LQR-Trees [Tedrake, 2010]

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Outline

Motivation

- Design robust algorithms for non-linear feedback motion planning
- Non-linear underactuted systems such as robot manipulator or bipedal walking
- Computation of planning regions of attraction (funnels) for non-linear underactuated dynamical systems
- Applicable to real robots

Definition of Lyapunov Functions

For a given dynamical system

$$\dot{x} = f(x), f(0) = 0$$

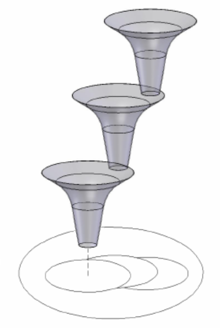
a Lyapunov function is $V(x)$, $V \in C$ where

- $V(x) > 0$, positive definite
- $\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} < 0$, negative definite

If conditions met in some state space ball B_r, then origin is a.s.

Sequential Composition of Lyapunov Functions

- Each funnel acts like a valid Lyapunov function
- A.s. of each Lyapunov falls in the region of attraction of the next lower level
- The lowest function stabilizes in the goal point



Sequential composition of funnels [Burridge, 1999]

Sums of Squares

We want to check inequalities and validate Lyapunov functions using sums-of-squares (SoS) method [Parrilo, 2000]

- $x^4 + 2x^3 + 3x^2 - 2x + 2 \geq 0, \;\; \forall x \in \mathbb{R}$, by employing SoS

$$x^4+2x^3+3x^2-2x+2 = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^{\intercal} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = X^{\intercal} A X$$

- Eigenvalues of A are $\lambda_1 = 3.88$, $\lambda_2 = 1.65$, $\lambda_1 = 0.47$, so the inequality stands $\forall x \in \mathbb{R}$

Sums of Squares Properties

General structure of (SoS) for a 4-*th* order polynomial is

$$fx^4 + 2ex^3 + (d + 2c)x^2 + 2bx + a = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^{\intercal} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

- Extend to multivariable polynomials
- Check non-negativity by searching positive semidefinite matrix

Feedback Synthesis by SoS Optimization

Given a system $\dot{x} = f(x) + g(x)u$ we want to generate

- Feedback control law $u = \pi(x)$
- Lyapunov fcn $V(x)$, s.t. $V(x) > 0$, $\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}\dot{x} < 0$

BUT, this is a difficult problem as the set of $V(x)$, $\pi(x)$ may not be convex sets

- Rely on LQR synthesis
- Design a series of locally-valid controllers
- Compose these controllers utilizing feedback motion planning

The Minimum Principle

The first order or necessary condition for optimality is called *Maximum (Minimum) Principle*

- Given a function we want to minimize $f(x, y, z)$ on a level surface (constraint) $g(x, y, z)$ we get

$$\nabla f = \lambda \nabla g$$

- To convert a constrained problem to an unconstrained we construct the Hamiltonian function

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^{\mathsf{T}} f(x, u, t)$$

Goal Stabilization

- For a given non-linear dynamical system

$$\dot{x} = f(x, u, t), \;\; x \in \mathbb{R}^n, \;\; u \in \mathbb{R}^m$$

- Set goal state x_G, where $f(x_G, u_G) = 0$, and $\bar{x} = x - x_G$, $\bar{u} = u - u_G$
- Linearize around (x_G, u_G), $\dot{\bar{x}} \approx A\bar{x}(t) + B\bar{u}(t)$
- Infinite horizon LQR minimum energy cost-to-go fcn (performance index)

$$J_\infty = \frac{1}{2}\int_0^\infty [\bar{x}^\intercal(t) Q \bar{x}(t) + \bar{u}^\intercal(t) R \bar{u}(t)] dt,$$

$$Q = Q^\intercal \geq 0, R = R^\intercal > 0$$

Hamiltonian System

Set the Hamiltonian

$$H(x,u,t) = L(x,u,t) + \lambda^\intercal f(x,u,t)$$

- State equation

$$\dot{x} = \frac{\partial H}{\partial \lambda}$$

- Costate equation

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial f^\intercal}{\partial x}\lambda + \frac{\partial L}{\partial x}$$

- Stationarity condition

$$0 = \frac{\partial H}{\partial u} = \frac{\partial f^\intercal}{\partial u}\lambda + \frac{\partial L}{\partial u}$$

Riccati equation and Optimal Control Law

- Infinite horizon LQR problem results the optimal cost

$$J^*(\bar{x}) = \frac{1}{2}\bar{x}^\intercal S\bar{x}$$

- $S > 0$, w/ the solution given from ARE

$$0 = Q - SBR^{-1}B^\intercal S + SA + A^\intercal S$$

- Optimal feedback closed loop control policy

$$\bar{u}^* = -R^{-1}B^\intercal S\bar{x} = -Kx$$

Goal State Convergence

The domain of attraction of the LQR over some sub-level set

$$B_G(\rho) = \{x | 0 \leq V(x) \leq \rho\}$$

To guarantee a.s. we require $V(x)$ to be a valid Lyapunov function

- $V(x) > 0, \;\; x \in B_G(\rho)$
- $\dot{V}(x) < 0, \;\; x \in B_G(\rho)$

Assign $V(x) = J^*(\bar{x}) = \frac{1}{2}\bar{x}^\intercal S \bar{x}$

- By definition positive definite
- $\dot{V}(x) = \dot{J}(\bar{x}) = \frac{dV}{dx}\frac{dx}{dt} = 2\bar{x}^\intercal S \dot{x} = 2\bar{x}^\intercal S f(x_G + \bar{x}, u_G - K\bar{x})$

Lyapunov Verification Using SoS

We require

$$\dot{J}^*(\bar{x}) < 0, \quad \forall \bar{x} \neq 0 \in B_G(\rho), \quad \dot{J}^*(0) = 0$$

- First, modify the inequality from negative to non-positive

$$\dot{J}^*(\bar{x}) \leq -\epsilon ||\bar{x}||_2^2, \quad \forall \bar{x} \in B_G(\rho), \quad \epsilon \in \mathbb{R}^+$$

- Second, include the constraint with Lagrange multiplier $h(\cdot)$

$$\dot{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2$$

Lagrange Mulitplier Searching

- If $f^{(cl)}(x,u) = f(x, u_G - K(x - x_g))$, search for $h(\cdot)$ polynomial with sufficient order for $\dot{J}^*(\bar{x})$, using SoS

$$\begin{aligned} \text{find} \quad & h(\bar{x}) \\ \text{subject to} \quad & \dot{J}^*(\bar{x}) + h(\bar{x})(\rho - J^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \end{aligned}$$

- If $f^{(cl)}(x) \approx \hat{f}^{(cl)}(\bar{x})$, where $\hat{f}^{(cl)}(\bar{x})$ is the Taylor expansion (algebraic approximation) and $\dot{\hat{J}}(\bar{x}) = 2\bar{x}^{\intercal} S \hat{f}^{(cl)}(\bar{x})$

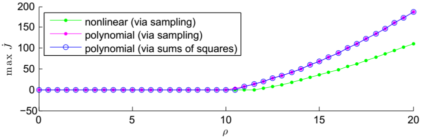
$$\begin{aligned} \text{find} \quad & h(\bar{x}) \\ \text{subject to} \quad & \dot{\hat{J}}(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \end{aligned}$$

Optimization for ρ

Set a convex optimization problem for the region of attraction

$$\begin{aligned} \max \quad & \rho \\ \text{subject to} \quad & \dot{\hat{J}}^*(\bar{x}) + h(\bar{x})(\rho - \hat{J}^*(\bar{x})) \leq -\epsilon ||\bar{x}||_2^2 \\ & h(\bar{x}) \geq 0 \\ & \rho > 0 \end{aligned}$$

- At each step the Lagrange multiplier searching is performed
- If the program is feasible ρ increased



Polynomial verification of damped single pendulum [Tedrake, 2010]

Trajectory Optimization

Time-Invariant LQR Verification

Conclusions

References

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Thank You!