PHYS0070 - Analytical Mechanics

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Contents

Introduction

> I keep the subject constantly before me, and wait 'till the first dawnings open slowly, by little and little, into a full and clear light.
>
> Isaac Newton

How come a spinning top doesn't fall? Why wrap dental floss around your fingers before you clean your teeth? How can NASA engineers gravity slingshot a space probe to travel fast enough to escape our solar system? Analytical Mechanics can empower you to answer these questions and more regarding phenomena governed by Newton's Laws of Motion. It will ask you to draw liberally on your math skills as you learn new means to apply them. After all, to paraphrase Galileo: "Mathematics is the language in which the universe was written."

- Course description

Chapter 1

Vectors and Kinematics

1.1 Vectors

We can denote vectors one of two ways. Firstly, we can express them component-wise:

$$\begin{cases} F_x = ma_x \\ F_y = ma_y \\ F_z = ma_z \end{cases}$$

But we can also express them in either bolded letters or letters with arrows over them:

$$\mathbf{F} = m\mathbf{a} \text{ or } \vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

Recall the three important vector operations:

$$\begin{aligned} &\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta \\ &|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin\theta \\ &\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \end{aligned}$$

And at last, recall the basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, which are unit vectors in the positive x, positive y, and positive z directions respectively.

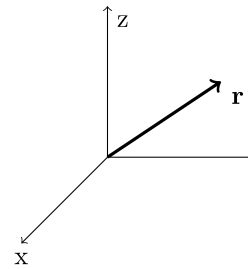
1.2 The Position Vector

In order to describe an object's position more expressively, we use a vector we call the *position vector*. This is a vector drawn from an arbitrary origin (which depends on the reference frame of choice) to the particle itself. That is, we turn the 3-tuple (x, y, z) into $\mathbf{r}(x, y, z)$, where

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Then the displacement vector $\mathbf{s}$ has

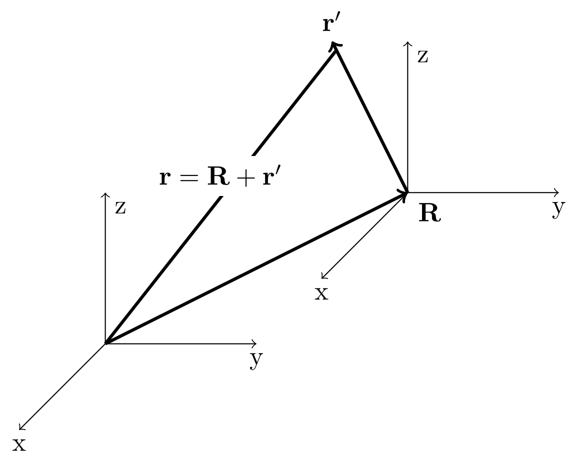
$$\begin{cases} s_x = x_2 - x_1 \\ s_y = y_2 - y_1 \\ s_z = z_2 - z_1 \end{cases}$$



$\mathbf{s}$ contains information about $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to each other.

If we have the position of an object relative to a coordinate system x', y', z', and want to describe the motion of that object relative to another coordinate system (perhaps one that is moving, or oriented differently), we can just draw a vector $\mathbf{R}$ from the origin x, y, z to the origin x', y', z', then add that to the vector from the origin of x', y', z' to the object (this is just the position vector $\mathbf{r}'$). That is:

$$\mathbf{r} = \mathbf{R} + \mathbf{r}'$$



1.3 Velocity and Acceleration Vectors

Recall that average velocity is

$$\overline{\mathbf{v}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

Then the instantaneous velocity is

$$v = \lim_{\Delta t \to 0} = \frac{x(t + \Delta t) - x(t)}{\Delta t} \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$$

Note that we typically reserve the notation $\dot{x}$ for the derivative with respect to *time*.

We can apply the same notion to vectors. As a change $\Delta\mathbf{r} = \mathbf{r}_2 = \mathbf{r}_1$ becomes infinitesimally small, it approaches the vector tangent to the motion (see figure 1.1). This is equivalent to the operation

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= \mathbf{v} \\
&= \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{k}}
\end{aligned}
$$

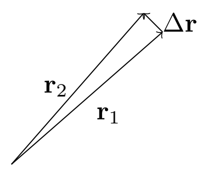


Figure 1.1: Vector differentiation

By the same process, we obtain $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$.

Note that

$$
\begin{aligned}
\frac{\Delta\mathbf{r}}{\Delta t} \approx \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \Rightarrow \Delta\mathbf{r} &\approx \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\Delta t \\
\Delta\mathbf{r} &\approx \mathbf{v}\Delta t
\end{aligned}
$$

Where $\Delta\mathbf{r}$ is a small (but not infinitely small) change in $\mathbf{r}$.

1.3.1 Why Can We Differentiate Each Component?

Our expression for the acceleration vector is a bit misleading. Whether we can simply differentiate each component actually depends on the basis vectors we choose. Say we have general basis vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. We must treat these as functions of time, as they're not necessarily constant. By the product rule:

$$
\frac{\mathrm{d}}{\mathrm{d}t}(x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 + z\hat{\mathbf{e}}_3) = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{e}}_1 + x\frac{\mathrm{d}\hat{\mathbf{e}}_1}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{e}}_2 + y\frac{\mathrm{d}\hat{\mathbf{e}}_2}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{e}}_3 + z\frac{\mathrm{d}\hat{\mathbf{e}}_3}{\mathrm{d}t}
$$

In the case of section 1.3, we chose basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, which are *constant*. Then, the derivatives $\frac{\mathrm{d}\hat{\mathbf{i}}}{\mathrm{d}t}, \frac{\mathrm{d}\hat{\mathbf{j}}}{\mathrm{d}t}, \frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t}$ are all *zero*. This resolves the above equation to

$$
\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{e}}_1 + x(0) + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{e}}_2 + y(0) + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{e}}_3 + z(0) = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{e}}_1 + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{e}}_2 + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{e}}_3
$$

We shall see that this won't always be the case when we chose different basis vectors.

1.4 Formal Solutions to Kinematic Equations

Start in one dimension, and let $a = a(t)$ vary with time for generality. To obtain velocity at a time t_1,

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} = a(t) &\Rightarrow \mathrm{d}v = a(t)\,\mathrm{d}t \\ \int_{v_0}^{v+1} &= \int_{t_0}^{t_1} a(t)\,\mathrm{d}t \\ v(t_1) - v(t_0) &= \int_{t_0}^{t_1} a(t)\,\mathrm{d}t \\ v(t_1) &= \int_{t_0}^{t_1} a(t)\,\mathrm{d}t + v_0\end{aligned}$$

Since vectors can be integrated component-wise, this is equivalent to

$$\boxed{\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{a}(t')\,\mathrm{d}t'}$$

Where we just turned this into an indefinite integral in order to obtain $\mathbf{v}$ at any chosen time t. Displacement is no different:

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(t')\,\mathrm{d}t'}$$

Note that it is convention to let $t_0 = 0$.

With uniform acceleration $\mathbf{a} =$ constant, and letting $t_0 = 0$

$$\begin{aligned}v(t) &= \int_0^t a\,\mathrm{d}t' + v_0 \\ v(t) &= at \\ x(t) &= \int_0^t v(t)\,\mathrm{d}t' + x_0 \\ x(t) &= \int_0^t at + v_0\,\mathrm{d}t + x_0 \\ x(t) &= v_0 t + \frac{1}{2}at^2 + x_0\end{aligned}$$

Which is an important kinematic equation:

$$\boxed{x(t) = x_0 + v_0 t + \frac{1}{2}at^2}$$

1.5 Polar Coordinates

Sometimes a motion is defined by its radius (distance from origin) at a certain angle. Then we use polar coordinates, which are defined to be

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \theta &= \arcsin\left(\frac{y}{x}\right) \end{aligned}$$

Example Then, describing uniform circular motion in the xy plane with basis vectors $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}$:

$$\mathbf{r} = r(\cos\theta\hat{\mathbf{\imath}} + \cos\theta\hat{\mathbf{\jmath}})$$

1.5.1 Polar Basis Vectors

Notice that the expression in the above example is somewhat complicated—these expressions can get even more complicated once we consider motions in ellipses and whatnot. Then we would like to express position, velocity, and acceleration vectors in terms of bases that are always pointing *tangentially* to the motion and *radially* for the origin.

We define the basis vector $\hat{\mathbf{r}}$ to *always* point radially to the object (so, in the direction of $\mathbf{r}$). Then we define $\hat{\theta}$ to point perpendicular to $\hat{\mathbf{r}}$. In other words,

$$\boxed{\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{\imath}} + \sin\theta\hat{\mathbf{\jmath}}}$$

$$\boxed{\hat{\theta} = -\sin\theta\hat{\mathbf{\imath}} + \cos\theta\hat{\mathbf{\jmath}}}$$

Before we proceed, note that by definition $\hat{\mathbf{r}}$ depends on time. The direction along which it points is time-dependent. Then, we cannot simply differentiate our position vector component-wise—we must treat the basis vector as a function of time and apply the product rule. Let's then find $\dot{\hat{r}}$.

$$\begin{aligned} \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} &= -\dot{\theta}\sin\theta\hat{\mathbf{\imath}} + \dot{\theta}\cos\theta\hat{\mathbf{\jmath}} \\ &= \dot{\theta}\hat{\theta} \\ \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} &= -\dot{\theta}\cos\theta\hat{\mathbf{\imath}} - \dot{\theta}\sin\theta\hat{\mathbf{\jmath}} \\ &= -\dot{\theta}\hat{\mathbf{r}} \end{aligned}$$

Then the position vector is just $\hat{\mathbf{r}}$ scaled up by the distance from the origin r:

$$\boxed{\mathbf{r} = r\hat{\mathbf{r}}}$$

Which means that, by the product rule:

$$\begin{aligned} \mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= (r\hat{\mathbf{r}})' \\ &= \dot{r}\hat{\mathbf{r}} + r\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} \end{aligned}$$

And at last

$$\begin{aligned}\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{\mathbf{r}} + \theta r\hat{\theta}\right) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\theta} + \ddot{\theta}r\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} - r\dot{\theta}\hat{\mathbf{r}} \\ &= \left(\ddot{r} - \dot{\theta}^2 r\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}\end{aligned}$$

In summary:

$$\boxed{\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}}$$

$$\boxed{\mathbf{a} = \left(\ddot{r} - \dot{\theta}^2 r\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}}$$

Note that here $\ddot{r}$ does *not* represent the total radial acceleration. Instead, it represents the derivative of the change in the *radius* of the motion. If the radius of the motion is changing $\ddot{r} \neq 0$, but for situations such as uniform circular motion (where the radius is constant), $\ddot{r} = 0$, and $\dot{r} = 0$.

1.5.2 Alternative Derivation

We illustrate an alternative derivation of the derivative of $\hat{\mathbf{r}}$

$\hat{\mathbf{r}}$ is constant in length, so it only changes in direction. This means $\hat{\mathbf{r}}$'s derivative is a vector that points perpendicular to it. Then note that a small change in $\hat{\mathbf{r}}$ is effectively an arc length (see figure 1.2). This means that the magnitude of $\Delta\hat{\mathbf{r}}$ is

$$|\Delta\hat{\mathbf{r}}| = |r|\theta$$

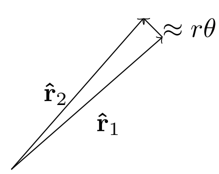


Figure 1.2

But since $\hat{\mathbf{r}}$ is a basis vector, this is just $\left|\dot{\hat{r}}\right| = \theta$. Then we take $\theta = \omega\Delta t$ so that we can take $\lim_{\Delta t \to 0}$ to obtain the derivative:

$$\begin{aligned}|\Delta\hat{\mathbf{r}}| &= \omega\Delta t \\ \left|\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}\right| &= \omega \\ &= \dot{\theta}\end{aligned}$$

So,

$$\boxed{\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \dot{\theta}\hat{\theta}}$$

Chapter 2

Newton's Laws

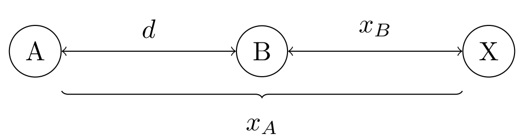
2.1 Newton's Laws

1. Inertial systems exist.
2. $\mathbf{F} = m\mathbf{a}$.
3. $\mathbf{F}_a = -\mathbf{F}_b$

Law one is just the statement that objects in motion stay in motion. Law two needs not be elaborated on. Law three is the statement that every force has an equal and opposite reaction.

2.2 Non-Inertial Reference Frames

Newton's second law only holds for inertial reference frames. We examine the case of an accelerating reference frame.



Let the distance between an observer A and an observer B be d. Let the distance between B and an object x be x_B, and the distance between that object and A x_A. Take B to be accelerating. Then it follows that

$$x_A(t) = d(t) + x_B(t)$$

Take A to be an inertial observer. Then if we differentiate both sides twice with respect to time and multiply everything by the mass of object x,

$$M\ddot{x}_A = M\ddot{x}_B + M\ddot{d}$$

Now, since we know A is an inertial observer, we know the actual force on x is indeed $M\ddot{x}_A$. But this is evidently equal to the force on x from the perspective of B, $M\ddot{x}_B$, *plus* the acceleration of B with respect to A times the mass M. This second term is the so-called **fictitious force**, and, solving for the force on x from the perspective of B:

$$M\ddot{x}_B = M\ddot{x}_A - M\ddot{d}$$

Once again, since we know $M\ddot{x}_A$ is the *actual* force on x, and since we can easily perform this component-wise on vectors, we obtain that

$$\boxed{\mathbf{F}_{\text{net}} = \mathbf{F}_{\text{actual}} - \mathbf{F}_{\text{fict}}}$$

$$\boxed{\mathbf{F}_{\text{fict}} = M\ddot{\mathbf{R}}}$$

Where $\mathbf{R}$ is the position of the non-inertial reference frame relative to an inertial reference frame.

2.3 Applying Newton's Laws

2.3.1 The Kleppner and Kolenkow Approach

1. Isolate the masses, treat them as particles.
2. Draw force diagrams for each mass.
3. Show coordinate system on force diagrams.
4. Write equations of motion, broken into x, y, z components.
5. Write any constraint equations.
6. Solve!

Some additional tips:

- For pulley systems, take the total length of the string set this equal to the sums of the portions of strings along pulleys and heights/displacements of particles of system, then differentiate twice for acceleration constraint. Since the total length should be constant, this should be some expression equal to zero.
- (Referring to example 2.10) Watch out when dividing by trigonometric functions with varying angles, as these values could be zero at certain angles.

2.4 Dynamics in Polar Coordinates

When dealing with rotational motion, and forces constant along the radial direction, we typically want to use polar basis vectors to simplify the equations of motion.

Example We take the "whirling block" example from the textbook. Mass A spins around on a table, then B is dropped and pulls A inward. Our initial conditions are a radius r_0 and angular velocity ω_0. Take T to be the tension along the string. Then the equations of motion are

$$\begin{aligned} F_{\text{A, net, radial}} &= -T = m_A(\ddot{r} - r\dot{\theta}^2) \\ F_{\text{A, net, angular}} &= 0 = m_A(r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ F_{\text{B, net}} &= m_B g - T = M_B \ddot{z} \end{aligned}$$

With constant string length, our acceleration constraint is

$$\ddot{r} = -\ddot{z}$$

Now we solve for B's acceleration. Using the above equation

$$\begin{aligned}
m_B g + m_A\left(\ddot{r} - r\ddot{\theta}^2\right) &= m_B \ddot{z} \\
m_B g - m_A \ddot{z} - m_A r\dot{\theta}^2 &= M_B \ddot{z} \\
m_B g - r\dot{\theta}^2 m_A &= (m_A + m_B)\ddot{z} \\
\frac{m_B g - r\dot{\theta}^2 m_A}{m_A + m_B} &= \ddot{z}
\end{aligned}$$

Then, plugging in the initial conditions:

$$\ddot{z} = \frac{m_B g - r_0 \omega_0^2 m_A}{m_A + m_B}$$

Chapter 3

Forces and Equations of Motion

3.1 Fundamental Forces

Fundamental forces are forces that cannot be expressed in simpler terms:

- Gravitational
- Electromagnetic
- Weak interactions
- Strong interactions

3.2 Gravitational Force

The gravitational force on mass a due to mass b is

$$\boxed{\mathbf{F} = -\frac{Gm_a m_b}{r^2}\hat{\mathbf{r}}_{a,b}}$$

Where $\hat{\mathbf{r}}_{a,b}$ is the vector pointing from a to b.

It follows that

$$\mathbf{F}_{a,b} = -\mathbf{F}_{b,a}$$

Notice that the quantity m_a used as a proportion for the gravitational force need not be the same mass derived from $F = ma$; this is in fact only true because of the **equivalence principle**.

3.3 Phenomenological Forces

Phenomenological physics aims to fit quantitative models to observable phenomena without explicitly investigating the (often complex) causes behind said phenomena. This is in contrast to fundamental theories of physics, which attempts to provide a foundation for and explain larger-scale models by starting with the simplest and most reduced building blocks (for example, gravity is a fundamental theory, as it is not "expressible in simpler terms"). Phenomenology is an *ad hoc* approach which disregards the fundamentals in favor of more convenient and accessible models which may be more readily used.

Friction is an excellent example of a phenomenological approach to physics. Where one could, in principle, calculate the effect of every individual electromagnetic interaction between atoms of two surfaces, taking into account the microscopic ruggedness of said surfaces in order to determine the amount of force opposite the motion of the objects, such a calculation would be unnecessarily formidable. Instead, physicists choose to simplify such phenomena at a macroscopic level by measuring the effects of friction and defining convenient quantities such as the coefficient of friction.

3.3.1 Tension

Strings are composed of microscopic sections interacting via contact forces. In the absence of external forces, and considering a string to be massless, we obtain that the *tension force is equal in every location of the string*. Then we are free to choose any point along the string and draw force vectors pointing along the string with equal magnitudes.

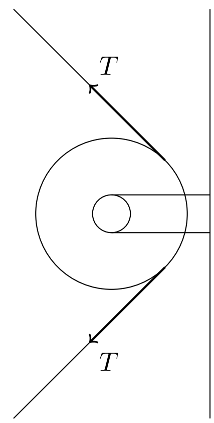


Figure 3.1: Tension force acting along a string/rope on a pulley.

For example, in figure 3.1, despite the rope lying on a pulley and pointing at angled directions, the tension force is the same.

3.3.2 Normal Force

The normal force is a macroscopic phenomenological force which is due to microscopic electromagnetic interactions between objects. Intuitively, it is the force that prevents objects from accelerating through solid surfaces; it is therefore, in most cases, *at least* equal and opposite to the force component perpendicular to the surface.

3.3.3 Friction

Friction opposes the relative motion of bodies in contact. There are two types of friction:

- **Static friction** prevents bodies from moving if they are stationary. It is exactly opposite to an applied force which would make the object move in absence of friction. (It must vary because, if it was constant, we could push with a smaller force and the object would have a nonzero net force, causing an acceleration in the direction of friction.) Static friction is then equal and opposite to the object's motion, until a certain threshold of static friction force is reached and the object starts moving.

  This threshold is commonly described by the normal force N and a given constant μ_s specific to the surfaces of the objects in question:

  $$\boxed{F_{\text{fr, static}} = \mu_s N}$$

If the applied force is greater than this quantity, the object begins moving. This is occasionally referred to as the *limiting friction*.

- **Kinetic friction** acts opposite to a body *in motion*. It is likewise described by a given coefficient μ_k (which depends on the materials of the bodies in question) and the normal force:

$$\boxed{F_{\text{fr, kinetic}} = \mu_k N}$$

In contrast to static friction, kinetic friction is constant—it does not depend on the magnitude of the applied forces.

3.4 Differential Equations

Differential equations are equations involving derivatives of functions. The unknown is the function of which the derivatives are taken, and we "solve" a differential equation by finding that function.

Differential equations describe how a system changes in a small period of time. Solving a differential equation allows us to determine the state of the system at every possible point in time (excluding solutions that are not defined on the entire domain).

A convenient way of constructing differential equations is by considering a quantity at two different points in time or in space, $t + \Delta t$ and t, or $x + \Delta x$ and x. Then we subtract

$$\begin{aligned} F(t + \Delta t) - F(t) &= F + \Delta F - F \\ &= \Delta F \\ t + \Delta t - t &= \Delta t \end{aligned}$$

At this point, if we are able to find an expression for

$$\frac{\Delta F}{\Delta t}$$

we can "take the limit" on both sides and arrive at

$$\frac{\mathrm{d}T}{\mathrm{d}t}$$

Then, we may solve the differential equation. In this course, we will typically only encounter "separable" differential equation, meaning that we can multiply both sides by $\mathrm{d}t$ and integrate.

Note: When finding $\frac{\Delta F}{\Delta t}$, we might encounter some product of the form $\Delta F \cdot \Delta t$. You can imagine this to be a very small quantity that goes to zero. So any $\Delta F \Delta t = 0$. (There's a rigorous mathematical proof behind this but it looks a bit complicated so I omit it here.)

Example This is exercise 4.13 from the text.

Let V be the velocity of the freight car, v_0 be the velocity of the sand relative to the freight car, and M_0 be the mass of the freight car. Consider the momentum, the velocity, and the mass at t and $t + \Delta t$.

$$\begin{aligned} P(t) &= V(t)M(t) + (v_0 + V(t))\Delta m \\ P(t + \Delta t) &= V(t + \Delta t)(M(t) + \Delta m) \end{aligned}$$

We subtract these quantities to find ΔP

$$\begin{aligned} \Delta P &= V(t + \Delta t)(M(t) + \Delta m) - V(t)M(t) - (v_0 + V(t))\Delta m \\ \Delta P &= \Delta V M(t) + \Delta V \Delta m - v_0 \Delta m \end{aligned}$$

$\Delta V \Delta m$ is very small and thus negligible, and $\mathrm{d}P/\mathrm{d}t$ is zero for there are no external forces.

$$
\begin{aligned}
0 &= \frac{\mathrm{d}V}{\mathrm{d}t} M(t) - v_0 \frac{\mathrm{d}M(t)}{\mathrm{d}t} \\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{v_0}{M(t)} \frac{\mathrm{d}M(t)}{\mathrm{d}t} \\
\mathrm{d}V &= \frac{v_0}{M(t)} \mathrm{d}M(t) \\
V(t) &= v_0 \ln |M(t)| - v_0 \ln |M_0| \\
V(t) &= \ln \left| \frac{M(t)}{M_0} \right|
\end{aligned}
$$

Notice that $M(t) = \frac{\mathrm{d}m}{\mathrm{d}t} t + M_0$, where we are given $\mathrm{d}m/\mathrm{d}t = 10\,\mathrm{kg/s}$. Then

$$
\begin{aligned}
V(t) &= v_0 \ln \left| \frac{\mathrm{d}M}{\mathrm{d}t} \frac{t}{M_0} + 1 \right| \\
V(t) &= (5\,\mathrm{m/s}) \ln \left| 10\,\mathrm{kg/s} \cdot \frac{100\,\mathrm{s}}{1000\,\mathrm{kg}} + 1 \right| \\
&= 2.02733\,\mathrm{m/s}
\end{aligned}
$$

3.5 Simple Harmonic Motion

Very often in physics, we encounter forces proportional to a displacement. Such forces are of the form

$$F = -kx$$

Where k is some given constant. Using Newton's second law, we obtain the equation of motion

$$m\ddot{x} = -kx$$

Putting everything in one side and dividing by m, we obtain

$$\boxed{\ddot{x} + \frac{k}{m} x = 0} \tag{3.1}$$

which is a *second order differential equation*. We commonly write $\omega^2 = k/m$ for reasons that will become clear.

Solutions to such second order differential equations are of the form

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{3.2}$$

Where A and B are constants which can be solved for if we have more information. λ_1 and λ_2 are given by

$$\lambda = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where a, b, and c are the coefficients of $\ddot{x}$, $\dot{x}$ and x respectively. In the case of equation (3.1), we have $a = 1$, $b = 0$, $c = k/m$. Then

$$
\begin{aligned}
\lambda &= \pm \frac{\sqrt{-4\frac{k}{m}}}{2} \\
&= \pm i \sqrt{\frac{k}{m}}
\end{aligned}
$$

Note the imaginary number i due to the expression in the square root being negative. Substituting into equation (3.2), we have

$$x(t) = Ae^{i\sqrt{k/m}t} + Be^{-i\sqrt{k/m}t}$$

Now we want to get rid of that i. Use Euler's identity, which states $e^{ix} = \cos x + i\sin x$:

$$\begin{aligned} x(t) &= A\Big(\cos\sqrt{k/m}t + i\sin\sqrt{k/m}t\Big) + B\Big(\cos\Big(-\sqrt{k/m}t\Big) + \sin\Big(-\sqrt{k/m}t\Big)\Big) \\ x(t) &= A\cos\sqrt{k/m}t + Ai\sin\sqrt{k/m}x + B\cos\Big(-\sqrt{k/m}x\Big) + B\sin\Big(-\sqrt{k/m}t\Big) \end{aligned}$$

Use the trig identity $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$:

$$\begin{aligned} x(t) &= A\cos\sqrt{k/m}t + Ai\sin\sqrt{k/m}t + B\cos\sqrt{k/m}t + Bi\sin\sqrt{k/m}t \\ &= (A+B)\cos\sqrt{k/m}t + (A+B)i\sin\sqrt{k/m}t \end{aligned}$$

Since A and B are just constants which we will solve for later, we can call $A + B = C_1$, and we can combine the imaginary number i into $A + B$ to obtain C_2:

$$x(t) = C_1\cos\sqrt{k/m}x + C_2\sin\sqrt{k/m}x$$

Which is a periodic motion with angular velocity $\sqrt{k/m}$. This is why we called $\sqrt{k/m} = \omega$: if we consider this to be the x coordinate of a particle moving in uniform circular motion, ω gives the angular velocity!

So, for any equation of motion of the form

$$\ddot{x} + ax = 0$$

The solution is

$$\boxed{x(t) = C_1\cos\omega t + C_2\sin\omega t}$$

Where $\omega = \sqrt{a}$, and where, with some initial value such as $x(0) = 0$, $\dot{x}(0) = 0$ we can solve for C_1 and C_2.

3.5.1 Small Angle Approximations

Occasionally, we might want to analyze a motion only for a small change in x or in θ. In that case, the following identities come in handy.

We may perform a Taylor series expansion of $\sin\theta$ to find

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

Generally, when we take small θ, we only keep the *first order term*, ignoring terms with θ^2 or above. Then

$$\boxed{\sin\theta \approx \theta \text{ for small } \theta}$$

And likewise

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\boxed{\cos\theta \approx 1 \text{ for small } \theta}$$

And any higher order θ become

$$\begin{aligned} &\theta^2 \approx 0 \\ &\dot{\theta}^2 \approx 0 \\ &\ddot{\theta}^2 \approx 0 \text{ for small } \theta \end{aligned}$$

3.6 Viscosity

A body moving through a fluid experiences a force directed opposite to its motion and proportional to its velocity. That is,

$$\mathbf{F}_v = -c\mathbf{v}$$

where c is a constant that depends on the body's geometry and the fluid itself.

The equation of motion is them

$$m\ddot{x} = -c\dot{x}$$

This brings us to an important technique: when we have an equation of motion with a force proportional to $\dot{x} = v$, it is easiest to solve it by treating v as the unknown and writing the equation in the Form

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -cv$$

We've turned a second order differential equation (with a double time derivative) into a first order, *separable* differential equation. Solving:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{c}{m}v$$
$$-\frac{m}{c}\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}t} = 1$$

We denote $m/c = \tau$. Integrating

$$\int_{v_0}^{v} -\tau \cdot \frac{1}{v}\,\mathrm{d}v = t$$
$$-\tau \ln v + \tau \ln v_0 = t$$
$$\ln\frac{v}{v_0} = -\frac{t}{\tau}$$

Raise both sides to the power of e to get rid of the logarithm,

$$\frac{v}{v_0} = e^{-\frac{t}{\tau}}$$

Giving us the velocity

$$\boxed{v(t) = v_0 e^{-\frac{t}{\tau}}}$$

We may integrate this again to obtain the position with respect to time:

$$x(t) = -v_0\tau e^{-\frac{t}{\tau}} + x_0$$

Chapter 4

Momentum

4.1 Dynamics of Particle Systems

Recall $\mathbf{F} = M\mathbf{a}$. We write this

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}$$

where $\mathbf{P} \equiv M\mathbf{a}$ is a quantity called the **momentum**.

Consider N particles with masses $m_i, \ldots, m_N$. The position of the i-th particle is $\mathbf{r}_i$ and the force on it is $\mathbf{f}_i$. Then Newton's second law gives

$$\mathbf{f}_i = \frac{\mathrm{d}\frac{\mathrm{d}}{\mathrm{d}p}_i}{\mathrm{d}t}$$

We split the force on particle i into two terms: $\mathbf{f}_i^{\text{int}} + \mathbf{f}_i^{\text{ext}}$. The internal force is the force due to all other particles *in the system*. The external force is due to forces *outside* the system. Then

$$\mathbf{f}_i^{\text{int}} + \mathbf{f}_i^{\text{ext}} = \frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t}$$

If we are considering the entire system, we just add the forces and momenta of every individual particle. Using summation notation:

$$\sum_{i=1}^{N} \mathbf{f}_i^{\text{int}} + \sum_{i=1}^{N} \mathbf{f}_i^{\text{ext}} = \sum_{i=1}^{N} \frac{\mathrm{d}\mathbf{p}_j}{\mathrm{d}t}$$

According to Newton's third law, $\mathbf{f}_i^{\text{int}} = -\mathbf{f}_j^{\text{int}}$ for two particles i and j. Then, the sum of the internal forces becomes zero! We then have

$$\sum_{i=1}^{N} \mathbf{f}_i^{\text{ext}} = \sum_{i=1}^{N} \frac{\mathrm{d}\mathbf{p}_j}{\mathrm{d}t}$$

We may pull the $\mathrm{d}/\mathrm{d}t$ out of the sum on the right hand side. Then, we call the sum of the momenta $\mathbf{P}$, and the sum of the external forces $\mathbf{F}_{\text{ext}}$.

$$\boxed{\mathbf{F}_{\text{ext}} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}}$$

And, with no external forces applied, we have

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = 0$$

Meaning that the total momentum of the system does not change with time, and it is *conserved.* This means we have the relationship

$$\sum_i m_{i0} v_{i0} = \sum_i m_{i1} v_{i1}$$

Where $i0$ denotes the i-th particle at time zero, and $i1$ is the i-th particle at time 1.

4.2 Center of Mass

With $\mathbf{F} = \mathrm{d}\mathbf{P}/\mathrm{d}t$, we have a result identical to a single particle, but generalized to an entire system. We may write

$$\mathbf{F} = M\ddot{\mathbf{R}}$$

Where M is the total mass of the system, and $\mathbf{R}$ is the **center of mass** vector. With

$$M\ddot{\mathbf{R}} = \frac{\mathrm{d}\frac{\mathrm{d}}{\mathrm{d}P}}{\mathrm{d}t} = \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i$$

Integrating twice, we have that

$$\boxed{\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i}$$

The center of mass vector is then a weighted position of the system. We sum the position of each particle, weighted by the mass of that particle m_i, then divide by the total mass M.

Note that the center of mass doesn't really capture the orientation of the body. We may only use it to describe translational motion. Rotational motion of a body is more complicated and will be described later.

4.2.1 Extended Body

We may divide a continuous body into N discrete elements. $\mathbf{r}_i$ then denotes the position of the i-th element, and m_i its mass.

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i$$

Let the size of each mass element become very small, and the number of elements go to infinity. This is the definition of an integral, and the sum becomes:

$$\mathbf{R} = \lim_{N\to\infty}\sum_{i=1}^{N} m_i \mathbf{r}_i = \int_V r\,\mathrm{d}m$$

Where $\int_V$ denotes an integral over the entire volume of the region.

The main conceptual difficulty in finding the center of mass of a body is determining $\mathrm{d}m$. In the case of a long rod of uniform mass, for example, we can use the fact that a fraction of the rod's length will be that same fraction of the rod's mass:

$$\frac{\Delta m}{M} = \frac{\Delta l}{L}$$

Taking the limit,

$$\frac{\mathrm{d}m}{M} = \frac{\mathrm{d}x}{L} \Rightarrow \mathrm{d}m = \frac{M}{L}\,\mathrm{d}x$$

And we may integrate from one end of the rod to the next using $\mathrm{d}x$, the position along the rod.

More generally, however, we observe that the density δ is given by

$$\delta = \frac{M}{V}$$

So that

$$\delta V = M \Rightarrow \delta \, \mathrm{d}V = \mathrm{d}M$$

Letting us express the center of mass as

$$\mathbf{R} = \frac{1}{M} \int_V \mathbf{r} \delta \, \mathrm{d}V$$

Where we may integrate each component of $\mathbf{r}$ individually:

$$\mathbf{R} = \frac{1}{M} \int_V x\delta \, \mathrm{d}V \, \hat{\mathbf{i}} + \frac{1}{M} \int_V y\delta \, \mathrm{d}V \, \hat{\mathbf{j}} + \frac{1}{M} \int_V z\delta \, \mathrm{d}V \, \hat{\mathbf{k}}$$

δ can even be a function, in the case that the mass is not uniformly distributed. $\mathrm{d}V$ depends on how many dimensions we're working in, but is usually easily determined. For example, in three dimensions, this integral becomes

$$\mathbf{R} = \frac{1}{M} \iiint_V \mathbf{r} \delta \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

Note that the center of mass of a **system of extended bodies** may be determined by finding the weighted average of the center of masses of each extended body:

$$\mathbf{R} = \frac{M_1 \mathbf{R}_1 + M_2 \mathbf{R}_2}{M_1 + M_2}$$

Where

$$\mathbf{R}_1 = \frac{1}{M_1} \int \mathbf{r}_1 \, \mathrm{d}m$$
$$\mathbf{R}_2 = \frac{1}{M} \int \mathbf{r}_2 \, \mathrm{d}m$$

4.3 Center of Mass Coordinates

Often a problem can be simplified by a clever choice of coordinates. For example, it is useful to define a coordinate system with the origin at the center of mass. In the last section, we defined the center of mass coordinate as

$$\mathbf{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{r}_i$$

Then we may separate the position of a particle into two components:

$$\boxed{\mathbf{r} = \mathbf{r}' + \mathbf{R}}$$

Where $\mathbf{R}$ is the vector from the origin to the center of mass, and $\mathbf{r}'$ is the vector from the center of mass to the particle. A geometric representation is depicted in figure 4.1.

Then, we may find the position of the particle relative to the center of mass by

$$\mathbf{r}' = \mathbf{r} - \mathbf{R}$$

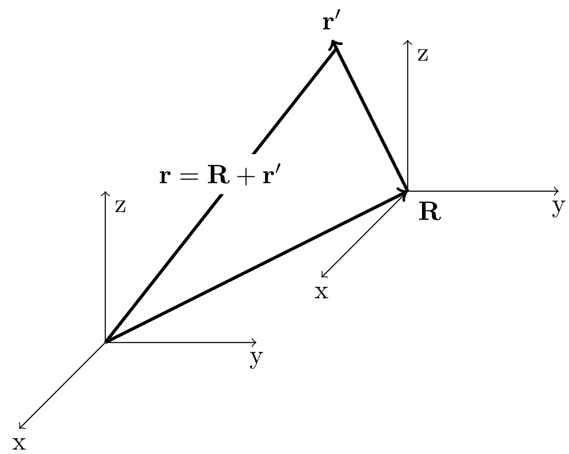


Figure 4.1

4.4 Impulse

If $\mathbf{F} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}$, it follows that we can integrate $\mathbf{F}$ to find $\mathbf{P}$:

$$\int_0^t \mathbf{F}\,\mathrm{d}t = \mathbf{P}(t) - \mathbf{P}_0$$

Then the change in momentum is given by the integral of the force with respect to time. Intuitively, you can think of the change in momentum as the total amount of force you apply over a period in time.

Note that this integral form applies only to systems the mass of which does not change. If the mass changed, then the momentum would be a product $\mathbf{P} = m(t)\mathbf{v}(t)$. In the case of varying mass, we have to set up a differential equation by considering the mass of the system at t and $t + \Delta t$.

Chapter 5

Energy

5.1 Introduction

Energy gives us an alternate approach to finding the motion of a system.

We want to be able to determine the velocity of an object given the force acting on it. However, we most often encounter force as a function of position. To work around this and obtain an expression for velocity in terms of position, we first use Newton's second law:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F(x) \Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t} = F(x)$$

We then integrate with respect to x.

$$m\int_{x_a}^{x_b} \frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}x = \int_{x_a}^{x_b} F(x)\,\mathrm{d}x$$

The right side can be integrated normally, but the left side requires some adjustments. Using

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \Rightarrow \mathrm{d}x = v\,\mathrm{d}t$$

We proceed as follows:

$$m\int_{t_a}^{t_b} \frac{\mathrm{d}v}{\mathrm{d}t}v\,\mathrm{d}t = m\int_{t_a}^{t_b} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v^2\right)\mathrm{d}t$$

Where we use the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v^2\right) = v\frac{\mathrm{d}v}{\mathrm{d}t}$ by the chain rule. Then

$$\begin{aligned} m\int_{t_a}^{t_b} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v^2\right)\mathrm{d}t &= m\int_{t_a}^{t_b} \mathrm{d}\frac{1}{2}v^2 \\ &= m\cdot\frac{1}{2}v^2\Big|_{t_a}^{t_b} \end{aligned}$$

Thus,

$$\frac{1}{2}mv^2\Big|_{t_a}^{t_b} = \int_{x_a}^{x_b} F(x)\,\mathrm{d}x$$
$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x_b} F(x)\,\mathrm{d}x$$

where we can use indefinite upper limits and express some v in terms of x:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x} F(x)\,\mathrm{d}x$$

We then define the **kinetic energy**:

$$\boxed{\frac{1}{2}mv^2 \equiv K}$$

5.2 Work Energy Theorem

From the preceding section, we have

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x} F(x)\,\mathrm{d}x$$

we call the right hand side the **work** done by the force F. Then, substituting K for kinetic energy, and defining the work along the path from a to b W_{ba}:

$$\boxed{W_{ba} = K_b - K_a} \tag{5.1}$$

Where the unit for work is the joule:

$$1\ \mathrm{N}\cdot\mathrm{m} = 1\ \mathrm{J} = 1\ \mathrm{kg}\cdot\frac{\mathrm{m}^2}{\mathrm{s}^2}$$

Intuitively, you can think of the work as the total force applied in the direction of the motion. In the case of one-dimensional motion, this means we simply integrate with respect to x. However, in more than one dimension, we must take the integral

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

Where $\mathbf{F}$ is the force at a point (x, y, z), and $\mathrm{d}\mathbf{r}$ is a small change in the position. The force field only contributes to the work done if its force is exerted in the direction of motion. This can be found by projecting the force due to $\vec{\mathbf{F}}$ at that point onto a small change in position (hence the dot product). Most problems we encounter will only involve one dimension, but it's important to keep in mind that the definition of work requires that the force be done in the *same direction* as the particle's motion.

5.3 Potential Energy

A force is considered to be conservative when the work done by it does not depend on the particular path the motion takes, only on the endpoints. For example, if work is done on the particles moving in the paths given by 5.1, and that work is due to a conservative force, all of the paths will have the same work done on them.

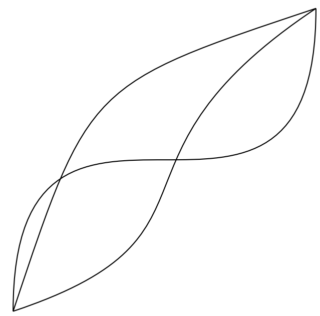


Figure 5.1: Several paths taken by a particle moving from A to B.

Mathematically, this means the work is

$$\int_a^b \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \text{function}(\mathbf{r}_b) - \text{function}(\mathbf{r}_a)$$

The above equation simply states that the work done just depends on the endpoints of the motion—where it starts and where it ends. We define this "function" to be $-U(\mathbf{r})$ (the reason for the negative sign is explained in section 5.6). Then

$$\int_a^b \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -U(\mathbf{r}_b) + U(\mathbf{r}_a)$$

Or, in one dimension:

$$\boxed{\int_a^b F(x)\,\mathrm{d}x = -U(a) + U(b)} \tag{5.2}$$

This function U is the **potential energy** of the system. It can be thought of as the amount of kinetic energy the system *can* have; i.e., how quickly the system can possibly move. Typically, it is either given to us, or we may integrate the force with respect to position to determine it.

5.4 Conservation of Mechanical Energy

In the preceding section, we defined the potential energy. Recall that the work energy theorem, as given in section 5.2, is

$$W_{ba} = K_b - K_a = \int_a^b F(x)\,\mathrm{d}x$$

for conservative forces. Then, if the force F is conservative, we may use (5.2) to write

$$W_{ba} = K_b - K_a = -U_b + U_a$$

We can rearrange the terms to yield

$$K_a + U_a = K_b + U_b$$

This means that the sum of the potential and kinetic energies is constant for any two points a and b. This is the so-called **conservation of mechanical energy**. We call the sum of the kinetic and potential energies the **total energy** E:

$$\boxed{K_a + U_a = K_b + U_b = E = \text{constant}}$$

Note that this equation is valid up to a constant. That means that what matters is *change* in kinetic and potential energies, not necessarily the actual value of energy. We may define the potential energy to have value $U = 0$ at whichever point is most convenient. Usually this is at ground level, or at the equilibrium position of a spring.

Also note that this is only valid for *conservative forces*. See section 5.7 for the case of non-conservative forces.

5.5 Relationship Between Energy and Force

It follows that if the potential energy is negative the integral of the force with respect to x, the force should be negative the derivative of the potential energy. Verifying this, let us consider a small increment in x:

$$\begin{aligned} U_b - U_a &= -\int_a^b F(x)\,\mathrm{d}x \\ U(x+\Delta x) - U(x) &= -\int_x^{x+\Delta x} F(x)\,\mathrm{d}x \end{aligned}$$

With a sufficiently small Δx, $F(x)$ is nearly constant and

$$\begin{aligned} \Delta U &\approx -F(x)[(\Delta x + x) - x] \\ \Delta U &\approx -F(x)\Delta x \\ \frac{\Delta U}{\Delta x} &\approx -F(x) \end{aligned}$$

Taking the limit, we arrive at the important relationship

$$\boxed{-\frac{\mathrm{d}U}{\mathrm{d}x} = F(x)}$$

5.6 Energy Diagrams

Energy diagrams are an excellent way of intuitively understanding the motion of a particle or system. Energy diagrams typically plot the potential energy in the vertical axis and the position x in the horizontal axis. Some key properties of energy diagrams are

1. One can think of the energy diagram as a "hill"—if you were to place a ball at a sloped region of the hill, the ball would roll down. Likewise, in a sloped region of the energy diagram, the system experiences a force proportional to the slope at that point, and the system will "roll" down to a stable position (see points x_1 and x_2 on figure 5.2). In fact, we define the potential energy to be the negative of the integral of the force so that we may interpret the potential energy function graphically like this.

2. Equilibria are points at which the system experiences no forces. Intuitively, this is a location in the analogous "hill" where the ball would not move unless disturbed. With calculus, this corresponds to a place where the slope of the energy diagram has slope zero, and $\frac{\mathrm{d}U}{\mathrm{d}x} = 0$ (notice this implies the system experiences no force). **Stable equilibria** are the "bottom" of the hill, where the system would experience a restoring force (see $x = 0$ on figure 5.2). **Unstable equilibria** are the top of the hill, where a small force would lead to the system "rolling off" the hill.

3. Because the maximum possible potential energy of the system is simply the total energy, the motion of the system is confined to the region of the energy diagram where $U \leq E$.

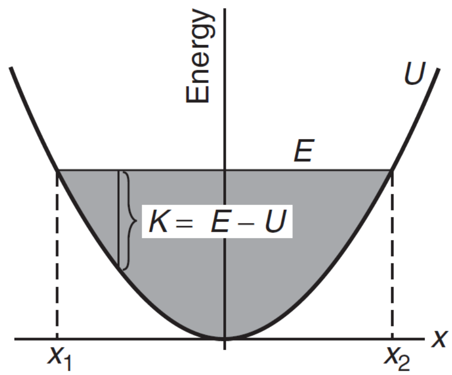


Figure 5.2: An energy diagram for a simple harmonic oscillator.

5.7 Non-conservative Forces

Not all forces are conservative—forces that are *not* the gradient or derivative of some potential function are non-conservative, and the work done by them depend on the path taken. Thus, conservation of energy need not apply.

We can separate the total force into components

$$\mathbf{F} = \mathbf{F}^c + \mathbf{F}^{nc}$$

Then the work energy theorem (which we take to be true regardless of whether the force is conservative) gives

$$\begin{aligned} W^{\text{total}} &= \int_a^b \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_a^b \mathbf{F}^c \cdot \mathrm{d}\mathbf{r} + \int_a^b \mathbf{F}^{nc} \cdot \mathrm{d}\mathbf{r} \\ &= -U_b + U_a + W^{\text{nc}} \end{aligned}$$

The work-energy theorem also states $W^{\text{total}} = K_b - K_a$. Then

$$\boxed{-U_b + U_a + W^{\text{nc}} = K_b - K_a}$$

This can also be expressed

$$K_b + U_b - (K_a + U_a) = W^{\text{nc}}$$

or, alternatively,

$$\boxed{E_b - E_a = W^{\text{nc}}}$$

meaning that the change in total energy of the system is equal to the work done by non-conservative forces along a given path.

Chapter 6

Topics in Dynamics

6.1 Small Oscillations in Bound Systems

When a local minimum occurs in a potential energy function, we expect the system to experience harmonic motion about that minimum if slightly perturbed. The shape of the potential function for a harmonic oscillator $U(x) = \frac{1}{2}kx^2$ is parabolic; it follows that any parabola-like shape in the graph of a potential function will lead to similar motion.

We explore this oscillatory motion by using a Taylor expansion about the potential minimum.

$$U(r) = U(r_0) + (r - r_0)\frac{\mathrm{d}U}{\mathrm{d}r}\bigg|_{r_0} + \frac{1}{2}(r - r_0)^2\frac{\mathrm{d}^2U}{\mathrm{d}r^2}\bigg|_{r_0} + \cdots$$

Since r_0 is a minimum, the first derivative at that point is zero and the second term vanishes. Further, $U(r_0)$ is just some constant. Then

$$U(x) = U(r_0) + \frac{1}{2}(r - r_0)^2\frac{\mathrm{d}^2U}{\mathrm{d}r^2}\bigg|_{r_0}$$

If the potential energy of a harmonic oscillator is

$$U(x) = C + \frac{1}{2}k(x - x_0)^2$$

We want to make these two equations equivalent. To do so, set

$$\boxed{k = \frac{\mathrm{d}^2U}{\mathrm{d}r^2}^2\bigg|_{r_0}}$$

6.2 Collisions

Conservation of momentum yields that the momentum of the system before and after a collision should be

$$\mathbf{P}_i = \mathbf{P}_f$$

If the collision is between two bodies, we have

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$$

The above comprises three equations; with unknown $\mathbf{v}_1'$ and $\mathbf{v}_2'$, we have six unknowns. Therefore, in the three-dimensional case, more information is needed. However, if we reduce the motion to one dimension (see section 6.2.2), the problem is easy.

Also, if we are given the angles of the trajectories, and if we restrict the motion to two-dimensions, the problem is also solvable. Simply split the, say, final velocity into $v_f \cos\theta\hat{\mathbf{i}} + v_f \sin\theta\hat{\mathbf{j}}$.

6.2.1 Elastic and Inelastic Collisions

Elastic collisions are collisions wherein kinetic energy is conserved. This amounts to

$$K_i = K_f$$

and usually means the two bodies "bounce" perfectly.

Inelastic collisions are collisions wherein kinetic energy is not conserved. Note that this does *not* mean that total energy is not conserved, rather that some of the kinetic energy is converted to other types of energy:

$$K_i = K_f + Q \Rightarrow K_i - Q = K_f$$

where if Q is positive, there is loss in kinetic energy, and if Q is negative, there is an increase in kinetic energy (occasionally called superelastic collisions).

6.2.2 One-Dimensional Collisions

In one-dimensional motion, conservation of momentum and energy are sufficient to solve for the final velocities:

$$\text{Momentum: } m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$
$$\text{Energy: } \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 + Q$$

(In the case of elastic collisions, set $Q = 0$)

6.2.3 Collisions and Center of Mass Coordinates

Treat three-dimensional collision problems in the center of mass coordinate system to simplify the problem. This is because, by the definition of center of mass, the total momentum will be zero. The center of mass's origin lies on

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

Meaning that, if $\mathbf{r}_i = \mathbf{r}_{ic}$ are measured from the center of mass,

$$0 = \frac{m_1 \mathbf{r}_{1c} + m_2 \mathbf{r}_{2c}}{m_1 + m_2}$$
$$0 = m_1 \mathbf{r}_{1c} + m_2 \mathbf{r}_{2c}$$

And, differentiating,

$$0 = m_1 \mathbf{v}_{1c} + m_2 \mathbf{v}_{2c} \tag{6.1}$$

Now let us consider two particles of masses m_1 and m_2 with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. The center of mass velocity lies on the line joining $\mathbf{v}_1$ and $\mathbf{v}_2$ and is

$$\mathbf{V} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} \tag{6.2}$$

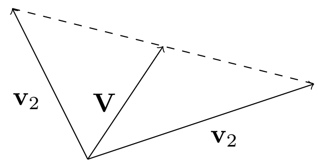


Figure 6.1: The center of mass velocity and the velocity vectors.

The center-of mass coordinates are, by section 4.3,

$$\mathbf{r} = \mathbf{r}_c + \mathbf{R}$$

Differentiating,

$$\begin{aligned} \mathbf{v} = \mathbf{v}_c + \mathbf{V} \Rightarrow \mathbf{v}_{1c} &= \mathbf{v}_1 - \mathbf{V} \\ &= \frac{m_1 + m_2}{m_1 + m_2}\mathbf{v}_1 - \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \\ &= \frac{m_1\mathbf{v}_1 m_2\mathbf{v}_1 - m_1\mathbf{v}_1 - m_2\mathbf{v}_2}{m_1 + m_2} \\ &= \frac{m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2) \end{aligned} \tag{6.3}$$

And

$$\begin{aligned} \mathbf{v}_{2c} &= \mathbf{v}_2 - \mathbf{V} \\ &= \frac{m_1 + m_2}{m_1 + m_2} - \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \\ &= \frac{m_1\mathbf{v}_2 + m_2\mathbf{v}_2 - m_1\mathbf{v}_1 - m_2\mathbf{v}_2}{m_1 + m_2} \\ &= \frac{-m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2) \end{aligned} \tag{6.4}$$

The vector $\mathbf{v}_1 - \mathbf{v}_2$, along which $\mathbf{v}_{2c}, \mathbf{v}_{1c}$ point, connects the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. $\mathbf{v} \equiv vbv_1 - \mathbf{v}_2$ is the relative velocity of the particles.

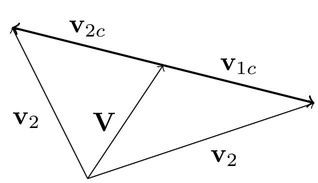


Figure 6.2: The center of mass velocity and the velocity vectors.

The momenta in the center of mass system are

$$\begin{aligned} \mathbf{p}_{1c} &= m_1\mathbf{v}_{1c} \\ &= \frac{m_1 m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2) \\ &= \mu\mathbf{v} \\ \mathbf{p}_{2c} &= m_2\mathbf{v}_{c2} \\ &= \frac{-m_1 m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2) \\ &= -\mu\mathbf{v} \end{aligned}$$

And $\mu \equiv m_1 m_2/(m_1+m_2)$ is called the **reduced mass**, a commonly occurring quantity. Notice that the total momentum in the center of mass system (the sum of these two momenta) is zero, confirming what we previously computed.

Conservation of energy in the center of mass system yields

$$\frac{1}{2}m_1 v_{1c}^2 + \frac{1}{2}m_2 v_{2c}^2 = \frac{1}{2}m_1 v_{1c}'^2 + \frac{1}{2}m_2 v_{2c}'^2 \tag{6.5}$$

Since the total momentum in the center of mass system is zero, we have

$$m_1 v_{1c} - m_2 v_{2c} = m_1 v_{1c}' - m_2 v_{2c}' = 0$$

Eliminate v_{2c}, v_{2c}' from (6.5) to obtain

$$\frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2^2}\right)v_{1c}^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2^2}\right)v_{1c}'^2$$
$$v_{1c} = v_{1c}'$$

Likewise yielding $v_{2c} = v_{2c}'$. In the center of mass system, the initial and final velocities are the same, the motions simply change directions.

Take particle m_2 to be initially at rest. Then (6.2) becomes

$$\mathbf{V} = \frac{m_1}{m_1+m_2}\mathbf{v}_1$$

And the center of mass system velocities, (6.3) and (6.4), become

$$\mathbf{v}_{1c} = \frac{m_2}{m_1+m_2}\mathbf{v_1}$$
$$\mathbf{v}_{2c} = -\frac{m_1}{m_1+m_2}\mathbf{v}_1$$

Chapter 7

Angular Momentum, Fixed Axis Rotation

7.1 Introduction

This chapter focuses on the rotational analogues of linear phenomena we have discussed previously:

Translational	**Rotational**
Force	Torque
Linear momentum	Angular momentum
Center of mass	Moment of inertia

The driving principle behind this is that an extended body's motion can be completely described by a rotational component and a translational component. we've already discussed the translational component of such bodies—now we turn to the rotational component.

7.2 Angular Momentum of a Particle

A particle with momentum $\mathbf{p}$ at position $\mathbf{r}$ in a given coordinate system has **angular momentum**

$$\boxed{\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}}$$

With units $\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}$. Note that this vector depends on the position of the particle as well as its linear momentum, and that it points perpendicular to its momentum as well as its position with respect to the origin. Then the magnitude of the angular momentum is proportional to the mass of the particle, its distance from the origin, its velocity, and the angle between the velocity and the position. Since

$$\mathbf{r} \times \mathbf{p} = mrv \sin \alpha$$

the angular momentum is at a maximum when $\mathbf{r}$ and $\mathbf{v}$ are perpendicular and decreases the more those vectors point along the same direction.

The vector $\mathbf{L}$ points along the "axis of rotation"; that is, the axis around which the particle is moving. You may use the right hand rule to determine the direction of $\mathbf{L}$.

Decompose $\mathbf{r}$ into a component $r_\perp$ perpendicular to the particle's trajectory and a component $r_\parallel$ parallel to the particle's trajectory:

$$\begin{aligned} r_\perp &= r\sin(\pi-\phi) = r\sin\phi \\ L_z &= (\mathbf{r}\times\mathbf{p})_z = rp\sin\phi = r_\perp p \end{aligned} \tag{7.1}$$

Assuming that the particle's motion is restricted to the x- y plane, $L_z = |\mathbf{L}|$. Then equation (7.1) completely describes the angular momentum, and, if θ is the angle between the radius vector and the velocity vector,

$$L = rp\sin\theta$$

If the motion is *not* restricted to a plane, we may still find the components L_x, L_y, L_z by explicitly computing $\mathbf{r}\times\mathbf{p}$.

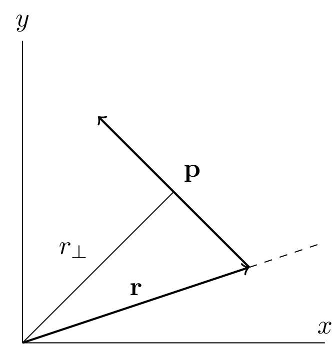


Figure 7.1

Note that we *cannot convert between angular and linear momentum.* These are separate quantities.

7.3 Fixed Axis Rotation

By fixed axis, we mean that the *direction* of the axis of rotation remains the same, thought the axis itself is allowed to translate.

We may choose the axis of rotation to be along the z direction. In fixed axis rotation, a solid extended body has particles that remain the same distance from the axis of rotation. We use ρ to denote the distance from the axis of rotation, and r to denote the distance from the *origin*:

$$\rho = \sqrt{x_j^2 + y_j^2}$$

Consider a body rotating around the z axis, such that, for particle j,

$$|\mathbf{v}_j| = |\dot{\mathbf{r}}_j| = \omega\rho_j \tag{7.2}$$

where we just use a result from rotational kinematics (see section 1.5.1). The angular momentum of the j-th particle is

$$\mathbf{L}_j = \mathbf{r}_j \times m_j\mathbf{v}j$$

We're only concerned with the angular momentum along the axis of rotation, in this case, in the z direction. Then we consider only the component of $\mathbf{r}_j$ which is in the same plane as $\mathbf{b}$. This is precisely ρ.

$$L_{j,z} = m_j v_j \rho_j$$

With equation (7.2):

$$\boxed{L_{j,z} = m_j \rho_j^2 \omega}$$

The total angular momentum is the sum of the individual angular momenta of each particle

$$L_Z = \sum_j L_{j,z} = \sum_j m_j \rho_j^2 \omega \tag{7.3}$$

7.4 Moment of Inertia

We may write equation (7.3)

$$L_z = \sum_j I\omega$$

By defining the important quantity

$$I \equiv \sum_j m_j \rho_j^2$$

For a continuous body, this becomes

$$\boxed{I = \int \rho^2 \,\mathrm{d}m = \int (x^2 + y^2)\,\mathrm{d}m}$$

This it the **moment of inertia** of this body. The moment of inertia is specific to the geometry and mass of a particular body. It is the rotational analogue of mass.

7.5 Parallel Axis Theorem

The parallel axis theorem states that we may find the moment of inertia I about an arbitrary axis as long as we know the moment of inertia I_0 about an axis parallel to I and through the center of mass.

If the distance between the axes is l and the mass of the body M, we have

$$\boxed{I = I_0 + Ml^2}$$

Proof: Consider the moment of inertia of the body around an axis along the z direction. Our radius vector from the z axis to the j-th particle is then

$$\boldsymbol{\rho}_j = x_j \hat{\mathbf{i}} + y_j \hat{\mathbf{j}}$$

and the moment of inertia is

$$I = \sum_j m_j \rho_j^2 \tag{7.4}$$

The vector from the z axis to the center of mass is

$$\mathbf{R}_\perp = X\hat{\mathbf{i}} + Y\hat{\mathbf{j}}$$

The vector from the axis crossing the center of mass to the particle j is $\boldsymbol{\rho}'_j$, and the moment of inertia about the center of mass is

$$I_0 = \sum_j m_j \rho_j'^2$$

And finally, we notice that

$$\boldsymbol{\rho_j} = \boldsymbol{\rho}'_j + \mathbf{R}_\perp$$

We substitute this in equation (7.4):

$$\begin{aligned} I &= \sum_j m_j \rho_j^2 \\ &= \sum_j m_j \left(\boldsymbol{\rho}'_j + \mathbf{R}_\perp\right)^2 \\ &= \sum_j m_j \left(\rho'^2_j + 2\boldsymbol{\rho}'_j \cdot \mathbf{R}_\perp + R_\perp^2\right) \end{aligned}$$

The definition of center of mass yields

$$\sum_j m_j \boldsymbol{\rho}'^2_j = \sum_j m_j \left(\boldsymbol{\rho}_j - \mathbf{R}_\perp\right) = M(\mathbf{R}_\perp - \mathbf{R}_\perp) = 0$$

And notice that the magnitude of $\mathbf{R}_\perp$ is the distance between the two axes. This yields

$$\begin{aligned} \sum_j m_j \left(\rho'^2_j + R_\perp^2\right) &= \sum_j m_j \rho'^2_j + \sum_j m_j R_\perp^2 \\ &= I_0 + Ml^2 \end{aligned}$$

7.6 Torque

The **torque** is the rotational analogue of force. It is commonly denoted $\boldsymbol{\tau}$ or $\mathbf{N}$. When applying a force to rotate an object, the radius from the axis of rotation will actually affect the motion as well. Think of opening a door from the handle as opposed to from the hinge—it takes much more effort to open it from the hinge. Also, if you pushed on the door on the edge and toward the hinge, the door would not move. Thus, we scale force by radius to obtain the torque, and we take into consideration the direction of the applied force.

The torque is defined

$$\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}}$$

where the cross product ensures the direction of the force is accounted for, and the magnitude of $\mathbf{r}$ ensures the radial proportionality is accounted for.

As with angular momentum, we may find the torque in different ways. We can consider the component of the radius vector perpendicular to the force:

$$\tau = r_\perp \cdot F$$

Or we can consider the component of the force perpendicular to the radius

$$\tau = r \cdot F_\perp$$

Thus, as with angular momentum, if θ is the angle between the radius vector and the applied force vector, we have

$$\tau = Fr\sin\theta$$

Note that, when a system is allowed to undergo both translational motion and rotational motion, we must consider *both* the torques and the forces acting on the body. We consider force to change the object's translational motion, and the torque to change the object's rotational motion. Since torque is proportional to force, then an applied force might translate as well as rotate the body—the translation occurs in the way we've explored so far, and the rotation is governed by the above laws.

Note that, by this logic, for a body to be in equilibrium, the total force *and* the total torque must be zero.

7.6.1 Torque and Angular Momentum

Just as force determines the rate of change of linear momentum, torque determines the rate of change of angular momentum.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p}\right) + \left(\mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)$$

The first term vanishes because $\dot{r}$ and $\mathbf{p}$ point in the same direction. Likewise, recall that $\dot{\mathbf{p}} = \mathbf{F}$. Then

$$\boxed{\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}}$$

7.6.2 Conservation of Angular Momentum

Notice that the relation we just obtained implies that if there is no torque acting on the system, the angular momentum. We can use this to our advantage. For example, if we restrict our rotation to a plane, we obtain

$$mr_0v_0 = mr_1v_1$$

Likewise, we determine that

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$$
$$\int_{t_0}^{t_1} \boldsymbol{\tau}\,\mathrm{d}t = \Delta\mathbf{L}$$

Note that we *cannot convert between angular and linear momentum.* These are separate quantities.

7.7 Dynamics of Fixed Axis Rotation

We proceed with the experimental facts that *internal torques cancel* and only *external torques affect the system.*

Consider a body rotating with angular velocity ω around the z-axis. Then

$$L_z = I\omega$$

Since $\boldsymbol{\tau} = \mathrm{d}\mathbf{L}/\mathrm{d}t$,

$$\begin{aligned}\tau_z &= \frac{\mathrm{d}}{\mathrm{d}t} \\ &= I\frac{\mathrm{d}\omega}{\mathrm{d}t} \\ &= I\alpha\end{aligned}$$

Where we define the **angular acceleration**

$$\alpha \equiv \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Notice the analogue between force and torque, moment of inertia and mass, and acceleration and angular acceleration:

$$\tau = I\alpha \Leftrightarrow F = ma$$

7.7.1 Center of Mass and Torque

The torque on a body in a uniform gravitational field is $\mathbf{R} \times \mathbf{W}$ where $\mathbf{R}$ is a vector from the origin (the pivot point) to the center of mass and $\mathbf{W}$ is the weight vector. It then follows that if $\mathbf{R}$ and $\mathbf{W}$ are parallel, the torque is zero and the system will not tip over— i.e., it is in equilibrium. For this to be the case, $\mathbf{R}$ must point vertically to the center of mass, or, in other words, the pivot point must be directly below the center of mass.

7.7.2 Kinetic Energy of Body in Pure Rotation

We start with the familiar formula for kinetic energy, summed over all particles in the system:

$$\begin{aligned} K &= \sum_j \frac{1}{2} m_j v_j^2 \\ &= \sum_j \frac{1}{2} m_j \rho_j^2 \omega^2 \\ &= \frac{1}{2} I \omega^2 \end{aligned}$$

With

$$\begin{aligned} v_j &= \rho_j \omega \\ I &= \sum_j m_j \rho_j^2 \end{aligned}$$

Note that this quantity represents the same type of energy as our previous formulas for kinetic and potential energies. Rotational energy and linear kinetic energy are the same type of energy.

7.8 Motion with Translation and Rotation

Consider fixed axis rotation, but allow the axis itself to translate. Let the z axis be the axis of rotation. Show that the angular momentum L_z can be expressed as a sum of the angular momentum $I_0\omega$ due to rotation of the body about its center of mass and the angular momentum due to the motion of the center of mass with respect to the origin of the inertial coordinate system:

$$\boxed{L_z = I_0\omega + (\mathbf{R} \times M\mathbf{V})_z} \tag{7.5}$$

where $\mathbf{R}$ is a vector from the origin to the center of mass, $V = \dot{\mathbf{R}}$, and I_0 is the moment of inertia about the center of mass. Notice this is essentially the sum of the moment of inertia of the body itself plus the moment of inertia of the center of mass about a larger axis.

Proof: Consider the body to be an aggregation of N particles with masses m_j located at $\mathbf{r}_j$ with respect to the origin of an *inertial* system. Then

$$\mathbf{L} = \sum_{j=1}^{N} (\mathbf{r}_j \times m_j \mathbf{r}_j)$$

with the center of mass at

$$\mathbf{R} = \frac{\sum_j m_j \mathbf{r}_j}{M}$$

Use the center of mass coordinates $\mathbf{r}'_j = \mathbf{r}_j - \mathbf{R}$ and combine that with the equation for $\mathbf{L}$,

$$\begin{aligned}\mathbf{L} &= \sum_j \left(\mathbf{R} + \mathbf{r}'_j\right) \times m_j \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_j\right) \\ &= \mathbf{R} \times \sum_j m_j \dot{\mathbf{R}} + \sum_j m_j \mathbf{r}'_j \times \dot{\mathbf{R}} + \mathbf{R} \times \sum_j m_j \dot{\mathbf{r}}'_j + \sum_j m_j \mathbf{r}'_j \times \dot{\mathbf{r}}'_j \end{aligned} \tag{7.6}$$

Due to the definition of the center of mass coordinates, the middle two terms are zero:

$$\begin{aligned}\sum_j m_j \mathbf{r}'_j &= \sum_j m_j (\mathbf{r}_j - \mathbf{R}) \\ &= \sum_j m_j \mathbf{r}_j - M\mathbf{R} \\ &= 0\end{aligned}$$

The first term is

$$\mathbf{R} \times \sum_j m_j \dot{\mathbf{R}} = \mathbf{R} \times M\dot{\mathbf{R}}$$

Since we are considering angular momentum purely along the z axis, we may reduce $\mathbf{r}'_j$ to its component perpendicular to the z axis, $\boldsymbol{\rho}'_j$. With $\dot{\rho}'_j = \omega\rho$, we have

$$\left(\sum_j m_j \mathbf{r}'_j \times \dot{\mathbf{r}}'_j\right)_z = \left(\sum_j m_j \rho'_j \times \dot{\boldsymbol{\rho}}'_j\right)_z = \sum_j m_j \rho'_j \omega$$

which is just $I_0\omega$. Then, at last, equation (7.6) becomes

$$L_z = I_0\omega + (\mathbf{R} + M\mathbf{V})_z$$

7.8.1 Torque on Moving Body

Torque can be divided into two terms:

$$\begin{aligned}\boldsymbol{\tau} &= \sum_j \mathbf{r}_j \times \mathbf{f}_j \\ &= \sum_j \left(\mathbf{r}'_j + \mathbf{r}\right) \times \mathbf{f}_j \\ &= \sum_j \left(\mathbf{r}'_j \times \mathbf{f}_j\right) + \mathbf{R} \times \mathbf{F}\end{aligned}$$

The first term is the torque around the center of mass due to various external forces, and the second term is the torque on the center of mass due to the *total* external force.

With fixed axis rotation, we may reduce this to one dimension:

$$\boxed{\tau_z = \tau_0 + (\mathbf{R} \times \mathbf{F})_z}$$

Where τ_0 is the z component of the torque around the center of mass. Equation (7.5) for L_z gives

$$\begin{aligned}\frac{\mathrm{d}L_z}{\mathrm{d}t} &= I_0 \frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R} \times M\mathbf{V})_z \\ &= I_0\alpha + (\mathbf{R} + M\mathbf{a})_z\end{aligned}$$

where α is angular acceleration and $\mathbf{a}$ is linear acceleration. With $\tau_z = \mathrm{d}L/\mathrm{d}t$, we combine the above two equations for τ_z:

$$\begin{aligned} t_0 + (\mathbf{R} \times \mathbf{F})_z &= I_0\alpha + (\mathbf{R} \times M\mathbf{a})_z \\ &= I_0\alpha + (\mathbf{R} \times \mathbf{F})_z \end{aligned}$$

yielding $\tau_0 = I_0\alpha$, which states that the rotational motion about the center of mass depends only on the torque about the center of mass (it does not depend on center of mass).

We may also separate the **kinetic energy** into rotational and translational components:

$$\begin{aligned} K &= \frac{1}{2}\sum_j m_j v_j^2 \\ &= \frac{1}{2}\sum m_j(\dot{\rho}_j + \mathbf{V})^2 \\ &= \frac{1}{2}\sum_j \dot{\rho}_j' + \sum_j m_j\dot{\rho}_j' \cdot \mathbf{V} + \frac{1}{2}\sum_j m_j V^2 \\ &= \boxed{\frac{1}{2}I_0\omega^2 + \frac{1}{2}MV^2} \end{aligned}$$

7.9 The Work-Energy Theorem and Rotation

Recall that

$$K_b - K_a = W_{ba} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} F \cdot \mathrm{d}\mathbf{r}$$

We may divide the work energy theorem into *translational* energy and *rotational* energy. For $K_b - K_a = W_{ba}$,

$$\boxed{K = \frac{1}{2}MV^2 + \frac{1}{2}I_0\omega^2}$$

Proof: Start with the translational component. Use the equation of motion for the center of mass:

$$\mathbf{F} = M\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}$$

The work done when the center of mass is displaced by $\mathrm{d}\mathbf{R} = \mathbf{V}\,\mathrm{d}t$ is

$$\mathbf{F} \cdot \mathrm{d}\mathbf{R} = M\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \cdot \mathbf{V}\,\mathrm{d}t = M\mathbf{V} \cdot \mathrm{d}\mathbf{V} = \mathrm{d}\left(\frac{1}{2}MV^2\right)$$

Integrating,

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot \mathrm{d}\mathbf{R} = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2$$

Turn to the kinetic energy component. With a fixed axis through the center of mass:

$$\tau_0 = I_0\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

We would like to put this in a form resembling kinetic energy, $\frac{1}{2}I_0\omega^2$. Thus, multiply both sides by $\mathrm{d}\theta = \omega\,\mathrm{d}t$:

$$\tau_0\,\mathrm{d}\theta = I_0\frac{\mathrm{d}\omega}{\mathrm{d}t}\omega\,\mathrm{d}t = I_0\omega\,\mathrm{d}\omega = \mathrm{d}\frac{1}{2}I_0\omega^2$$

Integrating,

$$\boxed{\int_{\theta_a}^{\theta_b} \tau_0 \, \mathrm{d}\theta = \frac{1}{2} I_0 \omega_b^2 - \frac{1}{2} I_0 \omega_a^2}$$

The integral on the left hand side is the work done by the applied torque. We've just divided the kinetic energy of the system into a component that accounts for translational motion and a component that accounts for the rotational motion. We've also shown that the initial and final velocity and rotational velocity of the system *both* evolve according to the familiar work-energy theorem—only we must consider the work done by the *torque* on the system when computing the rotational energy.

Chapter 8

Rigid Body Motion

8.1 Vectorial Angular Velocity

Angular velocity cannot be expressed in vector form because, for small angles, the transformation is nearly linear.

Although angular position cannot be expressed in vector form, we express the **angular velocity** with a vector:

$$\boxed{\boldsymbol{\omega} = \frac{\mathrm{d}\theta_x}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}\theta_y}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}\theta_z}{\mathrm{d}t}\hat{\mathbf{k}} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}}$$

Consider a rigid body rotating about an axis along the direction $\hat{\mathbf{n}}$. Choose the coordinate system such that the origin is on the axis. The coordinate system is fixed and inertial. The particles in the body move in circular motion around the axis. The vector $\mathbf{r}$ from the origin to each particle traces out a cone. ϕ is the angle between $\hat{\mathbf{n}}$ and $\mathbf{r}$. The tip of $\mathbf{r}$ moves on a circle of radius $r\sin\phi$. Let $\Delta\theta$ be a small angular displacement of the particle. We have a small displacement $\Delta\mathbf{r}$ which is the base of the isosceles triangle with angle ϕ and side length $r\sin\phi$.

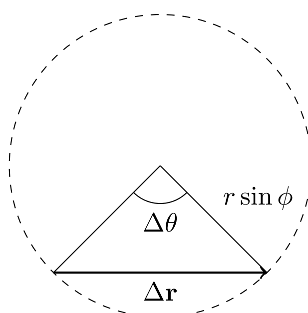


Figure 8.1

This yields

$$|\Delta\mathbf{r}| = 2r\sin\theta\sin\left(\Delta\theta/2\right)$$

Since $\Delta\theta$ small, we use the small angle approximation and

$$|\Delta\mathbf{r}| = r\sin\theta\delta\theta$$

Divide both sides by Δt and take the limit:

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| = r\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Notice that $|\mathrm{d}\mathbf{r}/\mathrm{d}t|$ is tangential to the circular motion of each particle. The direction of $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is perpendicular to the plane formed by $\hat{\mathbf{n}}$ and $\mathbf{r}$. Thus we have

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t}\sin\phi \text{ perpendicular to } \hat{\mathbf{n}} \text{ and } \hat{\mathbf{r}} = \hat{\mathbf{n}} \times \hat{\mathbf{r}}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

since the cross product of $\hat{\mathbf{n}}$ and $\mathbf{r}$ yields a vector perpendicular to the plane they form. If we call the vector in the $\hat{\mathbf{n}}$ direction with magnitude $\mathrm{d}\theta/\mathrm{d}t$ the vector $\boldsymbol{\omega}$, and use the notation $\mathrm{d}\mathbf{r}/\mathrm{d}t$, we have

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

Such that

$$\boxed{\boldsymbol{\omega} = \mathbf{r} \times \mathbf{v}}$$

Some remarks:

1. This is a vector that can indeed be decomposed into its components along different axes, and doing so is often advantageous.
2. $\boldsymbol{\omega}$ is not necessarily parallel to the angular momentum vector $\mathbf{L}$ as was the case with fixed axis rotation.
3. $\boldsymbol{\tau} = \mathrm{d}\mathbf{L}/\mathrm{d}t$ *still holds.*

What we've done, in effect, is chosen an axis of rotation and derived a velocity vector, and decided that, if we want the angular velocity to be vector-valued, it has to point along the axis of rotation for it to yield the proper velocity vector.

8.2 The Gyroscope

We will call the angular velocity of the the flywheel (the freely spinning wheel) ω_s, pointed along the axle, and $\mathbf{L}_s$ be the component of the angular momentum associated with that angular velocity.

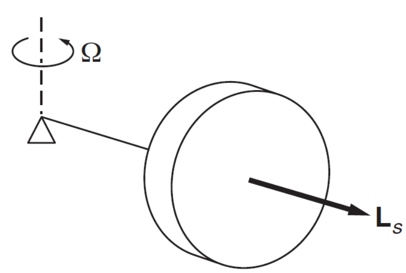


Figure 8.2: The gyroscope.

The gyroscope's precession is due to the combination of the fly-wheel's angular momentum and the torque that gravity applies. Observe figure 8.4. If the pivot is the "pylon" (the triangle in figure 8.4), and the radius vector $\mathbf{r}$ is the radius vector, we have

$$\tau = \mathbf{F}_g \times \mathbf{r}, \text{ pointing into the page}$$

We know that $\mathrm{d}\mathbf{L}/\mathrm{d}t = \tau$. Since the torque is perpendicular to the radius vector, it follows that the change in angular momentum will not be in length, but purely direction. And $\mathbf{L}_s$ will rotate, or precess.

The rate of change of $\mathbf{L}_s$ will be evaluated polar coordinates. Let θ be the precession angle.

$$\frac{\mathrm{d}\mathbf{L}_s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(L_s\hat{\mathbf{r}}) = \hat{\mathbf{r}}\frac{\mathrm{d}L_s}{\mathrm{d}t} + L_s\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}$$

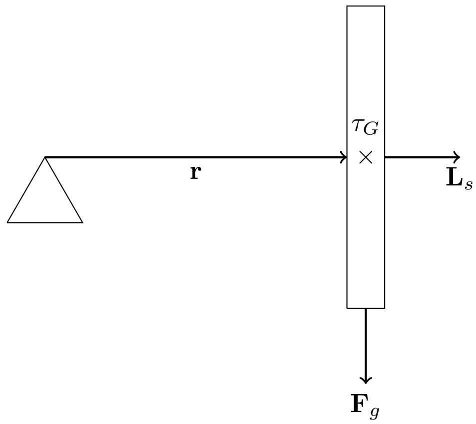


Figure 8.3: Side view of gyroscope.

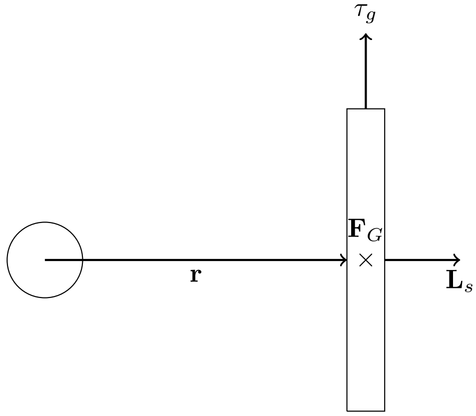


Figure 8.4: Top view of gyroscope.

Once again, the torque is perpendicular to $\mathbf{L}_s$, so the magnitude of the angular momentum, L_s, is constant, and its derivative vanishes. Using this and the fact that, by section 1.5.1, $\mathrm{d}\hat{\mathbf{r}}/\mathrm{d}t = \dot{\theta}\hat{\boldsymbol{\theta}}$,

$$\tau = \frac{\mathrm{d}\mathbf{L}_s}{\mathrm{d}t} = L_s \cdot \dot{\theta}\hat{\boldsymbol{\theta}} \Rightarrow \left|\frac{\mathrm{d}\mathbf{L}_s}{\mathrm{d}t}\right| = L_s\dot{\theta} = L_s\Omega$$

If we call $\dot{\theta} = \Omega$, the precessional angular velocity. Now, with gravity perpendicular to the radius vector (observe the figures once again).

$$\tau = Fr\sin\theta = lW \tag{8.1}$$

Set (8.2) and (8.1) equal to each other to obtain

$$\Omega L_s = lW \Rightarrow \Omega = \frac{lW}{I_0\omega_s}$$

More generally, then,

$$\boxed{\Omega = \frac{\tau}{I_0\omega_s}}$$