Problem Set 1

Georgios Kontoudis • gpkont@vt.edu
MATH4225 Elementary Real Analysis
Instructor: Dr. Jacob Fillman

Fall 2017

Exercise 1. Negate the statement: "For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. "

Solution. "There exists $\epsilon > 0$, such that for every $\delta > 0$, $|f(x) - f(y)| < \epsilon$ is not true whenever $|x - y| < \delta$." □

Exercise 2 (a). Show that there is no rational number r such that

$$r^2 = 32. \tag{1}$$

Solution. Suppose on the contrary that, $r \in \mathbb{Q}$ and $r^2 = 32$. Then, from the definition of rational numbers we get $r = \frac{a}{b}$, with $a, b \in \mathbb{Z}$, $b \neq 0$, and a, b share no common factors. Equation 1 implies

$$a^2 = 32b^2 \Rightarrow a^2 = 2^4 b^2. \tag{2}$$

This means that a^2 is even. Thus, a is even because if it was odd a^2 would be odd. Therefore, we have $a = 2d, d \in \mathbb{Z}$ and Equation 2 yields

$$2^2 d^2 = 2^4 b^2 \Rightarrow b^2 = 2^{-2} d^2. \tag{3}$$

For the same reason as above, b is even and since a, b cannot share common factors they cannot be both even. Thus, r is not rational $r \notin \mathbb{Q}$. □

Exercise 2 (b). Show that the set $E = \{r \in \mathbb{Q} : r^2 < 32\}$ has no least upper bound.

Solution. The set E is nonempty and is bounded above by any $x > \sqrt{32} \notin \mathbb{Q}$. We consider the set $F = \{x \in \mathbb{Q} : x > 0, x^2 > 32\}$ as the set of the upper bounds of E in $\mathbb{Q}$. We claim that F has no least element, which means that for every $p \in F$ there exists a $q \in F$ such that $q < p$. We associate every rational $p > 0$, by using the secant method, with

$$q = p - \frac{f(p)(32 - p)}{f(32) - f(p)} = p - \frac{p^2 - 2^5}{p + 2^5} = \frac{2^5 p + 2^5}{2^5 + p}. \tag{4}$$

Next, we employ the secant method's properties, as q has the structure

$$q = \frac{\alpha p + \alpha}{\alpha + p}. \tag{5}$$

If $p^2 - 2^5 < 0$, then from Equation 4 we get that $q > p$, and Equation 5 yields $q^2 < 2^5$, while for $p^2 - 2^5 > 0$ Equation 4 returns $q < p$ and Equation 5 yields $q^2 > 2^5$. Thus, E has no supremum (least upper bound) in $\mathbb{Q}$. □

Exercise 3. For $n \in \mathbb{N}$ with $n \geq 2$, $\sqrt{n}$ is irrational if and only if its prime factorization contains an odd power of a prime.

Solution. For the sake of argument let $\sqrt{n}$ be in $\mathbb{Q}$, then $\sqrt{n} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. If we square both sides then we get $n = \frac{a^2}{b^2} \Rightarrow b^2 n = a^2$. We employ the fundamental theorem of arithmetic and we obtain $a^2 = p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_k^{2\alpha_k}$, where every unique element has even power. Similarly, b^2 has even powers. Thus, $n = q_1^{2\beta_1} q_2^{2\beta_2} \dots q_k^{2\beta_k}$, which means that $n = l^2$ where $l = q_1^{\beta_1} q_2^{\beta_2} \dots q_k^{\beta_k}$. Therefore, if the prime factorization of $\sqrt{n}$ has only even power of primes, then it is rational. □

Exercise 4. If $r \neq 0$ is a rational number and x is irrational, prove that $r + x$ and rx are irrational.

Solution. Assume, to contrary that $r + x$ and rx are rational, then we get $r + x = r + x \Rightarrow x = r + x - r$ which means that x is rational. Similarly, $rx = rx \Rightarrow x = \frac{rx}{r}$ which means that x is rational. Thus, x is irrational so the addition $r + x$ and the product rx are irrational too. □

Exercise 5. Let

$$E = \{11 + (-1)^n \left(3 - \frac{5}{n^3}\right) : n \in \mathbb{N}\}. \tag{6}$$

Identify $\inf E$ and $\sup E$.

Solution. The set E for even numbers yields

$$E_{even} = \{14 - \frac{5}{n^3} : n = 2k, k \in \mathbb{N}\}, \tag{7}$$

while for odd numbers we get

$$E_{odd} = \{8 + \frac{5}{n^3} : n = 2k + 1, k = 0, k \in \mathbb{N}\}. \tag{8}$$

Let the candidate for supremum be 14. We first need to check if 14 is an upper bound, which means that for all $e \in E$, $e \leq 14$. If $e \in E$, we get for the odd numbers

$$8 + \frac{5}{n^3} \leq 14 \Rightarrow \frac{5}{6} \leq n^3, \tag{9}$$

which is true for all odd numbers $n = 2k + 1, k = 0, k \in \mathbb{N}$. Similarly, if $e \in E$, we get for the even numbers

$$14 - \frac{5}{n^3} \leq 14 \Rightarrow \frac{5}{n^3} \geq 0, \tag{10}$$

which is also true for all even numbers $n = 2k, k \in \mathbb{N}$. Thus, 14 is an upper bound for all $e \in E$.

Next, let the candidate for infimum be 8. We need to check if 8 is a lower bound, which means that for all $e \in E$, $e \geq 8$. If $e \in E$, we get for the odd numbers

$$8 + \frac{5}{n^3} \geq 8 \Rightarrow \frac{5}{n^3} \geq 0, \tag{11}$$

which is true for all odd numbers $n = 2k + 1, k = 0, k \in \mathbb{N}$. Similarly, if $e \in E$, we get for the even numbers

$$14 - \frac{5}{n^3} \geq 8 \Rightarrow \frac{5}{6} \leq n^3, \tag{12}$$

which is also true for all even numbers $n = 2k, k \in \mathbb{N}$. Thus, 8 is a lower bound for all $e \in E$.

Before we continue with the supremum and infimum, we observe that $3 - \frac{5}{n^3} > 0$ for all $n \in \mathbb{N}$ except $n = 1$ which yields $e = 13$, that is neither a lower bound nor an upper bound. Assume, to the contrary that $e = 13$ is an upper bound, which means that for all $e \in E$, $e \leq 13$. Then, if $e \in E$, we get for the even numbers, $(14 - \frac{5}{n^3}) \leq 13 \Rightarrow \frac{5}{n^3} \geq 1$ which is false for all even numbers $n = 2k, k \in \mathbb{N}$, and thus 13 is not an upper bound. Assume, to the contrary that $e = 13$ is a lower bound, which means that for all $e \in E$, $e \geq 13$. Then, if $e \in E$, we get for the odd numbers, $(8 + \frac{5}{n^3}) \geq 13 \Rightarrow n^3 \leq 1$ which is false for all odd numbers $n = 2k + 1, k \in \mathbb{N}$, and thus 13 is not a lower bound.

Then, we will show that sup E=14 and inf E=8. We observe that $\frac{5}{n^3} < 1$ for all $n = \mathbb{N}, n \neq 1$, which means that the set E converges to 14 for all even values $n = 2k, k \in \mathbb{N}$ and to 8 for all odd values $n = 2k + 1, k \neq 0, k \in \mathbb{N}$. So we wil study even numbers for the supremum and odd numbers for the infimum.

Since $e = 14$ is an upper bound of E, $(14 - \frac{5}{n^3}) \leq 14$ for all $n = 2k, k \in \mathbb{N}$. Assume, to the contrary that $e < 14$. Thus, $14 - e > 0$ and by the Archimedean Property, there exists an n^3 for all $n = 2k, k \in \mathbb{N}$ such that $n^3(14 - e) > 5$ that yields

$$e < 14 - \frac{5}{n^3} \in E, \tag{13}$$

which contradicts that e is an upper bound. Therefore, sup E=14.

For the infimum, since $e = 8$ is a lower bound of E, $(8 + \frac{5}{n^3}) \geq 8$ for all $n = 2k, k \in \mathbb{N}$. Assume, to the contrary that $e > 8$. Thus, $e - 8 > 0$ and by the Archimedean Property, there exists an n^3 for all $n = 2k + 1, k \in \mathbb{N}$ such that $n^3(e - 8) > 5$ that yields

$$e > 8 + \frac{5}{n^3} \in E, \tag{14}$$

which contradicts that e is a lower bound. Therefore, inf E=8.

$\square$

Exercise 6 (a). Many math texts adopt the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. Discuss why this is reasonable convention.

Solution. Any real number is an upper bound of the empty set, so $-\infty$ can be the least, and thus the supremum. Similarly, the infimum of an empty set is $+\infty$, since any real number is a lower bound of the empty set, and thus $+\infty$ would be the greatest. $\square$

Exercise 6 (b). Adopting this convention, show that, for $E \subset \mathbb{R}$, one has $\inf E \leq \sup E$ *if and only if* $E \neq \emptyset$.

Solution. For non-empty sets we inherit the definition of upper bound and lower bound. We call $u \in \mathbb{R}$ an upper bound for E if $x \leq u$ for all $x \in E$. We call $u \in \mathbb{R}$ a lower bound for E if $x \geq u$ for all $x \in E$. For a nonempty set $E \neq \emptyset$, that has two or more elements we get $\inf E \leq x \leq y \leq \sup E$. Note that for a single element in a set we have $\inf E = \sup E$. $\square$