

Logics for Artificial Intelligence:

Assignment 1

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1. (a) Prove that $Z \cap (X \cup Y) = (Z \cap X) \cup (Z \cap Y)$

$$\begin{aligned}
 \text{Take } \alpha \in Z \cap (X \cup Y) &\Leftrightarrow \alpha \in Z \text{ and } \alpha \in X \cup Y \\
 &\Leftrightarrow \alpha \in Z \text{ and } (\alpha \in X \text{ or } \alpha \in Y) \\
 &\Leftrightarrow (\alpha \in Z \text{ and } \alpha \in X) \text{ or } (\alpha \in Z \text{ and } \alpha \in Y) \\
 &\Leftrightarrow \alpha \in Z \cap X \text{ or } \alpha \in Z \cap Y \\
 &\Leftrightarrow \alpha \in (Z \cap X) \cup (Z \cap Y)
 \end{aligned}$$

- (b) (2.13.4)

$$\begin{aligned}
 \text{Take } \alpha \in X \Delta (Y \Delta Z) &\Leftrightarrow \alpha \in X \text{ and } \alpha \notin (Y \Delta Z) \text{ or } \alpha \in (Y \Delta Z) \text{ and } \alpha \notin X \\
 &\Leftrightarrow \alpha \in X \text{ and } \alpha \notin Y \text{ and } \alpha \in Z \text{ or} \\
 &\alpha \notin X \text{ and } \alpha \in Y \text{ and } \alpha \notin Z \text{ or} \\
 &\alpha \notin X \text{ and } \alpha \notin Y \text{ and } \alpha \in Z \text{ or} \\
 &\alpha \in X \text{ and } \alpha \in Y \text{ and } \alpha \notin Z
 \end{aligned}$$

$$\begin{aligned}
 \text{Now take } \alpha \in (X \Delta Y) \Delta Z &\Leftrightarrow \alpha \in (X \Delta Y) \text{ and } \alpha \notin Z \text{ or } \alpha \in Z \text{ and } \alpha \notin (X \Delta Y) \\
 &\Leftrightarrow \alpha \in X \text{ and } \alpha \notin Y \text{ and } \alpha \notin Z \text{ or} \\
 &\alpha \notin X \text{ and } \alpha \in Y \text{ and } \alpha \notin Z \text{ or} \\
 &\alpha \notin X \text{ and } \alpha \notin Y \text{ and } \alpha \in Z \text{ or} \\
 &\alpha \in X \text{ and } \alpha \in Y \text{ and } \alpha \in Z
 \end{aligned}$$

$$\therefore X \Delta (Y \Delta Z) = (X \Delta Y) \Delta Z$$

(2.13.5) Consider:

$$\begin{aligned}
(X \cup Y) \setminus (X \cap Y) &= ((X \cup Y) \setminus X) \cup ((X \cup Y) \setminus Y) \\
&= (Y \setminus X) \cup (X \setminus Y) \\
&= X \Delta Y
\end{aligned}$$

$$\therefore X \Delta Y = (X \cup Y) \setminus (X \cap Y)$$

- (c) Note that $2^{\cup_{i \in I} X_i} \not\subseteq \cup_{i \in I} 2^{X_i}$. This can be easily seen through example. Consider $X_i = \{\{1, 2\}, \{3\}\}$. Now $\{1, 2, 3\} \in 2^{\cup_{i \in I} X_i}$ but $\{1, 2, 3\} \notin \cup_{i \in I} 2^{X_i}$. Therefore $2^{\cup_{i \in I} X_i} \not\subseteq \cup_{i \in I} 2^{X_i}$.
- (d) Prove that $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$

$$\begin{aligned}
\text{Take } (x, y) \in X \times (Y \cup Z) &\Leftrightarrow x \in X \text{ and } y \in Y \cup Z \\
&\Leftrightarrow x \in X \text{ and } y \in Y \text{ or } x \in X \text{ and } y \in Z \\
&\Leftrightarrow (x, y) \in (X \times Y) \cup (X \times Z)
\end{aligned}$$

(e) (4.2.1)

$$\begin{aligned}
(R^{-1})^{-1} &= (\{(y, x) \in Y \times X : xRy\})^{-1} \\
&= \{(x, y) \in X \times Y : xRy\} \\
&= R
\end{aligned}$$

(4.2.2)

$$\begin{aligned}
\text{dom}(R^{-1}) &= \{y \in Y : (\exists x \in X) xRy\} \\
&= \text{range}(R) \\
\text{range}(R^{-1}) &= \{x \in X : (\exists y \in Y) xRy\} \\
&= \text{dom}(R)
\end{aligned}$$

2. Interpretations are represented by an ordered pair, (p, q) , where the first element is the truth value of p and the second the truth value of q .

- (a) $\text{Mod}(p \vee \neg p) = \{(T, T), (T, F), (F, T), (F, F)\}$
(b) $\text{Mod}(p \vee q) = \{(T, T), (T, F), (F, T)\}$
(c) $\text{Mod}(p \vee \neg q) = \{(T, T), (T, F), (F, F)\}$
(d) $\text{Mod}(\neg p \vee q) = \{(T, T), (F, T), (F, F)\}$

- (e) $\text{Mod}(\neg p \vee \neg q) = \{(T, F), (F, T), (F, F)\}$
- (f) $\text{Mod}(p) = \{(T, T), (T, F)\}$
- (g) $\text{Mod}(q) = \{(T, T), (F, T)\}$
- (h) $\text{Mod}(p \leftrightarrow q) = \{(T, T), (F, F)\}$
- (i) $\text{Mod}(\neg(p \leftrightarrow q)) = \{(T, F), (F, T)\}$
- (j) $\text{Mod}(\neg q) = \{(T, F), (F, F)\}$
- (k) $\text{Mod}(\neg p) = \{(F, T), (F, F)\}$
- (l) $\text{Mod}(p \wedge q) = \{(T, T)\}$
- (m) $\text{Mod}(p \wedge \neg q) = \{(T, F)\}$
- (n) $\text{Mod}(\neg p \wedge q) = \{(F, T)\}$
- (o) $\text{Mod}(\neg p \wedge \neg q) = \{(F, F)\}$
- (p) $\text{Mod}(p \wedge \neg p) = \emptyset$

3. There are no formulas that are not logically equivalent to one of the formulas in question 2. This is obvious as the union over the models in question 2 is exactly the power set of all possible interpretations for (p,q). Thus any formula with variables p and q will be logically equivalent to one of the formulas in question 2.

4. First consider:

$$\begin{aligned} \{A \wedge B \rightarrow C\} &\Leftrightarrow \neg(A \wedge B) \vee C \\ &\Leftrightarrow \neg A \vee \neg B \vee C \end{aligned}$$

Now consider:

$$\begin{aligned} (A \rightarrow C) \vee (B \rightarrow C) &\Leftrightarrow \neg A \vee C \vee \neg B \vee C \\ &\Leftrightarrow \neg A \vee \neg B \vee C \end{aligned}$$

So clearly:

$$\{A \wedge B \rightarrow C\} \models (A \rightarrow C) \vee (B \rightarrow C)$$

However consider below where (A,B,C) is an ordered tuple representing

the respective truth values:

$$(T, F, F), (T, F, T) \in \{A \wedge B \rightarrow C\}$$

$$\text{But: } (T, F, F) \notin A \rightarrow C$$

$$\text{Therefore: } \{A \wedge B \rightarrow C\} \not\models A \rightarrow C$$

and

$$(F, T, F), (F, T, T) \in \{A \wedge B \rightarrow C\}$$

$$\text{But: } (F, T, F) \notin B \rightarrow C$$

$$\text{Therefore: } \{A \wedge B \rightarrow C\} \not\models B \rightarrow C$$

5. (a) Consider $Mod(\alpha) \cap Mod(\beta)$. These are the interpretations where both α and β are satisfied, i.e. α and β are both satisfied, hence $Mod(\alpha) \cap Mod(\beta) = Mod(\alpha \wedge \beta)$.
- (b) The models of $\neg\alpha$ are all interpretations where α is false, or all interpretations where α is not true. In other words all interpretations in $W - Mod(\alpha)$.
- (c) Note that if α is valid then it is true in all interpretations. Therefore it is true for all interpretations in $Mod(K)$. Therefore $Mod(K \cup \{\alpha\}) = Mod(K)$ and is therefore satisfiable.
- (d) If $\alpha \equiv \beta$ then all interpretations that are true for α are also true for β , and vice versa. Therefore:

$$\begin{aligned} Mod(\alpha) = Mod\beta &\Leftrightarrow Mod(\alpha) \subseteq Mod(\beta) \text{ and } Mod(\beta) \subseteq Mod(\alpha) \\ &\Leftrightarrow \alpha \models \beta \text{ and } \beta \models \alpha \end{aligned}$$

(e)

$$\begin{aligned} \alpha \leftrightarrow \beta \text{ is valid} &\Leftrightarrow \alpha \models \beta \text{ and } \beta \models \alpha \\ &\Leftrightarrow Mod(\alpha) \subseteq Mod(\beta) \text{ and } Mod\beta \subseteq Mod(\alpha) \\ &\Leftrightarrow Mod(\alpha) = Mod(\beta) \\ &\Leftrightarrow \alpha \equiv \beta \end{aligned}$$

6. (a) Consider $\alpha \in \mathcal{L}$. Then:

$$\begin{aligned} Mod(\alpha) = Mod(\alpha) &\Rightarrow Mod(\alpha) \subseteq Mod(\alpha) \\ &\Rightarrow \alpha \models \alpha \end{aligned}$$

(b) Consider $Mod(K \cup \{B\})$:

$$Mod(K \cup \{B\}) = Mod(K) \cap Mod(\{B\}) \subseteq Mod(K) \subseteq Mod(\alpha)$$

Therefore $K \cup \{B\} \models \alpha$

(c) If $p, \neg p \in K$ then K is unsatisfiable and $Mod(K) = \emptyset$. Since \emptyset is contained in all sets $Mod(K) = \emptyset \subseteq Mod(\gamma)$ for every $\gamma \in \mathcal{L}$. Therefore $K \models \gamma$ for every $\gamma \in \mathcal{L}$.