Homework 1

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Problem 1. Show that the set of real matrices

$$SO(2) = \{R \in \mathbb{R}^{2\times 2} | R^{-1} = R^{\top}, det(R) = 1\},$$

is a group under matrix multiplication. Is this an Abelian group?

Solution. A group is called Abelian if it satisfies all the group properties and the commutative property.

First, we study the **closure** axiom for multiplication $R_1 \times R_2 \in \mathbb{R}^{2\times 2}$ for all $R_1, R_2 \in \mathbb{R}^{2\times 2}$. Indeed, the property holds as the multiplication of two matrices 2×2 results a 2×2 matrix. Note that this property is valid only for square matrices. More specifically let $R_1, R_2 \in SO(2)$ and the multiplication yields

$$\begin{aligned} R_1 R_2 &= \begin{bmatrix} cos\theta_1 & sin\theta_1 \\ -sin\theta_1 & cos\theta_1 \end{bmatrix} \begin{bmatrix} cos\theta_2 & sin\theta_2 \\ -sin\theta_2 & cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 & cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2 \\ -sin\theta_1 cos\theta_2 - cos\theta_1 sin\theta_2 & -sin\theta_1 sin\theta_2 + cos\theta_1 cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} cos(\theta_1 + \theta_2) & sin(\theta_1 + \theta_2) \\ -sin(\theta_1 + \theta_2) & cos(\theta_1 + \theta_2) \end{bmatrix} = R_{12}. \end{aligned}$$

Since in the set of real numbers $\mathbb{R}$ the closure property holds for addition, then $\theta_1 + \theta_2 \in \mathbb{R}$ for any $\theta_1, \theta_2 \in \mathbb{R}$. Therefore, $R_{12} \in SO(2)$ for any $R_1, R_2 \in SO(2)$.

Next, we study the **associativity** of multiplication in $SO(2)$. Let $R_f, R_g, R_h \in SO(2)$ where $R_f = f_{il}$, $R_g = g_{lk}$, and $R_h = h_{kj}$, then we get

$$\begin{aligned} ((R_f R_g) R_h)_{ij} &= \sum_{p=1}^{k} (R_f R_g)_{ip} h_{pj} \\ &= \sum_{p=1}^{k} (\sum_{q=1}^{l} f_{iq} g_{qp}) h_{pj} \\ &= \sum_{p=1}^{k} \sum_{q=1}^{l} f_{iq} g_{qp} h_{pj}. \end{aligned}$$

Similarly,

$$\begin{aligned}
(R_f(R_gR_h))_{ij} &= \sum_{q=1}^{l} f_{iq}(R_gR_h))_{qj} \\
&= \sum_{q=1}^{q} f_{iq}(\sum_{p=1}^{k} g_{lp}h_{pj}) \\
&= \sum_{q=1}^{q}\sum_{p=1}^{k} f_{iq}g_{lp}h_{pj} \\
&= \sum_{p=1}^{k}\sum_{q=1}^{l} f_{iq}g_{qp}h_{pj}.
\end{aligned}$$

This means that any matrix multiplication can be associative and thus holds in $SO(2)$.

Regarding the **identity** element, we need to show that there exists an element $I \in SO(2)$ such that $RI = IR = R$. Let $R = r_{ij}$, $I = i_{ij}$, and $b_{ij} = RI_{ij}$. We shall prove that $b_{ij} = r_{ij}$ and $r_{ij} = b_{ij}$. The definition of identity matrix determines $i_{ij} = 0$ for all $i \neq j$, $i_{ii} = 1$, and $det(I) = 1$. The matrix multiplication yields

$$b_{ij} = \sum_{n=1}^{m} r_{in}i_{nj} = r_{ij}.$$

Similarly,

$$r_{ij} = \sum_{n=1}^{m} b_{in}i_{nj} = b_{ij}.$$

Thus, the identity element property holds in $SO(2)$.

Since the $det(R) \neq 0$, $det(R) = 1$ and $R \in \mathbb{R}^{2\times 2}$ then $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is **invertible**

$$R^{-1} = \frac{1}{det(R)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We want to prove that $R^{-1}R = RR^{-1} = I$, so

$$RR^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix},$$

since $det(R) = 1 \Rightarrow ad - bc = 1$, therefore

$$RR^{-1} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

Similarly,

$$R^{-1}R = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

But the latter does not corresponds to $SO(2)$ as $R^{-1} \neq R^T$. Let $R \in SO(2)$ then $R^{-1} = R^T$. We shall prove that $RR^T = R^T R = I$

$$
\begin{aligned}
RR^\top &= \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \\
&= \begin{bmatrix} cos^2\theta + sin^2\theta & -sin\theta cos\theta + cos\theta sin\theta \\ -sin\theta cos\theta + cos\theta sin\theta & sin^2\theta + cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
R^\top R &= \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \\
&= \begin{bmatrix} cos^2\theta + sin^2\theta & cos\theta sin\theta - sin\theta cos\theta \\ cos\theta sin\theta - sin\theta cos\theta & sin^2\theta + cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.
\end{aligned}
$$

Thus, the inverse element property holds for all $R \in SO(2)$. Since, the CAII properties hold the $SO(2)$ is a group multiplication.

To prove that $SO(2)$ is also **commutative** let $R_1, R_2 \in SO(2)$ and from the closure property we know that

$$
R_1 R_2 = \begin{bmatrix} cos(\theta_1 + \theta_2) & sin(\theta_1 + \theta_2) \\ -sin(\theta_1 + \theta_2) & cos(\theta_1 + \theta_2) \end{bmatrix} = R_{12}.
$$

Similarly,

$$
\begin{aligned}
R_2 R_1 &= \begin{bmatrix} cos\theta_2 & sin\theta_2 \\ -sin\theta_2 & cos\theta_2 \end{bmatrix} \begin{bmatrix} cos\theta_1 & sin\theta_1 \\ -sin\theta_1 & cos\theta_1 \end{bmatrix} \\
&= \begin{bmatrix} cos(\theta_1 + \theta_2) & sin(\theta_1 + \theta_2) \\ -sin(\theta_1 + \theta_2) & cos(\theta_1 + \theta_2) \end{bmatrix} = R_{12}.
\end{aligned}
$$

Thus, $SO(2)$ is commutative as $R_1 R_2 = R_2 R_1$ and an abelian group. □

Problem 2. Jia Guo

Problem 3. Jia Guo

Problem 4. Use Lyapunov's direct method to show that the origin is an assymptotically stable equilibrium for

$$
\begin{aligned}
\dot{x_1} &= -tan(x_1) + x_2 \\
\dot{x_2} &= -tan(x_2) - x_1.
\end{aligned}
$$

Solution. For the given system we choose a Lyapunov function candidate

$$
V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0,
$$

which is positive definite $V > 0$ for all $x_1, x_2 \in \mathbb{R} - \{0\}$ and $V(0,0) = 0$. The rate of change in V yields

$$\begin{aligned}\dot{V} &= x_1\dot{x_1} + x_2\dot{x_2} \\ &= x_1(-tan(x_1) + x_2) + x_2(-tan(x_2) - x_1) \\ &= -(tan(x_1)x_1 + tan(x_2)x_2).\end{aligned}$$

The x_1, x_2 have to be confined in the interval of one period, that includes the origin, so that the function is locally Lipschitz. Then, $tan(x)x > 0$ for all $x_1, x_2 \in D - \{0\}$ and $\dot{V} < 0$ results a valid Lyapunov function. Thus, the origin is an asymptotically stable equilibrium.

□