Electromagnetism

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Chapter 1

The Electric Field

The entirety of electrostatics is the result of one equation, the experimentally determined Coulomb's law:

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}}$$

this is the force on a particle of charge Q due to a particle with charge q, with $\boldsymbol{\mathscr{r}}$ pointing from q to Q. ϵ_0 is a constant called the **permittivity of free space**, and the factor of 4π is included so that some calculations turn out nicely later on. The vector $\boldsymbol{\mathscr{r}}$ is defined to be

$$\boxed{\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'}$$

where $\mathbf{r}$ is the vector from the origin to Q. and $\mathbf{r}'$ is the particle from the origin to the particle q. Then $\boldsymbol{\mathscr{r}}$ is the vector from q to Q. (see figure 1.1)

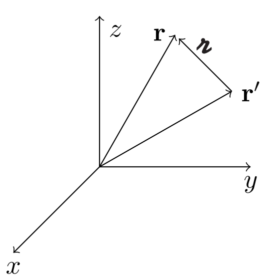

Figure 1.1: The vector $\boldsymbol{\mathscr{r}}$.

1.1 The Electric Field

Coulomb's law follows the **law of superposition**, which states that for several points charges $q_1, q_2, \ldots, q_n$,

$$\begin{aligned}\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots &= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2} + \frac{q_1 Q}{\mathscr{r}_1^2} + \cdots\right)\\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}^2 + \cdots\right)\end{aligned}$$

We've factored out Q; this leads us to define the **Electric field** for the group of charges $q_1, q_2, \ldots, q_n$: the force per unit charge, such that all we have to do is multiply by the charge we want to place, and give a position vector as an input:

$$\mathbf{F} = Q\mathbf{E}$$

where, for discrete point charges,

$$\boxed{\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i}$$

1.1.1 Continuous Charge Distributions

If the charge is some continuous distribution, we can some up all the infinitesimal charge elements dq to obtain the electric field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}q$$

Now, if we know the charge density—the amount of charge per unit volume at a given point $\mathbf{r}'$—we can can write $\mathrm{d}q = \rho(\mathbf{r}')\,\mathrm{d}\tau'$ for a volume element $\mathrm{d}\tau$:

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau'}$$

Sometimes the charge might be distributed in a thin line, and we can instead integrate along a line using a linear charge density. Or the charge might be distributed in a thin shell or surface, and we have a surface charge density. In these circumstances, we have

Distribution Type	dq
Linear	$\lambda\,\mathrm{d}l'$
Surface	$\sigma\,\mathrm{d}a'$
Volume	$\rho\,\mathrm{d}\tau'$

Note that these are simply idealistic representations—charges are indeed three-dimensional, but if they are distributed in a shell thin enough, a surface approximates the charge distribution sufficiently well.

1.2 Gauss's Law

With a point charge q at the origin, the flux of $\mathbf{E}$ through a spherical surface of radius r is

$$\oint\mathbf{E}\cdot\mathrm{d}\mathbf{a} = \int\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right)\cdot(r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\cdot 4\pi q = \frac{q}{\epsilon_0}$$

What has happened is the flux of the electric field through the sphere decreases at the same rate (because $1/r^2$) as the surface area of the sphere grows. A different surface might have larger surface area but its normal vectors won't be parallel to the field and it will have a smaller flux.

Suppose instead of one point charge we have a set of charges and

$$\mathbf{E} = \sum_{i=1}^{n}\mathbf{E}_i$$

The flux through a surface that encloses them all is thus

$$\oint\mathbf{E}\cdot\mathrm{d}\mathbf{a} = \sum_{i=1}^{n}\left(\oint\mathbf{E}_i\cdot\mathrm{d}\mathbf{a}\right) = \sum_{i=1}^{n}\left(\frac{q_i}{\epsilon_0}\right)$$

For the charge inside the solid sphere—we get a linear decrease because the volume inside the charge increases with r^3 but the inverse square law (since we can treat the solid sphere as a point mass) increases with $1/r^2$.

And thus

$$\boxed{\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}}$$

We can apply the divergence theorem to see

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \int (\nabla \cdot \mathbf{E})\, \mathrm{d}\tau$$

And we can rewrite the total charge enclosed in terms of the charge density:

$$Q_{\text{enc}} = \int \rho\, \mathrm{d}\tau$$

And Gauss's law becomes

$$\int (\nabla \cdot \mathbf{E})\, \mathrm{d}\tau = \int \left(\frac{\rho}{\epsilon_0}\right) \mathrm{d}\tau$$

This is true for any volume or surface, so the integrands are equal:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

1.3 Curl of E

Consider again a point charge at the origin:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

The electric field is conservative. Let's compute a path integral from a point a to a point b.

$$\begin{aligned}
\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} &= \int_{\mathbf{a}}^{\mathbf{b}} \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}\right) \cdot \left(\mathrm{d}r\, \hat{\mathbf{r}} + r\, \mathrm{d}\theta\, \hat{\boldsymbol{\theta}} + r \sin\theta\, \mathrm{d}\phi\, \hat{\boldsymbol{\phi}}\right) \\
&= \int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\, \mathrm{d}r \\
&= -\frac{1}{4\pi\epsilon_0} \frac{q}{r}\Big|_{r_a}^{r_b} \\
&= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b}\right)
\end{aligned}$$

If the path is closed, $r_a = r_b$ and

$$\boxed{\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0}$$

Using Stokes' theorem one sees:

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = 0$$

This is true for any surface, so

$$\boxed{\nabla \times \mathbf{E} = 0}$$

To show this for a charge distribution, one first notices

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots$$

where $\mathbf{E}_i$ is the field due to the i-th particle. The curl operator is in fact linear, so

$$\nabla \times \mathbf{E} = \nabla \times \mathbf{E}_1 + \nabla \times \mathbf{E}_2 + \cdots = 0$$

1.4 Electric Potential

The electric field is a conservative field, which means we can express it as a potential scalar function by

$$\boxed{V(\mathbf{r}) \equiv -\int_O^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l}}$$

where O is some reference point that one is given (typically it is infinity). Just as with potential energy, this is due to the arbitrary constant term at the end of this derivative. Conversely,

$$\mathbf{E} = -\nabla V$$

Since $\mathbf{E}$ is the force per unit charge, and the spatial derivative of the force is potential energy, V is the potential energy per unit charge.

Note that because integration is linear, the potential obeys the superposition principle: $V = V_1 + V_2 + \dots$.

Additionally, one sees from Gauss's law that

$$\nabla \cdot \mathbf{E} \text{ and } \nabla V = \mathbf{E} = \nabla^2 \mathbf{E} = -\frac{\rho}{\epsilon_0}$$

1.4.1 Continuous Charge Distribution

Let us compute the potential of a point charge. We let $O = r = \infty$.

$$V(r) = -\int_O^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{1}{4\pi\epsilon_0} \int_\infty^r \frac{q}{r'^2}\, \mathrm{d}r' = \frac{1}{4\pi\epsilon_0}\Big|_\infty^r = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

If the charge is not at the origin we find

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

The superposition principle yields, for a distribution of particles

$$V(\mathbf{r}) = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_i}{\mathscr{r}_i}$$

For a continuous distribution this becomes an integral

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, \mathrm{d}\tau'}$$

1.5 Boundary Conditions

At a surface charge, the electric field experiences a discontinuity. This is to be expected—the thin shell of charges should produce a field that changes once you cross the surface. Let us find the difference between the field above and below a surface. We will draw a box surface with sides parallel to the surface and infinitesimally above it. The sides perpendicular to the surface are then very thin. If the surface charge is σ at that infinitesimal section of surface area, then Gauss's law yields

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

Where A is the area of the the side of the box parallel to the surface. The sides perpendicular are very small, and in the limit contribute nothing. This leaves us with

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

Where we've evaluated the integral and divided by the area A. The perpendicular component of the electric field then experiences a discontinuity of σ/ϵ_0. The parallel component, on the other hand, is zero, for consider a loop that extends just above and just below the surface:

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0$$

The portion of the loop perpendicular to the surface can be made arbitrarily short, yielding

$$E^{\parallel}_{\text{above}} l - E^{\parallel}_{\text{below}} l = 0 \Longrightarrow \mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}$$

Then there is no discontinuity in the parallel component, and one can write

$$\boxed{\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}}$$

where $\hat{\mathbf{n}}$ is the vector normal to the surface. The gradient of the potential then also has this discontinuity:

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

Now, if we apply the dot product of $\hat{\mathbf{n}}$ to both sides, we obtain the directional derivative of V along the normal to the surface (this will often be simple, like $\partial/\partial x$ or $\partial/\partial r$):

$$\boxed{\left.\frac{\partial V}{\partial n}\right|_{\text{above}} - \left.\frac{\partial V}{\partial n}\right|_{\text{below}} = -\frac{\sigma}{\epsilon_0}} \tag{1.1}$$

Where by these evaluations above and below, we are handling the discontinuity of the derivative of V; we should take the limit as the potential approaches the boundary from whichever side is below or above.

1.6 Work and Energy

1.6.1 Work to Move a Charge

If the electric field is the force per unit charge, we see that the work it takes to move a charge is

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} = -Q\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{b})]$$

Which makes sense, since V is the potential energy per unit charge. One sees

$$V(\mathbf{a}) - V(\mathbf{b}) = \frac{W}{Q} \tag{1.2}$$

To bring a charge Q from infinity (which we defined as our reference point) and put it at a point $\mathbf{r}$, one has

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

1.6.2 Energy of a Point Charge Distribution

Suppose we are bringing together a collection of point charges from infinity, where the field is too weak for the charges fields' to affect each other. The first charge takes no work because there is no existing field. Bringing in q_2 will take $q_2 V_1(\mathbf{r}_2)$, with V_1 being the potential due to q_1 and $\mathbf{r}_2$ is the final position of q_2:

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}} \right)$$

We assume charges we've brought in won't move now. Then we bring in q_3, expending work $q_3 V_{1,2}(\mathbf{r_3})$, where $V_{1,2}$ is the potential due to q_1 and q_2:

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}} \right)$$

The work to bring in four charges is the sum

$$W_4 = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}} + \right)$$

We take the product of each unique pair of charges and divide by their distances. We write

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

Where the second sum sign is notated as such to ensure we don't count the same pair of charges twice. We can remedy this, alternatively, by allowing the count of the same pair twice but dividing by two.

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq 1}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

We pull out the charge q_i:

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n} q_i V(\mathbf{r}_i) \right)$$

Then the term in parentheses is the potential at a point $\mathbf{r}_i$ due to all other charges but q_i, yielding

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{1.3}$$

1.6.3 Energy of a Continuous Charge Distribution

For a charge density, we consider the contribution to the potential by any volume charge element to be sufficiently small that V is unchanged by considering the potential without that charge element. Then, analogously to (1.3), we obtain

$$W = \frac{1}{2}\int \rho V \, \mathrm{d}\tau$$

One can rewrite this in a more elegant fashion. We will eliminate ρ and V. Use Gauss's law to eliminate ρ:

$$W = \frac{\epsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V \, \mathrm{d}\tau$$

Use integration by parts (7.1):

$$W = \frac{\epsilon_0}{2}\left[-\int \mathbf{E} \cdot (\nabla V) \, \mathrm{d}\tau + \oint V\mathbf{E} \cdot \mathrm{d}\mathbf{a}\right]$$

With $\nabla V = -\mathbf{E}$,

$$W = \frac{\epsilon_0}{2}\left(\int_V E^2 \, \mathrm{d}\tau + \oint_S V\mathbf{E} \cdot \mathrm{d}\mathbf{a}\right)$$

We can expand the volume we integrate over to an arbitrary size, for anywhere that that $\rho = 0$ should not contribute anything. This works out in the above integral, for the product $V\mathbf{E}$ is proportional to $1/r^3$ and the surface area only increases with r^2, so the contribution by the surface integral decreases and the contribution by the volume integral constantly increases because it is indeed a volume integral of a squared quantity. Thus we have justified, in two ways, a safe integration over all of space:

$$\boxed{W = \frac{\epsilon_0}{2}\int E^2 \, \mathrm{d}\tau}$$

1.7 Conductors

In conductors, charges are free to move wherever.

1. $\mathbf{E} = 0$ inside conductors—in the presence of an electric field, the positive charges are pushed as far away from the field as possible and the negative charges are as attracted to the field as possible. This actually ends up cancelling out the electric field inside a conductor.

2. $\rho = 0$ inside a conductor. This is because if $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ and $\mathbf{E} = 0$ inside the conductor, the charge density must naturally be zero.

3. Net charges reside on the surface. The charge cannot be in the conductor, so it must be on the surface. You can think of this as being due to the charges trying to push each other as far away as possible—this farthest place would indeed be the surface.

4. Conductors are equipotentials. Any two points inside the conductor must have equal potential, for otherwise there would be an electric field.

5. $\mathbf{E}$ is perpendicular to the surface outside a conductor, for if there was a tangential component, there would be charge inside the conductor attracted to that component and that charge would cancel the component out.

6. Grounded conductors are *maintained* at $V = 0$, and they produce no electric field. If $\mathbf{E} \neq 0$, and we bring a charge in from infinity (where by definition we say $V = 0$), then the line integral would be nonzero due to $\mathbf{E}$ and $V \neq 0$!

Note that if you hold a positive charge near a conductor, that charge will attract the negative charges in the conductor to the surface closest to the charge and there will be attraction between the conductor and the charge. The charge has **induced** a charge in the conductor.

1.7.1 Surface Charge

Using (1.1), if the field inside the conductor is zero (i.e., the inside of a conductor is an equipotential), we find

$$\begin{aligned} \mathbf{E} &= \frac{\sigma}{\epsilon_0}\mathbf{n} \\ \sigma &= -\epsilon_0 \frac{\partial V}{\partial n} \end{aligned}$$

Which allows us to find the surface charge density on a conductor.

1.7.2 Force on a Conductor

Momentarily ignore the fact we are working with a conductor. With an electric field, a surface charge will experience a force. The force per unit area is $\sigma\mathbf{E} \equiv \mathbf{f}$, but with the discontinuity of the field at the surface, this value is actually ambiguous. Let's compute it as follows: consider a patch of surface charge small enough that it is approximately flat and of approximately constant charge density. The field consists of the electric field due to that patch and the electric field due to everything else:

$$\mathbf{E} = \mathbf{E}_{\text{patch}} + \mathbf{E}_{\text{other}}$$

The patch cannot exert a force on itself. Then the force on the path is due to $\mathbf{E}_{\text{other}}$. The patch produces a field $\sigma/2\epsilon_0$ on either side pointing normal to the surface. Then

$$\mathbf{E}_{\text{above}} = \mathbf{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}\mathbf{E}_{\text{below}} = \mathbf{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

Now we solve for $\mathbf{E}_{\text{other}}$:

$$\mathbf{E}_{\text{other}} = \frac{1}{2}(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}) = \mathbf{E}_{\text{average}}$$

Now, in the case of a conductor, the field below is zero, so the field outside is σ/ϵ_0 and

$$\mathbf{E} = \frac{1}{2\epsilon_0}\sigma\hat{\mathbf{n}} \Longrightarrow \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}}$$

1.8 Capacitors

Suppose two conductors, one with charge $+Q$ and another which charge $-Q$. With V constant and not necessarily zero on both conductors, we can talk about the potential difference between them:

$$V = V_+ - V_- = -\int_-^+ \mathbf{E}\cdot \mathrm{dl}$$

We know that

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}^2}\,\mathrm{d}\tau$$

and that doubling Q doubles ρ which doubles $\mathbf{E}$. This is due to the second uniqueness theorem. The field you get from total charge $2Q$ is the same as you should get if the field has strength $2\mathbf{E}$, for both these scenarios satisfy the same boundary conditions.

If **E** is proportional to Q, so is V. This constant of proportionality is called the **capacitance** of the system:

$$\boxed{C \equiv \frac{Q}{V}}$$

Capacitance depends on the geometry of the situation. You can think of it as the amount of charge you can fit on the conductor for a certain voltage difference.

Example: Consider a parallel plate capacitor consisting of metal surfaces of area A a distance d apart from each other, and with $+Q$ above and $-Q$ below. Assume the surface charge density is uniform. It is then $\sigma = Q/A$. Then the field is $(1/\epsilon_0)Q/A$, and the potential difference can be found by integrating from one plate to another

$$V = \int_0^d \frac{Q}{A\epsilon_0}\, \mathrm{d}z = \frac{Q}{A\epsilon_0} d$$

Which yields $C = \frac{A\epsilon_0}{d}$. Then for parallel-plate capacitors the amount of charge you can fit on the conductors per voltage difference increases with the area of the capacitor.

For n ¿ -2, you get a finite result for the integral which vanishes as the radius of the integration sphere goes to zero. For n = -2, you get a constant (from the Dirac delta function) For n ¡ -2, the volume integral of the divergence will diverge as the radius of the integration sphere goes to zero

Chapter 2

Potentials

2.1 Laplace's Equation in Electrostatics

We have previously shown that

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

This is Poisson's equation "with a source." Such an equation is quite difficult to solve; however, if we are looking for a solution in a domain where there is no charge (and this occurs more often than you'd think: the inside of a hollow conductor, the upper half of space, so on), then $\rho = 0$ and this reduces to to Laplace's equation:

$$\boxed{\nabla^2 V = 0}$$

The study of Laplace's equation is quite involved, and techniques to solve it can be found here. In this chapter we focus on details of Laplace's equation specific to electromagnetism.

2.2 Uniqueness Theorems

Uniqueness theorems give us extremely useful constraints on the solution of a particular differential equation. In the case of Laplace's equation, if we have specified values that the solution must take at a boundary (boundary conditions, in the form of a function), then the uniqueness theorem says that the solution that satisfies those boundary conditions is the *only* solution for that boundary. That means that, no matter the technique used to find it, as long as a function satisfies the boundary conditions in addition to Laplace's equation, that function is *the* solution.

2.2.1 First Uniqueness Theorem

First uniqueness theorem: The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on some boundary surface S.

Proof: Suppose two solutions to Laplace's equation.

$$\nabla^2 V_1 = 0 \text{ and } \nabla^2 V_2 = 0$$

both of which have the same value at the surface S. Take their difference

$$V_3 \equiv V_1 - V_2$$

This is a solution to Laplace's equation:

$$\nabla V_3 = \nabla V_1 - \nabla V_2 = 0$$

Which means that it must obey the maximum principle. But V_1 and V_2 have the same value at the surface; their difference at the surface is zero, and the value of V_3 at the surface is zero. Then, by the maximum principle, the maximum and minimum of V_3 are both zero and $V_1 = V_2$.

Remark: an improvement can be made on this theorem. Consider we are working with Poisson's equation instead of Laplace's equation—i.e., there is some inside the domain, and, with $\rho \neq 0$, we have two solutions satisfying boundary conditions and

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} \text{ and } \nabla^2 V_2 - \frac{\rho}{\epsilon_0}$$

(Note these ρ are the same). This yields

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0$$

Then V_3 again satisfies Laplace's equation and we again derive $V_1 = V_2$.

Corollary: The potential in a volume $\mathcal{V}$ is uniquely determined if the charge density throughout the region and the value of V on the boundaries is specified.

2.2.2 Second Uniqueness Theorem

Occasionally we will be given the charge on conducting surfaces at the boundaries of the solution's domain. We do not mean the charge *distribution*, only the *total* charge, which is free to arrange itself in any way it pleases. We might also be given a charge distribution inside the domain itself.

Second uniqueness theorem: in a volume $\mathcal{V}$ surrounded by conductors and maintaining a specified charged density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given.

Proof: Suppose two fields satisfy the conditions. Both obey Gauss's law:

$$\nabla \cdot \mathbf{E}_1 = \frac{\rho}{\epsilon_0} \text{ and } \nabla \cdot \mathbf{E}_2 = \frac{\rho}{\epsilon_0}$$

And both obey Gauss's law in integral form, where the surfaces enclose each conductor (i denotes the i-th conductor in the case that there are multiple):

$$\oint \mathbf{E}_1 \cdot \mathrm{d}\mathbf{a} = \frac{Q_i}{\epsilon_0} \text{ and } \oint \mathbf{E}_2 \cdot \mathrm{d}\mathbf{a} = \frac{Q_i}{\epsilon_0}$$

We again examine the difference

$$\mathbf{E}_2 \equiv \mathbf{E}_1 - \mathbf{E}_2 \Longrightarrow \nabla \cdot \mathbf{E}_3 = 0, \text{ and } \oint \mathbf{E}_3 \cdot \mathrm{d}\mathbf{a} = 0$$

We don't know how the charges Q_i arrange themselves, but we do know that each conductor is an equipotential and thus V_3 is some constant C_i over each conducting surface. We invoke the product rule to see

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3 (\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2$$

because $\nabla V_3 = -\mathbf{E}_3$. We integrate over $\mathcal{V}$ and apply the divergence theorem on the left hand side:

$$\int \nabla \cdot (V_3 \mathbf{E}_3) \, \mathrm{d}\tau = \oint V_3 \mathbf{E}_3 \cdot \mathrm{d}\mathbf{a} = -\int (E_3)^2 \, \mathrm{d}\tau$$

Since V_3 is a constant over each surface, it can be brought outside the integral. All that remains is $\oint \mathbf{E}_3 \cdot \mathrm{d}\mathbf{a}$, which we know is zero. Thus

$$\int (E_3)^2 \, \mathrm{d}\tau = 0$$

Because the integrand is squared, there can never be a negative contribution and thus $E_3 = 0$ everywhere, implying $\mathbf{E}_1 = \mathbf{E}_2$.

2.3 Electric Dipoles

Electric dipoles consist of two charges of opposite charge very close to one another. Henceforth we consider two types of dipoles: **physical** dipoles, which are a finite distance apart from one another, and **perfect** dipoles, which are an idealization of this scenario, two opposite charges in the same point.

For a physical dipole, we consider the potential a large distance away from the two charges; let the charges be a distance d apart from each other, and have charge $\pm q$. Then the potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

The law of cosines yields

$$\mathscr{r}_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

We let $r \gg d$, so the third term is negligible. The binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \approx \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

Which has

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \approx \frac{d}{r^2}\cos\theta$$

Which yields

$$V(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

If we have a configuration of four point charges, we would have a quadrupole; eight would make an octopole, and one would make a monopole. This is the approximate potential for a physical dipole, and the actual potential for a perfect dipole. For a perfect dipole, we let $d \to 0$, and we must let $q \to \infty$ because otherwise the charges would cancel each other out.

An alternate formula can be given by the multipole expansion that is derived in section 2.4. In the case of a perfect dipole, the dipole term is the only one that persists. In the case of a physical dipole, all other terms fall to zero as the distance from the dipole increases.

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\rho(\mathbf{r}')\,\mathrm{d}\tau'$$

α is the angle between $\mathbf{r}'$ and $\mathbf{r}$,

$$r'\cos\alpha = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}$$

and the dipole can be written

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}} \cdot \int \mathbf{r}'\rho(\mathbf{r}')\,\mathrm{d}\tau'$$

We call the integral the **dipole moment**, for it is a quantity which does not depend on $\mathbf{r}$, and thus one that is intrinsic to the system.

$$\boxed{\mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}')\,\mathrm{d}\tau'}$$

The dipole moment of a collection of point charges is analogously

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i$$

And the dipole's potential is

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}} \tag{2.1}$$

2.3.1 Electric Field of a Dipole

We calculate the electric field of a perfect dipole. Choose coordinates such that $\mathbf{p}$ is at the origin and points in the z direction. The potential at r, θ is then

$$V_{\text{dip}}(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

The spherical gradient is

$$\nabla V = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} - \frac{1}{4\sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{\phi}}$$

Which yields

$$\boxed{\mathbf{E}_{\text{dip}} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\hat{\boldsymbol{\theta}}}\right)}$$

But this can be put in a coordinate-free form. We are given

$$\begin{aligned}\mathbf{E}_{\text{dip}}(r,\theta) &= \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)\\ &= \frac{1}{4\pi\epsilon_0 r^3}\left(2p\cos\theta\hat{\mathbf{r}} + p\sin\theta\hat{\boldsymbol{\theta}}\right)\end{aligned}$$

It suffices to show that

$$2p\cos\theta\hat{\mathbf{r}} + p\sin\theta\boldsymbol{\theta} = 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}$$

θ is the angle between $\mathbf{p}$ and $\hat{\mathbf{r}}$, so

$$2(\mathbf{p}\cdot\hat{\mathbf{r}}) = 2p\cos\theta\hat{\mathbf{r}}$$

Then it remains to show

$$\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p} = p\sin\theta\hat{\boldsymbol{\theta}} \tag{2.2}$$

$\mathbf{p}$, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ lie in a plane, so we may investigate in two dimensions. One notices

$$\hat{\mathbf{r}} = \begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix},\ \hat{\boldsymbol{\theta}} = \begin{bmatrix}\cos\theta\\ -\sin\theta\end{bmatrix}$$

Returning to (2.2),

$$
\begin{aligned}
\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p} &= p\sin\theta\hat{\boldsymbol{\theta}} \\
\hat{\mathbf{r}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}) - \hat{\mathbf{p}} &= \sin\theta\hat{\boldsymbol{\theta}} \\
\begin{bmatrix}\sin\theta \\ \cos\theta\end{bmatrix}\cos\theta - \begin{bmatrix}0 \\ 1\end{bmatrix} &= \sin\theta\begin{bmatrix}\cos\theta \\ -\sin\theta\end{bmatrix} \\
\begin{bmatrix}\sin\theta\cos\theta \\ \cos^2\theta - 1\end{bmatrix} &= \begin{bmatrix}\sin\theta\cos\theta \\ -\sin^2\theta\end{bmatrix} \\
\begin{bmatrix}\sin\theta\cos\theta \\ -\sin^2\theta\end{bmatrix} &= \begin{bmatrix}\sin\theta\cos\theta \\ -\sin^2\theta\end{bmatrix}
\end{aligned}
$$

Then

$$
\boxed{\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]}
$$

2.4 Multipole Expansion

We've explored the case of perfect and physical dipoles while making mention of other terms that correspond to octpoles, quadripoles, so on. Here we see where these terms appear. We derive an expansion for *any* charge distribution in terms of pole terms. We know the potential of a continuous distribution is

$$
V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}\tau'
$$

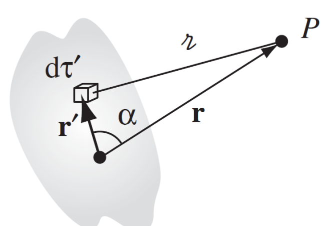

Figure 2.1

Using the law of cosines, one sees

$$
\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]
$$

We rewrite $\mathscr{r} = r\sqrt{1+\epsilon}$ where

$$
\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)
$$

For points far from the charge distribution, $r \gg r'$, leading to $\epsilon \ll 1$, which allows a binomial expansion.

$$
\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)
$$

in terms of r, r' and α,

$$\begin{aligned}
\frac{1}{\mathscr{r}} =& \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 + \dots\right] \\
=& \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \cdots\right]
\end{aligned}$$

The coefficients are actually Legendre polynomials (see section 7.2) in $\cos\alpha$. We can then rewrite

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

We substitute this back into our integral for V to find

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,\mathrm{d}\tau'}$$

Where the $1/r$ term is the monopole contribution, the $1/r^2$ term is the dipole contribution, the $1/r^3$ the quadripole, so on.

Chapter 3

Electric Fields in Matter

3.1 Induced Dipoles

When we place an atom in an electric field, the negatively charged electron cloud is drawn toward the field and the positively charged nucleus is pushed away. This produces a **polarized** atom. The atom will then have a dipole moment $\mathbf{p}$ pointing along the same direction as the field:

$$\mathbf{p} = \alpha \mathbf{E}$$

where α is the constant **atomic polarizability**.

Molecules are more complex, and their polarizability depends on the orientation of the molecule. This leads to the **polarizability** tensor:

$$A = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}$$

Which produces the matrix equation

$$\mathbf{p} = A\mathbf{E}$$

3.2 Alignment of Polar Molecules (Dipoles in Fields)

Some molecules are polar and have preexisting dipole moments. Suppose we place one such molecule in an uniform electric field. The force on the positive end is $q\mathbf{E}$ and the force on the negative end is $-q\mathbf{E}$; there is no net force. But these forces are applied at a radius from the center of mass, so there is a net torque. Letting the distance between the charges be d,

$$\begin{aligned} \mathbf{N} =& (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ =& [(\mathbf{d}/2) \times (q\mathbf{E})] + +[(-\mathbf{d}/2) \times (-q\mathbf{E})] \\ =& q\mathbf{d} \times \mathbf{E} \end{aligned}$$

Thus a dipole $q\mathbf{d}$ in a uniform field experiences a torque

$$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}}$$

In a nonuniform field, there will be a net force in addition to a torque:

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q\Delta\mathbf{E}$$

with $\Delta\mathbf{E} \equiv \mathbf{E}_+ - \mathbf{E}_-$. With a very short dipole one can treat $\Delta\mathbf{E}$ as a directional derivative along the dipole ($\delta\mathbf{E}$ is the difference in the electric field between either end of the dipole). Then $\Delta\mathbf{E} = (\mathbf{d} \times \nabla)\mathbf{E}$ and

$$\boxed{\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}}$$

3.3 Polarization

Instead of considering just one dipole in an electric field, we are now interested in several atoms, and the alignment of many molecules. We capture this effect by denoting $\mathbf{P}$ the dipole moment per unit volume. This measure is **polarization**.

This polarization will itself produce a field. We know the potential due to a dipole, (2.1). We have $\mathbf{p} = \mathbf{P}\,\mathrm{d}\tau'$.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

This integral is not particularly handy—we see that

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

Then

$$V = \frac{1}{4\pi\epsilon_0}\int \mathbf{P} \cdot \nabla'\left(\frac{1}{\mathscr{r}}\right)\mathrm{d}\tau'$$

We integrate by parts

$$V = \frac{1}{4\pi\epsilon_0}\left[\int \nabla' \cdot \left(\frac{\mathbf{P}}{\mathscr{r}}\right)\mathrm{d}\tau' - \int \frac{1}{\mathscr{r}}(\nabla' \cdot \mathbf{P})\,\mathrm{d}\tau'\right]$$

with the divergence theorem

$$V = \frac{1}{4\pi\epsilon_0}\oint \frac{1}{\mathscr{r}}\mathbf{P} \cdot \mathrm{d}\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}(\nabla' \cdot \mathbf{P})\,\mathrm{d}\tau'$$

The first term is an integration over a surface with $1/\mathscr{r}$. It looks like a surface charge:

$$\boxed{\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}}$$

The second term is an integral over a volume again with $1/\mathscr{r}$, suggesting a volume charge

$$\boxed{\rho_b \equiv -\nabla \cdot \mathbf{P}}$$

We call these **bound charges**. Then we have the potential due to the polarization:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma_b}{\mathscr{r}}\,\mathrm{d}a' + \frac{1}{4\pi\epsilon_0}\int \frac{\rho_b}{\mathscr{r}}\,\mathrm{d}\tau'$$

3.4 Electric Displacement

We've determined the field due to the polarization. Now we find the total field due to the **free** charge that is causing the polarization as well as the field due to the polarization itself. The charge density is

$$\rho = \rho_b + \rho_f$$

Gauss's law yields

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

We combine the divergence terms

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

We now define a useful quantity, the expression in parentheses, the **displacemenet D**:

$$\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}}$$

Then we have a Gauss's law-looking equation:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f \iff \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}} \tag{3.1}$$

It will turn out that $\mathbf{D}$ is easier to compute than the total electric field. It is hard to compute the total electric field because the field.

3.4.1 Boundary Conditions

First, one sees

$$\nabla \times \mathbf{D} = \epsilon_0 (\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}$$

for $\nabla \times \mathbf{E} = 0$ but $\nabla \times \mathbf{E} \neq 0$ not necessarily.

With (3.1), one can find

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$$

And the curl determined above yields

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$$

3.5 Linear Dialectrics

In many materials, the polarization is proportional to $\mathbf{E}$.

$$\boxed{\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}}$$

This constant χ_e is called the **electric susceptibility** of the medium. Then we find that the displacement is

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

Such that

$$\mathbf{D} = \epsilon \mathbf{E}$$

where

$$\epsilon \equiv \epsilon_0 (1 + \chi_e)$$

this is the the **permittivity** of the material. We also commonly denote

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

the **relative permittivity**.

3.5.1 Boundary Values

In a linear dialectric the bound charge density is proportional to the free charge density:

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f$$

We combine $\mathbf{D} = \epsilon \mathbf{E}$ with the boundary conditions for $\mathbf{D}$ we found before

$$\epsilon_{\text{above}} E^{\perp}_{\text{above}} - \epsilon_{\text{below}} E^{\perp}_{\text{below}} = \sigma_f$$

In terms of the potential,

$$\epsilon_{\text{above}} \frac{\partial V}{\partial n}\bigg|_{\text{above}} - \epsilon_{\text{below}} \frac{\partial V}{\partial n}\bigg|_{\text{below}} = -\sigma_f$$

where, recall, the potential is still continuous:

$$V_{\text{above}} = V_{\text{below}}$$

Chapter 4

Magnetostatics

4.1 Magnetic Fields and Magnetic Forces

We now start considering forces between charges in motion. Moving charges produce a **magnetic field B** which in no way interacts with electric charges.

The magnetic force on a charge Q moving with velocity $\mathbf{v}$ moving in a magnetic field $\mathbf{B}$ is the **Lorentz force law**

$$\boxed{\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})}$$

In the presence of electric *and* magnetic fields the net force on Q would be

$$\mathbf{F} = Q[E + (\mathbf{v} \cdot \mathbf{B})]$$

The Lorentz force law is an *axiom* of electromagnetism, it cannot be derived and is taken as an experimental result.

EXAMPLE: **Cyclotron motion**: typically, charged particles in magnetic fields exhibit circular motion. The magnetic force provides a centripetal force.

Notice that **magnetic forces do no work**. For take a bit of work done by the Lorentz force law:

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, \mathrm{d}t = 0$$

for $\mathbf{v} \times \mathbf{B}$ will always be perpendicular to $\mathbf{v}$. Then no kinetic energy can be added to the particle, and thus magnetic forces can only change the *direction* of a particle's motion, not its *speed*.

4.2 Currents

The **current** in a wire is the charge in a unit time passing by a given point. Analogously, this can be thought of as the mass of water passing by a given point in a river in unit time. Notice that negative charges moving in one direction is equivalent to positive charges moving in the opposite direction. Since one must take the product $q\mathbf{v}$, flipping the sign of q also flips the sign of $\mathbf{v}$ and thus many phenomena regarding moving charges are independent of whether we consider negative or positive charge moving.

Let us pretend it is the positive charge that moves, as this is the convention. Current is measured in couloumbs per second, or amperes:

$$1\ \mathrm{A} = 1\ \mathrm{C/s}$$

Suppose a line charge λ moving down a wire at speed v. Then over a period of time Δt, charge will have traversed a distance $v\Delta t$, such that the total charge having passed by a point is $\lambda v \Delta t$, and

$$I = \frac{\text{charge}}{\text{time}} = \frac{\lambda v \Delta t}{\Delta t} = \lambda v$$

As a matter of fact, one may choose to represent the direction of the velocity at that point as well,

$$\mathbf{I} = \lambda \mathbf{v}$$

Since this direction is captured by the direction of the wire, we typically do not express it is a vector quantity. In a many-dimensional case, it is not so simple, for there might be many different directions and many different surfaces through which the current could be measured.

The magnetic force on a segment of current-carrying wire is

$$\mathbf{F} = \int (\mathbf{v} \times \mathbf{B})\,\mathrm{d}q = \int (\mathbf{v} \times \mathbf{B})\lambda\,\mathrm{d}l = \int (\mathbf{I} \times \mathbf{B})\,\mathrm{d}l$$

If, as is common, $\mathbf{I}$ and $\mathrm{d}\mathbf{l}$ (the vector $\mathrm{d}l$) point in the same direction,

$$\boxed{\mathbf{F} = \int I(\mathrm{d}\mathbf{l} \times \mathbf{B})}$$

Typically the current is constant in magnitude along the wire and we may place I in front of the integral:

$$\mathbf{F}_{\text{mag}} = I$$

4.2.1 Charge Flow Over a Surface

When charge flows over an *surface*, one describes it by the **surface current density K**. We consider a ribbon of width $\mathrm{d}l_\perp$ running parallel to the flow. The current in this ribbon (think of the wire case) is $\mathrm{d}I\mathbf{I}$, and the current density is

$$\mathbf{K} \equiv \frac{\mathrm{d}\mathbf{I}}{\mathrm{d}l_\perp}$$

Thus K is the current per unit width, with the vector value pointing along the direction tangential to the surface along the direction of the current. If the surface charge density is σ and its velocity is $\mathbf{v}$ then

$$\mathbf{K} = \sigma \mathbf{v}$$

This works out because we are considering the total change in charge in one direction, thereby reducing the density by one dimension upon multiplying by $\mathbf{v}$. In general, $\mathbf{K}$ will vary from point to point over the surface due to variations in either σ, $\mathbf{v}$, or both.

The magnetic force on the surface current is then

$$\mathbf{F} = \int (\mathbf{v} \times \mathbf{B})\sigma\,\mathrm{d}a = \int (\mathbf{K} \times \mathbf{B})\,\mathrm{d}a$$

where we've exploited the fact that we must integrate over the entire surface, and thus utilize a surface charge density to express the integral in terms of $\mathbf{K}$.

Note that $\mathbf{B}$ is discontinuous at a surface current, so we must use the average field over both sides.

4.2.2 Charge Flow Over a Volume

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the **volume current density J**. Consider a tube of infinitesimal cross section $\mathrm{d}a_\perp$ running parallel to the current. If the current in this tube is $\mathrm{d}\mathbf{I}$ the volume current density is

$$\mathbf{J} \equiv \frac{\mathrm{d}\mathbf{I}}{\mathrm{d}a_\perp}$$

Thus J is the current per unit area, with the area being normal to the vector $\mathbf{J}$. If the volume charge density is ρ and the velocity is $\mathbf{v}$ then

$$\mathbf{J} = \rho\mathbf{v}$$

Again, this makes sense because we reduce by one dimension and obtain the unit area. The magnetic force on a volume charge is then

$$\mathbf{F} = \int (\mathbf{v} \times \mathbf{B})\rho\,\mathrm{d}\tau = \int (\mathbf{J} \times \mathbf{B})\,\mathrm{d}\tau$$

This suggests that total charge crossing a surface S can be written as

$$I = \int_S J\,\mathrm{d}a_\perp = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{a}$$

In particular, the charge per unit time leaving a volume V is (by the divergence theorem)

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{a} = \int_V (\nabla \cdot \mathbf{J})\,\mathrm{d}\tau$$

Because charge is conserved, whatever flows out comes at the expense of what remains inside. The charge density must decrease inside the volume:

$$\int_V (\nabla \cdot \mathbf{J})\,\mathrm{d}\tau = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}\tau = -\int_V \left(\frac{\partial\rho}{\partial t}\right)\mathrm{d}\tau$$

Where we switch to a partial derivative because ρ might also vary with position and thus be multivariate. Since this applies to any volume, one can get rid of the integrals to obtain

$$\nabla \cdot \mathbf{J} = -\frac{\partial\rho}{\partial t} \tag{4.1}$$

The is the mathematical statement of local charge conservation, the **continuity equation**. Then, analogously to the relationship between $q \sim \lambda\,\mathrm{d}l \sim \sigma\,\mathrm{d}a \sim \rho\,\mathrm{d}\tau$, we have

$$\sum_{i=1}^{n} q_i\mathbf{v}_i \sim \int_{\text{line}} \mathbf{I}\,\mathrm{d}l \sim \int_{\text{surface}} \mathbf{K}\,\mathrm{d}a \sim \int_{\text{volume}} \mathbf{J}\,\mathrm{d}\tau$$

4.3 Steady Currents

Stationary charges produce electric fields that are constant in time; hence electro*statics*. Steady currents (constant velocity, goes on indefinitely, no charge "piling up" anywhere) produce magnetic fields constant in time, leading to the theory of **magnetostatics**. Formally, magnetostatics is

$$\frac{\partial\rho}{\partial t} = 0, \quad \frac{\partial\mathbf{J}}{\partial t} = 0\text{-vector}$$

Then notice we get a Lagrange equation from (4.1):

$$\nabla \cdot \mathbf{J} = 0 \tag{4.2}$$

4.4 The Biot-Savart Law

The magnetic field of a steady line current is given by the **Biot-Savart law**:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}l' = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}$$

We integrate along the path of the current and in the direction of the flow. $\mathrm{d}\mathbf{l}'$ is an element of wire length, and $\boldsymbol{\mathscr{r}}$ is the vector from the source to the point $\mathbf{r}$. The constant μ_0 is the **permeability of free space**.

The units are such that the unit of $\mathbf{B}$ is newtons per ampere-meter, or **teslas** (T).

$$1\ \mathrm{T} = 1\ \mathrm{N/(A\cdot m)}$$

The Biot-Savart law is analogous to Couloumb's law in electrostatics, determining the strength of a field that you may place a test charge in. It is not derived from the Lorentz force law but it conveys the same idea: charges in motion produce magnetic forces.

For surface currents, the Biot-Savart law reads

$$\mathbf{B}(r) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}a'$$

For volume currents,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}\tau'$$

4.4.1 Magnetic Field of a Wire

$$B = \frac{\mu_0 I}{2\pi s} \tag{4.3}$$

4.5 Divergence of B

The magnetic field of an infinite straight wire consists of several concentric field lines following the right hand rule. It is clear that this is a field with nonzero curl. The integral of the magnetic field around a circular path of radius s centered at the wire is, with (4.3), is

$$\oint \mathbf{B}\cdot\mathrm{d}\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\,\mathrm{d}l = \frac{\mu_0 I}{2\pi s}\oint \mathrm{d}l = \mu_0 I$$

For the strength remains constant and we simply obtain the circumference of a circle of radius s from the integral. As a matter of fact, any loop enclosing the wire yields the same answer. With cylindrical coordinates (s,ϕ,z) and the current flowing along the z axis, we have $\mathbf{B} = (\mu_0 I/2\pi s)\hat{\boldsymbol{\phi}}$ and $\mathrm{d}l = \mathrm{d}s\,\hat{\mathbf{s}} + s\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\mathbf{z}}$. Then

$$\oint \mathbf{B}\cdot\mathrm{d}\mathbf{l} = \oint \Big((\mu_0 I/2\pi s)\hat{\boldsymbol{\phi}}\Big)\cdot\Big(\mathrm{d}s\,\hat{\mathbf{s}} + s\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\mathbf{z}}\Big) = \frac{\mu_0 I}{2\pi}\int_0^{2\pi}\frac{1}{s}\cdot s\,\mathrm{d}\phi = \mu_0 I$$

For the field only contributes in the ϕ direction anyway.

Suppose we have a bundle of straight wires. Each wire passing through the loop contributes $\mu_0 I$, as the position and shape of the loop with respect to the wire does not matter. The line integral is thus

$$\oint \mathbf{B}\cdot\mathrm{d}\mathbf{l} = \mu_0 I_{\mathrm{enc}} \tag{4.4}$$

with I_{enc} is the total current enclosed by the integration path. If the flow of charge is represented by a volume current density $\mathbf{J}$, the enclosed current is

$$I_{\text{enc}} = \int \mathbf{J} \cdot \mathrm{d}a$$

where the integral is over any capping surface of the loop. One applies Stokes' to (4.4) find

$$\int (\nabla \times \mathbf{B}) \cdot \mathrm{d}\mathbf{a} = \mu_0 \int \mathbf{J} \cdot \mathrm{d}\mathbf{a}$$

Which implies

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

4.5.1 Formal Derivation

The above derivation is fine for an intuitive understanding but it does not work for currents that are not straight. We now derive the curl through the Biot-Savart law. Recall that in the case of a volume current the law is

$$\mathbf{B}(\mathbf{r} = \frac{\mu_0}{4\pi}) \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

We recognize that

$$\begin{aligned} \mathbf{B} &\text{ is a function of } (x, y, z) \\ \mathbf{J} &\text{ is a function of } (x', y', z') \\ \boldsymbol{\mathscr{r}} &= (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \\ \mathrm{d}\tau' &= \mathrm{d}x' \, \mathrm{d}y' \, \mathrm{d}z' \end{aligned}$$

The integration is over the primed coordinates and the divergence and curl of $\mathbf{B}$ are with respect to unprimed coordinates.

Applying the divergence to the Biot-Savart law, we obtain

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \mathrm{d}\tau'$$

We invoke a product rule to find

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right)$$

But $\nabla \times \mathbf{J} = 0$ because $\mathbf{J}$ doesn't depend on the unprimed coordinates—it is a function of (x', y', z'). Meanwhile, $\nabla \times (\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2) = 0$ (a known fact) so

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Applying the curl to the Biot-Savart law,

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \mathrm{d}\tau' \tag{4.5}$$

We again expand the integrand using a product rule:

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \tag{4.6}$$

We ignore derivatives of $\mathbf{J}$ because $\mathbf{J}$ doesn't depend on unprimed coordinates. The first term is the divergence which leads to a Dirac delta:

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi\delta^3(\mathscr{r})$$

The second term integrates to zero. The derivative acts on $\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$, so we switch from ∇ to ∇' and add a minus sign:

$$-(\mathbf{J} \cdot \nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = (\mathbf{J} \cdot \nabla')\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}$$

Where we are applying $(\mathbf{J} \cdot \nabla')$ to each component individually as if it is a scalar multiplication. The x component is

$$(\mathbf{J} \cdot \nabla')\left(\frac{x - x'}{\mathscr{r}^3}\right) = \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3}\mathbf{J}\right] - \left(\frac{x - x'}{\mathscr{r}^3}\right)(\nabla' \cdot \mathbf{J})$$

Now for *steady* currents the divergence of $\mathbf{J}$ is zero (see section 4.3), so

$$\left[-(\mathbf{J} \cdot \nabla)\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^2}\right]_x = \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3}\mathbf{J}\right]$$

Then the contribution of the second term to the integral is

$$\int_V \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3}\mathbf{J}\right] \mathrm{d}\tau' = \oint_S \frac{(x - x')}{\mathscr{r}^3}\mathbf{J} \cdot \mathrm{d}\mathbf{a}'$$

(by the divergence theorem). We may make this region as large as we like, since it is simply the volume appearing in the Biot-Savart law and $\mathbf{J} = 0$ outside that volume anyway. As such, one may integrate such that the current on the boundary is zero, and this integral vanishes.

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}$$

Then one finds the second term of (4.6) is zero and (4.5) is

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi\delta^3(\mathbf{r} - \mathbf{r}') \, \mathrm{d}\tau' = \mu_0 \mathbf{J}(\mathbf{r})$$

4.6 Ampère's Law

The equation for the curl of $\mathbf{B}$,

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

is known as **Ampère's Law** (differential form). It can be converted to integral form by applying Stokes' theorem:

$$\int (\nabla \times \mathbf{B}) \cdot \mathrm{d}\mathbf{a} = \oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \int \mathbf{J} \cdot \mathrm{d}\mathbf{a}$$

We know that $\int \mathbf{J} \cdot \mathrm{d}\mathbf{a}$ is the total current passing through the surface, which we will call I_{enc} (the current enclosed by the "**Amperian loop**"). Then

$$\boxed{\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}}}$$

is the integral version of Ampère's law. This is the magnetic analogue of Gauss's law in electrostatics. Thus

$$\text{Couloumb's law} \Longrightarrow \text{Gauss's law}$$
$$\text{Biot-Savart law} \Longrightarrow \text{Ampère's law}$$

Just as with Gauss's law, Ampère's law is always true for steady currents, but it is not always useful. Only when we may exploit symmetry and pull B out of the integral $\oint \mathbf{B} \cdot d\mathbf{l}$ can we calculate the magnetic field from Ampère's law.

4.7 Boundary Conditions

The electric field has a discontinuity at a surface charge, and so does the magnetic field. Let's apply the divergence of the magnetic field in integral form:

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$

to a thin pillbox straddling the surface. This is a surface integral for a surface the sides of which contribute very little. Thus the amount going in through one side must be equal to the amount going out the other side!

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

For the tangential components, we draw an Amperian loop running perpendicular to the surface current above for a length l and below for a length l. Assuming $\mathbf{B}$ remains constant on either side along this length, we simply have to multiply by l and take the difference of either side:

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}}\right) l = \mu_0 I_{\text{enc}} = \mu_0 K l$$

Dividing both sides by l,

$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

Thus the component parallel to the surface but perpendicular to the current is discontinuous by $\mu_0 K$.

TODO DRAW AMPERIAN LOOP PARALLEL TO SHOW IT IS CONTINUOUS.

THen we have

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$$

4.8 Magnetostatics vs. Electrostatics

Now we can put together **Maxwell's equations** for electrostatics, the divergence and curl of the electric field:

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \epsilon_0 \\ \nabla \times \mathbf{E} = 0 \end{cases}$$

and Maxwell's equations for magnetostatics, the divergence and curl of $\mathbf{B}$:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \end{cases}$$

Along with the force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} + \mathbf{B})$$

Chapter 5

Magnetic Fields in Matter

5.1 Diamagnets, Paramagnets, Ferromagnets

Materials have tiny currents in them—electrons orbiting nuclei. These currents are so small that they can be regarded as magnetic dipoles. They also cancel each other out because of the non-uniform orientation of the atoms. However, just as a dialectric can be polarized when an electric field is applied, materials can become **magnetized** (magnetically polarizes) upon application of a magnetic field.

Electric polarization is usually in the same direction as $\mathbf{E}$, byt magnetization can be parallel to $\mathbf{B}$ (**paramagnets**), opposite to $\mathbf{B}$ (**diamagnets**), and some substances retain magnetization after the external field has been removed—**ferromagnets**.

5.2 Torques and Forces on Magnetic Dipoles

A magnetic dipole experiences a torque in a magnetic field just as an electric dipole does in an electric field.

Let us construct a rectangular current loop in a uniform field $\mathbf{B}$. (Any arbitrary loop can be built from infinitely many rectangular loops, for the inner sides of pairs of loops would cancel each other.) Center the loop at the origin, tilt it an angle θ from the z axis. Let $\mathbf{B}$ point in the z direction. The forces on the sloping sides (let their length be a) cancel each other out. The forces on the non-sloping sides (let their length be b) are equal and opposite and thus there is no translation. However, there is a generated torque.

$$\mathbf{N} = 2 \cdot \frac{a}{2} \sin\theta = aF \sin\theta \hat{\mathbf{x}}$$

the magnitude of the force is

$$F = \lambda v B \cdot b = IbB$$

and thus

$$\mathbf{N} = IabB \sin\theta \hat{\mathbf{x}} = mB \sin\theta \hat{\mathbf{x}}$$

or

$$\mathbf{N} = \mathbf{m} \cdot \mathbf{B}$$

where $m = Iab$ is the magnetic dipole moment of the loop (see equation Notice that this is the analogue to the electric case: $\mathbf{N} = \mathbf{p} \times \mathbf{E}$. Again one sees the torque is such that the dipole is parallel to the field.

In a uniform field, the net force on a current loop is zero; let us integrate over the loop to find the total force on it:

$$\mathbf{F} = I \oint (\mathrm{d}\mathbf{l} \times \mathbf{B}) = I \left(\oint \mathrm{d}\mathbf{l} \right) \times \mathbf{B} = 0$$

for $\oint d\mathbf{l} = 0$ (we can take $\mathbf{B}$ out because it is constant). In a a nonuniform field this is not true. For an infinitesimal loop with dipole moment $\mathbf{m}$ in a field $\mathbf{B}$, the force is

$$\boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})}$$

5.3 Effect of Magnetic Field on Atomic Orbit

Assume orbit of an electron is a circle of radius R. We can treat this as a steady current because the period is so short.

$$I = \lambda v = \frac{e}{2\pi R} \cdot v = -\frac{ev}{2\pi R} = \frac{-e}{T}$$

Where e is the charge of the electron, and with

$$\frac{2\pi}{T} = \omega = \frac{v}{R} \Longrightarrow T = \frac{v}{2\pi R}$$

The orbital dipole moment is then

$$\mathbf{m} = -\frac{1}{2} evR\hat{\mathbf{z}}$$

Where $\mathbf{m} = I\mathbf{a}$. The tilt on the entire orbit is minimal because of the increased mass, so the contribution to paramagnetism is minimal compared to the contribution due to the tilt of the electron's spin axis. However, this force does speed up or slow down the orbit of the electron, depending on the orientation of $\mathbf{B}$.

Normally the centripetal acceleration v^2/R is due to the electrical forces along ($m_e \equiv$ electron mass):

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} = m_e \frac{v^2}{R^2}$$

But in the presence of a magnetic field there's an additional force $-e(\mathbf{v} \times \mathbf{B})$. Suppose $\mathbf{B}$ is perpendicular to the plane of the orbit. Then

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} + e\bar{v}B = m_e \frac{\bar{v}^2}{R}$$

Under these conditions the new speed $\bar{v}$ is greater than v. Let us compare them by subtracting the equations of motion:

$$e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v} + v)(\bar{v} - v)$$

Assuming the difference is small, $\bar{v}$ and v are roughly the same

$$evB = \frac{m_e}{R}(2v)(\Delta v) \Longrightarrow \frac{ebR}{2m_e} = \Delta v$$

And thus there is a positive increase in speed. A change in orbital speed leads to a change in dipole moment:

$$\Delta\mathbf{m} = -\frac{1}{2} e(\Delta v)R\hat{\mathbf{z}} = -\frac{e^2 R^2}{4m_e}$$

The change in $\mathbf{m}$ is opposite the direction of $\mathbf{B}$ (this would be the case even if the orbit was the other direction). So these atoms pick up a little extra dipole moment, with increments antiparallel to the field. This is called **diamagnetism**. It is much weaker than paramagnetism, so it only occurs in atoms with even number of electrons, wherein paramagnetism does not occur.

5.4 Magnetization

Analogously to the electric dipole moment per unit volume $\mathbf{P}$, we have the magnetization

$$\mathbf{M} \equiv \text{magnetic dipole moment per unit volume}$$

5.5 Bound Currents

Suppose a piece of magnetized material with given $\mathbf{M}$. What magnetic field is produced by this object? The vector potential of a single dipole is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\boldsymbol{\imath}}}{\imath^2}$$

In the magnetized object, each volume element $\mathrm{d}\tau'$ carries a dipole moment $\mathbf{M}\,\mathrm{d}\tau'$. The vector potential is then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, \mathrm{d}\tau'$$

Now we choose to exploit the identity $\hat{\boldsymbol{\imath}}/\imath^2 = \nabla'(1/\imath)$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_9}{4\pi} \int \left[\mathbf{M}(\mathbf{r}') \times \left(\nabla' \frac{1}{\imath}\right)\right] \mathrm{d}\tau'$$

Integrating by parts, we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[\int \frac{1}{\imath} [\nabla' \times \mathbf{M}(\mathbf{r}')]\, \mathrm{d}\tau' - \int \nabla' \times \left[\frac{\mathbf{M}(\mathbf{r}')}{\imath}\, \mathrm{d}\tau'\right]\right]$$

Using identity (...) we can express the latter term as a surface integral

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{\imath} [\nabla' \times \mathbf{M}(\mathbf{r}')]\, \mathrm{d}\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\imath} [\mathbf{M}(\mathbf{r}') \times \mathrm{d}\mathbf{a}']$$

The first term looks like the potential of a volume current, with

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M}}$$

and the second looks like the potential of a surface current:

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}}$$

Which leads to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(\mathbf{r}')}{\imath}\, \mathrm{d}\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{\imath}\, \mathrm{d}a'$$

5.6 The Auxiliary Field

Just as in the electric case, we put together the magnetic field generated by the magnetized material and the magnetic field generated by other currents, **free currents**. We may write the total current as

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f$$

Using our expression for bound volume current and Ampère's law can be written

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

We subtract $\nabla \times \mathbf{M}$ from both sides to obtain

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} \times \mathbf{M} \right) = \mathbf{J}_f$$

We may write the quantity in parenthesis

$$\boxed{\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}} \tag{5.1}$$

Then $\mathbf{H}$ captures the total magnetic field as well as the magnetization; it is a purely mathematical quantity with no physical meaning. It is, loosely speaking, the "field" that produces the free current $\mathbf{J}_f$ alone (for we subtract $\mathbf{M}$).

Ampère's law then reads

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f} \iff \oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{f_{\text{enc}}} \tag{5.2}$$

Where $I_{f_{\text{enc}}}$ is the total free current passing through the Amperian loop.

Now we are capable of solving for things using the free current alone, without having to worry about the effect of the free current on the bound current.

5.6.1 Boundary conditions

We may rewrite the boundary conditions of section 4.7. In terms of $\mathbf{H}$ and the *free* current. One sees that the only parallel that remains is

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$$

And (5.2) yields

$$\mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

(See the equations in the referenced section; we draw a parallel between $\mathbf{K}$ and $\mathbf{K}_f$ and $\mathbf{H}$ and $\mathbf{B}$).

5.7 Linear and Nonlinear Media

In paramagnetic and diamagnetic materials, the magnetization is simply due to the presence of the magnetic field, and when $\mathbf{B}$ is removed $\mathbf{M}$ also disappears. For most substances, the magnetization is actually proportional to the field. One can then write

$$\mathbf{M} = \frac{1}{\mu_0}\chi_m \mathbf{B}$$

although convention is to write in terms of $\mathbf{H}$:

$$\begin{aligned} \mathbf{M} &= \frac{1}{\mu_0}\chi_m \cdot \mu_0(\mathbf{H} + \mathbf{M}) \\ \mathbf{M}(1 - \chi_m) &= \chi_m \mathbf{H} \end{aligned}$$

Yielding

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}}$$

where we've renamed the constant from $\chi_m/(1-\chi_m)$ to χ_m; this variable is the **magnetic susceptibility**. Materials obeying this equation are called **linear media**. Now we substitute this into (5.1) to see that

$$\begin{aligned}
\mathbf{H} &= \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \\
\mathbf{B} &= \mu_0(\mathbf{H} + \mathbf{M}) \\
&= \mu_0(\mathbf{H} + \chi_m \mathbf{H}) \\
&= \mu_0(1 + \chi_m)\mathbf{H}
\end{aligned}$$

Thus $\mathbf{B}$ is also proportional to $\mathbf{H}$ by

$$\mathbf{B} = \mu \mathbf{H}$$

Where as with the electrical analogue:

$$\mu \equiv \mu_0(1 + \chi_m)$$

With χ_m (no material), this just becomes $\mu = \mu_0$.

$$\frac{\mu_0 I}{4\pi} \oint \frac{\mathrm{d}\ell \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{2\,\mathrm{d}\theta \cos\phi}{R^2 + z^2}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{R}{\sqrt{z^2 + R^2}}$$

$$\frac{\mu_0 I}{4\pi} \frac{R^2 \cdot 2\pi}{R^2 + z^2} = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}$$

If $z \gg R$, then $\sqrt{R^2 + z^2} \approx z$ Then

$$\frac{\mu_0 I R^2}{2z^3}$$

Chapter 6

Electrodynamics

6.1 Ohm's Law

How fast charges move given a certain push depends on the material. For most substances, with $\mathbf{J}$ being the current density and $\mathbf{f}$ being the force per unit charge, and σ a constant called **conductivity**:

$$\mathbf{J} = \sigma \mathbf{f}$$

Usually we determine σ empirically. Often we are given $1/\sigma = \rho$, the **resistivity** (the lower the resistivity, the higher the current density per unit force).

The force that acts on the charges could be any force, but most times it is an electromagnetic force:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} + \mathbf{B})$$

Usually $\mathbf{v}$, the velocity of the charges, is small enough that the second term can be ignored and

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}$$

(notice we need not write $\mathbf{f} = Q\mathbf{E}$) because $\mathbf{f}$ is already the force per unit charge, $\mathbf{f} = \mathbf{F}/Q$. This is called **Ohm's Law**.

Recall that, supposedly, $\mathbf{E} = 0$ inside a conductor. This is still true in the electrostatic case: $\mathbf{J} = 0$ and $\mathbf{J}/\sigma = 0 = \mathbf{E}$. Also, for idealistic "perfect" conductors, $\sigma = \infty$ and $\mathbf{E} = \mathbf{J}/\sigma = 0$ even with nonzero current. Metals are good enough conductors that the electric field required to produce a current is practically none.

<u>Example:</u> Suppose a cylindrical resistor of cross-sectional area A and length L with conductivity σ. Suppose potential is constant at both ends, and the potential difference is V. The current that flows is then

$$I = AJ = \sigma AE$$

Assume the electric field is constant (think of two parallel plates). Then

$$V(L) - V(0) = V = \int_0^L E \, \mathrm{d}l = EL \Longrightarrow E = \frac{V}{L}$$

Thus

$$I = \sigma A \frac{V}{L} \tag{6.1}$$

<u>Example:</u> Suppose two long coaxial metal cylinders with radii a and b separated by a material with conductivity σ. If their potential difference is V, what current flows from one to the other in a length L.

First the field the cylinder produces. Assume its linear charge density to be λ. Draw a Gaussian surface—a cylinder of length L a radius s away from the center of the innermost metal cylinder. Gauss's law yields

$$\begin{aligned}\oint \mathbf{E}\cdot \mathrm{d}\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0}\\ 2\pi sLE &= \frac{\lambda L}{\epsilon_0}\\ \mathbf{E} &= \frac{\lambda \hat{\mathbf{s}}}{2\pi s\epsilon_0}\end{aligned}$$

The potential difference is

$$\begin{aligned}V(b)-V(a) &= \int_b^a \mathbf{E}\cdot \mathrm{d}\mathbf{l}\\ V &= \frac{\lambda}{2\pi\epsilon_0}\ln\left|\frac{a}{b}\right|\\ \frac{2\pi\epsilon_0 V}{\ln|a/b|} &= \lambda\end{aligned}$$

Substituting,

$$\mathbf{E} = \frac{V}{\ln|a/b|}\frac{\hat{\mathbf{s}}}{s}$$

Then, using the same surface to find the current at a radius s away from the center of the innermost cylinder, with $A = 2\pi sL$:

$$\begin{aligned}I = AJ &= A\cdot\frac{\sigma V}{\ln|a/b|}\frac{1}{s}\\ &= \frac{2\pi L\sigma}{\ln|a/b|}V \qquad (6.2)\end{aligned}$$

6.1.1 The More Familiar Ohm's Law

Equations (6.1) and (6.2) suggest we write Ohm's law as

$$\mathbf{V} = \mathbf{IR}$$

i.e., it suggests that the current from one point in a resistor to another is proportional to the potential difference. This holds because we can always express the electric field in terms of the potential—in fact, this is easier to control than the actual electric field. R is called the **resistance**, and it is a function of the conductivity and the geometry of the set up.

Materials for which this holds are called **ohmic materials**. For such materials with steady currents and uniform conductivity, we can use the fact from (4.2) to see

$$\nabla\cdot\mathbf{E} = \frac{1}{\sigma}\nabla\cdot\mathbf{J} = 0$$

From this we glean that there is no accumulation of charge anywhere inside the conductor and thus the charge density is zero and all unbalanced charge is on the surface. Also, it follows that we can use Laplace's equation to solve for the potential.

6.2 Electromotive Force

There are two forces driving current around a circuit, the force due to the source $\mathbf{f}_s$, which is confined to one part of the loop (e.g., the battery), and the electrostatic force which keeps the current smooth and uniform:

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

We would like to know the net effect of $\mathbf{f}_S$. Turns out it is determined by the line integral of the total force $\mathbf{f}$:

$$\boxed{\mathcal{E} = \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \oint \mathbf{f}_s \cdot \mathrm{d}\mathbf{l}}$$

Where we've used the fact that $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0$ for electrostatic fields. $\mathcal{E}$ is the **emf** of the circuit (short for **electromotive force**, although it is not really a force).

With an ideal source of emf, say, a resistanceless battery, one sees $\mathbf{J}/\sigma = 0 = \mathbf{f}$ with $\sigma = \infty$. Thus $\mathbf{E} = -\mathbf{f}_s$. Then, with $\mathbf{f}_s$ zero outside the line along the source (say it begins at a and ends at b), the emf is

$$\mathcal{E} = \oint \mathbf{f}_s \cdot \mathrm{d}\mathbf{l} = \int_a^b \mathbf{f}_s \cdot \mathrm{d}\mathbf{l}$$

But $\mathbf{f}_s = -\mathbf{E}$. Then this is

$$\mathcal{E} = -\int_a^b \mathbf{E} \cdot \mathrm{d}l = V$$

This source effectively maintains the voltage difference equal to the emf.

Being the line integral of $\mathbf{f}_s$, $\mathcal{E}$ can be interpreted as the work done unit charge by the source.

6.2.1 Motional emf

Generators exploit **motional emfs**, which occur when you move a wire through a magnetic field. Suppose a current loop with a resistor R partially permeated by a magnetic field. If the entire loop is pulled to the right at a speed v, the magnetic field exerts a force on each charge with magnitude in the vertical direction equal to qvB in the clockwise direction. The emf is

$$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot \mathrm{d}\mathbf{l} = vBh$$

where h is the height of the loop.

As the charges move up, the right hand rule leads to the magnetic field also producing a force to the left, against the motion of the wire. Call the charges' vertical velocity $\mathbf{u}$. Then the magnetic force has a component quB to the left. The person pulling the wire must thus pull with with a force per unit charge

$$f_{\text{pull}} = uB$$

The particle is moving in the direction of the resultant velocity $\mathbf{w} = \mathbf{v} + \mathbf{u}$; the distance it travels is $h/\cos\theta$. The work done per unit charge is thus

$$\int \mathbf{f}_{\text{pull}} \cdot \mathrm{d}\mathbf{l} = (uB)(\frac{h}{\cos\theta})\sin\theta = vBh = \mathcal{E}$$

Thus the work done per unit charge is exactly the emf. The emf was calculated by constructing an inertial coordinate system that follows the motion of the circuit, and the work done per unit charge was calculated using a laboratory coordinate system.

We derive a different way to express the emf generated in a moving loop. Let Φ be the flux of $\mathbf{B}$ through the (interior of the) loop.

$$\Phi \equiv \int \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

For the rectangular loop we have been discussing, we see

$$\Phi = Bhx$$

As the loop moves the flux decreases:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = Bh\frac{\mathrm{d}x}{\mathrm{d}t} = -Bhv$$

where $\mathrm{d}x/\mathrm{d}t = -v$. This is exactly the emf we derived earlier. Thus we find the **flux rule** for motional emf:

$$\boxed{\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}}$$

(Note that indeed the region of the loop that is being influenced by a magnetic field must be increasing, for if the whole loop is moving in the magnetic field, there are opposing currents that will cancel each other out.)

Proof: Consider a current loop at times t and $t + \mathrm{d}t$. Compute the flux at a time t using surface $\mathcal{S}$ and the flux at time $t + \mathrm{d}t$ using $\mathcal{S}$ plus the ribbon that connects the two loop positions. The change in flux is then

$$\mathrm{d}\Phi = \Phi(t + \mathrm{d}t) - \Phi(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

Focus on a point P. In time $\mathrm{d}t$, it moves to point P'. Let $\mathbf{v}$ be the velocity of the wire and $\mathbf{u}$ be the velocity of a charge along the wire. $\mathbf{w} = \mathbf{v} + \mathbf{u}$ is the total velocity relative to a stationary frame. The infinitesimal element of area on the ribbon can be written $\mathrm{d}\mathbf{a} = (\mathbf{v} \times \mathrm{d}\mathbf{l})\,\mathrm{d}t$. Therefore

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint \mathbf{B} \cdot (\mathbf{v} \times \mathrm{d}\mathbf{l})$$

With $\mathbf{w} = \mathbf{v} + \mathbf{u}$ and $\mathbf{u}$ parallel to $\mathrm{d}\mathbf{l}$ we may write this

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint \mathbf{B} \cdot (\mathbf{w} \times \mathrm{d}\mathbf{l})$$

We can write this triple product as

$$\mathbf{B} \cdot (\mathbf{w} \times \mathrm{d}\mathbf{l}) = -(\mathbf{w} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l}$$

Such that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\oint (\mathbf{w} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l}$$

$\mathbf{w} \times \mathbf{B}$ is the magnetic force per unit charge (for $\mathbf{w}$ is the total velocity of the charge), so $\mathbf{w} \times \mathbf{B} = \mathbf{f}_{\text{mag}}$ and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\oint \mathbf{f}_{\text{mag}} \cdot \mathrm{d}\mathbf{l}$$

and the integral of $\mathbf{f}_{\text{mag}}$ is defined to be the emf, so

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

6.3 Faraday's Law

A changing magnetic field induces an electric field. We know

$$\mathcal{E} = \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l}$$

But when we change the magnetic field without moving the charges at all, it happens that a current flows. Since there is no applied force $\mathbf{f}_s$ and $\mathbf{f} = \mathbf{f}_s + \mathbf{E}$, we presume that there must be some "induced" electric field such that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l}$$

Since $\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{a}$, $\mathrm{d}\Phi/\mathrm{d}t = \int \partial\mathbf{B}/\partial t \, \mathrm{d}\mathbf{a}$ and

$$\mathcal{E} = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{a}$$

Using Stokes' theorem, we obtain the differential form of **Faraday's law**:

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Of course, in the electrostatic case, there is no magnetic field and $\nabla \times \mathbf{E} = 0$

To get the directions of Faraday's law right, we use **Lenz's law**: nature abhors a change in flux. The induced current will flow in such a direction that the flux it produces tends to cancel the charge.

6.3.1 The Induced Electric Field

Faraday's law is a generalization of the property $\nabla \times \mathbf{E} = 0$ to a time-dependent scenario. The divergence of $\mathbf{E}$ is still given by Gauss's law. If $\mathbf{E}$ is a pure "Faraday field"—due only to the changing $\mathbf{B}$, with $\rho = 0$—then

$$\nabla \cdot \mathbf{E} = 0, \ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Which is mathematically equivalent to magnetostatics!

$$\nabla \cdot \mathbf{B} = 0, \ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

The analog to the Biot-Savart law is then

$$\mathbf{E} = -\frac{1}{4\pi} \int \frac{(\partial/\partial \mathbf{B}t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, \mathrm{d}\tau = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{B} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} \, \mathrm{d}\tau$$

We can use similar tricks as we did with Ampère's Law in integral form, only with Faraday's law:

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

6.4 Inductance

Suppose two loop sof wire. If there is a steady current I_1 around loop 1 it produces a magnetic field $\mathbf{B}_1$. Some of the field lines pass through loop 2; let Φ_2 be the flux of $\mathbf{B_1}$ through 2. The Biot-Savart law yields

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{\mathrm{d}\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

which gives that the field is proportional to the current I_1. Thus the flux through loop 2 is also proportional to I_1:

$$\Phi_2 = \int \mathbf{B}_1 \cdot \mathrm{d}\mathbf{a}_2 \Longrightarrow \Phi_2 = M_{21} I_1$$

With M_{21} being a constant known as the **mutual inductace** of the loops.

We derive a formula for the mutual inductance. Let us express the flux in terms of the vector potential and use Stokes' theorem:

$$\Phi_2 = \int \mathbf{B}_1 \cdot \mathrm{d}\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot \mathrm{d}\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot \mathrm{d}\mathbf{l}_2$$

According to the formula for the vector potential given a surface current:

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{\mathrm{d}\mathbf{l}_1}{\mathscr{r}}$$

And thus

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{\mathrm{d}\mathbf{l}_1}{\mathscr{r}} \right) \cdot \mathrm{d}\mathbf{l}_2$$

Thus

$$M_{21} = \frac{\mu_0}{4\pi} \oint \left(\oint \frac{\mathrm{d}\mathbf{l}_1}{\mathscr{r}} \right) \cdot \mathrm{d}\mathbf{l}_2$$

Notice this doesn't change if we do M_{12} instead. Rename $M_{21} = M$.

Suppose you vary the current in loop 1. The flux through loop 2 will vary as well, and Faraday's law gives

$$\mathcal{E}_2 = -\frac{\mathrm{d}\Phi_2}{\mathrm{d}t} = -M \frac{\mathrm{d}I_1}{\mathrm{d}t}$$

But the current is itself in the magnetic field it produces. Then it follows it is affected by its own magnetic field, and we have that the flux is again proportional to the current, this time choosing a constant L— the **self-inductance**:

$$\Phi = LI$$

L depends on the geometry of the loop. Again,

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -L \frac{\mathrm{d}I}{\mathrm{d}t}$$

This emf is *opposite* any change in current; it is thus called **back emf**. L plays a role analogous to mass in this scenario, as we will see in the next section.

6.5 Energy in Magnetic Fields

Takes energy to counteract the back emf and start a current flowing, but once you turn the current off the energy is recovered. The work done on a unit charge against the back emf in a closed loop around the circuit is $-\mathcal{E}$. The amount of charge per unit time passing down the wire is I. The total work done per unit time is then

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\mathcal{E} I = LI \frac{\mathrm{d}I}{\mathrm{d}t}$$

Suppose we start with zero current and build up to a current I. Then

$$W = \int_0^I LI' \frac{\mathrm{d}I'}{\mathrm{d}t} \, \mathrm{d}t = \int_0^I LI' \, \mathrm{d}I'$$

which yields

$$\boxed{W = \frac{1}{2} L I^2}$$

But recall that $\Phi = LI$ and also

$$\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{a} = \int (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = \oint \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

Thus

$$LI = \oint \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

Thus

$$W = \frac{1}{2} I \oint \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I}) \, \mathrm{d}l$$

Which can be generalized to volume currents by

$$W = \frac{1}{2} \int (\mathbf{A} \cdot \mathbf{J}) \, \mathrm{d}\tau$$

But we can use the fact that $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ to eliminate $\mathbf{J}$:

$$W = \frac{1}{2\mu_0} \int \mathbf{A} \cdot (\nabla \times \mathbf{B}) \, \mathrm{d}\tau$$

We can use integration by parts to obtain

$$\begin{aligned} W &= \frac{1}{2\mu_0} \left[\int B^2 \, \mathrm{d}\tau - \int \nabla \cdot (\mathbf{a} \times \mathbf{B}) \, \mathrm{d}\tau \right] \\ &= \frac{1}{2\mu_0} \left[\int B^2 \, \mathrm{d}\tau - \oint (\mathbf{A} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{a} \right] \end{aligned}$$

$\mathbf{J}$ is zero outside the region with the volume current, so we can increase the volume to the entire space. $\mathbf{A}$ and $\mathbf{B}$ increase as we move out, until the second integral becomes zero, and

$$\boxed{W = \frac{1}{2\mu_0} \int B^2 \, \mathrm{d}\tau}$$

6.6 Maxwell's Equations

6.6.1 The Problem

In this chapter we've obtained

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \end{cases}$$

Now, there is an inconsistency in these formulas. Recall that the divergence of the curl is always zero. Apply the divergence to the last equation:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \times \mathbf{J})$$

In magnetostatics, $\nabla \cdot \mathbf{J}$ is zero and thus things work out. However, in the electrodynamic case, this quantity is not necessarily zero, whereas the left hand side *is*. Then we are compelled to find what we must add to the left hand side such that the two sides of the equation are indeed equal when we take the divergence.

6.6.2 The Solution

To find the correction term, we first determine what the right hand side is. Use (4.2) to determine

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t} = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

Thus

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{B} + \mathbf{C}) &= \nabla \cdot \mathbf{J} \\ \nabla \cdot \mathbf{C} &= \nabla \cdot \left(-\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \end{aligned}$$

Which yields

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$$

In the magnetostatic case, $\partial \mathbf{E}/\partial t$ is zero and we reduce to the familiar law. But now with this correction term, *a changing electric field induces a magnetic field.*

Maxwell called this correction term the **displacement current**:

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

6.6.3 Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{6.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.5}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{6.6}$$

Along with the force law,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

this summarizes the entirety of classical electrodynamics. In fact, one sees we can obtain the dispersion relation by

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{B}) &= \mu_0 (\nabla \cdot \mathbf{J}) + \mu_0 \epsilon_0 \left(\nabla \cdot \frac{\partial \mathbf{E}}{\partial t}\right) \\ 0 &= \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}[\epsilon_0 \nabla \cdot \mathbf{E}] \\ \nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \end{aligned}$$

6.7 Magnetic Charge

In free space there is no charge density, so $\rho = 0$, and no current, so $\mathbf{J} = 0$. Thus Maxwell's equations become

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

In this form, $\mathbf{E}$ and $\mathbf{B}$, and $\mathbf{B}$ and $-\mu_0\epsilon_0\mathbf{E}$ can be interchanged to get from one equation to the other (the constants just ensure we are working with the same units).

Now we are left wondering why there is no quantity in $\nabla \cdot \mathbf{B}$ and $\nabla \times \mathbf{E} = -\, \partial \mathbf{B}/\partial t$ such that

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho_e}{\epsilon_0} \\ \nabla \times \mathbf{E} &= -\mu_0 \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= \mu_0 \rho_m \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}_e + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Where we have corresponding magnetic charge density ρ_m and a magnetic current $\mathbf{J}_m$.

6.8 Maxwell's Equations in Matter

6.9 Boundary Conditions

As we've seen before, the fields $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{D}$ are discontinuous at surface charges or currents. We can actually derive these discontinuities from Maxwell's equations in integral form:

$$\begin{aligned} &\oint \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{f_{\text{enc}}} \\ &\oint \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0 \\ &\oint \mathbf{E} \cdot \mathrm{dl} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \mathbf{B} \cdot \mathrm{d}\mathbf{a} \\ &\oint \mathbf{H} \cdot \mathrm{dl} = I_{f_{\text{enc}}} + \frac{\mathrm{d}}{\mathrm{d}t} \int \mathbf{D} \cdot \mathrm{d}\mathbf{a} \end{aligned}$$

We apply the first equation to a thin Gaussian pillbox extending into either side of a surface charge and obtain

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f a$$

The edge of the pillbox doesn't contribute anything in the limit, so we obtain that the component of $\mathbf{D}$ in the perpendicular direction is

$$\boxed{D_1^\perp - D_2^\perp = \sigma_f}$$

The same reasoning applied to the second equation yields:

$$\boxed{B_1^\perp - B_2^\perp = 0}$$

We use now the third equation. Construct a thin Amperian loop with one edge above the surface and the other edge below.

$$\mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

As the width of the loop goes to zero, the area goes to zero and thus so does the flux. Then we obtain

$$\boxed{\mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0}$$

We use the same reasoning to obtain from equation four:

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\mathrm{enc}}}$$

If $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface such that $\hat{\mathbf{n}} \times \mathbf{l}$ is normal to the Amperian loop and gives an area such that we can use $\mathbf{K}_f$

$$I_{f_{\mathrm{enc}}} = \mathbf{K}_f \cdot (\mathbf{n} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l}$$

Thus

$$\boxed{\mathbf{H}_1^{\|}\mathbf{H}_2^{\|} = \mathbf{K}_f \times \hat{\mathbf{n}}}$$

6.10 Alternating Currents

Let Q be the charge on the capacitor at a time t. The potential difference across the capacitor is V. This is the same at the voltage across the inductor L and resistor R. We know that $Q = CV$, and that, by definition $I = -\frac{\mathrm{d}Q}{\mathrm{d}t}$, and, finally, that the potential drop across the resistor is IR and the voltage drop across the inductor is (the same as the emf) $L\frac{\mathrm{d}I}{\mathrm{d}t}$. Then we get the following relationships:

$$I = -\frac{\mathrm{d}Q}{\mathrm{d}t},\ Q = CV,\ V = L\frac{\mathrm{d}I}{\mathrm{d}t} + RI$$

To solve for the current, we eliminate two of the variables.Then we get the following relationships:

$$I = -\frac{\mathrm{d}Q}{\mathrm{d}t},\ Q = CV,\ V = L\frac{\mathrm{d}I}{\mathrm{d}t} + RI$$

Emf is work done per unit charge, which is analogous to the potential energy per unit charge, which is exactly what the voltage difference is.

Chapter 7

Mathematical Tools

7.1 Useful Identities

(a)

$$\boxed{\int (\nabla T)\,\mathrm{d}\tau = \oint T\,\mathrm{d}\mathbf{a}}$$

(b) Start with the divergence theorem:

$$\int_V \nabla \cdot \mathbf{F}\,\mathrm{d}V = \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{a}$$

Let $\mathbf{F} = (\mathbf{v} \times \mathbf{c})$ where $\mathbf{c}$ is a constant vector.

$$\int_V \nabla \cdot (\mathbf{v} \times \mathbf{c})\,\mathrm{d}V = \oint_s (\mathbf{v} \times \mathbf{c}) \cdot \mathrm{d}\mathbf{a}$$

Performing a circular shift on the integrands of both sides:

$$\int_V \mathbf{c} \cdot (\nabla \times \mathbf{v})\,\mathrm{d}V = \oint \mathbf{c} \cdot (\mathbf{v} \times \mathrm{d}\mathbf{a})$$

Since $\mathbf{c}$ is a constant vector, we may move the integration to the variable vector in the dot product:

$$\mathbf{c} \cdot \int_V (\nabla \times \mathbf{v})\,\mathrm{d}V = \mathbf{c} \cdot \oint \mathbf{v} \times \mathrm{d}\mathbf{a}$$

If one allows $\mathbf{c}$ to be $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}$, and $\hat{\mathbf{k}}$, we see that each of the components of the above dot products is equal, and thus that

$$\boxed{\int_V (\nabla \times \mathbf{v})\,\mathrm{d}V = \oint \mathbf{v} \times \mathrm{d}\mathbf{a}}$$

(c) (Green's first identity) Again, start with the divergence theorem:

$$\int_V \nabla \cdot \mathbf{F}\,\mathrm{d}V = \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{a}$$

Let $\mathbf{F} = T\nabla U$. Then

$$\int_V \nabla \cdot (T\nabla U)\,\mathrm{d}V = \oint_S T\nabla U \cdot \mathrm{d}\mathbf{a}$$

Now apply the product rule for the nabla operator to the left side:

$$\boxed{\int_V \nabla T \cdot \nabla U + T\nabla^2 U = \oint T\nabla U \cdot \mathrm{d}\mathbf{a}}$$

(d) (Green's second identity) Green's first identity still applies if we switch T and U. So we may write

$$\oint_S T\nabla U \,\mathrm{d}\mathbf{a} = \int_V \nabla T \cdot \nabla U \,\mathrm{d}\mathbf{x} + \int_V T\nabla^2 U \,\mathrm{d}V$$
$$\oint_S U\nabla T \,\mathrm{d}\mathbf{a} = \int_V \nabla U \cdot \nabla T \,\mathrm{d}V + \int_V U\nabla^2 T \,\mathrm{d}V$$

Then subtract these two equations to obtain

$$\boxed{\int_S (U\nabla T - T\nabla U) \,\mathrm{d}V = \int_S (U\nabla T - T\nabla U) \cdot \mathrm{d}\mathbf{a}}$$

Note: This is not part of the exercise but I think it's worth mentioning. Sometimes it becomes easier to evaluate a surface flux integral if we notice that

$$\int_S \nabla f \cdot \mathrm{d}\mathbf{a} = \int_S \nabla f \cdot \hat{\mathbf{n}} \,\mathrm{d}S$$

$\nabla f \cdot \hat{\mathbf{n}}$ is just a directional derivative in the direction of the outer normal vector. Then we may write

$$\int_S \frac{\partial f}{\partial n} \,\mathrm{d}S$$

(e) Start with Stokes's theorem:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{a} = \oint \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Let $\mathbf{F} = \mathbf{c}T$, where $\mathbf{c}$ is a constant vector:

$$\int_S (\nabla \times \mathbf{c}T) \cdot \mathrm{d}\mathbf{a} = \oint \mathbf{c}T \cdot \mathrm{d}\mathbf{r}$$

Use the product rule for the curl operator on the left side, and the linearity of the inner product in the right side:

$$\int_S [T(\nabla \times \mathbf{c}) - (\mathbf{c} \times \nabla T)] \cdot \mathrm{d}\mathbf{a} = \mathbf{c} \cdot \oint T \,\mathrm{d}\mathbf{r}$$

With $\mathbf{c}$ constant, $\nabla \times \mathbf{c} = 0$ such that

$$-\int_S (\mathbf{c} \times \nabla T) \cdot \mathrm{d}\mathbf{a} = \mathbf{c} \cdot \oint T \,\mathrm{d}\mathbf{r}$$

Perform a cyclic shift on the left hand side:

$$-\mathbf{c} \cdot \int_S \nabla T \times \mathrm{d}\mathbf{a} = \mathbf{c} \cdot \oint T \,\mathrm{d}\mathbf{r}$$

Once again this implies that

$$\boxed{-\int_S \nabla T \times \mathrm{d}\mathbf{a} = \oint T \,\mathrm{d}\mathbf{r}}$$

f) Suppose we are trying to integrate the product rule

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

We use the divergence theorem to find

$$\int \nabla \cdot (f\mathbf{A})\,\mathrm{d}\tau = \int f(\nabla \cdot \mathbf{A})\,\mathrm{d}\tau + \int \mathbf{A} \cdot (\nabla f)\,\mathrm{d}\tau = \oint f\mathbf{A} \cdot \mathrm{d}\mathbf{a}$$

Rearranging terms,

$$\boxed{\int_V f(\nabla \cdot \mathbf{A})\,\mathrm{d}\tau = -\int_V \mathbf{A} \cdot (\nabla f)\,\mathrm{d}\tau + \oint_s f\mathbf{A} \cdot \mathrm{d}\mathbf{a}} \tag{7.1}$$

7.2 Legendre Polynomials

Legendre polynomials are orthogonal polynomials that occur frequently throughout mathematics and physics. We have seen they are used to expand the potential $1/\mathscr{r}$, they are used in multipole expansions, and they even occur in some classes of neural networks. Legendre polynomials can be defined in three ways.

7.2.1 Orthogonal Definition

The n-th Legendre polynomial is a polynomial of degree n $P_n(x)$ such that, over the integral $[-1, 1]$,

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = 0 \text{ if } n \neq m$$

If we enforce $P_n(1)$, then $P_0(x) = 1$ and we have already defined all other polynomials in this series, as they must necessariyl be orthogonal to this first polynomial!

7.2.2 Generating Function Definition

The polynomials can be defined as the coefficients in the expansion in t of the function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=1}^{\infty} P_n(x)t^n$$

This generation function is in fact the exact function that we obtained from our multipole expansion, except in $\cos\alpha$:

$$V(r, \theta) \propto \frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\alpha}}$$

7.2.3 Solution to Differential Equation

Lastly, the Legendre polynomials are solutions to the differential equation

$$(1 - x^2)P_n''(x) - 2xP - n'(x) + n(n+1)P_n(x) = 0$$