# Memorandum: Alterations to Gertler and Kiyotaki 2015

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## 1 Overview

The following outlines the key equations of Gertler and Kiyotaki (2015), a basic explanation of their significance, a proposed plan for their alteration (with some results still to come, and the details of endogenous equity issuance to outsiders unspecified), and the significance thereof.

## 2 Key Features

Equations (1) through (4) detail the fundamental assets and production technologies of the GK model: a fixed quantity of a durable asset "capital" set to unity which may be operated by either bankers or households, but more efficiently by bankers.

$$K_t^b + K_t^h = 1 \tag{1}$$

Capital operated by bankers $K_t^b$ requires no other input and generates output of a final good as well as returning the undepreciated capital :

$$\begin{aligned} &\textit{date}\ t \rightarrow \textit{date}\ t+1 \\ K_t^b\ \textit{capital} \rightarrow &\begin{cases} Z_{t+1}K_t^b & \textit{output}\ , \\ K_t^b & \textit{capital} \end{cases} \end{aligned} \tag{2}$$

with $Z_{t+1}$ an aggregate productivity shock.

Households operated with an additional management cost $f(K_t^h)$, paid for with period $t$ final goods, to produce the same output the following period:

$$\begin{aligned} &\textit{date}\ t \rightarrow \textit{date}\ t+1 \\ \begin{cases} K_t^b & \textit{capital} \\ f(K_t^h) & \textit{goods} \end{cases} \rightarrow &\begin{cases} Z_{t+1}K_t^b & \textit{output}\ , \\ K_t^b & \textit{capital} \end{cases} \end{aligned} \tag{3}$$

The management was taken to be increasing, convex, and costless when no household operation took place:

$$f(K^h_t) = \frac{\alpha}{2}(K^h_t)^2 \tag{4}$$

In the planned extension, the key features remain the same with the following exceptions:

1. $K^b_t = \int_i K^{b,i}_t = \sum_i K^{b,i}_t$ (likewise $K^h_t$), the latter in the case of discrete household-bank pairs. This is made explicit because the main purpose of the extension is to explore bank heterogeneity.

2. The production functions (2), (3), and (4) operate at the level of the individual household or bank, rather than the aggregate.

3. The productivity shock $Z_{t+1}$ is a combined firm-specific and aggregate shock: $Z^i_{t+1} = Z^{agg}_{t+1} S^i_{t+1}$. This is added because even with added frictions, if all agents are subject to the same shocks motivating dispersion in outcomes across agents is infeasible.

4. Banks continuing in operation must pay an adjustment cost when changing the quantity of capital operated, $h(K^{h,i}_t, K^{h,i}_{t-1})$. On the part of buyers this reflects higher oversight costs when purchasing loans or investment projects from other banks or households, and initiation costs such as closing costs on mortgages which are in excess of costs of ongoing servicing of existing loans and projects. Costs to sellers reflect due diligence costs to assure the buyer of the quality of the assets, collections or litigation costs and haircuts on loans not rolled over, and shutdown costs of non-continuing investment projects. These transaction costs are assumed to take the form$h(K^{b,i}_t, K^{b,i}_{t-1}) = \frac{\gamma}{2}(\frac{K^{b,i}_t - K^{b,i}_{t-1}}{K^{b,i}_{t-1}})^2 K^{b,i}_{t-1}$ .

# 3 Households

In the GK model households are paired with banks, and deposit savings at banks, receiving a promised rate of return if no bank run returns, and the promised rate of return times a recovery rate if one does.

$$R^i_{t+1} = \begin{cases} \bar{R^i}_{t+1} & no - run \\ x^i_{t+1}\bar{R^i}_{t+1} & run \end{cases} \tag{5}$$

Households have log utility, and face a period by period budget constraint

$$C^h_t + D_t + Q_t K^h_t + f(K^h_t) = Z_t W^h + R_t D_{t-1} + (Z_t + Q_t) K^h_{t-1}$$

The left side of the equation is uses of funds (consumption, new deposits, capital held for next period, and the management cost), the right is sources (an endowment proportional to the aggregate shock, deposit returns, gross return from capital held last period).

In addition to being modified to reflect explicitly distinct households and banks with potentially different rates of return, households may hold equity in their paired bank, receive dividends from it (a fraction $\mathit{\Omega}$ of period total dividends $\mathit{\Delta}$), and make payments $\Xi$ for newly issued shares (receive for repurchased). These will be discussed further in the section on bankers.

$$C^{h,i}_t + D^i_t + Q_t K^{h,i}_t + f(K^{h,i}_t) + \Xi^i_t = Z_t W^h + R_t D_{t-1} + (Z_t + Q_t) K^h_{t-1} + \mathit{\Omega}^i_{t-1} \Delta^i_t$$

Apart from indexing households, no changes result in the stochastic discount factors $\Lambda_{t,t+s} = \beta^s \frac{C^h_t}{C^h_{t+s}}$ of households or their required rates of return. However, note that because households may end up with different quantities of capital, and thus different marginal returns to holding capital themselves, they may differ in their SDFs and thus risk free rates. Added are relations for the return on household ownership of bank equity.

$$E_t(\Lambda^i_{t,t+1} R^e_{t+1}) = 1$$

$$R^e_{t+1} = \varOmega^i_{t+1}\Delta^i_t + \frac{\varOmega^i_{t+1}}{\varOmega^i_t} \frac{(Z^i_{t+1} + Q_{t+1})K^{b,i}_t - R^i_{t+1}D^i_t - h(K^{b,i}_{t+1}, K^{b,i}_t) - \Delta^i_t + \Xi^i_{t+1}}{Q_t K^{b,i}_t}$$

(Details of equity holdings subject to change.)

Note that the possibility of complete loss on bank equity may require a change from log utility if the possibility of a bank run is anticipated.

# 4 Banks

In the GK model, risk neutral bankers earn income from operations and exit with constant exogenous probability $(1-\sigma)$ to motivate dividend payments and avoid saving indefinitely to avoid financial constraints (exiting bankers consume their entire net worth); bankers also stopped operations if their net worth fell to 0. To offset exiting bankers, an equal number of entering bankers receive a per period endowment of $w^b$, Additionally, they face an incentive compatibility constraint (15) $\theta Q_t K^b_t \leq V_t$: the quantity of assets purchased in period $t$ which they can divert before being forced into bankruptcy the following period must be less than or equal to their continuation value

$$V_t = E_t\left[\sum_{s=1}^{\infty} \beta^i (1-\sigma)\sigma^{s-1} c^b_{t+s}\right],$$

the discounted value of expected terminal consumption.

I propose endogenizing consumption by including a dividend payment decision, as the behavior of retained earnings may be important not only for the presence or absence of a bank run, but also for the evolution of leverage ratios and total intermediation in the absence of one.

With this inclusion, and the recognition that when consumption is decided on in the current period by continuing agents rather than forced by the exit process, the value function becomes

$$V_t = E_t\left[\sum_{s=0}^{\infty} \beta^i u(c^{b,i}_{t+s})\right]$$

Additionally, $c^{b,i}_t$ is, rather than the individual banker's net worth, that banker's portion of dividends paid by his bank $c^{b,i}_t = (1 - \varOmega^i_{t-1})\Delta^i_t$. I am provisionally proceeding on the assumption that $u(.)$ will remain risk neutral, however, to motivate payment of dividends rather than deferring consumption indefinitely, I expect to need to apply a faster discount rate for bankers than households: for their $\beta$ to be lower than the household's.

In the GK model, the bank and banker were identical, and surviving banks followed a net worth process determined by current conditions and previously made portfolio decisions:

$$n_t = (Z_t + Q_t)k^b_{t-1} - R_t d_{t-1};$$

net worth equal to (current shock and capital price) times capital held coming into period, less deposits repaid with interest.

Without exogenous exit, and with capital decisions, the bank and banker are no longer identical. Bank equity after operations and financing goes similarly:

$$n_t^i = (Z_t^i + Q_t)k_{t-1}^{b,i} - R_t^i d_{t-1}^i - h(k_t^{h,i}, k_{t-1}^{h,i}) + \Xi_t^i - \Delta_t^i;$$

As above, bankers are assumed to consume their entire portion of dividends, so net worth evolves according to

$$m_i^i = (1 - \Omega_t^i)n_t^i$$

In GK, the value of assets of a bank were equal to the value of net worth plus deposits taken. The same is true in the extension, allowing for different definition of bank equity:

$$Q_t k_t^{b,i} = d_t^i + n_t^i$$

With the exogenous capital payments decisions and constant returns to scale in both objectives and constraints, GK have the banker's problem (here with unanticipated bank runs) being simply choosing their leverage to maximize expected net worth in the following period, equivalently their single period return on equity, subject to the incentive constraint, giving (18):

$$\frac{V_t}{n_t} = \psi_t = \max_{\phi_t} E_t \left\{\beta\left(1 - \sigma + \sigma\psi_{t+1}\right)\left[\left(R_{t+1}^b - R_{t+1}\right)\phi_t + R_{t+1}\right]\right\}$$

Where

$$R_{t+1}^b = \frac{(Z_t + Q_t)k_t^{b,} - R_t d_t}{n_t}$$

$$\phi_t = \frac{Q_t k_t^b}{n_t}$$

Subject to a scaling of the incentive constraint.

For simplicity, I present the setup for the banker's problem sequentially adding extensions.

I begin with indexing and adding transaction costs while keeping the exogenous dividend process and complete ownership of the bank's equity by the banker: $\Xi_t^i$, $\Delta_t^i$, and $\Omega_t^i$ are 0 for all $i$ and $t$.

$$V_t^i = E_t\left[\sum_{s=1}^{\infty}\beta^i(1-\sigma)\sigma^{s-1}c_{t+s}^{b,i}\right] = E_t\left[\sum_{s=1}^{\infty}\beta^i(1-\sigma)\sigma^{s-1}(n_{t+s}^{b,i} - h(k_{t+s}^{h,i}, k_{t+s-1}^{h,i}))\right]$$

$$= \beta E_t\left[(1-\sigma)(n_{t+1}^{b,i} - h(k_{t+1}^{h,i}, k_t^{h,i})) + \sigma V_{t+1}^i\right]$$

This is (having returned to the GK definition of $n$) not identical to the recursive formulation of GK (16). Unlike GK, I cannot simply maximize the expected growth rate $\frac{n_{t+1}}{n_t}$, but $\frac{n_{t+1} - h(k_{t+1}^{h,i}, k_t^{h,i})}{n_t}$. Apart from the change in the value function, the incentive compatibility constraint remains the same. I do not expect the solution to do so, first of all because

The maximization problem becomes

$$V_t^i = \max_{k_t^{b,i},d_t^i} \beta E_t \left\{(1-\sigma)(n_{t+1}^{b,i} - h(k_{t+1}^{h,i}, k_t^{h,i})) + \sigma V_{t+1}^i\right\}$$

Subject to the balance sheet constraint

$$n_{t+1}^{b,i} = \left(R_{t+1}^{b,i} - R_{t+1}^i\right)\phi_t^i + R_{t+1}^i$$

with

$$\phi_t^i = \frac{Q_t^i k_t^{b,i}}{n_t^i - h(k_t^{h,i}, k_{t-1}^{h,i})}$$

now the leverage measured against net worth after adjustment costs, and also subject to the incentive constraint

$$\theta Q_t^i K_t^{b,i} \leq V_t^i$$

Because $k_{t-1}^{b,i}$ and $d_{t-1}^i$ (thus for fixed shock and capital price $n_t^i$ as well) are predetermined, it may be that here too the maximization can be done in terms of $\phi_t^i$ alone, as in GK. This depends on capital held remaining an increasing function of leverage given starting resources.

Adding endogenous dividend decisions while maintaining complete ownership of the bank by the banker results in needing to reformat the problem to account for explicit equity decisions, which take place in the present period rather than all considered consumption being in the future, while removing the case-splitting that was the result of exit:

$$V_t^i = \max_{k_t^{b,i},d_t^i} E\left\{(Z_t^i + Q_t^i)k_{t-1}^{b,i} - R_t^i d_{t-1}^i - Q_t^i k_t^{b,i} - h(k_t^{h,i}, k_{t-1}^{h,i}) + d_t^i + \beta V_{t+1}^i\right\}$$

With

$$\Delta_t^i = (Z_t^i + Q_t^i)k_{t-1}^{b,i} - R_t^i d_{t-1}^i - Q_t^i k_t^{b,i} - h(k_t^{h,i}, k_{t-1}^{h,i}) + d_t^i$$

the dividends paid in each period (return on assets entering period less deposits repaid, value of assets held for next period, and transaction costs, plus deposits raised).

The former incentive constraint on deposits relative to purchased assets remains (with time reindexed), and is sufficient to ensure the banker is unable to divert assets or deposits to himself when combined with

$$\Delta_t^i \leq (Z_t^i + Q_t)k_{t-1}^{b,i} - R_t^i d_{t-1}^i - h(k_t^{h,i}, k_{t-1}^{h,i})$$

which has that dividends cannot be greater than the liquidation value of the bank's entering assets less transaction costs and deposits repaid: a banker can't issue a large number of new deposits and pay a huge dividend, then go out of business.

Reformatting into a decision in terms of the investment rate $I_t^i = \frac{k_t^{b,i} - k_{t-1}^{b,i}}{k_{t-1}^{b,i}}$ gives:

$$V_t^i = \max_{I_t^i,d_t^i} E\left\{(Z_t^i + Q_t)k_{t-1}^{b,i} - R_t^i d_{t-1}^i - Q_t(1+I_t^i)k_{t-1}^{b,i} - h(I_t^i, k_{t-1}^{h,i}) + d_t^i + \beta V_{t+1}^i\right\}$$

The capital/investment side of things can be solved out with q-theory methods given shock dynamics, future prices, and future deposits.

With risk neutrality and unanticipated bank runs, the first order condition is $R_t^i = \beta^{-1}$, which guarantees that in the case of unanticipated bank runs, banks will always issue as many deposits as is consistent with their incentive constraint (or none), unless their household's risk free rate is exactly equal to the banker's discount rate. In cases where households may anticipate a bank run, the potential for influencing the interest rate that must be paid may enable different results. If not, then an alternative utility function would be required to generate results that diverge from GK apart from slowing things down and/or being amplified due to transaction costs, without resorting to a design that generates divergent capital prices.

In the absence of risk spreads due to anticipated bank runs, roughly the same would hold including equity issuance, in that the banker would see a similar dividend and deposit relationship. The banker's problem is more complex, however:

$$V_t^i = \max_{k_t^{b,i}, d_t^i} E\left\{(1-\Omega_t^i)((Z_t^i + Q_t^i)k_{t-1}^{b,i} - R_t^i d_{t-1}^i - Q_t^i k_t^{b,i} - h(k_t^{h,i}, k_{t-1}^{h,i}) + d_t^i + \Xi_t^i + \beta V_{t+1}^i)\right\}$$

With the previous constraints and the outside ownership a yet to be determined function of the old outside ownership fraction, the money raised, and other variables:

$$\Omega_t^i = f(\Omega_{t-1}^i, \Xi_t^i, Q_t^i k_t^{b,i}, d_t^i, ...)$$

# 5 Aggregation and Equilibrium without Bank Runs

GN has the value of assets held by the banking system equal to system leverage times net worth:

$$Q_t K_t^b = \phi_t N_t$$

The same holds after an aggregation in the expanded model, integrating/summing each side reflecting the individual banks.

The resources available to bankers as a whole before financing decisions and transaction costs follows:

$$N_t = \sum_i n_t^i = \sum_i (Z_t^i + Q_t) k_{t-1}^{b,i} - R_t^i d_{t-1}^i + W^b$$

Where in the absence of a recent bank run, $W^b = 0$ as no new bankers are replacing failed banks. If keeping the exogenous entry and exit process, the resources would be more similar to GK's:

$$N_t = \sum_i n_t^i = \sigma\left[\sum_i (Z_t^i + Q_t) k_{t-1}^{b,i} - R_t^i d_{t-1}^i + W^b\right]$$

with $W^b = (1-\sigma)\, w^b$ the endowment rate.

GK has total banker consumption

$$C_t^b = (1-\sigma)\left[(Z_t + Q_t) k_{t-1}^b - R_t d_{t-1}\right]$$

For the extended model with exogenous entry and exit follows similarly, but subtracts transaction costs:

$$C_t^b = \sum_i c_t^i = (1-\sigma)\left[\sum_i (Z_t^i + Q_t)k_{t-1}^{b,i} - R_t^i d_{t-1}^i - h(k_t^{h,i}, k_{t-1}^{h,i})\right]$$

And with endogenous financing decisions includes new deposits taken and equity raised from households (again, the next step assumes equity raising is 0 and household ownership of bank equity is permanantly 0)

$$C_t^b = \sum_i c_t^i = \sum_i (1-\Omega_t^i)\left[(Z_t^i + Q_t)k_{t-1}^{b,i} - R_t^i d_{t-1}^i - h(k_t^{h,i}, k_{t-1}^{h,i}) + d_t^i + \Xi_t^i\right]$$

GK has total output as the sum of output from capital, the household's endowment, and entering bankers' endowment:

$$Y_t = Z_t + Z_t W^h + W^b$$

The same continues to hold, summing over households and bankers, with the above specifications for banker endowment.

GK has output equal to management costs and consumption of each type of agent:

$$Y_t = f(K_t^h) + C_t^h + C_t^b$$

With the addition of the transaction cost, the same remains true:

$$Y_t = C_t^h + C_t^b + \sum_i f(k_t^{h,i}) + h(k_t^{b,i}, k_{t-1}^{b,i})$$

# 6 Unanticipated Bank Runs

The condition for an unanticipated bank run to have the potential to occur in GK is that the recovery rate on deposits be less than 1, that the banking system has insufficient assets to cover all promised deposits when capital has a price $Q_t^*$ which obtains in a bank run:

$$x_t = \frac{(Q_t^* + Z_t)\,K_{t-1}^b}{R_t D_{t-1}} < 1$$

or

I have the same hold in my model on a bank by bank basis with the addition of adjustment costs:

$$x_t^i = \frac{(Q_t^* + Z_t^i)\,k_{t-1}^{b,i} - h(0, k_{t-1}^{b,i})}{R_t d_{t-1}} < 1$$

(Or, if the parameterization is such that *h'* can ever be above $Q_t^*$, the maximum net worth contingent on capital remaining above 0) This also has a different interpretation of $Q_t^*$ and the addition of adjustment costs: rather than a fixed price, it is a function of the joint survival or failure of banks. Consider a case in which there are two banks, initially identical but receiving different ideosyncratic shocks. One may have a recovery ratio below 1 in the event that the price of capital is as it would be with all banks failing, and the other might retain positive net worth even in that situation. In that case, the latter could not in fact fall to a run, and the equilibrium price of capital would be higher, and $Q_t^*$ set to that instead; this might push the weaker bank's net worth above the point where it is susceptible to a run, and it might not. In the latter case, a run equilibrium is possible for the weaker bank but not the stronger. For simplicity, I take it to be the lowest consistent price; an alternative would be to do a random selection over the range of prices between the full disintermediation price and the no run price, and keep the lowest consistent price above the random selection.

With appropriately defined leverage (resources less costs, dividends, and plus net equity issuance as applicable), alternative formulation (29) of the recovery ratio in terms of returns on bank assets (in event of a run), the risk free rate, and leverage, will continue to hold, but again on a bank-by-bank basis (and indexed as such):

$$x_t = \frac{R_t^{b*}}{R_t} \cdot \frac{\phi_{t-1}}{\phi_{t-1} - 1} < 1$$

with return on bank capital operated

$$R_t^{b*} \equiv \frac{Q_t^* + Z_t}{D_{t-1}}.$$

The liquidation price is analogous to that seen in (30) and (31) of GK:

$$1 = K_t^h$$

$$Q_t^* = E_t \left[ \sum_{s=1}^{\infty} \Lambda_{t,t+s} (Z_{t+s} - \alpha K_{t+s)}^h) \right] - \alpha$$

Which sets the price equal to the discounted dividends of capital held by households for the indefinite future, subject to the bank net worth process:

$$N_{t+1} = W^b + \sigma W_b,$$

$$N_{t+i} = \sigma[(Z_{t+s} + Q_{t+s}) K_{t+s-1}^b - R_{t+s} D_{t+s-1}] + W^b, \; for \; all \; s \geq 2$$

The same holds on an individual basis, with of course bank net worth following the appropriate process described above.

# 7 Anticipated Bank Runs

GK continue with the case of households anticipating the possibility of systemic bank run in time *t+1* with probability $p_t$. This results in a different first order condition for households.

$$1 = \bar{R}_{t+1} E_t[(1 - p_t)\Lambda_{t,t+1} + p_t \Lambda_{t,t+1}^* x_{t+1}],$$

that is, that the promised rate of return times the probability of receiving that amount scaled by the applicable SDF plus the recovery amount in the event of a run times the then-applicable SDF must equal 1. The household must be indifferent to holding the deposit subject to the anticipation that the banking system might fail.

I have the same continue to hold. The recovery rate is based on the promised rather than the risk free rate, as in GK,

$$x_{t+1} = \min\left[1, \frac{R^{b*}_{t+1}}{\bar{R}_{t+1}} \cdot \frac{\phi_t}{\phi_t - 1}\right] = \min\left[1, \frac{\left(Q^*_{t+1} + Z_{t+1}\right) k^b_t}{\bar{R}_{t+1} d_t}\right]$$

As in GK, this has the effect of changing the promised rate of return to

$$\bar{R}_{t+1} = \frac{1 - p_t E_t\left(\Lambda^*_{t,t+1} R^{b*}_{t+1}\right) \frac{\phi_t}{\phi_t - 1}}{\left(1 - p_t\right) E_t\left(\Lambda_{t,t+1}\right)}$$

(on a pair by pair basis), so that banks will take into account leverage decisions' effects on their rate of return and resources available in the event of survival. This feeds through to the appropriate maximization problem described above, with the same constraints, but updated cost of borrowed funds. As GK found, this gives a way for expectations of a run to negatively affect bank equity even in the absence of one materializing.

A similar change (as yet to be determined) obtains with outside holdings of bank equity, though with expected utility rather than a promised rate of return.

I plan to follow GK in the assumption that if a run on a bank next period is impossible, the probability to a run on that bank is 0, and that otherwise the probability of a run is a decreasing function of the expected recovery ratio $p_t = 1 - E_t(x_{t+1})$. The more likely a bank run is to occur, and the more severe in the event one does occur, the higher the agents assign to seeing the sunspot outcome. Note that in my case, this is a pair-by-pair probability, that is, the assessed probability of that bank failing. The possibility of other banks failing while the one under consideration survives, however, will enter into the agents' consideration via price effects of capital.

# 8 Bibliography

Gertler, Mark and Nobuhiru Kiyotaki. "Banking, Liquidity, and Bank Runs in an Infinite Horizon Economy." *American Economic Review*, vol. 105, no. 7, 2015, pp. 2011-2034.