PHYS0500 - Advanced Classical Mechanics

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Contents

Introduction

> We have already various treatises on Mechanics, but the plan of this one is entirely new. I hope that the manner in which I have tried to attain this object will leave nothing to be desired.
>
> Joseph Louis Lagrange

title

lorem ipsum

- author

Chapter 1

Oscillations

1.1 Simple Harmonic Oscillations, Energy

A system undergoes a force proportional to displacement:

$$F(x) = -kx$$

Using Newton's second law, we obtain the key equation of motion

$$\boxed{m\ddot{x} = -kx}$$

Which can be put in the form

$$\ddot{x} + \omega_0^2 x = 0$$

Solving this second order differential equation, we obtain solutions of the form

$$x(t) = A\cos\omega t + B\sin\omega t$$

Alternatively, we can express this as a single trigonometric function: suppose there is an angle δ such that $\tan\delta = B/A$. Then

$$\begin{aligned} x(t) &= A\cos\omega t + B\sin\omega t \\ &= \sqrt{A^2 + B^2}\cos\delta\cos\omega t + \sqrt{A^2 + B^2}\sin\delta\cos\omega t \end{aligned}$$

Then, by the angle subtraction identity:

$$\boxed{x(t) = A\cos(\omega t - \delta)}$$

Where we turned $\sqrt{A^2 + B^2}$ into an arbitrary constant. Note that we can also make the same argument to obtain $x(t) = A\sin(\omega t + \phi)$ for a different phase shift ϕ.

The **kinetic energy** of the system is then (we use the sine form of the motion)

$$\begin{aligned} F\frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m\omega_0^2 A\cos^2(\omega_0 t - \delta) \\ &= \frac{1}{2}kA^2\cos^2(\omega_0 t - \delta) \end{aligned}$$

The **potential energy** is related to work:

$$\mathrm{d}W = -F\,\mathrm{d}x = kx\,\mathrm{d}x$$

Integrating,

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2\left(\omega_0^t\delta\right)$$

With total energy $T + U = E$, we obtain

$$\begin{aligned} E &= \frac{1}{2}kA^2\left[\cos^2(w_0t - \delta) + \sin^2(\omega_0t - \delta)\right] \\ &= \frac{1}{2}kA^2 \end{aligned}$$

1.2 Harmonic Oscillations in Two Dimensions

We consider the vector form of the harmonic oscillator:

$$\mathbf{F} = -k\mathbf{r}$$

In polar and rectangular cooridnates:

$$\begin{cases} F_x = -kr\cos\theta = -kx \\ F_y = -kr\sin\theta = -ky \end{cases}$$

Then the equations of motion are

$$\begin{cases} \ddot{x} + \omega_0^2x = 0\ddot{y} + \omega_0^2y = 0 \end{cases}$$

The solutions to which are

$$\begin{cases} x(t) = A\cos(\omega_0t - \alpha)y(t) = A\cos(\omega_0t - \beta) \end{cases}$$

The equations of motion then trace out oscillatory motion in separable components.

1.3 Damped Oscillations

Previously, we've dealt with the case $m\ddot{x} = -kx$. Now we introduce a term to represent a damping force. Assume this damping force is a function of velocity, as with air drag.

$$\boxed{m\ddot{x} = -b\dot{x} - kx}$$

We now turn to solving this second-order differential equation:

$$\begin{gathered} m\ddot{x} + b\dot{x} + kx = 0 \\ \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \end{gathered}$$

Denote the damping parameter $\beta = \frac{b}{2m}$, as well as the familiar $\omega_0^2 = \frac{k}{m}$:

$$\ddot{x} + 2\beta\dot{x} + \omega_0 x = 0$$

The characteristic equation gives roots

$$r_1 = -b \pm \sqrt{\beta^2 - \omega_0^2}$$

And the general solution is then

$$\begin{aligned} x(t) &= A_1 \exp\left[\left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)t\right] + A_2 \exp\left[\left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)t\right] \\ &= A_1 e^{-\beta t} \exp\left[\sqrt{\beta^2 - \omega_0^2}t\right] + A_2 e^{-\beta} \exp\left[-\sqrt{\beta^2 - \omega_0^2}t\right] \end{aligned}$$

Factoring out the $e^{-\beta}$,

$$\boxed{x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]} \tag{1.1}$$

Now we consider three cases of this general solution.

1.3.1 Underdamped Motion

Consider

$$\boxed{\omega_0^2 > \beta^2 \Rightarrow \beta^2 - \omega_0^2 < 0}$$

Define

$$\omega_1^2 \equiv \omega_0^2 - \beta^2$$

Since $\beta^2 - \omega_0^2$ is negative, (1.1) becomes:

$$\begin{aligned} x(t) &= e^{-\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_2 t}\right] \\ &= e^{-\beta t}[A_1 \cos\omega_1 t + iA_1 \sin\omega_1 t + A_2 \cos(-\omega_1 t) + iA_2 \sin(-\omega_1 t)] \\ &= e^{-\beta t}[A\cos(\omega_1 t - \delta)] \\ &= Ae^{-\beta t}\cos(\omega_1 t - \delta) \end{aligned}$$

ω_1 can be interpreted as the angular frequency of the motion, although the fact that the motion is not completely period means there is, strictly speaking, no angular frequency. The amplitude, given by $Ae^{-\beta t}$, decreases over time; the amplitude alone, considered as a function, is termed the **envelope**.

The ratio of amplitudes of of successive maxima is

$$\frac{Ae^{-\beta T}}{Ae^{-\beta(T+T_1)}} = e^{\beta T_1}$$

Where the first of any pair of maxima occurs at $t = T$, and $T_1 = 2\pi/\omega_1$. This quantity is called the **decrement of motion**.

Note that there is loss in energy. The rate of energy loss is proportional to the square of the velocity, so the lsos is not uniform.

1.3.2 Critically Damped Motion

If damping force is sufficiently large,

$$\beta^2 = \omega_0^2 \Rightarrow \beta^2 - \omega_0^2 = 0$$

Then (1.1) becomes

$$\begin{aligned} x(t) &= e^{-\beta t}\left[A_1 e^{0t} + A_2 t e^{0t}\right] \\ &= e^{-\beta t}[A_1 + A_2 t] \end{aligned}$$

Then this motion is not too slow, nor too fast, so to speak. This corresponds to a smoother return to the equilbrium position.

1.3.3 Overdamped Motion

Finally, we have the case

$$\beta^2 > \omega_0^2 \Rightarrow \beta^2 - \omega_0^2 > 0$$

Then (1.1) becomes

$$x(t) = e^{-\beta t}[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}]$$

Where we defined $\omega_2 = \beta^2 - \omega_0^2$. This quantity is by no means an angular frequency, since this motion simply asymptotically approaches the equilibrium position.

1.4 Driven Oscillations

The simplet case of driven oscillation is one where an external driving force varies harmonically with time. Then the forces on the system are

$$F = -kx - b\dot{x} + F_0 \cos \omega t$$

Where the second term of the right hand side is the damping force and the third term is the driving force. Let us begin solving:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

Once again letting $\beta = b/2m$, $x_0^2 = k/m$,

$$\ddot{x} + 2\beta + \omega_0^2 x = A \cos \omega t$$

This is an imhomogeneous second order linear differential equation. As such, we can find the nullspace/completementary solution $x_c(t)$, and the particular solution $x_p(t0)$, and the general solution will be some linear combination of these.

The complementary solution is the same as the undriven damped oscillation from section 1.3:

$$x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]$$

Try, by guessing, the particular solution

$$x_p(t) = D \cos(\omega t - \delta)$$

(In general, you will want to guess that the particular solution is something similar to the inhomogeneity. For example, if the inhomogeneity is $C\cosh\omega t$, guess some $A\cosh\omega t$.) Plug into the equation of motion to test. We differentiate and expand $\cos(\omega t-\delta)$ and $\sin(\omega t-\delta)$:

$$\left(A - D\left[\left(\omega_0^2-\omega^2\right)\cos\delta + 2\omega\beta\sin\delta\right]\cos\omega t\right) - \left(D\left[\left(\omega_0^2-\omega^2\right)\sin\delta - 2\omega\beta\cos\delta\right]\right)\sin\omega t = 0$$

Notice that $\cos\omega t$ and $\sin\omega t$ are linearly independent, so this will only be satisfied if the coefficients go to zero.

$$\begin{aligned}
D\left[\left(\omega_0^2-\omega^2\right)\sin\delta - 2\omega\beta\cos\delta\right] &= 0\\
\left(\omega_0^2-\omega^2\right)\sin\delta &= 2\omega\beta\cos\delta\\
\frac{\omega_0^2-\omega^2}{2\omega\beta} &= \frac{\cos\delta}{\sin\delta}\\
\tan\delta &= \frac{2\omega\beta}{\omega_0^2-\omega^2}
\end{aligned}$$

Then we know the phase shift is

$$\tan\delta = \frac{2\omega\beta}{\omega_0^2-\omega^2}$$

From this, we may solve for $\sin\delta, \cos\delta$, after which we substitute into the coefficient for the $\cos\omega t$ term, which we also know to be zero:

$$\begin{aligned}
0 &= A - D\left[\left(\omega_0^2-\omega^2\right)\cos\delta + 2\omega\beta\sin\delta\right]\cos\omega t\\
A &= D\left[\left((\omega_0^2-\omega^2)\cdot\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}} + \frac{2\omega\beta\cdot 2\omega\beta}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}\right)\right]\\
A &= D\left[\frac{(\omega_0^2-\omega^2)^2+(2\omega\beta)^2}{\sqrt{(\omega_0^2+\omega^2)^2+4\omega^2\beta^2}}\right]\\
A &= D\sqrt{(\omega_0^2+\omega^2)^2+4\omega^2\beta^2}
\end{aligned}$$

And we obtain the coefficient of the particular solution:

$$\boxed{D = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}} \tag{1.2}$$

For

$$\boxed{x_p(t) = D\cos(\omega t-\delta)}$$

With

$$\boxed{\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2-\omega^2}\right)}$$

Note that this is a different phase shift than that of the oscillating motion. δ instead represents the phase difference between the driving force and the resultant motion. Note that this varies with ω for a fixed ω_0.

The general solution is then

$x(t) = x_c(t) + x_p(t)$

$x_c(t)$ (the complementary solution) represents **transient** effects—the die out and eventually give way to the motion purely due to the driving force. After $t = \frac{1}{\beta}$, $e^{-\beta t}$ is just e, so, in symbols:

$$x(t \gg 1/\beta) = x_p(t)$$

1.5 Resonance Phenomena

We notice that the amplitude, as given by (1.2), varies on ω. Now we turn to finding ω_R, the frequency at which the amplitude of the particular solution is at a maximum. This is called the **amplitude resonance frequency**.

Set

$$\frac{\mathrm{d}D}{\mathrm{d}\omega} = 0$$

Then

$$\frac{\mathrm{d}D}{\mathrm{d}\omega} = -\frac{A\big(2(\omega_0^2 - \omega) \cdot 2\omega + 8\omega\beta\big)}{2((\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2)^{3/2}}$$

Then we find that

$$\boxed{\omega_R = \sqrt{\omega_0^2 - 2\beta^2}}$$

As the damping coefficient increases, ω_R decreases.

We customarily describe the degree of damping in the oscillating system by the **quality factor**

$$Q \equiv \frac{\omega_R}{2\beta}$$

If Q is large, with little damping, the shape of the resonance curve approaches that of an undamped oscillator. If the damping is very large, Q is small and the resonance is destroyed.

1.5.1 Kinetic Energy Resonance

We look for the value of ω for which the kinetic energy is a maximum.

Recall that $T = (1/2)m\dot{x}^2$. Use

$$\dot{x} = \frac{-A\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} \sin(\omega t - \delta)$$

The kinetic energy becomes

$$T = \frac{mA^2}{2} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2} \sin(\omega t - \delta) \sin^2(\omega t - \delta)$$

The kinetic energy varies periodically, so let us find the average kinetic energy:

$$\operatorname{avg} T = \frac{mA^2}{2} \cdot \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2} \operatorname{avg}\big(\sin^2(\omega t - \delta)\big)$$

Over one period, the average of $\sin^2\theta$ is $\frac{1}{2}$. Then

$$\text{avg}\, T = \frac{mA^2}{4} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 - 4\omega^2\beta^2}$$

We label the critical angular velocity ω_E. Then

$$\frac{\mathrm{d}}{\mathrm{d}\omega}(\text{avg}\, T) = 0$$

And, solving, we obtain

$$\boxed{\omega_E = \omega_0}$$

That is, the kinetic energy resonance frequency occurs at the natural undamped frequency of oscillations of the system.

1.6 Principle of Superposition

The oscillations we've described follow the differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\frac{\mathrm{d}}{\mathrm{d}t} + b\right)x(t) = A\cos\omega t$$

This differential operator is linear. We may generalize this to

$$\mathcal{L}x(t) = F(t)$$

Recall that then

$$\mathcal{L}(x_1 + x_2) = \mathcal{L}x_1 + \mathcal{L}x_2$$

So if we have two solutions x_1, x_2 to two different force functions $F_1(t)$, $F_2(t)$, we can add these to obtain

$$\mathcal{L}c_1x_1 + c_2x_2 = c_1F_1(t) + c_2F_2(t)$$

Which means the differential equation with a sum of force functions has, as a solution, the sum of the individual solutions.

Extend the argument to a set $x_n(t)$ of solutions to $F_n(t)$:

$$\mathcal{L}\left(\sum_{n=1}^{N} a_n x_n(t)\right) = \sum_{n=1}^{N} c_n F_n(t)$$

Which can be simplified by defining

$$x(t) = \sum c_n x_n$$
$$F(t) = \sum c_n F_n$$

Then, if each of these $F_n(t)$ has simple harmonic dependencies on time—e.g. $a_n\cos(\omega_n t - \phi_n)$—we know the corresponding solution $x_n(t)$ has the form described previously.

Then, if some arbitrary force function $F(t)$ can be expressed as a series of such harmonic terms, the complete solution can also be written as a series of harmonic terms—i.e., the solution and the force function should be written as their Fourier series.

1.7 Analogy Between Electrical and Mechanical Quantities

Mechanical		**Electrical**	
x	Displacement	q	Charge
$\dot{x}$	Velocity	$\dot{q} = I$	Current
m	Mass	L	Inductance
b	Damping resistance	R	Resistance
$1/k$	Mechanical Compliance	C	Capacitance
F	Amplitude of impressed force	$\mathcal{E}$	Amplitude of impressed emf

Chapter 2

Calculus of Variations

2.1 Calculus of Variations

Our interest is determining the path that gives an extremum solution; that is, out of all possible "trajectories" from a point x_1 to a point x_2 we want to determine the trajectory that minimizes a certain quantity.

In symbols, we want to determine a $y(x)$ such that the following integral is an extremum:

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} \, \mathrm{d}x$$

where f is a function of $y(x)$, its first derivative with respect to x, and an independent variable x. With the limits of integration fixed, the entire integral—which is a functional—depends solely on $y(x)$. We begin by varying $y(x)$ until an extremum is found.

Give all possible functions y a parametric representation $y = y(\alpha, x)$ such that $y(0, x) = y(x)$ is a function that gives an extremum for J. Then we write

$$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$

Where η is some function of x with contiuous derivatives which vanishes at the limits of integration: $\eta(x_1) = \eta(x_2) = 0$. Then J becomes a function of this parameter:

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\} \, \mathrm{d}x$$

2.2 Euler's Equation

If $y(0, x)$ is an extremum (we suppose that it is), the following holds

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$$

We proceed with this assumption, and solve for a $y(0, x) = y(x)$ such that this is the case:

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f\{y, y'; x\} \, \mathrm{d}x$$

The limits of integration are fixed, so the differential operator can be applied to the integrand:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} \right) \mathrm{d}x \tag{2.1}$$

Since $y(\alpha, x) = f(0, x) + \alpha\eta(x)$,

$$\frac{\partial y}{\partial x} = \eta(x); \ \frac{\partial y'}{\partial \alpha} = \frac{\mathrm{d}\eta}{\mathrm{d}x}$$

(2.1) becomes

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{\mathrm{d}\eta}{\mathrm{d}x} \right) \mathrm{d}x \tag{2.2}$$

Integrate $\frac{\partial f}{\partial y}\frac{\mathrm{d}\eta}{\mathrm{d}x}$ by parts:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{\partial \eta}{\partial x}\,\mathrm{d}x = \frac{\partial f}{\partial y'}\eta(x)\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\eta(x)\,\mathrm{d}x \right)$$

$\eta(x_1) = \eta(x_2) = 0$, so , plugging back into (2.2),

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right)\eta(x) \right] \mathrm{d}x \ \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right)\eta(x)\,\mathrm{d}x$$

y and y' are still functions of α, so the integrand itself must vanish for the derivative of J to be zero at $\alpha = 0$. Thus we arrive at the **Euler equation**

$$\boxed{\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0}$$

Once we evaluate these partial derivatives and plug them into the Euler equation, we have a differential equation the solution to which is the minimum path! Additionally, no η or α are involved in the equation, so we only have to solve for y.

An application of calculus of variations is finding a **geodesic**— the shortest path on a given surface. One must simply find $\mathrm{d}s$ in the given coordinate system and substitute the constraining variables of the surface in order to solve for the geodesic. Note that it is often most useful to substitute the differential form; e.g. if given $z(r)$ in cylindrical coordinates, substitute $\mathrm{d}z = z'(r)\,\mathrm{d}r$.

2.3 Second Form of Euler's Equation

We use this form when f does not depend on x and $\frac{\partial f}{\partial x}$ is thus zero. For any $f(x, y'; x)$:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial f}{\partial x} \\ &= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x} \end{aligned} \tag{2.3}$$

And

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'} \right) = y''\frac{\partial f}{\partial y} + y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right)$$

Then, substituting from (2.3) for $y'' \cdot \partial f/\partial y$.,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'} \right) = \frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right)$$

Grouping, the last two terms become

$$y'\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y}\right)$$

This is precisely the first form of the Euler equation (up to a negative sign). Then they must go to zero, and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right)=\frac{\mathrm{d}f}{\mathrm{d}x}-\frac{\partial f}{\partial x}$$
$$\frac{\partial f}{\partial x}+\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right)-\frac{\mathrm{d}f}{\mathrm{d}x}=0$$

When f doesn't explicitly depend on x, $\partial f/\partial x=0$, and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f-y'\frac{\partial f}{\partial y}\right)=0$$

Which means the expression is a constant and

$$\boxed{f-y'\frac{\partial f}{\partial y'}=C}$$

2.4 Calculus of Variations with Several Variables

Commonly, f is a functional of several dependent variables:

$$f=f\{y_1(x),y_1'(x),y_2(x),y_2'(x),\ldots,x\}$$

or

$$f=fy_i(x),y_i'(x);x,\ i=1,2,3,\ldots$$

Then we have

$$y(\alpha,x)=y_i(0,x)+\alpha\eta_i(x)$$

And, proceeding analogously,

$$\frac{\partial J}{\partial\alpha}=\int_{x_1}^{x_2}\sum_i\left(\frac{\partial f}{\partial y_i}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i}\right)\eta_i(x)$$

Which yields a set of differential equations:

$$\boxed{\frac{\partial f}{\partial y_i}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i}=0\ i\in\mathbb{N}}$$

2.5 Euler's Equation with Auxiliary Condition

We are given a constraint equation of the form

$$g(y,z;x)=0$$

Then our functional is some

$$f=f(y,y';z,z';x)$$

Then we obtain
$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} \right) \frac{\partial z}{\partial \alpha} \right] \mathrm{d}x$$

Note that the variations in y and z are not independent; we use g:

$$\mathrm{d}g = \left(\frac{\partial g}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial \alpha} \right) \mathrm{d}\alpha = 0 \tag{2.4}$$

Where $\partial\alpha/\partial x = 0$. Now
$$\begin{cases} y(\alpha, x) = y(x) + \alpha\eta_1(x) \\ z(\alpha, x) = z(x) + \alpha\eta_2(x) \end{cases}$$

From this equation we may determine $\partial y/\partial\alpha$ and $\partial z/\partial\alpha$. Then we plug into (2.4):

$$\frac{\partial g}{\partial y}\eta_1(x) = -\frac{\partial g}{\partial z}\eta_2(x)$$

And, plugging into our expression for $\partial J/\partial\alpha$:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right) \eta_1 + \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} \right) \eta_2 \right] \mathrm{d}x$$

Factor $\eta_1(x)$ out of the brackets and write

$$\frac{\eta_2}{\eta_1} = -\frac{\partial g/\partial y}{\partial g/\partial z}$$

we obtain
$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right) - \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} \right) \frac{\partial g/\partial y}{\partial g/\partial z} \right] \mathrm{d}x$$

As per $\partial J/\partial\alpha = 0$, the expression in the square brackets must vanish. We obtain:

$$\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} \right) \left(\frac{\partial g}{\partial y} \right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} \right) \left(\frac{\partial g}{\partial z} \right)^{-1}$$

Because y and z are both functions of x, these two sides must be equal to some function of x, which we define to be $-\lambda(x)$. Then
$$\begin{cases} \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} = 0 \\ \frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} = 0 \end{cases}$$

We use these two equations and the equation of constraint to solve for the three unknown equations.

For several independent variables and an arbitrary number of constraints, we obtain

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'} + \sum_j \lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0}$$

With $g_j(y_i; x) = 0$.

Chapter 3

Lagrangian and Hamiltonian Mechanics

3.1 Hamilton's Principle

The principle, as written in words: of all possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between kinetic and potential energies.

(In the language of generalized coordinates, we can say that the system will evolve such that the path between two points in the configuration space is minimized.)

Mathematically:

$$\delta \int_{t_1}^{t_2} T - U \, \mathrm{d}t$$

This expression means that the integral must be an extremum, not minimum, but in all dynamical processes, this ends up being a minimum.

Then, given rectangular coordinates and a conservative force field, we have $T(\dot{x}_i)$, $U = U(x_i)$, we define a quantity called the **Lagrangian**:

$$L = T - U$$

Which leads to

$$\delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i; t) \, \mathrm{d}t = 0$$

Where we made the following conversions from calculus of variations:

$$\begin{aligned} x &\to t \\ y_i(x) &\to \\ y_i'(x) &\to \\ f &\to L \end{aligned}$$

Euler's equation becomes what is known as the **Euler-Lagrange Equation**:

$$\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}_i}$$

3.2 Generalized Coordinates

Now we take advantage of the flexibility in specifying coordinates that is inherent to the Euler-Lagrange equation. We need not worry where the origin is, or about what parts of the system are expressed in what coordinate system.

Consider n discrete particles, some of which may form a rigid body. We must then use n radius vectors $\mathbf{r}_i$. This means $3n$ quantities. With m constraints, we have $3n - m$ independent quantities. We define this quantity to be the **degrees of freedom** of the system:

$$\boxed{s = 3n - m}$$

We don't need to choose s rectangular coordinates or s curvilinear coordinates or any cooridnate system at all. We may simply chose any s independent parameters

Then **generalized coordinates** are any set of quantities that completely describes the state of the system. They are typically written

$$q_1, q_2, \ldots, q_n$$

We typically notate such a list of generalized coordinates with q_i or q_j.

A set of s generalized coordinates is called a **proper set** of generalized coordinates. We try to choose the coordinates such that the resulting equations of motion are simple enough to be solved or, at most, interpeted.

Then, for object α, rectangular coordinate axis i, we have conversions from generalized coordinates (e.g., polar coordinates) to rectangular coordinates:

$$x_{\alpha,i} = x_{\alpha,i}(q_j)$$

And likewise

$$\dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(q_j, \dot{q}_j t)$$

With s independent generalized coordinates, we can represent the state of the system as a point in an s-dimensional space called the **configuration space**. Each dimension corresponds to one of the q_j coordinates.

3.3 The Euler-Lagrange Equation in Generalized Coordinates

Note that the Lagrangian is a scalar quantity, so it must be invariant under coordinate transformations. As long as we define the Lagrangian to be the difference between the kinetic and potential energies, we may use different generalized coordiantes.

Then the statement of the problem is

$$\boxed{\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,\mathrm{d}t}$$

And we made the transformations

$$\begin{aligned} x &\to t \\ y_i(x) &\to q_j(t) \\ y_j'(x) &\to \dot{q}_j(t) \\ f &\to L \end{aligned}$$

And the Euler-Lagrange equations become

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} = 0}$$

For $j = 1, 2, 3, \ldots, s$.

Some **conditions**:

1. The forces acting on the system must be derivable from a potential, or from several potentials. So, no non-conservative forces.
2. The equations of constraint must be relations that connect the coordinates of the particles and may be functions of time. That is, they must be of form

$$f_k(x_{\alpha,i}, t) = 0 \;\; k = 1, 2, \ldots m$$

If constraints can be expressed as in condition 2, they are termed **holonomic**. If the equations don't explictly depend on time, they are said to be **fixed**, or **scleronomic**. Moving constraints are called **rheonomic**.

3.4 E-L Equations with Undetermined Multipliers

Constraints of the form

$$f(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) = 0$$

are considered **nonholonomic**. Equations that can be integrated for relationships among coordinates are termed **semiholonomic**.

In section 2.5, we showed what Euler's equation looks like given constraints. The analog in Lagrangian mechanics is:

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0}$$

for s generalized coordinates q_j, m constraints f_k. This, and the constraint equations

$$f_k(q_j, t) = 0$$

The corresponding Lagrangian is

$$L = T - U + \lambda(t) \cdot f$$

3.5 Conservation Theorems

3.5.1 Conservation of Energy

We require that time be homogeneous. A particle moving under no forces in a certain inertial reference frame may not, later, have a different velocity. Then a Lagrangian describing a closed system cannot depend explicitly on time. That is, in shifting time by $t' = t + \Delta t$, the Lagrangian remains unchanged. Let's show this by starting with

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

Where there is no $\partial L/\partial t$ because this quantity is zero as per the requirement of no time dependence.

The Euler-Lagrange equation gives

$$\frac{\partial L}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}$$

Making this substitution,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_j \dot{q}_j \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

This is the result of the product rule. Then we may group

$$\frac{\partial L}{\partial t} - \sum_j \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

This quantity is a constant in time. Denote it

$$L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = -H = \text{constant}$$

If the potential energy U does not depend on the velocities $\dot{x}_{\alpha,i}$ or time, then $U(x_{\alpha,i})$. If the relationship between the generalized coordinates and the rectangular coordinates does not depend on time, we have that $x(q_j)$, then $U = (q_j)$ and $\partial U/\partial \dot{q}$. This means that

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

And our expression for $-H$ has

$$(T-U) - \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = -H$$

The second term on the left hand side is exactly twice the kinetic energy.

$$T - U - 2T = -H \Rightarrow T + U = E = H = \text{constant}$$

3.5.2 Conservation of Linear Momentum

Space is homogeneous in an inertial reference frame. Two objects a certain distance from each other will, provided they are under the influence of no forces, maintain that distance. We explore the Lagrangian's invariance with respect to coordinate translation.

Consider an infinitesimal displacement $\mathbf{r}_\alpha \to \mathbf{r}_\alpha + \delta\mathbf{r}$. For simplicity, consider only one particle, and use rectangular coordinates. With $\mathbf{r} = \sum_i \delta x_i \hat{\mathbf{e}}_i$,

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = 0$$

Where we've taken the change in the Lagrangian to be zero, as aforementioned.

The δx_i are not explicitly or implicitly dependent on time. Then

$$\delta \dot{x}_i = \delta \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\delta x_i = 0$$

Leading to

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i = 0$$

Each δx_i is an independent displacement, so δL vanishes identically only if each partial derivative of L is

$$\frac{\partial L}{\partial x_i} = 0$$

According to the Euler-Lagrange equation, this means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}_i} = \text{constant}$$

Meaning that, continuing to enforce that $U \neq U(\dot{x}_i)$,

$$\begin{aligned}\frac{\partial (T-U)}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} &= \frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{2}m\sum_j \dot{x}_j^2\right) \\ &= m\dot{x}_i \\ &= p_i = \text{constant}\end{aligned}$$

3.5.3 Conservation of Angular Momentum

In an inertial reference frame, space is *isotropic*: the orientation of the system does not affect the dynamical laws of the system. In other words, the laws are rotationally invariant. Specifically, the Lagrangian of a closed system does not change under an infinitesimal rotation.

Consider a small rotation by angle $\delta\theta$ about a given axis. The radius vector undergoes the change $\mathbf{r} \to \mathbf{r} + \delta\mathbf{r}$, where, taking the vector $\delta\boldsymbol{theta}$ to point along the axis of rotation,

$$\delta\mathbf{r} = \delta\boldsymbol{\theta} \times \mathbf{r} \tag{3.1}$$

The velocity vectors also change on rotation of the system, such that

$$\delta\dot{\mathbf{r}} = \delta\boldsymbol{\theta} \times \dot{\mathbf{r}} \tag{3.2}$$

Consider only a single particle, express Lagrangian in rectangular coordinates. The infinitesimal rotation leads to, by differentiating

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = 0$$

We know that $p_i = \partial L/\partial \dot{x}_i$, and $\dot{p}_i = \partial L/\partial x_i$ (see section 3.6). Then the above equation becomes

$$\delta L = \sum_i \dot{p}_i \delta x_i + \sum_i p_i \delta \dot{x}_i = 0$$

Noticing that both terms are dot products, we have that

$$\dot{\mathbf{p}} \cdot \delta\mathbf{r} + \mathbf{p} \cdot \delta\dot{\mathbf{r}} = 0$$

Using (3.1) and (3.2), we have

$$\dot{\mathbf{p}} \cdot (\delta\theta \times \mathbf{r}) + \mathbf{p} \cdot (\delta\boldsymbol{\theta} \times \dot{\mathbf{r}}) = 0$$

Permute:

$$\begin{aligned}\delta\boldsymbol{\theta}\cdot(\mathbf{r}\times\dot{\mathbf{p}})+\delta\boldsymbol{\theta}\cdot(\dot{\mathbf{r}}\times\mathbf{p})&=0\\ \delta\boldsymbol{\theta}[(\mathbf{r}\times\dot{\mathbf{p}})+(\dot{\mathbf{r}}\times\mathbf{p})]&=0\end{aligned}$$

Notice that the terms in the brackets are the result of a differentiation:

$$\delta\boldsymbol{\theta}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\mathbf{p})=0$$

Since $\delta\boldsymbol{\theta}$ is arbitrary, we must have that

$$\frac{\mathrm{d}}{\mathrm{d}t}=0\Rightarrow\mathbf{r}\times\mathbf{p}=\text{constant}=\mathbf{L}$$

3.5.4 Summary

Notice these are all results of certain symmetries. This is due to Noether's Theorem showing conservation principles always arise from symmetries.

Characteristic of inertial frame	**Property of Lagrangian**	**Conserved Quantity**
Time homogeneous	Not explicit function of time	Total energy
Space homogeneous	Invariant to translation	Linear momentum
Space isotropic	Invariant to rotation	Angular momentum

3.6 Hamiltonian Dynamics

In principle, the guiding idea behind Hamiltonian dynamics is that we would like to go from a function of position and velocity (the Lagrangian) to a function of conserved quantities (the total energy; momentum). The Hamiltonian is then a function on the **phase space** of momentum (velocity) and position.

It can be shown that we obtain, from the Lagrangian, (see section 3.5.2) the rectangular momenta

$$p_i=\frac{\partial L}{\partial\dot{x}_i}$$

These are precisely the momenta we are used to, they are not a new quantity. However, we may apply the same procedure to obtain the generalized analogue of the traditional momenta. The **generalized momenta** are defined

$$\boxed{p_j\equiv\frac{\partial L}{\partial\dot{q}_j}}$$

And we have, from the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_j}=\frac{\partial L}{\partial q_j}\Rightarrow\boxed{\dot{p}_j=\frac{\partial L}{\partial q_j}}$$

By this definition of generalized momenta, we have

$$\boxed{H=\sum_j p_j\dot{q}_j-L}\tag{3.3}$$

Which is the Legendre transform.

And from these we have

$$\boxed{\dot{q}_k = \frac{\partial H}{\partial p_k} \quad -\dot{p}_k = \frac{\partial H}{\partial q_k}}$$

This quantity H is called the **Hamiltonian**, and it is equal to the total energy of the system if

- The equations connecting the rectangular and generalized coordinates are time-independent.
- The potential energy is velocity independent.

See section 3.5

3.6.1 Method

1. Set up the Lagrangian and use it to find p_j and $\dot{p}_j$.
2. Set up the Hamiltonian and eliminate the $\dot{q}_j$ using the results from step 1 so that you have an expression in p_j and q_j.
3. Use (3.3) to find the equations of motion. This might require some differentiating to find $\ddot{q}_j$ in terms of $\dot{p}_j$.

Chapter 4

Central Force Motion

4.1 Reduced Mass

Describe a system of two particles; this constitutes six coordinates, $\mathbf{r}_1$ and $\mathbf{r}_2$. Alternatively, choose $\mathbf{R}$ and $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$. Restrict our attention to systems without frictional losses, and such that the potential function is a function of position only. The Lagrangian is

$$L = \frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{}|$$

We let the origin of the coordinate system be the center of mass: $\mathbf{R} \equiv 0$. Then

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = 0$$

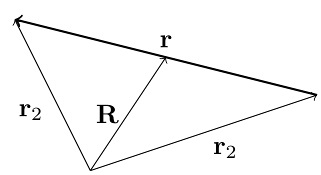

Figure 4.1

This equation, combined with our definition $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, yields

$$\begin{cases} \mathbf{r}_1 = \dfrac{m_2}{m_1 + m_2} \mathbf{r} \\ \mathbf{r}_2 = -\dfrac{m_1}{m_1 + m_2} \mathbf{r} \end{cases} \tag{4.1}$$

Substitute these into the Lagrangian to obtain

$$\boxed{L = \frac{1}{2} \mu |\dot{\mathbf{r}}|^2 - U(r)}$$

Where μ is the **reduced mass**:

$$\boxed{\mu \equiv \frac{m_1 m_2}{m_1 + m_2}}$$

We've reduced a two-body problem to a one-body problem of a "particle" of mass μ moving in an arbitrary central potential field. Once the motion of $\mathbf{r}$ is determined, the motions of $\mathbf{r}_1, \mathbf{r}_2$ may be determined using (4.1).

Notice that the vector $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ is the position of $\mathbf{r}_1$ relative to $\mathbf{r}_2$. It then follows that the "central force" acting on the "particle" is just the force due to m_2 on m_1, and the particle itself is m_1 with a different mass.

4.2 First Integrals of the Motion

The system just described possesses spherical symmetry. Under such conditions, the angular momentum is conserved (Noether's theorem):

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \text{constant}$$

With $\mathbf{r}$ and $\mathbf{p}$ lying in a plane together (as should be the case with this conservation), we may convert to polar coordinates without losing generality. Use

$$\boxed{L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)} \tag{4.2}$$

The Lagrangian is cyclic in θ, so the angular momentum, conjugate to the coordinate θ, is conserved (as should be expected):

$$\begin{aligned} \dot{p}_\theta &= \frac{\partial L}{\partial \theta} = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} \equiv l \end{aligned}$$

This is a first integral of the motion:

$$\boxed{l \equiv \mu r^2 \dot{\theta}}$$

There is no place to use conservation of linear momentum—we've eliminated the linear motion of the center of mass—so we use the only conservation law left, conservation of energy. We've limited ourselves to conservative systems, so we have $T + U = E =$ constant such that

$$\frac{1}{2}\mu\left(\dot{r}^2 + r\right) + U(r)$$

or, distributing and substituting the angular momentum.

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)}$$

4.3 Equations of Motion

Solving the above first integral, we have

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{\frac{2}{\mu}(E - U) - \frac{l^2}{\mu^2 r^2}} \tag{4.3}$$

We may solve for $\mathrm{d}t$ to obtain $t(r)$ then invert to obtain $r(t)$. However, we turn to investigating the motion in terms of r and θ. Write

$$\mathrm{d}\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r = \frac{\dot{\theta}}{\dot{r}}\,\mathrm{d}r$$

We can then substitute $\dot{\theta} = l/\mu r^2$ and (4.3) to obtain

$$\boxed{\theta(r) = \int \frac{\pm(l/r^2)\,\mathrm{d}r}{\sqrt{2\mu\Big(E - U - \frac{l^2}{2\mu r^2}\Big)}}} \tag{4.4}$$

Which usually requires the substitution $u = l/r$ then a trig substitution to integrate. This can only be integrated for specific forms of the force law. If the force law is some $F(r) \propto r^n$, there are solutions expressible in sines and cosines (circular functions) only for $n = 1, -2, -3$.

Alternative approach to the problem: use the Euler-Lagrange equation for r using the Lagrangian from (4.2):

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} &= 0 \\ \mu\Big(\ddot{r} - r\dot{\theta}^2\Big) &= -\frac{\partial U}{\partial r} = F(r) \end{aligned}$$

We want to put this in a more adequate form. The following change of variable is useful here and in many other situations:

$$u = \frac{1}{r}$$

Such that

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}}$$

We know $\dot{\theta} = l/\mu r^2$, so

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{\mu}{l}\dot{r}$$

Differentiating again,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = \frac{\mathrm{d}}{\mathrm{d}\theta}\Big(-\frac{\mu}{l}\dot{r}\Big) = \frac{\mathrm{d}t}{\mathrm{d}\theta}\frac{\mathrm{d}}{\mathrm{d}t}\Big(-\frac{\mu}{l}\dot{r}\Big) = -\frac{\mu}{l\dot{\theta}}\ddot{r}$$

With the same substitution for $\dot{\theta}$, we have

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -\frac{\mu^2}{l^2}r^2\ddot{r}$$

Solve for $\ddot{r}$ and $r\dot{\theta}^2$ in terms of u.

$$\begin{cases} \ddot{r} = -\dfrac{l^2}{\mu^2}u^2\dfrac{d^2u}{d\theta^2} \\ r\dot{\theta}^2 = \dfrac{l^2}{\mu^2}u^2 \end{cases}$$

Substitute into our Euler-Lagrange equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(1/u)$$

Such that, returning to r,

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)}$$

Which is particularly useful to find $r = r(\theta)$.

4.4 Orbits in a Central Field

Observe

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}$$

This equation vanishes at

$$E - U(r) - \frac{l^2}{2\mu r^2} = 0$$

Which implies a *turning point* of the motion has been reached. There are two roots to this equation, which we will term r_{max} and r_{min}. Then the motion is confined to the annular region $r_{\text{min}} \leq r \leq r_{\text{max}}$. Some combinations of $U(r)$, l, and E produce one root, such that the orbit is circular.

If the motion is periodic, the orbit is **closed**, and after a finite number of excursions between r_{max} and r_{min}, the motion repeats itself. If this is not the case, the orbit is **open**.

If we want to test the closedness or openness of the orbit, we solve (4.4) for a transit from r_{min} to r_{max} and back to r_{min} to see how much the angle changes. Symmetry allows us to integrate from r_{min} to r_{max} and just double this value.

$$\Delta\theta = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{(l/r^2)\,\mathrm{d}r}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}$$

It follows that the path is closed such that after a rational number of times traversing from the inner and outer radii, we have a change in angle of 2π. Then the path is closed only if

$$\Delta\theta = 2\pi \cdot \frac{a}{b}$$

where a, b are integers.

If the potential is some $U(r) \propto r^{n+1}$, then a closed non-circular path is only possible if $n = -2, 1$.

4.5 Effective Potential

Consider a particle moving in a central potential. Newton's laws yield

$$m\ddot{\mathbf{r}} = -\nabla U$$

In a general coordinate system, we obtain three coupled differential equations. To simplify such a system, we switch to spherical coordinates. Now we focus solely on the radial component of such a motion:

$$m\ddot{\mathbf{r}} \cdot \hat{\mathbf{e}}_r = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r\hat{\mathbf{e}}_r) \cdot \hat{\mathbf{e}}_r$$

Recall that $\mathbf{a} = (\ddot{r} - \dot{\theta}^2 r)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$. Then

$$m\ddot{\mathbf{r}} \cdot \hat{\mathbf{e}}_r = m\ddot{r} - m\dot{\theta}^2 r$$

Use the fact that the angular momentum (which is conserved) is $L = m\dot{\theta}r^2$:

$$m\ddot{\mathbf{r}} \cdot \hat{\mathbf{e}}_r = m\ddot{r} - \frac{L^2}{mr^3}$$

It then follows that

$$m\ddot{r} - \frac{L^2}{mr^3} = -\nabla U$$

Which can be thought of as $F_{\text{net}} + F_{\text{fict}} = F_{\text{actual}}$. The L^2/mr^3 term is the fictitious **centrifugal force** which results from the angular velocity component we must consider when computing radial acceleration. Then, solving for $m\ddot{r}$ the **effective force** is

$$\boxed{\mathbf{F}_{\text{eff}} = m\ddot{r} = -\nabla U(\mathbf{r}) + \frac{L^2}{mr^3}\hat{\mathbf{r}}}$$

As a scalar function, this is $F_{\text{eff}} = -\,\mathrm{d}U/\mathrm{d}r + L^2/mr^3$ Integrating, we obtain the **effective potential**:

$$\boxed{U_{\text{eff}}(\mathbf{r}) = U(\mathbf{r}) + \frac{L^2}{2mr^2}}$$

Note that we do not differentiate or integrate L because it is conserved. We have effectively reduced a two-dimensional problem (with derivatives of θ and r) to a one-dimensional problem.

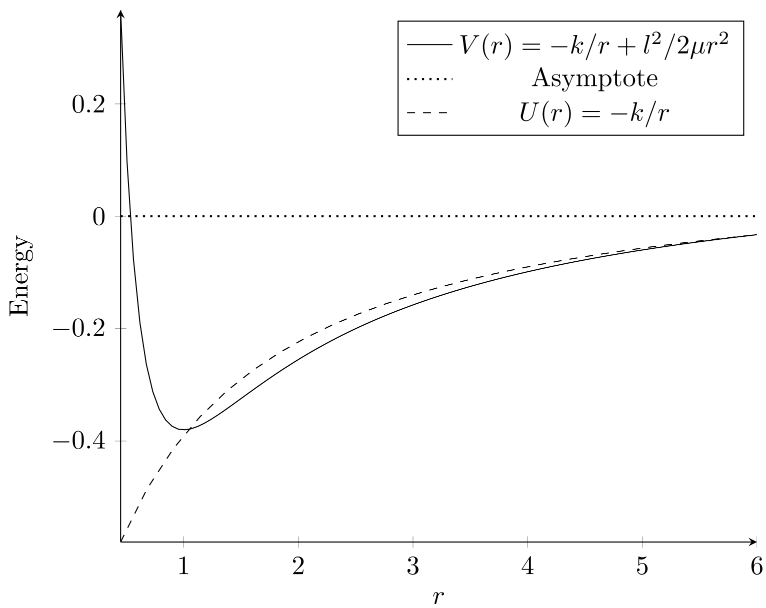

Figure 4.2

Inspect figure 4.2. There is an asymptote at $E = 0$:

- If a particle has energy less than 0 and is moving toward the force center $r = 0$, it will hit the potential wall and the oscillate around that local minimum, bounded. The bounding values are the **apsidal** distances, the turning points of the orbit: $r_{\text{min}} \leq r \leq r_{\text{max}}$.
- A particle may have a stable orbit at that minimum r_0, and the orbit will be circular.
- Finally, the particle might have $E \geq 0$ such that after it encounters that potential wall, it is pushed to infinity without bound.

This potential barrier near $r = 0$ is due to the centrifugal force— the tangential velocity of the motion becomes so large at small radii that the particle will tend to move outward. Notice that without the centrifugal term of the effective potential, $U(r)$ has no potential barrier at $r = 0$ and the particle would fall into the force center.

4.5.1 Small Oscillations

Occasionally we might be asked to find the frequency of oscillations about the stable orbit radius of a particle (imagine disturbing an orbiting particle ever so slightly along the radial direction). Then we may perform a Taylor expansion of U_{eff} about the stable radius r_0, and take the second order term as an analogue of $1/2kx^2$:

$$\frac{1}{2}kx^2 \sim \frac{1}{2}U''_{\text{eff}}(r-r_0)^2$$

4.6 Planetary Motion

From equation (4.4) we may solve for a particle moving in a central gravitational force. Change variable to $u \equiv l/r$ and integrate, then define $\theta = 0$ to be such that $r(\theta = 0)$ is a minimum,

$$\cos\theta = \frac{\frac{l^2}{\mu k}\cdot\frac{1}{r} - 1}{\sqrt{1+\frac{2El^2}{\mu k^2}}}$$

We define the constants

$$\alpha \equiv \frac{l^2}{\mu k}, \quad \varepsilon = \sqrt{2+\frac{2El^2}{\mu k}}$$

Such that the above equation becomes

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta}$$

ε is the **eccentricity**, and 2α is the **latus rectum**. This is a conic section. The value of the eccentricity (and the energy E, since ε is an expression of energy) classifies the orbit: where V_{min} the minimum value of

Eccentricity	Energy	Conic Section
$\varepsilon > 1$	$E > 0$	Hyperbola
$\varepsilon = 1$	$E = 0$	Parabola
$0 < \varepsilon < 1$	$V_{\text{min}} < E < 0$	Ellipse
$\varepsilon = 0$	$E = V_{\text{min}}$	Circle

the effective potential.

4.6.1 Kepler's Laws

1. In planetary motion, the orbits are ellipses with major and minor axes $2a$ and $2b$ such that

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|} \tag{4.5}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}} \tag{4.6}$$

And then the apsidal distances are

$$r_{\text{min}} = a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon}$$
$$r_{\text{max}} = a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon}$$

2. A small segment of area is a triangle with height $r\,\mathrm{d}\theta$ and base r, such that

$$\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$$

Divide by $\mathrm{d}t$ to obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2\mu} = \text{constant} \tag{4.7}$$

3. Rewrite (4.7)

$$\mathrm{d}t = \frac{2\mu}{l}\,\mathrm{d}A$$

The entire are of the ellipse is swept out by one complete period τ,

$$\int_0^\tau \mathrm{d}t = \frac{2\mu}{l}\int_0^A \mathrm{d}A$$
$$\tau = \frac{2\mu}{l}A$$

The area of an ellipse is πab. Using (4.5) and (4.6),

$$\tau = \pi k\sqrt{\frac{\mu}{2}}\cdot|E|^{-3/2}$$

Alternatively, note that $b = \sqrt{\alpha a}$ from (4.5) and (4.6). With $\alpha \equiv l^2/\mu k$, we write

$$\boxed{\tau^2 = \frac{4\pi^2\mu}{k}a^3}$$

Chapter 5

Dynamics of Particle Systems

5.1 Energy of Systems

The energy of a system of particles has a quite satisfying expression. Consider a system of particles moving from some initial configuration of $\mathbf{r}_\alpha$ to another configuration of the coordinates $\mathbf{r}_\alpha$. Then the work-energy theorem yields

$$W_{12} = \sum_\alpha \int_1^2 \mathbf{F}_\alpha \cdot \mathrm{d}\mathbf{r}_\alpha$$

Which can be expressed

$$W_{12} = \sum_\alpha \int_1^2 \mathrm{d}\left(\frac{1}{2} m_\alpha v_\alpha^2\right) = T_2 - T_1$$

where

$$T = \sum_\alpha = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2$$

Use center of mass coordinates $\dot{\mathbf{r}}_\alpha = \dot{\mathbf{r}}_\alpha + \dot{\mathbf{R}}$ such that

$$\begin{aligned}
\dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha = v_\alpha^2 &= \left(\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{R}}\right) \cdot \left(\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{R}}\right) \\
&= \dot{\mathbf{r}}'_\alpha \cdot \dot{\mathbf{r}}'_\alpha + 2\left(\dot{\mathbf{r}}'_\alpha \dot{\mathbf{R}}\right) + \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} \\
&= v_\alpha'^2 + 2\left(\dot{\mathbf{r}}'_\alpha \cdot \dot{\mathbf{R}}\right) + V^2
\end{aligned}$$

Substituting into the expression for kinetic energy:

$$T = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2$$

$$= \sum_\alpha \frac{1}{2} m_\alpha v_\alpha'^2 + \sum_\alpha V^2 + \dot{\mathbf{R}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha m_\alpha \mathbf{r}'_\alpha$$

But $\sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0$. Thus

$$\boxed{T = \sum \frac{1}{2} m_\alpha v_\alpha'^2 = \frac{1}{2} M V^2}$$

And we can split the total kinetic energy of the system into the kinetic energy of the center of mass plus the kinetic energies of the particles relative to the center of mass.

5.2 Elastic Collisions of Two Particles

Using solely the conservation theorems we cannot predict the angles between, say, the initial and final velocity vectors of one of the particles. For that we need to specify the force field. This is because we have six unknowns and four equations (one for each component of cons. of momentum and the additional energy conservation). We derive the relationships that can be obtained without specification of the force field.

Simplify these derivations by using conservation a laboratory (LAB) and center of mass (CM) coordinate system. We assume m_1 begins in motion and m_2 at rest.

$$
\begin{aligned}
m_1 &= \text{mass of the moving particle} \\
m_2 &= \text{mass of the struck particle}
\end{aligned}
$$

Primed quantities are in reference to the CM system:

$$
\begin{aligned}
\mathbf{u}_1 &= \text{Initial velocity of } m_1 \text{in LAB} \\
\mathbf{v}_1 &= \text{Final velocity of } m_1 \text{in LAB} \\
\mathbf{u}_1' &= \text{Initial velocity of } m_1 \text{in CM} \\
\mathbf{v}_1' &= \text{Final velocity of } m_1 \text{in CM}
\end{aligned}
$$

and the same definitions for $\mathbf{u}_2 = 0, \mathbf{v}_2, \mathbf{u}_2', \mathbf{v}_2'$.

$$
\begin{aligned}
T_0 &= \text{Total kinetic energy in LAB system} \\
T_0' &= \text{Total kinetic energy in CM system} \\
T_1 &= \text{Final kinetic energy of} m_1 \text{in LAB system} \\
T_1 &= \text{Final kinetic energy of} m_1 \text{in CM system}
\end{aligned}
$$

and the same definitions for T_2, T_2'.

$$
\begin{aligned}
\mathbf{V} &= \text{velocity of center of mass in the LAB system} \\
\psi &= \text{angle through which } m_1 \text{ is deflected in LAB system} \\
\zeta &= \text{angle through which } m_2 \text{ is deflected in LAB system} \\
\theta &= \text{angle through which } m_1, m_2 \text{ are deflected in CM}
\end{aligned}
$$

We know $m\mathbf{r}_1 + m_2\mathbf{r}_2 = M\mathbf{R}$, which means that

$$
m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = M\mathbf{V} \tag{5.1}
$$

Since $\mathbf{u}_2 = 0$, and $M = m_1 + m_2$, we have

$$
\mathbf{V} = \frac{m_1\mathbf{u}_1}{m_1 + m_2}
$$

Because m_2 is at rest initially, the initial center of mass speed of m_2 is

$$
\begin{aligned}
\mathbf{u}_2 = \mathbf{u}_2' + \mathbf{V} \Rightarrow \mathbf{u}_2' &= -\mathbf{V} \\
u_2' &= \frac{m_1 u_1}{m_1 + m_2} = V
\end{aligned} \tag{5.2}
$$

where we ignored signs since we are only considering the scalar quantity.

With the collision elastic, we can use energy and momentum conservation to obtain $u_1' = v_1'$, $u_2' = v_2'$ (this is done in section 6.2.3 of the PHYS0070 notes).

Center of mass coordinates, by definition, yield

$$m_1 u_1' - m_2 u_2' = 0 \Rightarrow m_1 u_1' = m_2 u_2'$$

This equation along with (5.2) and (5.1) yields

$$u_1 = u_1' + u_2'$$

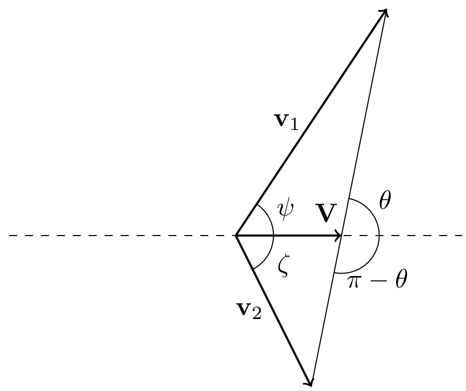

Figure 5.1: Final condition in the LAB system.

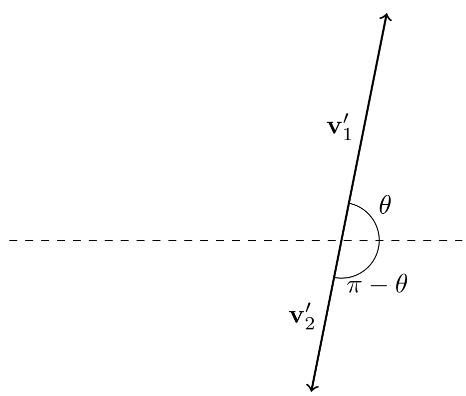

Figure 5.2: Final condition in the LAB system.

Observe figures 5.1 and 5.2, notice that

$$\begin{cases} v_1' \sin\theta = v_1 \sin\psi \\ v_1' \cos\theta + V = v_1 \cos\psi \end{cases}$$

Divide these equations to obtain

$$\tan\psi = \frac{v_1' \sin\theta}{v_1' \cos\theta + V} = \frac{\sin\theta}{\cos\theta + \frac{V}{v_1'}} \tag{5.3}$$

With $V = m_1 u_1/(m_1 + m_2)$ and $v_1' = m_2 u_2/(m_1 + m_2)$, we have

$$\frac{V}{v_1'} = \frac{m_1 u_1/(m_1 + m_2)}{m_2 u_1/(m_1 + m_2)} = \frac{m_1}{m_2} \tag{5.4}$$

Which means (5.3) becomes

$$\boxed{\tan\psi = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}}$$

At this point we must consider the two cases

$$\begin{cases} V/v_1' < 1 \Rightarrow V < v_1' \\ V/v_1' > 1 \Rightarrow V > v_1' \end{cases}$$

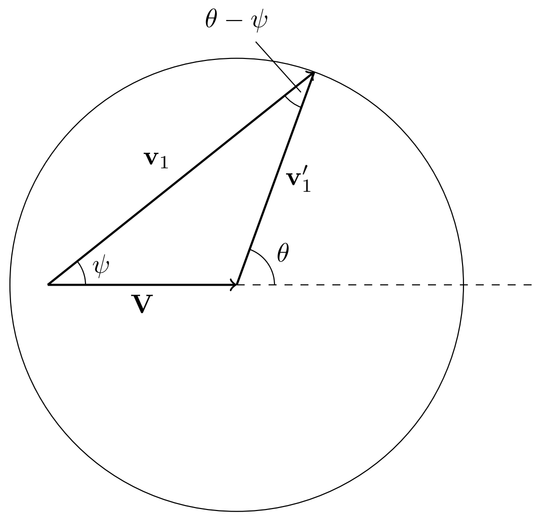

Figure 5.3: $V < v_1'$

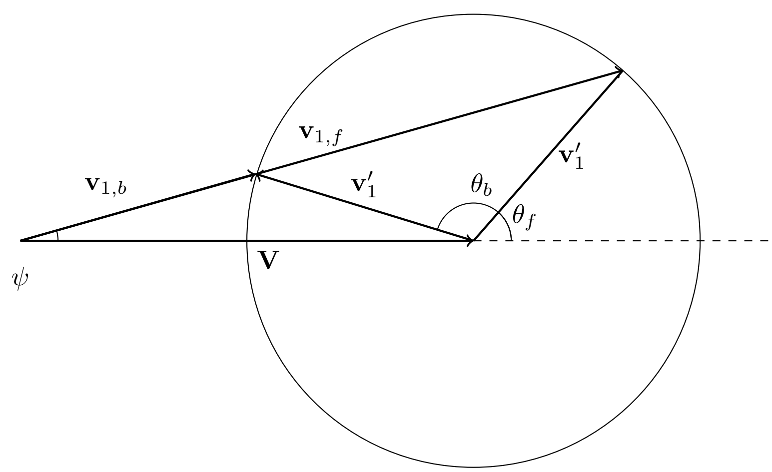

Figure 5.4: $V > v_1'$

Observe figures **??** and . For $\mathbf{v}_1 = \mathbf{V} + \mathbf{v}_1'$ to hold, we have three different scenarios:

- If $m_1 < m_2$, then $v < V_1'$ and the above holds for only one θ.
- If $m_1 > m_2$, then $V > v_1'$ and the above holds for two angles: θ_b, θ_f.
- If $m_2 = m_1$,

$$\tan\psi = \frac{\sin\theta}{\cos\theta + 1} = \tan\frac{\theta}{2} \Rightarrow \boxed{\psi = \frac{\theta}{2}}$$

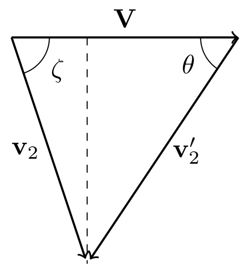

Figure 5.5: Final state of the mass m_2.

Now turn to finding the angle of deflection of m_2, ζ. Inspect 5.5:

$$\begin{cases} v_2 \sin\zeta = v_2' \sin\theta \\ v_2 \cos\zeta = V - v_2' \cos\theta \end{cases}$$

Divide these equation:

$$\tan\zeta = \frac{v_1' \sin\theta}{V - v_2' \cos\theta} = \frac{\sin\theta}{\frac{V}{v_2'} - \cos\theta}$$

But $V = v_2'$. Then

$$\tan\zeta = \frac{\sin\theta}{1 - \cos\theta} = \cot\frac{\theta}{2}$$

Such that $\tan\zeta = \tan(\pi/2 - \theta/2)$ and

$$\boxed{2\zeta = \pi - \theta = \phi}$$

If $m_1 = m_2$, $\theta = 2\psi$ and $\zeta + \psi = \frac{\pi}{2}$.

5.3 Scattering Cross-Sections

The scattering cross section effectively gives us a probability distribution for a particle to scatter at a given angle in a specified force field. The **impact parameter** parameter b is the closest distance the particles m_1 and m_2 would have from each other if there was no forces among them. This quantity cannot be precisely known in nuclear interactions and thus we resort to a probability distribution.

Define the **intensity** or **flux density** I to be the number of particles passing in unit time through a unit area normal to the direction of the beam. With a small enough force law, the motion of the particle following the collision will asymptotically approach a line which makes an angle θ with the initial velocity vector. We usually do this to identical particles initially at rest with mass m_2 (since each particle will be deflected after the collision) but for our purpose, we consider one static particle.

Define a **differential scattering cross section** in the CM system $\sigma(\theta)$ for the scattering into an element of solid angle $\mathrm{d}\Omega'$ at a particular θ.

$$\sigma(\theta) = \frac{N \text{ of interactions that lead to scattering into } \mathrm{d}\Omega' \text{ at angle } \theta}{\text{Number of incident particles per unit area}}$$

If $\mathrm{d}N$ is the number of particles scattered into $\mathrm{d}\Omega'$ per unit time, then the probability of scattering into $\mathrm{d}\Omega'$ for a unit time are of the incident beam is

$$\sigma(\theta)\,\mathrm{d}\Omega' = \frac{\mathrm{d}N}{I} \text{ or } \sigma(\theta) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'} = \frac{1}{I}\frac{\mathrm{d}N}{\mathrm{d}\Omega'}$$

(multiply by $\mathrm{d}\Omega'$ to consider that entire solid angle element). This has units area per steradian. If the scattering has axial symmetry, we can integrate over the azimuthal angle 2π to obtain 2π and the element of solid angle $\mathrm{d}\Omega'$ is

$$\mathrm{d}\Omega' = 2\pi \sin\theta \, \mathrm{d}\theta$$

For a central force, the number of particles with impact parameters within a range $\mathrm{d}b$ at a distance b must correspond to the number of particles scattered into the angular range $\mathrm{d}\theta$ at angle θ. Thus

$$I \cdot 2\pi b \, \mathrm{d}b = -I \cdot \sigma(\theta) \cdot 2\pi \sin\theta \, \mathrm{d}\theta \tag{5.5}$$

Then

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \tag{5.6}$$

Consider the incident particle to be of mass μ, and the angle this particle's motion makes with its initial trajectory line to be 2Θ. Last chapter we obtained the small change in Θ:

$$\Delta\Theta \int_{r_{\min}}^{r_{\max}} \frac{(l/r^2)\,\mathrm{d}r}{\sqrt{2\mu\sqrt{E - U - (l^2/2\mu r^2)}}}$$

By definition,

$$\theta = \pi - 2\Theta \tag{5.7}$$

and we define the particle's max distance to be $r_{\max} = \infty$. Further, we take the initial energy of the system to be $T_0' = \mu u_1^2$, and, once again entertaining the particle's motion in case there was no interaction, we have that

$$l = \mu u_1 b \Rightarrow b\sqrt{2\mu T_0}$$

And thus we have

$$\Theta = \int_{r_{\min}}^{\infty} \frac{(b/r^2)\,\mathrm{d}r}{\sqrt{1 - (b^2/r^2) - (U/T_0')}}$$

We've also used $E = T_0'$ because at $r = \infty$ the potential energy is zero and the total energy is purely the kinetic energy. This equation along with (5.7) gives the dependence $\theta(b)$, which can be inverted to find $b(\theta)$. With this information along with $U(r)$ and T_0', we can calculate the differential cross section from (5.6) *in the center of mass system*, since we've been considering m_2 fixed. For $m_2 \gg m_1$ this is quite close to the LAB system.

In the case that this is not true, we explore the general relations. The total number of particles scattered into a unit solid angle must be the same in the LAB system as in the CM system, and we have

$$\sigma(\theta)\,\mathrm{d}\Omega' = \sigma(\psi)\,\mathrm{d}\Omega'$$
$$\sigma(\theta) \cdot 2\pi \sin\theta \, \mathrm{d}\theta = \sigma(\psi) \cdot 2\pi \sin\psi \, \mathrm{d}\psi$$

where θ and ψ are the same scattering angle measured in the CM and LAB systems respectively, and same with $\mathrm{d}\Omega'$ and $\mathrm{d}\Omega$. Thus

$$\sigma(\psi) = \sigma(\theta) \cdot \frac{\sin\theta}{\sin\psi}\frac{\mathrm{d}\theta}{\mathrm{d}\psi}$$

Evaluate the derivative $\mathrm{d}\theta/\mathrm{d}\psi$. Start by inspecting figure 5.2. From the sine law:

$$\frac{sin(\theta - \psi)}{\sin\psi} = \frac{V}{v_1'} \tag{5.8}$$

Use equation (5.4) to obtain

$$\frac{sin(\theta - \psi)}{\sin\psi} = \frac{m_1}{m_2} \equiv x \tag{5.9}$$

Find the differential dx and set it equal to zero

$$\mathrm{d}x = 0 = \frac{\partial x}{\partial \psi}\,\mathrm{d}\psi + \frac{\partial x}{\partial \theta}\,\mathrm{d}\theta$$

Take the partial derivatives and collect terms

$$\frac{\mathrm{d}\theta}{\mathrm{d}\psi} = \frac{\sin(\theta-\psi)\cos\psi}{\cos(\theta-\psi)\sin\psi} + 1$$

Expand $\sin(\theta-\psi)$ and simplify to obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}\psi} = \frac{\sin\theta}{\cos(\theta-\psi)\sin\psi}$$

Substituting back into our expression for $\sigma(\psi)$:

$$\sigma(\psi) = \sigma(\theta)\cdot\frac{\sin^2\theta}{\cos(\theta-\psi)\sin^2\psi} \tag{5.10}$$

Multiply both sides of our expression for x in(5.9) by $\cos\psi$ and add $\cos(\theta-\psi)$ to obtain

$$x\cos\psi + \cos\left(\theta-\psi\right)$$

Expand $\sin(\theta-\psi)$ and $\cos(\theta-\psi)$ on the left-hand side to obtain

$$\frac{\sin\theta}{\sin\psi} = x\cos\psi + \cos(\theta-\psi)$$

Substitute this result into (5.10) to obtain

$$\sigma(\psi) = \sigma(\theta)\cdot\frac{\left[x\cos\psi+\cos(\theta-\psi)\right]^2}{cos(\theta-\psi)}\ x<1$$

Again use (5.9)

$$\cos(\theta-\psi) = \sqrt{1-x^2\sin^2\psi}$$

Such that

$$\boxed{\sigma(\psi) = \sigma(\theta)\cdot\frac{\left[x\cos\psi+\sqrt{1-x^2\sin^2\psi}\right]^2}{\sqrt{1-x^2\sin^2\psi}}}$$

Where we may use (5.9) to obtain

$$\boxed{\theta = \sin^{-1}\left(x\sin\psi\right)+\psi}$$

Now we have the scattering cross-section in the LAB system for a given angle ψ measured in the LAB system (as opposed to in the CM system).

Example Calculate the differential cross section $\sigma(\theta) =$ and the total cross section σ_t for the elastic scattering of a particle from an impenetrable sphere of radius a.

Using equation (5.5) for a single particle of mass μ being scattered by a force center, we will consider a particle of radius r colliding with a particle of radius a, then take the limit as $r\to 0$.

$$I\cdot 2\pi b\,\mathrm{d}b = -I\cdot\sigma(\theta)\cdot 2\pi\sin\theta\,\mathrm{d}\theta$$

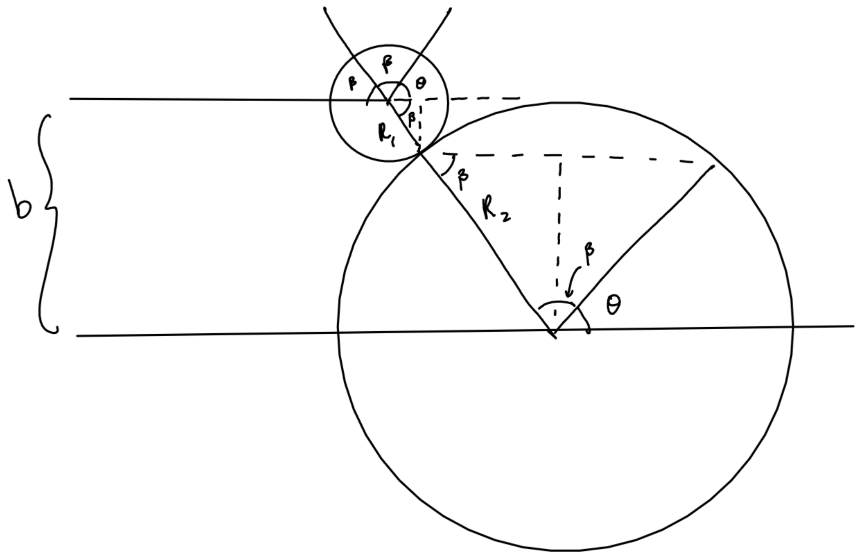

Figure 5.6: Note that $R_1 = r$ and $R_2 = a$.

Consider unit flux density $I = 1$. Find the relationship between b and θ.
By figure 5.6, notice that

$$b = r\sin\beta + a\sin\beta$$

And also that

$$2\beta + \theta = \pi \Rightarrow \beta = \frac{\pi}{2} - \frac{\theta}{2}$$

Then

$$b = (r+a)\cos\frac{\theta}{2}\,\mathrm{d}b = -\frac{1}{2}(r+a)\sin\frac{\theta}{2}\,\mathrm{d}\theta$$

Substitute into (5.5)

$$2\pi(r+a)\cos\frac{\theta}{2}\cdot\left(-\frac{1}{2}\right)(r+a)\sin\frac{\theta}{2}\,\mathrm{d}\theta = \sigma(\theta)2\pi\sin\theta\,\mathrm{d}\theta$$

Use the trigonometric identity $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$:

$$-\frac{\pi}{2}(r+a)^2\sin\theta\,\mathrm{d}\theta = -\sigma(\theta)2\pi\sin\theta\,\mathrm{d}\theta$$
$$\frac{1}{4}(r+a)^2 = \sigma(\theta)$$

Take the limit as $r \to 0$:

$$\sigma(\theta) = \frac{1}{4}a^2$$

To find the total cross-section, integrate over Ω. Equation (5.5) gives the solid angle element $\mathrm{d}\Omega = 2\pi \sin\theta \, \mathrm{d}\theta$:

$$\begin{aligned}
\sigma_t &= \int_\Omega \sigma(\theta) \, \mathrm{d}\Omega \\
&= \int_0^\pi \frac{1}{4}(r+a)^2 \cdot 2\pi \sin\theta \, \mathrm{d}\theta \\
&= \frac{\pi}{2}(r+a)^2 [-\cos]_0^\pi \\
&= \frac{\pi}{2}(r+a)^2 (1-(-1)) \\
&= \pi (r+a)^2
\end{aligned}$$

Let $r \to 0$:

$$\sigma_t = \pi a^2$$

Chapter 6

Motion in Non-Inertial Reference Frames

Consider two sets of coordinate axes. Let one set be "fixed," or inertial, and the other be an arbitrary set that is in motion with respect to the inertial system. Designate the latter axes as "rotating." Use x_i' in the fixed system and x_i in the rotating system. A point P has

$$\mathbf{r}' = \mathbf{R} + \mathbf{r} \tag{6.1}$$

where $\mathbf{r}'$ is the radius vector in the fixed system, $\mathbf{r}$ is the radius vector in the rotating system, and $\mathbf{R}$ is the vector from the origin of the fixed system to the origin of the rotating system.

Represent some infinitesimal displacement as a pure rotation (indistinguishable for very small displacements) about an axis called the **instantaneous axis of rotation**. If the x_i system undergoes an infinitesimal rotation $\delta\boldsymbol{\theta}$, the motion of P can be expressed by an infinitesimal angular velocity

$$(\mathrm{d}\mathbf{r})_{\text{fixed}} = \mathrm{d}\boldsymbol{\theta} \times \mathbf{r}$$

where we're taking P to be at rest in the rotating reference frame. Divide this equation by $\mathrm{d}t$ to obtain the rate of change of $\mathbf{r}$ as measured in the fixed coordinate system:

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\text{fixed}} = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} \times \mathbf{r}$$

Defining $\boldsymbol{\omega} \equiv \boldsymbol{\theta}t$, we have

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{fixed}} = \boldsymbol{\omega} \times \mathbf{r}$$

Now allow P to have nonzero velocity $\mathrm{d}\mathbf{r}/\mathrm{d}t_{\text{rotating}}$ with respect to the x_i system. Then, since the above equation effectively gives the velocity relative to the fixed axes of $\mathbf{R}$, (6.1) suggests we simply should add the velocity with respect to the x_i rotating system:

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\text{rotating}} + \boldsymbol{omega} \times \mathbf{r} \tag{6.2}$$

This relationship is, in fact, true for any vector:

$$\boxed{\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{Q}} \tag{6.3}$$

If the rotating coordinate frame itself is translating as well as rotating, we have

$$\left(\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right)_{\text{fixed}} + \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\text{fixed}}$$

And using equation (6.3)

$$\left(\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right)_{\text{fixed}} + \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\mathbf{r}$$

Define

$$\begin{aligned}
\mathbf{v}_f \equiv \dot{\mathbf{r}}_f &\equiv \left(\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}\right)_{\text{fixed}} \\
\mathbf{V} \equiv \dot{\mathbf{R}}_f &\equiv \left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right)_{\text{fixed}} \\
\mathbf{v}_r \equiv \dot{\mathbf{r}}_r &\equiv \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\text{rotating}}
\end{aligned}$$

we may write

$$\boxed{\mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \boldsymbol{\omega}\times\mathbf{r}} \tag{6.4}$$

where $\boldsymbol{\omega}$ is the angular velocity of the rotating axes.

6.1 Centrifugal and Coriolis Forces

We know $\mathbf{F} = m\mathbf{a}$ is only valid in inertial reference frames. Then we must have some

$$\mathbf{F} = m\mathbf{a}_f = m\left(\frac{\mathrm{d}\mathbf{f}_f}{\mathrm{d}t}\right)_{\text{fixed}}$$

We want to see what this law looks like when we are measuring the position from the perspective of a rotating reference frame. Let's solve for this acceleration. Differentiating (6.4), we have

$$\left(\frac{\mathrm{d}\mathbf{f}_f}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\right)_{\text{fixed}} + \left(\frac{\mathrm{d}\mathbf{v}_r}{\mathrm{d}t}\right)_{\text{fixed}} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\text{fixed}}$$

The first term is just the acceleration of the origin of the rotating reference frame

$$\ddot{\mathbf{R}}_f \equiv \left(\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\right)_{\text{fixed}}$$

Evaluate the second term using (6.3):

$$\left(\frac{\mathrm{d}\mathbf{v}_r}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\mathbf{v}_r}{\mathrm{d}t}\right)_{\text{rotating}} + \omega\times\mathbf{v}_r = \mathbf{a}_r + \boldsymbol{\omega}\times\mathbf{v}_r$$

Where $\mathbf{a}_r$ is the acceleration of the particle in the rotating reference frame. The last term can be obtained using (6.2):

$$\begin{aligned}
\boldsymbol{\omega}\times\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{\text{fixed}} &= \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \\
&= \boldsymbol{\omega}\times\mathbf{v}_r + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})
\end{aligned}$$

Finally, combining all of these equations, we have

$$\mathbf{F}_{\text{actual}} = m\mathbf{a}_f = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

We wanted the apparent motion with respect to the rotating coordinate system; this is $m\mathbf{a}_r$. Solving for this,

$$\boxed{\mathbf{F}_{\text{eff}} \equiv m\mathbf{a}_r = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r}$$

This can be broken down as follows:

- $\mathbf{F}$ is the sum of the forces acting on the particle as measured in the *fixed* system.
- $-m\ddot{\mathbf{R}}_f$ is the translational motion of the rotating system relative to the rotating system.
- $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ is the rotational motion of the rotating system relative to the rotating system.
- $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the centrifugal force, which, if you work out the cross products, points outward along the radial direction.
- $-2m\boldsymbol{\omega} \times \mathbf{v}_r$ is the **Coriolis force**, which arises only when the particle has a velocity in the rotating reference frame.

6.2 Motion Relative to the Earth

The non-inertial motion of the earth is dominated by its rotation about its own axis. We place an inertial reference frame $x'y'z'$ at the center of the earth and a moving reference frame xyz on the surface of the earth. The forces as measured in the inertial reference frame are

$$\mathbf{F} = \mathbf{S} + m\mathbf{g}_0$$

where $\mathbf{S}$ is the sum of the forces other than gravity, and $\mathbf{g}_0$ is the gravitational field vector:

$$\mathbf{g}_0 = -G\frac{M_E}{R^2}\hat{\mathbf{e}}_R$$

We treat the earth as perfectly spherical, and analyze the effective force

$$\mathbf{F}_{\text{eff}} \equiv m\mathbf{a}_r = \mathbf{S} + m\mathbf{g}_0 - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

Let the Earth's angular velocity $\boldsymbol{\omega}$ be along the inertial system's z' direction. This quantity changes slightly enough that we neglect $\dot{\boldsymbol{\omega}} \times \mathbf{r}$.

Equation (6.3) gives, since $\mathbf{R}$ is stationary with respect to the rotating coordinate system:

$$\ddot{\mathbf{R}}_f = \boldsymbol{\omega} \times \dot{\mathbf{R}}_f = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$$

Which yields

$$\mathbf{F}_{\text{eff}} = \mathbf{S} + m\mathbf{g}_0 - m\boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{R})] - 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

The second and third terms are the gravitational force and the centrifugal force . Since we are observers on earth, we experience the sum of these two forces— a value slightly less than the force due to earth's gravitational field. We can call this

$$\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\omega} \times [\boldsymbol{\omega} \times \mathbf{r} + \mathbf{R}]$$

Since we are considering that the gravitational field vector is constant, we are looking at motion near the surface of the earth; in turn, this means $r \ll R$, and the centrifugal force is dominated by $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$. For motion far from the surface of the earth, we'd also have to consider variations in g. Nevertheless, the effective force becomes

$$\boxed{\mathbf{F}_{\text{eff}} = \mathbf{S} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \mathbf{v}_r}$$

Chapter 7

Dynamics of Rigid Bodies

7.1 Introduction

We define a rigid body as a collection of particles whose relative distances are fixed. In actual, real bodies, particles vibrate and in general move relative to each other. However, our model is adequate in the macroscopic scale.

We will have two coordinate systems: a coordinate system that is inertial and fixed, and a coordinate system fixed with respect to the body. We will also find that the orientation of the body can be completely described by three angles called the Eulerian angles.

With the rotation of the body and the position of the body's center of the mass, we can completely describe the motion of arbitrary, extended bodies.

7.2 The Inertia Tensor

The advantage of the inertia tensor is that it completely captures the moment of inertia of the object about any axis of rotation you might choose. Once we choose an axis of rotation, we simply perform a matrix product and that yields the appropriate moment of inertia.

Consider a rigid body composed of n particles of masses m_α. The body rotates with angular velocity $\boldsymbol{\omega}$ about some fixed point with respect to the body coordinate system. If this point has velocity $\mathbf{V}$, the velocity of the particle α with respect to the fixed system is expressed by equation (6.4). However, the rigid body requires that the constituent particles are not moving relative to the rotating axes. Then

$$\mathbf{v}_r = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\text{rotating}} = 0$$

And (6.4) becomes

$$\mathbf{v}_\alpha = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

The velocity $\mathbf{v}_\alpha$ is measured with respect to the fixed system.

The kinetic energy of a particle α is given by

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$$

such that the total kinetic energy is

$$T = \frac{1}{2} \sum_\alpha m_\alpha (\mathbf{V} + \omega \times \mathbf{r}_\alpha)^2$$

We expand this to obtain

$$T = \frac{1}{2}\sum_\alpha m_\alpha V^2 + \sum_\alpha m_\alpha \mathbf{V}\cdot\boldsymbol{\omega}\times\mathbf{r}_\alpha + \frac{1}{2}\sum_\alpha m_\alpha(\boldsymbol{\omega}\times\mathbf{r}_\alpha)^2$$

Now enforce that the origin of the body coordinate system *coincide with the center of mass.* First, take $\mathbf{V}$ and $\boldsymbol{\omega}$ out of the summation on the second term.

$$\sum_\alpha m_\alpha \mathbf{V}\cdot\boldsymbol{\omega}\times\mathbf{r}_\alpha = \mathbf{V}\cdot\boldsymbol{\omega}\times\left(\sum_\alpha m_\alpha \mathbf{r}_\alpha\right)$$

Now observe that $\sum_\alpha \mathbf{r}_\alpha = M\mathbf{R}$, which disappears because we are measuring $\mathbf{r}_\alpha$ from the center of mass. The kinetic energy can then be written

$$T = T_{\text{trans}} + T_{\text{rot}}$$

where

$$\boxed{T_{\text{trans}} = \frac{1}{2}\sum_\alpha m_\alpha V^2 = \frac{1}{2}MV^2}$$

And $T_{\text{rot}} = \frac{1}{2}\sum_\alpha m_\alpha(\boldsymbol{\omega}\times\mathbf{r}_\alpha)^2$. We resolve this vector-valued component using the identity

$$(\mathbf{A}\times\mathbf{B})^2 = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2$$

Such that

$$T_{\text{rot}} = \frac{1}{2}\sum_\alpha m_\alpha\left[\omega^2 r_\alpha^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)^2\right]$$

Express this quantity in terms of the components of the vectors, $\omega_i, r_{\alpha,i} = x_{\alpha,i}$.

$$T_{\text{rot}} = \frac{1}{2}\sum_\alpha m_\alpha\left[\left(\sum_i \omega_i^2\right)\left(\sum_k x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \omega_j x_{\alpha,j}\right)\right]$$

We want to pull out some of the summations. Write $\omega_i = \sum_j \omega_j \delta_{ij}$ such that

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\sum_\alpha\sum_{i,j} m_\alpha\left[\omega_i\omega_j\delta_{ij}\left(\sum_k x_{\alpha,k}^2\right) - \omega_i\omega_j x_{\alpha,i}x_{\alpha,j}\right] \\ &= \frac{1}{2}\sum_{i,j}\omega_i\omega_j\sum_\alpha m_\alpha\left(\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}V\right) \end{aligned}$$

Define the ijth element of the sum over α to be

$$\boxed{I_{ij} \equiv \sum_\alpha m_\alpha\left(\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i}, x_{\alpha,j}\right)} \tag{7.1}$$

If instead of a discrete set of particles, we have a continuous mass of density $\rho(\mathbf{r})$, we may turn this into an integral

$$\boxed{I_{ij} = \int_V \rho(\mathbf{r})\left(\delta_{ij}\sum_k x_k^2 - x_i x_j\right)} \tag{7.2}$$

And we have the generalized version of the familiar $I\omega^2/2$,

$$\boxed{T_{\text{rot}} = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j}$$

Equations (7.1), as well as (7.2) are the entries of a three-by-three matrix. With $r_\alpha^2 = \sum_i x_{i,\alpha}^2$ and $(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) = (x_\alpha, y_\alpha, z_\alpha)$, we have

$$I = \begin{bmatrix} \sum_\alpha m_\alpha\left(r_\alpha^2 - x_\alpha^2\right) & -\sum m_\alpha x_\alpha y_\alpha & -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ -\sum_\alpha m_\alpha y_\alpha x_\alpha & \sum_\alpha m_\alpha\left(r_\alpha^2 - y_\alpha^2\right) & -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ -\sum_\alpha m_\alpha z_\alpha x_\alpha & -\sum_\alpha z_\alpha y_\alpha & \sum_\alpha m_\alpha\left(r_\alpha^2 - z_\alpha^2\right) \end{bmatrix}$$

The diagonal entries are the **moments of inertia** about their respective axes, and the off-diagonal elements are the **products of inertia**.

7.2.1 Properties of the Inertia Tensor

1. I is symmetric.
2. Its eigenvectors correspond to the principal axes of inertia, and since I is symmetric, these axes are orthogonal.
3. The eigenvalues correspond to the principal moments of inertia.
4. The rotation that diagonalizes I is exactly the rotation that transforms to the principal axes of inertia. We have
$$I = QI'Q^{-1} = QI'Q^T$$
where Q is the isometry (orthonormal matrix) that diagonalizes the inertia tensor matrix, and I' is the inertia tensor for the principal axes.

7.3 Angular Momentum

With respect to some fixed point O fixed in the body coordinate system, the angular momentum of the body is

$$\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha$$

We choose O based on the following criteria:

1. If one or more points is fixed in the fixed coordinate system, choose O to coincide with one such point.
2. If no such point is fixed, choose O to be the center of mass.

The linear momentum relative to the body coordinate system is $\mathbf{p}_\alpha$:

$$\mathbf{p}_\alpha = m_\alpha \mathbf{v}_\alpha = m_\alpha \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

Hence the angular momentum is

$$\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)$$

We use the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2\mathbf{B} - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$$

Such that

$$\mathbf{L} = \sum_\alpha m_\alpha \left[r_\alpha^2 \boldsymbol{\omega} - \mathbf{r}_\alpha (\mathbf{r}_\alpha \cdot \boldsymbol{\omega}) \right]$$

Let us try to express this with the moment of inertia tensor. Some manipulation of indices yields

$$L_i = \sum_j \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)$$

Where ew once again have the inertia tensor under the sum over α. Then we have the matrix product

$$\boxed{\mathbf{L} = I\boldsymbol{\omega}}$$

Since, generally, the inertia tensor is non-diagonal, the angular momentum *does not* coincide with the angular velocity vector. This is only the case if the rotation is taking place along the principal axes.

We can multiply the above equation for L_i by $\omega_i/2$ and sum over i to obtain

$$\frac{1}{2} \sum_i \omega_i L_i = \frac{1}{2} \sum_{i,j} \omega_i \omega_j = \boxed{T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}}$$

7.4 Principal Moments

A significant simplification occurs if the inertia tensor is diagonal. Then

$$\begin{aligned} L_i &= \sum_j I_i \delta_{ij} \omega_j = I_i \omega_i \\ T_{\text{rot}} &= \frac{1}{2} \sum_{i,j} I_i \delta_{ij} \omega_i \omega_j = \frac{1}{2} \sum_i I_i \omega_i^2 \end{aligned}$$

To diagonalize the inertia tensor, we must find a set axes called the **principal axes of inertia**. To find these axes, we diagonalize the inertia matrix. Since the matrix is symmetrical, the eigenvectors are orthogonal and the diagonalization takes the form

$$I = Q \Lambda Q^T = Q I' Q^T$$

Where I' is the diagonal inertia tensor we're looking for. It follows that Q^T is precisely the linear transformation that leads to the desired axes, and that the columns of Q give the directions of these axes (the above matrix product transforms to these coordinates, then applies I', then back to the original coordinates, leaving us with I). An angular velocity along such an axis yields angular momentum parallel to the angular velocity.

$$I' = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

I_i are the **principal moments of inertia**. The values the principal moments of inertia take clue us into the symmetry of the body.

- $I_1 = I_2 = I_2$: complete spherical symmetry. Termed a **spherical top**.
- $I_1 = I_2 \neq I_3$: symmetry along one axis. Termed a **symmetric top**.
- $I_1 \neq I_2 \neq I_3$: no symmetry. Termed an **asymmetric top**.

7.5 General Parallel Axis Theorem

It's not always easiest to compute the inertia tensor with the center of mass as the origin. Thus we choose some other axes X_i with same orientation as our chosen coordinate system but with origin Q which is not the center of mass. We will compute the inertia tensor with respect to this coordinate system and call it J, and hope to transform it to our center of mass coordinates inertia tensor I. Start with (7.1):

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,i} X_{\alpha,j} \right)$$

Let the vector connecting Q with O, the center of mass origin, be $\mathbf{a}$, and if $\mathbf{r}$ is the position of a particle in the center of mass system, let that position in the X_i system be

$$\mathbf{R} = \mathbf{a} + \mathbf{r}$$

with components

$$X_i = a_i + x_i$$

Substituting these in our expression for J_{ij},

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k \left(2x_{\alpha,k} a_k + a_k^2 \right) - \left(a_i x_{\alpha,j} + a_j x_{\alpha,i} + \alpha_j \alpha_i \right) \right)$$

The first summation is simply I_{ji}, so we regroup

$$J_{ij} = I_{ij} + \sum_\alpha + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j \right) + \sum_\alpha m_\alpha \left(2\delta_{ij} \sum_k x_{\alpha,k} a_k - a_i x_{\alpha,j} - a_j x_{\alpha,i} \right)$$

Each term in the last summation has some

$$\sum_\alpha m_\alpha x_{\alpha,k} = 0$$

because we chose the origin of x_i to be the center of mass. Then we have

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j \right)$$

And our defined quantities are

$$\sum_\alpha m_\alpha = M, \;\; \sum_k a_k^2 \equiv a^2$$

Such that, when we solve for I_{ij}, we have

$$\boxed{I_{ij} = J_{ij} - M\left(a^2 \delta_{ij} - a_i a_j \right)}$$

Then the desired inertia tensor I with respect to the center of mass can be found from any J if we have the vector $\mathbf{a}$ from the origin of J to the center of mass.

7.6 Eulerian Angles

The eulerian angles are ϕ, θ, ψ, where

- ϕ is the precession angle (about the fixed vertical axis).
- θ is the angle of inclination from the fixed vertical axis.
- ψ is the rotational angle about the body's symmetry axis.

The components of the angular velocity vector in terms of the Eulerian angles are

$$\begin{aligned}
\omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\
\omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\
\omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}
\end{aligned}$$

We obtain the Eulerian angles by making three rotations about different axes.

1. We rotate counterclockwise through an angle ϕ about the x_3' axis. Call this the x_i'' system.
$$\lambda_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
2. Counterclockwise through an angle θ about the x_1'' axis (where the x_1 axis was taken after the first rotation). Call this the x_i''' system.
$$\lambda_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$
3. Counter clockwise through an angle ψ about the x_3''' axis (the axis x_3' was taken to after the second rotation). Call this the x_i system.
$$\lambda_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the total rotation matrix is the matrix product

$$\lambda = \lambda_\psi \lambda_\theta \lambda_\phi$$

7.7 Euler's Equations for a Rigid Body

7.7.1 Force-Free

In the absence of forces, the potential energy is zero and the Lagrangian is purely the rotational kinetic energy. Choose x_i to *coincide with the principal axes.* Then

$$T = \frac{1}{2}\sum_i I_i \omega_i^2$$

Choose the Eulerian angles (section 7.6) as generalized coordinates, so that the Lagrangian for the coordinate ψ is

$$\frac{\partial T}{\partial \psi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\psi}} = 0$$

Which we may express

$$\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi} - \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}} = 0$$

Differentiate the components of $\boldsymbol{\omega}$ (also in section 7.6) with respect to ψ and $\dot{\psi}$.

$$\begin{cases} \partial\omega_1/\partial\psi = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = \omega_2 \\ \partial\omega_2/\partial\psi = -\dot{\phi}\sin\theta\sin\psi - \dot{\theta}\cos\psi = -\omega_1 \\ \partial\omega_2/\partial\psi = 0 \end{cases} , \quad \begin{cases} \partial\omega_1\Big/\partial\dot{\psi} = \partial\omega_2\Big/\partial\dot{\psi} = 0 \\ \partial\omega_2\Big/\partial\dot{\psi} = 1 \end{cases}$$

From our kinetic energy, we have $\partial T/\partial\omega_i = I_i\omega_i$ such that the Euler-Lagrange equation becomes

$$I\omega_1\omega_2 + I_2\omega_2(-\omega_1) - \frac{\mathrm{d}}{\mathrm{d}t}I_3\omega_3 = 0$$
$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

We could have chosen any axis as the x_3 axis, so we permute over the principal axes to obtain **Euler's equations** for force-free motion. Although $\dot{\omega_3}$ is the Lagrange equation of motion for ψ, $\dot{\omega_1}, \dot{\omega_2}$ are *not* the equations of motion for θ and ϕ.

$$\boxed{\begin{cases} (I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0 \\ (I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 = 0 \\ (I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \end{cases}}$$

7.7.2 In a Force Field

Start with the fundamental relation for torque:

$$\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{\text{fixed}} = \mathbf{N}$$

Recall that

$$\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{\text{body}} + \boldsymbol{\omega}\times\mathbf{L} = \mathbf{N}$$

The x_3 body axis component of this equation is

$$\dot{L}_3 + \omega_1 L_2 - \omega_2 L_1 = N_3$$

Since we chose the x_i axes to coincide with the principal axes, we have

$$L_i = I_i\omega_i$$

Such that

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3$$

Again we may write this for every component:

$$\boxed{\begin{cases} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3 \end{cases}}$$

Note that this motion depends on the body's geometry only through the principal moments of inertia. Then we can represent such a motion through an **equivalent ellipsoid** that has the exact same principal moments of inertia.

7.8 Force-Free Motion of a Symmetric Top

Consider a body with $I_1 = I_2 \neq I_3$. The force-free Euler equations become

$$\begin{cases} (I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0 \\ (I_3 - I_1)\omega_3\omega_1 - I_1\dot{\omega}_2 = 0 \\ I_3\dot{\omega}_3 = 0 \end{cases}$$

In the absence of forces, the motion of the center of mass is inertial; thus, focus solely on the rotation of the body. For the nontrivial case, consider $\boldsymbol{\omega}$ does not lie along a principal axis.

From the last of the above equations,

$$\omega_3 = \text{constant}$$

Write the remaining equations

$$\begin{cases} \dot{\omega}_1 = -\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2 \\ \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_1 \end{cases}$$

The terms in the parenthesis are constants. Define

$$\Omega \equiv \frac{I_3 - I_1}{I_1}$$

Such that

$$\begin{cases} \dot{\omega}_1 + \Omega\omega_2 = 0 \\ \dot{\omega}_2 - \Omega\omega_1 = 0 \end{cases}$$

This is a negative inverse relationship; we should think to use an imaginary substitution. Multiply the second equation by i and add it to the first.

$$(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega(\omega_1 + i\omega_2) = 0$$

Define $\eta \equiv \omega_1 + i\omega_2$, this simplifies to

$$\dot{\eta} - i\Omega\eta = 0$$

A first order differential equation with solution

$$\eta = Ae^{i\Omega t}$$
$$\omega_1 + i\omega_2 = A\cos\Omega t + iA\sin\Omega t$$

Such that

$$\begin{cases} \omega_1(t) = A\cos\Omega t \\ \omega_2(t) = A\sin\Omega t \end{cases}$$

These are the parametric equations for a circle. Then the projection of $\boldsymbol{\omega}$ onto the x_1x_2 plane is a circle.

Let us confirm that the magnitude of $\boldsymbol{\omega}$ is constant

$$|\boldsymbol{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 + \omega_3^2}$$

which is indeed constant with ω_3 constant. Then the angular velocity vector traces out a cone in space over time (to an observer in the body coordinate system). The vector precesses around the x_3 axis with rate Ω, so the body symmetry axis is the cone's symmetry axis. We call this the **body cone**.

We are considering force-free motion, so $\mathbf{L}$ is constant in time and fixed in the fixed coordinate system. Since we specified that the center of mass of the body is fixed, the rotational kinetic energy of the body also remains constant:

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \text{constant}$$

Since ω moves and $\mathbf{L}$ is stationary, for the above to be constant, the projection of $\boldsymbol{\omega}$ onto $\mathbf{L}$ must be constant. Then $\mathbf{L}$, ω and the x_3 axis all lie in a plane.

If we designate x_3' (fixed coordinates) to coincide with $\mathbf{L}$, in the fixed axis system we see $\boldsymbol{\omega}$ likewise tracing out a cone around the x_3' axis. Call this the space cone. Since, from the perspective of the body system, $\boldsymbol{\omega}$ must be precessing, we can imagine the body cone and the space cone are rolling on each other.

7.8.1 Symmetric Top with One Point Fixed

Consider a symmetric top with its tip held fixed rotating in a gravitational field. Choose the origin of the fixed coordinate system to coincide with the origin of the body coordinate system. Choose this origin to be the fixed tip. The distance to the center of mass is purely along the x_3 body axis; denote this h. Let the top's mass be M. Assume $I_1 = I_2$ and $I_2 \neq I_3$.

The kinetic energy of this body is

$$T = \frac{1}{2}\sum_i I_i\omega_i^2 = \frac{1}{2}I_1\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}I_3\omega_3^2$$

The angular velocity components in terms of the Eulerian angles are

$$\begin{aligned}
\omega_1^2 &= \left(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi\right)^2 \\
&= \dot\phi^2\sin^2\theta\sin^2\psi + 2\dot\phi\dot\theta\sin\theta\sin\psi\cos\psi + \dot\theta^2\cos^2\psi \\
\omega_2^2 &= \left(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi\right)^2 \\
&= \dot\phi^2\sin^2\theta\cos^2\psi - 2\dot\phi\dot\theta\sin\theta\sin\psi\cos\psi + \dot\theta^2\sin^2\psi
\end{aligned}$$

Such that

$$\begin{aligned}
\omega_1^2 + \omega_2^2 &= \dot\phi^2\sin^2\theta + \dot\theta^2 \\
\omega_3^2 &= \left(\dot\phi\cos\theta + \dot\psi\right)^2
\end{aligned}$$

And the energy becomes

$$T = \frac{1}{2}I_1\left(\dot\phi^2\theta\dot\theta^2\right) + \frac{1}{2}I_3\left(\dot\phi\cos\theta + \dot\psi\right)^2$$

The potential energy is $Mgh\cos\theta$ (the gravitational force acting on the center of mass), so the Lagrangian becomes

$$L = \frac{1}{2}I_1\left(\dot{\phi}^2\theta\dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - Mgh\cos\theta$$

The Lagrangian is cyclic in ϕ and in ψ. Thus the momenta conjugate to these coordinates are constants:

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = \left(I_1\sin^2\theta + I_3\cos^2\theta\right)\dot{\phi} + I_3\dot{\psi}\cos\theta = \text{constant} \tag{7.3}$$

$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = \text{constant} \tag{7.4}$$

Because the coordinates are angles, these momenta are angular momenta. Solve the above two equations for $\dot{\phi}$ and $\dot{\psi}$. (7.4) yields

$$\dot{\psi} = \frac{p_\psi - I_3\dot{\phi}\cos\theta}{I_3}$$

Substituting into (7.3),

$$\begin{aligned} p_\phi &= (I_1\sin^2\theta + I_3\cos^2\theta)\dot{\phi} + \left(p_\psi - I_3\dot{\phi}\cos\theta\right)\cos\theta \\ &= \left(I_1\sin^2\theta\right)\dot{\phi} + p_\psi\cos\theta \end{aligned}$$

So that

$$\boxed{\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}} \tag{7.5}$$

Use this expression for $\dot{\phi}$ in our above expression for $\dot{\psi}$ to obtain

$$\boxed{\dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I_1\sin^2\theta}} \tag{7.6}$$

Assume the system is conservative. Then the total energy is a constant:

$$E = \frac{1}{2}I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\omega_3^2 + Mgh\cos\theta = \text{constant}$$

Use the Eulerian expression for ω_3 to write (7.4) as

$$p_\psi = I_3\omega_3 \Rightarrow I_3\omega_3^2 = \frac{p_\psi^2}{I_3} = \text{constant}$$

(This should make sense; ψ is the rotation about the body's symmetry axis.)

It follows that $E - I_3\omega_3^2/2$ is also a constant. Define this E' such that

$$E' \equiv \frac{1}{2}I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + Mgh\cos\theta$$

Now substitute our expression for $\dot{\phi}$ to obtain

$$E' = \frac{1}{2}I\dot{\theta}^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$$

Which we can write

$$\boxed{E' = \frac{1}{2}I_1\dot{\theta}^2 + V(\theta)} \tag{7.7}$$

Where $V(\theta)$ is an effective potential

$$\boxed{V(\theta) \equiv \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + Mgh\cos\theta} \tag{7.8}$$

We can then use (7.7) to solve for the motion of the inclination angle θ:

$$t(\theta) = \int \frac{\mathrm{d}\theta}{\sqrt{(2/I_1)[E' - V(\theta)]}}$$

Which can be inverted to obtain $\theta(t)$ then substituted into (7.6) and (7.5) for the motions of each Eulerian angle. Such a procedure is, however, quite complicated.

7.8.2 Qualitative Analysis

The effective potential in (7.8) has asymptotic behavior at two values of θ: θ_1 and θ_2. These represent the turning points of the inclination angle. Then the inclination is generally confined to $\theta_1 \le \theta \le \theta_2$. If the energy is $E' = V_{\min}$, θ is limited to a minimum value θ_0 and we have steady precession at fixed inclination.

Let us try to find the value of θ_0. Set the derivative of $V(\theta)$ equal to zero

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = \frac{- - \cos\theta_0 (p_\phi - p_\psi \cos\theta_0)^2 + p_\psi \sin^2\theta_0 (p_\phi - p_\psi \cos\theta_0)}{I_1 \sin^3\theta_0} - Mgh\sin\theta_0 = 0$$

Define $\beta \equiv p_\phi - p_\psi \cos\theta_0$ such that the above becomes

$$(\cos\theta_0)\beta^2 - \left(p_\psi \sin^2\theta_0\right)\beta + \left(MghI_1 \sin^4\theta_0\right) = 0$$

Which, solving as a quadratic,

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MghI_1\cos\theta_0}{p_\psi^2}}\right)$$

β must be a real quantity. Thus, the radicand must be positive. Let's explore the case that $\theta < \pi/2$. Then for the radicand to be positive:

$$p_\psi^2 \ge 4MghI_1\cos\theta_0 \Rightarrow \omega_3 \ge \frac{2}{I_3}\sqrt{MghI_1\cos\theta_0}$$

Where we used $p_\psi = I_3\omega_3$. This is the condition for steady precession.

We want to find the precession for this steady inclination. Use (7.6) to obtain

$$\dot{\psi}_0 = \frac{\beta}{I_1 \sin^2\theta_0} \tag{7.9}$$

Since there are two values of β, we have two possible precessions, a fast precession $\dot{\phi}_{0(+)}$ and a slow precession $\dot{\phi}_{0(-)}$. With ω_3 large or p_ψ large (a fast top), the second term in β's radicand is small and we can expand to obtain

$$\begin{cases} \dot{\phi}_{0(+)} \approx \dfrac{I_3\omega_3}{I_1\cos\theta_0} \\ \dot{\phi}_{0(-)} \approx \dfrac{Mgh}{I_3\omega_3} \end{cases}$$

In the case $\theta_0 > \pi/2$, the radicand is always positive and there is no condition on ω_3. Then note the radical is greater than unity and the two cases of precessional angular velocity have *opposite* directions.

For the general case of $\theta_1 < \theta < \theta_2$, β in (7.9) may change signs as θ varies; whether it does depends on p_ψ, p_ϕ:

- $\dot{\phi}$ does to change sign: monotonic and periodic oscillation of θ between θ_1 and θ_2.
- $\dot{\phi}$ changes sign: the precessional angular velocity has opposite signs at θ_1 and θ_2. We get a looping-like motion of hte symmetry axis.
- In the case
$$(p_\phi - p_\psi \cos\theta)|_{\theta=\theta_1} = 0$$
and $\dot{\phi}(\theta_1) = 0$, $\dot{\theta}(\theta_1) = 0$: we get a cusplike motion.

This variation in θ is called **nutation**.

Remark: In addition to the analysis performed above, it is often useful to find the motion of vectors such as $\mathbf{L}$ and ω by using equation (6.3) and the relationship $\mathbf{L} = I\boldsymbol{\omega}$ which tells us that if, in a certain frame, $\boldsymbol{\omega}$ is constant, $\mathbf{L}$ must also be.

Chapter 8

Coupled Oscillations

8.1 Two Coupled Harmonic Oscillators

Consider two masses each attached to a spring with constant κ. Add a coupling spring between the two masses with constant κ_{12}. Restrict the motion to one dimension. Use the generalized coordinates x_1 and x_2, measured from the springs' equilibria.

The force on m_1 is $-\kappa x_1 - \kappa_{12}(x_1 - x_2)$ and the force on m_2 is $-\kappa x_2 - \kappa_{12}(x_2 - x_1)$. Consider $m_1 = m_2 = M$. Then

$$\begin{cases} M\ddot{x}_1 + (\kappa + \kappa_{12})x_1 - \kappa_{12}x_2 = 0 \\ M\ddot{x}_2 + (\kappa + \kappa_{12})x_2 - \kappa_{12}x_1 = 0 \end{cases}$$

We expect the motion to be oscillatory, so we attempt a solution of the from

$$\begin{cases} x_1(t) = B_1 e^{i\omega t} x_2(t) = B_2 e^{i\omega t} \end{cases}$$

The coefficients are allowed to be imaginary, we we will later have to take solely the real component of this motion. Substitute these solutions into the equations of motion:

$$-M\omega^2 B_1 e^{i\omega t} + (\kappa + \kappa_{12})B_1 e^{i\omega t} - \kappa_{12}B_2 e^{i\omega t} - M\omega^2 B_2 e^{i\omega t} + (\kappa + \kappa_{12})B_2 e^{i\omega t} - \kappa_{12}B_1 e^{i\omega t} \tag{8.1}$$

We collect terms and divide everything by the exponential:

$$\begin{cases} (\kappa + \kappa_{12} - M\omega^2)B_1 - \kappa_{12}B_2 = 0 \\ -\kappa_{12}B_1 + \left(\kappa + \kappa_{12} - M\omega^2\right)B_2 = 0 \end{cases}$$

We want a nontrivial solution to these linear equations: the coefficients B_i should not be zero. Then the corresponding matrix should be singular and the determinant should be

$$\begin{vmatrix} \kappa + \kappa_{12} - M\omega^2 & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} - M\omega^2 \end{vmatrix} = 0$$

The expansion of this determinant yields

$$\left(\kappa + \kappa_{12} - M\omega^2\right)^2 - \kappa_{12}^2 = 0$$

Let us solve for ω. Taking the square root of both sides:

$$\kappa + \kappa_{12} - M\omega^2 = \pm\kappa_{12}$$

$$\omega = \pm\sqrt{\frac{\kappa + \kappa_{12} \pm \kappa_{12}}{M}}$$

Then there are two of what we call **characteristic frequencies**, or **eigenfrequencies**.

$$\omega_1 = \pm\sqrt{\frac{\kappa + 2\kappa_{12}}{M}},\ \omega_2 = \pm\sqrt{\frac{\kappa}{M}} \tag{8.2}$$

The general solution is then

$$\begin{aligned} x_1(t) &= B_{11}^+ e^{i\omega_1 t} + B_{11}^- e^{-\omega_2 t} + B_{12}^+ e^{i\omega_2 t} + B_{12}^- e^{-i\omega_2 t} \\ x_2(t) &= B_{21}^+ e^{i\omega_1 t} + B_{21}^- e^{-\omega_2 t} + B_{22}^+ e^{i\omega_2 t} + B_{22}^- e^{-i\omega_2 t} \end{aligned}$$

These amplitudes are not independent. Substituting into (8.1) yields that when $\omega = \omega_1$ $B_1 1 = -B_{21}$ and when $\omega = \omega_2$ $B_{12} = B_{22}$. Thus

$$\begin{aligned} x_1(t) &= B_1^+ e^{i\omega_1 t} + B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} \\ x_2(t) &= -B_1^+ e^{i\omega_1 t} - B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} \end{aligned}$$

The normal coordinates (section 8.4) are

$$\begin{cases} \eta_1 = x_1 - x_2 \\ \eta_2 = x_1 + x_2 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2}(\eta_2 + \eta_1) \\ x_2 = \frac{1}{2}(\eta_2 - \eta_1) \end{cases}$$

Substitute these coordinate changes into the equations of motion to find

$$\begin{cases} M(\ddot{\eta}_1 + \ddot{\eta}_2) + (\kappa + 2\kappa_{12})\eta_1 + \kappa\eta_2 = 0 \\ M(\ddot{\eta}_1 - \ddot{\eta}_2) + (\kappa + 2\kappa_{12})\eta_1 - \kappa\eta_2 = 0 \end{cases}$$

Subtract these two equations to yield

$$\begin{cases} M\ddot{\eta}_1 + (\kappa + 2\kappa_{12})\eta_1 + \kappa\eta_2 = 0 \\ M\ddot{\eta}_2 + \kappa\eta_2 = 0 \end{cases}$$

Notice that these coordinates are *uncoupled*, and thus independent. We compute η_1 and η_2 using our previously obtained solutions:

$$\begin{cases} \eta_1(t) = C_1^+ e^{i\omega_1 t} + C_1^- e^{-\omega_1 t} \\ \eta_2(t) = C_2^+ e^{i\omega_2 t} + C_2^- e^{-i\omega_2 t} \end{cases}$$

with the same eigenfrequencies as before.

Let's impose some initial conditions to analyze the motion.

- If we let the initial positions to be perfectly opposite to another and the initial velocities to be perfectly opposite as well, we have

  $$\begin{aligned} x_1(0) &= -x_2(0) \\ \dot{x}_1(0) &= -\dot{x_2}(0) \\ \eta_2(0) &= 0 \\ \dot{\eta_2}(0) &= 0 \end{aligned}$$

  Which leads to $C_2^\pm = 0$ and $\eta_2 = 0$. The particles' positions will then always cancel each other out and the motion is the **antisymmetrical** mode of oscillation.

- If we let the initial positions to be the same relative to their equilibria and the initial velocities to be the same, we have

$$\begin{aligned} x_1(0) &= x_2(0) \\ \dot{x}_1(0) &= \dot{x_2}(0) \\ \eta_1(0) &= 0 \\ \dot{\eta_1}(0) &= 0 \end{aligned}$$

  Which leads to $C_1^{\pm} = 0$ and $\eta_1 = 0$. The particles' positions (relative to their equilibrium positions) will then always be the same, and this is the **symmetrical** mode of oscillation.

The general motion will be a linear combination of these symmetrical and antisymmetrical modes.

Note that if we hold m_2 fixed and allow m_1 to oscillate the frequency would be

$$\omega_0 = \sqrt{\frac{\kappa + \kappa_{12}}{M}}$$

The effect of the coupling is to split this natural frequency into a higher and a lower frequency. such that $\omega_1 > \omega_0 > \omega_2$.

Generally, for a system of n identical coupled oscillators, there will be $n/2$ characteristic frequencies greater than ω_0 and $n/2$ characteristic frequencies lesser than ω_0; for odd n, ω_0 itself is one of the characteristic frequencies.

8.2 Weak Coupling

In reference to the preceding section, explore the case when the force constant of the coupling spring is small compared to the oscillating springs: $\kappa_{12} \ll \kappa$. Using equation (8.2),

$$\omega_1 = \sqrt{\frac{\kappa + 2\kappa_{12}}{M}} = \sqrt{\frac{\kappa}{M}}\sqrt{1 + \frac{2\kappa_{12}}{\kappa}} = \sqrt{\frac{\kappa}{M}}\sqrt{1 + 4\varepsilon}, \text{ where } \varepsilon \equiv \frac{\kappa_{12}}{2\kappa} \ll 1$$

Which, after a Taylor expansion, becomes

$$\omega_1 \approx \sqrt{\frac{\kappa}{M}}(1 + 2\varepsilon)$$

With the other mass held fixed, the natural frequency of the oscillator is

$$\begin{aligned} \omega_0 = \sqrt{\frac{\kappa + \kappa_{12}}{M}} &\approx \sqrt{\frac{\kappa}{M}}(1 + \varepsilon) \\ \frac{\omega_0}{1 + \varepsilon} &\approx \sqrt{\frac{\kappa}{M}} \\ \frac{\omega_0(1 - \varepsilon)}{1 - \varepsilon^2} &\approx \sqrt{\frac{\kappa}{M}} \\ \omega_0(1 - \varepsilon) &\approx \sqrt{\frac{\kappa}{M}} \end{aligned}$$

since ε^2 is very small.

Then the characteristic frequencies are approximately

$$\begin{aligned}\omega_1 &\approx \sqrt{\kappa/M}(1+2\varepsilon)\\ &\approx \omega_0(1-\varepsilon)(1+2\varepsilon)\\ &\approx \omega_0(1+\varepsilon) \end{aligned}\tag{8.3}$$

$$\begin{aligned}\omega_2 &= \sqrt{\frac{\kappa}{M}}\\ &\approx \omega_0(1-\varepsilon)\end{aligned}\tag{8.4}$$

Now displace oscillator 1 a distance D and release from rest. The initial conditions become

$$\begin{aligned}x_1(0) &= D\\ x_2(0) &= 0\\ \dot{x}_1(0) &= 0\\ \dot{x}_2(0) &= 0\end{aligned}$$

We substitute these initial conditions into our solution in the preceding section to obtain

$$B_1^+ = B_1^- = B_2^+ = B_2^- \frac{D}{4}$$

and $x_1(t)$ becomes

$$\begin{aligned}x_1(t) &= \frac{D}{4}\left[\left(e^{i\omega_1 t}+e^{-i\omega_1 t}\right)+\left(e^{i\omega_2 t}+e^{-i\omega_2 t}\right)\right]\\ &= \frac{D}{2}(\cos\omega_1 t+\cos\omega_2 t)\\ &= D\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\end{aligned}$$

With (8.3) and (8.4), we have

$$\begin{cases}x_1(t) = (D\cos\varepsilon\omega_0 t)\cos\omega_0 t\\ x_2(t) = (D\sin\varepsilon\omega_0 t)\sin\omega_0 t\end{cases}$$

So the amplitudes of these two oscillations vary very slowly (due to the term $D\cos\varepsilon\omega_0 t$, with ε very small), but out of phase such that when one has maximum amplitude the other has zero amplitude. The two oscillators effectively slowly *exchange energy*. Given our initial conditions of x_1 displaced and x_2 at equilibrium, when $t = \pi/(2\varepsilon\omega_0)$, all energy has been transferred from m_1 to m_2.

8.3 General Problem

Consider a conservative system described in terms of generalized coordinates q_k and time t. Specify that there is a stable equilibrium state such that

$$\begin{aligned}q_k &= g_{k0}\\ \dot{q}_k &= 0\\ \ddot{q}_k &= 0\end{aligned}$$

Every nonzero term in $\mathrm{d}/\mathrm{d}t\,(\partial L/\partial\dot{q}_k)$ has to contain either $\dot{q}_k$ or $\ddot{q}_k$. Then the Euler-Lagrange equation only has derivatives of the form

$$\left.\frac{\partial L}{\partial q_k}\right|_0 = \left.\frac{\partial T}{\partial q_k}\right|_0 - \left.\frac{\partial U}{\partial q_k}\right|_0 = 0 \tag{8.5}$$

Assume relationship between generalized coordinates and rectangular coordinates does not involve time. Then the kinetic energy is

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k$$

Note that m_{jk} is allowed to be a function of the coordinates. .

This does not include q_k, so

$$\left.\frac{\partial U}{\partial q_k}\right|_0 = 0$$

Which means that the potential energy's derivative must likewise be zero, by equation (8.5):

$$\left.\frac{\partial U}{\partial q_k}\right|_0 = 0 \tag{8.6}$$

Specify that the generalized coordinates be measured from the equilibrium position such that $q_{k0} = 0$.

Expand potential energy about the equilibrium position:

$$U(q_1, q_2, \ldots, q_n) = U_0 + \sum_k \left.\frac{\partial U}{\partial q_k}\right|_0 + \frac{1}{2}\sum_{j,k}\left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_0 q_j q_k + \cdots$$

Second term vanishes by (8.6), and we may choose $U_0 = 0$. Restrict q_k to be small such that third-order terms and above are negligible. Then

$$U = \frac{1}{2}\sum_{j,k} A_{jk} q_j q_k$$

Where we've defined

$$A_{jk} \equiv \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_0$$

With order of integration irrelevant, $A_{jk} = A_{kj}$.

If m_{jk} and A_{jk} are diagonal, our q_k are the *normal coordinates* that supposedly simplify the motion.

We continue regardless of normal coordinates. From the Euler-Lagrange equation,

$$\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{q}_k} = 0$$

U is a function only of the coordinates and T only of the velocities. Then

$$\frac{\partial U}{\partial q_k} + \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial T}{\partial \dot{q}_k} = 0$$

and the derivatives are (since m)

$$\frac{\partial U}{\partial q_k} = \sum_j A_{jk} q_j$$
$$\frac{\partial T}{\partial \dot{q}_k} = \sum_j m_{jk}\dot{q}_j$$

And the equations of motion become

$$\sum_j (A_{jk} + m_{jk}\ddot{q}) = 0$$

Problems will often be given in this form. From these equations of motion you will easily be able to infer the matrices A and M.

Assume solutions of the form $q_j(t) = a_j e^{i(\omega t-\delta)}$. Substitute:

$$\sum_j \left(A_{jk} - \omega^2 m_{jk}\right) a_j = 0$$

where we've factored out the exponents. In matrix form:

$$\boxed{\left(A - \omega^2 M\right) a = 0}$$

For the solution to be non-trivial, the matrix applied to the vector a must be singular, and we have the condition

$$\left|A - \omega^2 M\right| = 0$$

Which yields an equation one can solve for ω. There will be, in general, n roots, which we may label ω_r^2. These are the so-called **characteristic frequencies** or **eigenfrequencies**. The amplitude vector a_r is then the eigenvector associated with ω_r^2. Since A is symmetric, these eigenvectors are mutually orthogonal. Then the solutions are a linear combination of the eigenfrequencies:

$$\boxed{q_j(t) = \sum_r a_{jr} e^{i(\omega_r t - \delta_r)}}$$

8.3.1 Alternate Derivation of Matrix Form

The equations of motion of a coupled oscillation system may be expressed in matrix form

$$M\ddot{x} = -Ax$$

Where M is the matrix of the masses of the system, x is a vector $(x(t)_1, x(t)_2, \ldots, x(t)_n)$, and A holds the coefficients of the oscillators. For example, in section 8.1, we would have

$$M = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix},\ A = \begin{bmatrix} \kappa + \kappa_{12} & -\kappa \\ -\kappa & \kappa + \kappa_{12} \end{bmatrix}\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We now replace each x_i in the x vector with $x_i = a_i e^{i(\omega t+\phi)}$. This means we can rewrite

$$z = e^{i(\omega t+\phi)} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

since this presupposes every element of the system is oscillating in the same phase and frequency, these are the normal coordinates! Our matrix equation becomes

$$M\ddot{z} = -Az$$

where, if we differentiate,$\ddot{z} = -\omega^2 z$ such that

$$M\omega^2 z = Az$$

Recall that each z has an exponential in front, cancel these exponentials to be left solely with the vector of amplitudes a_i:

$$\omega^2 Ma = Aa$$

Move everything to the right hand side my multiplying both sides by M^{-1}

$$\omega^2 a = M^{-1}Aa$$
$$(M^{-1}A - \omega^2 I)a = 0$$

It follows that the eigenvalue is ω^2 and a is the eigenvector of $M^{-1}A$. We can then multiply both sides by M to eliminate $M{-}1$:

$$(A - \omega^2 M)a = 0$$

Once again, for this to have a non-trivial solution, we require

$$\boxed{\left|A - \omega^2 M\right| = 0}$$

8.4 Normal Coordinates

We've shown the eigenvectors a_i are orthogonal. We may normalize them to form an orthonormal set, but then we must introduce a constant scale factor α_r:

$$q_j(t) = \sum_r \alpha_r a_{jr} e^{i(\omega_r t - \delta_r)}$$

where $a_j r$ are the normalized components of the eigenvector r. Simplify the notation by incorporating the phase into the coefficient out front:

$$q_j(t) = \sum_r \beta_r a_{jr} e^{i\omega_r t}$$

Define the quantity

$$\boxed{\eta_r(t) \equiv \beta_r e^{i\omega_1 t}}$$

Such that

$$\boxed{q_j(t) = \sum_r a_{jr}\eta_r(t)}$$

The η_r are, by definition, undergoing motion in only one frequency. These are the so-called **normal coordinates**. They satisfy equations of the form

$$\ddot{\eta}_r + \omega_r^2 \eta_r = 0$$

There are n independent equations of the above form, completely separable. The orthonormality condition placed on the vector a is

$$\sum_{j,k} m_{jk} a_{jr} a_{ks} = \delta_{rs}$$