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SUBJECT:  GK Steady State with Idiosyncratic Productivities -- Probability Structure
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1 Overview

This discusses the probability structure used in my analysis of the steady-state of a GK extension in which there is a continuum of banks which receive different productivity levels each period, and make maximizing decisions conditional on their own transition probabilities. It covers the structure I have been using, conjectures regarding what will be necessary to keep it well behaved while scaling it up, and ends with a brief section concerning how the incentive compatibility constraint affects other features of the parameterization and equilibrium.

2 The Structure Used So Far

I have used two levels of productivity with transition probabilities as follows: there is a γ probability of remaining at the same level, and a $(1-\gamma)$ probability of drawing a new level, which will be the high state with probability p^h.

I chose this structure because the analytical investigation indicated that problems in finding equilibria arise mainly due to characteristics of the highest state: it is there that the potential for infinite-valued banks arises (or rather, for a probability structure in which the probability of being in higher states next period does not decrease with the level of the current state, the highest state will be the first to become explosively valued).

Similarly, only one low-level state is necessary to allow defaults and interbank lending to take place.

To begin, I choose an average level of the productivity of the capital in the economy, because jointly raising the whole productivity distribution (along with parameters I'd prefer to keep roughly constant relative to productivity: household capital management cost and endowments) would have no effect apart from increasing the price of capital by the same factor. Following GK, I choose approx. 0.0126. I then choose the fraction by which the high state's productivity should exceed the average and a probability of selecting the high state in the $(1-\gamma)$ fraction that a bank does not remain in its previous state. These two choices pin down the productivity level and probability of drawing the low state: higher productivity in the high state pushes down productivity in the low state, as does higher probability of the high state. γ, the autocorrelation of productivity, is independent of the other choices.

The equilibria of interest feature default when a bank transitions from high to low productivity. This requires that the rate of return on assets in the low state be insufficient to pay deposits: $\phi * (Z^l + Q)/Q < R^h * (\phi^h - 1)$, where h denotes levels associated with high states and l low states, Z the productivity level, Q the asset price, R the deposit rate, and ϕ the leverage of the bank, $(\phi - 1)$ thus the fraction of bank assets funded by deposits.

Consider initially the case of $Z^l = 0$, or the net rate of return in the low state being 1. Then for a bank to exactly 0 equity (and thus not push up the rate on deposits) when the rate of time preference is $\beta = 0.99$, its leverage would have to be approximately 100. Increasing leverage from there results in losses and pushes up R, but relatively slowly, as it is linked to the probability of having a high to low transition. Likewise, even with infinite leverage the gross rate of return on deposits would be 1, so absent a very high (greater than 0.5) probability of the high to low transition, the interest rate would remain low, under about 1.0202 (which if taken as granted would require leverage of near 50 to result in default in the first place). Besides not wanting to calibrate to such high leverage, steady state leverage this high will imply very high market to book values of banks for reasonable loan spreads (GK calibrated to 0.0010):

$\psi_h = [(spread_{hh}) * \phi_h + R_h](1 - \sigma + \sigma * \psi_h)[\gamma + (1-\gamma)(1-p_h)]$

and if $[(spread_{hh}) * \phi_h + R_h] * [\gamma + (1-\gamma)(1-p_h)] > 1$ the value will be infinite as can be seen by an iterative process up updating ψ holding the rest constant. With ϕ of 100, R_h assumed still 1.01, and taking the GK spread as a low estimate of the spread given higher than average physical return per unit of assets, $[\gamma + (1-\gamma)(1-p_h)]$ must be under about 0.5, implying even with complete mean reversion ($\gamma = 0$) the

physical return on capital in the high state has to be over 2 times the average to generate even as little as 0 net return in the bad state.

Simply put, you can't get default in steady state without unreasonably high leverage unless there are actual losses in bad states, and even this level of productivity in the low state requires very high probability or very high productivity in the high state.

A sufficiently negative net return in the low state reduces the required leverage for default, but requires either raising the productivity of the high state or increasing its probability. With full mean reversion, this is not an insurmountable problem, as inspection has shown that $\gamma = 0$ with sufficiently high probability in the high state (0.99) and level of the high state (> 0.66 times the average) gives results with the appropriate default structure and steady state aggregates comparable to GK.

Moving away from IID, however, introduces parameterizations in which no equilibrium is found at all that conforms to a default structure desired, though sufficiently extreme parameterizations likewise can be found which will result in the desired survival outcomes, which have higher leverage, total capital held by banks, and price of capital.

What should be noted here is the high concentration at the high end required to get bad enough low states to generate defaults. There isn't much scope to make a richer distribution of states that have meaningful mass without spreading the high state. However, maintaining the average productivity means filling out the region between the two points requires increasing the productivity of some portion of the high distribution. If expected future productivity strictly increases with current productivity, then an increasing spread on loans on the highest end (which also loosens the leverage constraint) can be destabilizing, as behavior is dominated by the most extreme state. This will require not only the potential to fall to a somewhat less high state, but also larger expected losses in default.

For this reason, I believe that for a probability structure with significant persistence and a full support from lowest state to highest, a highly asymmetrical distribution is necessary with a long left tail and most of the probability mass close to the average.

3 Effects of the IC Constraint on Equilibrium

The IC constraint, $\theta\phi \leq \psi$ governs the degree of leverage a bank in a given state operates with. As θ falls, leverage ϕ is allowed to increase, which in expectation also increases the market to book ratio of the bank ψ. The increased leverage at the bank will push up the amount of capital operated by banks collectively, and by the household's first order condition for capital push up the price of capital, reducing the loan spreads available to banks at all productivity levels, a general equilibrium force opposing the tendency of higher allowed leverage to increase ψs. Likewise, the increase in leverage (weakly) increases expected losses on deposits, pushing up interest rates paid on deposits, reducing the lending spread from the other end, and thus mitigates the increase in ψ as well.

An increase in bank system leverage could be accommodated without increasing the amount of capital operated by banks through increasing the rate of banker exit and/or reducing the endowment of new bankers.

A more weakly binding IC constraint is expected to increase the degree to which the highest productivity banks dominate an equilibrium with many productivity levels and persistence. This is because higher amounts of bank capital operated drive down returns and spreads for all banks, resulting in the cutoff point between which a bank prefers to invest in capital itself rather than lend in the interbank market to shift to a higher productivity level.