Homework 2

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ME5774 Nonlinear Systems Theory
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Fall 2017

Problem 1. Determine if the equilibrium $x_1 = x_2 = 0$ is unstable, stable, asymptotically stable, exponentially stable for

$$\dot{x_1} = x_2 - x_1^3 + x_1^5$$
$$\dot{x_2} = -x_1,$$

where $\mathbf{x} = [x_1 \quad x_2]^\top \in \mathbb{R}^2$. State the strongest claim you can make and provide support for the conclusion. You can consider only local properties.

Solution. Let us consider a candidate Lyapunov function

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$

that is positive definite $V > 0$ for all $x_1, x_2 \in \mathbb{R}^2 - \{0\}$ and $V(0,0) = 0$. The rate of change in V yields

$$\begin{aligned} \dot{V} &= x_1\dot{x_1} + x_2\dot{x_2} \\ &= x_1(x_2 - x_1^3 + x_1^5) + x_2(-x_1) \\ &= x_1^2(x_1^4 - x_1^2). \end{aligned}$$

The x_1, x_2 have to be confined in $D = \{\mathbf{x} \in \mathbb{R}^2 : -1 < x_1 < 1\}$ that includes the origin, so the function is locally Lipschitz. Then, $x_1^4 - x_1^2 < 0$ for all $x_1, x_2 \in D - \{0\}$ and $\dot{V} < 0$ results a valid Lyapunov function. The origin is an asymptotically stable equilibrium, yet V is radially unbounded,

$$V(x_1, x_2) \to \infty \text{ as } \|\mathbf{x}\| \to \infty.$$

Thus the origin is a global asymptotically stable equilibrium. □

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Problem 4. For

$$\dot{x_1} = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x_2} = -x_1 + x_2 - x_2(x_1^2 + x_2^2),$$

where $\mathbf{x} = [x_1 \quad x_2]^\top \in \mathbb{R}^2$. Show that the equilibrium $x_1 = x_2 = 0$ is unstable using Chetaev's theorem.

Solution. For the given system we choose a Lyapunov function candidate

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$

which is positive definite $V > 0$ for all $x_1, x_2 \in \mathbb{R} - \{0\}$ and $V(0,0) = 0$. The rate of change in V yields

$$\begin{aligned}
\dot{V} &= x_1\dot{x_1} + x_2\dot{x_2} \\
&= x_1(x_1 + x_2 - x_1(x_1^2 + x_2^2)) + x_2(-x_1 + x_2 - x_2(x_1^2 + x_2^2)) \\
&= (x_1^2 + x_2^2)(1 - x_1^2 - x_2^2).
\end{aligned}$$

Since $x_1^2 + x_2^2 > 0$ for all $\mathbf{x} \in \mathbb{R}^2$, then in order to satisfy $\dot{V} > 0$ we need $1 - x_1^2 - x_2^2 > 0$ to hold. This means that $\|\mathbf{x}\| < 1$. Then, we define the set $B_r = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 0.9\} \subset U$, where $U = \{\mathbf{x} \in B_r : V(\mathbf{x}) > 0\}$. Therefore, by employing Chetaev's theroem we conclode that $\mathbf{x} = 0$ is an unstable equilibrium. $\square$