Homework 4
Math 270A, Fall 2019, Prof. Roman Vershynin

Problem 1 (Generalization of Borel-Cantelli)

Let $E_1, E_2, \dots$ be events on the same probability space. Assume that

$$\mathbb{P}(E_n) \to 0 \quad \text{and} \quad \sum_n \mathbb{P}(E_n \cap E_{n-1}^c) < \infty.$$

Show that

$$\mathbb{P}(E_n \text{ occur i.o.}) = 0.$$

sol: define $A_n = \{E_n \cap E_{n-1}^c\}$. Then we have $\sum_n \mathbb{P}\{A_n\} < \infty$. By Borel Cantelli lemma: $\mathbb{P}\{A_n \;\; i.o.\} = 0$, Now suppose $\mathbb{P}\{E_n \;\; i.o.\} > 0$, consequently $\mathbb{P}\left\{\{A_n i.o.\}^C \cap \{E_n i.o.\}\right\} = \mathbb{P}\{E_n i.o.\} > 0$. Pick any $\omega \in \{E_n \;\; i.o.\} \cap \{A_n i.o.\}^C$, $\exists n_k$ s.t. $\omega \in E_{n_k} \forall k$, and $\exists N(\omega)$ s.t. $\omega \notin A_n \;\; \forall n > N(\omega)$. For $n_k > N(\omega)$, since $\omega \notin A_{n_k}$, we must have $\omega \in E_{n_k - 1}$. Then apply this argument to recursively, we have $\omega \in E_n \;\; \forall n > N(\omega)$.
Then we haveP $\{\cup_{N=1}^{\infty} \{\omega | \omega \in E_n \;\forall n > N\}\}$=P $\{E_n \;\; i.o.\} > 0$, which means $\mathbb{P}\{\omega | \omega \in E_n \;\; \forall n > M\} > 0$ for some M. Then $\mathbb{P}\{E_n\} > \mathbb{P}\{\omega | \omega \in E_n \;\; \forall n > M\}$ for any $n > M$, which contradicts to $\mathbb{P}\{E_n\} \to 0$. Hence, by contradiction, we have $\mathbb{P}\{E_n \;\; i.o.\} = 0$.

Problem 2 (Extreme values)

Let $X_1, X_2, \dots$ be i.i.d. random variables with the standard exponential distribution, i.e.

$$\mathbb{P}\{X_i > x\} = e^{-x}, \quad x \geq 0.$$

(a) Show that

$$\limsup_n \frac{X_n}{\log n} = 1 \text{ a.s.}$$

(b) Let $M_n := \max_{1 \leq k \leq n} X_k$. Show that

$$\limsup_n \frac{M_n}{\log n} = 1 \text{ a.s.}$$

sol: $\sum_n \mathbb{P}\{X_n > \ln n\} + \sum_n \frac{1}{n} = \infty$. Since X_n are independent, by second Borel-Cantelli lemma, we have $\mathbb{P}\{X_n > \ln n \;\; i.o.\} = 1$ Then we have $\limsup_n \frac{X_n}{\ln n} \geq 1$ a.s. Now pick any $\epsilon > 0$, $\mathbb{P}\left\{\frac{X_n}{\ln n} > 1 + \epsilon\right\} = \frac{1}{n^{1+\epsilon}}$. Then, $\sum_n \mathbb{P}\left\{\frac{X_n}{\ln n} > 1 + \epsilon\right\} = \sum_n \frac{1}{n^{1+\epsilon}} < \infty$. By Borel-Cantelli lemma, $\mathbb{P}\left\{\frac{X_n}{\ln n} > 1 + \epsilon \;\; i.o.\right\} = 0$, which means
$\mathbb{P}\left\{\omega | \exists N(\omega) \;\; \frac{X_n}{\ln n} \leq 1 + \epsilon \;\; \forall n > N(\omega)\right\} = 1$. Then $\limsup_n \frac{X_n}{\ln n} \leq 1 + \epsilon$ a.s. Then take $\epsilon = \frac{1}{m}$, we have $\cap_{m=1}^{\infty} \mathbb{P}\left\{\limsup_n \frac{X_n}{\ln n} \leq 1 + \frac{1}{m}\right\} = \mathbb{P}\left\{\limsup_n \frac{X_n}{\ln n} \leq 1\right\} = 1$. Hence, we

conclude $\limsup_n \frac{X_n}{\ln n} = 1$ a.s.
(b) Firstly we observe $\frac{M_n}{\ln n} \geq \frac{X_n}{\ln n} \Rightarrow \limsup_n \frac{M_n}{\ln n} \geq \limsup_n \frac{X_n}{\ln n} = 1$ a.s.
Now suppose $\limsup_n \frac{M_n}{\ln n} > 1$. Since $\frac{1}{\ln n} \to 0$, consequently $\frac{X_N}{\ln n} \to 0$ for any fixed N, there has to be two subsequences n_k and $m_k \leq n_k$ s.t. $\frac{M_{n_k}}{\ln n_k} > 1$ and $M_{n_k} = X_{m_k}$. Then, we find $\frac{X_{m_k}}{\ln m_k} \geq \frac{X_{m_k}}{\ln n_k} > 1$, which contradicts to $\limsup_n \frac{X_n}{\ln n} = 1$. Hence, $\limsup \frac{M_n}{\ln n} \leq 1 \Rightarrow \limsup \frac{M_n}{\ln n} = 1$

Problem 3 (Generalization of Kolmogorov three-series theorem)

Let

$$\psi(x) := \begin{cases} x^2 & \text{when } |x| \leq 1 \\ |x| & \text{when } |x| \geq 1. \end{cases}$$

Let $X_1, X_2, \ldots$ be independent mean zero random variables. Show that if $\sum_n \mathbb{E}\,\psi(X_n) < \infty$, then $\sum_n X_n$ converges a.s.

sol: Let $Y_n = X_n \mathbb{1}_{\{|X_n| \leq 1\}}$. Note that

$$\mathbb{P}\{|X_n| = 1\} = \int_{\{|X_n \geq 1|\}} 1 d\omega \leq \int_{\{|X_n \geq 1|\}} |X_n| d\omega \leq \mathbb{E}\,\psi(X_n)$$

Hence, $\sum_n \mathbb{P}\{|X_n| \geq 1\} \leq \sum_n \mathbb{E}\,\psi(X_n) < \infty$. By Borel-Cantelli lemma, a.s. $X_n(\omega) \leq 1$ (i.e. $X_n(\omega) = Y_n(\omega)$) for all but finitely many n. Since X_n are zero mean, we have $\int_{\{|X_n| \leq 1\}} X_n d\omega = -\int_{\{|X_n| \geq 1\}} X_n d\omega$. Then,

$$\sum_n \mathbb{E}\,Y_n \leq \sum_n \int_{\{|X_n| \geq 1\}} |X_n| d\omega \leq \sum_n \mathbb{E}\,\psi(X_n) < \infty$$

Also,

$$\sum_n Var(Y_n) = \sum_n \int_{\{|X_n| \leq 1\}} X_n^2 d\omega \leq \sum_n \mathbb{E}\,\psi(X_n) < \infty$$

Then, by Kolmogorov's Two-Series Theorem, $\sum_n Y_n(\omega)$ converge a.s. Then, a.s. $\sum_n X_n(\omega) = \sum_{n=1}^{N(\omega)} X_n(\omega) + \sum_{n=N(\omega)+1}^{\infty} Y_n(\omega)$ converges.

Problem 4 (Failure of the strong law of large numbers)

Construct a sequence of independent mean zero random variables $X_1, X_2, \ldots$ such that

$$\frac{1}{n} \sum_{k=1}^{n} X_k \to \infty \text{ a.s.}$$

Why does not this example contradict the strong law of large numbers?

sol: let X_n s.t. $X_n = -n^5 + n^3$ with probability $\frac{1}{n^2}$ and $X_n = n^3$ with probability $1 - \frac{1}{n^2}$. Then we see that $\mathbb{E}\,X_n = 0 \ \ \forall n$.

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{X_n = -n^5 + n^3\right\} = \sum_{n}^{\infty} \frac{1}{n^2} < \infty$$

By Borel-Cantelli lemma, we have a.s. $X_n(\omega) = n^3$ for all but finitely many n. Then for any such ω, $\exists N(\omega) s.t.$ $X_n(\omega) = n^3$ for all $n > N(\omega)$. Then, $\frac{1}{n}\sum_{k=1}^{n} X_k(\omega) = \frac{1}{n}\Big(\sum_{k=1}^{N(\omega)} X_k(\omega) + \sum_{k=N(\omega)+1}^{n} X_k(\omega)\Big) \geq \frac{-O(N(\omega)^5)+(N(\omega)+1)^2+\cdots+n^2}{n} \geq n - \frac{O(N(\omega)^5)}{n} \to \infty$ as n $\to \infty$. Hence, we have $\frac{1}{n}\sum_{k=1}^{n} X_k(\omega) \to \infty$ a.s.

This does not contradict to SLLN since the version of nonidentical SLLN requires $\sum_n Var(X_n) < \infty$, which is clearly false in this example.

Problem 5 (Recurrent events)

Suppose disasters occur at random times X_i apart from each other. Precisely, k-th disaster occur at time $T_k := X_1 + \cdots + X_k$ where X_i are i.i.d. random variables taking positive values and with finite mean μ. Let

$$N(t) := \max\{n : \ T_n \leq t\}$$

be the number of disasters that have occurred by time t. Prove that

$$N(t) \to \infty \quad \text{and} \quad \frac{N(t)}{t} \to \frac{1}{\mu}$$

almost surely as $t \to \infty$.

(Hint: check that $N(t) < n$ iff $T_n > t$, and $T_{N(t)} \leq t < T_{N(t)+1}$. Use the strong law of large numbers for T_n/n.)

sol:
lemma: if $X_n \to C > 0$ a.s., then $\frac{1}{X_n} \to \frac{1}{C}$ a.s.
$\forall \omega$ s.t. $\lim_{n\to\infty} X_n(\omega) = C$, $\exists N(\omega)$ s.t. $X_n(\omega) > \frac{C}{2}$ $\forall n > N(\omega)$. Then we have $|\frac{1}{X_n(w)} - \frac{1}{C}| = \frac{|X_n(\omega)-C|}{|C\cdot X_n(\omega)|} \leq \frac{2|X_n(\omega)-C|}{C^2} \to 0$. The proof of lemma is completed.

Hence, it suffices to show $\frac{t}{N(t)} \to \mu$ a.s. Now consider $\frac{T_n}{n}$, by SLLN, $\frac{T_n}{n} \to \mu$ a.s. Then, $\forall t$, $T_{N(t)} \leq t < T_{N(t)+1}$, so $\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{T_{N(t)+1}}{N(t)} = \frac{T_{N(t)}}{N(t)} + \frac{X_{N(t)+1}}{N(t)}$. Since, X_n are real-valued, then, for a.s. ω , $T_N(\omega) < \infty$ for any $N \in \mathbb{N}$ (i.e. $T_N(\omega) < t$ for t large enough). Hence, we see that $\lim_{t\to\infty}$ N(t)=∞.
Then it left to show $\frac{X_{n+1}}{n} \to 0$ a.s. we can let $Y_n = X_n \cdot \mathbb{1}_{\{|X_n|<n\}}$, we show $\frac{Y_{n+1}}{n} \to \mu_n = \mathbb{E}[\frac{Y_{n+1}}{n}]$ a.s. by Borel Cantelli lemma, and $\mu_n \to 0$, so $\frac{Y_{n+1}}{n} \to 0$ a.s. Then, again by Borel Cantelli lemma $\frac{Y_{n+1}}{n} = \frac{X_{n+1}}{n}$ for all but finitely many n a.s. (The detail of this technique is in Problem 9). We conclude $\lim_{t\to\infty} \frac{X_{N(t)+1}}{N(t)} = 0$ a.s.
Finally ,since both $\frac{T_{N(t)}}{N(t)}$ and $\frac{T_{N(t)}}{N(t)} + \frac{X_{N(t)+1}}{N(t)}$ goes to μ a.s. we have $\frac{t}{N(t)} \to \mu$ a.s.

Problem 6 (Convergence of series)

Let $X_1, X_2, \ldots$ be independent random variables. Show that $\sum_n X_n$ converges in probability if and only if $\sum_n X_n$ converges almost surely.

(Hint: to prove that the sequence of partial sums S_n is Cauchy, argue as in the proof of Kolmogorov's two-series theorem but use Etemadi's maximal inequality.)

sol: If $\sum_n X_n$ converges in probability, then there is a random variable Y s.t. $S_n \to Y$ in probability.
Now fix $\delta > 0$ and any $N \in \mathbb{N}$, define $w_n = sup_{m>n}|S_m - S_n|$, By Etemadi's maximal inequality:

$\mathbb{P}\left\{max_{n\leq m\leq n+N}|X_n + \cdots + X_m| > \delta\right\} \leq 3max_{n\leq m\leq n+N}\mathbb{P}\left\{|X_n + \cdots + X_{n+N}| > \delta\right\} < \epsilon$
Take $N \to \infty$, we have

$$\mathbb{P}\left\{sup_{m>n}|S_m - S_n| > \delta\right\} \leq 3sup_{m>n}\mathbb{P}\left\{|Y - S_n| > \delta\right\} < \epsilon$$

For large enough n. Then, we conclude that $w_n \to 0$ in probability. However, w_n is monotonic decreasing by defition, we have $w_n \to 0$ a.s. Hence, $\sum_n X_n$ converges a.s.

If $S_n \to Y$ a.s. for some random variable Y. Then $\forall \delta > 0$, $\cap_{n=1}^{\infty}\mathbb{P}\left\{|Y - S_n| > \delta\right\} = 0 \Rightarrow \quad \forall\epsilon > 0, \cap_{n=N}^{\infty}\mathbb{P}\left\{|Y - S_n| > \delta\right\} < \epsilon$ for some N $\Rightarrow S_n \to Y$ in probability.

Problem 7 (Divergence of series with positive i.i.d. terms)

Let $X_1, X_2, \ldots$ be i.i.d. random variables taking non-negative values, and such that $\mathbb{P}\left\{X_i > 0\right\} > 0$. Prove that

$$\sum_n X_n = \infty \text{ a.s.}$$

sol:

$$\mathbb{P}\left\{X > 0\right\} = \cup_{n=1}^{\infty}\mathbb{P}\left\{X > \frac{1}{n}\right\} > 0 \Rightarrow \mathbb{P}\left\{X > \frac{1}{N}\right\} > 0 \text{ for some N}$$

Then, we have

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{X_n > \frac{1}{N}\right\} = \infty$$

Since X_n are independent, by Second Borel-Cantelli lemma, we have $\mathbb{P}\left\{X_n > \frac{1}{N} \;\; i.o.\right\} = 1$ Then, for a.s. ω, $\exists n_k$ s.t. $X_{n_k}(\omega) > \frac{1}{N}$. Since X_n are non-negative, we have

$$\sum_n X_n\omega \geq \sum_k X_{n_k}(\omega) = \sum_k \frac{1}{N} = \infty$$

Problem 8 (Diophantine approximation)

Call number $x \in [0,1]$ *badly approximable* (by rationals) if there exists $c = c(x) > 0$ and $\varepsilon = \varepsilon(x) > 0$ such that for any $p, q \in \mathbb{N}$ we have

$$\left|x - \frac{p}{q}\right| > \frac{c}{q^{2+\varepsilon}}.$$

Prove that almost all numbers in $[0,1]$ are badly approximable (i.e. all except a set of Lebesgue measure zero).

(Hint: fix c, ε. For each q, consider the set E_q of numbers x that satisfy the reverse inequality. Use Borel-Cantelli lemma for these sets.)

sol: Fix c, ϵ. Let $E_q = \cup_p \{x : |x - \frac{p}{q}| < \frac{c}{q^{2+\epsilon}}\}$ We note that if $p \geq (c+1)q$, we have $|\frac{p}{q} - x| \geq |c+1-x| \geq c > \frac{c}{q^{2+\epsilon}}$. This means if $p \geq (c+1)q$, then $|\frac{p}{q} - x| > \frac{c}{q^{2+\epsilon}}$ for any $x \in$ [0,1].

Hence,$\forall q$, $p \leq (c+1)q$. Then $m(E_q) \leq \sum_p m\{x : |x - \frac{p}{q}| < \frac{2c}{q^{2+\epsilon}}\} \leq 2(c+1)q \cdot \frac{c}{q^{2+\epsilon}} = \frac{2c(c+1)}{q^{1+\epsilon}}$. Then,

$$\sum_{q=1}^{\infty} m(E_q) = \sum_{q=1}^{\infty} \frac{2c(c+1)}{q^{1+\epsilon}} < \infty$$

Let A={x: x in finite many E_q}$\cap \mathbb{Q}^C$. By Borel Cantelli lemma, $m(A) = 1$. Then $\forall x \in A$, since x is in finitely many E_q, if any, so there are finite many p and their corresponding q, and we can take the minimum $\left(|x - \frac{p}{q}|q^2\right)$, which is positive because x is irrational. Take $0 < c(x) < min_{p,q}\left(|x - \frac{p}{q}|q^2\right)$ among those finitely many E_q and p. Clearly, $c(x) \leq c$. Then we have $\forall p, q$, $|x - \frac{p}{q}| > \frac{c(x)}{q^{2+\epsilon}}$

Problem 9 (Random harmonic series)

Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean μ. Prove that

$$\frac{1}{\ln n} \sum_{k=1}^{n} \frac{X_k}{k} \to \mu \text{ a.s.}$$

(Hint: work along the subsequence 2^{2^n}.)

sol: W.O.L.G, we assume that X_n are nonnegative.
It is known (intergral test) that

$$\ln n \leq \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1$$

For a fixed $\delta > 0$, take a subseqence $n_l = 2^{(1+\delta)^l}$, then have

$$\mu = \frac{\mu \cdot \ln n_l}{\ln n_l} \leq \frac{1}{\ln n_l} \mathbb{E}[\sum_{k=1}^{n_l} \frac{X_k}{k}] \leq \frac{1}{\ln n_l}(\ln n_l \cdot \mu + \mu) \to \mu \text{ as } l \to \infty$$

Now we define $Y_n = X_n \cdot \mathbb{1}_{\{|X_n| < n\}}$. By Dominated Convergence Theorem, $\mathbb{E}[Y_n] \to \mathbb{E}[X_n]$. Then $\forall \epsilon > 0$ $\exists N$ s.t. $\mathbb{E}[X_n] - E[Y_n] < \epsilon$ for $n > N$.
Let $S_n = \frac{1}{\ln n} \sum_{k=1}^{n} \frac{Y_k}{k}$. We have $\mathbb{E}[\frac{1}{\ln n} \sum_{k=1}^{n} \frac{X_k}{k} - S_n] \leq \frac{1}{\ln n}(\sum_{k=1}^{N} \frac{\mathbb{E}[X_k] - \mathbb{E}[Y_k]}{k} + \sum_{k=1}^{n} \frac{\epsilon}{k}) \leq \frac{1}{\ln n}(\sum_{k=1}^{N} \frac{\mathbb{E}[X_k] - \mathbb{E}[Y_k]}{k} + \epsilon(\ln n + 1)) = C\epsilon$ for some constant C (i.e. o(ϵ)).
Hence, we have $\mathbb{E}[S_n] = \mu_n \to \mu$.

For any fixed $\epsilon > 0$, we note that $\sum_m \frac{\mathrm{Var}(Y_m)}{m^2} \le \sum_m \frac{\mathbb{E}[Y_m^2]}{m^2} \le 2\,\mathbb{E}[X] = 2\mu$ as we showed in the class.

$$\mathbb{P}\left\{|S_{n_l} - \mu_{n_l}| > \epsilon\right\} \le \frac{1}{\epsilon^2(1+\delta)^l(\ln 2)^2}\sum_{k=1}^{n_l}\frac{\mathbb{E}[Y_k^2]}{k^2} \le \frac{2\mu}{\epsilon^2(1+\delta)^l(\ln 2)^2}$$

$$\sum_{l=1}^{\infty}\mathbb{P}\left\{|S_{n_l} - \mu_{n_l}| > \epsilon\right\} \le \sum_{l=1}^{\infty}\frac{2\mu}{\epsilon^2(\ln 2)^2(1+\delta)^l} < \infty$$

Then, by Borel Cantelli lemma, we have $S_{n_l} \to \mu_{n_l}$ a.s., so $S_{n_l} \to \mu$ a.s (triangle ineqaulity). Now consider $n_l < N < n_{l+1}$, we have that

$$\frac{1}{\ln n_{l+1}}\sum_{k=1}^{n_l}\frac{Y_k}{k} \le \frac{1}{\ln N}\sum_{k=1}^{N}\frac{Y_k}{k} \le \frac{1}{\ln n_l}\sum_{k=1}^{n_{l+1}}\frac{Y_k}{k}$$

$$\Rightarrow \frac{1}{1+\delta}\frac{1}{\ln n_l}\sum_{k=1}^{n_l}\frac{Y_k}{k} \le \frac{1}{\ln N}\sum_{k=1}^{N}\frac{Y_k}{k} \le (1+\delta)\frac{1}{\ln n_{l+1}}\sum_{k=1}^{n_{l+1}}\frac{Y_k}{k}$$

Then we take $\delta \to 0$, then we have:

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{Y_k}{k} \to \mu \text{ a.s.}$$

Finally, we note that $\sum_n X_n \cdot \mathbb{1}_{\{|X|>n\}} \le X \le \left(\sum_n X \cdot \mathbb{1}_{\{|X|>n\}}\right) + 1$, so have $\sum_n \mathbb{P}\{|X| < n\} < \infty$ if and only if $\mathbb{E}[X] < \infty$. Then we conclud, by Borel Cantelli lemma, $\sum_n \mathbb{P}\{|X_n| > n\} < \infty$ impplied that $Y_n = X_n$ for all but finitely many n for a.s. fixed ω. Then difference divided by $\ln n$ goes 0 as $n \to \infty$. Then, we have

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{X_k}{k} \to \mu \text{ a.s.}$$