LQR-Trees [Tedrake, 2010]

George Kontoudis, Shriya Shah

ME5984 Motion Planning Analysis
Spring 2017

Mechanical Engineering Department, Virginia Tech

October 2, 2018

Outline

Motivation

- Design robust algorithms for non-linear feedback motion planning
- Non-linear underactuated systems such as robot manipulator or bipedal walking
- Computation of planning regions of attraction (funnels) for non-linear underactuated dynamical systems
- Applicable to real robots

Definition of Lyapunov Functions

For a given dynamical system

$$\dot{x} = f(x), f(0) = 0$$

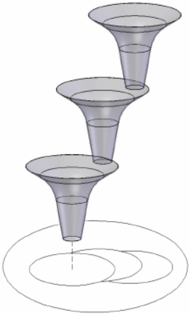
a Lyapunov function is $V(x)$, $V \in C$ where

- $V(x) > 0$, positive definite
- $\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} < 0$, negative definite

If conditions met in some state space ball B_r, then origin is a.s.

Sequential Composition of Lyapunov Functions

- Each funnel acts like a valid Lyapunov function
- A.s. of each Lyapunov falls in the region of attraction of the next lower level
- The lowest function stabilizes in the goal point



Sequential composition of funnels [Burridge, 1999]

Sums of Squares

We want to check inequalities and validate Lyapunov functions using sums-of-squares (SoS) method [Parrilo, 2000]

- $x^4 + 2x^3 + 3x^2 - 2x + 2 \geq 0, \;\; \forall x \in \mathbb{R}$, by employing SoS

$$x^4+2x^3+3x^2-2x+2 = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^\intercal \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = X^\intercal A X$$

- Eigenvalues of A are $\lambda_1 = 3.88$, $\lambda_2 = 1.65$, $\lambda_1 = 0.47$, so the inequality stands $\forall x \in \mathbb{R}$

Sums of Squares Properties

General structure of (SoS) for a 4-*th* order polynomial is

$$fx^4 + 2ex^3 + (d+2c)x^2 + 2bx + a = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^\intercal \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

- Extend to multivariable polynomials
- Check non-negativity by searching positive semidefinite matrix

Feedback Synthesis by SoS Optimization

Given a system $\dot{x} = f(x) + g(x)u$ we want to generate

- Feedback control law $u = \pi(x)$
- Lyapunov fcn $V(x)$, s.t. $V(x) > 0$, $\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}\dot{x} < 0$

BUT, this is a difficult problem as the set of $V(x)$, $\pi(x)$ may not be convex sets

- Rely on LQR synthesis
- Design a series of locally-valid controllers
- Compose these controllers utilizing feedback motion planning

The Minimum Principle

The first order or necessary condition for optimality is called *Maximum (Minimum) Principle*

- Given a function we want to minimize $f(x, y, z)$ on a level surface (constraint) $g(x, y, z)$ we get

$$\nabla f = \lambda \nabla g$$

- To convert a constrained problem to an unconstrained we construct the Hamiltonian function

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^{\mathsf{T}} f(x, u, t)$$

Goal Stabilization

- For a given non-linear dynamical system

$$\dot{x} = f(x, u, t), \ \ x \in \mathbb{R}^n, \ \ u \in \mathbb{R}^m$$

- Set goal state x_G, where $f(x_G, u_G) = 0$, and $\bar{x} = x - x_G$, $\bar{u} = u - u_G$
- Linearize around (x_G, u_G), $\dot{\bar{x}} \approx A\bar{x}(t) + B\bar{u}(t)$
- Infinite horizon LQR minimum energy cost-to-go fcn (performance index)

$$J_\infty = \int_0^\infty [\bar{x}^\intercal(t) Q \bar{x}(t) + \bar{u}^\intercal(t) R \bar{u}(t)] dt,$$

$$Q = Q^\intercal \geq 0, R = R^\intercal > 0$$

Hamiltonian System

Set the Hamiltonian

$$H(x,u,t) = L(x,u,t) + \lambda^\intercal f(x,u,t)$$

- State equation
$$\dot{x} = \frac{\partial H}{\partial \lambda}$$
- Costate equation
$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f^\intercal}{\partial x}\lambda + \frac{\partial L}{\partial x}$$
- Stationarity condition
$$0 = \frac{\partial H}{\partial u} = \frac{\partial f^\intercal}{\partial u}\lambda + \frac{\partial L}{\partial u}$$

Riccati equation and Optimal Control Law

- Infinite horizon LQR problem results the optimal cost

$$J^*(\bar{x}) = \bar{x}^\intercal S \bar{x}$$

- $S > 0$, w/ the solution given from ARE

$$0 = Q - SBR^{-1}B^\intercal S + SA + A^\intercal S$$

- Optimal feedback closed loop control policy

$$\bar{u}^* = -R^{-1}B^\intercal S\bar{x} = -K\bar{x}$$

Goal State Convergence

The domain of attraction of the LQR over some sub-level set

$$B_G(\rho) = \{x | 0 \leq V(x) \leq \rho\}$$

To guarantee a.s. we require $V(x)$ to be a valid Lyapunov function

- $V(x) > 0, \;\; x \in B_G(\rho)$
- $\dot{V}(x) < 0, \;\; x \in B_G(\rho)$

Assign $V(x) = J^*(\bar{x}) = \frac{1}{2}\bar{x}^\intercal S \bar{x}$

- By definition positive definite
- $\dot{V}(x) = \dot{J}(\bar{x}) = \frac{dV}{dx}\frac{dx}{dt} = 2\bar{x}^\intercal S \dot{x} = 2\bar{x}^\intercal S f(x_G + \bar{x}, u_G - K\bar{x})$

Lyapunov Verification Using SoS

We require

$$\dot{J}^*(\bar{x}) < 0, \quad \forall \bar{x} \neq 0 \in B_G(\rho), \quad \dot{J}^*(0) = 0$$

- First, modify the inequality from negative to non-positive

$$\dot{J}^*(\bar{x}) \leq -\epsilon ||\bar{x}||_2^2, \quad \forall \bar{x} \in B_G(\rho), \quad \epsilon \in \mathbb{R}^+$$

- Second, include the constraint with Lagrange multiplier $h(\cdot)$

$$\dot{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2$$

Lagrange Multiplier Searching

- If $f^{(cl)}(x,u) = f(x, u_G - K(x - x_g))$, search for $h(\cdot)$ polynomial with sufficient order for $\dot{J}^*(\bar{x})$, using SoS

$$\begin{aligned} \text{find} \quad & h(\bar{x}) \\ \text{subject to} \quad & \dot{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \end{aligned}$$

- If $f^{(cl)}(x) \approx \hat{f}^{(cl)}(\bar{x})$, where $\hat{f}^{(cl)}(\bar{x})$ is the Taylor expansion (algebraic approximation) and $\dot{\hat{J}}(\bar{x}) = 2\bar{x}^\intercal S \hat{f}^{(cl)}(\bar{x})$

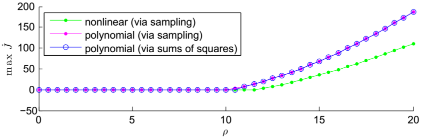
$$\begin{aligned} \text{find} \quad & h(\bar{x}) \\ \text{subject to} \quad & \dot{\hat{J}}(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \end{aligned}$$

Optimization for ρ

Set a convex optimization problem for the region of attraction

$$\begin{aligned} \max \quad & \rho \\ \text{subject to} \quad & \dot{\hat{J}}^*(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \\ & \rho > 0 \end{aligned}$$

- At each step the Lagrange multiplier searching is performed
- If the program is feasible ρ increased



Polynomial verification of damped single pendulum [Tedrake, 2010]

Trajectory Optimization

- Trajectory design w/ RRT or other motion planning technique
  1. Don't guarantee stability w/ any initial condition
  2. Need to design a new trajectory every time
  3. Can deal problems w/ up to 5 states
- Increase set of states out of ρ to reach goal
- Stabilize the trajectory using LQR optimal controller
- Initialize out of the domain of attraction and optimize the cost function

$$J = \int_{t_0}^{t_f} [1 + u_0^T R u_0] dt$$

Discarded Sampling Points

- Trajectory opitmization algorithms are local
- Employed to test validity of sampling points
- If trajectory optimization fails the point is discarded
- Algorithms have specific time restriction to improve faster sampling
- Final tree constraint specifies whether the computed trajectory will connect w/ the tree

$$\min_{x' \in \mathfrak{T}_{k-1}} ||x_0(t_f) - x'||_2 = 0,$$

$\mathfrak{T}_k$: Set of all points in the tree after k-iterations

Trajectory Stabilization

- For a given non-linear dynamical system

$$\dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \in [t_0, t_f]$$

- A nominal trajectory $x_0(t)$, $u_0(t)$ yields $\bar{x}(t) = x(t) - x_0(t)$, $\bar{u}(t) = u(t) - u_0(t)$
- Linearize around the trajectory $(x_0(t), u_0(t))$, $\dot{\bar{x}} \approx A(t)\bar{x}(t) + B(t)\bar{u}(t)$
- Finite time horizon LQR minimum energy cost-to-go fcn (performance index)

$$J(\bar{x}', t') = \bar{x}^\mathsf{T}(t_f)S(t_f)\bar{x}(t_f) + \int_{t'}^{t_f} [\bar{x}^\mathsf{T}(t)Q\bar{x}(t) + \bar{u}^\mathsf{T}(t)R\bar{u}(t)]dt,$$

$$Q_f = Q_f^\mathsf{T} > 0, \quad Q = Q^\mathsf{T} \geq 0, \quad R = R^\mathsf{T} > 0, \quad \bar{x}(t)' = \bar{x}'$$

Riccati equation and Optimal Control Law

- Infinite horizon LQR problem results the optimal cost

$$J^*(\bar{x}, t) = \bar{x}^\intercal S(t)\bar{x}$$

- $S(t) = S(t)^\intercal > 0$, w/ the solution given from RE

$$\dot{S} = Q - SBR^{-1}B^\intercal S + SA + A^\intercal S$$

- Optimal feedback closed loop control policy

$$\bar{u}^*(t) = -R^{-1}B^\intercal(t)S(t)\bar{x}(t) = -K(t)\bar{x}(t)$$

TV-LQR Verification

- For trajectory stabilization a bounded goal domain is defined (not a.s.)

$$B_f = \{x | 0 \leq V(x, t_f) \leq \rho_f\}$$

- Search for time-varying domains

$$B(\rho(\cdot), t) = \{x | 0 \leq V(x, t) \leq \rho(t)\}$$

- This sublevel set should guarantee for the closed-loop system

$$x(t) \in B(\rho(\cdot), t) \Rightarrow x(t_f) \in B_G \quad \forall t \in [t_0, t_f]$$

Function of Region of Attraction $\rho(t)$

Time-Invariant case

$$J^*(\bar{x}) \leq \rho, \quad \rho \in \mathbb{R}^+$$

$$\dot{J}(\bar{x}) \leq 0, \quad \dot{J}(0) = 0$$

- $J^*(\bar{x}) = V(x)$: Lyapunov fcn
- ρ: domain of attraction (T-I)

Time-Variant case

$$J^*(\bar{x}, t) \leq \rho(t)$$

$$\dot{J}(\bar{x}, t) \leq \dot{\rho}(t), \quad \dot{J}(x_0, t) = 0$$

- $J^*(\bar{x}, t) = V(x, t)$: Lyapunov fcn
- At every time instant we assign a Lyapunov fcn
- $\rho(t)$: domain of attraction (T-V)
- Conditions assure that $V(x, t)$ decreases faster than $\rho(t)$ along the trajectory

Time-Varying Lyapunov Function

- We assign the positive definite j^* as our Lyapunov fcn

$$V(x,t) = J^*(\bar{x},t) = \bar{x}^\intercal S(t)\bar{x}$$

- We get the bounded goal domain

$$B_f = \{x | 0 \leq \bar{x}^\intercal S(t)\bar{x} \leq \rho_f\}$$

- The time derivative of the assigned Lyapunov fcn yields

$$\dot{J}^*(\bar{x},t) = \bar{x}^\intercal \dot{S}(t)\bar{x} + 2\bar{x}^\intercal S(t) f(\hat{x}_0(t) + \bar{x}, \hat{u}_0(t) - K(t)\bar{x})$$

Selection of $\rho(t)$

- We desire the largest domain of attraction $\rho(t)$
- Initially we approximate $\rho(t)$ w/ a linear polynomial

$$\rho_k(t) = \beta_{1k} t + \beta_{0k}$$

$$\rho(t) = \begin{cases} \rho_k(t), & \forall t \in [t_k, t_{k+1}) \\ \rho_f, & t = t_f \end{cases}$$

- We require for the approximation of domain of attraction

$$\rho_k(t_{k+1}) = \beta_{1k} t_{k+1} + \beta_{0k} \leq \rho(t_{k+1})$$

$$J^*(\bar{x}, t) = \rho_k(t) \Rightarrow \dot{\hat{J}}^*(\bar{x}, t) \leq \dot{\rho}_k(t) = \beta_{1k} \quad \forall t \in [t_k, t_{k+1}),$$

where $\dot{\hat{J}}^*$ is the Taylor expansion of the dynamics

Lagrange Multiplier Searching

Approximately verify the second condition of Lyapunov fcns w/ SoS

$$
\begin{aligned}
\text{find} \quad & h_1(\bar{x}, t), h_2(\bar{x}, t), h_3(\bar{x}, t), \\
\text{subject to} \quad & \dot{\hat{J}}^*(\bar{x}, t) - \dot{\rho}_k(t) + h_1(\bar{x}, t)(\rho_k(t) - J^*(\bar{x}, t)) + \\
& \qquad + h_2(\bar{x}, t)(t - t_k) + h_3(\bar{x}, t)(t_{k+1} - t) \leq 0, \\
& h_2(\bar{x}, t) \geq 0, \\
& h_3(\bar{x}, t) \geq 0
\end{aligned}
$$

- $h_1(\bar{x}, t)$ should be eliminated if the equality constraint holds
- The Lagrange multipliers should be polynomials of sufficient order to counteract $\dot{\hat{J}}^*(\bar{x}, t)$

Optimization for $\rho(t)$

Set a convex optimization problem for the region of attraction

$$\begin{aligned}
\max_{\beta_{\cdot k}} \quad & \rho_k(t_k) = \beta_{1k} t + \beta_{0k}, \qquad k = N-1, \dots, 1 \\
\text{subject to} \quad & \rho_k(t_{k+1}) \le \rho(t_{k+1}) \\
& \dot{\hat{J}}^*(\bar{x}, t) - \dot{\rho}_k(t) + h_1(\bar{x}, t)(\rho_k(t) - J^*(\bar{x}, t)) + \\
& \qquad + h_2(\bar{x}, t)(t - t_k) + h_3(\bar{x}, t)(t_{k+1} - t) \le 0, \\
& h_2(\bar{x}, t) \ge 0, \\
& h_3(\bar{x}, t) \ge 0
\end{aligned}$$

LQR-Tree Algorithm Characteristics

- Randomized backward growing tree technique in the same fashion as RRT
- Large web of stabilized controllers for the motion planning, given various initial conditions
- Information for the funnels allows for immediately discard sample points which are inside the verified region

Objective: *Find the set of stabilizable controllers for all possible initial conditions from which the specific final goal state can be reached.*

LQR-Tree Algorithm

Goal: Stabilize (x_G, u_G)

1. Stabilize x_G, u_G locally w/ TI-LQR
   + Compute region of attraction ρ_c
2. Pick a random point x_s
   If already in the funnel discard
3. Perform trajectory optimization w/ constraints $x(0) = x_s$, $\text{dist}(x(t_f), (\text{tree})) = 0$
   + Compute feedback cost-to-go TV-LQR
   + Compute largest funnel to land in the goal region
4. Repeat

LQR-Tree Pseudo-code

Algorithm 1 LQR-Tree (f,x_G,u_G,Q,R) [Tedrake, 2010]

[A,B] $\Leftarrow$ linearization of f(x,u) around x_G, u_G
[K,S] $\Leftarrow$ TI-LQR(A,B,Q,R)
$\rho_c \Leftarrow$ domain of attraction computation
T.init$\{x_G, u_G,$ S,K, ρ_c, NULL$\}$
for $k = 1$ to K **do**
    $x_{rand} \Leftarrow$ random sample
    if $x_{rand} \in C_k$ **then**
        continue
    end if
    $[t, x_0(t), u_0(t)]$ from trajectory optimization
    if $x_0(t_f) \notin \mathcal{T}_k$ **then**
        continue
    end if
    [K(t), S(t)] $\Leftarrow$ TV-LQR(A(t),B(t),Q,R)
    $\rho_c(t) \Leftarrow$ polynomial domain of attraction computation
    $i \Leftarrow$ pointer to branch in tree
    T.add-branch $(x_0(t), u_0(t), S(t), K(t), \rho_c, i)$
end for

Overview

- Lyapunov functions and SOS
- LQR, its design and verification
- Overview of Motion planning
- RRT algorithm
- Trajectory Optimization and Time varying LQR
- LQR trees algorithm
- LQR trees in action (Simulation)

Conclusions

- Computation of Lyapunov fcns w/ SoS let us build feedback motion planning algorithms
- Deal with non-linear systems w/ up to 20 states
- Offline computation of the problem
- Need to know the dynamics of the system

References

R. Tedrake, I. Manchester, M. Tobenkin, and J. Roberts
LQR-trees: Feedback motion planning via sums-of-squares verification
International Journal of Robotics Researchs, 1038–1052, SAGE Publications, 2010.

R. Burridge, A. Rizzi, and D. Koditschek
Sequential composition of dynamically dexterous robot behaviors
International Journal of Robotics Researchs, 534–555, SAGE Publications, 1999.

P.Parrilo
Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization
Ph.D. Theis, MIT, 2000.

Thank You!