A Highly-Accurate Approach to Multivariate Time Series Forecasting via Calender-Aware Tensor Decomposition with Wavelet Denoising Supplementary File

I. Introduction

THIS is the supplementary file for the paper entitled " A Highly-Accurate Approach to Multivariate Time Series Forecasting via Calender-Aware Tensor Decomposition with Wavelet Denoising". Supplementary theoretical derivations and experimental results are put into this file.

Theorem S1 (Explicit calendrical-prior injection on the timestamp torus via Tucker-2 decomposition)**.** *Let each timestamp be encoded as an angle vector $\tau_t \in \mathbb{R}^O$ after standard calendrical scalings with wrap-around (e.g., hour $\mapsto 2\pi\,\frac{hour}{24}$, weekday $\mapsto 2\pi\,\frac{idx}{7}$), so that the intrinsic domain is the O-torus $\mathbb{T}^O = (\mathbb{R}/2\pi\mathbb{Z})^O$. For each time t define the timestamp-lifted channel–interaction matrix*

$$\mathcal{X}_t = \sum_{o=1}^{O} \tau_t^{(o)}\,\mathcal{W}_o \in \mathbb{R}^{F\times F}, \quad \mathcal{W}_o \in \mathbb{R}^{F\times F}, \tag{S1}$$

stack $\{\mathcal{X}_t\}_{t=1}^{L}$ into $\mathcal{X} \in \mathbb{R}^{F\times F\times L}$, and perform a Tucker-2 decomposition (time mode fixed to identity):

$$\mathcal{G} = \mathcal{X} \times_1 U^{(1)\top} \times_2 U^{(2)\top}, \qquad U^{(1)}, U^{(2)} \in \mathbb{R}^{F\times r}\ . \tag{S2}$$

Let $\mathcal{G}_t = \mathcal{G}(:,:,t)$ and $g_t = vec(\mathcal{G}_t) \in \mathbb{R}^{r^2}$. Given $\Omega \in \mathbb{R}^{r^2\times D_\omega}$, define the periodic embedding

$$E_t = \big[\sin(\Omega^\top g_t),\ \cos(\Omega^\top g_t)\big] \in \mathbb{R}^{2D_\omega}. \tag{S3}$$

Then the following hold.

(A) ***Torus-characterization (explicit periodicity).*** *There exists $M \in \mathbb{R}^{D_\omega\times O}$, determined by $(\Omega, U^{(1)}, U^{(2)}, \{\mathcal{W}_o\})$, such that*

$$E_t = \big[\sin(M\tau_t),\ \cos(M\tau_t)\big]. \tag{S4}$$

Hence E_t consists of explicit periodic functions on the torus coordinates. Moreover, if the rows of M are restricted (or quantized) to integer vectors in $\mathbb{Z}^O$, then each coordinate becomes a real character of $\mathbb{T}^O$.

(B) ***Subspace alignment, sufficiency, and per-step identifiability.*** *Let $W_{(1)} = [vec(\mathcal{W}_1), \ldots, vec(\mathcal{W}_O)] \in \mathbb{R}^{F^2\times O}$ and $d := \mathrm{rank}(W_{(1)}) \le O$. If $U^{(1)}, U^{(2)}$ are chosen so that $\mathrm{span}(U^{(2)}\otimes U^{(1)})$ contains $\mathrm{Im}(W_{(1)})$, then there exists $A \in \mathbb{R}^{r^2\times O}$ with $\mathrm{rank}(A) = d$ such that*

$$g_t = A\,\tau_t \qquad (t = 1, ..., L), \tag{S5}$$

and the map $\tau_t \mapsto g_t$ is a linear sufficient statistic for any predictor built on Tucker-2 decomposition of $\mathcal{X}_t$. Moreover, because the time mode is the identity, $\mathcal{G}_t$ aligns per step with τ_t (no cross-time mixing), preserving the phase of calendrical cycles at each t.

(C) ***Composite seasonality.*** *If a row $m^\top$ of M has multiple nonzeros, then $\sin(m^\top\tau_t)$ and $\cos(m^\top\tau_t)$ realize composite seasonality (e.g., weekday×hour, month×weekday). Thus T^2IM can express mixed calendrical interactions via mixed characters of $\mathbb{T}^O$.*

Proof. We proceed in three steps.

Step 1 (Algebraic reduction to torus characters). By $vec(AXB) = (B^\top \otimes A)vec(X)$,

$$g_t = vec\big(U^{(1)\top}\mathcal{X}_t U^{(2)}\big) = (U^{(2)\top} \otimes U^{(1)\top})\,vec(\mathcal{X}_t). \tag{S6}$$

Since $vec(\mathcal{X}_t) = W_{(1)}\tau_t$, we have

$$g_t = (U^{(2)\top} \otimes U^{(1)\top})\,W_{(1)}\,\tau_t =: A\,\tau_t, \qquad A \in \mathbb{R}^{r^2\times O}. \tag{S7}$$

Thus

$$\Omega^\top g_t = \Omega^\top A\,\tau_t =: M\tau_t, \qquad M = \Omega^\top A \in \mathbb{R}^{D_\omega \times O}. \tag{S8}$$

Therefore

$$E_t = \big[\sin(M\tau_t),\ \cos(M\tau_t)\big], \tag{S9}$$

whose entries are real and imaginary parts of $\exp(\mathrm{i}\,m^\top \tau_t)$ with m equal to rows of M. Because τ_t lies on $\mathbb{T}^O$, these are precisely (real) characters of the torus evaluated at τ_t, establishing (A).

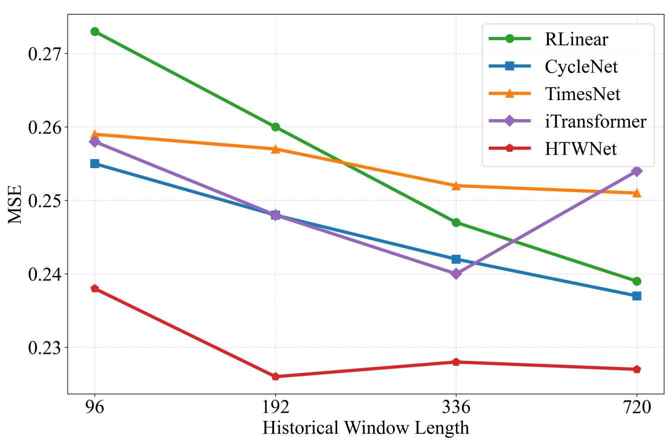
Step 2 (Subspace alignment and sufficiency). Let $d := \mathrm{rank}(W_{(1)})$, so $vec(\mathcal{X}_t) \in \mathrm{Im}(W_{(1)})$, a d-dimensional linear subspace of $\mathbb{R}^{F^2}$. Choosing $U^{(1)}, U^{(2)}$ so that $\mathrm{span}(U^{(2)} \otimes U^{(1)}) \supseteq \mathrm{Im}(W_{(1)})$ makes $A = (U^{(2)\top} \otimes U^{(1)\top})W_{(1)}$ satisfy $\mathrm{rank}(A) = d$ (no information loss). Hence $g_t = A\tau_t$ is a linear, lossless reparameterization of the timestamp signal within the interaction subspace. Because the Tucker factorization keeps the time mode equal to identity, $\mathcal{G}_t$ depends only on the *current* τ_t (no mixing across t), so calendrical phase is preserved *per step*. Any predictor that depends on $\mathcal{X}_t$ through Tucker-2 decomposition must factor through g_t, making g_t a sufficient statistic, which proves (B).

Step 3 (Composite seasonality). Let $m^\top$ be a row of M with support on indices $\{o_1, \ldots, o_s\} \subseteq \{1, \ldots, O\}$, $s \geq 2$. Then $m^\top \tau_t = \sum_{j=1}^{s} m_{o_j} \tau_t^{(o_j)}$ couples multiple calendrical angles. The functions $\sin(m^\top \tau_t)$ and $\cos(m^\top \tau_t)$ are mixed characters of $\mathbb{T}^O$ and implement multiplicative interactions of the corresponding seasonalities (e.g., weekday$\times$hour). Hence T^{2}IM expresses composite seasonality, proving (C). □

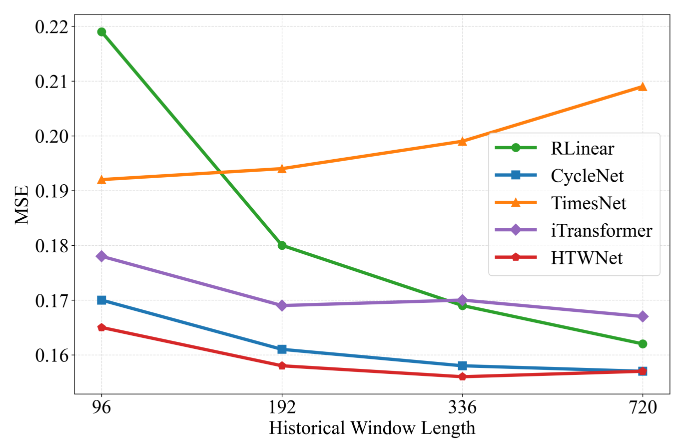
Corollary S2 (Universal approximation on the timestamp torus)**.** *Let $f^\star : \mathbb{T}^O \to \mathbb{R}^q$ be continuous. For any $\varepsilon > 0$, there exist D_ω, parameters $(\Omega, U^{(1)}, U^{(2)}, \{\mathcal{W}_o\})$ (thus M), gating γ (selecting finitely many coordinates of E), and a linear head W_{out} such that the T^{2}IM predictor $\hat{f}$ satisfies*

$$\sup_{\tau \in \mathbb{T}^O} \|\hat{f}(\tau) - f^\star(\tau)\|_2 \leq \varepsilon. \tag{S10}$$

Proof. *Real characters $\{\cos(m^\top \tau), \sin(m^\top \tau)\}$ form a trigonometric basis on $\mathbb{T}^O$, finite sums uniformly approximate any continuous function (Stone–Weierstrass on compact groups). Choose the rows of M to be the desired frequency vectors m (achievable via $\Omega, U^{(1)}, U^{(2)}, \{\mathcal{W}_o\}$ so that $M = \Omega^\top (U^{(2)\top} \otimes U^{(1)\top})W_{(1)}$), gate the corresponding coordinates, and pick W_{out} to match coefficients.* □



(a) Weather



(b) Electricity

Fig. S1. Improved model performance with a longer historical window.

TABLE S(I)
FULL RESULTS OF HTWNET AND BASELINES.

Dataset		ETTm1		ETTm2		ETTh1		ETTh2		Electricity		Weather		Traffic	
Metric		MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
RLinear	96	0.355	0.376	0.182	0.265	0.386	0.395	0.318	0.363	0.201	0.281	0.192	0.232	0.649	0.389
	192	0.387	0.392	0.246	0.304	0.437	0.424	0.401	0.412	0.201	0.283	0.240	0.271	0.601	0.366
	336	0.424	0.415	0.307	0.342	0.479	0.446	0.436	0.442	0.215	0.298	0.292	0.307	0.609	0.369
	720	0.487	0.450	0.407	0.398	0.481	0.470	0.442	0.454	0.257	0.331	0.364	0.353	0.647	0.387
FITS	96	0.355	0.375	0.183	0.266	0.386	0.396	0.295	0.350	0.200	0.278	0.166	0.213	0.651	0.391
	192	0.392	0.393	0.247	0.305	0.436	0.423	0.381	0.396	0.200	0.280	0.213	0.254	0.602	0.363
	336	0.424	0.415	0.307	0.342	0.478	0.444	0.426	0.438	0.214	0.295	0.269	0.294	0.609	0.366
	720	0.487	0.449	0.407	0.399	0.502	0.495	0.431	0.446	0.255	0.327	0.346	0.343	0.647	0.385
CycleNet	96	0.325	0.363	0.166	0.248	0.378	0.391	0.285	0.335	0.141	0.234	0.170	0.216	0.480	0.314
	192	0.366	0.382	0.233	0.291	0.426	0.419	0.373	0.391	0.155	0.247	0.222	0.259	0.482	0.313
	336	0.396	0.401	0.293	**0.330**	0.464	0.439	0.421	0.433	0.172	0.264	0.275	0.296	0.476	0.303
	720	0.457	0.433	0.395	0.389	0.461	0.460	0.453	0.458	0.210	0.296	0.349	0.345	0.503	0.320
TiDE	96	0.364	0.387	0.207	0.305	0.479	0.464	0.400	0.440	0.237	0.329	0.202	0.261	0.805	0.493
	192	0.398	0.404	0.290	0.364	0.525	0.492	0.528	0.509	0.236	0.330	0.242	0.298	0.756	0.474
	336	0.428	0.425	0.377	0.422	0.565	0.515	0.643	0.571	0.249	0.344	0.287	0.335	0.762	0.477
	720	0.487	0.461	0.558	0.524	0.594	0.558	0.874	0.679	0.284	0.373	0.351	0.386	0.719	0.449
MSD-Mixer	96	0.304	0.351	0.169	0.259	0.377	0.391	0.284	0.345	0.152	0.254	**0.148**	0.212	0.500	0.324
	192	**0.344**	0.375	0.232	0.300	0.427	0.422	**0.362**	0.392	0.165	0.263	**0.200**	0.262	0.506	0.324
	336	**0.370**	**0.395**	**0.292**	0.337	0.469	0.443	0.399	0.428	0.173	0.273	**0.256**	0.310	0.528	0.341
	720	**0.427**	**0.428**	0.392	0.398	0.485	0.475	0.426	0.457	0.201	0.299	**0.327**	0.362	0.561	0.369
SOFTS	96	0.325	0.361	0.180	0.261	0.381	0.399	0.297	0.347	0.143	0.233	0.166	0.208	**0.376**	**0.251**
	192	0.375	0.389	0.246	0.306	0.435	0.431	0.373	0.394	0.158	0.248	0.217	0.253	**0.398**	**0.261**
	336	0.405	0.412	0.319	0.352	0.480	0.452	0.410	0.426	0.178	0.269	0.282	0.300	**0.415**	**0.269**
	720	0.466	0.447	0.405	0.401	0.499	0.488	0.411	0.433	0.218	0.305	0.356	0.351	**0.447**	**0.287**
TimesNet	96	0.338	0.375	0.187	0.267	0.384	0.402	0.340	0.374	0.168	0.272	0.172	0.220	0.593	0.321
	192	0.374	0.387	0.249	0.309	0.436	0.429	0.402	0.414	0.184	0.289	0.219	0.261	0.617	0.336
	336	0.410	0.411	0.321	0.351	0.491	0.469	0.452	0.452	0.198	0.300	0.280	0.306	0.629	0.336
	720	0.478	0.450	0.408	0.403	0.521	0.500	0.462	0.468	0.220	0.320	0.365	0.359	0.640	0.350
ModernTCN	96	0.317	0.362	0.173	0.255	0.386	0.394	0.292	0.340	0.173	0.260	0.155	0.203	0.550	0.355
	192	0.363	0.389	0.235	0.296	0.436	0.423	0.377	0.395	0.181	0.267	0.202	0.247	0.527	0.337
	336	0.403	0.412	0.308	0.344	0.479	0.445	0.424	0.434	0.196	0.283	0.263	0.293	0.537	0.342
	720	0.461	0.443	0.398	0.394	0.481	0.469	0.433	0.448	0.238	0.316	0.334	0.343	0.570	0.359
Crossformer	96	0.404	0.426	0.287	0.366	0.423	0.448	0.745	0.584	0.219	0.314	0.158	0.230	0.522	0.290
	192	0.450	0.451	0.414	0.492	0.471	0.474	0.877	0.656	0.231	0.322	0.206	0.277	0.530	0.293
	336	0.532	0.515	0.597	0.542	0.570	0.546	1.043	0.732	0.246	0.337	0.272	0.335	0.558	0.305
	720	0.666	0.589	1.730	1.042	0.653	0.621	1.104	0.763	0.280	0.363	0.398	0.418	0.589	0.328
PatchTST	96	0.329	0.367	0.175	0.259	0.414	0.419	0.302	0.348	0.195	0.285	0.177	0.218	0.544	0.359
	192	0.367	0.385	0.241	0.302	0.460	0.445	0.388	0.400	0.199	0.289	0.225	0.259	0.540	0.354
	336	0.399	0.410	0.305	0.343	0.501	0.466	0.426	0.433	0.215	0.305	0.278	0.297	0.551	0.358
	720	0.454	0.439	0.402	0.400	0.500	0.488	0.431	0.446	0.256	0.337	0.354	0.348	0.586	0.357
iTransformer	96	0.334	0.368	0.180	0.264	0.386	0.405	0.297	0.349	0.148	0.240	0.174	0.214	0.395	0.268
	192	0.377	0.391	0.250	0.309	0.441	0.436	0.380	0.400	0.162	0.253	0.221	0.254	0.417	0.276
	336	0.426	0.420	0.311	0.348	0.487	0.458	0.428	0.432	0.178	0.269	0.278	0.296	0.433	0.283
	720	0.491	0.459	0.412	0.407	0.503	0.491	0.427	0.445	0.225	0.319	0.358	0.349	0.467	0.302
GLAFFLinear	96	0.363	0.386	0.177	0.256	0.390	0.400	0.307	0.351	0.191	0.275	0.216	0.258	0.487	0.271
	192	0.388	0.392	0.246	0.299	0.440	0.426	0.403	0.405	0.176	0.270	0.245	0.274	0.500	0.280
	336	0.425	0.415	0.316	0.349	0.488	0.454	0.443	0.438	0.197	0.289	0.293	0.307	0.535	0.301
	720	0.503	0.460	0.407	0.397	0.488	0.474	0.415	0.436	0.252	0.331	0.366	0.354	0.580	0.317
TimeLinear	96	0.325	0.364	0.167	0.249	0.378	0.391	0.285	0.335	0.140	0.234	0.166	0.212	0.459	0.293
	192	0.365	0.381	0.233	0.291	0.424	0.418	0.373	0.390	0.155	0.247	0.218	0.256	0.467	0.298
	336	0.395	0.401	0.295	0.331	0.463	0.438	0.398	0.418	**0.169**	0.265	0.272	0.294	0.481	0.305
	720	**0.456**	0.433	0.395	0.390	0.464	**0.456**	**0.377**	**0.412**	**0.198**	**0.290**	0.347	0.342	0.512	0.320
HTWNet	96	**0.301**	**0.341**	**0.161**	**0.240**	**0.363**	**0.387**	**0.276**	**0.327**	**0.134**	**0.227**	**0.148**	**0.192**	0.418	0.264
	192	0.347	**0.374**	**0.230**	**0.287**	**0.411**	**0.413**	0.368	**0.388**	**0.153**	**0.243**	**0.200**	**0.238**	0.446	0.275
	336	0.382	0.399	0.295	0.331	**0.445**	**0.434**	**0.388**	**0.411**	**0.169**	**0.261**	0.261	**0.285**	0.458	0.279
	720	0.433	0.431	**0.389**	**0.387**	**0.457**	0.458	0.404	0.429	0.206	0.293	0.347	**0.341**	0.502	0.296