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SUBJECT:   Sunspot process and dynamic programming
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Bank run sunspots and DP formulation

In determining the behavior of the bank run/financial accelerator model it is necessary to define a process for assigning what banks are subject to a run in each period, and incorporating that process into the dynamic programming problems the bankers and households solve.

I propose three potential methods to handle this addition, which may roughly be distinguished by the combination of aggregate and indiosyncratic components to the sunspot shocks.

Fundamental productivity shocks are considered to be handled identically across the methods that follow.

0.1 Purely Idiosyncratic

In this formulation, a bank-household pair has a fundamental state vector (Kb, RD, g, Z) of the quantity of capital operated by banks, the deposits owed to the household (inclusive of interest), the household's holdings of the risk free asset, **and local and aggregate productivity: single state?**. *No, or not exactly; that was an implicit merging of the two notationally, but with independent driving processes. Knowing the resulting composite level of productivity would not, in general, be sufficient to know the distribution of the composite in future periods. Keeping it as a single term was meant (in the paper and here) to keep the notation a little more compact.*

Given the aggregate state, a recovery ratio $X^i \in [0,1]$ exists in the event of a run. **The sunspot?** *Strictly speaking, in all states regardless of the outcome, it's just uninteresting most of the time, where X=1. X<1 if: there is a fundamental bank failure or if the sunspot occurs.* $\xi^i \sim U(0,1)$ is drawn, and if $X^i < \xi^i$ the bank experiences a run. Additionally, if any pairs exist for which, though the recovery ratio is positive, the bank cannot repay deposits owed even in the absence of a run sunspot (bank failure is fundamental), that pair experiences a run. **It will be interesting to see how frequent it is, as it might not be an equilibrium outcome depending on the risk tolerances of various agents and the range of fundamental shocks.** *As a fundamental I would expect it to be fairly rare unless the underlying productivity processes have high variance and/or persistence. And even then, that should induce risk-limiting behaviors.*

Thus the states are fully defined by (Kb, RD, g, Z, br), and agents can optimize their choices of Kb, RD, and g subject to the known current state and the productivity transition process, as well as the probability of a bank run occuring in each (Kb, RD, g, Z) combination.

This method generates a degree of symmetry across fundamental states, in that for all states where a run could occur some pairs will experience one, and the fraction that do is decreasing in the fundamental strength of the banks. **Are you using some form of LLN reasoning here?** *LLN in assuming a continuum of pairs, and some ceteris paribus in that it's more directly about choosing potential leverage levels given the same productivity states rather than actual leverage given an entering state. Higher expected returns on the asset lead to higher leverage and, other things equal, lower risk of default, so that muddies the issue over all states.* Additionally, it has the benefit that (assuming the idiosyncratic shock is IID, which I find desirable because it keeps more focus on stresses arising from fundamentals) the state space is increased as little as possible relative to GK. However, this comes at the cost of there never actually being (for a large number of pairs) larger or smaller runs, given a fundamental state, defined as a full distribution of endogenous state variables and productivities.

0.2 Purely Aggregate

The same fundamental state vector applies when the sunspot is purely aggregate. However, an economywide bank run sunspot ξ^{agg} is added, which is necessary for determining the recovery ratio. $X^i \in [0,1]$ is now a function of (Kb, RD, g, Z, ξ^{agg}). Given this state, a bank is similarly subject to a run if $X^i < \xi^{agg}$, as well as if in the state a failure is fundamental.

Thus the states are fully defined by (Kb, RD, g, Z, ξ^{agg}), though for computational purposes finding an equilibrium will require a bank run/no-run state as well, and agents can optimize their choices of Kb, RD, and g subject to the known current state (Z, ξ^{agg}) and the productivity and sunspot transition processes with known transition probabilities.

This method has the ability to generate different combinations of bank runs depending on the sunspot that arises. As a result it can also demonstrate the potential for nonlinear responses to sunspots of varying severities, as an increasingly severe sunspot can have a direct effect of pushing some banks into the run region, and pull in more due to effects on the price of the risk free asset and capital. The downsides are the addition of a new state variable (assuming ξ^{agg} to be Markov rather than IID, as this must be more directly and precisely observable by agents in the economy and would do less to drive cross sectional variation independent of fundamentals than would be the case with a Markov idiosyncratic sunspot shock), and that all banks of a given fundamental strength will stand or fall together, which may be counterfactual and in any case eliminates confidence in individual banks as a factor.

I think it is fine to make ξ^{agg} into a Markov process, but do not really see why having an aggregate shock requires Markov dynamics or why these could not have been introduced in the setup above. It certainly is one way to have "expected bank runs."

I agree. I think Markov is a good option for persistence in household confidence, but there's no reason a different discrete process, or even IID, couldn't be used

0.3 Combination Aggregate and Idiosyncratic

The states are the same as in the purely aggregate case, but an idiosyncratic shock ξ^i is once again present (for example as a zero-mean triangular or normal distribution). While the recovery ratio $X^i \in [0,1]$ is still a function of (Kb, RD, g, Z, ξ^{agg}), banks now run if $X^i < \xi^{agg} + \xi^i$ (if neither survival nor failure is a fundamental outcome). Don't understand this last statement.

Agents' optimization is similar to the purely aggregate method, but they now incorporate probabilities of both bank failure and survival in each (fundamentals, aggregate sunspot) state, rather than knowing with certainty one outcome or the other would obtain.

This method offers the richest set of outcomes, allowing for broad or narrow runs with the same fundamental state vector, while at the same time allowing for similarly situated banks to fail or survive depending on differences in local expectations. As in both of the previous cases, the **probability of bank failure** decreases as the fundamentals of the bank become stronger. **As you move forward with this work, I think that it is useful to think about the forecasting implications of the various models that you are developing and plan to develop. So you would want to be precsie about the conditioning infromation and even the sort of regressions that might be used for predicting bank failure.** The same nonlinear run effect as in the purely aggregate case would be expected to obtain. The state space is effectively no larger than the purely aggregate sunspot case given the same assumptions as above about the respective sunspot shocks.

Dynamic programming problems (combination)

Letting that ratio $\frac{V_t}{n_t} \equiv \psi_t$, the bank's problem becomes

$$\psi_t^i = \max_{\phi_t^i} E_t \left\{ \beta \left(1 - \sigma + \sigma \psi_{t+1}^i\right) \left[(R_{t+1}^{b,i} - R_{t+1}^i)\phi_t^i + R_{t+1}^i \right] \right\}$$
$$= \max_{\phi_t^i} E_t \left\{ \mu_t^i \phi_t^i + \nu_t^i \right\}$$

with the reframed incentive constraint

$$\theta \phi_t^i \leq (=) \psi_t^i = \mu_t^i \phi_t^i + \nu_t^i \quad (1)$$

where

$$R^{b,i}_{t+1} = \frac{(Z^i_{t+1} + Q^i_{t+1})}{Q^i_t}$$

Then with the combined aggregate-idiosyncratic productivity process Ψ and aggregate sunspot process Ξ the maximization can be written in a discrete form as

$$\begin{aligned} \psi^i_t &= \max_{\phi^i_t} \sum_{Z', \xi agg', \xi i'} \left(p(Z'^{agg'}, \xi^{i'} | Z, \xi^{agg'}, \xi^i) \right) \\ & \left\{ \beta \left(1 - \sigma + \sigma \psi^i_{t+1} \left(S', \phi \right) \right) \left[(R^{b,i}_{t+1} \left(S', \phi \right) - R^i_{t+1}) \phi^i_t + R^i_{t+1} \right] \right\} \end{aligned}$$

where for any state-shock combination resulting in a run the state value is simply 0 and S' is the external state including the global distribution of bank and household positions and the new shocks. The summation over the idiosyncratic sunspot shocks may be collapsed into an expression for the probability banks survive given the productivity shocks and aggregate sunspot if the idiosyncratic sunspot shock's distribution is defined.

The household's problem is similarly adjusted. Beginning with:

$$V^{h,i}_t = \max_{Kh, RD, g} E_t \left(u(C^{h,i}_t) + V(.) \right)$$

and the income = expenditure constraint:

$$\begin{aligned} & C^{h,i}_t + D^i_t + Q^i_t K^{h,i}_t + f(K^{h,i}_t) + p_t(g^i_{t+1} - w^i G) + h(g^i_{t+1}, w^i G) \\ = \; & Z^i_t W^h w^i + R^i_t D^i_{t-1} + (Z^i_t + Q^i_t) K^{h,i}_{t-1} + g^i_t \end{aligned}$$

It then becomes

$$\begin{aligned} V^{h,i}_t &= \max_{Kh, RD, g} u(C^{h,i}_t \left(S, Kh, RD, g \right)) \\ & \sum_{Z', \xi agg', \xi i'} \left(p(Z'^{agg'}, \xi^{i'} | Z, \xi^{agg'1}, \xi^i) \right) \left(V(S', Kh, RD, g) \right) \end{aligned}$$

Similarly, the size of the summation can be reduced by breaking into next-period run and no-run states, with probabilities of each arising (given Z' and ξ^{agg}') a function of the distribution of ξ^i.

Luke – this is good progress. I think that you will want to work to streamline the notation. Is the bottom line that a sunspot driven bank run (a) affects rewards to the parties so that it has welfare consequences; (b) affects various state variables going forward (so setting off a version of the transition dynamics that you looked at previously) and (c) that "markov sunspots" may lead to "expected bank runs."?

I would say (a) and (b) are entirely accurate, and (c) is basically correct, but I'd put it that there's a continuum of expectedness over sunspot states given fundamental states, more than a straight expected/unexpected situation.