Homework 1

Problem 1: Show that the set of real matrices

$$SO(2) = \{ R \in \mathbb{R}^{2\times 2} \mid R^{-1} = R^T, \det(R) = +1 \}$$

is a group under matrix multiplication.
is this group Abelian?
Note: Any element $\boldsymbol{R} \in SO(2)$ can be parameterized by an angle θ: $\boldsymbol{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Problem 2: Show that the vector space of real valued n-dimensional functions defined on an interval $[a, b]$ which are piecewise continuous on this interval is an inner product space with respect to the following operation:

$$\langle f, g \rangle = \int_a^b f^T(t)\, g(t)\, dt$$

Here $f : [a, b] \to \mathbb{R}^n$ and $g : [a, b] \to \mathbb{R}^n$ are element (i.e. vectors) in the aforementioned vector space.
Note: you do not need to prove that the space is indeed vectorial.

Problem 3: Use the comparison lemma to find an upper bound on $v(t)$ where

$$\dot{v} \leq -v^3, \quad v(t_0) \leq v_0$$

Problem 4: Use Lyapunov's direct method to show that the origin is an asymptotically stable equilibrium for:

$$\dot{x}_1 = -\tan(x_1) + x_2$$

$$\dot{x}_2 = -\tan(x_2) - x_1$$

Note: "tan" is the tangent function.