Memorandum: Alterations to Gertler and Kiyotaki 2015

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1 Overview

The following memo details recent progress in my extension of GK2015 to heterogeneous agents and the steps to be taken in the last weeks before the draft is to be submitted. It includes the updates to the GK model as they currently stand, alternatives which have been discarded or tabled, preliminary theoretical results, and planned empirical comparisons. Continuing from the mid-July memo is a section on remaining concerns for extending the methodology from a computational standpoint.

2 Key Features

Equations (1) through (4) detail the fundamental assets and production technologies of the GK model: a fixed quantity of a durable asset "capital" set to unity which may be operated by either bankers or households, but more efficiently by bankers.

$$K_t^b + K_t^h = 1 \tag{1}$$

Capital operated by bankers K_t^b requires no other input and generates output of a final good as well as returning the undepreciated capital :

$$\begin{gathered} \textit{date}\, t \rightarrow \textit{date}\, t+1 \\ K_t^b\ \textit{capital} \rightarrow \begin{cases} Z_{t+1}K_t^b & \textit{output}\ , \\ K_t^b & \textit{capital} \end{cases} \end{gathered} \tag{2}$$

with Z_{t+1} an aggregate productivity shock.

Households operated with an additional management cost $f(K_t^h)$, paid for with period t final goods, to produce the same output the following period:

$$\begin{gathered} \textit{date}\, t \rightarrow \textit{date}\, t+1 \\ \begin{cases} K_t^h & \textit{capital} \\ f(K_t^h) & \textit{goods} \end{cases} \rightarrow \begin{cases} Z_{t+1}K_t^h & \textit{output}\ , \\ K_t^h & \textit{capital} \end{cases} \end{gathered} \tag{3}$$

The management was taken to be increasing, convex, and costless when no household operation took place:

$$f(K_t^h) = \frac{\alpha}{2}(K_t^h)^2 \tag{4}$$

In the planned extension, the key features remain the same with the following exceptions:

1. $K_t^b = \int_i K_t^{b,i} = \sum_i K_t^{b,i}$ (likewise K_t^h), the latter in the case of discrete household-bank pairs. This is made explicit because the main purpose of the extension is to explore bank heterogeneity. **The implementation has been to have two bank-household pairs. I expect that this could be extended to a larger number with relative ease, but at this point a distribution is likely out of the question for at least the initial draft.**

2. The production functions (2), (3), and (4) operate at the level of the individual household or bank, rather than the aggregate.

3. The productivity shock Z_{t+1} is a combined firm-specific and aggregate shock:

   (a.1): $Z_{t+1}^i = Z_{t+1}^{agg} S_{t+1}^i$.

   This is added because even with added frictions, if all agents are subject to the same shocks motivating dispersion in outcomes across agents is infeasible. The aggregate shock and the idiosyncratic shock need not follow the same processes. **The current implementation eschews an aggregate portion of the shock, and has an AR(1) idiosyncratic shock.**

4. Households have an additional source of income beyond their own capital holdings, deposits, and the productivity-linked endowment: **a risk free endowment proportional to the pair's mass/proportion of capital belonging to the pair**, $G^i = w^i G$. Households can buy or sell this endowment (but cannot take a short position–in practice, I have so far had to parameterize endowments such that this requirement does not bind), subject to a convex transaction cost $h(g_{t+1}^i)$. This convex cost is used to generate a unique optimal decision, where otherwise there is potential for a multiplicity of steady states. In the results presented later, I have taken $h(g_{t+1}^i) = \frac{\alpha}{200}(\frac{g_{t+1}^i}{w^i G} - 1)^2 w^i G$, a CRS function far weaker than the household management cost of capital; in the case presented, the maximum proportion of gross output the transaction cost consumes is on order 1 part in 10,000,000 of gross output, so the quantitative differences between the representative bank and two-bank systems are not driven by this cost.

5. Capital is assumed, in contrast to households' risk free income, to be permanently fixed within a given pair.

6. **The household's untility function in the presented quantitative results is log, but can be generalized.**

3 Households

Households are permanently paired with banks where they deposit savings, receiving a promised rate of return if no bank run returns, and the promised rate of return times a recovery rate if one does.

$$R^i_{t+1} = \begin{cases} \bar{R^i}_{t+1} & no-run \\ x^i_{t+1}\bar{R^i}_{t+1} & run \end{cases} \tag{5}$$

Households face a period by period budget constraint

$$C^{h,i}_t + D^i_t + Q_t K^{h,i}_t + f(K^{h,i}_t) + p_t(g^i_{t+1} - w^i G) + h(g^i_{t+1}, w^i G) = Z^i_t W^h + R^i_t D^i_{t-1} + (Z^i_t + Q^i_t)K^{h,i}_{t-1} + g^i_t \tag{6}$$

The left side of the equation is uses of funds (consumption, new deposits, capital held for next period, risk free asset holdings, and the management cost), the right is sources (an endowment proportional to the aggregate shock, deposit and risk free asset returns, gross return from capital held last period). The households' optimal holdings follow these conditions:

$$E(\Lambda^i_{t,t+1} R^i_{t+1}) = 1 \tag{7}$$

$$\Lambda^i_{t,t+s} = \beta^s \frac{u(C^{h,i}_{t+s})}{u(C^{h,i}_t)} \tag{8}$$

$$E(\Lambda^i_{t,t+1} R^{h,i}_{t+1}) = 1 \tag{9}$$

$$R^{h,i}_t = \frac{(Z^i_{t+1} + Q_{t+1})}{Q^i_t + f'(K^{h,i}_t)} \tag{10}$$

The presence of a tradable risk free asset guarantees a uniform SDF and risk free rate across household-bank pairs, *provided an interior optimum*. Otherwise, a Lagrange multiplier will appear in the risk free asset's price equation for at least one household.

4 Banks

In the GK model, risk neutral bankers earn income from operations and exit with constant exogenous probability $(1-\sigma)$ to motivate dividend payments and avoid saving indefinitely to avoid financial constraints (exiting bankers consume their entire net worth); bankers also stopped operations if their net worth fell to 0. I have modified this assumption to operate within each bank: there are a large numer of bankers at each bank, with equal ownership shares, and each period a fraction $(1-\sigma)$ exit and take with them their portion of the bank's net worth after costs in cash. **In my current implementation, I am retaining the CRS GK bank model substantially unaltered. The reason for this decision will be discussed in alternative models.**

5 Aggregation and Equilibrium without Bank Runs

The value of assets held by the banking system is equal to the sum over all banks, or equivalently the sum of their continuing net worths times their leverage $\phi^i_t = \frac{Q^i_t k^{b,i}_t}{n^i_t}$:

$$Q_t K_t^b = \sum Q_t k_t^{b,i} = \sum \phi_t^i n_t^i$$

The same holds after an aggregation in the expanded model, integrating/summing each side reflecting the individual banks.

The net worth of continuing bankers as a whole follows:

$$N_t = \sum_i n_t^i = \sum((Z_t^i + Q_t)k_{t-1}^{b,i} - R_t^i d_{t-1}^i + W^{b,i})\sigma$$

Where in the absence of a recent bank run on i (in which case everything else would be 0), $W^{b,i} = 0$ as no new bankers are replacing failed banks.

Thus banker consumption is:

$$C_t^b = \sum_i c_t^i = (1-\sigma)\left[\sum_i (Z_t^i + Q_t)k_{t-1}^{b,i} - R_t^i d_{t-1}^i + W^{b,i}\right]$$

Output is the sum of output from capital, the household's endowment (also bankers' endowment if after a run):

$$Y_t = Z_t + Z_t W^h$$

With the addition of the transaction cost, uses of output are:

$$Y_t = C_t^h + C_t^b + \sum_i f(k_t^{h,i}) + h(g_{t+1}^i, w^i G)$$

Lastly, the quantities of risk free asset held across all households must sum to the aggregate stock:

$$G = \sum_i g_t^i$$

6 Unanticipated Bank Runs (Substantially unchanged from 7/16)

The bank by bank recovery ratio with the addition of adjustment costs is:

$$x_t^i = \frac{\left(Q_t^* + Z_t^i\right) k_{t-1}^{b,i}}{R_t d_{t-1}} < 1$$

Q_t^* is a function of the joint survival or failure of banks. Consider a case in which there are two banks, initially identical but receiving different ideosyncratic shocks. One may have a recovery ratio below 1 in the event that the price of capital is as it would be with all banks failing, and the other might retain positive net worth even in that situation. In that case, the latter could not in fact fall to a run, and the equilibrium price of capital would be higher, and Q_t^* set to that instead; this might push the weaker bank's net worth above the point where it is susceptible to a run, and it might not. In the latter case, a run equilibrium is possible for the weaker bank but not the stronger. Any fixed point where the price is the result of a set of banks failing and no others (this process will be described later), which results in that same set failing and no others, can be the outcome of a run.

With appropriately defined leverage $\phi_t^i = \frac{Q_t^i k_t^{b,i}}{n_t^i}$, the recovery ratio in terms of returns on bank assets (in event of a run), the risk free rate, and leverage is:

$$x_t^i = \frac{R_t^{b,i*}}{R_t^i} \cdot \frac{\phi_{t-1}^i}{\phi_{t-1}^i - 1} < 1$$

with return on bank capital operated

$$R_t^{b,i*} \equiv \frac{Q_t^* + Z_t^i}{Q_{t-1}^i}.$$

The liquidation price is analogous to that seen in (30) and (31) of GK:

$$1 = k_t^{h,i}$$

$$Q_t^* = E_t \left[\sum_{s=1}^{\infty} \Lambda_{t,t+s}^i (Z_{t+s}^i - \alpha k_{t+s}^{h,i}) \right] - \alpha$$

Which sets the price equal to the discounted dividends of capital held by households for the indefinite future, subject to the bank net worth process:

$$n_{t+1}^i = (1+\sigma) W_b^i,$$

$$n_{t+i} = \sigma[(Z_{t+s}^i + Q_{t+s}) k_{t+s-1}^{b,i} - r_{t+s}^i d_{t+s-1}^i],\ for\ all\ s \geq 2$$

7 Anticipated Bank Runs

GK continue with the case of households anticipating the possibility of systemic bank run in time *t+1* with probability p_t. This results in a different first order condition for households.

$$1 = \bar{R}_{t+1} E_t[(1-p_t)\Lambda_{t,t+1} + p_t \Lambda_{t,t+1}^* x_{t+1}],$$

that is, that the promised rate of return times the probability of receiving that amount scaled by the applicable SDF plus the recovery amount in the event of a run times the then-applicable SDF must equal 1. The household must be indifferent to holding the deposit subject to the anticipation that the banking system might fail.

The addition of more banks that may or may not jointly fail results in a different probability distribution that is relevant. This is described as follows:

- ~~For a given set of state variables and shocks, order the banks by $\bar{Q}_t^i$: the price they require for them to survive with positive net worth.~~
- ~~Generate the set of prices $\tilde{Q}_t^i$: the price of capital in the event all banks j with $\bar{Q}_t^j \geq \bar{Q}_t^i$ fail~~
- ~~Then the possible prices are those $\tilde{Q}_t^i$ such that $Q_t^{i-1} \leq \tilde{Q}_t^i < Q_t^i$, plus the price induced by all banks failing if it is below the survival price of all banks, plus the no-run equilibrium price if high enough for all banks to survive.~~

- **Because capital is no longer mobile, it is no longer the case that prices are as restricted with such a pronounced fixed point relationship. The price of capital is different, but related, in each pair.**

- **I propose, as the fastest potential implementation strategy, taking the beliefs based on the recovery ratio in the event all banks run jointly (or, if that is impossible, the maximum set) rather than breaking into every possible combination in some ordering method. If I proceed with 2 banks, there's not much difference.**

The recovery rate is based on the promised rather than the risk free rate, similar to GK (34),

$$x^{i,j}_{t+1} = \min\left[1, \frac{R^{b,i,j}_{t+1}}{\bar{R}^{i}{}_{t+1}} \cdot \frac{\phi^i_t}{\phi^i_t - 1}\right] = \min\left[1, \frac{\left(Q^j_{t+1} + Z^i_{t+1}\right) k^{b,i}_t}{\bar{R}^{i}{}_{t+1} d^i_t}\right]$$

Because there are a number of outcomes that can occur in addition to either the entire banking system failing or nothing failing, the promised rate of return must satisfy an equivalent to (33):

$$1 = \bar{R}^i_{t+1} E_t[(\sum_{j=0}^{B} p^{i,j}_t \Lambda^{i,j}_{t,t+1} x^{i,j}_{t+1}]$$

where

$$\Lambda^{i,j}_{t,t+1} = \beta u'(c^{h,i,j}_{t+1})/u'(c^{h,i}_t)$$

is the SDF in period $t+1$ when sunspot event j is realized: the j weakest (subject to realized shocks) banks face runs, with 0 being the no-run equilibrium and $j{=}B$ a systemic run. Then if in a given state bank i is the lth strongest bank, we find the following **(see attached page 15 for derivation):**

$$\bar{R}_{t+1} = \frac{1 - \frac{\phi^i_t}{\phi^i_t - 1} E_t\left(\sum_{j=l}^{B} p^j_t \Lambda^{i,j}_{t,t+1} R^{b,i,j}_{t+1}\right)}{E_t\left(\sum_{j=0}^{l-1} \Lambda^{i,j}_{t,t+1}\right)}$$

(on a pair by pair basis), so that banks will take into account leverage decisions' effects on their rate of return and resources available in the event of survival. This feeds through to the appropriate maximization problem described above, with the same constraints, but updated cost of borrowed funds and thus law of motion subtracting the ex-ante risky rate from the realized rate of return rather than a risk free rate.

As GK found, this gives a way for expectations of a run to negatively affect bank equity even in the absence of one materializing.

I plan to follow GK in the assumption that if a run on a bank next period is impossible, the probability to a run on that bank is 0, and that otherwise the probability of a run is a decreasing function of the expected recovery ratio $p_t = 1 - E_t(x_{t+1})$. The more likely a bank run is to occur, and the more severe in the event one does occur, the higher the agents assign to seeing the sunspot outcome. Note that in my case, this is a pair-by-pair probability, that is, the assessed probability of that bank failing. The possibility of other banks failing while the one under consideration survives, however, will enter into the agents' consideration via price effects of capital.

The probability of a bank failing is determined as follows:

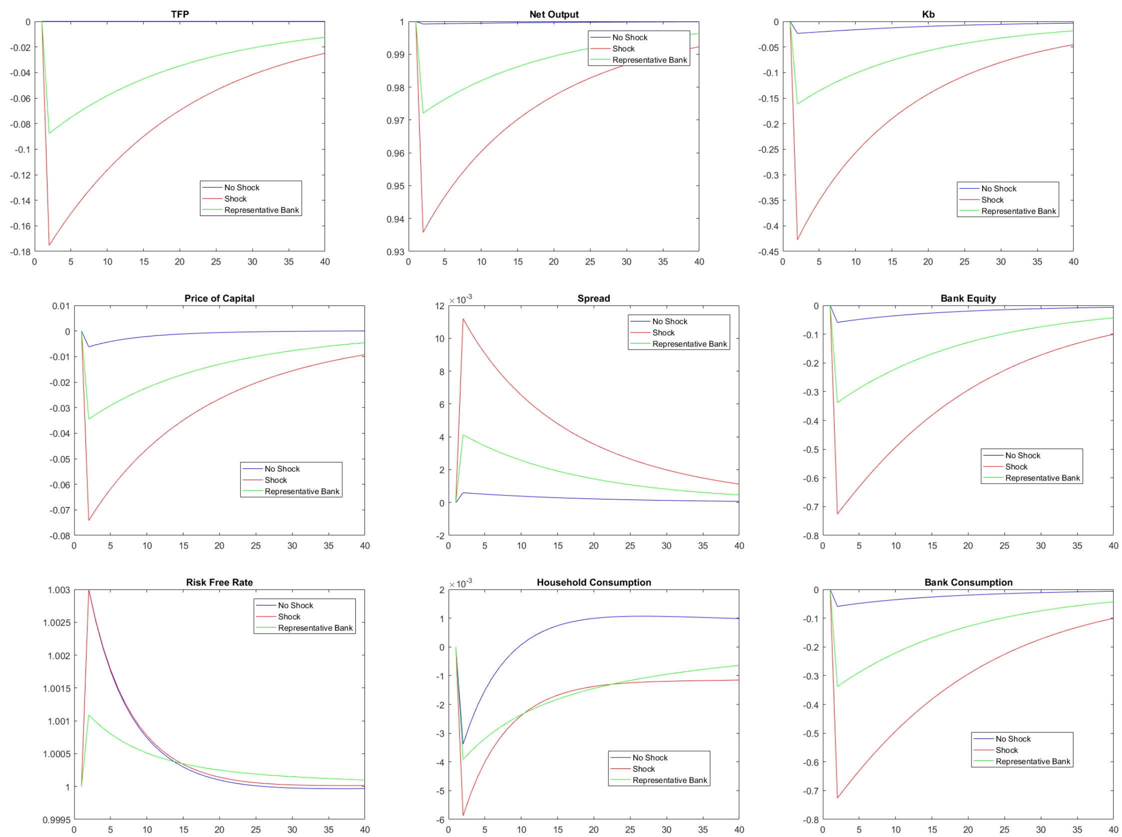
- Determine the recovery ratios of all banks in the event that all are hit with sunspot shocks. If all recovery ratios are below 1, subtract from 1 to give the probability for each bank.

- If any recovery ratios are 1 or above, run again with only those banks which did not survive receiving run sunspots. Repeat until the set of banks is stable.

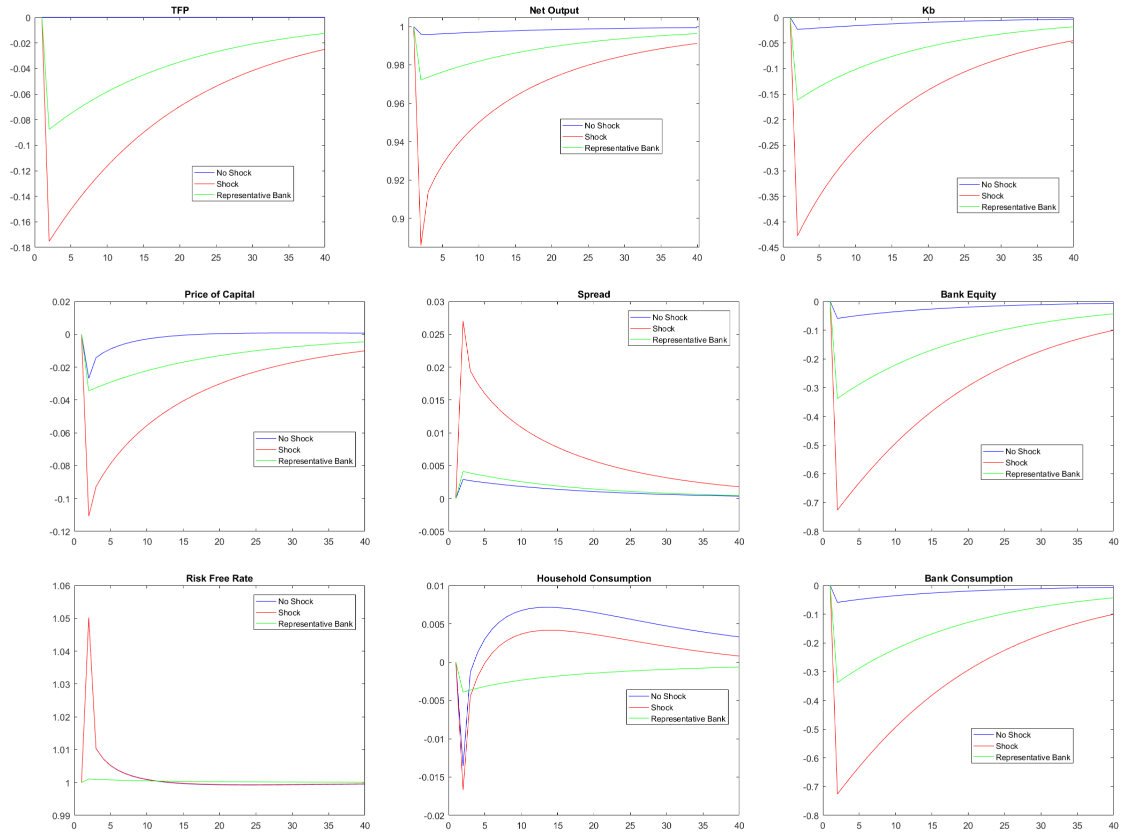
8 Example Theoretical Results

The following are modeled after Figure 3 and 4 in GK, showing reactions to a large negative idiosyncratic shock to one bank-household pair, with and without a bank run. For comparison, a representative bank under a shock half the size of the negative shocked bank is included, and does not suffer a run in either case.

8.1 No Run



8.2 With Run at Affected Bank



9 Solution Methods (Still something of a concern, unchanged from 7/16)

The online appendix to GK includes a description for a solution method for the dynamics of the system, piecewise working backwards from a return to steady state with perfect foresight of shocks but not, in general, capital, consumption, etc. Provided a solution to the optimization problems faced by the banks/working decision rule accounting for the adjustment costs and how they affect the value function, and thus leverage constraint, certain problems remain.

First, the number of endogenous state variables is multiplied by the number of banks, plus the prior period's capital holdings for each bank and holdings of the risk free asset: where GK needed essentially the leverage of the bank to solve the optimization problem, now both capital and deposits matter independently. SDFs must be calculated for B times as many households, and in principle there are up to B + 1 possible numbers of defaults in each period, given everything else, and the recovery rates in each one depend on the recovery rates in the others and the perceived probability of each sunspot occurring, in addition to those for each time step from then on.

The GK methodology depends on the fact that after a run, capital holdings are entirely determined with banks holding none and households holding everything. This enables the relatively simple calculation of a saddle path back from the steady state to the known points during and after the run, including for the case of anticipated runs. This of course does not apply to a model with the possibility of partial runs where in general multiple outcomes could have arisen, and in turn what those outcomes were depended on the history to that point.

For a given run event at time t+i after the starting point, either you need to check the entire feasible state space including jump variables for what is an equilibrium that could take a saddle path back to the steady state, then out of those determine which could be supported by optimizing agents prior to that point. And this would need to be done for every time step to calculate the probabilities.

10 Alternative Models Not Used

The model described in the 7/16 memo and subsequent methodological Bank Problem memos (no tradable risk free asset, mobile capital with adjustment costs to banks changing capital), adjusted so that an individual continuing banker at a given bank does not see the full continuation value of the bank in his maximization problem, but scaled down by (N - W)/N: the endowment is handled with entering bankers bringing an endowment and taking a proportional ownership stake in the bank. This resulted in a very good matching of the GK results for a single bank in the absence of a bank run, getting similar parameters for a given desired steady state and very similar responses to shocks: price of capital tending to fall more than TFP, output amplification of shocks, etc. Reason this was tabled: I have been unable to get this model to yield a valid transition path once a run sunspot is added.

The model described in the 7/16 memo and subsequent Bank Problem memos (no tradable risk free asset, mobile capital with adjustment costs to banks changing capital): This generated in a single-bank system no amplification of shocks, and prices of capital generally responding less than one for one with the productivity shock. It could handle bank runs in a single bank system, but I have not been able to get it to yield valid results in multibank scenarios involving nonsystematic runs. A comprehensive search for compatible shock combinations might find some, if that is the only extant problem.

Very small values for the risk free assets available for sale in the above-described model, or extremely small values for the transaction cost in risk free assets: the lack of an equilibrating force results (in attempts so far) in running up against the constraint and staying there forever. This might be corrected with a different choice of how to implement the constraint in code, but for now providing a fairly small (1/5 of labor income in steady state has been more than enough, in combination with the effect of moderately reducing the underlying variability of household income) amount but enough to avoid the constraint allows for, e.g., a 17.5% productivity shock combined with a bank run in one pair, and no productivity shock and no run in the other.

11 Steps Forward

A full solution to this model is likely to be intractable, and I am considering ways to simplify it. The pieces I am least willing to compromise on are heterogeneity and at least some bank-household stickiness in terms of moving deposits around (given the presence of some, I suspect perfect stickiness leaves the problem the simplest).

The simplifying steps I am considering so far are:

- Assume a very small number of banks, probably 2, maybe 3

- Take as simple as possible a cost function for the household's management cost and bank's transaction costs

- Assume following the productivity shock there is a limited period of time in which agents anticipate a bank run may occur, before the probability falls to 0 exogenously: along the lines of "if we can make it through the next year, it's smooth sailing"

- Greatly simplify the assigned probability of default, making it less a function of future and more a function of current factors

- Get rid of or modify the incentive compatibility constraint (however, the restriction that there is a fixed amount of capital to be distributed could in principle bind as well), for example by using the GK policy extension of instituting a fixed maximum leverage ratio

Other concerns:

- Perfect competition: pushes in direction of always maximizing leverage subject to incentive compatibility constraint. This tends to make all banks behave more similarly (and is part of why I believe the cost function is important in maintaining differentiation), but am interested in 1) preserving differences in choices, including leverage ratios and 2) banks taking into account the decisions their choices have on other banks and the probability of a general run. However, removing perfect competition would require calculating (for example) price derivatives in terms of changes in investment, when the price is an outcome of solving for the saddle path. Unless there is a good way to insert this into the probabilities of default alone, moving away from perfect competition likely isn't an option.

Empirical comparisons:

- Large, failed US banks circa Lehman (Citi, or Fannie/Freddie, Lehman itself as in GK) vs. survivors or community bank index. Potential for transition to anticipated no-run equilibrium starting 2009Q1 post AIG, stress tests.

- German vs. periphery banks, circa Greek crisis, or around time of "whatever it takes" as transition to no anticipation of runs.

12 Bibliography

Gertler, Mark and Nobuhiru Kiyotaki. "Banking, Liquidity, and Bank Runs in an Infinite Horizon Economy." *American Economic Review*, vol. 105, no. 7, 2015, pp. 2011-2034.
Online Appendix: https://assets.aeaweb.org/assets/production/articles-attachments/aer/app/10507/20130665_app.pdf