第 4 章量子电路

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本 章内容：

1. 量子计算的基础模型：量子电路模型。
2. 存在一部分通用的门，使得任意的量子计算可以用这些门表示。

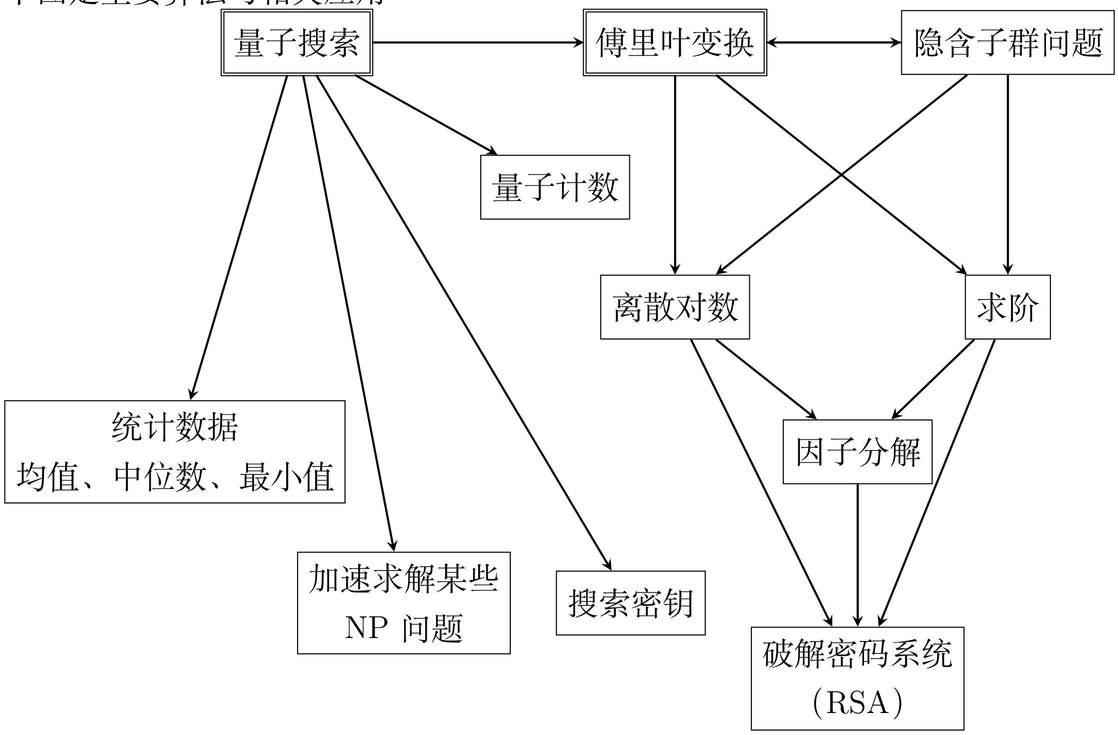
Contents

4.1 量子算法

量子计算机的目标是运行新的算法，使得在经典计算机上需要大量资源去解决的问题成为可能。

已有两类算法实现了这个目标：
1. 基于 Shor 提出的量子傅里叶变换，包括求解因子分解和离散对数问题的著名算法，相比最优的经典算法仍然呈现指数加速。
2. 基于 Grover 提出的量子搜索算法。相对于最优秀的经典算法平方量级加速显著。

下图是主要算法与相关应用



4.2 单量子比特运算

1.3.1 节已有简要的介绍，现在进行一些复习。首先一个单量子比特是由两个复参数构成的向量 $|\psi\rangle = a|0\rangle + b|1\rangle$，满足 $|a|^2 + |b|^2 = 1$。量子比特的操作必须保证范数要求，所以用 2×2 的酉矩阵表示。

证明：

首先，线性算子可以通过矩阵来定义。设某一个算子操作是 A。
则 A 作用在一个单量子比特上的结果是 $A|\psi\rangle$
记为 $|\phi\rangle = A|\psi\rangle$
由于 $|\phi\rangle$ 仍然需要满足范数的要求，所以 $\langle\phi|\phi\rangle = 1$
代入 $|\phi\rangle$ 的定义有 $1 = \langle\phi|\phi\rangle = (A|\psi\rangle, A|\psi\rangle) = A^\dagger A(|\phi\rangle, |\phi\rangle) = A^\dagger A$
同理可得 $A^\dagger A = AA^\dagger = 1$
所以必须是酉矩阵。

1. 重要的矩阵

泡利矩阵：

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

其他重要矩阵：阿达玛门 H，相位门 S，$\pi/8$ 门 T。

$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; S \equiv \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}; T \equiv \begin{bmatrix} 1 & 0 \\ 0 & exp(i\pi/4) \end{bmatrix}$$

重要等式：$H = (X+Z)/\sqrt{2}$ 和 $S = T^2$

2. 布洛赫球面表示

单量子比特 $a|0\rangle + b|1\rangle$ 可视为单位球面上的点 (θ, φ)。对应的关系为 $a = cos(\theta/2)$，$b = e^{i\varphi} sin(\theta/2)$。对应的球面向量（即布洛赫向量）为 $(cos\varphi sin\theta, sin\varphi sin\theta, cos\theta)$

习题 4.1 计算泡利矩阵的特征向量，并找到布洛赫球面上对应不同泡利矩阵的归一化特征向量的点。

X 的特征向量是 $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ 和 $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$。

对应于布洛赫球面的向量为 $\theta = 90°, \varphi = 0°, (1,0,0)$ 和 $\theta = 90°, \varphi = 180°, (-1,0,0)$

Y 的特征向量是 $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$ 和 $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix}$。

对应于布洛赫球面的向量为 $\theta = 90°, \varphi = 90°, (0,1,0)$ 和 $\theta = 90°, \varphi = 270°, (0,-1,0)$

Z 的特征向量是 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 和 $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$。

对应于布洛赫球面的向量为 $\theta = 0°, (0,0,1)$ 和 $\theta = 180°, (0,0,-1)$

3. 旋转算子

泡利矩阵出现在指数上的时候，是关于 $\hat{x}, \hat{y}, \hat{z}$ 的旋转算子。定义为以下方程：

$$R_x(\theta) \equiv e^{-i\theta X/2} = cos\frac{\theta}{2}I - isin\frac{\theta}{2}X = \begin{bmatrix} cos\frac{\theta}{2} & -isin\frac{\theta}{2} \\ -isin\frac{\theta}{2} & cos\frac{\theta}{2} \end{bmatrix}$$

$$R_y(\theta) \equiv e^{-i\theta Y/2} = cos\frac{\theta}{2}I - isin\frac{\theta}{2}Y = \begin{bmatrix} cos\frac{\theta}{2} & -sin\frac{\theta}{2} \\ sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{bmatrix}$$

$$R_z(\theta) \equiv e^{-i\theta Z/2} = cos\frac{\theta}{2}I - isin\frac{\theta}{2}Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

习题 4.2 x 为实数，A 为满足 $A^2 = I$ 的矩阵，证明：

$$exp(iAx) = cos(x)I + isin(x)A$$

证明：

$$\begin{aligned} exp(iAx) &= \sum_{k=0}^{\infty} \frac{1}{k!}(iAx)^k \\ &= \sum_{k=0}^{\infty} (\frac{1}{2k!}(iAx)^{2k} + \frac{1}{(2k+1)!}(iAx)^{2k+1}) \\ &= \sum_{k=0}^{\infty} \frac{1}{2k!}(iAx)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(iAx)^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{1}{2k!}(-1)^k A^{2k} x^{2k} + \sum_{k=0}^{\infty} \frac{i}{(2k+1)!}(-1)^k x^{2k+1} A^{2k+1} \\ &= I \sum_{k=0}^{\infty} \frac{1}{2k!}(-1)^k x^{2k} + iA \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k x^{2k+1} \\ &= cos(x)I + isin(x)A \end{aligned}$$

习题 4.3

$$T = exp(i\pi/8) \begin{bmatrix} exp(i\pi/8) & 0 \\ 0 & exp(-i\pi/8) \end{bmatrix}$$

习题 4.4

$$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$R_x(\theta)=\begin{bmatrix}cos\frac{\theta}{2} & -isin\frac{\theta}{2}\\-isin\frac{\theta}{2} & cos\frac{\theta}{2}\end{bmatrix}$$

$$R_z(\theta)=\begin{bmatrix}e^{-i\theta/2} & 0\\0 & e^{i\theta/2}\end{bmatrix}$$

$R_z(\theta)$ 不改变矩阵元素的模长。所以关于 $\hat{x}$ 的旋转算子用的是

$$R_x(\pi/2)=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\-i & 1\end{bmatrix}$$

$$\begin{aligned}R_z(\pi/2)R_x(\pi/2)R_z(\pi/2)&=e^{-i\frac{\pi}{4}}\begin{bmatrix}1 & 0\\0 & i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\-i & 1\end{bmatrix}e^{-i\frac{\pi}{4}}\begin{bmatrix}1 & 0\\0 & i\end{bmatrix}\\&=\frac{e^{-i\frac{\pi}{2}}}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\end{aligned}$$

4. 绕任意单位向量的旋转

$\hat{n}=(n_x,n_y,n_z)$。定义关于 $\hat{n}$ 轴转角为 θ 的旋转的形式为

$$R_n(\theta)\equiv e^{-i\theta\hat{n}\vec{\sigma}/2}=cos\frac{\theta}{2}I-isin\frac{\theta}{2}(n_xX+n_yY+n_zZ)$$

其中 $\vec{\sigma}=(X,Y,Z)$

习题 4.5 证明 $(\hat{n}\vec{\sigma})^2 = I$。并由此式证明上式。

证明:

$$\begin{aligned}(\hat{n}\vec{\sigma})^2 =&(n_xX+n_yY+n_zZ)(n_xX+n_yY+n_zZ)\\ =&n_x^2X^2+n_y^2Y^2+n_z^2Z^2\\ &+n_xn_y(XY+YX)+n_zn_y(ZY+YZ)+n_xn_z(XZ+ZX)\\ =&(n_x^2+n_y^2+n_z^2)I\end{aligned}$$

上式根据

$$\begin{aligned}XY =&iZ, YX=-iZ, YZ=iX, ZY=-iX, ZX=iY, XZ=-iY\\ X^2 =&Y^2=Z^2=I\end{aligned}$$

得到，由此 4.8 可证。

习题 4.6 证明单一量子比特状态由布洛赫向量 $\vec{\lambda}$ 表示的时候，旋转算子 $R_{\hat{n}}(\theta)$ 对于该状态的作用是在布洛赫球面上关于 $\hat{n}$ 轴旋转角度 θ 的效果。

证明:

假设在布洛赫球面上关于 $\hat{n}$ 轴旋转角度 θ 的效果的算子为 $R(\hat{n},\theta)$

$$\begin{aligned}|\psi\rangle =&cos(\frac{\alpha}{2})|0\rangle+e^{i\varphi}sin(\frac{\alpha}{2})|1\rangle\\ &(sin(\alpha)cos(\varphi), sin(\alpha)sin(\varphi), cos(\alpha))\\ R(\hat{n},\theta) =&R_z(\varphi)R_y(\alpha)R_z(\theta)R_y(-\alpha)R_z(-\varphi)\\ =&(cos(\varphi/2)I-isin(\varphi/2)Z)(cos(\alpha/2)I-isin(\alpha/2)Y)\\ &(cos(\theta/2)I-isin(\theta/2)Z)(cos(\alpha/2)I+isin(\alpha/2)Y)\\ &(cos(\varphi/2)I+isin(\varphi/2)Z)\\ =&(A_1-B_1)(A_2-B_2)(A_3-B_3)(A_2+B_2)(A_1+B_1)\\ =&A_1^2A_2^2A_3-A_1^2A_2^2B_3+2A_1^2A_2B_2B_3+2A_1A_2^2B_2B_3\\ &-A_1^2A_3B_2^2-A_2^2A_3B_1^2+A_1^2B_2B_3B_2+A_2^2B_1B_3B_1+4A_1A_2B_1B_3B_2\\ &-2A_2B_1B_2B_3B_1-2A_1B_1B_2B_3B_2+A_3B_1B_2^2B_1-B_1B_2B_3B_2B_1\end{aligned}$$

$$
\begin{aligned}
=&cos(\varphi/2)^2cos(\alpha/2)^2cos(\theta/2)I - icos(\varphi/2)^2cos(\alpha/2)^2sin(\theta/2)Z\\
&- 2cos(\varphi/2)^2cos(\alpha/2)sin(\alpha/2)Ysin(\theta/2)Z\\
&- 2cos(\varphi/2)cos(\alpha/2)^2sin(\alpha/2)Ysin(\theta/2)Z\\
&+ cos(\varphi/2)^2cos(\theta/2)(sin(\alpha/2)Y)^2 + cos(\alpha/2)^2cos(\theta/2)(sin(\varphi/2)Z)^2\\
&- icos(\varphi/2)^2sin(\alpha/2)Ysin(\theta/2)Zsin(\alpha/2)Y\\
&- i(cos(\alpha/2))^2sin(\varphi/2)Zsin(\theta/2)Zsin(\varphi/2)Z\\
&- 2cos(\alpha/2)sin(\varphi/2)Zsin(\alpha/2)Ysin(\theta/2)Zsin(\varphi/2)Z\\
&- 2cos(\varphi/2)sin(\varphi/2)Zsin(\alpha/2)Ysin(\theta/2)Zsin(\alpha/2)Y\\
&+ cos(\theta/2)sin(\varphi/2)Z(sin(\alpha/2)Y)^2sin(\varphi/2)Z\\
&- isin(\varphi/2)Zsin(\alpha/2)Ysin(\theta/2)Zsin(\alpha/2)Ysin(\varphi/2)Z\\
&- 4icos(\varphi/2)cos(\alpha/2)sin(\varphi/2)Zsin(\theta/2)Zsin(\alpha/2)Y\\
=&cos(\varphi/2)^2cos(\alpha/2)^2cos(\theta/2)I\\
&- icos(\varphi/2)^2cos(\alpha/2)^2sin(\theta/2)Z\\
&- 2icos(\varphi/2)^2cos(\alpha/2)sin(\alpha/2)sin(\theta/2)X\\
&- 2icos(\varphi/2)cos(\alpha/2)^2sin(\alpha/2)sin(\theta/2)X\\
&+ cos(\varphi/2)^2cos(\theta/2)(sin(\alpha/2))^2I + cos(\alpha/2)^2cos(\theta/2)(sin(\varphi/2))^2I\\
&+ icos(\varphi/2)^2sin(\alpha/2)sin(\theta/2)sin(\alpha/2)Z\\
&- i(cos(\alpha/2))^2sin(\varphi/2)sin(\theta/2)sin(\varphi/2)Z\\
&+ 2icos(\alpha/2)sin(\varphi/2)sin(\alpha/2)sin(\theta/2)sin(\varphi/2)X\\
&+ 2cos(\varphi/2)sin(\varphi/2)sin(\alpha/2)sin(\theta/2)sin(\alpha/2)I\\
&+ cos(\theta/2)sin(\varphi/2)(sin(\alpha/2))^2sin(\varphi/2)I\\
&+ isin(\varphi/2)sin(\alpha/2)sin(\theta/2)sin(\alpha/2)sin(\varphi/2)Z\\
=&cos(\varphi/2)^2cos(\alpha/2)^2cos(\theta/2)I\\
&+ 2cos(\varphi/2)sin(\varphi/2)sin(\alpha/2)sin(\theta/2)sin(\alpha/2)I\\
&+ cos(\theta/2)sin(\varphi/2)(sin(\alpha/2))^2sin(\varphi/2)I\\
&+ cos(\varphi/2)^2cos(\theta/2)(sin(\alpha/2))^2I + cos(\alpha/2)^2cos(\theta/2)(sin(\varphi/2))^2I\\
&- 2icos(\varphi/2)^2cos(\alpha/2)sin(\alpha/2)sin(\theta/2)X\\
&- 2icos(\varphi/2)cos(\alpha/2)^2sin(\alpha/2)sin(\theta/2)X\\
&+ 2icos(\alpha/2)sin(\varphi/2)sin(\alpha/2)sin(\theta/2)sin(\varphi/2)X\\
&+ icos(\varphi/2)^2sin(\alpha/2)sin(\theta/2)sin(\alpha/2)Z\\
&- i(cos(\alpha/2))^2sin(\varphi/2)sin(\theta/2)sin(\varphi/2)Z\\
&+ isin(\varphi/2)sin(\alpha/2)sin(\theta/2)sin(\alpha/2)sin(\varphi/2)Z\\
&- icos(\varphi/2)^2cos(\alpha/2)^2sin(\theta/2)Z\\
&- 4icos(\varphi/2)cos(\alpha/2)sin(\varphi/2)sin(\theta/2)sin(\alpha/2)Y
\end{aligned}
$$

$$=I(cos\frac{\theta}{2})-isin(\frac{\theta}{2})(sin\alpha cos\varphi X+sin\alpha sin\varphi Y+cos\alpha Z)$$
$$=R_n(\theta)$$

习题 4.7 证明 XYX = -Y , 并以此证明 $XR_y(\theta)X=R_y(-\theta)$

证明：由于 XYX = iZX = i(iY) = -Y

$$\begin{aligned}XR_y(\theta)X&=X(cos(\frac{\theta}{2})I-isin(\frac{\theta}{2})Y)X\\&=cos(\frac{\theta}{2})XX-isin(\frac{\theta}{2})XYX\\&=cos(\frac{-\theta}{2})I-isin(\frac{-\theta}{2})Y\\&=R_y(-\theta)\end{aligned}$$

习题 4.8 证明对于实数 α 和 θ，三维实单位向量 $\hat{n}$，任意一单量子比特酉算子可表示为

$$U=exp(i\alpha)R_{\hat{n}}(\theta)$$

证明:

U $=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$$\begin{aligned}UU^\dagger&=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}\bar{a}&\bar{c}\\\bar{b}&\bar{d}\end{bmatrix}\\&=\begin{bmatrix}a\bar{a}+b\bar{b}&a\bar{c}+b\bar{d}\\c\bar{a}+d\bar{b}&c\bar{c}+d\bar{b}\end{bmatrix}\\&=I\end{aligned}$$

$$a\bar{a}+b\bar{b}=1$$
$$a\bar{c}+b\bar{d}=0$$
$$c\bar{c}+d\bar{d}=1$$
$$a=e^{ip}cos\gamma$$
$$b=e^{iq}sin\gamma$$
$$c=e^{im}sin\beta$$
$$d=e^{in}cos\beta$$

$$e^{ip}cos\gamma e^{-im}sin\beta + e^{iq}sin\gamma e^{-in}cos\beta = 0$$

$$e^{i(p-m-q+n)}cos\gamma sin\beta + sin\gamma cos\beta = 0$$

$$p - m - q + n = \pi$$

$$sin\gamma cos\beta - cos\gamma sin\beta = 0$$

$$sin(\gamma - \beta) = 0$$

$$\gamma = \beta + k\pi$$

$$U = \begin{bmatrix} e^{ip}cos(\beta+k\pi) & e^{i(p-m+n-\pi)}sin(\beta+k\pi) \\ e^{im}sin(\beta) & e^{in}cos(\beta) \end{bmatrix}$$

$$= \begin{bmatrix} e^{i(p+k\pi)}cos(\beta) & e^{i(p-m+n-\pi+k\pi)}sin(\beta) \\ e^{im}sin(\beta) & e^{in}cos(\beta) \end{bmatrix}$$

$$= \begin{bmatrix} e^{i(p+k\pi)}cos(\beta) & -e^{i(p-m+n+k\pi)}sin(\beta) \\ e^{im}sin(\beta) & e^{in}cos(\beta) \end{bmatrix}$$

$$= e^{i\alpha}\begin{bmatrix} e^{i(a-b)}cos(\beta) & -e^{i(c-d)}sin(\beta) \\ e^{i(d-c)}sin(\beta) & e^{i(b-a)}cos(\beta) \end{bmatrix}$$

$$R_{\hat{n}}(\theta) = cos\frac{\theta}{2}I - isin\frac{\theta}{2}(n_xX + n_yY + n_zZ)$$

$$R_{\hat{n}}(\theta) = cos\frac{\theta}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - isin\frac{\theta}{2}(n_x\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + n_y\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + n_z\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})$$

$$= cos\frac{\theta}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - in_xsin\frac{\theta}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - in_ysin\frac{\theta}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - in_zsin\frac{\theta}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= cos\frac{\theta}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - in_zsin\frac{\theta}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - in_xsin\frac{\theta}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + n_ysin\frac{\theta}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} e^{i(a-b)}cos(\beta) & -e^{i(c-d)}sin(\beta) \\ e^{i(d-c)}sin(\beta) & e^{i(b-a)}cos(\beta) \end{bmatrix} = A + B$$

$$A = \begin{bmatrix} e^{i(a-b)}cos(\beta) & 0 \\ 0 & e^{i(b-a)}cos(\beta) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -e^{i(c-d)}sin(\beta) \\ e^{i(d-c)}sin(\beta) & 0 \end{bmatrix}$$

$$cos\frac{\theta}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (A + \bar{A})/2$$

$$- in_zsin\frac{\theta}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = (A - \bar{A})/2$$

$$
\begin{aligned}
-in_x sin\frac{\theta}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} &= (B-\bar{B})/2 \\
n_y sin\frac{\theta}{2}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} &= (B+\bar{B})/2
\end{aligned}
$$

$$
\begin{aligned}
cos\frac{\theta}{2} &= cos(a-b)cos\beta \\
n_z sin\frac{\theta}{2} &= sin(b-a)cos\beta \\
n_x sin\frac{\theta}{2} &= sin(c-d)sin\beta \\
n_y sin\frac{\theta}{2} &= cos(c-d)sin\beta
\end{aligned}
$$

(2).

$$
\begin{aligned}
H &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \\
&= \begin{bmatrix}cos\frac{\pi}{4} & -e^{i\pi}sin\frac{\pi}{4}\\ sin\frac{\pi}{4} & e^{i\pi}cos\frac{\pi}{4}\end{bmatrix} \\
&= e^{i\frac{\pi}{2}}(cos\frac{\pi}{2}I - isin\frac{\pi}{2}(\frac{\sqrt{2}}{2}X+\frac{\sqrt{2}}{2}Z)))
\end{aligned}
$$

(3).

$$
\begin{aligned}
S &= \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} \\
&= \begin{bmatrix}cos0 & -sin0\\ e^{i\pi/2}sin0 & e^{i\pi/2}cos0\end{bmatrix} \\
&= e^{i\frac{\pi}{4}}(cos\frac{\pi}{4}I - isin\frac{\pi}{4}Z)
\end{aligned}
$$

习题 4.9

习题 4.10

证 明:

$$
\begin{aligned}
&R_x(\beta)R_y(\gamma)R_x(\delta)\\
&=\begin{bmatrix} cos(\frac{\beta}{2}) & -isin(\frac{\beta}{2})\\ -isin(\frac{\beta}{2}) & cos(\frac{\beta}{2})\end{bmatrix}\begin{bmatrix} cos(\frac{\gamma}{2}) & -sin(\frac{\gamma}{2})\\ sin(\frac{\gamma}{2}) & cos(\frac{\gamma}{2})\end{bmatrix}\begin{bmatrix} cos(\frac{\delta}{2}) & -isin(\frac{\delta}{2})\\ -isin(\frac{\delta}{2}) & cos(\frac{\delta}{2})\end{bmatrix}\\
&=\begin{bmatrix} cos(\frac{\beta}{2}) & -isin(\frac{\beta}{2})\\ -isin(\frac{\beta}{2}) & cos(\frac{\beta}{2})\end{bmatrix}\begin{bmatrix} cos(\frac{\gamma}{2})cos(\frac{\delta}{2})+isin(\frac{\gamma}{2})sin(\frac{\delta}{2}) & -icos(\frac{\gamma}{2})sin(\frac{\delta}{2})-sin(\frac{\gamma}{2})cos(\frac{\delta}{2})\\ sin(\frac{\gamma}{2})cos(\frac{\delta}{2})-icos(\frac{\gamma}{2})sin(\frac{\delta}{2}) & -isin(\frac{\gamma}{2})sin(\frac{\delta}{2})+cos(\frac{\gamma}{2})cos(\frac{\delta}{2})\end{bmatrix}\\
&=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
a_{11}&=cos(\frac{\gamma}{2})cos(\frac{\delta}{2})cos(\frac{\beta}{2})+isin(\frac{\gamma}{2})sin(\frac{\delta}{2})cos(\frac{\beta}{2})\\
&-isin(\frac{\gamma}{2})cos(\frac{\delta}{2})sin(\frac{\beta}{2})-cos(\frac{\gamma}{2})sin(\frac{\delta}{2})sin(\frac{\beta}{2})\\
&=cos(\frac{\gamma}{2})cos(\frac{\delta+\beta}{2})+isin(\frac{\gamma}{2})sin(\frac{\delta-\beta}{2})\\
a_{12}&=-icos(\frac{\gamma}{2})sin(\frac{\delta}{2})cos(\frac{\beta}{2})-sin(\frac{\gamma}{2})cos(\frac{\delta}{2})cos(\frac{\beta}{2})\\
&-sin(\frac{\gamma}{2})sin(\frac{\delta}{2})sin(\frac{\beta}{2})-icos(\frac{\gamma}{2})cos(\frac{\delta}{2})sin(\frac{\beta}{2})\\
&=-icos(\frac{\gamma}{2})sin(\frac{\delta+\beta}{2})-sin(\frac{\gamma}{2})cos(\frac{\delta-\beta}{2})\\
a_{21}&=-icos(\frac{\gamma}{2})cos(\frac{\delta}{2})sin(\frac{\beta}{2})+sin(\frac{\gamma}{2})sin(\frac{\delta}{2})sin(\frac{\beta}{2})\\
&+sin(\frac{\gamma}{2})cos(\frac{\delta}{2})cos(\frac{\beta}{2})-icos(\frac{\gamma}{2})sin(\frac{\delta}{2})cos(\frac{\beta}{2})\\
&=-icos(\frac{\gamma}{2})sin(\frac{\delta+\beta}{2})+sin(\frac{\gamma}{2})cos(\frac{\delta-\beta}{2})\\
a_{22}&=-cos(\frac{\gamma}{2})sin(\frac{\delta}{2})sin(\frac{\beta}{2})+isin(\frac{\gamma}{2})cos(\frac{\delta}{2})sin(\frac{\beta}{2})\\
&-isin(\frac{\gamma}{2})sin(\frac{\delta}{2})cos(\frac{\beta}{2})+cos(\frac{\gamma}{2})cos(\frac{\delta}{2})cos(\frac{\beta}{2})\\
&=cos(\frac{\gamma}{2})cos(\frac{\delta+\beta}{2})-isin(\frac{\gamma}{2})sin(\frac{\delta-\beta}{2})
\end{aligned}
$$

$$
U=exp(i\alpha)\begin{bmatrix} e^a cos\theta & -e^b sin\theta\\ e^c sin\theta & e^d cos\theta\end{bmatrix}(ps: a+d=c+b)
$$

$$e^a cos\theta = cos(\frac{\gamma}{2})cos(\frac{\delta+\beta}{2}) + isin(\frac{\gamma}{2})sin(\frac{\delta-\beta}{2})$$
$$e^b sin\theta = icos(\frac{\gamma}{2})sin(\frac{\delta+\beta}{2}) + sin(\frac{\gamma}{2})cos(\frac{\delta-\beta}{2})$$
$$e^c sin\theta = -icos(\frac{\gamma}{2})sin(\frac{\delta+\beta}{2}) + sin(\frac{\gamma}{2})cos(\frac{\delta-\beta}{2})$$
$$e^d cos\theta = cos(\frac{\gamma}{2})cos(\frac{\delta+\beta}{2}) - isin(\frac{\gamma}{2})sin(\frac{\delta-\beta}{2})$$

$$
\begin{aligned}
cos(a)cos\theta &= cos(\frac{\gamma}{2})cos(\frac{\delta+\beta}{2}) \\
sin(a)cos\theta &= sin(\frac{\gamma}{2})sin(\frac{\delta-\beta}{2}) \\
cos(b)sin\theta &= sin(\frac{\gamma}{2})cos(\frac{\delta-\beta}{2}) \\
sin(b)sin\theta &= cos(\frac{\gamma}{2})sin(\frac{\delta+\beta}{2}) \\
cos(c)sin\theta &= sin(\frac{\gamma}{2})cos(\frac{\delta-\beta}{2}) \\
sin(c)sin\theta &= -cos(\frac{\gamma}{2})sin(\frac{\delta+\beta}{2}) \\
cos(d)cos\theta &= cos(\frac{\gamma}{2})cos(\frac{\delta+\beta}{2}) \\
sin(d)cos\theta &= -sin(\frac{\gamma}{2})sin(\frac{\delta-\beta}{2}) \\
tan(a) &= tan(\frac{\gamma}{2})\frac{sin((\delta-\beta)/2)}{cos((\delta+\beta)/2)} \\
tan(b) &= \frac{1}{tan(\frac{\gamma}{2})}\frac{sin((\delta+\beta)/2)}{cos((\delta-\beta)/2)} \\
tan(c) &= -\frac{1}{tan(\frac{\gamma}{2})}\frac{sin((\delta+\beta)/2)}{cos((\delta-\beta)/2)} \\
tan(d) &= -tan(\frac{\gamma}{2})\frac{sin((\delta-\beta)/2)}{cos((\delta+\beta)/2)} \\
a + d &= b + c = 0 \\
cos(a)cos\theta &= cos(\frac{\gamma}{2})cos(\frac{\delta+\beta}{2}) \\
sin(a)cos\theta &= sin(\frac{\gamma}{2})sin(\frac{\delta-\beta}{2}) \\
cos(b)sin\theta &= sin(\frac{\gamma}{2})cos(\frac{\delta-\beta}{2}) \\
sin(b)sin\theta &= cos(\frac{\gamma}{2})sin(\frac{\delta+\beta}{2})
\end{aligned}
$$

so:

$$\begin{aligned}
cos^2(\gamma/2) &= cos^2(a)cos^2(\theta) + sin^2(b)sin^2(\theta) \\
tan(\frac{\delta-\beta}{2}) &= \frac{sin(a)}{cos(b)tan(\theta)} \\
tan(\frac{\delta+\beta}{2}) &= \frac{sin(b)tan(\theta)}{cos(a)} \\
tan(\delta) &= \frac{\frac{sin(a)}{cos(b)tan(\theta)} + \frac{sin(b)tan(\theta)}{cos(a)}}{1 - \frac{sin(a)}{cos(b)tan(\theta)}\frac{sin(b)tan(\theta)}{cos(a)}} \\
&= \frac{sin(a)cos(a) + sin(b)cos(b)tan^2(\theta)}{cos(a)cos(b)tan(\theta) - sin(a)sin(b)tan(\theta)} \\
tan(\beta) &= \frac{\frac{sin(b)tan(\theta)}{cos(a)} - \frac{sin(a)}{cos(b)tan(\theta)}}{1 + \frac{sin(a)}{cos(b)tan(\theta)}\frac{sin(b)tan(\theta)}{cos(a)}} \\
&= \frac{sin(b)cos(b)tan^2(\theta) - sin(a)cos(a)}{cos(a)cos(b)tan(\theta) + sin(a)sin(b)tan(\theta)} \\
U &= exp(i\alpha)\begin{bmatrix} e^a cos\theta & -e^b sin\theta \\ e^{-b} sin\theta & e^{-a} cos\theta \end{bmatrix} \\
&= exp(i\alpha)R_x(\beta)R_y(\gamma)R_x(\delta) \\
\beta &= arctan(\frac{sin(b)cos(b)tan^2(\theta) - sin(a)cos(a)}{cos(a)cos(b)tan(\theta) + sin(a)sin(b)tan(\theta)}) \\
\delta &= arctan(\frac{sin(a)cos(a) + sin(b)cos(b)tan^2(\theta)}{cos(a)cos(b)tan(\theta) - sin(a)sin(b)tan(\theta)})(+\pi(if\, need)) \\
cos^2(\gamma/2) &= cos^2(a)cos^2(\theta) + sin^2(b)sin^2(\theta)
\end{aligned}$$

正负号根据下面这个等式成立操作

$$\begin{aligned}
cos(a)cos\theta &= cos(\frac{\gamma}{2})cos(\frac{\delta+\beta}{2}) \\
cos(b)sin\theta &= sin(\frac{\gamma}{2})cos(\frac{\delta-\beta}{2})
\end{aligned}$$

4.11 假设 $\hat{m}, \hat{n}$ 是互不平行的三维实单位向量，证明对于任意的单量子比特酉算子，存在合适的 $\alpha, \beta_k, \gamma_k$ 使得

$$U = e^{i\alpha}R_{\hat{n}}(\beta_1)R_{\hat{m}}(\gamma_1)R_{\hat{n}}(\beta_2)R_{\hat{m}}(\gamma_2)\cdots$$

证明:

引理 1: $\hat{n} \perp \hat{q} \Longleftrightarrow Tr(R_{\hat{q}}(\theta)\hat{n}\cdot\vec{\sigma}) = 0 \qquad \theta \neq 0$

证明：假设 $\hat{n} = (n_x, n_y, n_z), \hat{q} = (q_x, q_y, q_z)$

已知 $Tr(X) = Tr(Y) = Tr(Z) = 0$

$$\begin{aligned}
&Tr(R_{\hat{q}}(\theta)\hat{n}\cdot\vec{\sigma}) = 0 \\
\Leftrightarrow &Tr([cos(\frac{\theta}{2})I - isin(\frac{\theta}{2})(q_xX + q_yY + q_zZ)](n_xX + n_yY + n_zZ)) = 0 \\
\Leftrightarrow &Tr(cos(\frac{\theta}{2})(n_xX + n_yY + n_zZ)) + \\
&Tr(-isin(\frac{\theta}{2})(q_xX + q_yY + q_zZ)(n_xX + n_yY + n_zZ))) = 0 \\
\Leftrightarrow &Tr(-isin(\frac{\theta}{2})(q_xn_x + q_yn_y + q_zn_z)) = 0
\end{aligned}$$

引理 2:

$$R_{\hat{n}}(\pi)R_{\hat{m}}(\gamma)R_{\hat{n}}(\pi)R_{\hat{m}}(-\gamma) = R_{\hat{q}}(\theta)$$

则 $n \cdot q = 0 \quad or \quad \theta = 0$

证明:

$$\begin{aligned}
Tr(R_{\hat{q}}(\theta)\hat{n}\cdot\vec{\sigma}) &= Tr(R_{\hat{n}}(\pi)R_{\hat{m}}(\gamma)R_{\hat{n}}(\pi)R_{\hat{m}}(-\gamma)\hat{n}\cdot\vec{\sigma}) \\
&= Tr(R_{\hat{n}}(\pi)R_{\hat{m}}(\gamma)R_{\hat{n}}(\pi)R_{\hat{m}}(-\gamma)R_{\hat{n}}(\pi)) \\
&= Tr(R_{\hat{m}}(\gamma)R_{\hat{n}}(\pi)R_{\hat{m}}(-\gamma)) \\
&= Tr(R_{\hat{n}}(\pi)) \\
&= 0
\end{aligned}$$

则 $\theta = 0 \quad or \quad n \cdot q = 0$

定理 4.1 : 任意单量子比特的酉操作, 存在 $\alpha, \beta, \gamma, \delta$, 使得 $U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$

找到酉变换 V 使得 $V(\hat{n}\vec{\sigma})V^{\dagger} = Z$，并且 $V(\hat{q}\vec{\sigma})V^{\dagger} = Y$

则根据定理 1 $V^{\dagger}UV = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$

$$\begin{aligned}
U &= V(V^{\dagger}UV)V^{\dagger} \\
&= e^{i\alpha}VR_z(\beta)R_y(\gamma)R_z(\delta)V^{\dagger} \\
&= e^{i\alpha}R_n(\beta)R_q(\gamma)R_n(\delta)
\end{aligned}$$

推论 4.2 ： 设 U 是作用在单个量子比特上的一个酉门，则单量子比特上存在酉算子 A,B,C，使得 $ABC = I$，并且 $U = e^{i\alpha}AXBXC$，其中 α 是某个全局相位因子。

证　明: 沿用定理 4.1 记号：设 $A \equiv R_z(\beta)R_y(\gamma/2), B \equiv R_y(-\gamma/2)R_z(-(\delta+\beta)/2), C \equiv R_z((\delta-\beta)/2)$

$$
\begin{aligned}
ABC &= R_z(\beta)R_y(\gamma/2)R_y(-\gamma/2)R_z(-(\delta+\beta)/2)R_z((\delta-\beta)/2) \\
&= I \\
XBX &= XR_y(-\gamma/2)R_z(-(\delta+\beta)/2)X \\
&= XR_y(-\gamma/2)XXR_z(-(\delta+\beta)/2)X \\
&= R_y(\gamma/2)R_z((\delta+\beta)/2) \\
AXBXC &= R_z(\beta)R_y(\gamma/2)R_y(\gamma/2)R_z((\delta+\beta)/2)R_z((\delta-\beta)/2) \\
&= R_z(\beta)R_y(\gamma)R_z(\delta)
\end{aligned}
$$

即满足 $ABC = I$，并且 $U = e^{i\alpha}AXBXC$

习题 4.12 求出阿达玛门 H 对应的 A, B, C 和 α

$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$H = e^{i\pi/2}R_z(0)R_y(\pi/2)R_z(\pi)$$

$$
\begin{aligned}
&\alpha = \pi/2 \\
&A \equiv R_z(0)R_y(\pi/4) \\
&B \equiv R_y(-\pi/4)R_z(-\pi/2) \\
&C \equiv R_z(\pi/2)
\end{aligned}
$$

习题 4.13 恒等关系：$HXH = Z, HYH = -Y, HZH = X, HTH = R_x(\frac{\pi}{4})$

习题 4.15（单量子比特运算组合） 布洛赫表示对旋转结合提供了一种可见效果的方法。

1. 如果先绕轴 $\hat{n_1}$ 旋转角度 β_1 ，再绕轴 $\hat{n_2}$ 旋转角度 β_2 ，则整个旋转过程可表示为绕轴 $\hat{n_{12}}$ 旋转角度 β_{12} ，其中

$$c_{12} = c_1 c_2 - s_1 s_2 \hat{n_1} \cdot \hat{n_2}$$
$$s_{12}\hat{n_{12}} = s_1 c_2 \hat{n_1} + c_1 s_2 \hat{n_2} + s_1 s_2 \hat{n_2} \times \hat{n_1}$$

这里 $c_i = cos(\beta_i/2), s_i = sin(\beta_i/2), c_{12} = cos(\beta_{12}/2), s_{12} = sin(\beta_{12}/2)$

2. 证明若 $\beta_1 = \beta_2$ 且 $\hat{n_1} = \hat{z}$ 则可简化为

$$c_{12} = c^2 - s^2 \hat{z} \cdot \hat{n_2}$$
$$s_{12}\hat{n_{12}} = sc(\hat{z} + \hat{n_2}) + s^2 \hat{n_2} \times \hat{z}$$

$c = c_1, s = s_1$

1. 证明:

$$\begin{aligned}
R_{\hat{n}_2}(\beta_2)R_{\hat{n}_1}(\beta_1) =& R_{\hat{n}_{12}}(\beta_{12}) \\
right =& cos(\beta_{12}/2)I - isin(\beta_{12}/2)(n_{12x}X + n_{12y}y + n_{12z}Z) \\
=& c_{12}I - is_{12}(\hat{n_{12}}\vec{\sigma}) \\
left =& (cos(\beta_2/2)I - isin(\beta_2/2)(n_{2x}X + n_{2y}y + n_{2z}Z)) \\
& (cos(\beta_1/2)I - isin(\beta_1/2)(n_{1x}X + n_{1y}y + n_{1z}Z)) \\
&= (c_2 I - is_2\hat{n}_2\vec{\sigma})(c_1 I - is_1\hat{n}_1\vec{\sigma}) \\
&= c_2 c_1 I - i(c_1 s_2 \hat{n}_2 + c_2 s_1 \hat{n}_1)\vec{\sigma} - s_1 s_2 \hat{n}_2\vec{\sigma}\hat{n}_1\vec{\sigma} \\
\hat{n}_2\vec{\sigma}\hat{n}_1\vec{\sigma} &= (n_{2x}n_{1x} + n_{2y}n_{1y} + n_{2z}n_{1z})I + i(n_{2x}n_{1y} - n_{2y}n_{1x})Z \\
&+ i(n_{2y}n_{1z} - n_{2z}n_{1y})X + i(n_{2z}n_{1x} - n_{2x}n_{1z})Y
\end{aligned}$$

得证。