HETEROGENEOUS RISK PREFERENCES IN FINANCIAL MARKETS

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Abstract. This paper builds a continuous time model of N heterogeneous agents whose CRRA preferences differ in their level of risk aversion and considers the Mean Field Game (MFG) in the limit as N becomes large. I add to the previous literature on preference heterogeneity in financial markets by studying the short run dynamics of financial variables and the limit in N . I find that agents dynamically self select into one of three groups depending on their preferences: leveraged investors, diversified investors, and saving divestors, driven by a wedge between the market price of risk and the risk free rate. Additionally, I find that changing the number of preference types has a non-trivial effect on the solution. Finally, I find through numerical solution that the model predicts a correlation between dividend yields and the stochastic discount factor, a non-linear response of volatility to shocks, and both pro- and counter-cyclical leverage cycles depending on the assumptions about the distribution of preferences.

Introduction

Each day, trillions of dollars worth of financial assets change hands. Being simply a piece of paper, a financial security gives its bearer the right to a stream of future dividends and capital gains for the infinite future. The price of this abstract object is so difficult to determine that if you ask two analysts for an exact price they will generally disagree. This fact has been well documented in studies such as Andrade et al. (2014) or Carlin et al. (2014). These observations are in direct contrast to a representative agent model of financial markets. Take for instance the aspect of trade in financial assets previously mentioned. With a single agent there can be no exchange because there is no counter party. We look for a set of prices to make the representative agent indifferent to consuming everything, holding the entire capital

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stock, etc. In order to have exchange in an economic model we must introduce two or more agents who are heterogeneous in some way.

In this paper I focus on heterogeneity in risk preferences and consider how the degree of heterogeneity effects model predictions. In particular I study the limit as the number of preference types grows. The majority of the theoretical work on heterogeneous risk preferences focuses on two agents (e.g. Dumas (1989); Coen-Pirani (2004); Guvenen (2006); Bhamra and Uppal (2014); Chabakauri (2013, 2015); Gârleanu and Panageas (2015); Cozzi (2011)). My work most closely resembles that of Cvitanić et al. (2011), who study an economy populated by N agents who differ in their risk aversion parameter, their rate of time preference, and their beliefs. However, they focus on issues of long run survival. I build on their results by studying how changes in the distribution of preferences affect the short run dynamics of the model, while focusing on a single aspect of heterogeneity: risk aversion. Additionally, I take their work to the limit as $N \to \infty$ and characterize a Markovian equilibrium in only a single state variable. This allows me to study how financial variables evolve in the short run.

The model is characterized by a degenerate equilibrium. That is there is no "stationary" equilibrium in the traditional sense. As the shock evolves, so does the model and in the long run it will degenerate either to the lower or to the upper bound of the solution, depending on whether the driving shock converges to $-\infty$ or $+\infty$, respectively. In fact, the non-stationary nature of the problem may be the very characteristic that brings it closer to the real world. I think few people would claim that interest rates and dividend yields are stationary processes (see Figure 1), but that they have exhibited clear downward trends since the 1980s. This paper finds that these trends are consistent with an economy populated by agents with heterogeneous risk preferences.

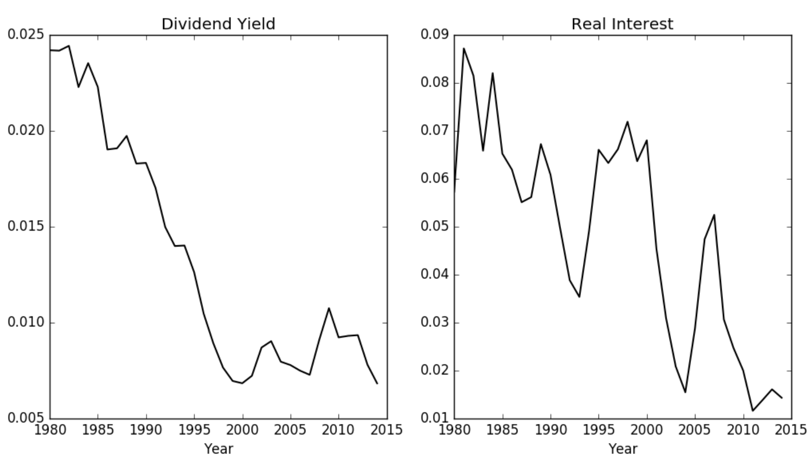


FIGURE 1. Dividend yields and real interest rates for U.S. since 1980. Source: Dividend yield taken, FRED. Real interest rate, IMF, GDP deflator implied inflation.

In the numerical solutions presented in Section 4, we will see that the dividend-price ratio exhibits a similar downward trend as the economy grows. This indicates predictability of stock returns up to some deterministic drift. This result is consistent with those of Campbell and Shiller (1988b) and Campbell and Shiller (1988a), who find that the returns on stocks can be predicted as a function of dividend yield. This could also explain the result of Mankiw (1981), who rejects the permanent income hypothesis on the basis that asset price co-movements with the stochastic discount factor are forecast-able When agents exhibit heterogeneous preferences, the stochastic discount factor does not correspond to a specific agent in every period, but to a time varying level of risk aversion. This level is inversely correlated with dividends: a rise in dividends implies a fall in the market clearing preference levels. For an economy with deterministic drift, asset prices are rising faster than dividends at the same time and move more than one to one with dividends. This excess volatility and time variation in the stochastic discount factor produce a slightly predictable dividend yield.

If we think of individual agents as each having a supply and demand function for risky assets and risk free bonds, it is possible to think of a model of heterogeneous risk preferences as one of market break down. Each agent populates a single theoretical market, but only one market can clear, as in models of the leverage cycle, e.g. Geanakoplos (2010).[1] The market which clears is the one corresponding to the agent who is indifferent between buying or selling their shares or bonds. However, this model will produce both pro- and counter-cyclical leverage cycles, as opposed to the theory proposed in leverage cycles driven by beliefs. This fact is driven by two factors: complete markets and the volatility of the supply of credit. Given that markets are complete, a negative shock will cause all agents to increase their portfolio weights, but given a fall in wealth the change in the value of their borrowing will be heterogeneous. Risk neutral agents reduce their borrowing while risk averse agents increase borrowing (or dis-save). The net result will be a reduction in the aggregate supply of credit. However, asset prices fall simultaneously and to a greater degree. Leverage being the ratio of the value of borrowing to the total value of equity, total leverage rises. This actually makes sense in the complete markets case, as agents will need to smooth their consumption and their borrowing is simply a residual of their consumption/portfolio choice. In order to produce a pro-cyclical leverage cycle as seen in Geanakoplos (2010), one must either introduce constraints or consider a distribution of preferences such that a very small mass borrows from a deep pool of lenders. Then the supply of credit in the market becomes more volatile, as we'll see in Section 4.

In addition, the formulas for the risk free rate and the market price of risk derived in this paper resemble greatly those in Basak and Cuoco (1998). In that paper, two agents participate in the economy, but one is restricted from participating in the financial market while the participating agent determines the value of financial assets. However in this paper, contrary to the limited participation literature, the clearing markets for stocks and bonds do not have to correspond to the same agent, nor does the corresponding agent even need to exist in the economy. We will see in Sections 2.2 and 2.4 that two moments of the distribution of consumption shares determine the market clearing preference levels, which then determine the interest

[1] I also should thank John Geanakoplos, since his work inspired this paper.

rate and market price of risk. Additionally, these values will vary over time and will be endogenously determined.

Models of a continuum of agents are not necessarily new, with games featuring a continuum of agents harkening back to Aumann (1964). However, the study of such models in continuous-time stochastic settings has recently garnered a large amount of attention thanks to a series of papers by Jean-Michel Lasry and Pierre-Louis Lions (Lasry and Lions (2006a), Lasry and Lions (2006b), Lasry and Lions (2007)). These authors studied the limit of N-player stochastic differential games as $N \to \infty$ and agents' risk is idiosyncratic, dubbing the system of equations governing the limit a "Mean Field Game" (MFG). Their work has then been applied to macroeconomics in works such as Moll (2014); Achdou et al. (2014) and Kaplan et al. (2016). However, these papers focus on idiosyncratic risk and do not study the problem of aggregate shocks. Recent work, such as Carmona et al. (2014), Carmona and Delarue (2013), Chassagneux et al. (2014), and Cardaliaguet et al. (2015) to name but a few, has focused on the issue of analyzing equilibria in MFG models with common noise, a category encompassing classic macroeconomic models such as Krusell and Smith (1998). The approach is often to use a stochastic Pontryagin maximum principle to derive a system of forward-backward stochastic differential equations governing the solution. This approach is clear from a mathematical perspective, but very difficult to formulate for more complex economic models.

This paper takes a different approach, solving the model with common noise using Girsanov theory in the style of Harrison and Pliska (1981) and Karatzas et al. (1987). Given the solution to the consumption problem, one recognizes that the stochastic discount factor (SDF) can be written as a function of a single state variable. This allows one to search for a Markovian equilibrium in the style of Chabakauri (2013) and Chabakauri (2015). The solution is characterized by mean field dependence through the control, as opposed to the state. This points towards a new way to consider control in the mean field setting when the heterogeneity is static.

The paper is organized as follows: in Section 1, I construct a continuous time model of financial markets populated by a finite number of agents who differ in their preferences towards risk. Section 2 characterizes the equilibrium. Section 3 discusses the extension to a continuum of types. Section 4 provides numerical results for different numbers of types and two assumptions about the distribution of preferences. Section 5 concludes. The more technical analysis and proofs have been relegated to the appendix.

1. The Model

Consider a continuous time Lucas (1978) economy populated by a number, N, of heterogeneous agents indexed by $i \in \{1, 2, ..., N\}$. Each agent has constant relative risk aversion (CRRA) preferences with relative risk aversion γ_i:

$$U_i(c_{it}) = \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} \qquad \forall i \in \{1, 2, ..., N\}$$

Additionally, agents will begin with a possibly heterogeneous initial wealth, $X_{i0} = x_i$. Agents' initial conditions will be distributed according to a density $(\gamma_i, x_i) \sim g(\gamma, x)$. [2,3]

Agents can continuously trade in shares of a per capita dividend process, which follows a geometric Brownian motion (GBM):

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_t \tag{1.1}$$

where μ_D and σ_D are constants. Risky share prices and risk-free bond prices follow a GBM and an exponential process, respectively:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \tag{1.2}$$

$$\frac{dS_t^0}{S_t^0} = r_t dt \tag{1.3}$$

Denote by X_{it} an individual's wealth at time t and by π_{it} the share of an individual's wealth invested in the risky stock[4].

1.1. **Budget Constraints and Individual Optimization.** All agents are initially endowed with a share in the average tree, as well as an initial bond position. An individual agent's constrained maximization subject to instantaneous changes in wealth can be written as:

$$\begin{aligned} \max_{\{c_{iu}, \pi_{iu}\}_{u=t}^{\infty}} \quad & \mathbb{E} \int_t^{\infty} e^{-\rho(u-t)} \frac{c_{iu}^{1-\gamma_i}}{1-\gamma_i} du \\ \text{s.t.} \quad & dX_{it} = \left[X_{it} \left(r_t + \pi_{it} \left(\mu_t + \frac{D_t}{S_t} - r_t \right) \right) - c_{it} \right] dt \\ & + \pi_{it} X_{it} \sigma_t dW_t \end{aligned}$$

where the constraint represents the dynamic budget of an individual

1.2. **Admissibility.** If an agent's optimal policy implies that they could hold more debt than equity, driving their wealth into negative territory, they could borrow infinitely. This case is ruled out in the real world and we should thus limit our attention to a smaller set of admissible policies. This issue was first treated in the continuous time setting by Karatzas et al. (1986) and I follow their assumptions

[2]In this paper, I will consider $\gamma \in [1, \overline{\gamma}]$ for ease of exposition and for simulations I will take $g(\gamma, x) = g(\gamma)\delta_{x^*}$, where δ_x is the Dirac delta function, such that agents begin with homogeneous wealth. These assumptions can be relaxed, but can induce numerical difficulties.

[3]Assuming this type of random initial condition and preference level provides the independence necessary to study asymptotics. In the sense of Lasry and Lions (2007), this is similar to the independent noise on individual state variables. Independence in this noise allows for the propagation of chaos, while here independence in the initial condition and preference parameter allow for the application of the Law of Large Numbers, essentially the same idea. These assumptions can be thought of economically as a sample. The economy as a whole is too large to measure, so an econometrician must study a random sample of individuals.

[4]Throughout the paper the notation is suppressed where possible for readability, but one should remember that the index i implies dependence both on the initial condition in x and the preference parameter γ.

here. Assume shares π_{it} and consumption c_{it} measurable, adapted, real valued processes such that

$$\int_0^\infty \pi_{it}^2 dt < \infty \quad \text{a.s.}$$
$$\int_0^\infty c_{it} dt < \infty \quad \text{a.s.}$$

Then we can define the set of admissible policies by the following:

Definition 1. *A pair of policies (π_{it}, c_{it}) is said to be admissible for the initial endowment $x_i \geq 0$ for agent i's optimization problem if the wealth process X_{it} satisfies*

$$X_{it} \geq 0, \quad \forall\, t \in [0, \infty) \quad a.s.$$

1.3. **Equilibrium.** Each agent will be considered to be a price taker. This implies an Arrow-Debreu type equilibrium concept.

Definition 2. *An equilibrium in this economy is defined by a set of processes $\{r_t, S_t, \{c_{it}, X_{it}, \pi_{it}\}_{i=1}^N\}$ $\forall$ t, given preferences and initial endowments, such that $\{c_{it}, X_{it}, \pi_{it}\}$ solve the agents' individual optimization problems and the following set of market clearing conditions is satisfied:*

$$\frac{1}{N}\sum_i c_{it} = D_t \ , \ \frac{1}{N}\sum_i (1-\pi_{it})X_{it} = 0 \ , \ \frac{1}{N}\sum_i \pi_{it}X_{it} = S_t \tag{1.4}$$

I consider Markovian equilibria where the problem can be written as a function of some finite number of state variables.

2. Equilibrium Characterization

To solve this problem I first use the martingale method to show how the SDF can be written as a function of a single state variable. I then use the hamilton-jacobi-bellman (HJB) equation to derive a system of ordinary differential equations which determine the portfolio and stock price volatility. Finally, asset prices are found to satisfy an integral equation which can be numerically approximated.

2.1. **The Static Problem.** Following Karatzas and Shreve (1998) we can define the SDF as

$$H_t = exp\left(-\int_0^t r_u du - \int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du\right) \tag{2.1}$$

where

$$\theta_t = \frac{\mu_t + \frac{D_t}{S_t} - r_t}{\sigma_t} \tag{2.2}$$

represents the market price of risk. This implies that the stochastic discount factor also follows a diffusion of the form

$$\frac{dH_t}{H_t} = -r_t dt - \theta_t dW_t \tag{2.3}$$

Following Proposition 2.6 from Karatzas et al. (1987), given an admissible pair (π_{it}, c_{it}) we can rewrite each agent's dynamic problem as a static one beginning at time $t = 0$

$$\begin{aligned} \max_{\{c_{iu}\}_{u=0}^{\infty}} \quad & \mathbb{E}\int_0^{\infty} e^{-\rho u}\frac{c_{iu}^{1-\gamma_i}-1}{1-\gamma_i}du \\ \text{s.t.} \quad & \mathbb{E}\int_0^{\infty} H_u c_{iu} du \leq x_i \end{aligned}$$

If we denote by Λ_i the Lagrange multiplier in individual i's problem, then the first order conditions can be rewritten as

$$c_{it} = \left(e^{\rho t}\Lambda_i H_t\right)^{\frac{-1}{\gamma_i}} \tag{2.4}$$

which holds for every agent in every period. It is important to point out that the Lagrange multiplier is constant in time and a function only of the preference parameter and initial condition: $\Lambda_i = \Lambda(\gamma_i, x_i)$. This will be a key fact in deriving the convergence in N.

Given each agent's first order conditions, we can derive an expression for consumption as a fraction of per-capita dividends.

Proposition 1. *One can define the consumption of individual, i, at any time, t, as a share ω_{it} of the per-capita dividend, D_t, such that*

$$c_{it} = \omega_{it} D_t \tag{2.5}$$

$$\textit{where } \omega_{it} = \frac{N\left(\Lambda_i e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(\Lambda_j e^{\rho t} H_t\right)^{\frac{-1}{\gamma_j}}} \tag{2.6}$$

This expression recalls the results in Basak and Cuoco (1998) or Cuoco and He (1994), where ω_{it} acts like a time-varying Pareto-Negishi weight. In those works, however, participation is driven by an imperfection in the information structure or some exogenous constraint. Here the choice of participation is driven by preferences towards risk. The value of the stochastic discount factor is equal across agents, but differs in its weight for each agent as they differ in risk aversion. This leads one to think that perhaps it would be better to think of this as an incomplete market. If markets were fully complete, there would be a risky asset for each agent, but here agents are forced to bargain over a single asset.

To derive an expression for the risk free rate and the market price of risk, we will need the following lemma about the drift and diffusion of agents' consumption processes:

Lemma 1. *If we model an agent's consumption as a GBM with time varying drift and diffusion coefficients μ_{it}^c and σ_{it}^c, then we have*

$$\begin{aligned} r_t &= \rho + \mu_{it}^c\gamma_i - (1+\gamma_i)\gamma_i\frac{(\sigma_{it}^c)^2}{2} \\ \theta_t &= \sigma_{it}^c\gamma_i \end{aligned}$$

which hold simultaneously for all i.

These formulas are very similar to those one would find in a standard representative agent model. However, these expressions hold simultaneously for all agents, meaning that the growth rate and volatility of consumption for each agent must adjust, while for a representative agent they would be replaced by the drift and diffusion of the dividend process. In order to better understand how these values adjust, rewrite Lemma 1 in terms of μ^c_{it} and σ^c_{it} and differentiate to get

$$\frac{\partial \mu^c_{it}}{\partial \theta_t} = \frac{1+\gamma_i}{\gamma_i^2}\theta_t \tag{2.7}$$

$$\frac{\partial \mu^c_{it}}{\partial r_t} = \frac{1}{\gamma_i} \tag{2.8}$$

$$\frac{\partial \sigma^c_{it}}{\partial \theta_t} = \frac{1}{\gamma_i} \tag{2.9}$$

$$\frac{\partial \sigma^c_{it}}{\partial r_t} = 0 \tag{2.10}$$

Equations (2.7) and (2.8) imply that the growth rate of every individual's consumption is increasing in both the market price of risk and in the interest rate. All things being equal, holding portfolios and preferences constant, a higher market price of risk implies greater returns. Thus, any given agent will earn more on their portfolio and can expect a higher (or less negative) growth rate in consumption. However, the magnitude of this effect depends both on the prevailing market price of risk and the agent's preferences.

In particular, consider Equation (2.8). Every agent's expected growth rate in consumption is increasing in the interest rate. This makes sense for agents who are net lenders, as they see greater returns on their savings, but this is counter-intuitive for agents who are net borrowers. It implies that, despite having to pay a higher interest rate on their borrowing they prefer to grow their consumption more quickly. This is driven by a wealth effect. An increase in the interest rate lowers the stochastic discount factor, reducing the price of consumption today and in the future. A higher interest rate implies a lower present value of lifetime consumption, whether an agent is a lender or borrower. This makes the budget constraint less binding for both. Because markets are complete, agents borrow solely to finance their consumption choices. So the loosening of the budget constraint will cause an increase in consumption growth rates for all agents despite their financial position.

Finally, Equations (2.9) and (2.10) imply that diffusion in consumption is increasing in the market price of risk, but this effect is decreasing in γ_i, while changes in the interest rate have no effect on consumption volatility. First, Equation (2.10) states that a change in the risk free rate has no effect on the volatility of any agent's consumption, except for its indirect effect on the market price of risk. Agents need to be compensated for volatility in their consumption stream and that compensation comes only from risky assets. Second, Equation (2.9) is decreasing in γ_i as a more risk averse agent will respond less to changes in the market; more risk averse agents desire a smoother consumption path. However, why consumption co-moves positively with the market price of risk is unclear. In order to understand this effect, we need to understand the determinants of the market price of risk.

2.2. **The Risk-Free Rate and Market Price of Risk.** Given Lemma 1, we can derive expressions for the market price of risk and the risk free rate:

Proposition 2. *The interest rate and market price of risk are fully determined by the sufficient statistics* $\xi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}$ *and* $\phi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i^2}$ *such that*

$$r_t = \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2}\frac{\xi_t + \phi_t}{\xi_t^3}\sigma_D^2 \tag{2.11}$$

$$\theta_t = \frac{\sigma_D}{\xi_t} \tag{2.12}$$

Proposition 2 is in terms of only certain moments of the empirical joint distribution of consumption shares and risk aversion: $\xi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}$ and $\phi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i^2}$. These represent the first and second moments of the distribution of elasticity of intertemporal substitution (EIS) with respect to consumption shares. In other words, an agent's preferences only effect the market clearing interest rate and market price of risk up to their amount of participation in the market for consumption.

In (2.12), we can see that the market price of risk in the heterogeneous economy is equal to the market price of risk that would prevail in a representative agent economy populated by an agent whose elasticity of inter-temporal substitution is equal to the consumption weighted average in our economy. This is because the market price of risk is determined by agents choosing the diffusion of their consumption. In the face of shocks each agent will increase or decrease their consumption such that the diffusion of their consumption is equal the market price of risk scaled down by their risk aversion (see Lemma 1).

Looking at (2.11), the first two terms are very reminiscent of the interest rate in a representative agent economy populated by the same agent that would determine the market price of risk. That is if we were to use a representative agent model where the agent's CRRA parameter satisfied $\frac{1}{\gamma} = \xi_t$ we would find the same market price of risk and nearly the same interest rate. We can rewrite Equation (2.11) as the interest rate that would prevail in our hypothetical economy plus an extra term:

$$r_t = \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2}\frac{\xi_t + 1}{\xi_t^2}\sigma_D^2 - \frac{1}{2}\frac{1}{\xi_t}\left(\frac{\phi_t}{\xi_t^2} - 1\right)\sigma_D^2$$

If it were the case that $\phi_t = \xi_t^2$, then this additional term would be zero and the interest rate and market price of risk in this model could be exactly matched by those in an economy populated by a representative agent with time varying risk aversion, similar to the model of habit formation by Campbell and Cochrane (1999). However, we can apply the discrete version of Jensen's inequality to show that $\phi_t > \xi_t^2, \quad \forall\, t < \infty$. This causes the additional term to be strictly negative. The risk free rate is then lower than it would be in an economy populated by a representative agent. This introduces a sort of "heterogeneity wedge", which I'll define as $\frac{\phi_t}{\xi_t^2} > 1$, between the price of risk and the price for risk free borrowing. the larger the difference between ξ_t^2 and ϕ_t the greater the wedge. This wedge is also one plus the coefficient of variation squared in the effective distribution of EIS. The more diverse the consumption shares of individual agents with respect to the elasticity of intertemporal substitution, the greater the wedge. The driving force behind the heterogeneity wedge is the market segmentation that occurs when agents differ in their preferences towards risk.

2.3. **Market Segmentation.** When this economy is populated by two or more agents who have different values of γ, the markets for risky and risk free assets will never clear at the same level and will generate a market segmentation involving three distinct groups. Define $\{\gamma_{rt}, \gamma_{\theta t}\}$ to be the RRA parameters in a representative agent economy that would produce the same interest rate and market price of risk, respectively:

$$r_t = \rho + \gamma_{rt}\mu_D - \gamma_{rt}(1+\gamma_{rt})\frac{\sigma_D^2}{2}$$
$$\theta_t = \gamma_{\theta t}\sigma_D$$

Equating these expressions to those in Proposition 2 we can solve for these preference levels, such that

$$\gamma_{rt} = \frac{\mu_D}{\sigma_D^2} - \frac{1}{2} - \sqrt{\left(\frac{\mu_D}{\sigma_D^2}\right)^2 - \frac{\mu_D}{\sigma_D^2}\left(1+\frac{2}{\xi_t}\right) + \frac{\xi_t+\phi_t}{\xi_t^3} + \frac{1}{4}}$$
$$\gamma_{\theta t} = \frac{1}{\xi_t}$$

Finally, with a bit of algebra, it can be shown that $\gamma_{rt} < \gamma_{\theta t}$, $\forall t < \infty$. This implies that the markets for risky and risk-free assets do not coincide in finite t. Additionally, it shows that the two markets overlap (see Figure 2). This implies a sort of market segmentation with three groups: leveraged investors, diversifying investors, and saving divestors.

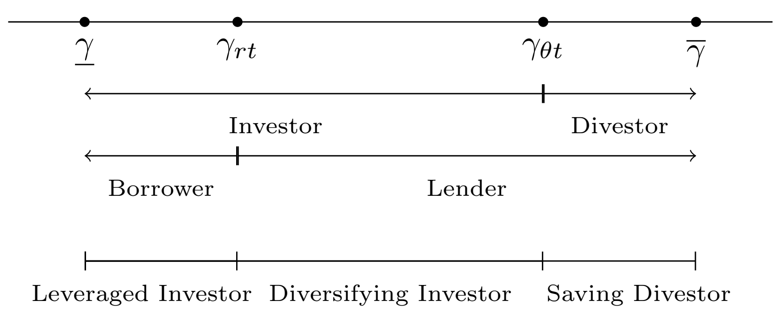


FIGURE 2. The market is segmented depending on an agent's preferences relative to the representative agent solution. The markets for risky and risk-free assets do not coincide and there are three segments. Agents with low risk aversion are simultaneously borrowing and investing. Agents with middling risk aversion are lending and investing. Agents with high risk aversion are saving and divesting.

The three market segments in this economy represent buyers and sellers of risky and risk-free assets. Agents who have low risk aversion will sell bonds in order to buy a larger share in the risky asset. Agents with a middling level of risk aversion will be purchasing both bonds and shares in the risky asset. They do this by capitalizing their gains in the risky asset. As we'll see in Section 4, as the low risk aversion agents dominate the market, they drive up asset prices, producing high returns. The diversifying investors capitalize these gains in both the risky and the risk-free assets. Finally, agents with high risk aversion will be purchasing bonds and shrinking their share in the risky asset. In general, these agents are simply exchanging with the low risk aversion agents their risky shares for bonds. This causes the risky asset to be concentrated amongst the low risk aversion investors as the economy grows and pushes up asset prices, again as we'll see in Section 4.

2.4. **Consumption Weight Dynamics.** We can study the dynamics of an agent's consumption weight by applying Itô's lemma to the expression given in Proposition 1.

Proposition 3. *Assuming consumption weights also follow a geometric Brownian motion such that*

$$\frac{d\omega_{it}}{\omega_{it}} = \mu_{it}^{\omega} dt + \sigma_{it}^{\omega} dW_t$$

an application of Itô's lemma to (2.6) *gives expressions for* μ_{it}^{ω} *and* σ_{it}^{ω}*:*

$$\mu_{it}^{\omega} = (r_t - \rho)\left(\frac{1}{\gamma_i} - \xi_t\right) \tag{2.13}$$
$$+ \frac{\theta_t^2}{2}\left[\left(\frac{1}{\gamma_i^2} - \phi_t\right) - 2\xi_t\left(\frac{1}{\gamma_i} - \xi_t\right) + \left(\frac{1}{\gamma_i} - \xi_t\right)\right]$$

$$\sigma_{it}^{\omega} = \theta_t\left(\frac{1}{\gamma_i} - \xi_t\right) \tag{2.14}$$

Consider first the case where an agent's preferences coincide with the weighted average, ie $\gamma_i = \gamma_{\theta t} = \frac{1}{\xi_t}$ (as in Section 2.3). In (2.14), which describes how an agent's consumption weight co-varies with the risk process, $\sigma_{it}^{\omega} = 0$. If an agent has the same EIS as the market then they will not desire to vary their consumption weight in the face of shocks. As in the analysis of the previous sub-section, this is because the agent is perfectly in agreement with the market. However, notice that in this case $\mu_{it}^{\omega} = \theta_t^2\left(\frac{1}{\gamma_{\theta t}^2} - \phi_t\right) = \sigma_D^2\left(1 - \frac{\phi_t}{\xi_t^2}\right)$, by Proposition 2. This is an indicator of the speed with which the economy is moving through this equilibrium. Although the agent is instantaneously satisfied with the current market price of risk, they are deterministically moving out of this position. The speed with which this is occurring is driven by the heterogeneity wedge, $\frac{\phi_t}{\xi_t^2}$. When this wedge is high, the rate at which the marginal agent moves out of the marginal position is greater.

Next consider the case where an agent is more patient than the weighted average, that is $\gamma^i > \gamma_{\theta t}$. Then $\sigma_{it}^{\omega} < 0$ and agent i's weight is negatively correlated to the market. This implies that if an agent is more patient than the average, or alternatively more risk averse, then their consumption share will increase when there are negative shocks and decrease when there are positive shocks. This is a prudence motive and these agents can be thought of as playing a "buy low, sell high" strategy for consumption. They do not want to grow their consumption faster than the economy, but to pad their position against future shocks. They smooth consumption over time, providing a very stable consumption path. For this reason, their decisions are driven not by a desire to increase their consumption today, but to insure themselves against shocks in the distant future.

Conversely, if an agent is less risk averse than the average, ie $\gamma^i < \gamma_{\theta t}$, their consumption shares covary positively with the market. These agents are essentially buying high and selling low, a strategy that will cause their wealth to be highly volatile. An agent with a lower risk aversion has a higher elasticity of intertemporal substitution and, thus, can be thought of as less patient. Given a shock to the dividend process, the expected growth rate remains constant, but the level shifts permanently because of the martingale property of the Brownian motion. Since less patient agents see the current output of the dividend as more important than its long-run behavior, present shocks have a greater effect on their personal price. Thus, a negative shock causes them to reduce their price and in turn their consumption shares, while a positive shock causes them to increase their price and consumption share. These are the day-traders, riding booms and busts to try to make a quick buck while not losing their shirts. Although they may benefit in the short fun, their consumption will be more volatile than the economy.

The analysis of (2.13) is quite difficult for the case of $\gamma_i \neq \gamma_{\theta t}$. The first term is the product of two separate terms: one involving the interest rate and rate of time preference, the other the agent's position in the distribution. If the interest rate is above the rate of time preference, the first term is positive. If the interest rate differs from the rate of time preference then the agent should desire to shift consumption across time periods, either from today to tomorrow or vice versa. However, the direction will be determined by their preference. If $\gamma_i > \gamma_{\theta t}$ then the product will be negative and this first term will contribute negatively to their growth rate μ^{ω}_{it}. The opposite is true when $\gamma_i < \gamma_{\theta t}$. The combined effect of these two terms is to say that if an agent is less patient than the average and the interest rate is greater than their rate of time preference, they will want to grow their consumption faster than the rate of growth in the economy, while if they are more patient than the average then they will tend to grow their consumption more slowly than the rate of growth in the economy. This effect is only partial, however, and it is necessary to take into consideration the second term.

The second term is quite a bit more complex. The term in brackets is a sort of quadratic in deviations from the weighted average of risk aversion. Whether this term is positive or negative depends in a complicated way on ξ_t and ϕ_t [5]. It is sufficient to note that, when the distribution is not too skewed, there exists a level of risk aversion such that if an agent is above this the second term in (2.13) is negative and that this level of risk aversion is not equal to $\gamma_{\theta t}$ or γ_{rt}. This is related to the deterministic nature of the shifting distribution of asset holdings. Although these two preference levels represent the instantaneous market clearing levels, they do not reflect how the distribution is evolving over time. This has important consequences for rates of return, which are driven by changes in the consumption weights over time.

2.5. **Asset Prices and Portfolios.** Asset prices and portfolios can be derived via a combination of the HJB and the martingale method. In fact it can be shown (and verified) that the individual maximization problem can be formulated as a function of only two state variables, individual wealth and the dividend process. Recall the first order condition Equation (2.4) and substitute this into the market clearing condition for consumption:

$$\frac{1}{N}\sum_i \left(e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}} = D_t$$

[5] It can be shown that the roots of this quadratic are $\frac{1}{\gamma_i} = \xi_t - \frac{1}{2} \pm \sqrt{\frac{1}{4} + \phi_t - \xi_t^2}$. Because $\phi_t > \xi_t^2$, there will always be two real roots. However, whether these are positive or negative depends on the values of ξ_t and ϕ_t.

Through a bit of algebra, it can be shown that the above implies that $\omega_{it} = f_i(D_t)$, where $f_i(\cdot)$ is an implicit function satisfying

$$\frac{1}{N}\sum_j \lambda_{ji}^{\frac{-1}{\gamma_j}} f_i(z)^{\frac{\gamma_i}{\gamma_j}} z^{\frac{\gamma_j-\gamma_i}{\gamma_j}} = 1 \text{ where } \lambda_{ji} = \frac{\Lambda_j}{\Lambda_i} \text{ is constant}$$

for all individuals i. Because the sdf is determined as a function of r_t and θ_t, which are in turn determined by $\{\omega_{it}\}_{i=1}^N$, this shows that the sdf is a function only of the dividend process D_t. Given this, it is natural to look for a solution to this problem in the two state variables (X_{it}, D_t). However, given the tractability of CRRA preferences, the value functions are seperable and the dependence on wealth disappears, giving a solution in a single state variable. This solution is given in Proposition 4:

Proposition 4. *Assuming there exists a Markovian equilibrium in D_t, the individuals' wealth-consumption ratios, given by $V_i(D_t)$ satisfy ODE's given by*

$$\begin{aligned} 0 =& 1 + \frac{\sigma_D^2 D_t^2}{2} V''(D_t) + \left[\frac{1-\gamma_i}{\gamma_i}\theta_t\sigma_D + \mu_D\right] D_t V'(D_t) \\ &+ \left[(1-\gamma_i)r_t - \rho + \frac{1-\gamma_i}{2\gamma_i}\theta_t^2\right]\frac{V(D_t)}{\gamma_i} \end{aligned} \tag{2.15}$$

which satisfy boundary conditions

$$V_i(0) = \frac{\gamma_i}{\rho - (1-\gamma_i)\left(\frac{\theta(0)^2}{2} + r(0)\right)} \qquad V_i(\infty) = \frac{\gamma_i}{\rho - (1-\gamma_i)\left(\frac{\theta(\infty)^2}{2} + r(\infty)\right)} \tag{2.16}$$

while portfolios are given as functions of $V_i(D_t)$:

$$\pi_{it} = \frac{1}{\gamma_i \sigma_t}\left(\gamma_i \sigma_D D_t \frac{V'(D_t)}{V(D_t)} + \theta_t\right) \tag{2.17}$$

And finally σ_t is given by

$$\sigma_t = \sigma_D\left(1 + D_t \frac{\mathcal{S}_t'(D_t)}{\mathcal{S}_t(D_t)}\right) \tag{2.18}$$

where

$$\mathcal{S}_t(D_t) = \frac{1}{N}\sum_i V_i(D_t)\omega_{it} \tag{2.19}$$

One may notice in Proposition 4 that there is no dependence on a stochastic distribution, as is common in models of heterogeneous agents with common shocks. In fact, individual optimization problems are nearly decoupled, with the only dependence coming through the pricing variables r_t, θ_t, and σ_t, which depend only on the dividend. This is driven by the static nature of heterogeneity. Given the fixed initial distribution of preferences and wealth, individual heterogeneity remains fixed and each agent's decision problem evolves as a standard control problem. This provides great tractability to the solution and, in fact, it is possible to write the price dividend ratio $\mathcal{S}_t$ as a semi-analytic integral equation:

Proposition 5. *The price dividend ratio can be expressed as*

$$\mathcal{S}_t = \frac{f_i(D_t)^{\gamma_i}}{\sigma_D\sqrt{2b}}\int_0^\infty \left[f_i(D_t e^u)^{-\gamma_i} e^{\psi_- u} + f_i(D_t e^{-u})^{-\gamma_i} e^{-\psi_+ u}\right] du \tag{2.20}$$

where $\psi_{\pm} = \frac{a \pm \sqrt{2b}}{\sigma_D}$, $b = \rho + \frac{a^2}{2\sigma_D^2} - (1-\gamma_i)\left(\mu_D - \frac{\sigma_D^2}{2}\gamma_i\right)$, *and* $a = \mu_D - \frac{\sigma_D^2}{2}(2\gamma_i - 1)$ *for all* i.

These equations recall the results in Chabakauri (2013, 2015)[6], but characterizes the equilibrium in a slightly different way. In particular, instead of taking consumption weights as the state variable (which would make the dimension of the problem very large), we can keep the dimension low by studying equilibrium taking only the dividend as a state. As stated in the previously mentioned work and reiterated here, this is not to say this is the only Markovian equilibrium (and in fact represents a proof to the contrary), but can be shown by the same steps to represent an optimum, showing at least existence.

3. Extension to Infinite Types

Consider now the limiting case as $N \to \infty$. This corresponds to a special type of mean field game with common noise, where the idiosyncratic volatility is degenerate. That is, although there are two degrees of randomness in the model corresponding to the random initial condition and the Brownian motion, there is no idiosyncratic risk process. Agents' states evolve idiosyncratically because of their heterogeneous preferences, but are subject only to a common noise. This implies for a given level of wealth and a given preference level, γ, all agents will have the same control. This fact is similar to symmetry in permutations of the state in Lasry and Lions (2007) and other papers on mean field games, but one can think of the preference parameter as being a degenerate state variable, i.e. $d\gamma = 0$. Additionally, because the constraint is determined by initial wealth, one can consider heterogeneity in the initial condition as being the key driver of the mean field. This characteristic makes the model dependent on the initial condition and the realization of the Brownian motion. Because of this the mean field will be with respect to the control and the determinant distribution will be over the initial condition.

If we take the consumption weights, $\omega_{it} = \omega_t(\gamma_i, x_i)$, we have a function of an empirical mean:

$$\omega_t(\gamma_i, x_i) = \frac{N\left(\Lambda(\gamma_i, x_i)e^{\rho t}H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(\Lambda(\gamma_j, x_j)e^{\rho t}H_t\right)^{\frac{-1}{\gamma_j}}}$$

By the strong law of large numbers (assuming the variance in consumption across agent types is bounded), the empirical average converges to the mean with respect to the distribution of the initial condition:

$$\omega_t(\gamma_i, x_i) \underset{N\to\infty}{\to} \omega_t(\gamma, x) = \frac{\left(\Lambda(\gamma, x)e^{\rho t}H_t\right)^{\frac{-1}{\gamma}}}{\int \left(\Lambda(\gamma, x)e^{\rho t}H_t\right)^{\frac{-1}{\gamma}} dG(\gamma, x)}$$

Although rather logical when viewed through the lens of work on a continuum of agents à la Aumann (1964), the market clearing condition for consumption weights implies something intriguing about their relationship to the initial distribution. If we think of $\omega_t(\gamma, x)$ as a ratio of probability measures, then the consumption weights act as the Radon-Nikodym derivative of a stochastic measure with respect

[6]...and I have to express my infinite gratitude for the insight provided by the author.

to the distribution of the initial condition. That is, define $\omega_t(\gamma, x) = \frac{dP_t(\gamma,x)}{dG(\gamma,x)}$. Then we have

$$\int \omega_t(\gamma, x) dG(\gamma, x) = \int \frac{dP_t(\gamma, x)}{dG(\gamma, x)} dG(\gamma, x) = \int dP_t(\gamma, x) = 1$$

The evolution of this distribution would be difficult to describe directly, but the expressions in Proposition 3 give the dynamics of this stochastic distribution. So $\omega_t(\gamma, x)$ allows one to calculate exactly the evolution of this stochastic distribution by use of a change of measure. Alternatively, one can think of $\omega_t(\gamma, x)$ as a sort of importance weight, where as the share of risky assets is concentrated towards one area in the support, the weight of this area grows in the determination of asset prices.

Additionally, the Radon-Nikodym interpretation allows one to compare the continuous types to finite types. Say for instance we would like to discretize the above expression for the market clearing condition on $\omega_t(\gamma, x)$ using a Riemann sum with an evenly space partition (e.g. a midpoint rule):

$$\int \omega_t(\gamma, x) dG(\gamma, x) \approx \frac{(\overline{\gamma} - \underline{\gamma})(\overline{x} - \underline{x})}{JK} \sum_{k=1}^{K} \sum_{j=1}^{J} \omega_t(\gamma_k, x_j) g(\gamma_k, x_j)$$

This looks quite similar to the market clearing conditions in the finite type model (Equation (1.4)). So, could we construct a finite types sample that matches this approximation, at least initially? Make the identification $N = JK$ and notice that since $\omega_t(\gamma, x)$ is a geometric Brownian motion, such that $\omega_t(\gamma, x) = \omega_0(\gamma, x)\hat{\omega}_t(\gamma, x)$ where $\hat{\omega}_t(\gamma, x)$ is a stochastic process. If we define the initial condition on omega as $\omega_0(\gamma, x) = \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})g(\gamma,x)}$, then the above market clearing condition becomes

$$\int \omega_t(\gamma, x) dG(\gamma, x) \approx \frac{1}{N} \sum_{k=1}^{K} \sum_{j=1}^{J} \hat{\omega}_t(\gamma_k, x_j)$$

This market clearing condition looks exactly like the condition in Equation (1.4). However, this has particular implications about the Radon-Nikodym derivative. From the definition of the Radon-Nikodym derivative we can write

$$P_t(A) = \int_A \omega_t(\gamma, x) dG(\gamma, x) = \int_A \omega_t(\gamma, x) g(\gamma, x) d\gamma dx$$

Substituting the imposed definition of $\omega_t(\gamma, x)$ we have

$$P_t(A) = \int_A \frac{\hat{\omega}_t(\gamma, x)}{(\overline{\gamma} - \underline{\gamma})(\overline{x} - \underline{x})} d\gamma dx$$

Now, since $\hat{\omega}_0(\gamma, x) = 1$, the above implies

$$P_0(A) = \int_A \frac{1}{(\overline{\gamma} - \underline{\gamma})(\overline{x} - \underline{x})} d\gamma dx$$

Thus the initial condition of the stochastic measure $P_0(A)$ is a uniform distribution.

All of this to say that if one attempts to approximate the continuous model by a finite model (in particular an evenly spaced grid) not taking into account the initial distribution $g(\gamma, x)$, one can only generate a certain initial condition: the product distribution $\omega_0(\gamma, x)g(\gamma, x) = \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})}$. One could also attempt to use a Monte-Carlo scheme, sampling many agents from the initial distribution. However, the variance will be large for low values of N and convergence will be determined by the particular sample, as we'll see in Section 4.

The continuous types model encompasses the discrete types model completely, in that if the true distribution $g(\gamma, x)$ were a discrete distribution one could get identical results. On the other hand, the continuous types model seems more complex at first glance. Although this seeming addition of complexity provides little in the way of new economic insight beyond an arbitrary number of agents, it does provide several nice explicit modeling tools. First, the joint distribution of initial wealth and risk aversion is explicitly modeled. In a model of finite types over an evenly spaced grid one can only model a product distribution such that $\omega_0(\gamma, x)g(\gamma, x)$ is uniform, while other assumptions about finite types will produce similar unexpected difficulties. Second, but closely related, is the computational simplification provided by the continuum. For finite types one must simulate many agents in order to match some distribution of preferences. Because the main drivers of financial variables in this model are moments of the distribution of risk preferences, one can simulate quadrature points to approximate a continuous distribution, whereas to do the same for the finite types model would require many simulated agents. This fact will become quite apparent in Section 4. Finally, the continuum points to coherently model the distribution of risk preferences if one believes there to be many preference types. It is hard to believe that, given how much heterogeneity there exists in all aspects of our society, there could exist only two types of risk preferences.

4. Simulation Results and Analysis

In this section, I review the simulation strategy as well as some simulation results and compare them. The underlying assumption in all of these simulations is that I am attempting to approximate a continuous distribution of types (for a recent survey on estimating risk preferences see Barseghyan et al. (2015) and for evidence on heterogeneity see Chiappori et al. (2012)). One could argue that this is the goal of any model of heterogeneous risk preferences with finite types, as in Dumas (1989), Chabakauri (2015), or Chabakauri (2013), but that one assumes finite types for tractability and with the hope that the results generalize. This section will try to convince you that, quantitatively, the results for the continuous types model are not the same as those for the finite types model and that the continuous types model is in fact more tractable than the finite types model. For details on the solution method, see Appendix B.

For all of the simulations, I will hold the following group of parameters fixed at the given values: $\mu_D = 0.03$, $\sigma_D = 0.06$, and $\rho = 0.01$. These settings correspond to a yearly parameterization. Solutions are represented over the state space, $D_t \in [0, \infty)$, which is truncated for clarity as most of the action is in the lower regions. Additionally, assume that all agents begin with the same initial wealth to make the results more easily studied. Given that markets are complete, this has no effect on the outcomes of the model, on aggregate.

4.1. Increasing Agents.

Increasing Agents. Initially, assume preferences are distributed uniformly over $[1.5, 10.0]$, making an evenly spaced finite types model a logical approximation.

This along with the single wealth type amounts to

$$dG(\gamma, x) = \frac{\delta_{x*} d\gamma dx}{\overline{\gamma} - \underline{\gamma}}$$

Consider changing the number of agents over the support. I'll consider the cases where $N = 2, 5, 10$ and 100. The motivation behind these to consider whether results for two agents generalize to many types. You will see the change in the number of types has a significant change in the level of all aggregate variables, implying that misspecification of the support of the distribution of risk preferences has a non-trivial effect on the model's short run predictions about market variables.

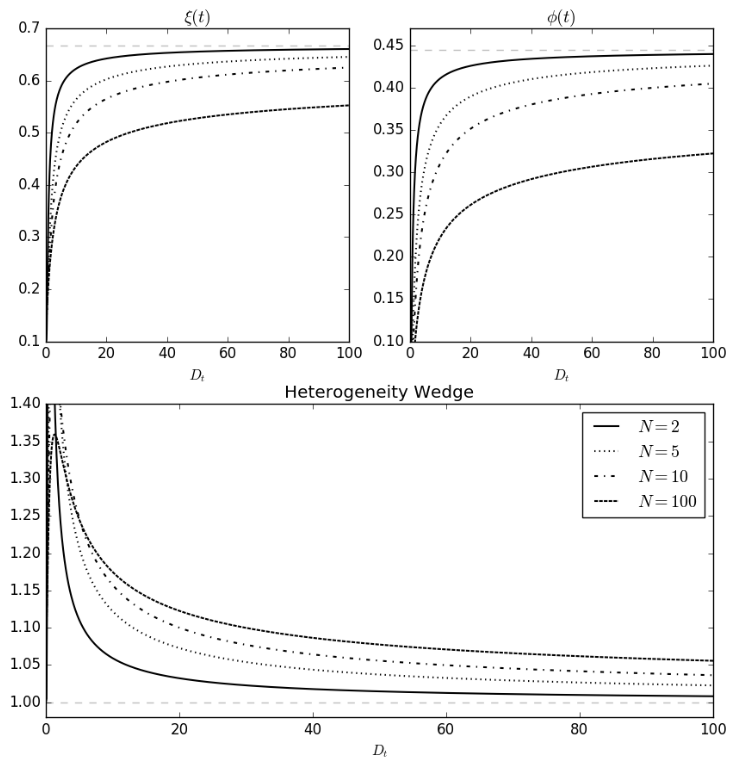


FIGURE 3. Sufficient statistics for the distribution of the risk aversion with finite types, where $\xi_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega_{it}}{\gamma_i}$ and $\phi_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega_{it}}{\gamma_i^2}$, and the heterogeneity wedge is defined as $\frac{\phi_t}{\xi_t^2}$. N corresponds to the number of types.

Consider first the distribution of consumption shares across individuals, which can be summed up by the sufficient statistics ξ_t and ϕ_t, as shown in Figure 3. You will immediately notice that the level, slope, and curvature of both of these measures changes substantially for different numbers of preference types over the given support. Additionally you'll notice that the heterogeneity wedge, defined as $\frac{\phi(t)}{\xi(t)^2}$ is also changing. This is driven by the introduction of a greater mass of agents, making it harder for a single agent to dominate, and by the fact that these are moments of non-linear transformations of γ_i. By Jensen's inequality we know that, although we do not change the expected value of γ_i, we do change the expected value of any non-linear transformation. Beyond this, the more agents there are, the more total mass there is in the economy. Although the averages are what is important for levels, changing the mass of agents over the support changes the rate of convergence, as in order for the most risk neutral agent to dominate they must accumulate a greater consumption share to bring ξ_t and ϕ_t to the same long run level. Additionally, both variables fall as we increase the number of agents and are converging very slowly for larger numbers of agents.

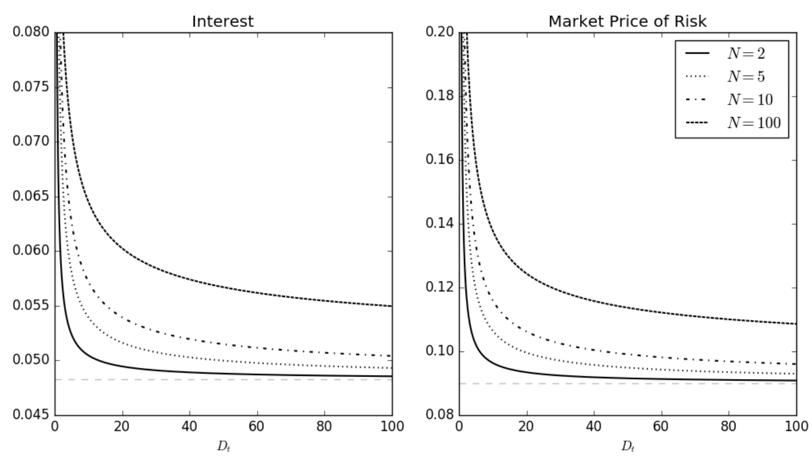


FIGURE 4. Interest rate and market price of risk for different numbers of agents.

These facts combine to cause the interest rate and market price of risk to be higher for more agents. In Figure 4 you can see that the underlying assumptions of the model imply that changing the support of the distribution of risk aversion causes a shift in levels, slope, and curvature of the interest rate and market price of risk. For two types the convergence is incredibly fast, in that the two key drivers of the sdf have almost zero volatility for very low values of the dividend process. As we increase the number of agents, both variables rise and their convergence becomes slower, implying a more long-lived volatility in asset pricing variables. This is consistent with what we observe in financial markets, in that the interest rate and market price of risk do indeed vary with output.

Higher levels of the interest rate and market price of risk drive down asset prices, while variance in these parameters increases asset price volatility. This can be seen in Figure 5, where we note that dividend yield is the inverse of the price-dividend ratio. When there are more preference types over a given support, this drives up the average EIS, reducing asset prices. Additionally, as it takes longer for a single preference type to dominate the economy, small shocks create larger changes in asset prices, increasing the volatility.

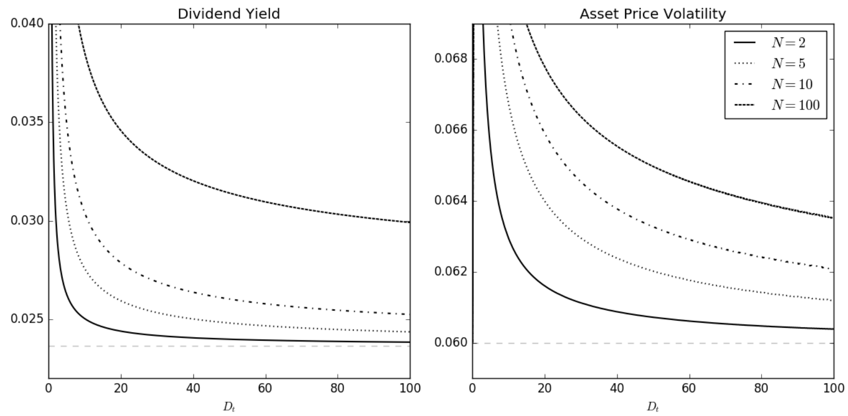


FIGURE 5. Dividend yield and volatility for different numbers of agents.

Beyond simply the level, it is interesting to note the non-linear response of volatility to changes in the dividend. In particular, a negative shock to the dividend process increases volatility, while a positive shock reduces volatility. This points to one possible explanation to the short-coming often noted in the celebrated work of Black and Scholes (1973), namely constant volatility. There has been much interest in mathematical finance in studying models with time-varying or stochastic volatility (see Lorig and Sircar (2016) for a nice review or Guyon and Henry-Labordère (2013) for a thorough mathematical treatment of several approaches), but few attempts to micro-found such models. This model produces such stochastic volatility and, in particular, greater downside volatility. Perhaps this represents a micro-foundation for the observed "volatility smile" so important in options pricing.

On the bond side, we see a non-zero and stochastic supply of credit, creating "leverage cycles". As you can see in Figure 6, increasing the number of agents increases leverage. This is driven by two forces. First, more preference types means a larger supply of bonds, so total borrowing increases. At the same time, asset prices are falling, reducing the value of agents' collateral and further increasing borrowers' leverage. This implies that this type of heterogeneity could make macro-financial models more realistic in their predictions about leverage. However, notice that a rise in D_t generates a fall in leverage and vice-versa. This implies a countercyclical leverage cycle, the opposite of that postulted by Geanakoplos (2010). This is driven here by a difference in the response of borrowing and asset prices. Both are positively correlated with the dividend, but since asset prices are more volatile a fall in the dividend causes a fall in borrowing and a greater fall in asset prices (see Figure 7), which increases total leverage which is defined as total borrowing over total wealth.

In summation, if one believes there is a continuum of types then the way that one discretizes or bins this distribution into a finite support has a substantial effect on the model's outcome. This is the driving interest in studying continuous types model, as explicitly modeling the distribution will provide stability to simulations with a comparable computational cost.

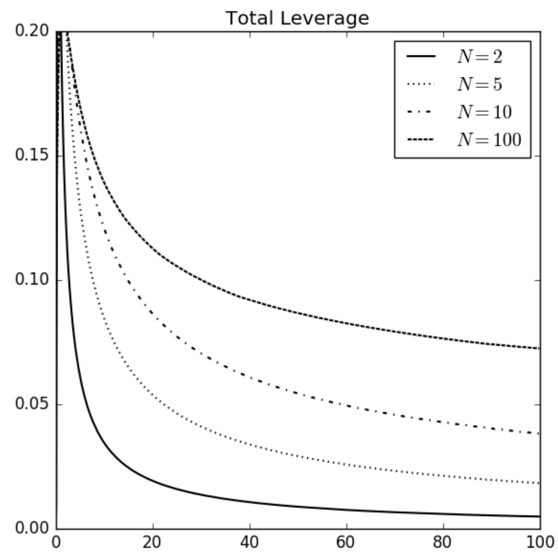


FIGURE 6. Total financial leverage for different numbers of agents.

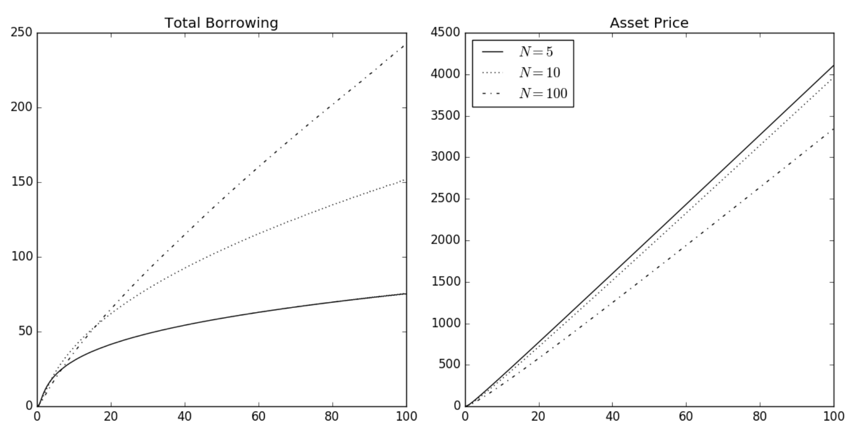


FIGURE 7. Total borrowing and asset prices for different numbers of agents.

4.2. **Finite Types versus Continuous Types.** For the continuous types model we must approximate the integrals in some way, here choosing to use a trapezoidal rule for comparability. As an example, consider the definition of $\xi(t)$ and its associated quadrature approximation:

$$\xi_t = \int \frac{\omega(t,\gamma,x)}{\gamma} dG(\gamma,x) \approx \frac{\overline{\gamma}-\underline{\gamma}}{2(M-1)} \sum_{m=1}^{M-1} \left[\frac{\omega_t(\gamma_m)g(\gamma_m)}{\gamma_m} + \frac{\omega_t(\gamma_{m+1})g(\gamma_{m+1})}{\gamma_{m+1}} \right]$$

where (γ_m) is an evenly spaced grid. For finite types, changing the number of simulated points changes the distribution $g(\gamma)$ in the model, while for the continuous types solution, changing the number of quadrature points does not change the assumptions about $g(\gamma)$, but only affects the accuracy of the approximation. This will be the key feature that differentiates the two solutions. Although the qualitative features will be similar, the robustness of the continuous types solution will be far superior to that of finite types and we will have greater freedom in the definition of the distribution of preferences.

For continuity, consider the same uniform distribution as before. In this case the results look almost identical as the finite types case. In Figure 10 you can see that the interest rates are very similar. This is driven by the integral approximations and the fact that we are using a uniform distribution over preferences. In this case, the above integral approximation become

$$\xi_t \approx \frac{1}{2(M-1)} \sum_{m=1}^{M-1} \left[\frac{\omega_t(\gamma_m)}{\gamma_m} + \frac{\omega_t(\gamma_{m+1})}{\gamma_{m+1}} \right]$$

which is almost identical to the definition of ξ_t under finite types. This is tightly linked to the point made in Section 3, in that we could *only* match a uniform initial distribution. To see this we can look at robustness results for different assumptions about the distribution of preferences and different methods of approximation.

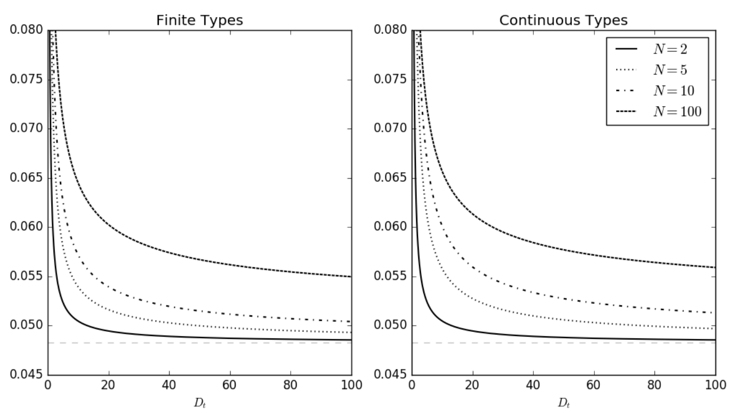
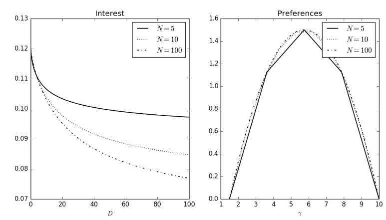
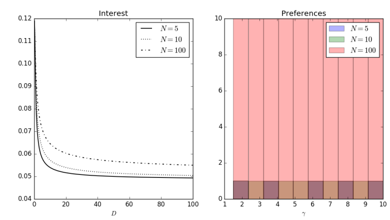


FIGURE 8. Interest rates for different numbers of agents and quadrature points, respectively, under the assumption of a uniform distribution of preferences.

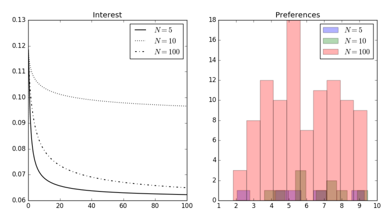
4.3. **Robustness.** Consider now the case where preferences follow a Beta(2, 2) distribution over the same support. In this case, we know exactly how to calculate the solution to the continuous types model, but could take several approaches to solving the finite types model in order to approximate this setting. First, we could consider taking the same naive uniform approximation, fixing a uniform distribution of consumption weights over the same support. Second, we could consider initializing the consumption weights to match the same initial condition in the continuous types case, i.e. $\omega_0(\gamma) = g(\gamma)$. Finally, we could attempt to use a monte-carlo approximation, drawing many agents from the distribution $g(\gamma)$. As a representative variable, the interest rate found using each of these methods, along with that obtained from the continuous types model and the associated preference distributions, are presented in Figure 9.



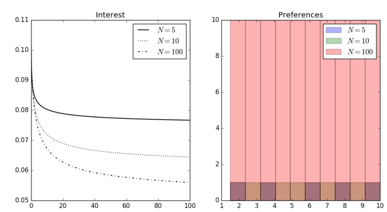
(a) Continuous Types



(b) Uniform



(c) Monte Carlo



(d) Initial Condition

FIGURE 9. Interest rate and operative preference distribution for continuous types and several finite types approximations under a Beta(2, 2) distribution.

You'll notice that the solutions are substantially different. First, as one would expect, a uniform distribution of agents is a poor approximation to a non-uniform distribution of preferences. Second, the monte-carlo approximation converges to the continuous types solution very slowly, as evidenced by the histogram representing the sample of agents. Finally, one might think that changing the initial condition in $\omega_t(\gamma_i)$ to match the distribution of preferences one could recover the same solution. However, as was pointed out in Section 3, this simultaneously changes the assumptions about the distribution $g(\gamma)$, producing a noticeably different interest rate.

Finally, it is also interesting to note how the assumption of a Beta distribution over preferences changes the outcome for financial variables. You'll see in Figure 10 that not only is volatility substantially higher for a longer period of time, but leverage is as well. Additionally, we find an inflection point in leverage as a function of the dividend. For lower values of D_t, a negative shock reduces leverage, implying a pro-cyclical leverage cycle. This difference with the uniform distribution case is driven by the volatility in borrowing (Figure 11). In order to produce a pro-cyclical leverage cycle, total borrowing must fall more quickly than total wealth. Total borrowing is more volatile because the interest rate is more volatile. A negative shock increases the interest rate, making borrowing more expensive, and the greater this change, the greater the reduction in borrowing. For large enough changes in the interest rate, the change in total borrowing will outweigh the change in asset prices.

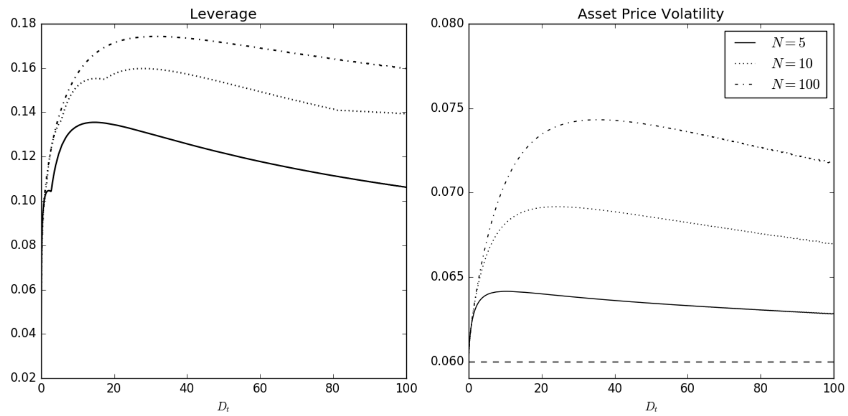


FIGURE 10. Leverage and volatility for different numbers of quadrature points assuming continuous types distributed beta(2, 2).

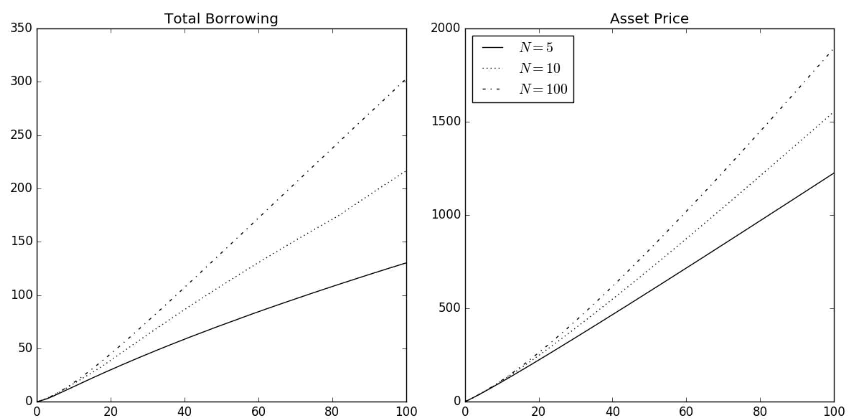


FIGURE 11. Total borrowing and asset prices for different numbers of quadrature points.

5. Conclusion

In this paper I have studied how the distribution of risk preferences affects financial variables, consumption shares, and portfolio decisions. The distribution of risk preferences has a large effect on financial variables driven mainly by consumption weighted averages of the EIS. The implication is that the amount of participation by individuals in the market for consumption, not the market for risky assets, determines to what degree their preferences affect price. In fact, the evolution of individual shares is determined by each agent's relative position to consumption weighted averages of the EIS and its square. Given the heterogeneity in preferences, markets for risk free bonds and risky assets clear at different levels, implying three groups. Leveraged investors have low risk aversion and borrow in order to grow their share in consumption. Saving divestors are highly risk averse and lend in order to shrink their share in consumption. Somewhere in the middle we have the diversifying investor, who is growing their share in consumption and simultaneously lending by buying bonds.

Outcomes are driven by a heterogeneity wedge[7] which describes how different are the market clearing risk free rate and market price of risk. This value can also be thought of as the squared coefficient of variation plus one in the weighted distribution of EIS. When this wedge is high, corresponding to two very different marginal investors and/or a diverse group of investors, asset prices are low, interest rates are high, and dividend yields are high. Conversely, when this wedge is low, corresponding to a concentration of consumption shares towards a single agent, asset prices are high, interest rates are low, and dividend yield is low.

The model can produce both pro- and counter-cyclical leverage cycles depending on the distribution of preferences. The cyclicality of the leverage cycle is driven by the volatilities of total borrowing and asset prices. When total borrowing is more volatile than asset prices, then leverage cycles are pro-cyclical, and vice-versa. In order for total borrowing to be volatile, there needs to be a large mass of lenders and a very risk neutral group of borrowers. This will generate volatility in the interest rate and, in turn, volatility in borrowing.

Additionally, dividend yield in this model is co-moves negatively with the growth rate in dividends. This implies a predictable component in stock market returns. A negative shock to this economy implies a shift of the distribution of consumption shares towards more risk averse agents. This reduces asset prices and predicts a faster growth rate in the asset prices in the future. We know from the numerical solution that economies with a lower weighted average of EIS will have a higher rate of return on risky assets. Papers such as Campbell and Shiller (1988a), Campbell and Shiller (1988b), Mankiw (1981), and Hall (1979) drew differing conclusions about the standard model of asset prices, but, broadly speaking, they all deduced that there was some portion of asset prices that was slightly predictable as a function of the growth rate in aggregate consumption. In the model presented here, we can take a step towards explaining this predictability as the dividend yield co-moves with the SDF.

Finally, I've shown how to extend the finite types model to one of a continuum of types. The results are reminiscent of theoretical work on Mean Field Games (MFG) with common noise, such as Carmona et al. (2014). However, this model takes a novel approach to solving such a MFG model by applying the Martingale Method, a typical tool in mathematical finance. The feature which makes this particular model so tractable is the dependence on the initial condition. This, in turn, is driven by market completeness. Agents seek to grow their consumption at some rate relative to the growth rate in the economy and do so by accumulating financial assets. They can accumulate assets by borrowing essentially without limit[8]. An interesting direction for future research would be to carry this approach over to incomplete markets, as in Chabakauri (2015), to study how borrowing constraints would affect the accumulation of assets and market dynamics.

[7]Defined as $\frac{\phi(t)}{\xi(t)^2} = \frac{\sum \frac{\omega^i(t)}{\gamma_i^2}}{\left[\sum \frac{\omega^i(t)}{\gamma_i}\right]^2}$ or $\frac{\int \frac{\omega(t,\gamma,x)}{\gamma^2} dF(\gamma,x)}{\left[\int \frac{\omega(t,\gamma,x)}{\gamma} dF(\gamma,x)\right]^2}$ in the continuous types case.

[8]CRRA preferences rule out the chance of default and agents are always able to borrow.

Appendix A. Proofs

Proof of Proposition 1. Taking ratios of consumption first order conditions for two arbitrary agents, i and j we find

$$\frac{c_{it}}{c_{jt}} = \Lambda_j^{\frac{1}{\gamma_j}} \Lambda_i^{\frac{-1}{\gamma_i}} \left(H_t e^{\rho t}\right)^{\frac{1}{\gamma_j} - \frac{1}{\gamma_i}}$$

To solve for the consumption weight of an individual i, take the market clearing condition in consumption and divide through by agent i's consumption

$$\begin{aligned} & \frac{1}{N}\sum_{j=1}^{N} c_{jt} = D_t \\ \Leftrightarrow \quad & \frac{\frac{1}{N}\sum_{j=1}^{N} c_{jt}}{c_{it}} = \frac{D_t}{c_{it}} \\ \Leftrightarrow \quad & c_{it} = \frac{c_{it}}{\frac{1}{N}\sum_{j=1}^{N} c_{jt}} D_t \\ \Leftrightarrow \quad & c_{it} = \left(\frac{N\left(e^{\rho t}\Lambda_i H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(e^{\rho t}\Lambda_j H_t\right)^{\frac{-1}{\gamma_j}}} \right) D_t \\ \Leftrightarrow \quad & c_{it} = \omega_{it} D_t \end{aligned}$$

□

Proof of Lemma 1. Modeling consumption as a geometric Brownian motion implies that for every agent i the consumption process can be described by the stochastic differential equation

$$\frac{dc_{it}}{c_{it}} = \mu_{it}^c dt + \sigma_{it}^c dW_t \tag{A.1}$$

Armed with this knowledge, take the first order condition for an arbitrary agent i's maximization problem, solve for H_t, and apply Itô's lemma:

$$\begin{aligned} H_t &= \frac{1}{\Lambda_i} c_{it}^{-\gamma_i} e^{-\rho t} \\ \Rightarrow \frac{dH_t}{H_t} &= \left(-\rho - \gamma_i \mu_{it}^c + \gamma_i(1+\gamma_i)\frac{(\sigma_{it}^c)^2}{2}\right) dt - \left(\gamma_i \sigma_{it}^c\right) dW_t \end{aligned}$$

Now, match coefficients to those in (**??**) to find

$$\begin{aligned} r_t &= \rho + \gamma_i \mu_{it}^c - \gamma_i(1+\gamma_i)\frac{(\sigma_{it}^c)^2}{2} \\ \theta_t &= \gamma_i \sigma_{it}^c \end{aligned}$$

Solving for μ_{it}^c and σ_{it}^c gives

$$\begin{aligned} \mu_{it}^c &= \frac{r_t - \rho}{\gamma_i} + \frac{1+\gamma_i}{\gamma_i^2}\frac{\theta_t^2}{2} \\ \sigma_{it}^c &= \frac{\theta_t}{\gamma_i} \end{aligned}$$

□

Proof of Proposition 2. Recall the definition of consumption dynamics in (A.1) and the market clearing condition for consumption in (1.4). Apply Itô's lemma to the market clearing condition:

$$
\begin{aligned}
&\frac{1}{N}\sum_i c_{it} = D_t \Rightarrow \frac{1}{N}\sum_i dc_{it} = dD_t \\
\Leftrightarrow &\frac{1}{N}\sum_i^N \left(c_{it}\mu_{it}^c dt + c_{it}\sigma_{it}^c dW_t\right) = D_t\mu_D dt + D_t\sigma_D dW_t \\
\Leftrightarrow &\frac{\frac{1}{N}\sum_i \left(c_{it}\mu_{it}^c dt + c_{it}\sigma_{it}^c dW_t\right)}{D_t} = \mu_D dt + \sigma_D dW_t \\
\Leftrightarrow &\frac{1}{N}\sum_i \omega_{it}\mu_{it}^c dt + \frac{1}{N}\sum_i \omega_{it}\sigma_{it}^c dW_t = \mu_D dt + \sigma_D dW_t
\end{aligned}
$$

By matching coefficients we find

$$
\begin{aligned}
\mu_D &= \frac{1}{N}\sum_i \omega_{it}\mu_{it}^c \\
\sigma_D &= \frac{1}{N}\sum_i \omega_{it}\sigma_{it}^c
\end{aligned}
$$

Now use Lemma 1 to substitute the values for consumption drift and diffusion, then solve for the interest rate and the market price of risk to find

$$
\begin{aligned}
\theta_t &= \frac{\sigma_D}{\xi_t} \\
r_t &= \frac{\mu_D}{\xi_t} + \rho - \frac{1}{2}\frac{\xi_t + \phi_t}{\xi_t^3}\sigma_D^2
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_t &= \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i} \\
\phi_t &= \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i^2}
\end{aligned}
$$

□

Proof of Proposition 3. Assume that consumption weights follow a geometric Brownian motion given by

$$
\frac{d\omega_{it}}{\omega_{it}} = \mu_{it}^\omega dt + \sigma_{it}^\omega dW_t \tag{A.2}
$$

Recall the definition of consumption weights in (2.6) and gather terms:

$$
\begin{aligned}
\omega_{it} &= \frac{\left(\Lambda^i e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(\Lambda_j e^{\rho t} H_t\right)^{\frac{-1}{\gamma_j}}} \\
\Leftrightarrow \omega^i(t) &= \left[\sum_j \Lambda_j^{\frac{-1}{\gamma_j}} \Lambda_i^{\frac{1}{\gamma_i}} \left(e^{\rho t} H_t\right)^{\frac{1}{\gamma_i} - \frac{1}{\gamma_j}}\right]^{-1}
\end{aligned} \tag{A.3}
$$

Recall the definition of Itô's lemma, where $\omega^i(t)$ is a function of $H_0(t)$ and t:

$$d\omega_{it} = \frac{\partial \omega_{it}}{\partial t}dt + \frac{\partial \omega_{it}}{\partial H_t}dH_t + \frac{1}{2}\frac{\partial^2 \omega_{it}}{\partial H_t^2}(dH_t)^2$$

Substituting for dH_t and using the Itô box calculus to see that $(dH_t)^2 = H_t^2\theta_t^2 dt$, we see that

$$\frac{d\omega_{it}}{\omega_{it}} = \frac{1}{\omega_{it}}\left(\frac{\partial \omega_{it}}{\partial t} - r_t H_t \frac{\partial \omega_{it}}{\partial H_t} + H_t^2\theta_t^2 \frac{1}{2}\frac{\partial^2 \omega_{it}}{\partial H_t^2}\right)dt - \theta_t \frac{1}{\omega_{it}}\frac{\partial \omega_{it}}{\partial H_t}dW_t$$

Matching coefficients with those in (A.2) it is clear that

$$\begin{aligned}
\mu^\omega_{it} &= \frac{1}{\omega_{it}}\left(\frac{\partial \omega_{it}}{\partial t} - r_t H_t \frac{\partial \omega_{it}}{\partial H_t} + H_t^2\theta_t^2 \frac{1}{2}\frac{\partial^2 \omega_{it}}{\partial H_t^2}\right)\\
\sigma^\omega_{it} &= -\theta_t \frac{1}{\omega_{it}}\frac{\partial \omega_{it}}{\partial H_t}
\end{aligned}$$

Differentiating the expression in (A.3), carrying out some painful algebra, and simplifying gives

$$\begin{aligned}
\mu^\omega_{it} &= (r_t - \rho)\left(\frac{1}{\gamma_i} - \xi_t\right) + \frac{\theta_t^2}{2}\left[\left(\frac{1}{\gamma_i^2} - \phi_t\right) - 2\xi_t\left(\frac{1}{\gamma_i} - \xi_t\right) + \left(\frac{1}{\gamma_i} - \xi_t\right)\right]\\
\sigma^\omega_{it} &= \theta_t\left(\frac{1}{\gamma_i} - \xi_t\right)
\end{aligned}$$

□

Proof of Propostition 4. Assume there exists a Markovian equilibrium in D_t. Then an individual's Hamilton-Jacobi-Bellman (HJB) equation writes

$$\begin{aligned}
0 = \max_{c_{it},\pi_{it}} \Bigg\{ & e^{-\rho t}\frac{c_{it}^{1-\gamma_i} - 1}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it}\left(r_t + \pi_{it}\sigma_t\theta_t\right) - c_{it}\right]\frac{\partial J_{it}}{\partial X_{it}}\\
&+\mu_D D_t \frac{\partial J_{it}}{\partial D_t} + \sigma_D\sigma_t\pi_{it}D_t X_{it}\frac{\partial^2 J_{it}}{\partial X_{it}\partial D_t} + \frac{1}{2}\left[X_{it}^2\pi_{.it}^2\sigma_t^2\frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2 \frac{\partial^2 J_{it}}{\partial D_t^2}\right]\Bigg\}
\end{aligned} \tag{A.4}$$

subject to the transversality condition $\mathbb{E}_t J_{it} \to 0$ for all i s.t. $\gamma_i > \underline{\gamma}$, as the agent with the lowest risk aversion will dominate in the long run (Cvitanić et al. (2011)). First order conditions imply

$$c_{it} = \left(e^{-\rho t}\frac{\partial J_{it}}{\partial X_{it}}\right)^{\gamma_i} \tag{A.5}$$

$$\pi_{it} = -\left(X_{it}\sigma_t \frac{\partial^2 J_{it}}{\partial X_{it}^2}\right)^{-1}\left[\theta_t\frac{\partial J_{it}}{\partial X_{it}} + \sigma_D D_t \frac{\partial^2 J_{it}}{\partial X_{it}\partial D_t}\right] \tag{A.6}$$

Assume that the value function is seperable as

$$J_{it}(X_{it}, D_t) = e^{-\rho t}\frac{X_{it}^{1-\gamma_i}V_i(D_t)^{\gamma_i}}{1-\gamma_i} \tag{A.7}$$

Substituting Equation (A.7) into Equations (A.5) and (A.6) gives

$$c_{it} = \frac{X_{it}}{V_i(D_t)} \tag{A.8}$$

$$\pi_{it} = \frac{1}{\gamma_i\sigma_t}\left(\gamma_i\sigma_D D_t \frac{V'(D_t)}{V(D_t)} + \theta_t\right) \tag{A.9}$$

which shows that $V_i(D_t)$ is the wealth-consumption ratio as a function of the dividend. Next, substitute Equations (A.7) to (A.9) into Equation (A.4) and simplify to find

$$\begin{aligned} 0 =& 1 + \frac{\sigma_D^2 D_t^2}{2} V''(D_t) + \left[\frac{1-\gamma_i}{\gamma_i}\theta_t\sigma_D + \mu_D\right] D_t V'(D_t) \\ &+ \left[(1-\gamma_i)r_t - \rho + \frac{1-\gamma_i}{2\gamma_i}\theta_t^2\right]\frac{V(D_t)}{\gamma_i} \end{aligned} \tag{A.10}$$

Which gives an ode for the wealth-consumption ratio. On the boundaries $D = 0$ and $D \to \infty$, the most risk averse and the least risk averse agent dominates, respectively (Cvitanić et al. (2011)). The boundary conditions correspond to the value functions for individuals when prices are set by these dominant agents.

Define the price-dividend ratio as a function of the single state variable: $\mathcal{S}_t(D_t) = \frac{S_t}{D_t}$. Apply Itô's lemma to $D_t\mathcal{S}_t = S_t$ and match coefficients to find

$$\begin{aligned} \mu_t &= D_t^2 + \frac{(\sigma_D D_t)^2}{2}\frac{\mathcal{S}_t''(D_t)}{\mathcal{S}_t(D_t)} D_t + D_t\mu_D + \frac{\mathcal{S}_t'(D_t)}{\mathcal{S}_t(D_t)}(\sigma_D D_t)^2 \\ \sigma_t &= \sigma_D\left(1 + D_t\frac{\mathcal{S}_t'(D_t)}{\mathcal{S}_t(D_t)}\right) \end{aligned}$$

Taking the market clearing condition for wealth, rewrite $\mathcal{S}_t(D_t)$ as a function of D_t:

$$\begin{aligned} S_t &= \frac{1}{N}\sum_i X_{it} \\ \Leftrightarrow \frac{S_t}{D_t} &= \frac{1}{N}\sum_i \frac{X_{it}}{D_t} \\ \Leftrightarrow \mathcal{S}_t(D_t) &= \frac{1}{N}\sum_i \frac{X_{it}}{c_{it}}\frac{c_{it}}{D_t} \\ &= \frac{1}{N}\sum_i V(D_t)\omega_{it} \end{aligned}$$

which gives $\mathcal{S}_t(D_t)$ given that $\omega_{it} = f_i(D_t)$

□

Proof of Propostition 5. The price-dividend ratio is given by the discounted sum of future cash flows:

$$\mathcal{S}_t = \frac{S_t}{D_t} = \mathbb{E}_t\int_t^\infty \frac{H_\tau}{H_t}\frac{D_\tau}{D_t}d\tau$$

Substitute for H_t using Equation (2.4) for any i, then re-arrange

$$= \left(\frac{c_{it}}{D_t}\right)^{\gamma_i}\mathbb{E}_t\int_t^\infty e^{-\rho(\tau-t)}\left(\frac{c_{i\tau}}{D_\tau}\right)^{-\gamma_i}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_i}d\tau$$

Now substitute $c_{it}/D_t = \omega_{it} = f_i(D_t)$ and $D_s = D_t \exp\left\{\left(\mu_D - \frac{\sigma_D^2}{2}\right)(s-t) + \sigma_D(W_s - W_t)\right\}$ for all $s \geq t$.

$$= f_i(D_t)^{\gamma_i}\mathbb{E}_t \int_t^\infty \exp\left\{-\rho(\tau - t) + (1-\gamma_i)\left[\left(\mu_D - \frac{\sigma_D^2}{2}\right)(\tau - t) + \sigma_D(W_\tau - W_t)\right]\right\}$$
$$\cdot f_i\left(D_t \exp\left\{\left(\mu_D - \frac{\sigma_D^2}{2}\right)(\tau - t) + \sigma_D(W_\tau - W_t)\right\}\right)^{-\gamma_i} d\tau$$

Introduce a new variable $z = \frac{W_\tau - W_t}{\sqrt{\tau - t}} \sim \mathcal{N}(0,1)$, then change variables to adjust bounds from $[t, \infty)$ to $[0, \infty)$.

$$= \frac{f_i(D_t)^{\gamma_i}}{\sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty \exp\left\{-\rho\tau + (1-\gamma_i)\left[\left(\mu_D - \frac{\sigma_D^2}{2}\right)\tau + \sigma_D\sqrt{\tau}z\right] - \frac{z^2}{2}\right\}$$
$$\cdot f_i\left(D_t \exp\left\{\left(\mu_D - \frac{\sigma_D^2}{2}\right)\tau + \sigma_D\sqrt{\tau}z\right\}\right)^{-\gamma_i} d\tau dz$$

Complete the square:

$$= \frac{f_i(D_t)^{\gamma_i}}{\sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty f_i\left(D_t \exp\left\{\left(\mu_D - \frac{\sigma_D^2}{2}\right)\tau + \sigma_D\sqrt{\tau}z\right\}\right)^{-\gamma_i}$$
$$\cdot \exp\left\{-\frac{1}{2}(z - \sigma_D\sqrt{\tau}(1-\gamma_i))^2\right\} \exp\left\{-\rho\tau + (1-\gamma_i)\left[\mu_D - \frac{\sigma_D^2}{2}\gamma_i\right]\tau\right\} d\tau dz$$

Define a change of variables such that $u = \left(\mu_D - \frac{\sigma_D^2}{2}\right)\tau + \sigma_D\sqrt{\tau}z$ and a constant $a = \mu_D - \frac{\sigma_D^2}{2}(2\gamma_i - 1)$. After simplifying:

$$= \frac{f_i(D_t)^{\gamma_i}}{\sigma_D\sqrt{2\pi}} \int_{-\infty}^\infty f_i\left(D_t e^u\right)^{-\gamma_i} \exp\left\{\frac{a}{\sigma_D^2}u\right\}$$
$$\cdot \int_0^\infty \exp\left\{-\rho\tau - \frac{u^2}{2\sigma_D^2\tau} - \frac{a^2}{2\sigma_D^2} + (1-\gamma_i)\left[\mu_D - \frac{\sigma_D^2}{2}\gamma_i\right]\tau\right\} \frac{1}{\sqrt{\tau}} d\tau du$$

The second integral can be solved explicitly using the formulas 3.471.9 and 8.469.3 from Gradshteyn and Ryzhik (2000), as in Longstaff and Wang (2012); Chabakauri (2015). In particular, write

$$= \frac{f_i(D_t)^{\gamma_i}}{\sigma_D\sqrt{2\pi}} \int_{-\infty}^\infty f_i\left(D_t e^u\right)^{-\gamma_i} \exp\left\{\frac{a}{\sigma_D^2}u\right\} G(u) du$$

such that

$$G(u) = \int_0^\infty \exp\left\{-\rho\tau - \frac{u^2}{2\sigma_D^2\tau} - \frac{a^2}{2\sigma_D^2} + (1-\gamma_i)\left[\mu_D - \frac{\sigma_D^2}{2}\gamma_i\right]\tau\right\} \frac{1}{\sqrt{\tau}} d\tau$$
$$= \sqrt{\frac{\pi}{b}} \exp\left\{-\frac{\sqrt{2b}}{\sigma_D}|u|\right\} \text{ where } b = \rho + \frac{a^2}{2\sigma_D^2} - (1-\gamma_i)\left[\mu_D - \frac{\sigma_D^2}{2}\gamma_i\right]$$

Substituting and simplifying one finds

$$\mathcal{S}_t = \frac{f_i(D_t)^{\gamma_i}}{\sigma_D\sqrt{2b}} \int_{-\infty}^\infty f_i\left(D_t e^u\right)^{-\gamma_i} \exp\left\{\frac{au - \sigma_D\sqrt{2b}|u|}{\sigma_D^2}\right\} du$$

Finally, define $\psi_\pm = \frac{a \pm \sqrt{2b}}{\sigma_D}$, split the integral at zero, evaluate the absolute value, and introduce a change of variable to set the domain of integration to $[0, \infty)$ for both integrals to find

$$\mathcal{S}_t = \frac{f_i(D_t)^{\gamma_i}}{\sigma_D \sqrt{2b}} \int_0^\infty \left[f_i(D_t e^u)^{-\gamma_i} e^{\psi_- u} + f_i(D_t e^{-u})^{-\gamma_i} e^{-\psi_+ u} \right] du$$

which gives the result. □

Appendix B. Numerical Methods

B.1. **Integral Approximation for Asset Prices.** Asset prices can be calculated in several ways in this model (and are in order to assess numerical accuracy of each portion of the solution). The most direct approach is to calculate the integral in Proposition 5:

$$\mathcal{S}_t = \frac{f_i(D_t)^{\gamma_i}}{\sigma_D \sqrt{2b}} \int_0^\infty \left[f_i(D_t e^u)^{-\gamma_i} e^{\psi_- u} + f_i(D_t e^{-u})^{-\gamma_i} e^{-\psi_+ u} \right] du \tag{B.1}$$

The numerical approximation proceeds in two steps, first approximating the function $f_i(D_t)$ and then approximating the integral. The function $f_i(D_t)$ satisfies

$$\frac{1}{N} \sum_j \lambda_{ji}^{\frac{-1}{\gamma_j}} f_i(z)^{\frac{\gamma_i}{\gamma_j}} z^{\frac{\gamma_j - \gamma_i}{\gamma_j}} = 1 \text{ where } \lambda_{ji} = \frac{\Lambda_j}{\Lambda_i} \text{ is constant} \tag{B.2}$$

Solving directly for the Lagrange multipliers Λ_i is technically challenging when starting from a distribution of wealth. However, using the first order conditions Equation (2.4) we can note that $\Lambda_i = c_{it}^{-\gamma_i} e^{-\rho t} H_t^{-1} \ \forall \ i, t$, so when $t = 0$, $\Lambda_i = c_{i0}^{-\gamma_i} \ \forall \ i$. Given this, we can solve the problem up to the wealth-consumption ratio in Proposition 4 and recover the distribution of wealth implied by the initial consumption.

Another particularity arrises in the exponential term within the integral and the semi-infinite domain of integration. In order to calculate a quadrature approximation, one must take an upper bound that is high enough while not getting numerical overflow errors. One benefit of the particular problem is that the integrad converges fairly quickly, but has high curvature on the lower end of the support. Because of that, I use a non-uniform grid where the points are exponentially distributed. To generate such a grid, fix a number K of points in the grid you would like. Set $D_0 = 0$. Then recursively define points in the grid as

$$D_{k+1} = D_k + \min \left\{ h_{max}; \frac{h^*}{p(D_k; \lambda)} \right\}$$

where h_{max} is the maximum step size, h^* is the minimum step size, and $p(\cdot; \lambda)$ is an exponential pdf with parameter λ.

Aside from these two particularities, the approximation proceeds in a fairly straight forward way:

1. Fix an initial distribution of consumption and calculate the Lagrange multiplier weights, λ_{ji}. Choose an index i to use to approximate asset prices.
2. Generate an evenly spaced grid for values of D (Note: the upper bound of this grid must be much higher than the truncated upper bound for the integral, as you need to calculate $f_i(D_t e^u)$.)

(3) For each point in the grid, solve the non-linear equation in Equation (B.2) for $f_i(D_t)$. This paper uses the `hybr` (hybrid) method provided by `scipy.optimize.root`.
(4) Generate an interpolator $\hat{f}_i(D_t)$. This paper uses the `1dinterp` function provided by `scipy.interpolate`.
(5) Generate a non-uniform grid over the domain of integration u using the above algorithm.
(6) For each point in the grid, calculate the integrand in Equation (B.1).
(7) Approximate the integral by a trapezoid rule, given in general by

$$\begin{aligned} I &= \int_a^b \eta(x)dx \\ &\approx \frac{1}{2}\sum_{k=1}^{K-1}(x_{k+1}-x_k)(\eta(x_{k+1})+\eta(x_k)) \end{aligned}$$

(8) Finally approximate Equation (B.1) by multiplying by the remaining factor.

NOTE: This approach is less stable than using the ODE in Proposition 4, given the exponential factor in the integrand. For this reason, this approximation was not used in the results presented in the paper, but were used to verify the two solutions matched.

B.2. **ODE Solution by Finite Difference.** To solve for portfolios and wealth, one needs to solve the ode in Equation (2.15). Given that one can directly calculate the coefficients using the methods given in Appendix B.1, it is possible to directly calculate the finite difference approximation. In the following I suppress the i subscript for clarity. Using a central difference scheme (assuming an evenly spaced grid), the ode for a given i can be approximated as

$$0 = 1 + a(D_k)\frac{V_{k+1}-2V_k+V_{k-1}}{h^2} + b(D_k)\frac{V_{k+1}-V_{k-1}}{2h} + c(D_k)V_k$$

where D_k corresponds to the kth point in the grid, h the step size, $a(D_k) = \sigma_D^2 D_k^2/2$, $b(D_k) = ((1-\gamma_i)\theta(D_k)\sigma_D/\gamma_i + \mu_D)D_k$, $c(D_k) = ((1-\gamma_i)r(D_k) - \rho + (1-\gamma_i)\theta(D_k)^2/(2\gamma_i))/\gamma_i$. This can be rewritten as a system of linear equations:

$$0 = 1 + (x_k - y_k)V_{k-1} + (z_k - 2x_k)V_k + (x_k + y_k)V_{k+1}$$

where $x_k = a(D_k)/h^2$, $y_k = b(D_k)/2h$, and $z_k = c(D_k)$. Combining this system of equations with the boundary conditions in Equation (2.16) one gets a system of $K-2$ equations in $K-2$ unknowns which takes a highly spars structure (Note: This paper takes the approach of fixing $lim_{D\to\infty}V'(D) = 0$, or a reflecting boundary condition. This seems to provide more stability and is confirmed by numerical simulations, although may not be the "best" approximation). This paper uses `scipy.sparse` to build the matrix equation and solve for the value functions.

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