Logics for Artificial Intelligence:
Assignment 1

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1. (a) Prove that $Z \cap (X \cup Y) = (Z \cap X) \cup (Z \cap Y)$

$$\begin{aligned}
\text{Take } \alpha \in Z \cap (X \cup Y) &\Leftrightarrow \alpha \in Z \text{ and } \alpha \in X \cup Y \\
&\Leftrightarrow \alpha \in Z \text{ and } (\alpha \in X \text{ or } \alpha \in Y) \\
&\Leftrightarrow (\alpha \in Z \text{ and } \alpha \in X) \text{ or } (\alpha \in Z \text{ and } \alpha \in Y) \\
&\Leftrightarrow \alpha \in Z \cap X \text{ or } \alpha \in Z \cap Y \\
&\Leftrightarrow \alpha \in (Z \cap X) \cup (Z \cap Y)
\end{aligned}$$

(b)

(c) Note that $2^{\cup_{i \in I} X_i} \not\subseteq \cup_{i \in I} 2^{X_i}$. This can be easily seen through example. Consider $X_i = \{\{1,2\},\{3\}\}$. Now $\{1,2,3\} \in 2^{\cup_{i \in I} X_i}$ but $\{1,2,3\} \notin \cup_{i \in I} 2^{X_i}$. Therefore $2^{\cup_{i \in I} X_i} \not\subseteq \cup_{i \in I} 2^{X_i}$.

(d) Prove that $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$

$$\begin{aligned}
\text{Take } (x,y) \in X \times (Y \cup Z) &\Leftrightarrow x \in X \text{ and } y \in Y \cup Z \\
&\Leftrightarrow x \in X \text{ and } y \in Y \text{ or } x \in X \text{ and } y \in Z \\
&\Leftrightarrow (x,y) \in (X \times Y) \cup (X \times Z)
\end{aligned}$$

(e) (4.2.1)

$$\begin{aligned}
(R^{-1})^{-1} &= (\{(y,x) \in Y \times X : xRy\})^{-1} \\
&= \{(x,y) \in X \times Y : xRy\} \\
&= R
\end{aligned}$$

(4.2.2)

$$\begin{aligned} dom(R^{-1}) &= \{y \in Y : (\exists x \in X) xRy\} \\ &= range(R) \\ range(R^{-1}) &= \{x \in X : (\exists y \in Y) xRy\} \\ &= dom(R) \end{aligned}$$

2. Interpretations are represented by an ordered pair, (p, q), where the first element is the truth value of p and the second the truth value of q.

   (a) Mod$(p \vee \neg p) = \{(T, T), (T, F), (F, T), (F, F)\}$
   (b) Mod$(p \vee q) = \{(T, T), (T, F), (F, T)\}$
   (c) Mod$(p \vee \neg q) = \{(T, T), (T, F), (F, F)\}$
   (d) Mod$(\neg p \vee q) = \{(T, T), (F, T), (F, F)\}$
   (e) Mod$(\neg p \vee \neg q) = \{(T, F), (F, T), (F, F)\}$
   (f) Mod$(p) = \{(T, T), (T, F)\}$
   (g) Mod$(q) = \{(T, T), (F, T)\}$
   (h) Mod$(p \leftrightarrow q) = \{(T, T), (F, F)\}$
   (i) Mod$(\neg(p \leftrightarrow q)) = \{(T, F), (F, T)\}$
   (j) Mod$(\neg q) = \{(T, F), (F, F)\}$
   (k) Mod$(\neg p) = \{(F, T), (F, F)\}$
   (l) Mod$(p \wedge q) = \{(T, T)\}$
   (m) Mod$(p \wedge \neg q) = \{(T, F)\}$
   (n) Mod$(\neg p \wedge q) = \{(F, T)\}$
   (o) Mod$(\neg p \wedge \neg q) = \{(F, F)\}$
   (p) Mod$(p \wedge \neg p) = \emptyset$

3. There are no formulas that are not logically equivalent to one of the formulas in question 2. This is obvious as the union over the models in question 2 is exactly the power set of all possible interpretations for (p,q). Thus any formula with variables p and q will be logically equivalent to one of the formulas in question 2.

4. First consider:

$$\begin{aligned}\{A \wedge B \to C\} &\Leftrightarrow \neg(A \wedge B) \vee C \\ &\Leftrightarrow \neg A \vee \neg B \vee C\end{aligned}$$

Now consider:

$$\begin{aligned}(A \to C) \vee (B \to C) &\Leftrightarrow \neg A \vee C \vee \neg B \vee C \\ &\Leftrightarrow \neg A \vee \neg B \vee C\end{aligned}$$

So clearly:

$$\{A \wedge B \to C\} \models (A \to C) \vee (B \to C)$$

However consider below where (A,B,C) is an ordered tuple representing the respective truth values:

$$\begin{aligned}(T, F, F), (T, F, T) \in \{A \wedge B \to C\} \\ \text{But: } (T, F, F) \notin A \to C \\ \text{Therefore: } \{A \wedge B \to C\} \not\models A \to C \\ \text{and} \\ (F, T, F), (F, T, T) \in \{A \wedge B \to C\} \\ \text{But: } (F, T, F) \notin B \to C \\ \text{Therefore: } \{A \wedge B \to C\} \not\models B \to C\end{aligned}$$

5. (a) Consider $Mod(\alpha) \cap Mod(\beta)$. These are the interpretations where both α and β are satisfied, i.e. α and β are both satisfied, hence $Mod(\alpha) \cap Mod(\beta) = Mod(\alpha \wedge \beta)$.

   (b) The models of $\neg\alpha$ are all interpretations where α is false, or all interpretations where α is not true. In other words all interpretations in $W - Mod(\alpha)$.

   (c) Note that if α is valid then it is true in all interpretations. Therefore it is true for all interpretations in Mod(K). Therefore $Mod(K \cup \{\alpha\}) = Mod(K)$ and is therefore satisfiable.

   (d) If $\alpha \equiv \beta$ then all interpretations that are true for α are also true for β, and vice versa. Therefore:

$$\begin{aligned}Mod(\alpha) = Mod\beta &\Leftrightarrow Mod(\alpha) \subseteq Mod(\beta) and Mod(\beta) \subseteq Mod(\alpha) \\ &\Leftrightarrow \alpha \models \beta \text{ and } \beta \models \alpha\end{aligned}$$

(e)

6. (a) Consider $\alpha \in \mathcal{L}$. Then:

$$\begin{aligned} Mod(\alpha) = Mod(\alpha) &\Rightarrow Mod(\alpha) \subseteq Mod(\alpha) \\ &\Rightarrow \alpha \models \alpha \end{aligned}$$

(b) Consider $Mod(K \cup \{B\})$:
$Mod(K \cup \{B\}) = Mod(K) \cap Mod(\{B\}) \subseteq Mod(K) \subseteq Mod(\alpha)$
Therefore $K \cup \{B\} \models \alpha$

(c) If $p, \neg p \in K$ then K is unsatisfiable and $Mod(K) = \emptyset$. Since $\emptyset$ is contained in all sets $Mod(K) = \emptyset \subseteq Mod(\gamma)$ for every $\gamma \in \mathcal{L}$. Therefore $K \models \gamma$ for every $\gamma \in \mathcal{L}$.