

$$\int dt f'\left[\frac{m}{2}\frac{1}{f'^2}\left(\frac{d\phi}{dt}\right)^2 - \frac{\theta}{2\pi}\frac{1}{f'}\frac{d\phi}{dt}\right] = \int dt\left[\frac{m}{2}\frac{1}{\sqrt{g}}\left(\frac{d\phi}{dt}\right)^2 - \frac{\theta}{2\pi}\frac{d\phi}{dt}\right]$$

Observe that the θ term is then

$$\int dt f'\left(-\frac{\theta}{2\pi}\frac{1}{f'}\frac{d\phi}{dt}\right) = -\int dt\frac{\theta}{2\pi}\frac{d\phi}{dt} = -\int \frac{\theta}{2\pi}d\phi$$

which is indeed independent of the metric.

Now, using the considerations of section 6.7, we find that topological term does not contribute to the stress-energy tensor

$$\frac{\delta S}{\delta g_{\mu\nu}} = -\frac{m}{2}\frac{1}{g}\left(\frac{\delta\sqrt{-g}}{\delta g}\right)\frac{d\phi}{dt} = -\frac{m}{2}\frac{1}{g}\left(\frac{1}{2}\sqrt{g}g_{\mu\nu}\right)\frac{d\phi}{dt}$$

And the stress energy tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \frac{m}{2}\frac{g_{\mu\nu}}{g}\frac{d\phi}{dt}$$

and indeed T_{00} is the energy:

$$T_{00} = \frac{m}{2}\frac{f'}{f'^2}\frac{d\phi}{dt} = \frac{m}{2}\frac{d\phi}{dt}.$$

6.4.3 Comments on topological terms

Total derivatives Total derivative terms in the Lagrangian do not contribute to the classical dynamics (the Euler-Lagrange equations are unchanged). With a total derivative term $\partial_\mu F^\mu = dF$ we can use Stokes theorem:

$$S = \int d^d x\, \partial_\mu F^\mu = \int dF \Longrightarrow S = \int_{\partial_M} F$$

this is a boundary integral which typically will vanish if we enforce that F vanishes on the boundary. Of course, at the full quantum level, the manner in which gauge fields might wind around to the spacetime boundaries will be detected by the phase of the path integral (see e.g., Coleman, *Aspects of Symmetry*; for this reason they are sometimes referred to as boundary terms) even if they vanish at infinity as required, hence these total derivative terms are topological.

TODO: understand this.

Metric dependence of Hodge star It is worth noting that the presence of certain operations in the term automatically rules out the term as topological. For instance, while the wedge product $\wedge$ makes no reference to a metric and thus is allowed, while the Hodge star depends on the metric and thus it cannot be present in a topological term. This can be seen from its defining property: If $\alpha, \beta \in \bigwedge^k V$ for an n-dimensional vector space and $\omega \in \bigwedge^n V$ is an n-volume form, $\star\beta$ is unique $(n-k)$-form such that

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle \omega$$

(in fact, explicitly, it is defined for $\alpha \in \bigwedge^k V$ as

$$\star\alpha = \frac{\sqrt{g}}{(n-k)!k!}\alpha^{i_1,\ldots,i_k}\epsilon_{i_1,\ldots,i_n}dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n}.$$

which manifestly makes mention of the metric). One way we can understand this is that $\star$ needs knowledge of the metric to scale the $(n-k)$-form $\star\beta$ properly so that it is related to β in a way that is useful and unique.

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- Tong, *Gauge Theory*, §3.5.

6.5 Clifford Algebra Cheat Sheet

$$1{+}1: \qquad \gamma^0 = \sigma_z, \quad \gamma^1 = -i\sigma_y, \quad \gamma^5 = \gamma^0\gamma^1$$

TODO: add also all the identities I've found myself using

6.6 Quantum Integrability, Conserved Quantities

- Integrability vs solvability
- Relation to Bethe Ansatz, Yang-Baxter
- Relation to presence of conserved quantities.

6.7 Stress-Energy from Lagrangian

The stress-energy tensor can be defined as the variation in the action with respect to the metric (see e.g., Caroll, *Spacetime and Geometry* §4.3):

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g_{\mu\nu}} = -\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}(\mathcal{L}_{\mathrm{M}}\sqrt{-g})$$

where the subscript M denotes the action/Lagrangian of the matter fields. Had we considered the Einstein-Hilbert action along with the matter fields, we would've recovered the Einstein equation. Calculating this derivative will involve differentiating a determinant (g) with respect to to a particular entry of the matrix ($g_{\mu\nu}$). Here is how one does this: start with the matrix identity

$$\ln\det M = \operatorname{tr}\ln M$$

where considering a variation in M we find this equation becomes

$$\begin{aligned}\delta(\ln\det M) &= \delta(\operatorname{tr}\ln M)\\ \frac{1}{\det M}\delta\det M &= \ln\big(M^{-1}\delta M\big)\\ \delta(\det M) &= \det M\cdot\operatorname{tr}\big(M^{-1}\delta M\big)\end{aligned}$$

Identifying $\det M = \det g_{\mu\nu} = g$ and using the fact that $\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$:

$$\delta g = g\operatorname{tr}\big(g^{\lambda\mu}\delta g_{\mu\nu}\big) = g\operatorname{tr}\big(g^{\lambda\mu}(-1)g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}\big) = -g\operatorname{tr}(g_{\nu\sigma}\delta g^{\rho\sigma})$$

so that taking the trace we find

$$\delta g = -g\ g_{\mu\nu}\delta g^{\mu\nu}$$

In particular this means that

$$\frac{\partial g}{\partial g_{\mu\nu}} = -g g_{\mu\nu}, \qquad \frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\ g_{\mu\nu}.$$

Chapter 7

December

Contents

7.1 Off-Diagonal Matrix Elements for VMC

In Variational Monte Carlo (VMC) simulations we calculate observables by weighting a function $f(\alpha)$ by a Monte Carlo weight $\rho(\alpha)$ by a random walk through configurations α with Metropolis-Hastings acceptance criteria:

$$\langle \mathcal{O} \rangle = \sum_{\alpha} f(\alpha)\rho(\alpha), \qquad f(\alpha) = \sum_{\beta} \langle \alpha | \mathcal{O} | \beta \rangle \frac{\langle \beta | \psi \rangle}{\langle \alpha | \psi \rangle}, \quad \rho(\alpha) = \frac{|\langle \alpha | \psi \rangle|^2}{\langle \psi | \psi \rangle}$$

$\rho(\alpha)$ is accounted for by the acceptance probability

$$P(\alpha \to \alpha') = \begin{cases} 1 & \rho(\alpha') > \rho(\alpha) \\ \rho(\alpha')/\rho(\alpha) & \rho(\alpha') < \rho(\alpha) \end{cases}$$

while $f(\alpha)$ must be computed at a chosen sample frequency. The best case scenario is that $\mathcal{O}$ is diagonal in the chosen basis states α, β over which we are perfoming our random walk. Then we just have to calculate

$$f(\alpha) = \langle \alpha | \mathcal{O} | \alpha \rangle$$

However, oftentimes we will have to compute more complicated expectation values which will involve off-diagonal elements. In what follows we work out some examples by considering all possible off diagonal elements of spin-1/2 operators.

Consider $S^x = (S^+ + S^-)/2$. This operator's only nonvanishing matrix elements are

$$\langle\uparrow| S_x |\downarrow\rangle = \langle\downarrow| S_x |\uparrow\rangle = \frac{1}{2}$$

This is of course related to the fact that $S^+ + S^-$ acts by flipping the spin, e.g.,

$$(S^+ + S^-)|\uparrow\rangle = \cancel{S^+|\uparrow\rangle} + S^-|\uparrow\rangle = |\downarrow\rangle$$

For $S^y = -i(S^+ - S^-)/2$, the nonvanishing elements are

$$\langle\uparrow| S_y |\downarrow\rangle = \frac{i}{2}, \qquad \langle\downarrow| S_y |\uparrow\rangle = -\frac{i}{2}$$

Now consider these operators as they act on a tensor product Hilbert space. Let's say S_i^x acts on a site i:

$$\langle\alpha| S_i^x |\beta\rangle = \frac{1}{2}\langle\alpha| (S_i^+ + S_i^-) |\beta\rangle$$

We specialize to half-filling in a Mott insulator phase (no double-occupancy). Then there is either a spin up or spin down at the site i. Because the operator is understood to flip whatever spin is at that site, and because the tensor-product factorization guarantees us that the remaining sites in $|\alpha\rangle$ and $|\beta\rangle$ must have the same configuration for the matrix element to be nonvanishing, we conclude that the only nonvanishing matrix elements $\langle\alpha| S_i^x |\beta\rangle$ are those $|\beta\rangle$ related to $|\alpha\rangle$ only by the flip of the spin at site i. Note that this means there is only one such matrix element, and it is off-diagonal.

Now we apply the above reasoning to products of operators. We write

$$|\alpha\rangle = |\sigma_{\alpha 1}\rangle \otimes \cdots \otimes |\sigma_{\alpha N}\rangle$$

(and similarly for $|\beta\rangle$), and we introduce the "complement"

$$|\alpha^{\complement}_{ij,\ldots}\rangle \equiv |\alpha\rangle \text{ excluding the states on sites } ij\ldots$$

Quadratic operators We then find, assuming $i \neq j$

$$\begin{aligned}\langle\alpha| S_i^x S_j^x |\beta\rangle &= \langle\alpha^{\complement}_{ij}| \otimes \langle\sigma_{\alpha i}| \otimes \langle\sigma_{\alpha j}| S_i^x S_j^x |\sigma_{\beta i}\rangle \otimes |\sigma_{\beta j}\rangle \otimes |\beta^{\complement}_{ij}\rangle \\ &= \langle\alpha^{\complement}_{ij}|\beta^{\complement}_{ij}\rangle \langle\sigma_{\alpha i}| S_i^x |\sigma_{\beta i}\rangle \langle\sigma_{\alpha j}| S_j^x |\sigma_{\beta j}\rangle\end{aligned}$$

Orthonormality ensures that the factor $\langle\alpha^{\complement}_{ij}|\beta^{\complement}_{ij}\rangle$ vanishes unless the configuration is the same on all sites but i, j. We use the notation $\delta_{\alpha^{\complement}_{ij}\beta^{\complement}_{ij}}$ to denote this fact. Then we have

$$\begin{aligned}\langle\alpha| S_i^x S_j^x |\beta\rangle &= \delta_{\alpha\beta}\langle\sigma_{\alpha i}| S_i^x |\sigma_{\beta i}\rangle \langle\sigma_{\alpha j}| S_j^x |\sigma_{\beta j}\rangle \\ \langle\alpha| S_i^y S_j^y |\beta\rangle &= \delta_{\alpha\beta}\langle\sigma_{\alpha i}| S_i^y |\sigma_{\beta i}\rangle \langle\sigma_{\alpha j}| S_j^y |\sigma_{\beta j}\rangle\end{aligned}$$

If $i = j$ we cannot factor the Hilbert space as we did above, but the answer is trivial as $(S_i^{(x,y)})^2 = 1/4$. This is rooted in the fact that the site will always be maximally correlated with itself.

At half-filling ($N_\uparrow = N_\downarrow$), an additional simplification occurs: We are restricted to the sector of Hilbert space with fixed number of up and down electrons. Thus, in this sector, any operator flipping to a different number up or down spins individually will have vanishing matrix elements. For instance, S_i^x flips a single spin and takes $N_\uparrow \to N_\uparrow - 1$ and $N_\downarrow \to N_\downarrow + 1$ or $N_\uparrow \to N_\uparrow + 1$ and $N_\downarrow \to N_\downarrow - 1$. It therefore has vanishing elements in this sector. On the other hand, $S_i^x S_j^x$ has nonvanishing elements when it flips spin i from up to down and spin j from down to up and vice-versa. Note this also means that this operator must act on sites with opposite spins. Also note that because the operator will always flip two opposite spins, products of operators that depend on the original spin being flipped (namely, $S_i^y S_j^y$) will no longer depend on the original spins. This means that we're guaranteed $\langle S_i^y S_j^y \rangle = \langle S_i^x S_j^x \rangle$ since $(1/2)^2 = -(i/2)^2$, as should be expected from the symmetry of the problem.

Biquadratic operators In order to perform a Hamiltonian reconstruction using the correlation matrix, one needs knowledge of the biquadratic four-point correlations of the theory in addition to the two-point correlators worked out above; i.e., we need

$$\langle S_i^\alpha S_j^\alpha S_k^\beta S_\ell^\beta \rangle, \qquad \alpha, \beta \in \{x, y, z\}$$

While at first glance the above constitutes $3^2 = 9$ different operators corresponding to all possible pairs of directions, a large number of correlators are equal to each other provided the wavefunction is rotationally invariant. Namely, we have that

$$\langle S_i^x S_j^x S_k^x S_\ell^x \rangle = \langle S_i^y S_j^y S_k^y S_\ell^y \rangle = \langle S_i^z S_j^z S_k^z S_\ell^z \rangle$$

and

$$\begin{aligned} \langle S_i^x S_j^x S_k^z S_\ell^z \rangle = \langle S_i^y S_j^y S_k^z S_\ell^z \rangle = \langle S_i^x S_j^x S_k^y S_\ell^y \rangle \\ = \langle S_i^z S_j^z S_k^y S_\ell^y \rangle = \langle S_i^y S_j^y S_k^x S_\ell^x \rangle = \langle S_i^z S_j^z S_k^x S_\ell^x \rangle. \end{aligned}$$

This can be seen by considering the unitary operators $U[\mathcal{R}]$ that implement the elements of the internal SU(2) symmetry in this representation. Consider $\langle \psi | S_i^x S_i^x S_j^y S_k^y | \psi \rangle$. Assuming the Gutzwiller wavefunction transforms trivially under any $U[\mathcal{R}]$ such that $U[\mathcal{R}] |\psi\rangle = \psi$ (up to a phase), we need only identify the group element $\mathcal{R}'$ taking

$$U[\mathcal{R}'] S_i^y U[\mathcal{R}']^{-1} = S_i^z$$

and leaving S_i^x invariant, so that inserting $U[\mathcal{R}']^{-1} U[\mathcal{R}']$ between every pair of operators we find

$$\begin{aligned} \langle \psi | S_i^x S_i^x S_j^y S_k^y | \psi \rangle &= \langle \psi | U[\mathcal{R}']^{-1} S_i^x S_j^x S_k^y S_\ell^y U[\mathcal{R}'] | \psi \rangle \\ &= \langle \psi | S_i^x S_j^x S_k^z S_\ell^z | \psi \rangle . \end{aligned}$$

Similar arguments lead to the conclusion that provided the wavefunction transforms trivially under the entire symmetry group, the expectation value is invariant across the orbit of $S_i^x S_j^x S_k^y S_\ell^y$ under SU(2). This can be understood as a consequence of the geometric fact that any plane crossing the origin in $\mathbb{R}^3$ can be taken to any other plane.

Thus, in actuality we only need to compute two different four-point correlators. The former possess the matrix elements

$$\langle\alpha|\, S_i^z S_j^z S_k^z S_\ell^z \,|\beta\rangle = \frac{1}{16}\delta_{\alpha\beta}\sigma_i\sigma_j\sigma_k\sigma_\ell$$

for we chose a basis diagonal in S^z. The latter operators

$$\langle S_i^x S_j^x S_k^z S_\ell^z\rangle$$

have a more complicated matrix structure which depends on which operators act at different sites. Consider $i = j = k = l$. Then

$$\langle\alpha|\, S_i^x S_i^x S_i^z S_i^z \,|\beta\rangle = \langle\alpha|\, (S_i^x)^2 (S_i^z)^2 \,|\beta\rangle = \frac{1}{16}\delta_{\alpha\beta}$$

Now consider $i = j$ independent of the other sites. Then we have

$$\langle\alpha|\, (S_i^x)^2 S_k^z S_\ell^z \,|\beta\rangle = \frac{1}{4}\,\langle\alpha|\, S_k^z S_\ell^z \,|\beta\rangle$$

of if $k = \ell$

$$\langle\alpha|\, S_i^x S_j^x (S_k^z S_k^z)^2 \,|\beta\rangle = \frac{1}{4}\,\langle\alpha|\, S_i^x S_j^x \,|\beta\rangle = \frac{1}{4}\,\langle\alpha|\, S_i^z S_j^z \,|\beta\rangle$$

where we've used rotational invariance in the last equality. Thus if any pair of x or z operators act on the same site we just have to calculate the two point S^z correlator for the remaining two sites.

The case $i = k \neq j = \ell$ has

$$\begin{aligned}\langle\alpha|\, S_i^x S_j^x S_i^z S_j^z \,|\beta\rangle &= \delta_{\alpha_{ij}^{\complement}\beta_{ij}^{\complement}}\,\langle\sigma_{\alpha i}|\, S_i^x S_i^z \,|\sigma_{\beta i}\rangle\,\langle\sigma_{\alpha j}|\, S_j^x S_j^z \,|\sigma_{\beta j}\rangle \\ &= \frac{1}{4}\delta_{\alpha_{ij}^{\complement}\beta_{ij}^{\complement}}\,\sigma_{\beta i}\sigma_{\beta j}\,\langle\sigma_{\alpha i}|\, S_i^x \,|\sigma_{\beta i}\rangle\,\langle\sigma_{\alpha j}|\, S_j^x \,|\sigma_{\beta j}\rangle\end{aligned}$$

(it is not hard to see that there is only one allowed $|\beta\rangle$ configuration, the one obtained by flipping the spins on sites i and j. This means $\sigma_{\beta i} = -\sigma_{\alpha i}$). If $j = \ell \neq i \neq k$, we have

$$\langle\alpha|\, S_i^x S_j^x S_k^z S_j^z \,|\beta\rangle = \frac{1}{4}\delta_{\alpha_{ij}^{\complement}\beta_{ij}^{\complement}}\,\sigma_k\sigma_{\beta j}\,\langle\sigma_{\alpha i}|\, S_i^x \,|\sigma_{\beta i}\rangle\,\langle\sigma_{\alpha j}|\, S_j^x \,|\sigma_{\beta j}\rangle$$

If $i = k \neq j \neq \ell$ an analogous expression is obtained by swapping $i \leftrightarrow j$ and $k \leftrightarrow \ell$:

$$\langle\alpha|\, S_i^x S_j^x S_k^z S_\ell^z \,|\beta\rangle = \frac{1}{4}\delta_{\alpha_{ij}^{\complement}\beta_{ij}^{\complement}}\,\sigma_\ell\sigma_{\beta i}\,\langle\sigma_{\alpha i}|\, S_i^x \,|\sigma_{\beta i}\rangle\,\langle\sigma_{\alpha j}|\, S_j^x \,|\sigma_{\beta j}\rangle\,.$$

Lastly, if $i \neq j \neq k \neq \ell$:

$$\langle\alpha|\, S_i^x S_j^x S_k^z S_\ell^z \,|\beta\rangle = \frac{1}{4}\delta_{\alpha_{ij}^{\complement}\beta_{ij}^{\complement}}\,\sigma_k\sigma_\ell\,\langle\sigma_{\alpha i}|\, S_i^x \,|\sigma_{\beta i}\rangle\,\langle\sigma_{\alpha j}|\, S_j^x \,|\sigma_{\beta j}\rangle\,..$$

References

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7.2 2d CFT Basics

> That book should be banned from classrooms.
>
> *Antal Jevicki on Di Francesco*

Once upon a time I tried learning some CFT from Di Francesco as a final project for my intro QFT class. Unsurprisingly for those familiar with that book, that did not go well. More recently I picked up Blumenhagen and Plauschinn's CFT book and found that it presented the material in a nicely condensed fashion (although I still think the bulk of it would've gone over my head as I was not sufficiently comfortable with field theory generally back then).

The conformal group We all know that the derivative $-i\partial_x$ is the generator of x translations; we can see directly that this is the case by considering the exponential map $\exp(i\epsilon(-i\partial_x))$ acting on a function $f(x)$:

$$\begin{aligned} e^{i\epsilon(-i\partial_x)}f(x) = e^{\epsilon\partial_x}f(x) &= (1+\epsilon\partial_x+\frac{1}{2}\epsilon^2\partial_x^2+\cdots)f(x) \\ &= f(x)+\epsilon\partial_x f+\frac{1}{2}\epsilon^2\partial_x^2+\cdots \\ &= f(x+\epsilon) \end{aligned} \tag{7.1}$$

In the case of a two-dimensional conformal field theory, the transformations in question can be treated as holomorphic shifts in the complex coordinate $z \to z+\epsilon(z)$. Because $\epsilon(z)$ is holomorphic (at least, we assume, in some neighborhood of z), we can Laurent expand it

$$z' = z+\epsilon(z) = z+\sum_n \epsilon_n(-z^{n+1})$$

Here ϵ_n is some Laurent coefficient which can be treated as a possibly small transformation parameter. (The reason for the $n+1$ exponent convention will be made clear when we consider the Witt algebra.) Now consider some function of z and Taylor expanding it to the first order:

$$f(z') = f(z+\epsilon(z)) = f(z)+f'(z)\epsilon(z)+\cdots = f(z)+f'(z)\left[\sum_n \epsilon_n(-z^{n+1})\right]+\cdots$$

We necessarily have one generator for each Laurent coefficient, all of which are associated with a first order derivative. It's easiest to see by considering one Laurent mode:

$$f(z)+f'(z)\epsilon_n(-z^{n+1}) = e^{\partial_z(-z^{n+1})}(z)$$

Witt algebra Comparing against our translational shift eq. (7.1), we find the generators are

$$\ell_n = -z^{n+1}\partial_z,\ [\ell_m, \ell_n] = (m-n)\ell_{m+n}$$

the commutator is fairly easy to calculate by using standard derivative rules (the $n+1$ convention makes the commutator easier to remember). Note that this is a rather special situation: the algebra is infinite-dimensional, which, roughly speaking, is the reason $(1+1)$ CFTs tend to be solvable—an algebra of this size severely constrains the structure of any theory that enjoys it as a symmetry [UNDERSTAND THIS BETTER].

It is obvious that at $z=0$ the generators with $n \leq 2$ blow up. If we compactify to the Riemannian sphere $\mathbb{C} \cup \{\infty\}$ we are still in trouble because inverting $z \to 1/w$ we have a singularity at infinity as well. We conclude that the globally well-defined conformal transformations are generated by $\ell_{\pm 1,0}$, which turn out to generate the subgroup $\mathrm{SL}(\mathbb{C},2)/\mathbb{Z}_2$—the Möbius transformations.

- $\ell_1 = \partial_z$, which is manifestly the generator of translations $z \to z + c$.
- $\ell_0 = -z\partial_z$ acts on $f(z)$ for ϵ small as

  $$(1+\epsilon z\partial_z)f(z) = f(z) + \epsilon z \partial z f$$

  Which is the first order perturbation expansion of $f(z(1+\epsilon))$ about z, i.e., ℓ_0 generates transformations of the form $z \to cz$. Since $c \in \mathbb{C}$, this comprises dilations and rotations. Actually, $\ell + \bar{\ell}$ strictly corresponds to dilations and $\ell - \bar{\ell}$ to rotations (I think, double check this).
- $\ell_1 = -z^2\partial_z$, whose transformation is easiest seen by acting on z:

  $$\begin{aligned} e^{-\epsilon z^2\partial_z} z &= (1 - \epsilon z^2\partial_z + \tfrac{1}{2}\epsilon^2(z^2\partial_z)^2 + \cdots)z \\ &= (1 - \epsilon z^2\partial_z + \tfrac{1}{2}\epsilon^2 z^2\partial_z z^2\partial_z)z \\ &= (1 - \epsilon z^2\partial_z + \tfrac{1}{2}\epsilon^2 z^2 \cdot 2z\partial z)z \qquad\qquad = (1 - \epsilon z^2\partial_z + \epsilon^2 z^3\partial_z)z \\ &= (1 - \epsilon z^2 + \epsilon^2 z^2)z \end{aligned}$$

  we identify the terms in the parenthesis as the expansion for $1/(1-\epsilon z)$,

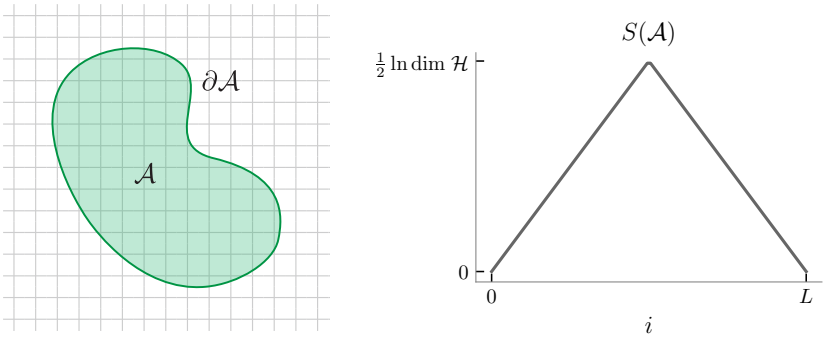
  $$e^{-\epsilon z^2\partial_z} z = \frac{z}{1-\epsilon z}.$$

  ℓ_1 thus generates SCTs.

In aggregate we have transformations of the form

$$z \to \frac{az+b}{cz+1}.$$

The presence of a central charge is rooted in the fact that the second group cohomology of $\mathrm{SL}(\mathbb{C},2)/\mathbb{Z}_2$ is nontrivial (see 2024 logs). We quotient out $\mathbb{Z}_2$, the center of $\mathrm{SL}(\mathbb{C},2)$, because scalar multiples of a transformation define the same automorphism on the Riemann sphere.



(a) (b)

7.2.1 Some Orbifold Theory

Twist operators take you between fixed points

7.3 Entanglement in QFT

7.3.1 Entanglement in Lattice Systems

It is tradition to begin discussion of entanglement in quantum field theories by considering how one might define entanglement on a lattice. From a conceptual point of view, lattice systems provide a natural way to define and visualize entanglement entropy in quantum field theories. From a calculational point of view, the presence of a lattice regularization makes entanglement measures better behaved—in this setting they are finite, whereas, as is typical for continuum QFTs, in the continuous case we find UV divergences which necessitate the introduction of some UV regularization, effectively limiting the fastest modes of which we are allowed to measure entanglement. (Actually, the origin of UV divergences in entanglement entropies is a subtle and interesting point, especially in the case of CFTs, which are normally free of such pathologies. We will discuss this in detail later.)

Entanglement entropy is typically understood as the entropy of the density matrix that results from a partial trace over a subsystem. The standard treatment, readily found in any quantum information theory text, is as follows: Assuming there is a physically sensible Hilbert space factorization at hand, partition this factorization into a subsystem $\mathcal{A}$ and its complement $\mathcal{A}^{\complement}$. Taking the total density matrix ρ and tracing over the complement degrees of freedom yields a new reduced density matrix $\mathrm{Tr}_{\mathcal{A}^{\complement}}\,\rho = \rho_{\mathcal{A}}$ whose spectrum is referred to as the **entanglement spectrum** (fig. 7.1a). This spectrum represents, of course, a mixed state whose entropy we can calculate; this is the **entanglement entropy**.

Lattice systems are a natural setting for this construction. In particular, quantum lattice systems are defined by factorizable many-body Hilbert space

$$\mathcal{H} = \bigotimes_i \mathcal{H}_i$$

whose product structure makes defining a partial trace uncomplicated: we simply consider a subsystem $\mathcal{A}$ consisting of a particular set of lattice sites, and trace over the degrees of freedom on the remaining lattice sites (fig. 7.1a). Equipped with a spatial intepretation for the Hilbert space factorization, we can also outline the boundary of our subsystem, $\partial\mathcal{A}$, and understand it as the interface for quantum information exchange between $\mathcal{A}$ and $\mathcal{A}^\complement$. For this reason it is referred to as the **entanglement surface**.

Aside: gauge theories

It is worth noting that for lattice gauge theories the above definition is not straightforward, and is the subject of ongoing research. The gauge field is formulated in terms of link variables that are situated *between* lattice sites; thus, a partition of the lattice does not pick out a unique, gauge invariant partition of link variables, and one must make some choice as to the the subsystem the variable belongs in.

Scaling laws It is natural to ask how the entanglement entropy scales with the size of $\mathcal{A}$. In one dimension, things are simple—$\mathcal{A}$ is some chain of contiguous lattice sites, and the only real freedom in our partition is the location of $\partial\mathcal{A}$, which is now a point. If we let this location be the site index i and measure the dependence of the entanglement entropy on the i (i.e., vary the fractional size of $\mathcal{A}$), we find a so-called **Page curve** which saturates the maximum entanglement entropy—half the dimensionality of the total Hilbert space of the system—at precisely halfway in the chain (fig. 7.1b).

In higher dimensions the situation can be more interesting. We consider the thermodynamic limit of a D (spatial) dimensional system, and a subsystem with linear size greater sufficiently larger than the lattice spacing: $L \gg a$. Under these conditions, most states in the Hilbert space will turn out to possess a volume-law scaling, wherein the entanglement entropy is nearly saturated, and $S(\mathcal{A})$ scales as the number of sites in the subsystem, $N_\mathcal{A}$:

$$S(\mathcal{A}) \propto N_\mathcal{A} \sim \frac{L^D}{a^D}$$

Most physically realistic states, however—say, the ground state of a Hamiltonian—are subject to constraints that tie the lattice geometry to the entanglement struture in the form of some locality constraint. This is to say that entanglement should primarily be found between

neighboring sites, or at least sites that are spatially near one another. This is the primary intuition behind **matrix product state** (MPS) ansatze and **tensor network theory** formulations, wherein one builds wavefunctions with geometrically sound entanglement structures by placing tensors at each lattice site and contracting tensor indices between neighboring sites, i.e., creating a bond. This class of wavefunctions occupies a relatively small corner of the Hilbert space and thus have seen much recent success as a circumvention around the curse of dimensionality; additionally, enforcing a degree of entanglement locality has proven a successful recipe for many-body variational ansatze.

The relevance of these states to entanglement scaling lies in the fact that we can think of calculating the entanglement entropy by counting the number of bonds cut by the boundary $\partial\mathcal{A}$. In this case it is clear that the entanglement entropy scales as the area of $\mathcal{A}$:

$$S(\mathcal{A}) \sim \frac{L^{D-1}}{a^{D-1}}.$$

It is commonly the case that gapped systems possess this entanglement scaling whereas gapless Hamiltonians in two dimensions tend to volume-law entanglement. Generally, there is a rich literature on entanglement scaling, network theory, and the like, and these points of view have grown increasingly central to modern many-body physics—especially when considering computational methods. We have also neglected to mention topological entanglement entropies, which have played an integral role in recent developments in condensed matter physics, but this is a topic for a different occasion.

Relativistic subsystems In order to apply the preceding discussion to relativistic systems, it is necessary to develop a notion of subsystem that plays well with a spacetime picture. Actually, the best way to do this is by leveraging the causal structure of the underlying spacetime, whereby we find several additional features which refine our understanding of a subsystem in field theory.

The natural extension of a lattice system like that of fig. 7.1a is a Cauchy slice Σ, where a subsystem is some subset of the slice $\mathcal{A}$. Causality leads to the constraint that no information from outside of $\mathcal{A}$ can enter a lightlike codimension-1 cones situated over A pointing in the positive and negative t direction; put differently, these are volumes where the content of the theory can be entirely determined by evolving the operator content of $\mathcal{A}$ by the appropriate (presumably Heisenberg) equations of motion. These are of course the future and past domains of dependence $D^{\pm}[\mathcal{A}]$ (or together $D[\mathcal{A}] = D^{+}[\mathcal{A}] \cup D^{-}[\mathcal{A}]$) the content of which, again, is entirely determined by what is on $\mathcal{A}$. We can also consider the region of spacetime that the content of $\mathcal{A}$ may have an influence on, i.e., the union of the light cones of all points on $\mathcal{A}$. These are the causal future and past $J^{\pm}[\mathcal{A}]$. Then the natural extension of our entanglement surface is simply the boudnary of $D[\mathcal{A}]$, which we will denote $\partial D[\mathcal{A}] = \partial\mathcal{A}$ for simplicity.

This is the appropriate notion of a subsystem for a relativistic system (fig. 7.2). Note that the choice of Cauchy surface is not unique: one can deform Σ while preserving the

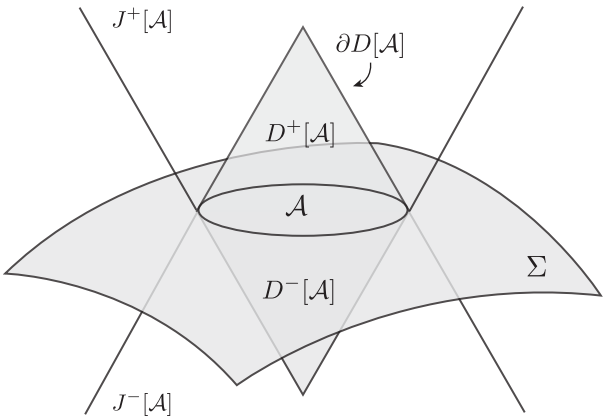


Figure 7.2: The causal struture of the subsystem $\mathcal{A}$. Σ is a Cauchy surface or slice, and $\partial D[\mathcal{A}]$ is the entangling surface. $D^{\pm}[\mathcal{A}]$ are the future and past domains of dependence—the regions of the manifold where the content of the theory can be entirely determined by the configuration on $\mathcal{A}$. $J^{\pm}[\mathcal{A}]$ are causal future and past, the region of spacetime that is affected by the configuration on $\mathcal{A}$.

domain of dependence of $\mathcal{A}$ and thus the entangling surface, so that, to be pedantic, the proper way to think about a subsystem is in terms of an equivalence class of Cauchy slices.

7.3.2 Path integral techniques

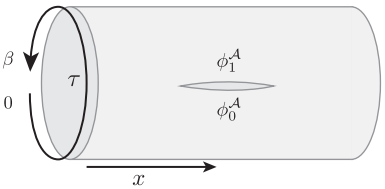
The entanglement entropy of continuum field theories is most conventiently calculated with a path integral formalism, usually because this admits a geometric picture which assigns clear interpretations to density matrix manipulations. To be specific, one identifies the matrix elements of a density matrix $\langle\phi_1|\,\rho\,|\phi_0\rangle$ with boundary conditions on the path integral, then visualizes these conditions as cuts on the spacetime manifold. It is most convenient to imagine this in an Euclidean setting by compactifing a complex time dimension $t \to i\tau \in [0, \beta]$, $\beta \sim \beta + 2\pi$, so that the vacuum of the theory may be studied by taking $\beta \to \infty$ (corresponding to flattening the temperature circle S^1 to $\mathbb{R}$). Now the path integral calculates matrix elements of the canonical ensemble: $\langle\phi_1|\,e^{-\beta H}/Z\,|\phi_0\rangle$. This is kosher so long as we're interested in steady states of the theory—as the spectrum of the Hamiltonian is what enters the canonical density matrix—or when considering an instantaneously static point in the spacetime manifold. [go back to understand this]

Let's make this formalism a little more explicit. Consider a $1+1$ Euclidean field theory, and let $\mathcal{A}$ be an interval of unspecified length living on some Cauchy slice which we can take to be $\tau = 0$. We could like to calculate the matrix elements of the reduced density matrix

$\langle \phi_1^{\mathcal{A}} | \rho_{\mathcal{A}} | \phi_0^{\mathcal{A}} \rangle$, which is easy to convince oneself amounts to setting boundary conditions on the interval (recall path integrals are essentially propagators) and fully integrating out the remainder of the field configurations. In fact, generally the partial trace has the geometric interpretation of closing a cut made by matrix elements, as we will see later. Anyway, this is represented by

$$\langle \phi_1^{\mathcal{A}} | \operatorname{Tr}_{\mathcal{A}^{\complement}} e^{\beta H}/Z | \phi_0^{\mathcal{A}} \rangle = \frac{1}{Z} \int_{\phi^{\mathcal{A}}=\phi_0^{\mathcal{A}}}^{\phi^{\mathcal{A}}=\phi_0^{\mathcal{A}}} \mathcal{D}\phi e^{S_{\mathrm{E}}[\phi]} = \frac{1}{Z} \int \mathcal{D}\phi e^{S_{\mathrm{E}}[\phi]}\, \delta(\phi_0, \phi_1)$$

$$= \frac{1}{Z}$$



(7.2)

(actually in this figure ϕ_0 and ϕ_1 should be switched) here the partition function Z has the interpretation of integrating over the entire spacetime without any cuts, and the delta function $\delta(\phi_0, \phi_1)$ provides a less schematic, more formal way of implementing the boundary conditions. As a sanity check, we verify the Euclidean action S_{E} is really nothing more than the Hamiltonian: we can see this by writing the action

$$i \int d^d x \mathcal{L}[\phi] = i \int d^d \left(\partial_t^2 \phi - V[\phi] \right)$$

(here $V[\phi]$ includes the spatial derivative terms) and seeing how it is transformed under $t \to -i\tau$,

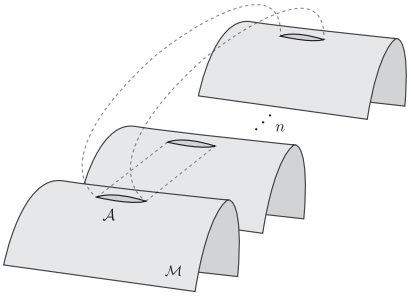
$$\int d^{d-1}x\, d\tau \left(-\partial_\tau^2 \phi - V[\phi] \right) = -\int d^{d-1}x\, d\tau (\partial_t^2 + V[\phi]) = -\int d^{d-1}x\, d\tau H[\phi].$$

Replica trick One computation that is made often in the entanglement entropy literature is that for the Rényi entropy via the replica trick. The idea is that while for the majority of field theories it is not quite clear how to compute the entanglement entropy, it is comparatively straightforward to compute the matrix elements of the n-th power of the reduced density matrix, the log of the trace of which is the n-th order Rényi entropy:

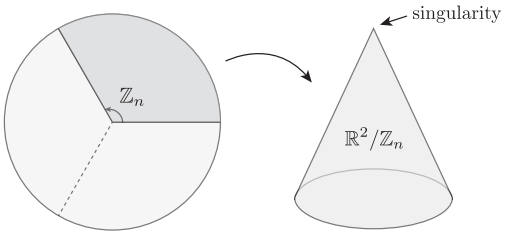
$$S_{\mathrm{A}}^{(n)} = \frac{1}{1-n} \ln \operatorname{Tr} \rho_{\mathcal{A}}^q$$

This is done by taking n copies of a path integral such as eq. (7.2) and implementing the matrix multiplication by identifying the i-th path integral's upper (ϕ_1^i) boundary condition with the $i+1$-th path integral's lower boundary condition (ϕ_0^{i+1}). E.g.,

$$\langle \phi_1^2 | \rho_{\mathcal{A}}^2 | \phi_0^1 \rangle = \int \mathcal{D}\phi_1^1 \langle \phi_1^2 | \rho_{\mathcal{A}} | \phi_1^1 \rangle \langle \phi_1^1 | \rho_{\mathcal{A}} | \phi_0^1 \rangle$$



(a) The n-fold replica orbifold glued cyclically along the slits $\mathcal{A}$.



(b) The cyclic orbifold $\mathbb{R}^2/\mathbb{Z}_n$ is geometrically a cone.

As we've stated above, by taking the trace over $\rho_{\mathcal{A}}^2$, we seal the last cut in the n replicas and create a space without boundary. A schematic of one such situation is displayed in fig. 7.3a. Note that after taking the trace, the replicated system exhibits a so-called **replica symmetry**: we can shift the replicas cyclically and obtain the same result, so that the system possess a $\mathbb{Z}_n$ "symmetry." As it turns out, the replica spacetime is now no longer formally a manifold: we will later see how this constrution is equivalent to defining the theory on an orbifold, which surprisingly leads to significant calculational advantage.

7.3.3 Thermofield Double, Modular Hamiltonian

There are two additional pieces of technology commonly employed in entanglement entropy calculations that are worth mentioning. The first is the **thermofield double**—which is a form of purification in the quantum information sense—a pure state constructed out of a thermal density matrix. The idea is to essentially write down a Schmidt decomposition out of two copies of a Hilbert space (hence "double") such that the partial trace over one of the copies yields the original canonical density matrix. Specifically the thermofield double is constructed by

$$\rho_{\mathrm{A}} = \frac{1}{Z}\sum_i e^{-\beta E_i}\,|i\rangle\langle i| \quad \Longrightarrow \quad |\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z}}\sum_i e^{-\frac{\beta}{2}E_i}\,|i\rangle\otimes|i\rangle$$

which lives in a double Hilbert space $\mathcal{H}_{\mathcal{A}_1}\otimes\mathcal{H}_{\mathcal{A}_2}$ That this field is indeed a purification of $\rho_{\mathcal{A}}$ can be seen by constructing the density matrix

$$\rho_{\mathrm{TFD}} = |\mathrm{TFD}\rangle\langle\mathrm{TFD}| = \frac{1}{Z}\sum_{ij} e^{-\frac{\beta}{2}(E_i+E_j)}\,|i\rangle\otimes|i\rangle\,\langle j|\otimes\langle j|$$

whose partial trace over $\mathcal{H}_{\mathcal{A}_2}$ is

$$\mathrm{Tr}_{\mathcal{A}_2}\,\rho_{\mathrm{TFD}}\,\frac{1}{Z} = \sum_k e^{-\frac{\beta}{2}(E_i+E_j)}\,\langle k|i\rangle\,\langle j|k\rangle\,|i\rangle\langle j| = \frac{1}{Z}\sum_k e^{-\beta E_k}\,|k\rangle\langle k|\,.$$

The thermofield double is a useful construct because it functions as a bridge between thermal entropy calculations and entanglement entropy calculations: the thermal entropy of $\rho_{\mathcal{A}}$ is exactly the entanglement entropy of $|\text{TFD}\rangle$ over the copy Hilbert space.

The second tool is the **modular Hamiltonian**, which is the name given to the operator $H_{\mathcal{A}}$ that satisfies

$$\rho_{\mathcal{A}} = \frac{1}{Z} e^{-H_{\mathcal{A}}}$$

i.e., it is the Hamiltonian whose spectrum gives $\rho_{\mathcal{A}}$ as a canonical density matrix (it is convention to absorb the inverse temperature into the definition of the Hamiltonian). Note that the von Neumann entropy can be retrieved as essentially the free energy corresponding to the modular Hamiltonian:

$$S_{\mathcal{A}} - \operatorname{Tr} \rho_{\mathcal{A}} \ln \rho_{\mathcal{A}} = \operatorname{Tr} \rho_{\mathcal{A}} H_{\mathcal{A}} + \operatorname{Tr} \rho \mathcal{A} \ln Z = \langle H_{\mathcal{A}} \rangle + \ln Z$$

This makes the modular Hamiltonian construction a sort of inverse problem of the thermofield double. This Hamiltonian is the subject of the famous **Bisognano-Wichmann theorem**, an algebraic QFT result which states that the modular Hamiltonian of certain field theories is nothing more than the integral of the generator of Lorentz boosts. Besides this landmark result, the modular Hamiltonian is frequently used to make nontrivial connections to bulk physics in holographic scenarios; the precise nature of these considerations lies outside the scope of the present discussion, but is nonetheless something I am interested in understanding further.

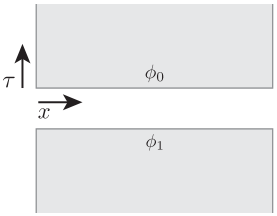
In condensed matter literature, the modular Hamiltonian is better known as the *entanglement Hamiltonian*. In this context, it has been applied to the problem of *Hamiltonian reconstruction*: the question of whether it is possible to entirely determine a Hamiltonian from one of its ground states. See for instance, Biao Lian's paper and references therein. A related approach which I studied extensively is the **correlation matrix** proposed by Qi and Ranard; it is an interesting line of inquiry—which I am currently exploring—to ask what the relationship between the correlation matrix and the modular Hamiltonian is, and whether these considerations can yield new insights into the gravity duals of holographic descriptions.

7.3.4 Example: the Rindler Wedge

We now make a heuristic argument for the form of the modular Hamiltonian (and thus indirectly the entanglement entropy) of the simplest possible field theoretic example: the vacuum of a $1+1$d field theory with $\mathcal{A}$ the positive half line, $x > 0$. In this case the domain of the dependence is the so-called **Rindler wedge**: the region in the $x > 0$ half of the x-t plane bounded from above by $x = t$ and below by $x = -t$. We will do this using the pictorial path integral formalism developed above; namely, we have an Euclidean thermal density matrix $\rho = e^{-\beta H}/Z$ whose $\beta \to \infty$ limit takes us to the vacuum $\rho = |0\rangle\langle 0|$ whose

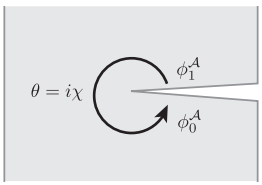
path integral is represented by

$$\langle\phi_1|\,\rho\,|\phi_0\rangle = \frac{1}{Z}\int_{\phi_0}^{\phi_1}\mathcal{D}\phi\; e^{S_{\mathrm{E}}[\phi]} = \frac{1}{Z}$$



(here the $\beta \to \infty$ limit has made the cylinder infinite in τ extent). By performing a partial trace, we integrate over the field configurations in the complement $\mathcal{A}^{\complement} = \{x \mid x < 0\}$, so that the path integral is

$$\left\langle\phi_1^{\mathcal{A}}\right|\rho_{\mathcal{A}}\left|\phi_0^{\mathcal{A}}\right\rangle = \left\langle\phi_1^{\mathcal{A}}\right|\mathrm{Tr}_{\mathcal{A}^{\complement}}\,\rho\left|\phi_0^{\mathcal{A}}\right\rangle =$$



Now, by inspecting this cut spacetime we can guess that the appropriate way to treat it is by switching to radial coordinates

$$x = r\cos\theta, \quad \tau = r\sin\theta$$

where θ is, as pictured, taking us from one boundary condition to another. In fact, we can go farther and immediately Wick rotate this coordinate system back to Minkowski signature. There is an ambiguity in what we choose as a time coordinate [I did not know this; investigate futher], and we can Wick rotate θ itself to $\theta = i\chi$. Under this transformation we have

$$x = r\cosh\chi, \qquad t = -i\tau = r\sinh\chi$$

where we've used $\cos ix = \cosh x$, and $-i\sin ix = \sinh x$. Now, the linear coordinates transform under boosts (written in the form of a rapidity η) as

$$\begin{cases} t' = \cosh\eta\; t + \sinh\eta\; x \\ x' = \sinh\eta\; t + \cosh\eta\; x \end{cases}$$

so that the radial coordinate is unchanged

$$\begin{aligned} r' = x'^2 - t'^2 &= (\sinh\eta\; t + \cosh\eta\; x)^2 - (\cosh\eta\; t + \sinh\eta\; x)^2 \\ &= \sinh^2\eta\; t^2 + 2\sinh\eta\cosh\eta\; xt + \cosh^2\eta\; x^2 \\ &\quad - \sinh^2\eta\; t^2 - 2\sinh\eta\cosh\eta\; xt - \cosh^2\eta\; x^2 \\ &= (\cosh^2\eta - \sinh^2\eta)x^2 + (\sinh^2\eta - \cosh^2\eta)t^2 = x^2 - t^2 \end{aligned}$$

and the Minkowski radial coordinates are shifted linearly:

$$\tanh\chi' = \frac{t'}{x'} = \frac{\cosh\eta\, t + \sinh\eta\, x}{\sinh\eta\, t + \cosh\eta\, x} = \frac{\cosh\eta\sinh\chi + \sinh\eta\cosh\eta}{\sinh\eta\sinh\chi + \cosh\eta\cosh\chi}$$
$$= \frac{\sinh(\chi+\eta)}{\cosh(\chi+\eta)} \Longrightarrow \tanh\chi' = \tanh(\chi-\eta).$$

We can therefore identify χ as none other than the rapidity of this coordinate system. If this seems unsavoury, recall that the boost is nothing more than a hyperbolic rotation, such that just as θ might parametrize a coordinate system that can be rotated, χ can parameterize a Lorenz frame. Just as the generator of time translations appears in the exponential of the canonical density matrix, the generator of boosts here then plays the role of a modular Hamiltonian with temperature $\beta = 2\pi$ (for we have rotated across the whole plane)

$$\rho_{\mathcal{A}} = \frac{1}{Z}e^{-2\pi K}$$

This is a manifestation of the aformentioned Bisognano-Wichmann theorem. In fact, this result gives us even more: The $\beta = 2\pi$ looks awfully familiar; indeed, let us consider the worldline of an observer confined to the Rindler wedge. One such worldline is a line of constant r:

$$r = \sqrt{x^2 - t^2} \Longrightarrow x = \sqrt{r^2 + t^2}$$

which one can verify is that of an observer with proper acceleration $a = 1/r$. Thus we have just recovered the Unruh effect—this observer sees a thermal density matrix with temperature $a/2\pi$ (the a comes from a redshift correction), the interpretation being that to the observer, the modes of the vacuum entangled with the complement of the Rindler wedge have decohered and given rise to a canonical ensemble. Indeed, it is not difficult to see how any observer strictly confined to the Rindler wedge must necessarily be accelerating.

It is worth briefly mentioning how

7.3.5 Cardy-Calabrese Formula

To illustrate how the replica trick is employed to calculate entanglement entropies, we derive the classic Cardy-Calabrese entropy for an interval on a $(1+1)$d CFT. We seek to compute the n-th order Rényi entropy by evaluating the partition function

$$\operatorname{Tr}\rho_{\mathcal{A}}^n = \frac{Z_n[\mathcal{A}]}{Z[\mathcal{M}]^n}. \tag{7.3}$$

Here $Z_n[\mathcal{A}]$ denotes the partition function over n copies of the spacetime manifold $\mathcal{M}$ glued along a cut on the subsystem $\mathcal{A}$. Here we take $\mathcal{A} = -\ell/2 < x < \ell/2$, a length-$\ell$ interval at $t = 0$, and $\mathcal{M}$ is the complex plane. The denominator $Z[\mathcal{M}]^n$ is the n-th power of partition function over $\mathcal{M}$ coming from the normalization over each subsystem partition

function—note that this implicitly includes the order-n branch along which the replicas are glued. The advantage of this construction is it admits a description in terms of orbifold theory, allowing us to import powerful tools from that formalism. In particular, we can view the numerator as a *cyclic orbifold CFT* $\mathcal{M}/Z_n$ with fixed points along the interval $\mathcal{A}$. In fact, in this language, we can more clearly see that the replica gluing procedure introduces conical singularities at the endpoints of the interval $\mathcal{A}$—we recognize that the quotient $\mathcal{M}/\mathbb{Z}_n$ is geometrically a cone with excess angle $2\pi(1-1/p)$ [Caramello]. These will turn out to induce a Weyl anomaly which lies at root of the UV divergence in the entanglement entropy.

The more practical advantage of this orbifold formulation is that it permits a direct calculation of the partition function. We do this by way of formal *twist operators* $\mathcal{T}_n(x,t)$, $\mathcal{T}_{-n}(x,t)$ whose insertions[1] create the endpoints of an order n branch point on the manifold and thus produce $\mathcal{M}/\mathbb{Z}_n$. The replica partition function is then exactly the insertion

$$Z_n[\mathcal{A}] = \langle \mathcal{T}(-\tfrac{\ell}{2},0)\mathcal{T}(\tfrac{\ell}{2},0)\rangle$$

Of course, the advantage of working with a CFT is we have precise knowledge of two point correlators. We find that assuming these are primary operators with conformal dimension $h_{\mathcal{T}} = \bar{h}_{\mathcal{T}}$

$$\langle \mathcal{T}(-\tfrac{\ell}{2},0)\mathcal{T}(\tfrac{\ell}{2},0)\rangle = \frac{1}{\ell^{4h_{\mathcal{T}}}}.$$

Now it remains to determine the conformal dimension $h_{\mathcal{T}}$. Here we reproduce the argument given in [Estiene]: we impose periodic boundary conditions with length L so that the theory is

$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{nc}{12}\right)$$

(see Di Francesco §11.3.2). Here L_0, $\bar{L}_0$ are elements of the two copies of the Virasoro algebra of this theory; in the radial quantization, their sum is the generator of time translations and thus is identified as the Hamiltonian. The presence of n replicas produces a central charge nc, where c is the central charge of the original theory. By construction, the twist operator $\mathcal{T}$ is associated with (i.e., creates/annihilates) the vacuum of the twisted theory, which with $L_0 |h_{\mathcal{T}}\rangle = h |h_{\mathcal{T}}\rangle$ has energy

$$E = \frac{2\pi}{L}\left(2h_{\mathcal{T}} - \frac{nc}{12}\right).$$

At the same time, this theory can be unwrapped to a single theory with period nL with Hamiltonian

$$H = \frac{2\pi}{nL}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$$

whose vacuum posseses vanishing conformal weight, and thus has energy

$$E = -\frac{2\pi}{nL}\frac{c}{12}$$

[1] When people refer to an "insertion" of an operator $\mathcal{O}$ at a spacetime point (x,t) they mean a calculation of the expectation value $\mathcal{O}(x,t)$ by way of "inserting" the operator into the partition function.

Equating the two energies then obtains

$$\frac{4\pi}{L}\left(2h_{\mathcal{T}} - \frac{nc}{12}\right) = \frac{4\pi}{nL}\frac{c}{24}$$
$$2h_{\mathcal{T}} - \frac{nc}{12} = \frac{c}{24n} \quad \Longrightarrow \quad h_{\mathcal{T}} = \frac{c}{24}(n - \tfrac{1}{n}).$$

Thus the partition function is

$$Z_n[\mathcal{A}] = \langle \mathcal{T}(-\tfrac{\ell}{2}, 0)\mathcal{T}(\tfrac{\ell}{2}, 0)\rangle = \frac{1}{\ell^{\frac{c}{6}(n-\frac{1}{n})}}.$$

We have evaluated the numerator of eq. (7.3) and should in principle be set to state the Rényi entropy of the interval of a CFT. However, an additional subtlety arises from the denominator $Z[\mathcal{M}]^n$: as stated above, the branch cut created in the replica trick produces conical singularities in the underlying manifold $\mathcal{M}$ located at the endpoints of the interval. These are geometric singularities (i.e., singularities in the metric) which induce a diverging curvature and thus must be regularized in our integration of the partition function—the obvious choice is to omit a disc of radius ϵ around the singularity. Any such regularization will, however, introduce a length scale to our theory and will induce a conformal (or Weyl) anomaly in the CFT. Put differently, while the lack of a length scale in CFTs on better-behaved (flat, nonsingular) spacetimes generally leads to a finite partition function, any regularization introduced to handle singularities will break scale invariance and thereby lead to a conformal anomaly which lies at the root of the UV-divergences in entanglement entropy calculations of this sort. Thus, we write

$$\operatorname{Tr}\rho_{\mathcal{A}}^n = \frac{Z_n[\mathcal{A}]}{Z[\mathcal{M}]^n}. = \left(\frac{\ell}{\epsilon}\right)^{\frac{c}{6}(n-\frac{1}{n})}$$

which leads naturally to the Rényi entropy

$$\begin{aligned} S_A^{(n)} &= \frac{1}{1-n}\ln \operatorname{Tr}\rho_{\mathcal{A}}^n = -\frac{1}{1-n}\cdot\frac{c}{6}\left(n - \frac{1}{n}\right)\ln\frac{\ell}{\epsilon} + \cdots \\ &= \frac{n+1}{n^2-1}\cdot\frac{1}{n}(n^2-1)\cdot\frac{c}{6}\ln\frac{\ell}{\epsilon} + \cdots \\ &= \frac{c}{6}\left(1 + \frac{1}{n}\right)\ln\frac{\ell}{\epsilon} + \cdots \end{aligned}$$

(the ellipses hide terms coming from the details of the regularization). This is the desired Cardy-Calabrese formula. The von Neumann entropy is simply the analytic continuation to $n = 1$:

$$S_{\mathcal{A}} = \frac{c}{3}\ln\frac{\ell}{\epsilon}.$$

While the continuation is trivial in this case, this is a peculiarity of the constrution under study that is typically absent in calculations for more complicated subsytem topologies. We

	Embedded in R^{d+1}	Embedded in Mink_{d+1}
Intrinsically Euclidean	S^d	Hyperbolic
Intrinsically Minkowski	dS_d	AdS_d

Table 7.1

can, however, modify the topology of the underlying spacetime without much difficulty; it turns out that, had we considered $\mathcal{M}$ compact in one dimension, say $\mathcal{M} = \mathcal{R} \times S^1$ with circumference ℓ_{S^1}, we would have found

$$S_{\mathcal{A}} = \frac{c}{3} \ln \left(\frac{\ell_{S^1}}{\pi \epsilon} \sin \frac{\ell}{\ell_{S^1}} \right)$$

and the change $\ell_{S^1} = i\beta$ then allows us to find the entanglement entropy of a thermal system with inverse temperature β.

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7.4 AdS/CFT, Roughly

In what follows we introduce famous AdS/CFT correspondence, first giving some motivation for the duality, then by describing the most general situation in which such correspondences exist.

7.4.1 Anti-de Sitter Space

We assume the reader has a degree of familiarity with conformal field theory and, and instead focus on providing background on the spacetime that admits the correspondence: anti-de Sitter space. It is most convenient to imagine d dimensional anti-de Sitter (AdS_d) as the metric for a surface embedded in some $d+1$ dimensional ambient space, much like how S^d might be embedded in $\mathbb{R}^{d+1}$ by subjecting the coordinates to the constraint $\sum x_i^2 = R^2$.

In fact, the analogy can be taken further: We can embed not only the sphere but also a hyperbolic space of constant negative curvature, the latter being subject to the constraint

$$\sum_{i=1}^{d-1} x_i^2 - x_d^2 = -R^2$$

and embedded not in $\mathbb{R}^{d+1}$ but in Mink_{d+1}. Then we can take the sphere to de Sitter space and the hyperbolic space anti-de Sitter space by Wick rotating one of the coordinates of the embedded surfaces. See table 7.1. Then AdS_d lives in a space with metric

$$ds^2 = -dx^2 + dx_1^2 + \cdots - dx_{d+1}^2$$

subject to the constraint

$$-x_0^2 + x_1^2 + \cdots - x_{d+1}^2 = -R^2$$

This is an extrinsic characterization of the space; any intrinsic characterization will depend on some choice of coordinates, the most common choice (and the choice that typically serves as the backdrop of AdS/CFT) are the **Poincaré coordinates**

$$ds^2 = -\frac{R^2}{z^2}\left(-dt^2 + \sum_{i=1}^{d-2} dx_i^2 + dz^2\right).$$

Here $z \in (0, \infty)$, and, if we consider a constant t slice such that $dt = 0$, we find that this is the metric of the Poincare half-plane. From this it is clear the sense in which the space possesses constant negative curvature; it is also clear that, by converting the half-plane to the Poincaré disk, one can obtain the famous model of AdS consisting of a temporal stack of hyperbolic disks. The coordinate system we have chosen does not, however, cover the entirety of AdS_d—the z coordinate finds the boundary of the AdS cylinder (the conformal boundary) at $z = 0$, but only goes as far as lightlike hyperplanes slicing across the cylinder at $z = \infty$. The region the coordinates cover is referred to as the **Poincaré patch**.

The holographic boundary theory of the bulk Poincaré patch theory will turn out to be Minkowski space Mink_{d-1}. There is an alternate coordinate system which does cover the entirety of the AdS_d; however, interestingly enough, the holographic theory induced by this coordinatization is not set on Minkowski space but instead on an Einstein static universe $\mathbb{R} \times S^{d-2}$. The details can be found in Rangamani and Takayanagi's book.

7.4.2 Some motivation for AdS/CFT

The anti-de Sitter/conformal field theory correspondence asserts that a theory of quantum gravity in asymptotically AdS spacetime (referred to as the bulk) can be entirely encoded in a a conformal field theory living on the AdS boundary (referred to as the boundary). There are additional constraints on the both theories that must be fulfilled for the duality to

hold, and various theories in either the bulk or the boundary fit are capable of fitting into the correpondence. Here we motivate AdS/CFT with the example that Juan Maldacena originally gave—$3+1$d $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) on the boundary and type IIB supergravity in the bulk. Crucial to the correspondence is the fact that both theories are limits of type IIB string theory in a sense that will be made exact shortly, and the observation made by Polchinski that in type II string theory there is an equivalence between $D3$-branes and extremal black p-branes.

The (charged) $D3$-brane in $\mathcal{R}^{1,9} = \text{Mink}_{10}$ produces the metric

$$ds^2 = H^{-1/2}(r)\, d\mathbf{x}^2 + H^{1/2}(r)(dr^2 + r^2 d\Omega_5^2), \qquad H(r) = 1 + \frac{R^4}{r^4}, \quad R = 4\pi g_s N \alpha'^2$$

where α' is the Regge slope, g_s is the string coupling, and N counts the number of $D3$-branes stacked here. First note that in the $r \to 0$ limit—i.e., close to the $D3$-brane, we have $H(r) \sim R^4/r^4$, and letting $r/R = r/z$:

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + d\mathbf{x}_3 + dz\right) + R^2 d\Omega_5^2$$

so this is asymptotically $\text{AdS}_5 \times S^5$.

At low energies–equivalent by dimensional analysis to $4\pi g_s N \ll 1$—the string endpoints on the $D3$-branes are known to be described by an $\text{SU}(N)$ $\mathcal{N}=4$ SYM with 't Hooft coupling $\lambda = 4\pi g_s N$. The condition $\lambda \ll 1$ is associated with a large N limit of SYM. From the equivalent point of view of the p-branes, the high curvature in the $r \to 0$ limit is associated low energy fluctuations about the $\text{AdS}_5 \times S^5$ background. It is this correspondence that motivates AdS/CFT.

GPKW prescription The correspondence can be made precise at the level of the Euclidean partition functions using the GPKW prescription, which states that when considering an operator $\mathcal{O}$ sourced by a current J in the boundary theory:

$$Z_{\text{CFT}}[J] = \int \mathcal{D}[\text{CFT fields}] \exp\left(S_{\text{CFT}} + \int \mathcal{O}(x)\mathcal{J}(x)\right)$$

we can equate this partition function with that for the string theory on the AdS background for a string field ϕ with boundary value J:

$$Z_{\text{CFT}}[J] = Z_{\text{string}}[\phi(J)]$$

But since we consider the classical gravitational theory, we can go one step further and simply equate this to the saddle-point solution

$$Z_{\text{CFT}}[J] = e^{-S_{\text{classical}}[\phi(J)]}.$$

7.4.3 Ryu-Takayanagi Formula

In context of entanglement entropy, the cornerstone of holographic methods is the Ryu-Takayanagi formula, which is a remarkable generalization of the Bekenstein-Hawking formula relating the area of the black hole horizon to the thermal entropy at the horizon. It suffers from the limitation that the boundary geometry must be static, but generalizations (the Hubeny-Ryu-Takayanagi formula) exist, and nonetheless it is a powerful result. The demonstration is quite involved and technical, so we opt to state the result and make some comments on how it is derived.

The Ryu-Takayanagi formula states that the entropy of a subsystem $\mathcal{A}$ living on the boundary theory is given by the area of the minimal surface whose boundary is the entanglement surface ∂A:

$$S(\mathcal{A}) = \frac{1}{4G_{\mathrm{N}}}\mathrm{area}\,(\min \mathcal{A})$$

Besides bearing $\partial\mathcal{A}$ as a boundary, the surface is additionally required to obey a so-called homology constraint stating the it must be homologous to $\mathcal{A}$; while it is tempting to think of this as meaning "related by a continuous deformation" in the sense of a homotopy class, homologous spaces are not always homotopic despite homotopic implying homologous.

The Ryu-Takayanagi formula is an example of a boundary calculation that is significantly easier to do in the bulk theory. This is rooted in arguments that the analytical continuation necessary to take us to the von Neumann entropy from the Rényi entropy is easier to perform in the bulk theory. At any rate, the argument is "simply" to consider an orbifold calculation of the partition function for the Rényi entropy, and use the GPKW prescription to contruct a brane homologous to $\mathcal{A}$. The string partition function is then solved by a saddle-point approximation which can be shown to be equivalent to a minimal surface condition. Details and extensions are found in Rangamani and Takayanagi's book.

7.5 Weyl/Conformal Anomalies

Chapter 8

Miscellany

$\mathrm{SO}(n+1)/\mathrm{SO}(n) = S^n$ (n-sphere), from Natase book on string theory.

The difference between Weyl and conformal invariance is that conformal transformation is an actual change of coordinates (like you are mapping from one set of coordinates to another, literally look at the equation!) that happens to result in a scaling of the metric, whereas Weyl transformation is a literal explicit change in the metric.

When people say pick a foliation of spacetime, they mean this is necessary in order to preserve causalityl.