Homework 4

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ME5774 Nonlinear Systems Theory
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Problem 1.

Solution. Consider the system

$$\dot{x_1} = \tanh x_1(ax_1 + x_2)$$
$$\dot{x_2} = bx_1x_2 + \frac{1}{1+x_2^2}u,$$

where $a = 2$, $b = 3$ are the determined values of the model. The proper form for integrator backstepping is

$$\dot{x_1} = 2x_1 \tanh x_1 + x_2 \tanh x_1 \tag{1}$$
$$\dot{x_2} = 3x_1x_2 + \frac{1}{1+x_2^2}u. \tag{2}$$

The system has an equilibrium at the origin $\mathbf{x}_e = [0 \quad 0]^\top$. Let us define $x_2 = \phi(x_1) = -2x_1 - x_1^2$. From Equation 4 we get the closed-loop system

$$\dot{x}_1 = -x_1^2 \tanh x_1.$$

Let us consider the Lyapunov function $V = \frac{1}{2}x_1^2 > 0$, where the rate of change yields

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 \\ &= x_1(-x_1^2 \tanh x_1) \\ &= -x_1^3 \tanh x_1.\end{aligned}$$

If $x_1 > 0$ then $\tanh x_1 > 0$, so $-x_1^3 \tanh x_1 < 0$. On the other hand, if $x_1 < 0$ then $\tanh x_1 < 0$, so $-x_1^3 \tanh x_1 < 0$. Next, if $x_1 = 0$ then $-x_1^3 \tanh x_1 = 0$. So $\dot{V} \leq -W(x_1)$, where $W(x_1) = x_1^3 \tanh x_1$ is a positive definite function for all $\mathbf{x} \in \mathbb{R}^2 - \{\mathbf{0}\}$. Since the Lyapunov function is radially unbounded, because $V \to \infty$ as $\|\mathbf{x}\| \to \infty$, we have a globally asymptotically stable equilibrium at the origin $\mathbf{x}_e$.

Let us consider the augmented Lyapunov function

$$V_a = V + \frac{1}{2}(x_2 - (-2x_1 - x_1^2))^2,$$

where by choosing the control u as

$$u = (1 + x_2^2)(-3x_1x_2 + \frac{\partial \phi}{\partial x_1}(2x_1 \tanh x_1 + x_2 \tanh x_1) - (\frac{\partial V}{\partial x_1} \tanh x_1) - K(x_2 - (-2x_1 - x_1^2))), \tag{3}$$

with $K > 0$ results

$$\begin{aligned} \dot{V}_a &\leq -W(x_1) - K(x_2 - \phi(x_1))^2 \\ &\leq -x_1^3 \tanh x_1 - K(x_2 - (-2x_1 - x_1^2))^2. \end{aligned}$$

Since the augmented Lyapunov function $V_a < 0$ for all $\mathbf{x} \in \mathbb{R}^2 - \{\mathbf{0}\}$, we get asymptotic stability of the origin $\mathbf{x}_e$ under this feedaback control u.

By using Equation 3 we get the closed-loop system in the physical space

$$\dot{x_1} = 2x_1 \tanh x_1 + x_2 \tanh x_1 \tag{4}$$

$$\dot{x_2} = (-2 - 2x_1)(2x_1 \tanh x_1 + x_2 \tanh x_1) - (x_1 \tanh x_1) - K(x_2 + 2x_1 + x_1^2). \tag{5}$$

In Figure 1 the states x_1, x_2 over time are depicted. Moreover, in Figure 2 the state x_1 over x_2 for various gains K is presented. □

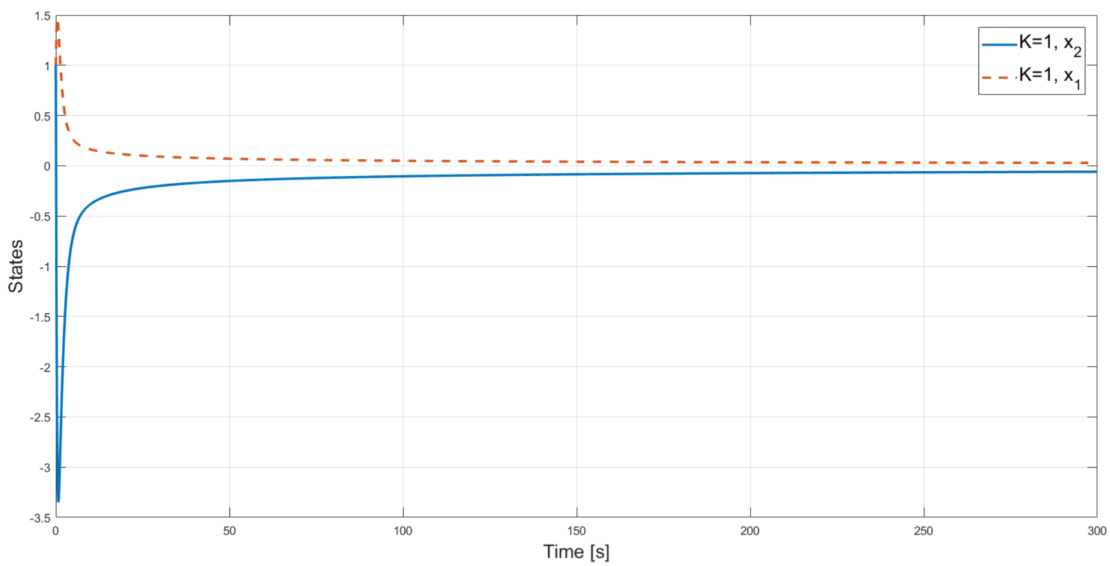

Figure 1: States x_1, x_2 over time using backstepping integral for gain $K = 1$.

Problem 2.

Solution. Consider the system

$$\begin{aligned} \dot{x_1} &= \tanh x_1(ax_1 + x_2) \\ \dot{x_2} &= bx_1x_2 + \frac{1}{1 + x_2^2}u, \end{aligned}$$

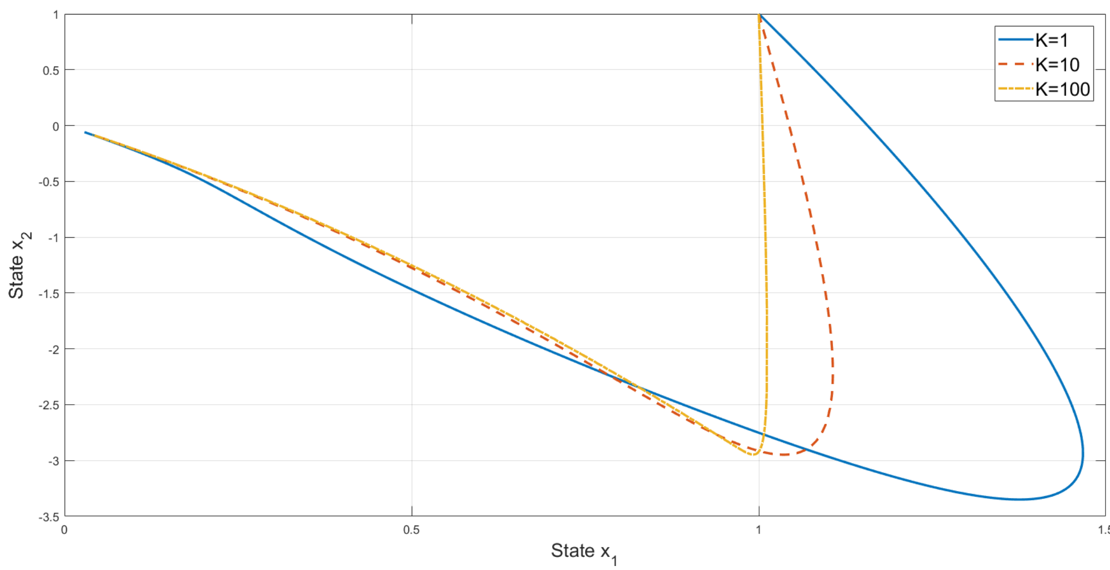

Figure 2: State x_1 over x_2 using backstepping integral for various gain K values.

but in this case the parameters a, b are uncertain. Since we are aware of the bounds $a \in [1,3]$, $b \in [2,4]$ we can extract the nominal parameters $\hat{a} = 2$, $\hat{b} = 3$. Let us set

$$a = \hat{a} + a - \hat{a}$$
$$b = \hat{b} + b - \hat{b}.$$

Then we can formulate the system's equation to the sliding mode control form

$$\dot{x_1} = \tanh x_1((\hat{a} + a - \hat{a})x_1 + x_2) = (\hat{a}x_1 + x_2)\tanh x_1 + (a - \hat{a})x_1 \tanh x_1$$
$$\dot{x_2} = (\hat{b} + b - \hat{b})x_1x_2 + \frac{1}{1 + x_2^2}u = \hat{b}x_1x_2 + \frac{1}{1 + x_2^2}(u + x_1x_2(1 + x_2^2)(b - \hat{b})),$$

where

$$f(x_1, x_2) = (\hat{a}x_1 + x_2)\tanh x_1, \qquad \delta_{x_1} = (a - \hat{a})x_1 \tanh x_1$$

$$f_a(x_1, x_2) = \hat{b}x_1x_2, \qquad G_a(x_2) = \frac{1}{1 + x_2^2}, \qquad \delta(x_2) = x_1x_2(1 + x_2^2)(b - \hat{b}).$$

Let us choose $x_2 = \phi(x_1) = -\hat{a}x_1 - \alpha|x_1| = -2x_1 - \alpha|x_1|$, where $\alpha > 1$. Next, considering the Lyapunov function $V = \frac{1}{2}x_1^2 > 0$ we compute the rate of change

$$\begin{aligned} \dot{V} &= x_1\dot{x}_1 \\ &= x_1(2x_1 - 2x_1 - \alpha|x_1|)\tanh x_1 \\ &= -x_1\alpha|x_1|\tanh x_1. \end{aligned}$$

In problem 1 we have proved that $x_1 \tanh x_1 \geq 0$. Since $\alpha > 1$ we get $-x_1\alpha|x_1|\tanh x_1| \leq 0$. Therefore, $\dot{V} > 0$ for all $\mathbf{x} \in \mathbb{R}^2 - \{\mathbf{0}\}$ which yields asymptotic stability at the origin. We choose $\alpha = 3$.

The control law will have the form

$$u = u_{eq} + (1 + x_2^2)v, \quad (6)$$

where

$$\begin{aligned} u_{eq} &= G_a^{-1}(x_1, x_2)(-f_a(x_1, x_2) + \frac{\partial \phi}{\partial x_1} f(x_1, x_2)) \\ &= (1 + x_2^2)(-(\hat{b}x_1x_2) + (-\hat{a} - \alpha sign(x_1))(\hat{a}x_1 + x_2 \tanh x_1)) \end{aligned} \quad (7)$$

Next we define $z = x_2 - \phi(x_1)$ and we get the rate of change $\dot{z} = v + \Delta$, where

$$\Delta = x_1x_2(b - \hat{b}) + (\hat{a} + \alpha sign(x_1))(a - \hat{a})x_1 \tanh x_1. \quad (8)$$

For finding a feedback control law that drives z in zero in finite time we work on the maximum value of Δ as follows

$$\begin{aligned} ||\Delta||_\infty &\leq ||x_1x_2(b - \hat{b})||_\infty + ||(\hat{a} + \alpha sign(x_1))(a - \hat{a})x_1 \tanh x_1||_\infty \\ &\leq |x_1||x_2||1| + (|2 + \alpha|)|1||x_1| \\ &\leq |x_1||x_2| + (|2 + \alpha|)|x_1| = \rho \end{aligned}$$

Note that we reduce $||sign(x_1)||_\infty = 1$ and $|| \tanh x_1||_\infty = 1$.

Next we define $\beta = \rho + c$, where $c > 0$ and we let

$$v = -\beta sign(x_2 - \phi(x_1)). \quad (9)$$

From the Equations 6,7,9 we get the feedback control law of the full system

$$\begin{aligned} u = \; & (1 + x_2^2)[(-\hat{b}x_1x_2) - (\hat{a} + \alpha sign(x_1))(\hat{a}x_1 + x_2 \tanh x_1)] + \\ & (1 + x_2^2)(-(|x_1||x_2| + |2 + \alpha||x_1| + c)sign(x_2 + \hat{a}x_1 + \alpha|x_1|)). \end{aligned}$$

Next, we replace the signum function with a saturation function to avoid chattering. The saturation function sat$(\frac{y}{\epsilon})$ approaches the signum function when $\epsilon \to 0$. In Figure 3 the states x_1, x_2 over time are depicted. Moreover, in Figure 4 the states x_1, x_2 are presented by employing the signum function and the saturation with various ϵ values. □

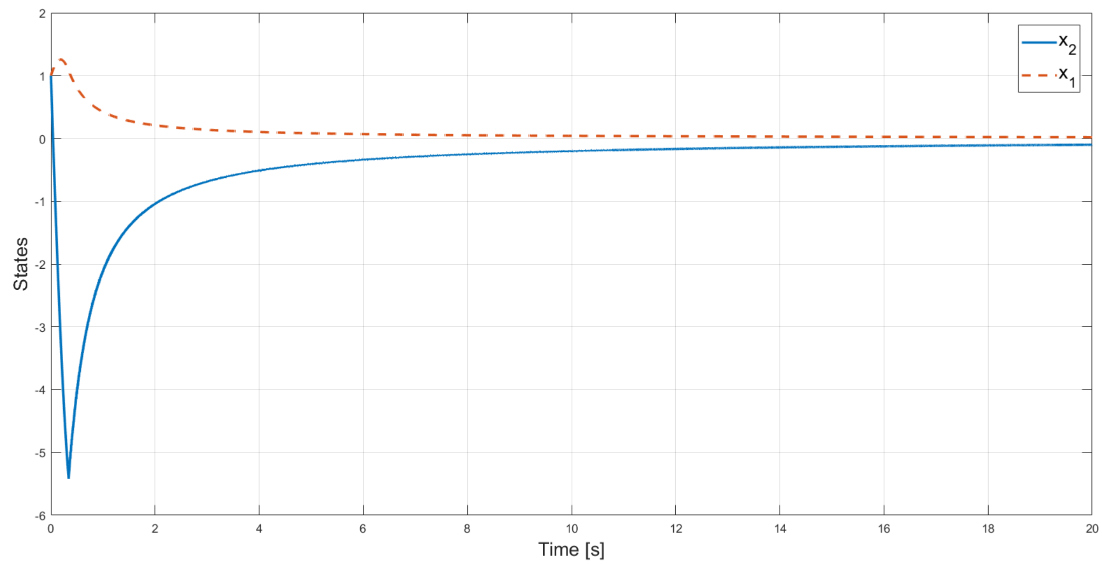

Figure 3: States x_1, x_2 over time of the closed-loop system using sliding mode control for gain $K = 0$.

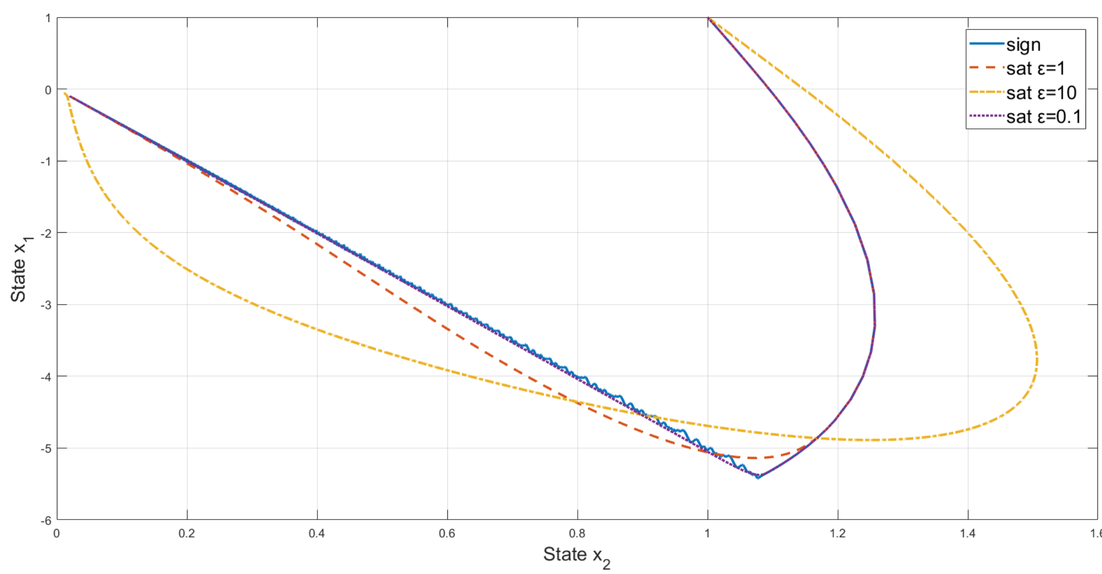

Figure 4: State x_1 over x_2 of the closed-loop system using sliding mode control for various saturation values by altering ϵ and gain $K = 0$.