Homework 1

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ME6544 Linear Control Theory
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Problem 1.1.1. RLC state space

Solution. By using the first Kirchoff's law $\sum_n i_n = 0$ we obtain

$$\begin{aligned} i_0 &= i_1 + i_2 \\ &= i_C + i_{R_2} \\ &= C\frac{dV_C}{dt} + \frac{V_{R_2}}{R_2}. \end{aligned} \tag{1}$$

Next, for the first closed loop the second Kirchoff's law $\sum_n u_n = 0$ yields

$$\begin{aligned} V_0 &= V_{R_1} + V_L + V_C \\ &= R_1 i_0 + L\frac{di_0}{dt} + V_C, \end{aligned} \tag{2}$$

and for the second closed loop

$$V_C = V_{R_2}. \tag{3}$$

Considering that the state vector is $\mathbf{x} = [i_0 \;\; V_C]^T$, the input is $u = V_0$, and the output is $y = V_{R_2}$, and by employing Equations 1, 3 we get

$$\begin{aligned} x_1 &= C\dot{x_2} + \frac{x_2}{R_2} \\ \dot{x_2} &= \frac{1}{C}x_1 - \frac{1}{CR_2}x_2. \end{aligned} \tag{4}$$

Then, Equation 2 results

$$\begin{aligned} u &= R_1 x_1 + L\dot{x_1} + x_2 \\ \dot{x_1} &= -\frac{R_1}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}u. \end{aligned} \tag{5}$$

Next, Equation 3 reveals the output

$$y = x_2. \tag{6}$$

By employing Equations 4, 5, 6 we obtain the state space form

$$\begin{aligned}\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} &= \begin{bmatrix}-\frac{R_1}{L} & -\frac{1}{L}\\ \frac{1}{C} & -\frac{1}{CR_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix} u\\ y &= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned}$$

□

Problem 1.1.2. RLC

Solution. The selected resistance values are $R = \{.1, .5, 3, 20\}$, the inductance values are $L = \{1, .3, 6, 15\}$, and the capacitance values are $C = \{10, 5, 1, 2\}$. In Figures 1, 2 the state trajectories $x_1(t)$, $x_2(t)$ for various R, L, C values are presented respectively. In Figure 3 the output trajectory $y(t)$ for various R, L, C values are depicted. In Figure 4 the phase of the states x_1, x_2 for various R, L, C values are illustrated.

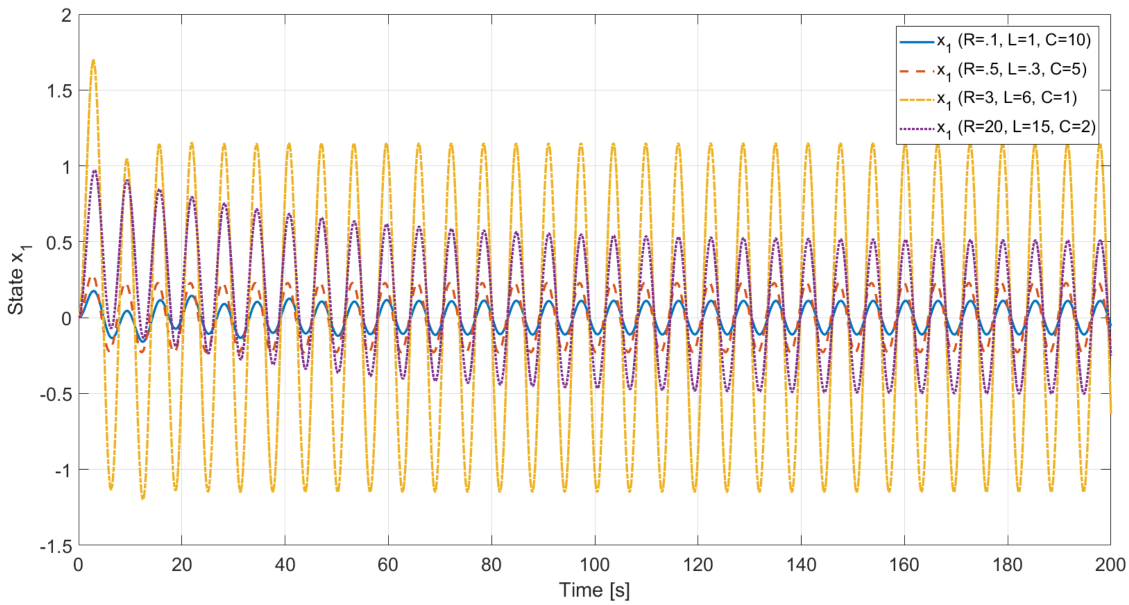

Figure 1: The state x_1 trajectory for various R, L, C values.

The state x_1 is only affected by the capacitance C as depicted in Figure 1. More specifically as C increases the magnitude of the state x_1 decreases. The periodic performance is imposed by the input. The state x_2 is affected by a combination of the capacitance C, the resistance R, and the inductance L as shown in Figure 2. More specifically, as the inductance L increases the magnitude of the state x_2 decreases. Next, as the fraction $\frac{R}{C}$ increases the magnitude of the state x_2 decreases. The periodic performance is imposed by the state x_1. The phase of the states x_1, x_2 performance is a combination of the aforementioned factors that affect the states x_1, x_2 as shown in Figure 4. Finally, the output y has similar performance with the state x_2, but it is also proportional with the resistance R as presented in Figure 3.

□

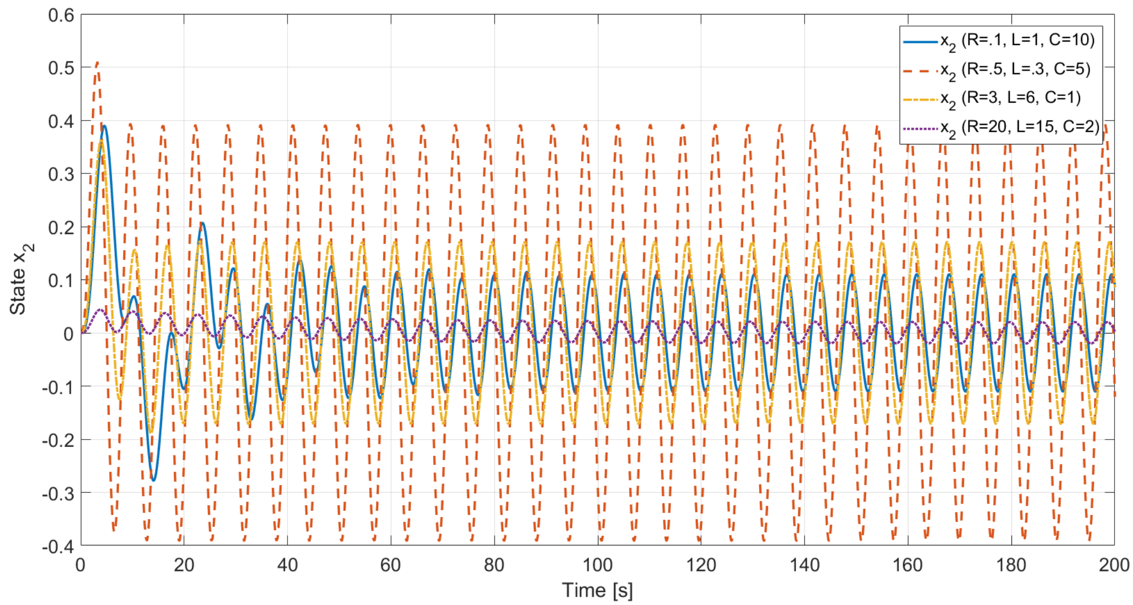

Figure 2: The state x_2 trajectory for various R, L, C values.

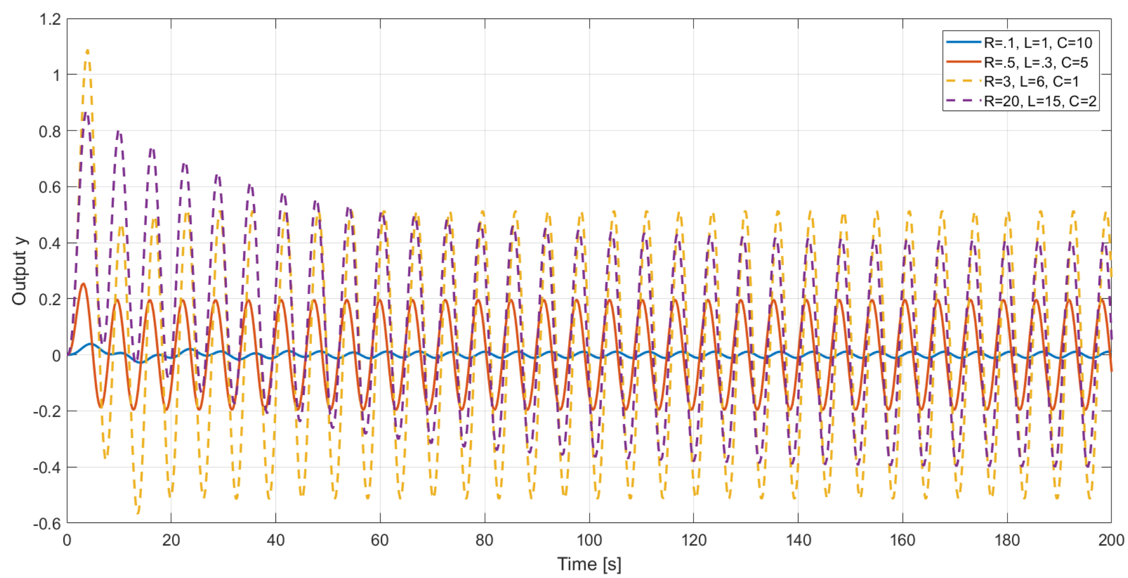

Figure 3: The output trajectories for various R, L, C values.

Problem 1.2. Van Der Pol Oscillator

Solution. The forced Van der Pol oscillator is described by

$$\ddot{y} - \mu(1 - y^2)\dot{y} + y - A\sin\omega t \quad = \quad 0. \tag{7}$$

It cannot be expressed in state space form because it includes non-linearities such as cross term multiplications and variables in raised powers. More specifically, it is a second order, autonomous, non-linear differential equation. Although, for the simulation we can derive the

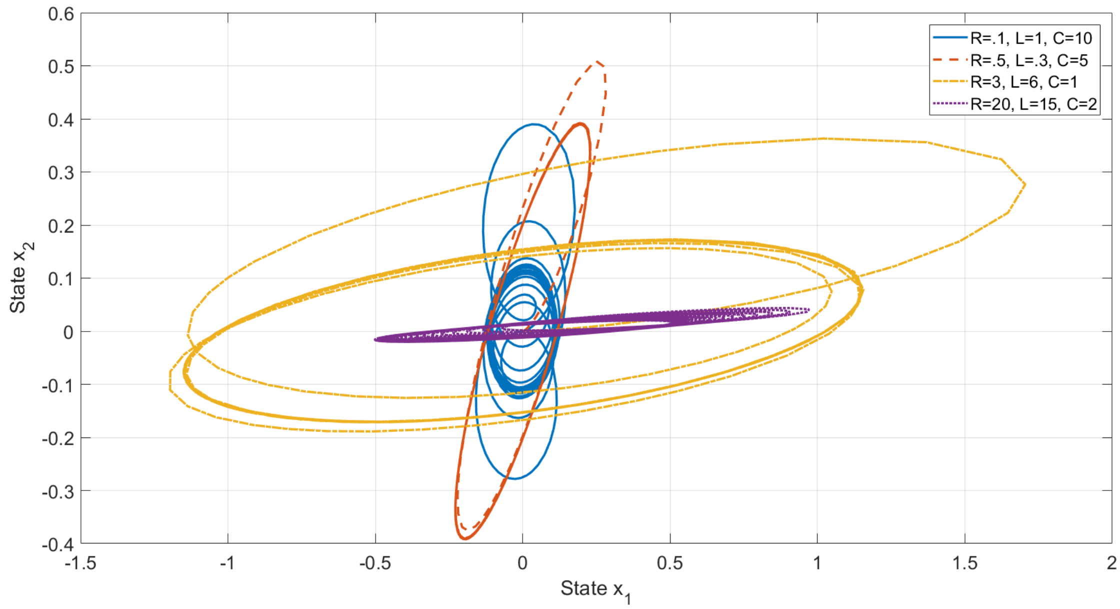

Figure 4: The phase of the states x_1, x_2 for various R, L, C values.

first order differential equations

$$\begin{aligned} \dot{y_1} &= y_2 \\ \dot{y_2} &= \mu(1 - y_1^2)y_2 - y_1 + u, \end{aligned} \tag{8}$$

where $u = A \sin \omega t$ is a periodic input. For the given initial conditions $y(0) = 0$, $\dot{y} = 0.5$ the trajectory $y(t)$ is shown in Figure 5, and the phase plane for $(\dot{y}, y)$ is presented in Figure 6.

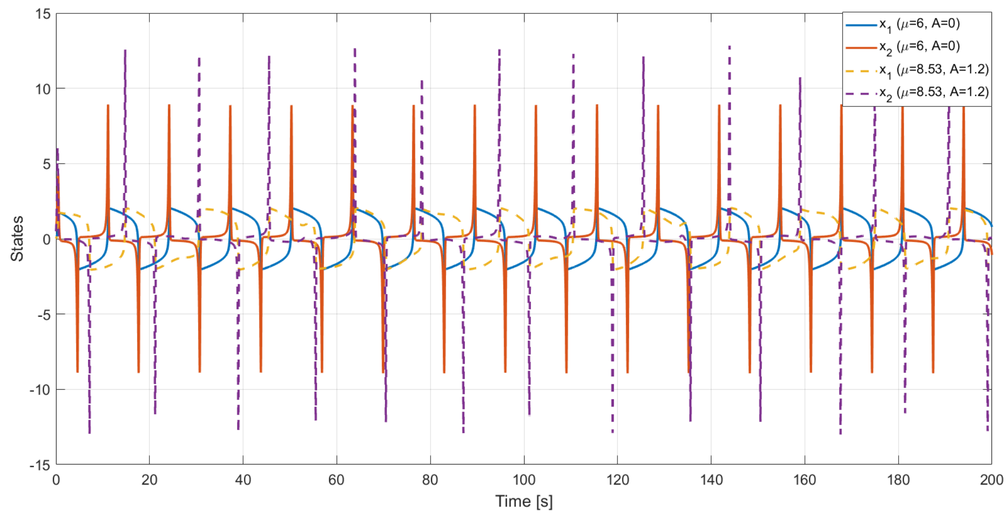

Figure 5: Van der Pol oscillator trajectories.

□

Problem 1.3. Block Diagram and State Evolution

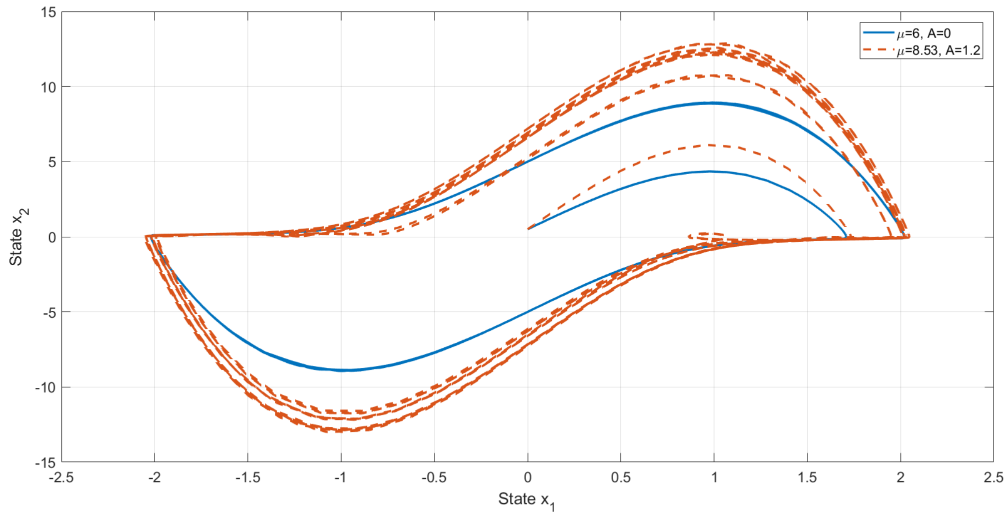

Figure 6: Van der Pol oscillator phase of states x_1, x_2.

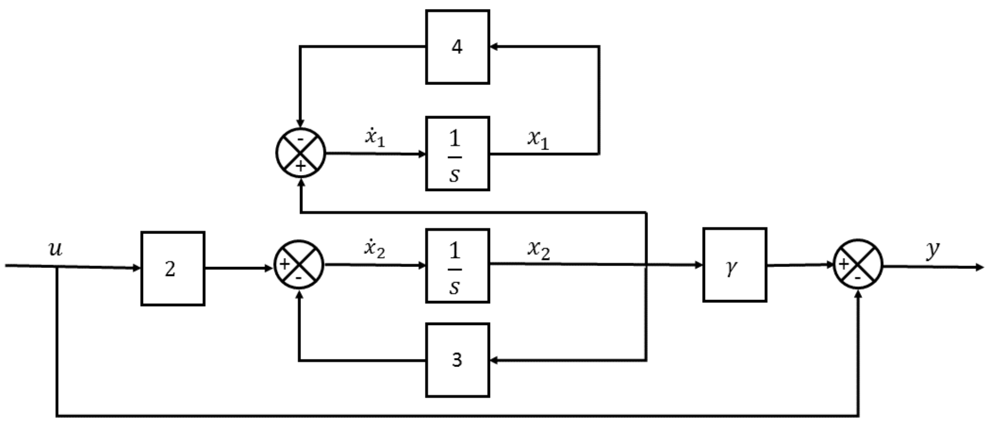

Figure 7: Block diagram of the given system.

Solution. The block diagram is presented in Figure 7. The given system relates the output with the input as

$$y = \gamma x_2 - u, \tag{9}$$

where u is the input, y is the output, and x_2 is a state. Ideally, we want to conceive a way to express the input and the output without any state correlation. For this reason, we compute the time derivative of Equation 9 which yields

$$\dot{y} = \gamma \dot{x_2} - \dot{u}. \tag{10}$$

Then, we substitute the second state from the state space description

$$\begin{aligned} \dot{y} &= \gamma(-3x_2 + 2u) - \dot{u} \\ &= -3\gamma x_2 + 2\gamma u - \dot{u}. \end{aligned} \tag{11}$$

Next, we employ Equation 9 to eliminate the state x_2, which results to a first order ordinary differential equation

$$\begin{aligned} \dot{y} = & \quad -3\gamma(\frac{y+u}{\gamma}) + 2\gamma u - \dot{u} \\ \dot{y} = & \quad -3y + -3u + 2\gamma u - \dot{u} \\ \dot{y} + 3y = & \quad u(2\gamma - 3) - \dot{u}. \end{aligned} \tag{12}$$

□

Problem 1.4. Transfer Function

Solution. For the $3D$ magnitude and phase plots we need to express the open loop transfer function with respect to the real and the imaginary parts. The given transfer function has one zero at $z = -6$ and 2 conjugate complex poles at $p_1 = -2.5 + j2.5981$, $p_2 = -2.5 - j2.5981$, which yields

$$\begin{aligned} H(s) &= \frac{s+6}{s^2 + 5s + 13} \\ H(s) &= \frac{s+6}{(s + 2.5 - 2.5981j)(s + 2.5 + 2.5981j)}. \end{aligned} \tag{13}$$

Then, we substitute to Equation 13 the complex number $s = a + bj$ to obtain the magnitude and the phase of the transfer function as depicted in Figures 8 and 9 respectively. In case we express the real and the imaginary axes in a logarithmic scale we obtain the $3D$ magnitude and $3D$ phase Bode diagrams as presented in Firgures 10 and 11 respectively. Finally, the classical Bode diagrams, which are an evaluation of the $3D$ magnitude and phase at $s = j\omega$ in a logarithmic scale, are shown in Figure 12.

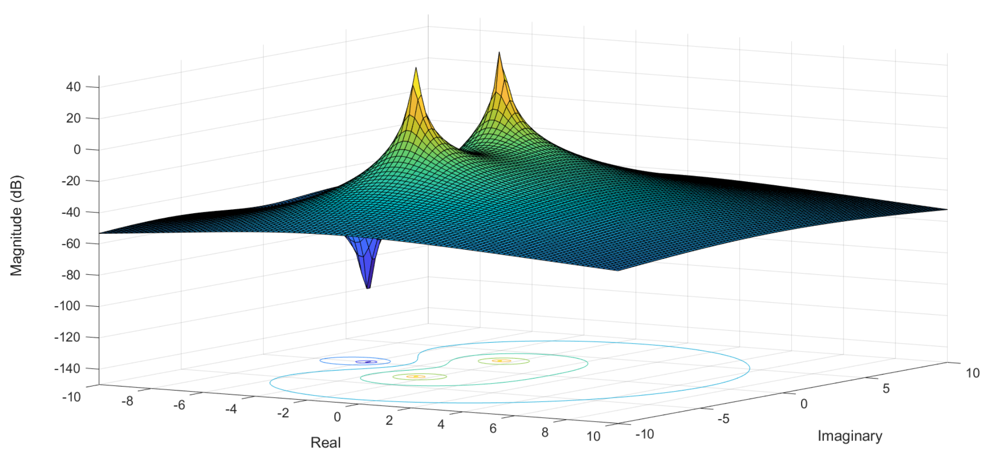

Figure 8: Magnitude of the open loop transfer function.

□

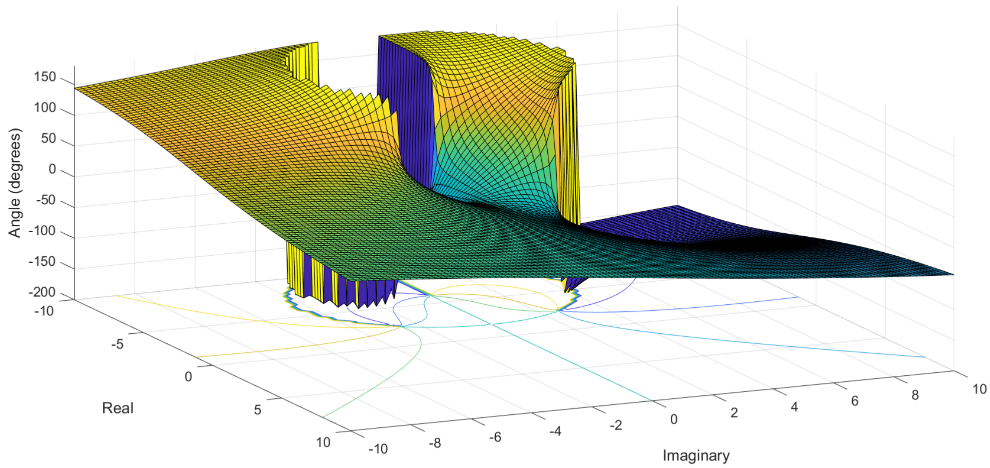

Figure 9: Phase of the open loop transfer function.

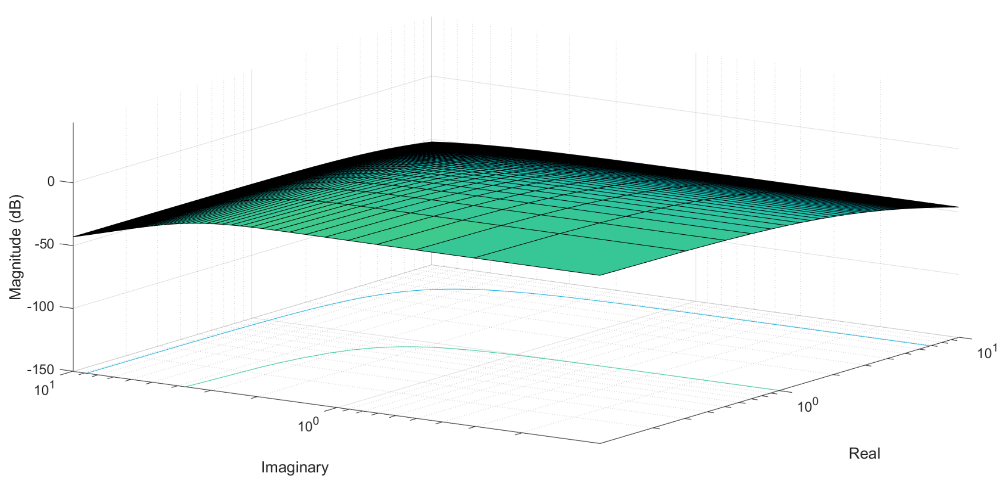

Figure 10: The $3D$ magnitude Bode diagram of the open loop transfer function.

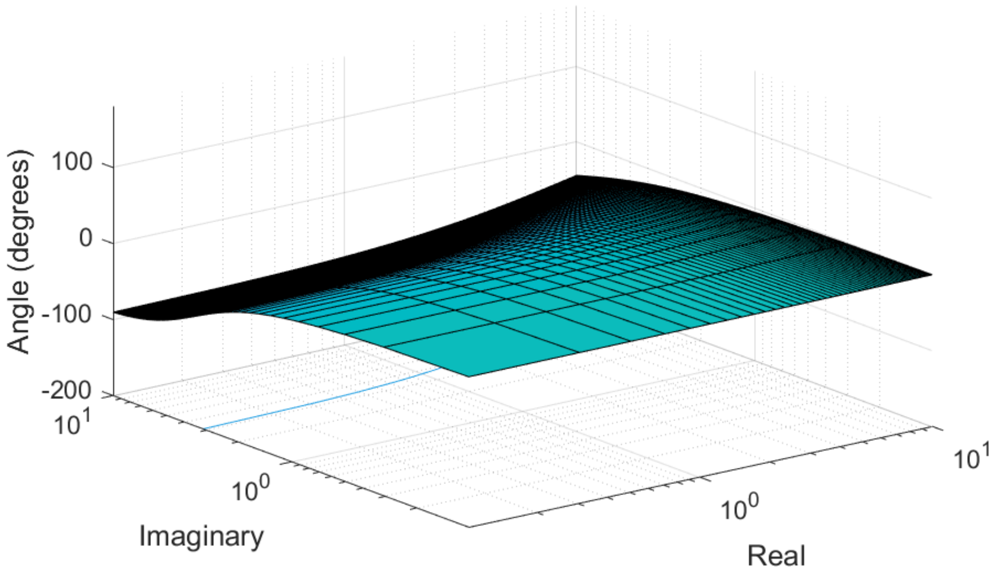

Figure 11: The $3D$ phase Bode diagram of the open loop transfer function.

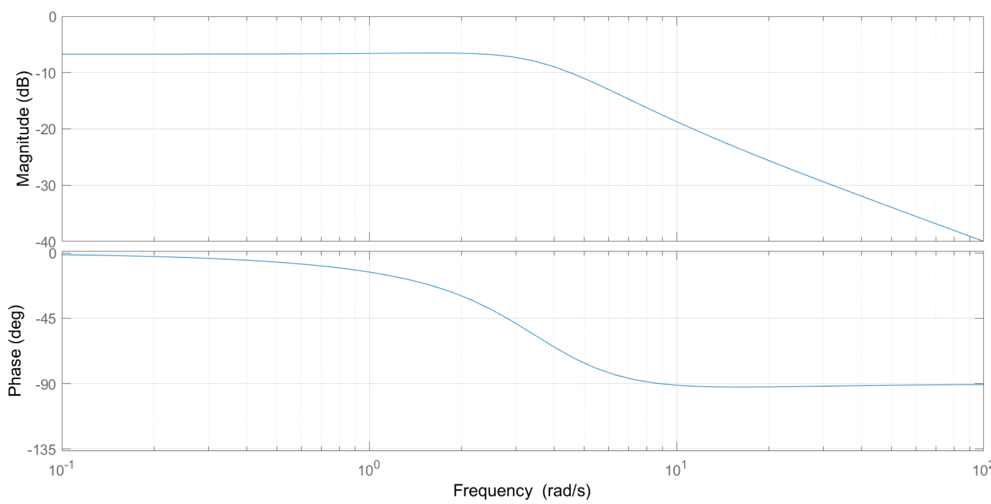

Figure 12: The Bode diagrams of the open loop transfer function.