Multivariable Calculus

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Contents

Chapter 1

Vectors and Matrices

1.1 Vectors

A *vector* is a mathematical object that has both magnitude and direction. That is, where a *scalar* value has only magnitude, a vector associates a direction with that magnitude.

Vectors can exist in as many dimensions as desired. Mathematicians often describe vectors as being comprised of *unit vectors* (vectors of length one) in the directions of each of the dimensions of the coordinate system. For example,

$$\vec{\mathbf{A}} = 2\hat{\mathbf{\imath}} + 3\hat{\mathbf{\jmath}} + 2\hat{\mathbf{k}}$$

would describe the vector that results from scaling the unit vector $\hat{\mathbf{\imath}}$ by 2, and adding it to $\hat{\mathbf{\jmath}}$ times 3 and $\hat{\mathbf{k}}$ times 2.

1.1.1 Simple Vector Operations

1. **Addition:** $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
2. **Subtraction:** $\vec{\mathbf{A}} - \vec{\mathbf{B}} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$. Visually equivalent to flipping the direction of $\vec{\mathbf{B}}$ and adding this vector to $\vec{\mathbf{A}}$.
3. **Scalar multiplication:** $c\vec{\mathbf{A}} = \langle ca_1, ca_2, ca_3 \rangle$
4. **Magnitude of a vector:** $|\vec{\mathbf{A}}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$

1.2 Dot Product

The dot product of two vectors

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{\mathbf{A}}||\vec{\mathbf{B}}| \cos\theta$$

can be used to find angles between vectors. It is equal to zero if vectors are perpendicular, greater than zero if vectors are pointing in about same direction, and less than zero if vectors are pointing in about opposite directions. It generally measures how much two vectors are pointing in the same direction.

Dot products can also be used to find $\text{proj}_{\vec{\mathbf{B}}}\vec{\mathbf{A}}$, the component of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$:

$$\text{projf}_{\vec{\mathbf{B}}}\vec{\mathbf{A}} = |\vec{\mathbf{A}}|\cos\theta = \frac{\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}}{|\vec{\mathbf{B}}|}$$

1.3 Determinants

Two dimensional determinants measure the area of the parallelogram formed by two vectors. Recall that the area of a parallelogram with side lengths A and B and angle θ is $AB\sin\theta$.

$$det(\vec{\mathbf{A}},\vec{\mathbf{B}}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = |\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin\theta = a_1b_2 - a_2b_1$$

Three dimensional determinants measure the volume of the parallelepiped formed by the three vectors. 3x3 determinants can be solved using Laplace expansions:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

1.4 Cross Product

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$|\vec{\mathbf{A}}\times\vec{\mathbf{B}}| = |\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin\theta$$

The cross product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ gives a vector the direction of which is perpendicular to the parallelogram formed by the two vectors, and the magnitude of which is the area of that parallelogram. It is also used in the formula for the volume of the parallelepiped formed by three vectors:

$$det(\vec{\mathbf{A}},\vec{\mathbf{B}},\vec{\mathbf{C}}) = \vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}})$$

1.4.1 Properties of the Cross Product

Below are some properties of the cross product.

$$\begin{aligned}
\vec{\mathbf{A}}\times\vec{\mathbf{B}} &= -\left(\vec{\mathbf{B}}\times\vec{\mathbf{A}}\right) \\
\left(k\vec{\mathbf{A}}\right)\times\vec{\mathbf{B}} &= k(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) \\
\vec{\mathbf{A}}\times\left(k\vec{\mathbf{B}}\right) &= k\left(\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right) \\
\vec{\mathbf{A}}\cdot\left(\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right) &= \left(\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right)\cdot\vec{\mathbf{C}} \\
\vec{\mathbf{A}}\times\left(\vec{\mathbf{B}}+\vec{\mathbf{C}}\right) &= \left(\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right)+\left(\vec{\mathbf{A}}\times\vec{\mathbf{C}}\right) \\
\vec{\mathbf{A}}\times\left(\vec{\mathbf{B}}\times\vec{\mathbf{C}}\right) &= \left(\vec{\mathbf{A}}\cdot\vec{\mathbf{C}}\right)\vec{\mathbf{B}}-\left(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}\right)\vec{\mathbf{C}}
\end{aligned}$$

1.5 Equation of Plane

$$ax + by + cz = d$$

Has normal vector $\vec{\mathbf{N}} = \langle a, b, c\rangle$.

To find the equation of a plane, consider an arbitrary point $P = (x, y, z)$, a given point $P_0 = (x_0, y_0, z_0)$ and a normal vector $\vec{\mathbf{N}} = \langle a, b, c\rangle$. This allows us to find the equation using the property $\overrightarrow{PP_0} \cdot \vec{\mathbf{N}} = 0$:

$$(x - x_0)a + (y - y_0)b + (z - z_0)c = 0$$

Solving further will give the above $ax + by + cz = d$, where d is a constant.

The distance between a point and a plane can be determined using the projection of a vector from a point on the plane to the given point P onto the normal vector of the plane $\vec{\mathbf{N}} = \langle a, b, c\rangle$:

$$D = \frac{\vec{\mathbf{N}} \cdot \vec{\mathbf{P}}}{|\vec{\mathbf{N}}|} = \frac{ap_1 + bp_2 + cp_3 - d}{\sqrt{a^2 + b^2 + c^2}}$$

Note that the constant d from the plane equation $ax + by + cz = d$ appears in the numerator.

1.5.1 Useful Formulas

The distance between a point S and a plane can be determined by projecting a vector from any point on the plane P to S onto the unit normal vector $\hat{\mathbf{n}}$:

$$d = |\mathbf{PS} \cdot \hat{\mathbf{n}}|$$

.

The angle between two planes is equal to the angle between the normal vectors of the two planes (see section 1.2).

1.6 Equation of Line in Space

To graph a line in space, parametrize x, y and z. Needed are: a point on line $P = (x_0, y_0, z_0)$ and a vector parallel to line $\vec{\mathbf{v}} = \langle a, b, c\rangle$. The equation of the line is thus

$$P(t) = P_0 + t\vec{\mathbf{v}} = \begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases}$$

The point at which a line intersects a plane can be determined by plugging $x, y, z(t)$ into the equation of the plane and solving for t.

1.6.1 More Useful Formulas

The distance between a point and a line in space can be considered $|\mathbf{PS}|\sin\theta$, the projection of a vector from a point on the line P to a point S onto a vector normal to the line:

$$\frac{|\mathbf{PS}||\vec{\mathbf{v}}|\sin\theta}{|\vec{\mathbf{v}}|} = \frac{|\mathbf{PS} \times \vec{\mathbf{v}}|}{|\vec{\mathbf{v}}|}$$

where $\vec{\mathbf{v}}$ is the vector parallel to the line.

1.7 Matrices, Linear Systems, Inverting Matrices

A three-by-three linear system may be expressed as a **matrix**:

$$\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \\ ix + jy + kz = l \end{cases} \Rightarrow \begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ h \\ l \end{bmatrix}$$

Perform matrix multiplication $A \cdot B$ by finding the dot product of the rows of A and the columns of B:

$$A \cdot B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 2+0+6 \\ 2+2+0 & 4+3+0 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 4 & 7 \end{bmatrix}$$

You can remember the dimensions of the final matrix by putting it under B and to the right of A.

Invert a matrix by:

1. Finding minors ("determinants" within matrix). Pick an element, and cross the row and column it is part of. The remaining uncrossed elements constitute the smaller determinant which you must find.
2. Finding cofactors. These are in a checkerboard pattern with a + in the top left. Multiply by the elements of the matrix by these cofactors.
3. Transposing the matrix. Switch its rows and columns.
4. Dividing by the determinant of the original matrix.

Which allows us to invert a linear system: $Ax = u \Rightarrow A^{-1}u = x$.

As the final step of inverting a matrix involves dividing by the determinant, a non-invertible matrix will be one with $\det(A) = 0$. We then arrive at the following property of linear systems $Ax = B$:

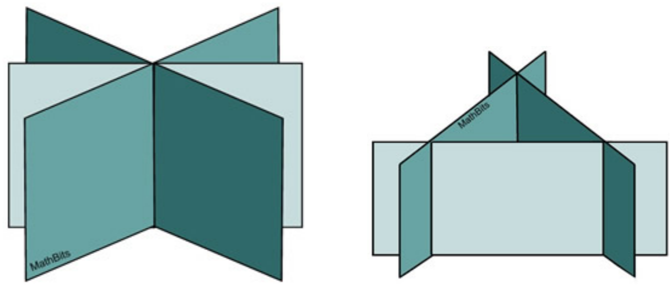
- If $det(A) \neq 0, x = A^{-1}B$
- If $det(A) = 0$, no solutions or ∞ many.

If $\det(A) = 0$, the three intersecting planes are parallel to the same direction—in other words, their normal vectors are coplanar. In this case, either the planes intersect in a line (hence infinitely many solutions), or the planes do not intersect (hence no solutions). In the special case $Ax = 0$, there exists a trivial solution where $x = 0$, but, if A is non-invertible, there must be other, non-trivial solutions.

1.8 Finding Parametric Equations

A parameterized vector-valued function is one of

$$\vec{\mathbf{F}} = x(t)\hat{\mathbf{\imath}} + y(t)\hat{\mathbf{\jmath}} + z(t)\hat{\mathbf{k}}$$



(a) Infinitely many solutions. (b) No solutions

Figure 1.1: Cases where a three-by-three matrix is not invertible.

A motion may be expressed by the parametric position vector $\vec{\mathbf{r}}$. The **acceleration, velocity, and tangent unit** (i.e., tangent to the motion) vectors of $\vec{\mathbf{r}}$ are, respectively,

$$\vec{\mathbf{v}} = \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} \quad \vec{\mathbf{a}} = \frac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2} \quad \hat{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

The magnitude of $\vec{\mathbf{v}}$—that is, the speed—is:

$$|\vec{\mathbf{v}}| = \frac{\mathrm{d}s}{\mathrm{d}t}$$

where s is the arclength along the path of motion. Giving the speed a unit direction vector, we get the following property:

$$\vec{\mathbf{v}} = \hat{\mathbf{T}}\frac{\mathrm{d}s}{\mathrm{d}t}$$

The position vector $\vec{\mathbf{r}}$ can be determined for an arbitrary motion by performing vector addition on a series of vectors starting from the origin, $\vec{\mathbf{r}} = \vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_1 + ...\vec{\mathbf{V}}_n$. Vectors $\vec{\mathbf{V}}_n$ may vary with time, and thus be parameterized.

For example, if one wants to parameterize the motion of a point on the edge of a rolling circle of radius a, start by drawing vectors tip-to-tail from the origin O to the point P. We draw from the origin to the bottom of the circle, $\overrightarrow{OA}$, then to the center $\overrightarrow{AC}$ and at last to the point $\overrightarrow{CP}$.

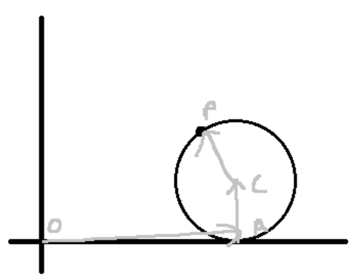


Figure 1.2: Diagram of the motion of a point on a rolling circle.

We then consider the motions of each of these vectors individually. $\overrightarrow{OA}$ increases horizontally according to the circumference along which the circle has rolled; thus it is $\langle a\theta, 0\rangle$. $\overrightarrow{AC}$ is simply the radius, pointed vertically: $\langle 0, a\rangle$. Finally, the point on the circle is a unit velocity motion along a circle, directed clockwise; thus, $\overrightarrow{CP} = \langle -a\sin\theta, -a\cos\theta\rangle$.

Adding all these vectors, we get $\vec{\mathbf{r}} = \overrightarrow{OP} = \langle a\theta - a\sin\theta, a - a\cos\theta\rangle$.

1.8.1 Properties and Formulas of Vector Differentiation

Calculus may be performed on vector-valued functions such as $\vec{\mathbf{r}}$. For example:

$$\frac{d\vec{\mathbf{v}}}{dt} = \frac{dx}{dt}\hat{\mathbf{\imath}} + \frac{dy}{dt}\hat{\mathbf{\jmath}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

Note that integration can also be performed on vector-valued functions:

$$\int_a^b \vec{\mathbf{r}}(t)\,\mathrm{d}t = \int_a^b x(t)\,\mathrm{d}t\,\hat{\mathbf{\imath}} + \int_a^b y(t)\,\mathrm{d}t\,\hat{\mathbf{\jmath}} + \int_a^b z(t)\,\mathrm{d}t\,\hat{\mathbf{k}}$$

And that the arclength (scalar) of a parametric curve is given by summing the infinitesimal distances between points:

$$\int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b |\vec{\mathbf{v}}|\,\mathrm{d}t$$

The **curvature** of $\vec{\mathbf{r}}(t)$ is the rate at which $\hat{\mathbf{T}}$ turns at unit arclength:

$$\kappa = \left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\right|$$

Note that this is a function that gives the curvature at a given point. Alternatively, if given a paremeterization t:

$$\kappa = \frac{1}{|\vec{\mathbf{v}}|}\left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}\right|$$

You can take this to mean that, since we are concerned with the change in $\hat{\mathbf{T}}$ with respect to arclength, if we take the derivative with respect to t, we must scale the curvature down according to how quickly the point traverses an arclength—hence $|\vec{\mathbf{v}}|$.

The circle of curvature, or osculating circle, of a curve has radius:

$$\text{Radius of curvature} = \frac{1}{\kappa}$$

Additional properties of vector-valued functions include:

$$\begin{aligned}
\frac{d}{dt}(\vec{\mathbf{u}} + \vec{\mathbf{v}}) &= \frac{d\vec{\mathbf{u}}}{dt} + \frac{d\vec{\mathbf{v}}}{dt} \\
\frac{d}{dt}(c \cdot \vec{\mathbf{u}}) &= c \cdot \frac{d\vec{\mathbf{u}}}{dt} \\
\frac{d}{dt}(h \cdot \vec{\mathbf{u}}) &= \frac{dh}{dt} \cdot \vec{\mathbf{u}} + h \cdot \frac{d\vec{\mathbf{u}}}{dt} \\
\frac{d}{dt}(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) &= \frac{d\vec{\mathbf{u}}}{dt} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \frac{d\vec{\mathbf{v}}}{dt} \\
\frac{d}{dt}(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) &= \frac{d\vec{\mathbf{u}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \frac{d\vec{\mathbf{v}}}{dt}
\end{aligned}$$

1.9 Frenet/TNB Frames and Torsion

Using a motion's tangent unit vector $\hat{\mathbf{T}}$, we can find two more vectors to form the TNB frame.

We begin by finding the normal unit vector $\hat{\mathbf{N}}$. We use the property that the derivative of a vector of constant length will always be orthogonal to the vector (because any change to this vector that *is not* perpendicular would cause a change in length. Think of the vector as being on a circle). Thus,

$$\hat{\mathbf{N}} = \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \frac{\mathrm{d}\hat{\mathbf{T}} \,/\, \mathrm{d}t}{\left|\mathrm{d}\hat{\mathbf{T}} \,/\, \mathrm{d}t\right|}$$

We now have a vector that points *along* the motion, and a vector that points in the direction in which the motion is *turning.* These vectors are in a plane; taking their cross product yields a third vector which is perpendicular to both, the binormal vector

$$\hat{\mathbf{B}} = \left|\hat{\mathbf{T}} \times \hat{\mathbf{N}}\right|$$

The **torsion** of the curve is the amount of "twisting" that a motion undergoes. Mathematically, we take it to be derivative of $\hat{\mathbf{B}}$ with respect to arclength; in other words, how much the plane formed by $\hat{\mathbf{N}}$ and $\hat{\mathbf{T}}$ is tilting:

$$\tau = -\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} \cdot \hat{\mathbf{N}}$$

Notice that $\frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s}$ points along the same direction as $\hat{\mathbf{N}}$, so projecting along $\hat{\mathbf{N}}$ gives its magnitude, or τ. We compute the torsion by

$$\tau = \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} \frac{1}{|\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}|^2}$$

1.9.1 Curvature

If we want to consider how tight a turn is in the motion, we can use a quantity called *curvature.* Curvature is determined by taking the magnitude of the change in $\hat{\mathbf{T}}$ for an increment in arclength. This should make sense, as, if the tangent unit vector's length is constant, its derivative should point in the direction of turning, and have a magnitude proportional to the amount of turning. We define curvature to be

$$\kappa = \left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s}\right| = \frac{1}{|\vec{\mathbf{v}}|}\left|\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}t}\right| = \frac{1}{r}$$

where r is the radius of the curve's osculating circle–i.e., the circle with the curvature of the curve at that point. Note that

$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \kappa\hat{\mathbf{N}}$$

Chapter 2

Partial Derivatives

2.1 Partial Derivatives

For $f(x, y, z)$, the partial derivative is

$$\frac{\partial f}{\partial x} = f_x \equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}$$

That is to say, the derivative of $f(x, y)$ with all variables but x held constant. This is computed by treating y, z as constants. The partial derivative has the geometric interpretation of slicing a solid with a plane parallel to the x axis and analyzing the function's "cross-section" along that plane. The "cross-section" will be reminiscent of a function of one variable, where the partial derivative represents the slope of that "cross-section" at a point.

Note that by this definition, even if y is a function of x, we implicitly assume that it does not vary with x and take the derivative with y as a constant regardless. The *total derivative*, to which the multivariable chain rule applies, does *not* make this assumption.

Example: Find f_x where $f(x, y) = x^3 y + y^2$.

$$\frac{\partial f}{\partial x} = 3x^2 y + 0$$

2.2 Tangent Plane

The tangent plane of a curve $f(x, y)$ at a point $(a, b, f(a, b))$ can be found using the partial derivatives f_x, f_y:

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

2.2.1 Extrema

Applications of tangent planes include finding the **extrema** of a multivariable function. At a local maximum or minimum, the tangent plane will be horizontal. I.e.,

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

at a critical point. However, a horizontal tangent plane can also occur at a *saddle point* (Figure 2.1).

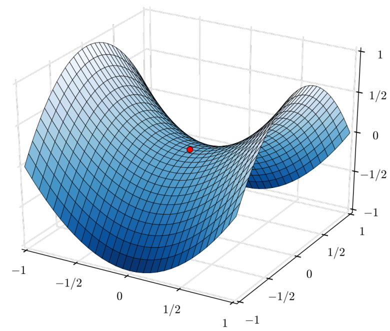


Figure 2.1: A saddle point.

To discern between a maximum, minimum, and a saddle point, use the **multivariable second derivative test**. Compute the second derivatives of f at a point (x_0, y_0):

$$A = f_{xx}(x_0, y_0),\ B = f_{xy}(x_0, y_0),\ C = f_{yy}(x_0, y_0)$$

Using these second derivatives:

$$\begin{aligned} AC - B^2 > 0 &\quad \text{max for } A < 0, \text{ min for } A > 0 \\ AC - B^2 < 0 &\quad \text{saddle} \\ AC - B^2 = 0 &\quad \text{cannot compute, degenerate} \end{aligned}$$

This rule is obtained from the Taylor expansion for a two-variable function, and from linear algebra. See section 5.1.

2.2.2 Extrema of a Function Along a Line

We can determine the extrema of $f(x, y)$ along a two-dimensional path by parameterizing the line. For example, if $y = -x$, use

$$\left\{ x(t) = t y(t) = -t \right.$$

Then, plug this parameterization into $f(x, y)$, to get a single-variable function $f(t)$. We can then find the maxima or minima of this function as we would in single variable calculus. We can also parameterize x, y and z to do this in space (where $f = f(x, y, z)$).

2.3 Taylor Series for a Multivariate Function

The Taylor series for a two-variable function $f(x, y)$ centered at the origin is:

$$\begin{aligned} f(x,y) &= f(0,0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}) + \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \\ \ldots &+ \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1} y \frac{\partial^n f}{\partial^{n-1} x \partial y} + \ldots + y^n \frac{\partial^n f}{\partial y^n} \right) \end{aligned}$$

2.4 Least-Squares Interpolation

Least-squares interpolation is a statistical method of finding a best-fit line for n data points. It measures the sum of the squares of the residuals for an arbitrary line with coefficients a, b, then minimizes that sum. In other words, it tries to find the line which is closest to the most points.

$$\text{Minimize: } D(a,b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$$

Do this by finding the horizontal tangent plane (see above):

$$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-x_i) = 0$$

$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-1) = 0$$

Distributing and dividing by the coefficient 2 gives:

$$\sum_{i=1}^{n} (x_i^2 a + x_i b - x_i y_i) = 0$$

$$\sum_{i=1}^{n} (x_i a + b - y_i) = 0$$

Finally, rearranging some terms and factoring out coefficients:

$$\left(\sum x_i^2\right) a + \left(\sum x_i\right) b = \sum x_i y_i$$

$$\left(\sum x_i\right) a + nb = \sum y_i$$

Where x_i, y_i are the given data points. Note that some phenomena might be related exponentially, in which case y_i might have to be linearized by finding $\ln(y_i)$.

2.5 Differentials

Multivariable calculus makes use of the total differential, i.e. the relationship of the function to all of the varibles changing together:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

which tells us how changes to x, y, z affect f.

When x, y, z are functions of, say, t, we can obtain the multivariable **chain rule** by dividing all terms by dt:

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

2.6 Chain Rule, Changes of Variables

Consider $f(x, y, z)$ and $x(u, v), y(u, v), z(u, v)$. Using the chain rule derived above, if we want to find, say, f_u:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

2.7 Gradient

The gradient of a function is

$$\boldsymbol{\nabla} f \equiv \langle f_x, f_y, f_z \rangle$$

where for every point (x, y, z) the gradient gives a vector pointing in the direction of steepest ascent, with the magnitude of the slope of that ascent. Note that for a three-dimensional curve, the gradient vector is two dimensional; that is, $\boldsymbol{\nabla} f$ is always one dimension less than f. Think of it as being projected on 2D plane below the 3D curve.

It can be used to approximate a change in a function:

$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z = \boldsymbol{\nabla} f \cdot \Delta \vec{\mathbf{r}}$$

It can also be used to find $\vec{\mathbf{N}}$ of the tangent plane to the surface $f(x, y, z)$ if f is considered a level curve, i.e. if is set equal to a constant c. Note that this method cannot determine whether the critical points are minima or maxima.

$$\boldsymbol{\nabla} f = \vec{\mathbf{N}} \text{ for } f(x, y, z) = c$$

This is because the gradients of level curves will always point in the normal direction, as that is the direction of steepest ascent at a point on a level curve.

Furthermore, the gradient can be used to find the **directional derivative**, i.e. the derivative in the direction $\hat{u}$:

$$\left.\frac{df}{dx}\right|_{\hat{u}} = \boldsymbol{\nabla} f \cdot \hat{u}$$

This has the geometric interpretation of projecting $\boldsymbol{\nabla} f$ onto $\hat{u}$ to find the component of the gradient vector in the desired direction.

2.8 Lagrange Multipliers

Consider a function $f(x, y, z)$ and a constraint $g(x, y, z) = c$. At an extremum, these functions have gradients proportional to one another by a factor λ:

$$\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g = \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \end{cases} \quad g(x, y, z) = c$$

Use this system of equations (which includes the constraint equation) to solve for the critical point(s) (x, y, z).

Justification: we can treat the constraint equation g as a level curve. $\boldsymbol{\nabla} g$ will be perpendicular to this curve at all times. The solutions to f must be intersecting the constraint curve g, so think of solving for the critical point as the extrema of the constraint curve projected onto the curve f. This will occur when f is at its highest point while still intersecting the constraint g and thus when the $\vec{\mathbf{N}}$ of both the constraint and f are parallel. See Figure 2.2.

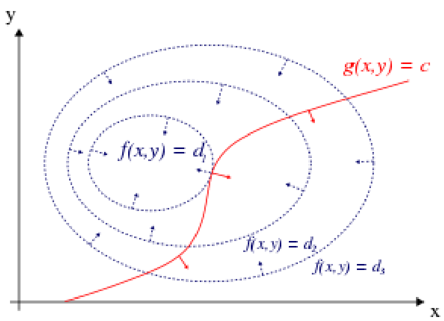


Figure 2.2: Visualization of Lagrange multipliers. The contour plot is in black, and the constraint in red.

Example Find the extrema of the function $f(x, y) - xy$ on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

Finding the gradient of both these functions, we obtain the system

$$\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g = \begin{cases} y = \frac{\lambda}{4} x \\ x = \lambda y \end{cases}$$

Substituting gives us

$$y = \frac{\lambda}{4} \lambda y = \frac{\lambda^2}{4} y$$

or $\lambda = \pm 2$ and $x = \pm 2y$ Substituting into the constraint equation:

$$\begin{aligned} \frac{(\pm 2y)^2}{8} + \frac{y^2}{2} &= 1 \\ 4y^2 + 4y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

from which we can easily obtain the x values ± 1.

2.9 Constrained Partial Derivatives

Consider $f(x, y, z)$ where $g(x, y, z) = c$. Given the constraint g, we cannot differentiate with respect to a variable while holding the other two constant. If we did that, variations in, say, x while holding y, z constant would make $g \neq c$. Either y or z must be allowed to vary.

Thus, we specify, for a constrained partial derivative, *one* variable to hold constant. If we choose z, this is notated by:

$$\left(\frac{\partial f}{\partial x}\right)_z$$

To solve for $\frac{\partial f}{\partial x}$, there are two methods. Method one is the **method of differentials**:

$$df = f_x dx + f_y dy + f_z dz$$

Holding z constant but allowing y to vary makes $dz = 0$. Furthermore, we can find the differentials of the constraint equation g in order to solve for dy. This is similar to partial differentiation.

$$df = f_x dx + f_y dy$$

$$dg = g_x dx + g_y dy + g_z dz = 0 \;\Rightarrow\; g_y dy = -g_x dx \;\Rightarrow\; dy = -\frac{g_x}{g_y} dx$$

Substituting dy gives us:

$$df = f_x dx - f_y \left(\frac{g_x}{g_y}\right) dx$$

Where we have only dx and df, so we can solve for $\frac{\mathrm{d}f}{\mathrm{d}x}$ easily.

Another method is the **chain rule method**:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial x}\right)_z + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial x}\right)_z + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial x}\right)_z$$

Note that $\frac{\partial x}{\partial z} = 0$, and that $\frac{\partial x}{\partial x} = 1$. The only partial derivative we need to find is $\frac{\partial y}{\partial x}$. Similarly to the method of differentials, we can partially differentiate the constraint equation with respect to x holding z constant. Think of partial differentiation.

Challenge problem: Let P be the point $(1, -1, 1)$ and assume $z = x^2 + y + 1$, and that $f(x, y, z)$ is a differentiable function for which $\nabla f(x, y, z) = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$. Let $g(x, z) = f(x, y(x, z), z)$; find ∇g at the point $(1, 1)$, i.e., $x = 1$, $z = 1$.

Chapter 3

Double Integrals and Line Integrals in the Plane

3.1 Double Integrals

3.1.1 Intuition

Where a single integral $\int f(x)\,dx$ gives the area under a graph, the double integral

$$\iint_R f(x,y)$$

gives the volume under a three dimensional curve for the two-dimensional region R. That is to say, where the limits of integration of a single integral are simply the two values between which the desired area lies, a double integral's limits of integration are given by a two dimensional shape or region "under" the three dimensional curve.

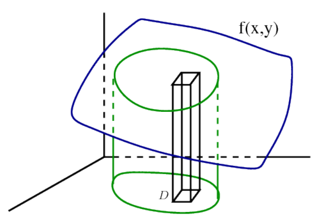


Figure 3.1: Geometric intuition for the double integral over a region D.

As opposed to summing infinitesimally small rectangles, Riemann sum of the double integral can be considered a sum of rectangular prisms of height $f(x_i, y_i)$ and base $\Delta x \Delta y$. See Figure 1.

Note that the limits of integration don't depend on the function at all—the function just gives the values you're integrating inside the region.

3.1.2 Evaluating Double Integrals

Once the region R is determined, the double integral can be set up as an *iterated integral*. That is to say, you can evaluate the expression as a single integral with respect to one variable *inside* a single integral with respect to different variable:

$$\iint_R F(x,y)dA = \int_{x_0}^{x_1} \left[\int_{y_0}^{y_1} f(x,y)\, dy\right] dx$$

where x is treated as a constant when evaluating the inner integral and vice versa. Think of $dxdy$ as notating the change in area dA.

Note that the difficulty in evaluating a double integral arises when determining the region R. If R is a rectangle, the limits of the iterated integral are simply the values of x and y which would enclose that rectangular region. However, a more complex region requires more thought. The convention is to focus on the region by examining solely the xy plane:

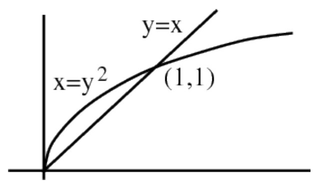


Figure 3.2: A region of double integration.

In the case of a "complicated" region such as Figure 2, think of taking vertical slices *from* the bottom curve $y = x$ *to* the top curve $y = \sqrt{x}$, then taking vertical slices in any desired limits (in this case from $x = 0$ to $x = 1$). This yields the double integral

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)$$

Occasionally, it might be advantageous to **change the order of integration**, in which case the limits would be determined by taking vertical slices from the "bottom" (leftmost) curve to the "top" (rightmost) curve, then evaluating the outer integral over the desired y domain:

$$\int_0^1 \int_{y^2}^{y} f(x,y)\, dxdy$$

The limits of outer integral can also be thought of as the "shadow" of the region cast onto either the x or the y axis. This analogy of "shadows" also extends nicely to triple integrals. See Unit 4.

3.1.3 Double Integrals in Polar Coordinates

For certain radially symmetric regions, a double integral might be more easily evaluated with a polar parametrization

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

When setting up a double integral with respect to r and θ note that a change in area $dA = dydx$ is not equivalent to $drd\theta$. This is because the change in area due to θ also depends on the distance from the origin r ($\Delta\theta$ will yield a larger change in area for greater rs. Think about it, or draw a picture). Thus, $dydx = rdrd\theta$. The double integral over a quadrant of the unit circle, for example, can be set up as

$$\int_0^{\frac{\pi}{2}} \int_0^1 f(r,\theta)\cdot r\, drd\theta$$

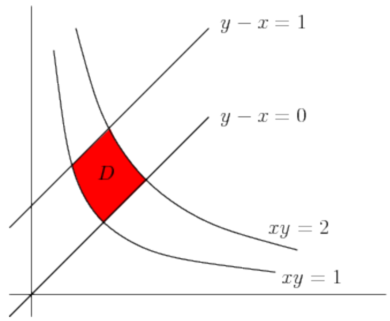


Figure 3.3: A region of integration with changes of variables $u = y - x$, $v = xy$.

3.1.4 Changes of Variables in Double Integrals

When performing changes of variables (either to simplify the limits of integration or integrand) with double integrals, the area of the region of integration will be scaled according to the change of variables. Say that

$$\begin{cases} u = 3x - 2y \\ v = x + 1 \end{cases}$$

The new coordinate system (u, v) will be a scaled form of (x, y). *Thus, $dudv = c \cdot dydx$ for a scaling factor* c. This scaling factor can be found using the determinant of a **Jacobian matrix**:

$$J = \frac{\partial(u, v)}{\partial x, y} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

Giving us $dudv = |J|\, dxdy$, where the scaling factor is positive regardless, hence the absolute value of the Jacobian.

Finding the **limits of integration** of a double integral with a change of variables is quite difficult. This is most easily done by graphing the region R and finding where u and v intersect the "edges" of this region. Use the values of x and y at the edges to establish certain equalities. In Figure 3.3, for example, $u \in (0, 1)$, $v \in (1, 2)$.

3.2 Applications of Double Integrals

The **area of a region** can be calculated with double integrals by

$$\text{area}(R) = \iint_R 1\, dA$$

Recall that $m = \delta \Delta A$ for a constant density δ. The **mass** of a region with varying density $\delta(x, y)$ can be calculated by

$$m = \iint_R \delta(x, y)\, dA$$

The **average value** of a three dimensional function can be calculated by dividing the volume under the curve by the area:

$$\bar{f} = \frac{1}{A} \iint_R f(x,y)\, dA$$

The **center of mass** of a flat object with varying density $\delta(x,y)$ can be calculated by adding the x or y positions and weighing them based on the mass of the object at that point (determined by an infinitesimal area dA multiplied by the density at that point $\delta(x,y)$):

$$\bar{x} = \frac{1}{m} \iint_R x \cdot \delta(x,y)\, dA$$

$$\bar{y} = \frac{1}{m} \iint_R y \cdot \delta(x,y)\, dA$$

The **polar moment of inertia** of a region is mr^2. Therefore, summing infinitesimally small masses at radii r, we have the polar moment of inertia about the origin for an object with density $\delta(x,y)$:

$$I_0 = \iint_R r^2 \delta\, dA$$

3.3 Vector Fields

A vector field $\vec{\mathbf{F}}$ has vertical and horizontal components which depend on a given (x,y):

$$\vec{\mathbf{F}} = M(x,y)\,\hat{\mathbf{\imath}} + N(x,y)\,\hat{\mathbf{\jmath}}$$

Think of it as a function which returns a vector. Vector fields are typically drawn by selecting points at regular intervals and drawing vectors at those points. However, vector fields return vectors for every point in the plane, and thus there are, in actuality, infinitely many vectors. See Figure 3.

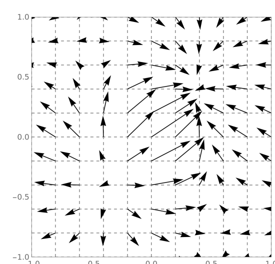


Figure 3.4: A vector field.

A vector field can represent several phenomena: the velocity of a fluid, a force field, an electrostatic field, or the gradient of a function.

3.4 Line Integrals

Say we would like to find the work on a particle moving in a trajectory C done by a force field $\vec{\mathbf{F}}$. The force field only contributes to the work done if its force is exerted in the direction of motion. For an infinitely small, straight segment ds, this could be found by projecting the force due to $\vec{\mathbf{F}}$ at that point onto the tangent unit vector $\hat{T}$. Alternatively, that straight segment can be thought of as an infinitely small change in position $d\vec{\mathbf{r}}$. Thus, the line integral is expressed as:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot \hat{T}\, ds$$

For a trajectory, or path, C.

For a **scalar function** $f(x, y, z)$ and a line or path $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{\imath}} + y(t)\hat{\mathbf{\jmath}} + z(t)\hat{\mathbf{k}}$,

$$\int_C f(x, y, z)\, \mathrm{d}s = \int_a^b f(x(t), y(t), z(t))\, |\vec{\mathbf{v}}(t)|\, \mathrm{d}t$$

where $|\vec{\mathbf{v}}(t)|$ gives a small change in arclength $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$. If we integrate $f(x, y, z) = 1$, we simply obtain the arclength of the line.

3.4.1 Evaluating Line Integrals

The aforementioned infinitely small change in position $d\vec{\mathbf{r}}$ can be broken up into its components $\langle dx, dy\rangle$, for an infinitely small change in the x and y directions respectively. Thus, the integral can be set up as

$$\int_C \langle M, N\rangle \cdot \langle dx, dy\rangle = \int_C M dx + N dy$$

We can take the dot product of $\langle M, N\rangle$ and $\mathrm{d}\vec{\mathbf{r}}$ because this is

$$\vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \left|\vec{\mathbf{F}}\right| |\mathrm{d}\vec{\mathbf{r}}| \cos\theta$$

where $|\mathrm{d}\vec{\mathbf{r}}|$ is the infinitesimal arclength we wish to integrate, $\left|\vec{\mathbf{F}}\right|$ is the magnitude of the field at that point, and $\cos\theta$ handles the projection.

Where, if the trajectory is given by $y = f(x)$, one can set up a single integral with respect to x by substituting into y and differentiating for $dy = f'(x)\, dx$. Note that the direction of the trajectory matters.

The line integral of the opposite direction of a trajectory is just $-\int_C$. Also note that a line integral can be broken up by sections of a trajectory: $\int_{C_1+C_2} = \int_{C_1} + \int_{C_2}$. And, most importantly, note that the *trajectory* path two points matters. For most $\vec{\mathbf{F}}$, two different paths between the same two points will yield different line integrals.

3.5 Conservative Fields

For certain special cases, the path taken between two points does *not* matter. I.e. any two paths will yield the same line integral as long as the path is between the same two points. This property is called **path independence** and it is a property of a **conservative field $\vec{\mathbf{F}}$**. Another property of a conservative field is that for a closed path (denoted $\oint_C$), the line integral is zero:

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$$

Namely, a field is conservative if $\vec{\mathbf{F}} = \boldsymbol{\nabla} f$; that is, if it is the gradient field of a function f.

The property of path independence then gives us the **fundamental theorem of calculus for line integrals**:

$$\int_C \boldsymbol{\nabla} f \cdot d\vec{\mathbf{r}} = f(x_1, y_1) - f(x_0, y_0)$$

Note that you'll have to determine both the x and y coordinates for the endpoints of the path.

3.5.1 Potential Functions

If $\vec{\mathbf{F}} = \boldsymbol{\nabla} f$, then f is considered the potential function. (This is the name given to f in mathematics. In physics, the potential is, in fact, $-f$).

$\vec{\mathbf{F}} = \boldsymbol{\nabla} f$ for $\langle M, N \rangle$ if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, because $f_{xy} = f_{yx}$. This is a property of the second partial derivatives of a function (see Unit 2).

To find the potential function f of $\vec{\mathbf{F}} = \boldsymbol{\nabla} f$, use the following method:

Step 1: Set up the horizontal and vertical components of $\vec{\mathbf{F}}$ as partial derivatives:

$$\begin{cases} f_x = M \\ f_y = N \end{cases}$$

Step 2: Integrate f_x with respect to x. The constant at the end of the antiderivative is, in fact, a function $g(y)$, since we did not integrate with respect to y:

$$f = \int f_x \, dx = x^2 y + x + g(y)$$

Step 3: Differentiate this new f with respect to y:

$$\frac{\partial f}{\partial y} = x^2 + g'(y)$$

Step 4: Set this expression equal to N, the y component of $\vec{\mathbf{F}}$, then solve for $g'(y)$. Then antidifferentiate $g'(y)$ to find $g(y)$.

3.6 Curl in a Plane

The **curl** of a vector field $\vec{\mathbf{F}}$ is

$$\operatorname{curl} \vec{\mathbf{F}} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

where F_x and F_y denote the x and y components of $\vec{\mathbf{F}}$ as opposed to the partial derivatives. We shall henceforth use this notation. This quantity measures the "rotation" of the vector field, with positive values indicating counterclockwise rotation, negative values indicating clockwise rotation, and a value of zero indicating no rotation (save for cases where the region is not simply connected, see section 7.1).

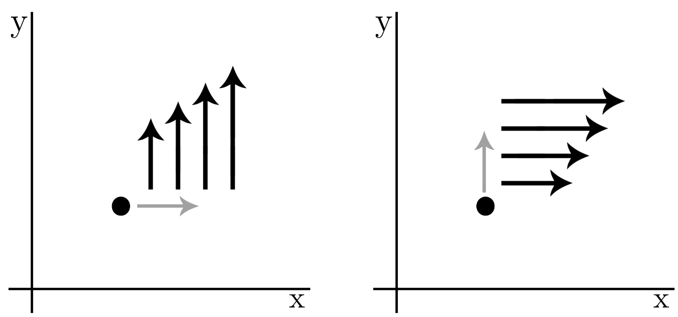
The curl is also zero when the vector field is conservative. This should make sense as $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$ for conservative vector fields.

The curl has two physical properties: it is twice the angular velocity for a velocity field, and it is equal to the torque when $\vec{\mathbf{F}}$ is a force field.

3.6.1 Intuition for Curl in a Plane

Think of $\frac{\partial F_y}{\partial x}$ as the change in the *vertical component* of $\vec{\mathbf{F}}$ if you take an infinitesimal step toward positive x. If $\frac{\partial F_y}{\partial x}$ is positive, the farther up the x direction, the faster the proverbial fluid is flowing. Likewise, $\frac{\partial F_x}{\partial y}$ can be thought of as the change in the *horizontal component* of $\vec{\mathbf{F}}$ for a small step in the positive y direction. If $\frac{\partial F_x}{\partial y}$ is positive, the farther up you travel in the coordinate plane, the stronger the fluid flow toward the right (the positive x direction), implying a clockwise rotation.

Thus, if the fluid has a large positive $\frac{\partial F_x}{\partial y}$ at the given point, $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ is *negative* and thus the curl is *clockwise*. Conversely, if the fluid has a large positive $\frac{\partial F_y}{\partial x}$ at the given point, $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ is *positive* and thus the curl is *counterclockwise*. See Figure 4.



(a) A positive $\dfrac{\partial F_y}{\partial x}$ (b) A positive $\dfrac{\partial F_x}{\partial y}$

Figure 3.5: Geometric intuition for curl. The gray arrow represents a small change in the x or y direction.

3.7 Green's Theorem

Green's theorem states that, if C is a closed curve enclosing a simply connected region R, then

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R \operatorname{curl} \vec{\mathbf{F}}\, dA$$

Notice that this implies that a conservative field also has a curl of zero.

Green's theorem can also be used to find the area of a region:

$$\mathrm{A}(R) = \iint_R 1\, dA = \oint_C x\, dy = \oint_C -y\, dx$$

Which can be proved using the fact that $\iint_R 1\, dA$ implies

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1 \;\Rightarrow\; \vec{\mathbf{F}} = \langle -\frac{y}{2}, \frac{x}{2} \rangle$$

3.7.1 Simply Connected Regions

A region R is considered simply connected if every point it encloses is also defined in $\vec{\mathbf{F}}$. That is to say, $\vec{\mathbf{F}}$ should be defined everywhere in the region. Think of continuously shrinking the region down to a point. If the region contains "hole," the shrinking would be impeded.

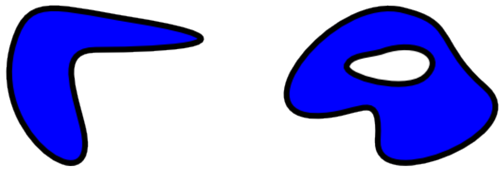


Figure 3.6: A comparison of two regions.

We can sidestep this when using Green's theorem by defining C_1 as the outer path and C_2 as an arbitrary inner path which excludes (by surrounding) the undefined portion of the domain from the region. We then compute $\int_{C_1} + \int_{C_2} = \iint \text{curl}\vec{\mathbf{F}}\ dA$.

3.8 Flux

Where above we defined the line integral for work, here we define the line integral for the **flux** of a $\vec{\mathbf{F}}$ through a path C. Where the work line integral involved the projection of a field onto the tangent vectors of a path, the flux line integral projects, for an infinitesimal straight line segment ds, $\vec{\mathbf{F}}$ onto the path's unit normal vector:

$$\int_C \vec{\mathbf{F}} \cdot \hat{n}\ ds$$

As the normal vector is perpendicular to the tangent vector, we can rotate $d\vec{\mathbf{r}}$ by 90 degrees:

$$\int_C \langle M, N\rangle \cdot \langle dy, -dx\rangle$$

Then evaluate as we would a work line integral.

Additionally, as this integral measures the field's movement perpendicular to the curve, we can interpret the flux as measuring the amount of "stuff" moving through the "membrane" traced by C in unit time (for, say, a velocity field). This quantity is measured **positively** for movement to the *right* of the path and **negatively** for movement to the *left*.

For two dimensional flux, this amount is measured in area per unit time. Think of Δs and the values of $\vec{\mathbf{F}}$ at the endpoints of Δs forming a parallelogram. The area of this parallelogram is $A = bh = ba\sin\theta = \Delta s(\hat{n}\cdot\vec{\mathbf{v}})$. As the sides are of units $\frac{m}{s}$ and m, the flux will be of unit m^2/s. The unit of three dimensional flux is then m^3/s, thus being a measurement of the volume of water passing through a surface every second.

3.8.1 Normal Form of Green's Theorem

There is an analogue to Green's Theorem for the flux line integral. For a simply connected region R:

$$\oint_C \vec{\mathbf{F}} \cdot \hat{n}\ ds = \iint_R \text{div}\,\vec{\mathbf{F}}\ dA$$

Where div$\vec{\mathbf{F}}$ is the divergence of the function at a point (see section 9).

The normal form of Green's theorem can be easily derived from the tangent form by setting $M = -Q$ and $P = N$:

$$\oint_C M dx + N dy = \iint_R N_x - M_y \, dA \;\Rightarrow\; \oint_R -Q dx + P dy = \iint_R P_x + Q_y \, dA$$

3.9 Divergence in a Plane

The **divergence** of a vector field $\vec{\mathbf{F}}$ is

$$\operatorname{div} \vec{\mathbf{F}} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$$

This quantity measures exactly what its name suggests—the degree to which a vector field is "diverging" at a given point. Alternatively, a point with positive divergence can be thought of as a "source" of fluid, whereas a point with negative divergence can be thought of as a "sink."

3.9.1 Intuition for Divergence in a Plane

Where the formula for curl can be thought of as measuring the change in the component for a step perpendicular to that component, the divergence can be thought of as a measurement of change in a component for a step in the direction of that component. More specifically, $\frac{\partial F_x}{\partial x}$ measures how much the x component of $\vec{\mathbf{F}}$ increases for an infinitesimal step in the x direction—i.e. how much the fluid is "separating" in the horizontal direction. The same holds true for the y component.

Chapter 4

Triple Integrals and Line Integrals in Space

4.1 Triple Integrals

4.1.1 Intuition

Where double integrals measured the volume under a surface for a region R, triple integrals, denoted

$$\iiint_D f(x,y,z)\,\mathrm{d}V \tag{4.1}$$

measure the total "quantity" of some function $f(x,y,z)$ by summing up its values at an infinitesimal volume $\lim_{\Delta x,y,z\to 0}\Delta x\Delta y\Delta z$. That is to say, the triple integral slices the region along planes parallel to the xy, xz, and yx planes.

As an example, think of $f(x,y,z)$ as representing the temperature of the room at every point in space. If we would like to add up the temperatures at each point to find a quantity such as the total kinetic energy of the particles in the room, we could sample "cubes" of space and take their temperatures, then sum the temperatures of each cube. The triple integral is then equivalent to letting the volume of these cubes approach zero and the amount approach infinity.

4.1.2 Evaluating Triple Integrals

As with double integrals, the evaluation of triple integrals can be performed through iterated integration.

$$\iiint_D f(x,y,z)\,\mathrm{d}V = \int_a^b \left[\int_c^d \left[\int_e^f f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}y\right]\mathrm{d}x \tag{4.2}$$

Also as with double integrals, the difficulty in evaluating triple integrals often lies in determining the region D. This is most conveniently done by "fixing" a point in the xy plane and considering where a line parallel to the z axis would enter and exit the region (or considering a different plane and different axis according to the order of integration). See Figure 4.1. Then, the limits of integration of the outer two integrals are simply the "shadow" cast by the region D; i.e., the usual region R that limits double integrals.

For example, if the region of integration is the volume above the paraboloid $z = x^2 + y^2$ and below the paraboloid $z = 4 - x^2 - y^2$, we must first consider a fixed xy which *enters* the region at $z = 4 - x^2 - y^2$ and *exits* at $z = x^2 + y^2$. Then, we set the z values of the paraboloids equal to each other to find the "shadow"

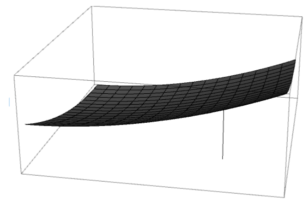


Figure 4.1: Visualization of a fixed point in the xy plane which enters at $z = 0$ and exits at $z = r^2$.

of the region; we find that this is the circle $x^2 + y^2 = 2$. Treating this circle as the region of integration of a double integral, we have the lower and upper bounds $\sqrt{2 - x^2}$ and $-\sqrt{2 - x^2}$ respectively. Finally, we set up the integral:

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} f(x, y, z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

Although this was not part of the 18.02 curriculum, changes of variables for triple integrals work identically to changes of variables for double integrals:

$$\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = |J|\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}z$$

Where $|J|$ is the absolute value of a three-by-three Jacobian.

4.1.3 Volume Element of Various Coordinate Systems

	Rectangular	**Cylindrical**	**Spherical**
$\mathrm{d}V =$	$\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$	$\mathrm{d}V = r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta$	$\rho^2 \sin\phi\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\theta$

4.2 Applications of Triple Integrals

The **mass** of a region D can be computed by summing the product infinitesimal volumes $\mathrm{d}V$ and densities $\delta(x, y, z)$:

$$\iiint_D \delta(x, y, z)\, \mathrm{d}V \tag{4.3}$$

Similarly, the **average value of a function** of three variables $f(x, y, z)$ in a region D can be determined by

$$\frac{1}{V} \iiint f(x, y, z)\, \mathrm{d}V \tag{4.4}$$

The **center of mass** $(\bar{x}, \bar{y}, \bar{z})$ of a region can be determined by weighing each value of x, y or z according to the mass of infinitesimal volumes $\delta(x, y, z) \cdot \mathrm{d}V$ then dividing by the volume of the region:

$$\bar{x} = \frac{1}{V} \iiint_D x\, \delta\, \mathrm{d}V \tag{4.5}$$

Recall the **moment of inertia** of a three-dimensional object, mr^2. The moment of inertia of a 3D region can be computed by summing the products of infinitesimal masses and their distance from the axis r. Taking, for example, the moment of inertia I_z around the z axis:

$$I_z = \iiint_D \left(x^2 + y^2\right) \delta\, \mathrm{d}V \tag{4.6}$$

Given the inverse square law $G\frac{mM}{r^2}$, **the gravitational attraction** on a point mass positioned at the origin can be computed using triple integrals. Assuming rotational symmetry, x and y components of the gravitational attraction vector $\vec{\mathbf{G}}$ can be ignored. Thus, the attraction can be treated as a scalar value (specifically, as the magnitude of a vector in the $\hat{\mathbf{k}}$ direction). Using spherical coordinates (Section 4.3), we take $r^2 = \rho^2$, treat m as 1 for a unit mass, take $M = \delta \cdot \mathrm{d}V$ and use $z = \rho \cos\phi$:

$$\iiint_D G\,\frac{\rho\cos\phi}{\rho^2}\,\delta\,\rho^2 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

Canceling the ρ and bringing out the constant G gives us

$$G\iiint_D \cos\phi\sin\phi\,\delta\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{4.7}$$

4.3 Spherical and Cylindrical Coordinates

Occasionally, it is advantageous to use a coordinate system that benefits from rotational symmetry. Where in two dimensions we had polar coordinates, we can easily extend this notion to three dimensions by adding a third-dimensional coordinate z. Thus, r, θ and z comprise the cylindrical coordinate system. Conversion from Cartesian coordinates to cylindrical coordinates is similar to conversion to polar coordinates:

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

Likewise, it is sometimes advantageous to integrate in a spherical coordinate system. Spherical coordinates are defined in terms of θ, the angle on the unit circle in the xy plane, ϕ, the angle *downward* from the z axis, ranging from 0 to π, and ρ, the radius (see Figure 4.2).

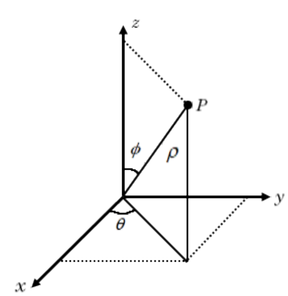


Figure 4.2: The spherical coordinate system.

The volume element of a triple integral in spherical coordinates can be determined by first considering the surface area of a section of a sphere of radius a. The bottom and top sides of the sphere are given by $a\sin\phi\Delta\theta$—they can be thought of as portions of size $\Delta\theta$ of a circle the radius of which depends on how far "up" or "down" the sphere we are (that is, $a\sin\phi$). The left and right portions of this section are given by $a\Delta\phi$, where these are portions of the circles traced by the sphere's longitude lines. This surface area element is then turned into a volume element by giving it a depth $\Delta\rho$. Thus, multiplying the latitudes $\rho\sin\phi\Delta\theta$ by the longitudes $\rho\Delta\phi$ and depth $\Delta\rho$, taking the limit, and summing:

$$\iiint_R f(\rho,\phi,\theta)\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{4.8}$$

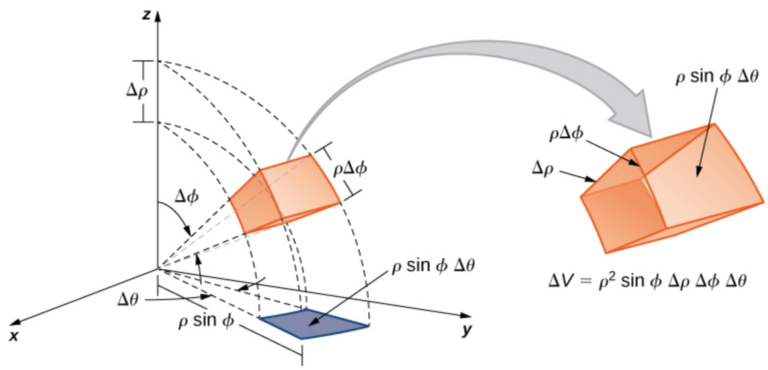


Figure 4.3: The spherical volume element.

4.3.1 Conversion to and from Spherical Coordinates

Converting from spherical to cylindrical or rectangular coordinates is fairly simple:

Spherical	ρ	θ	ϕ
Cylindrical	$r = \rho \sin\phi$	$\theta = \theta$	$z = \rho\cos\phi$
Rectangular	$x = r\cos\theta$	$y = r\sin\theta$	$z = \rho\cos\phi$

ρ can also be considered the distance from the origin, so we can take

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2 + z^2}\\ &= \sqrt{r^2 + z^2}\end{aligned}$$

4.4 Surface Integrals

The flux through a surface S due to a vector field $\vec{\mathbf{F}}$ is given by:

$$\iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, \mathrm{d}S \tag{4.9}$$

Where we are taking infinitesimal "polygons"—or sections of tangent planes—dS, which comprise S, determining their normal vectors $\hat{\mathbf{n}}$, and projecting $\vec{\mathbf{F}}$ onto those $\hat{\mathbf{n}}$.

Note that it is necessary to *orient* the surface; that is, we must choose a side of the surface out of which $\hat{\mathbf{n}}$ points. Conventionally, $\hat{\mathbf{n}}$ points *out* of closed surfaces.

Consider this a regular double integral where the evaluation is performed over an *area* (the area of a typical double integral, for example, being on the xy plane), despite the surface existing in $\mathbb{R}^3$. The area over which we are integrating is simply the surface S. Thus, we must *parametrize* this double integral in terms of two variables, restricting the integrand to points *on the surface* S. This is not to be confused with making a change in coordinate system.

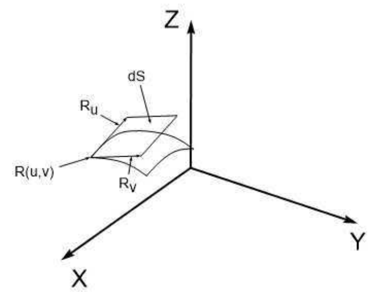


Figure 4.4: The surface area element of a parameterized surface.

4.4.1 Normal Vectors for Simple Surfaces

Surface	**Normal Vector $\hat{\mathbf{n}}$**	**Area Element**
Horizontal plane xy (can be generalized)	$\pm\,\hat{\mathbf{k}}$	$\mathrm{d}x\,\mathrm{d}y$
Spheres centered at 0	$\pm\,\frac{1}{a}\langle x,y,z\rangle$	$a^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$
Cylinders on z-axis	$\pm\,\frac{1}{a}\langle x,y,0\rangle$	$a\,\mathrm{d}z\,\mathrm{d}\theta$
General case	$\pm\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\rangle$	$\mathrm{d}x\,\mathrm{d}y$

The normal vector for a general surface, $\langle -f_x, -f_y, 1\rangle$ can be arrived at by taking the gradient of the level curve $w = f(x,y,z)$ (see Unit 2), or by taking the cross product of two vectors tangent to the surface $f(x,y)$ and perpendicular to the x and y axes, $\vec{\mathbf{u}} = \langle \Delta x, 0, f(x+\Delta x, y)\rangle$ and $\vec{\mathbf{v}} = \langle 0, \Delta y, f(x, y+\Delta y)\rangle$; taking the limit as Δx, Δy approach zero, we get

$$\begin{aligned}\vec{\mathbf{u}} &= \langle \mathrm{d}x\,, 0, f_x\,\mathrm{d}x\rangle \;=\; \langle 1, 0, f_x\rangle\,\mathrm{d}x\\ \vec{\mathbf{v}} &= \langle 0, \mathrm{d}y\,, f_y\,\mathrm{d}y\rangle \;=\; \langle 0, 1, f_y\rangle\,\mathrm{d}y\end{aligned}$$

The cross product of which gives us the area element above and, most importantly, the normal vector of a parallelogram formed by the two vectors (the "polygons" from the section above).

For a **surface which cannot be expressed as a function** $z = f(x,y)$ we take a surface whose points x, y, z are parameterized:

$$S : \vec{\mathbf{r}}(u,v) = \begin{cases} x = x(u,v)\\ y = y(u,v)\\ z = z(u,v)\end{cases}$$

and obtain the cross product of the infinitesimal changes in the position vector for steps the u and v directions (which gives us both the normal vector *and* the surface area element):

$$\pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial u}\times\frac{\partial\vec{\mathbf{r}}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v = \hat{\mathbf{n}}\,\mathrm{d}S \tag{4.10}$$

If **only the normal vector is given** (which might not necessarily be a unit vector), we can still determine $\hat{\mathbf{n}}\Delta S$; it simply remains for us to find the surface area element. We begin by projecting a section of the surface ΔS onto the xy plane and comparing the area of its shadow S. Taking the normal of this section to make an angle α with the vertical unit vector $\hat{\mathbf{k}}$, we can consider the scaling factor of ΔS to be $\cos\alpha$, as the angle that $\mathrm{d}S$ makes with the xy plane is the same as the angle between $\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$. Thus, we obtain

$$\begin{aligned}\Delta A &= \Delta S\cos\alpha\\ &= \Delta S\,\frac{\vec{\mathbf{N}}\cdot\hat{\mathbf{k}}}{|\vec{\mathbf{N}}||\hat{\mathbf{k}}|}\end{aligned}$$

Where the denominator $|\vec{\mathbf{N}}||\hat{\mathbf{k}}|$ simplifies to $|\vec{\mathbf{N}}|$ because the length of $\hat{\mathbf{k}}$ is one. Now, multiplying both sides by $\hat{\mathbf{n}}$ to obtain $\hat{\mathbf{n}}\,\mathrm{d}S$, and solving the equation above for ΔS,

$$\hat{\mathbf{n}}\Delta S = \frac{|\vec{\mathbf{N}}|\,\hat{\mathbf{n}}}{\vec{\mathbf{N}}\cdot\hat{\mathbf{k}}}\,\Delta A$$

Where $|\vec{\mathbf{N}}|\,\hat{\mathbf{n}}$ is simply the unit normal vector scaled up to the length of $\vec{\mathbf{N}}$ and, since $\vec{\mathbf{N}}$ and $\hat{\mathbf{n}}$ point in the same direction, this is just $\vec{\mathbf{N}}$.

Thus, taking the limit as ΔS, and ΔA approach zero:

$$\hat{\mathbf{n}}\,\mathrm{d}S = \pm\frac{\vec{\mathbf{N}}}{\vec{\mathbf{N}}\cdot\hat{\mathbf{k}}}\,\mathrm{d}x\,\mathrm{d}y \tag{4.11}$$

Remark: though we need to account for the area element of the surface $\mathrm{d}S$, taking the dot product of our surface's normal vector and, for example the xy plane, accounts for this area element. In other words, in the general case $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ will account for both the surface area element and the direction of the normal vector.

4.4.2 Physical Interpretation of Flux Surface Integral

If $\vec{\mathbf{F}}$ is a velocity field, we can consider of $\vec{\mathbf{F}}$ through a surface to be the *volume* of "stuff" moving through the surface in unit time. Considering that $\iint_S \vec{\mathbf{F}}\cdot\hat{\mathbf{n}}\,\mathrm{d}x\,\mathrm{d}y$ multiplies the integrand by an area element $\mathrm{d}x\,\mathrm{d}y$, we can assert that the units of our result (given that $\vec{\mathbf{F}}$ is a velocity field of units m/s) will be:

$$\frac{m}{s}\cdot m^2 = \frac{m^3}{s}$$

4.4.3 Surface Integrals of Scalar Functions

Where we have prior to now been considering surface integrals of vector fields, surface integrals can also be used to sum the values of a scalar function $f(x,y,z)$ on a surface S.

$$\iint_S f(x,y,z)\,\mathrm{d}S$$

Where we primarily concern ourselves with finding a way to scale from the area element $\mathrm{d}x\,\mathrm{d}y$ to the area of $\mathrm{d}S$, as the surface area of a slanted or curved surface is *not* simply the shadow of that surface on a plane, $\mathrm{d}x\,\mathrm{d}y$. Borrowing from the derivation of $\hat{\mathbf{n}}$ for an arbitrary function, $\langle -f_x, -f_y, 1\rangle$, we simply take the magnitude of this vector (as the magnitude of the cross product gives the area of the parallelogram formed by its two vectors). This is clearly $\sqrt{1+(f_x)^2+(f_y)^2}$. Thus, the surface integral of a scalar function f for a surface $h(x,y)$ is

$$\iint_S f(x,y,h(x,y))\sqrt{1+(h_x)^2+(h_y)^2}\,\mathrm{d}x\,\mathrm{d}y$$

Where if $f=1$, we get the surface area of a surface:

$$A = \iint_R \sqrt{(h_x)^2+(h_y)^2+1}\,\mathrm{d}x\,\mathrm{d}y$$

Note that if we have a surface parameterized as $\vec{\mathbf{r}} = x(u,v)\hat{\mathbf{\imath}} + y(u,v)\hat{\mathbf{\jmath}} + z(u,v)\hat{\mathbf{k}}$, we can express a surface integral as

$$\iint_R G(x,y,z)\,\mathrm{d}S = \iint_R G(x(u,v),y(u,v),z(u,v))\,|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\,\mathrm{d}u\,\mathrm{d}v$$

where $G(x,y,z)$ is the scalar function over which we integrate, and $|\vec{\mathbf{r}}_1 \times \vec{\mathbf{r}}_2|$ is the scaling factor for the surface area element. Also note that the surface area element above is simply a special case of $|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|$ where $u = x$ and $v = y$.

Furthermore, the surface area element of an implicit surface $f(x,y,z) = c$ (i.e. a level curve) can be determined by projecting the gradient of this surface (i.e., the vector normal to the surface) onto a unit vector $\hat{\mathbf{p}}$—which can be $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}$ or $\hat{\mathbf{k}}$ depending on the desired orientation of integration; in this case we choose $\mathrm{d}x\,\mathrm{d}y$—and scaling the gradient according to this projection:

$$\iint_S \frac{|\boldsymbol{\nabla} f|}{|\boldsymbol{\nabla} f \cdot \hat{\mathbf{p}}|}\,\mathrm{d}x\,\mathrm{d}y$$

Which can be thought of as scaling the magnitude of $\boldsymbol{\nabla} f$ by the *inverse* of how much it points perpendicularly to the xy plane; the more it points away, the larger we should count its *shadow* on the xy plane. Also note that this is analogous to equation (11) in Section 4.1.

4.4.4 Surface Area Elements of Different Types of Surfaces

For a surface defined **parametrically** by $\vec{\mathbf{r}}(u,v)$:

$$\iint_S |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\,\mathrm{d}x\,\mathrm{d}y$$

For a surface given implicitly, or a **level curve** $f(x,y,z) = c$:

$$\iint_S \frac{|\boldsymbol{\nabla} f|}{\left|\boldsymbol{\nabla} f \cdot \hat{\mathbf{k}}\right|}\,\mathrm{d}x\,\mathrm{d}y$$

And, finally, for a surface given explicitly in the form $z = f(x,y)$:

$$\iint_S \sqrt{(f_x)^2 + (f_y)^2 + 1}\;\mathrm{d}x\,\mathrm{d}y$$

4.5 Line Integrals in $\mathbb{R}^3$

Line integrals in space

$$\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_a^b \langle P, Q, R\rangle \cdot \langle \mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z\rangle \tag{4.12}$$

are straightforward extensions of line integrals in planes, where the integrand is parameterized according to the path C using $x(t)$, $y(t)$, $z(t)$. Then we use

$$\begin{cases} \mathrm{d}x = x'(t)\,\mathrm{d}t \\ \mathrm{d}y = y'(t)\,\mathrm{d}t \\ \mathrm{d}z = z'(t)\,\mathrm{d}t \end{cases}$$

To substitute the differentials and reduce the problem to a single integral.

The **fundamental theorem of calculus for line integrals** also holds for line integrals in space. That is,

$$\int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\vec{\mathbf{r}} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

4.5.1 Line Integrals of Scalar Functions

The line integral of a scalar function is fairly straightforward. Given a parameterization

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

we substitute our parameterized $x(t)$ into $f(x, y, z)$. Then, as with surface integrals, we must account for the scaling of our line's projection onto the plane, as we are integrating with respect to the arclength of the line. Recalling that the arclength is given by integrating $|\vec{\mathbf{v}}|$, the line integral of a scalar function is

$$\int_C f(x(t), y(t), z(t))\,|\vec{\mathbf{v}}|\,\mathrm{d}t = \int_C f(x(t), y(t), z(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

4.5.2 Conservative/Gradient Fields in 3D

We can test whether $\vec{\mathbf{F}}$ is a gradient field in a similar manner to two dimensional vector fields. We compare the partial derivatives of the components, this time looking for equalities in pairs of components:

$$\vec{\mathbf{F}} = \boldsymbol{\nabla} f \Leftrightarrow \begin{cases} P_y = Q_x \\ P_z = R_x \\ Q_z = R_y \end{cases}$$

Note that this test can also be used to determine whether P, Q, R are exact differentials of the form $\mathrm{d}f = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$. Also note that these are the expressions the differences of which are taken to find the components of $\boldsymbol{\nabla} \times \vec{\mathbf{F}}$, where the component that isn't accounted for by the expression (for example the z component in $P_y = Q_x$) is the axis of rotation of $\boldsymbol{\nabla} \times \vec{\mathbf{F}}$.

To find the potential function from which $\vec{\mathbf{F}}$ comes, we can proceed using a method similar (albeit slightly more complicated) to the one for two-dimensional vector fields.

$$\begin{cases} P = 2xy \\ Q = x^2 + z^3 \\ R = 3yz^2 - 4z^2 \end{cases}$$

Step 1: Integrate P, the x component, with respect to x. The constant of integration is a mystery function of y and z, $g(y, z)$

$$\int P\,\mathrm{d}x = f = x^2 y + g(y, z)$$

Step 2: Differentiate the f obtained from step 1 with respect to y. $g(y, z)$ becomes g_y, the partial derivative with respect to y of the mystery function. Set this equal to the y component and solve for g_y.

$$\frac{\partial f}{\partial y} = x^2 + g_y \Rightarrow x^2 + g_y = x^2 + z^3 \Rightarrow g_y = z^3$$

Step 3: Integrate g_y from above with respect to y to obtain $g(y,z)$ is equal to a known function plus a mystery function $h(z)$.

$$g(y,z) = \int z^3 \, \mathrm{d}y = yz^3 + h(z)$$

Step 4: Plug the $g(y,z)$ from step 3 into the original f obtained from step 1, differentiate with respect to z, then set this equal to the z component R in order to solve for $h'(z)$.

$$f = x^2y + g(y,z) = x^2y + yz^3 + h(z)$$

$$\frac{\partial f}{\partial z} = 3yz^2 + h'(z) \Rightarrow 3yz + h'(z) = 3yz^2 - 4z^3 \Rightarrow h'(z) = -4z^3$$

Step 5: Differentiate $h'(z)$ with respect to z, and plug the result into f.

$$\int -4z^3 \, \mathrm{d}z = -z^4 + c = h(z)$$

4.6 The Del Operator and 3D Curl

The symbol $\boldsymbol{\nabla}$ can be considered to stand for a vector the components of which are partial derivatives in those directions:

$$\boldsymbol{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

Thus, the **gradient** of a function f can be thought of as $\boldsymbol{\nabla}$ multiplied by a scalar value f:

$$\boldsymbol{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

The divergence of a vector field $\vec{\mathbf{F}}$ can be represented by the dot product of $\boldsymbol{\nabla}$ and $\vec{\mathbf{F}}$:

$$\boldsymbol{\nabla} \cdot \vec{\mathbf{F}} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Most importantly, the somewhat lengthy formula for the three-dimensional curl of a vector field $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ can be remembered by representing the curl as the cross product of $\boldsymbol{\nabla}$ and $\vec{\mathbf{F}}$:

$$\boldsymbol{\nabla} \times F = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_x - Q_z)\,\hat{\mathbf{i}} - (R_x - P_z)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}$$

Notice that this is a vector which points along the axis of rotation of the vector field; if the field has a non-zero curl the axis of rotation of which points in the $\hat{\mathbf{k}}$ direction, the difference between Q_x and P_y will be nonzero. This should be reminiscent of the formula for two-dimensional curl, the intuition for which extends nicely to three-dimensional curl (see Unit 3).

Note that the magnitude of $\nabla \times \vec{\mathbf{F}}$ for a velocity field is 2ω where ω is the angular velocity.

4.7 The Divergence Theorem

For a closed surface S and a region D bounded by S, the divergence theorem (also called Gauss' Theorem) says

$$\iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \iiint_D \nabla \cdot \vec{\mathbf{F}} \, \mathrm{d}V \tag{4.13}$$

if $\vec{\mathbf{F}}$ is defined and differentiable everywhere in D, and the flux is taken *out* of the surface. In words, it states that the flux of a vector field out of a closed surface S is equal to the triple integral of the divergence of a vector field in the region enclosed by the surface S.

Upon closer inspection, the above relation should make sense: it says that summing the divergence of a 3D region yields the amount of fluid leaving a region per unit time. The triple integral of the divergence is, in that way, analogous to the sum of all the "sources" and "sinks" in that region.

As with similar theorems in two dimensions, we can construct a surface which excludes an undefined or non-differentiable point in $\vec{\mathbf{F}}$. We simply define two surfaces, S_1 and S_2, where one bounds a cube and the other bounds, say, a sphere enclosing this cube. We then claim that the triple integral of the region between these two surfaces, designated D, is equal to the flux *out* of the sphere S_2 and *into* the cube S_2:

$$\iiint_D \nabla \cdot \vec{\mathbf{F}} \, \mathrm{d}V = \iint_{S_2} \vec{\mathbf{F}} \, \mathrm{d}\vec{\mathbf{S}} - \iint_{S_1} \vec{\mathbf{F}} \, \mathrm{d}\vec{\mathbf{S}}$$

Where we subtract the second surface integral because we are taking the flux *into* the cube, so the opposite of the flux out of the cube.

4.8 Stokes' Theorem

Stokes' theorem states that for a closed path and a defined and differentiable $\vec{\mathbf{F}}$,

$$\oint_C \vec{\mathbf{F}} \, \mathrm{d}\vec{\mathbf{r}} = \iint_S \left(\nabla \times \vec{\mathbf{F}} \right) \cdot \hat{\mathbf{n}} \, \mathrm{d}S \tag{4.14}$$

where S is any *capping surface* of C. A capping surface is any surface which can be bounded by C and continuously deformed. We orient the surface by considering "walking" along C with S to our left, and choosing $\hat{\mathbf{n}}$ so that it points upward. In words, Stokes' theorem states that the work line integral of a vector field along a path C is equal to the flux surface integral of the curl of the vector field.

Stokes' theorem is an extension of Green's Theorem to three-dimensions. In other words, Green's theorem is the special case of Stokes' theorem in a plane.

Recall that Green's theorem holds true for any *simply connected region*. This is also the case with Stokes' theorem; however, the definition of a simply connected region has some caveats in three dimensions. A line forming a loop in a simply connected region must be able to shrink down to a point. For example, if, in space, a single point is undefined, you could still shrink the loop "over" the undefined point; on the other hand, if the entire y axis is undefined, there is no way to cross this undefined point. See Figure 4.5.

From Green's theorem, we also arrive at a conclusion regarding Stokes' theorem and conservative fields. We can claim that, if $\vec{\mathbf{F}}$ is defined in a simply connected region and $\nabla \times \vec{\mathbf{F}} = 0$ then $\vec{\mathbf{F}} = \nabla f$.

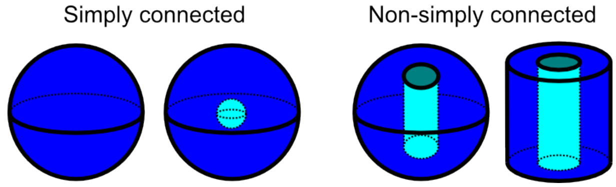


Figure 4.5: Simply connected and non-simply connected regions in 3D.

4.8.1 "Proof" of Stokes' Theorem

Given any S, decompose it into tiny, almost flat pieces, as if it making it into a polyhedron, as pictured in Figure 4.6. Summing the work done along the edges of each of these pieces counterclockwise, neighboring pieces will cancel the work done along each other. The only pieces which do not have neighbors are the ones at the edges of the surface. But the segments at the edges of the surface comprise the path C of which S is a capping surface.

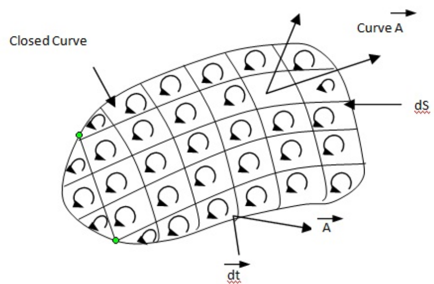


Figure 4.6: The surface S used to prove Stokes' theorem.

4.8.2 Surface Independence

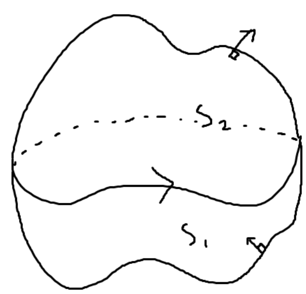


Figure 4.7: The surfaces S_1 and S_2.

For two capping surfaces of C, S_1 and S_2,

$$\oint_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \iint_{S_1} \left(\boldsymbol{\nabla} \times \vec{\mathbf{F}}\right) \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_{S_2} \left(\boldsymbol{\nabla} \times \vec{\mathbf{F}}\right) \cdot \mathrm{d}\vec{\mathbf{S}} \tag{4.15}$$

A fact that we can prove by recognizing that $S_1 - S_2$ is a closed surface (where we subtract S_2 to ensure that all normal vectors are pointing *out* of the closed surface). Recall that for a closed surface, the divergence theorem says

$$\iint_{S_1-S_2}\left(\boldsymbol{\nabla}\times\vec{\mathbf{F}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}S=\iiint_D \mathrm{div}\left(\boldsymbol{\nabla}\times\vec{\mathbf{F}}\right)\mathrm{d}S$$

But since the divergence of the curl of a vector field is always zero, (equation (5.1)) we can say

$$\iint_{S_1-S_2}\left(\boldsymbol{\nabla}\times\vec{\mathbf{F}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}S=\iint_{S_1}\left(\boldsymbol{\nabla}\times\vec{\mathbf{F}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}S-\iint_{S_2}\left(\boldsymbol{\nabla}\times\vec{\mathbf{F}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}S=0$$

$$\iint_{S_1}\left(\boldsymbol{\nabla}\times\vec{\mathbf{F}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}S=\iint_{S_2}\left(\boldsymbol{\nabla}\times\vec{\mathbf{F}}\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

Proving that Stokes' theorem holds for any surface S which is a capping surface of C.

4.9 Divergence-Less and Curl-less Fields

Curl-less fields: The following are equivalent:

1. $\nabla\times\mathbf{v}=0$ everywhere;
2. $\int_a^b \mathbf{v}\cdot\mathrm{d}\mathbf{r}$ is independent of path for any two endpoints;
3. $\oint \mathbf{v}\cdot\mathrm{d}\mathbf{r}=0$ for any closed loop;
4. $\mathbf{v}$ is the gradient of some scalar, $\mathbf{v}=\nabla t$.

Divergence-less fields: The following are equivalent:

1. $\nabla\cdot\mathbf{v}=0$ everywhere;
2. $\int_S \mathbf{v}\cdot\mathrm{d}\mathbf{A}$ is independent of surface for any given boundary line;
3. $\int_S \mathbf{v}\cdot\mathrm{d}\mathbf{A}=0$ for any closed surface;
4. $\mathbf{v}$ is the curl of some vector, $\mathbf{v}=\nabla\times\mathbf{w}$.

Chapter 5

Appendix

5.1 Derivation of the Second Derivative Rule

We proceed assuming the reader is familiar with linear algebra.

Recall positive definiteness: a positive definite matrix has positive eigenvalues, positive pivots, positive minors, a positive determinant, and is symmetric. Also recall the expression $x^T Ax$.

We put the second derivatives in a matrix.

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

As $f_{xy} = f_{yx}$, this is symmetric. Recall that if a matrix is positive definite, $x^T Ax > 0$. For a vector $\langle x, y \rangle$, this becomes a paraboloid with a minimum at zero:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$$

Its minimum is at zero because $x^T Ax > 0$. If the determinant of our second derivative matrix above is positive, and all of its second derivatives are positive, we "know" it is a positive definite matrix, and therefore we can assert that

$$ax^2 + 2bxy + cy^2 = f_{xx}x^2 + 2f_{xy}x = f_{yy}y^2$$

is a paraboloid with minimum at zero. Notice that this is the second term of the Taylor expansion for the given function. Therefore, we can claim that the function itself has a minimum at that point.

5.2 Misc. Properties

The flux of a the position vector field $\vec{\mathbf{F}} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$ outward through a closed surface S is 3 times the volume contained by that surface. By the divergence theorem (Section 4.7):

$$\begin{aligned} \iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \iiint_D \boldsymbol{\nabla} \cdot \vec{\mathbf{F}} \, \mathrm{d}V \\ &= 3 \iiint_D \mathrm{d}V \end{aligned}$$

Using $\boldsymbol{\nabla} \cdot \vec{\mathbf{F}} = 3$ and where $\iiint_D \mathrm{d}V$ is the volume of the region D.

The divergence of the curl of $\vec{\mathbf{F}}$ is always zero:

$$\operatorname{div}\left(\boldsymbol{\nabla} \times \vec{\mathbf{F}}\right) = 0 \tag{5.1}$$