Logics for Artificial Intelligence:
Assignment 1

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1. (a) Prove that $Z \cap (X \cup Y) = (Z \cap X) \cup (Z \cap Y)$

$$\begin{aligned} \text{Take } \alpha \in Z \cap (X \cup Y) &\Leftrightarrow \alpha \in Z \text{ and } \alpha \in X \cup Y \\ &\Leftrightarrow \alpha \in Z \text{ and } (\alpha \in X \text{ or } \alpha \in Y) \\ &\Leftrightarrow (\alpha \in Z \text{ and } \alpha \in X) \text{ or } (\alpha \in Z \text{ and } \alpha \in Y) \\ &\Leftrightarrow \alpha \in Z \cap X \text{ or } \alpha \in Z \cap Y \\ &\Leftrightarrow \alpha \in (Z \cap X) \cup (Z \cap Y) \end{aligned}$$

   (b)

   (c)

   (d) Prove that $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$

$$\begin{aligned} \text{Take } (x, y) \in X \times (Y \cup Z) &\Leftrightarrow x \in X \text{ and } y \in Y \cup Z \\ &\Leftrightarrow x \in X \text{ and } y \in Y \text{ or } x \in X \text{ and } y \in Z \\ &\Leftrightarrow (x, y) \in (X \times Y) \cup (X \times Z) \end{aligned}$$

2. Interpretations are represented by an ordered pair, (p, q), where the first element is the truth value of p and the second the truth value of q.

   (a) $\text{Mod}(p \vee \neg p) = \{(T, T), (T, F), (F, T), (F, F)\}$

   (b) $\text{Mod}(p \vee q) = \{(T, T), (T, F), (F, T)\}$

   (c) $\text{Mod}(p \vee \neg q) = \{(T, T), (T, F), (F, F)\}$

   (d) $\text{Mod}(\neg p \vee q) = \{(T, T), (F, T), (F, F)\}$

(e) $\text{Mod}(\neg p \vee \neg q) = \{(T, F), (F, T), (F, F)\}$

(f) $\text{Mod}(p) = \{(T, T), (T, F)\}$

(g) $\text{Mod}(q) = \{(T, T), (F, T)\}$

(h) $\text{Mod}(p \leftrightarrow q) = \{(T, T), (F, F)\}$

(i) $\text{Mod}(\neg(p \leftrightarrow q)) = \{(T, F), (F, T)\}$

(j) $\text{Mod}(\neg q) = \{(T, F), (F, F)\}$

(k) $\text{Mod}(\neg p) = \{(F, T), (F, F)\}$

(l) $\text{Mod}(p \wedge q) = \{(T, T)\}$

(m) $\text{Mod}(p \wedge \neg q) = \{(T, F)\}$

(n) $\text{Mod}(\neg p \wedge q) = \{(F, T)\}$

(o) $\text{Mod}(\neg p \wedge \neg q) = \{(F, F)\}$

(p) $\text{Mod}(p \wedge \neg p) = \emptyset$

3. There are no formulas that are not logically equivalent to one of the formulas in question 2. This is obvious as the union over the models in question 2 is exactly the power set of all possible interpretations for (p,q). Thus any formula with variables p and q will be logically equivalent to one of the formulas in question 2.

4. First consider:

$$\begin{aligned}\{A \wedge B \to C\} &\Leftrightarrow \neg(A \wedge B) \vee C \\ &\Leftrightarrow \neg A \vee \neg B \vee C\end{aligned}$$

Now consider:

$$\begin{aligned}(A \to C) \vee (B \to C) &\Leftrightarrow \neg A \vee C \vee \neg B \vee C \\ &\Leftrightarrow \neg A \vee \neg B \vee C\end{aligned}$$

So clearly:

$$\{A \wedge B \to C\} \models (A \to C) \vee (B \to C)$$

However consider below where (A,B,C) is an ordered tuple representing the respective truth values:

$$\begin{aligned}(T,F,F),(T,F,T) &\in \{A \wedge B \rightarrow C\}\\ \text{But: } (T,F,F) &\notin A \rightarrow C\\ \text{Therefore: } \{A \wedge B \rightarrow C\} &\not\models A \rightarrow C\\ &\text{and}\\ (F,T,F),(F,T,T) &\in \{A \wedge B \rightarrow C\}\\ \text{But: } (F,T,F) &\notin B \rightarrow C\\ \text{Therefore: } \{A \wedge B \rightarrow C\} &\not\models B \rightarrow C\end{aligned}$$

5.

6. (a) Consider $\alpha \in \mathcal{L}$. Then:

$$\begin{aligned}Mod(\alpha) = Mod(\alpha) &\Rightarrow Mod(\alpha) \subseteq Mod(\alpha)\\ &\Rightarrow \alpha \models \alpha\end{aligned}$$

(b) Consider $Mod(K \cup \{B\})$:
$Mod(K \cup \{B\}) = Mod(K) \cap Mod(\{B\}) \subseteq Mod(K) \subseteq Mod(\alpha)$
Therefore $K \cup \{B\} \models \alpha$

(c) If $p, \neg p \in K$ then K is unsatisfiable and $Mod(K) = \emptyset$. Since $\emptyset$ is contained in all sets $Mod(K) = \emptyset \subseteq Mod(\gamma)$ for every $\gamma \in \mathcal{L}$. Therefore $K \models \gamma$ for every $\gamma \in \mathcal{L}$.