Logics for Artificial Intelligence:
Assignment 1

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1. (a) Prove that $Z \cap (X \cup Y) = (Z \cap X) \cup (Z \cap Y)$

   Take $\alpha \in Z \cap (X \cup Y) \Leftrightarrow \alpha \in Z$ and $\alpha \in X \cup Y$
   $$\Leftrightarrow \alpha \in Z \text{ and } (\alpha \in X \text{ or } \alpha \in Y)$$
   $$\Leftrightarrow (\alpha \in Z \text{ and } \alpha \in X) \text{ or } (\alpha \in Z \text{ and } \alpha \in Y)$$
   $$\Leftrightarrow \alpha \in Z \cap X \text{ or } \alpha \in Z \cap Y$$
   $$\Leftrightarrow \alpha \in (Z \cap X) \cup (Z \cap Y)$$

   (b) (2.13.4)

   Take $\alpha \in X \triangle (Y \triangle Z) \Leftrightarrow \alpha \in X$ and $\alpha \notin (Y \triangle Z)$ or $\alpha \in (Y \triangle Z)$ and $\alpha \notin$
   $$\Leftrightarrow \alpha \in X \text{ and } \alpha \notin Y \text{ and } \notin Z \text{ or}$$
   $$\alpha \notin X \text{ and } \alpha \in Y \text{ and } \notin Z \text{ or}$$
   $$\alpha \notin X \text{ and } \alpha \notin Y \text{ and } \in Z \text{ or}$$
   $$\alpha \in X \text{ and } \alpha \in Y \text{ and } \in Z$$
   Now take $\alpha \in (X \triangle Y) \triangle Z \Leftrightarrow \alpha \in (X \triangle Y)$ and $\alpha \notin Z$ or $\alpha \in Z$ and $\alpha \notin (X \triangle Y)$
   $$\Leftrightarrow \alpha \in X \text{ and } \alpha \notin Y \text{ and } \notin Z \text{ or}$$
   $$\alpha \notin X \text{ and } \alpha \in Y \text{ and } \notin Z \text{ or}$$
   $$\alpha \notin X \text{ and } \alpha \notin Y \text{ and } \in Z \text{ or}$$
   $$\alpha \in X \text{ and } \alpha \in Y \text{ and } \in Z$$
   $\therefore X \triangle (Y \triangle Z) = (X \triangle Y) \triangle Z$

   (2.13.5) Consider:

$$(X \cup Y) \setminus (X \wedge Y) = ((X \cup Y) \setminus X) \cup ((X \cup Y) Y)$$
$$= (Y \setminus X) \cup (X \setminus Y)$$
$$= X \triangle Z$$
$$\therefore X \triangle Y = (X \cup Y) \setminus (X \cap Y)$$

(c) Note that $2^{\cup_{i \in I} X_i} \not\subseteq \cup_{i \in I} 2^{X_i}$. This can be easily seen through example. Consider $X_i = \{\{1, 2\}, \{3\}\}$. Now $\{1, 2, 3\} \in 2^{\cup_{i \in I} X_i}$ but $\{1, 2, 3\} \notin \cup_{i \in I} 2^{X_i}$. Therefore $2^{\cup_{i \in I} X_i} \not\subseteq \cup_{i \in I} 2^{X_i}$.

(d) Prove that $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$

Take $(x, y) \in X \times (Y \cup Z) \Leftrightarrow x \in X$ and $y \in Y \cup Z$
$$\Leftrightarrow x \in X \text{ and } y \in Y \text{ or } x \in X \text{ and } y \in Z$$
$$\Leftrightarrow (x, y) \in (X \times Y) \cup (X \times Z)$$

(e) (4.2.1)
$$(R^{-1})^{-1} = (\{(y, x) \in Y \times X : xRy\})^{-1}$$
$$= \{(x, y) \in X \times Y : xRy\}$$
$$= R$$

(4.2.2)

$$dom(R^{-1}) = \{y \in Y : (\exists x \in X)xRy\}$$
$$= range(R)$$
$$range(R^{-1}) = \{x \in X : (\exists y \in Y)xRy\}$$
$$= dom(R)$$

2. Interpretations are represented by an ordered pair, (p, q), where the first element is the truth value of p and the second the truth value of q.

(a) $\text{Mod}(p \vee \neg p) = \{(T, T), (T, F), (F, T), (F, F)\}$

(b) $\text{Mod}(p \vee q) = \{(T, T), (T, F), (F, T)\}$

(c) $\text{Mod}(p \vee \neg q) = \{(T, T), (T, F), (F, F)\}$

(d) $\text{Mod}(\neg p \vee q) = \{(T, T), (F, T), (F, F)\}$

(e) $\text{Mod}(\neg p \vee \neg q) = \{(T, F), (F, T), (F, F)\}$

(f) $\text{Mod}(p) = \{(T, T), (T, F)\}$

(g) $\text{Mod}(q) = \{(T, T), (F, T)\}$

(h) $\text{Mod}(p \leftrightarrow q) = \{(T, T), (F, F)\}$

(i) $\text{Mod}(\neg(p \leftrightarrow q)) = \{(T, F), (F, T)\}$

(j) $\text{Mod}(\neg q) = \{(T, F), (F, F)\}$

(k) $\text{Mod}(\neg p) = \{(F, T), (F, F)\}$

(l) $\text{Mod}(p \wedge q) = \{(T, T)\}$

(m) $\text{Mod}(p \wedge \neg q) = \{(T, F)\}$

(n) $\text{Mod}(\neg p \wedge q) = \{(F, T)\}$

(o) $\text{Mod}(\neg p \wedge \neg q) = \{(F, F)\}$

(p) $\text{Mod}(p \wedge \neg p) = \emptyset$

3. There are no formulas that are not logically equivalent to one of the formulas in question 2. This is obvious as the union over the models in question 2 is exactly the power set of all possible interpretations for (p,q). Thus any formula with variables p and q will be logically equivalent to one of the formulas in question 2.

4. First consider:

$$\{A \wedge B \to C\} \Leftrightarrow \neg(A \wedge B) \vee C$$
$$\Leftrightarrow \neg A \vee \neg B \vee C$$

Now consider:

$$(A \to C) \vee (B \to C) \Leftrightarrow \neg A \vee C \vee \neg B \vee C$$
$$\Leftrightarrow \neg A \vee \neg B \vee C$$

So clearly:

$$\{A \wedge B \to C\} \models (A \to C) \vee (B \to C)$$

However consider below where (A,B,C) is an ordered tuple representing

the respective truth values:

$$(T, F, F), (T, F, T) \in \{A \wedge B \to C\}$$
$$\text{But: } (T, F, F) \notin A \to C$$
$$\text{Therefore: } \{A \wedge B \to C\} \nvDash A \to C$$
$$\text{and}$$
$$(F, T, F), (F, T, T) \in \{A \wedge B \to C\}$$
$$\text{But: } (F, T, F) \notin B \to C$$
$$\text{Therefore: } \{A \wedge B \to C\} \nvDash B \to C$$

5. (a) Consider $Mod(\alpha) \cap Mod(\beta)$. These are the interpretations where both α and β are satisfied, i.e. α and β are both satisfied, hence $Mod(\alpha) \cap Mod(\beta) = Mod(\alpha \wedge \beta)$.

   (b) The models of $\neg\alpha$ are all interpretations where α is false, or all interpretations where α is not true. In other words all interpretations in $W - Mod(\alpha)$.

   (c) Note that if α is valid then it is true in all interpretations. Therefore it is true for all interpretations in Mod(K). Therefore $Mod(K \cup \{\alpha\}) = Mod(K)$ and is therefore satisfiable.

   (d) If $\alpha \equiv \beta$ then all interpretations that are true for α are also true for β, and vice versa. Therefore:

   $$Mod(\alpha) = Mod\beta \Leftrightarrow Mod(\alpha) \subseteq Mod(\beta) and Mod(\beta) \subseteq Mod(\alpha)$$
   $$\Leftrightarrow \alpha \models \beta \text{ and } \beta \models \alpha$$

   (e)

6. (a) Consider $\alpha \in \mathcal{L}$. Then:

   $$Mod(\alpha) = Mod(\alpha) \Rightarrow Mod(\alpha) \subseteq Mod(\alpha)$$
   $$\Rightarrow \alpha \models \alpha$$

   (b) Consider $Mod(K \cup \{B\})$:
   $Mod(K \cup \{B\}) = Mod(K) \cap Mod(\{B\}) \subseteq Mod(K) \subseteq Mod(\alpha)$
   Therefore $K \cup \{B\} \models \alpha$

   (c) If $p, \neg p \in K$ then K is unsatisfiable and $Mod(K) = \emptyset$. Since $\emptyset$ is contained in all sets $Mod(K) = \emptyset \subseteq Mod(\gamma)$ for every $\gamma \in \mathcal{L}$. Therefore $K \models \gamma$ for every $\gamma \in \mathcal{L}$.