Problem Set 1

1. Negate the following statement: "For every $\varepsilon > 0$, there exists $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$."

2. (a) Show that there is no rational number r such that $r^2 = 32$.

   (b) Show that the set $\{r \in \mathbb{Q} : r^2 < 32\}$ has no least upper bound in $\mathbb{Q}$.

3. Formulate and prove a necessary and sufficient condition on the prime factorization of $n \geq 2$ that ensures that the square root of n is rational. In other words, complete the blank in the following sentence and supply a proof: For $n \in \mathbb{N}$ with $n \geq 2$, $\sqrt{n}$ is irrational if and only if its prime factorization ____.

   Note: You may assume the fundamental theorem of arithmetic.

   Theorem (Fundamental Theorem of Arithmetic)**.** *For any $n \geq 2$ in $\mathbb{N}$, there are prime numbers $p_1, \ldots, p_k$ and exponents $a_1, \ldots, a_k \in \mathbb{N}$ so that*

   $$n = p_1^{a_1} \cdots p_k^{a_k} = \prod_{j=1}^{k} p_j^{a_j}.$$

   Moreover, this factorization is unique up to reordering the factors.

4. Exercise 1 from the text: If $r \neq 0$ is a rational number and x is irrational, prove that $r + x$ and rx are irrational.

5. Let

   $$E = \left\{ 11 + (-1)^n \left(3 - \frac{5}{n^3} \right) : n \in \mathbb{N} \right\}.$$

   Identify $\inf E$ and $\sup E$ (with proof).

6. Many math texts adopt the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

   (a) Discuss why this is a reasonable convention.

   (b) Adopting this convention, show that, for $E \subseteq \mathbb{R}$, one has $\inf E \leq \sup E$ *if and only if* $E \neq \emptyset$.