Logics for Artificial Intelligence: Assignment 1

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1. (a) Prove that $Z \cap (X \cup Y) = (Z \cap X) \cup (Z \cap Y)$

$$\begin{aligned}
\text{Take } \alpha \in Z \cap (X \cup Y) &\Leftrightarrow \alpha \in Z \text{ and } \alpha \in X \cup Y \\
&\Leftrightarrow \alpha \in Z \text{ and } (\alpha \in X \text{ or } \alpha \in Y) \\
&\Leftrightarrow (\alpha \in Z \text{ and } \alpha \in X) \text{ or } (\alpha \in Z \text{ and } \alpha \in Y) \\
&\Leftrightarrow \alpha \in Z \cap X \text{ or } \alpha \in Z \cap Y \\
&\Leftrightarrow \alpha \in (Z \cap X) \cup (Z \cap Y)
\end{aligned}$$

(b) (2.13.4)

$$\begin{aligned}
\text{Take } \alpha \in X \vartriangle (Y \vartriangle Z) \Leftrightarrow &\ \alpha \in X \text{ and } \alpha \notin (Y \vartriangle Z) \text{ or} \\
&\ \alpha \in (Y \vartriangle Z) \text{ and } \alpha \notin X \\
\Leftrightarrow &\ \alpha \in X \text{ and } \alpha \notin Y \text{ and } \notin Z \text{ or} \\
&\ \alpha \notin X \text{ and } \alpha \in Y \text{ and } \notin Z \text{ or} \\
&\ \alpha \notin X \text{ and } \alpha \notin Y \text{ and } \in Z \text{ or} \\
&\ \alpha \in X \text{ and } \alpha \in Y \text{ and } \in Z \\
\text{Now take } \alpha \in (X \vartriangle Y) \vartriangle Z \Leftrightarrow &\ \alpha \in (X \vartriangle Y) \text{ and } \alpha \notin Z \text{ or} \\
&\ \alpha \in Z \text{ and } \alpha \notin (X \vartriangle Y) \\
\Leftrightarrow &\ \alpha \in X \text{ and } \alpha \notin Y \text{ and } \notin Z \text{ or} \\
&\ \alpha \notin X \text{ and } \alpha \in Y \text{ and } \notin Z \text{ or} \\
&\ \alpha \notin X \text{ and } \alpha \notin Y \text{ and } \in Z \text{ or} \\
&\ \alpha \in X \text{ and } \alpha \in Y \text{ and } \in Z
\end{aligned}$$

$\therefore X \vartriangle (Y \vartriangle Z) = (X \vartriangle Y) \vartriangle Z$

(2.13.5) Consider:

$$\begin{aligned}(X \cup Y) \setminus (X \cap Y) &= ((X \cup Y) \setminus X) \cup ((X \cup Y)\ Y) \\ &= (Y \setminus X) \cup (X \setminus Y) \\ &= X \bigtriangleup Z\end{aligned}$$

$\therefore X \bigtriangleup Y = (X \cup Y) \setminus (X \cap Y)$

(c) Note that $2^{\cup_{i \in I} X_i} \not\subseteq \cup_{i \in I} 2^{X_i}$. This can be easily seen through example. Consider $X_i = \{\{1,2\},\{3\}\}$. Now $\{1,2,3\} \in 2^{\cup_{i \in I} X_i}$ but $\{1,2,3\} \notin \cup_{i \in I} 2^{X_i}$. Therefore $2^{\cup_{i \in I} X_i} \not\subseteq \cup_{i \in I} 2^{X_i}$.

(d) Prove that $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$

$$\begin{aligned}\text{Take } (x,y) \in X \times (Y \cup Z) &\Leftrightarrow x \in X \text{ and } y \in Y \cup Z \\ &\Leftrightarrow x \in X \text{ and } y \in Y \text{ or } x \in X \text{ and } y \in Z \\ &\Leftrightarrow (x,y) \in (X \times Y) \cup (X \times Z)\end{aligned}$$

(e) (4.2.1)

$$\begin{aligned}(R^{-1})^{-1} &= (\{(y,x) \in Y \times X : xRy\})^{-1} \\ &= \{(x,y) \in X \times Y : xRy\} \\ &= R\end{aligned}$$

(4.2.2)

$$\begin{aligned}dom(R^{-1}) &= \{y \in Y : (\exists x \in X) xRy\} \\ &= range(R) \\ range(R^{-1}) &= \{x \in X : (\exists y \in Y) xRy\} \\ &= dom(R)\end{aligned}$$

2. Interpretations are represented by an ordered pair, (p,q), where the first element is the truth value of p and the second the truth value of q.

   (a) Mod$(p \vee \neg p) = \{(T,T),(T,F),(F,T),(F,F)\}$
   (b) Mod$(p \vee q) = \{(T,T),(T,F),(F,T)\}$
   (c) Mod$(p \vee \neg q) = \{(T,T),(T,F),(F,F)\}$
   (d) Mod$(\neg p \vee q) = \{(T,T),(F,T),(F,F)\}$

(e) $\text{Mod}(\neg p \vee \neg q) = \{(T, F), (F, T), (F, F)\}$

(f) $\text{Mod}(p) = \{(T, T), (T, F)\}$

(g) $\text{Mod}(q) = \{(T, T), (F, T)\}$

(h) $\text{Mod}(p \leftrightarrow q) = \{(T, T), (F, F)\}$

(i) $\text{Mod}(\neg(p \leftrightarrow q)) = \{(T, F), (F, T)\}$

(j) $\text{Mod}(\neg q) = \{(T, F), (F, F)\}$

(k) $\text{Mod}(\neg p) = \{(F, T), (F, F)\}$

(l) $\text{Mod}(p \wedge q) = \{(T, T)\}$

(m) $\text{Mod}(p \wedge \neg q) = \{(T, F)\}$

(n) $\text{Mod}(\neg p \wedge q) = \{(F, T)\}$

(o) $\text{Mod}(\neg p \wedge \neg q) = \{(F, F)\}$

(p) $\text{Mod}(p \wedge \neg p) = \emptyset$

3. There are no formulas that are not logically equivalent to one of the formulas in question 2. This is obvious as the union over the models in question 2 is exactly the power set of all possible interpretations for (p,q). Thus any formula with variables p and q will be logically equivalent to one of the formulas in question 2.

4. First consider:

$$\begin{aligned} \{A \wedge B \to C\} &\Leftrightarrow \neg(A \wedge B) \vee C \\ &\Leftrightarrow \neg A \vee \neg B \vee C \end{aligned}$$

Now consider:

$$\begin{aligned} (A \to C) \vee (B \to C) &\Leftrightarrow \neg A \vee C \vee \neg B \vee C \\ &\Leftrightarrow \neg A \vee \neg B \vee C \end{aligned}$$

So clearly:

$$\{A \wedge B \to C\} \models (A \to C) \vee (B \to C)$$

However consider below where (A,B,C) is an ordered tuple representing the respective truth values:

$$\begin{aligned}(T,F,F),(T,F,T) &\in \{A \wedge B \to C\}\\ \text{But: } (T,F,F) &\notin A \to C\\ \text{Therefore: } \{A \wedge B \to C\} &\not\models A \to C\\ &\text{and}\\ (F,T,F),(F,T,T) &\in \{A \wedge B \to C\}\\ \text{But: } (F,T,F) &\notin B \to C\\ \text{Therefore: } \{A \wedge B \to C\} &\not\models B \to C\end{aligned}$$

5. (a) Consider $Mod(\alpha) \cap Mod(\beta)$. These are the interpretations where both α and β are satisfied, i.e. α and β are both satisfied, hence $Mod(\alpha) \cap Mod(\beta) = Mod(\alpha \wedge \beta)$.

   (b) The models of $\neg\alpha$ are all interpretations where α is false, or all interpretations where α is not true. In other words all interpretations in $W - Mod(\alpha)$.

   (c) Note that if α is valid then it is true in all interpretations. Therefore it is true for all interpretations in Mod(K). Therefore $Mod(K \cup \{\alpha\}) = Mod(K)$ and is therefore satisfiable.

   (d) If $\alpha \equiv \beta$ then all interpretations that are true for α are also true for β, and vice versa. Therefore:

$$\begin{aligned}\alpha \equiv \beta &\Leftrightarrow Mod(\alpha) = Mod\beta\\ &\Leftrightarrow Mod(\alpha) \subseteq Mod(\beta) \text{ and } Mod(\beta) \subseteq Mod(\alpha)\\ &\Leftrightarrow \alpha \models \beta \text{ and } \beta \models \alpha\end{aligned}$$

   (e)

$$\begin{aligned}\alpha \leftrightarrow \beta \text{ is valid } &\Leftrightarrow \alpha \models \beta \text{ and } \beta \models \alpha\\ &\Leftrightarrow Mod(\alpha) \subseteq Mod(\beta) \text{ and } Mod\beta \subseteq Mod(\alpha)\\ &\Leftrightarrow Mod(\alpha) = Mod(\beta)\\ &\Leftrightarrow \alpha \equiv Mod(\beta)\end{aligned}$$

6. (a) Consider $\alpha \in \mathcal{L}$. Then:

$$\begin{aligned}Mod(\alpha) = Mod(\alpha) &\Rightarrow Mod(\alpha) \subseteq Mod(\alpha)\\ &\Rightarrow \alpha \models \alpha\end{aligned}$$

   Therefore $\models$ is reflecive.

(b) Consider $Mod(K \cup \{B\})$:
$Mod(K \cup \{B\}) = Mod(K) \cap Mod(\{B\}) \subseteq Mod(K) \subseteq Mod(\alpha)$
Therefore $K \cup \{B\} \models \alpha$

(c) If $p, \neg p \in K$ then K is unsatisfiable and $Mod(K) = \emptyset$. Since $\emptyset$ is contained in all sets $Mod(K) = \emptyset \subseteq Mod(\gamma)$ for every $\gamma \in \mathcal{L}$. Therefore $K \models \gamma$ for every $\gamma \in \mathcal{L}$.