Homework 3
Math 270B, Winter 2020, Prof. Roman Vershynin

Problem 1 (Quadratic martingales)

Let $X_1, X_2, \dots$ be independent random variables with means μ_i and finite variances $\sigma_i = \mathbb{E}\, X_i^2$. Consider the sums $S_n := X_1 + \cdots + X_n$. Find sequences of real numbers (b_i) and (c_i) such that $S_n^2 + b_n S_n + c_n$ is a martingale (with respect to the σ-algebras generated by $X_1, \dots, X_n$).

sol: Since $\sigma_i^2 = \mathbb{E}[X_i^2]$, $\mu_i = 0, \forall i$. Then $\mathbb{E}[S_{n+1}^2 + b_n S_{n+1} + c_{n+1} | \mathcal{F}_n]$

$$= \mathbb{E}[(S_n + X_{n+1})^2 + b_n(S_n + X_{n+1}) + c_{n+1} | \mathcal{F}_n]$$

$$= S_n^2 + 2S_n \mathbb{E}[X_{n+1}|\mathcal{F}_n] + \mathbb{E}[X_{n+1}^2] + b_{n+1}S_n + c_{n+1}$$

Since S_n is a martingale, $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_{n+1} - S_n|\mathcal{F}_n] = 0$ a.s.

$$= S_n^2 + \sigma_i + b_{n+1}S_n + c_{n+1} = S_n^2 + b_n S_n + c_n$$

Then, we have that $c_{n+1} - c_n = -\sigma_{n+1}$, $b_{n+1} = b_n$. Then, let $c_1 = -\sigma_1$, $b_1 = 1$. We find sequences $c_n = \sum_{i=1}^n -\sigma_i$ and $b_n = 1$ make it a martingale.

Problem 2 (Maximum of two martingales)

(a) Show that if (X_n) and (Y_n) are martingales with respect to the same filtration, then $X_n \vee Y_n$ is a s ubmartingale.

(b) Give an example showing that $X_n \vee Y_n$ needs not be a martingale.

sol:
(a). By linearity of conditional expectation, we see that if X_n and Y_n are martingales, so are $X_n + Y_n$ and $X_n - Y_n$. Therefore, we have,

$$\mathbb{E}[X_{n+1} \vee Y_{n+1}|\mathcal{F}_n] = \frac{1}{2}\mathbb{E}[(X_{n+1}+Y_{n+1})+|X_{n+1}-Y_{n+1}|\Big|\mathcal{F}_n] = \frac{1}{2}[X_n+Y_n]+\frac{1}{2}\mathbb{E}[|X_{n+1}-Y_{n+1}|\Big|\mathcal{F}_n]$$

by conditional Jensen's inequality, $\mathbb{E}[|X_{n+1} - Y_{n+1}|\Big|\mathcal{F}_n] \geq \Big|\mathbb{E}[X_{n+1} - Y_{n+1}|\mathcal{F}_n]\Big| = |X_n - Y_n|$, so we have

$$\mathbb{E}[X_{n+1} \vee Y_{n+1}|\mathcal{F}_n] \geq \frac{1}{2}[X_n + Y_n] - \frac{1}{2}|X_n - Y_n| = X_n \vee Y_n$$

(b). Let X_n be a random walk (sum of n i.i.d Rademacher random variables) and Y_n be a trivial predictable martingale s.t. $Y_n = 10, \forall n$. Let $M_n = X_n \vee Y_n$. In problem 4, we showed that if M_n is a martingale, then $\mathbb{E}[M_n]$ are the same for any n. Clearly, $\mathbb{E}[M_n] = 10$ for $n \leq 10$. By CLT, $\frac{X_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1)$. We see that for large enough

n, $\mathbb{P}\{X_n > 10\} \sim \mathbb{P}\left\{N(0,1) > \frac{10}{\sqrt{n}}\right\} \to \frac{1}{2} > 0$. Hence, $\mathbb{E}[M_n] = \mathbb{E}[10 \cdot \mathbf{1}_{\{X_n \leq 10\}} + X_n \mathbf{1}_{\{X_n > 10\}}] > 10$. Hence, $\mathbb{E}[M_n]$ are not the same for any n, so M_n is not a martingale.

Problem 3 (An unbalanced martingale)

Give an example of a martingale (X_n) such that $X_n \to -\infty$ a.s.

sol: Let $\{Y_n\}$ be a sequence of independent random variables s.t. $\mathbb{P}\{Y_n = -1\} = 1 - \frac{1}{n^2}$ and $\mathbb{P}\{Y_n = -n^2 + 1\} = \frac{1}{n^2}$. We see that Y_n are zero mean, so that $X_n = \sum_{k=1}^n Y_k$ is a martingale with filtration $\mathcal{F}_n = \sigma(Y_1, \cdots, Y_n)$. As $\sum_{k=1}^\infty \mathbb{P}\{Y_k \neq -1\} = \sum_{k=1}^\infty \frac{1}{k^2} < \infty$, by Borel-Cantelli lemma, we have that for a.s. ω, $Y_k(\omega) = -1$ for all but finitely many k. Hence, for a.s. ω, $X_n(\omega) = \sum_{k=1}^n Y_k(\omega) \to -\infty$.

Problem 4 (One-sided bounded martingales)

Let (X_n) be a martingale that is bounded a.s. either above or below by some constant M. Show that $\sup_n \mathbb{E}|X_n| < \infty$.

$\forall n$, we have that $\mathbb{E}[\mathbb{E}[X_{n+1}|\mathcal{F}_n]] = \mathbb{E}[X_{n+1}]$ by the tower property. On the other hand, $\mathbb{E}[\mathbb{E}[X_{n+1}|\mathcal{F}_n]] = \mathbb{E}[X_n]$ by the definition of martingale. Therefore, $\mathbb{E}[X_n]$ are the same for any n. We have $\mathbb{E}[X_n] = C, \forall n$. Now, W.O.L.G, assume that X_n are bounded above by M. Then, as we decompose $X_n = X_n^+ - X_n^-$ as usual, we have that $\mathbb{E}[X_n^+] \leq M, \forall n$. Hence, $\mathbb{E}[X_n^-] = \mathbb{E}X_n^+ - \mathbb{E}[X_n] \leq M - C$, which implied that $E[|X_n|] = \mathbb{E}[X_n^+] + \mathbb{E}[X_n^-] \leq 2M - C, \forall n$.

Problem 5 (A product martingale vanishes)

Let $Z_1, Z_2, \ldots$ be nonnegative i.i.d. random variables with $\mathbb{E}\, Z_i = 1$ and $\mathbb{P}\{Z_i = 1\} < 1$. Show that, as $n \to \infty$,

$$\prod_{i=1}^n Z_i \to 0 \quad \text{a.s.}$$

sol: Let $X_n = \prod_{i=1}^n Z_i$ and $\mathcal{F}_n = \sigma(Z_1, \cdots, Z_n)$, then we see that X_n is a martingale. Also, $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] = \prod_{i=1}^n \mathbb{E}[Z_i]$=1 by non-negativeness and independence of Z_i. Then, by martingale converges theorem, we have that $X_n \to X$ for some random variable X. By Fatou's lemma, we have that $\mathbb{E}[X] \leq \lim_{n\to\infty} \mathbb{E}[X_n] = 1$. Now suppose on a set E of positive measure that X_n converges to X and $X > 0$ on E. Then, $Z_i(\omega) \neq 0, \forall i, \forall \omega \in E$. Hence, $\ln X_n$ and $\ln X$ is well-defined on E. Hence,

$$\forall \omega \in E, \ln X(\omega) = \sum_{i=1}^\infty \ln Z_i(\omega) \text{converges}$$

then $\lim_{n\to\infty} \ln Z_i(\omega) = 0 \Rightarrow \lim_{n\to\infty} Z_i(\omega) = 1$. However, since $\mathbb{P}\{Z_i = 1\} < 1$, we must have $\mathbb{P}\{|Z_i - 1| > \delta\} > 0$ for some $\delta > 0$. Then $\mathbb{P}\{|Z_i - 1| > \delta\} = \infty$, which means that, for a.s. ω, $|Z_i(\omega) - 1| > \delta$ infinitely often, which contradicts to $Z_i \to 1$ on a set of positive measure.

Problem 6 (Square integrable martingales)

Let (X_n) be a martingale and let $\Delta_n := X_n - X_{n-1}$ be the martingale differences. Prove that if $X_0 = 0$ and $\sum_{n=1}^{\infty} \Delta_n^2 < \infty$ then X_n converges in L^2 to some random variable X.

sol:

$$\forall n, \mathbb{E}[X_n^2] = \mathbb{E}[(\sum_{i=1}^{n} \Delta_i)^2] = \mathbb{E}[\sum_{i=1}^{n} \Delta_i^2] + \mathbb{E}[2 \sum_{1 \le i < j \le n} \Delta_i \Delta_j]$$

$$\le \sum_{i=1}^{\infty} \mathbb{E}[\Delta_i^2] + 2 \sum_{1 \le i < j \le n} \mathbb{E}[\Delta_i \mathbb{E}[\Delta_j | \mathcal{F}_i]] = \sum_{i=1}^{\infty} \mathbb{E}[\Delta_i^2] < \infty$$

Since $\mathbb{E}[\Delta_j | \mathcal{F}_i] = 0$ a.s.
Therefore, we have that $\limsup_n \mathbb{E}[|X_n|^2] < \infty$, by the L^p convergence theorem (Thm 5.4.5), we have that $X_n \to X$ a.s. and in L^2 for some random variable X.

Problem 7 (Branching process)

Construct a branching process (Z_n) in the usual way. Namely, let X be a random variable with mean μ and variance σ^2; it specifies the distribution of the offspring. Set

$$Z_{n+1} := X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)}$$

to be the size of the population at time $n+1$, where all $X_i^{(k)}$ are i.i.d. random variables distributed identically with X.

(a) Show that $X_n := Z_n / \mu^n$ defines a martingale.

(b) Show that

$$\mathbb{E}[Z_{n+1}^2 | \mathcal{F}_n] = \mu^2 Z_n^2 + \sigma^2 Z_n.$$

(c) Deduce that (X_n) is bounded in L^2 if an only $\mu > 1$.

(d) Show that when $\mu > 1$, the L^2-limit X of X_n (assume it exists) satisfies

$$\mathrm{Var}(X) = \frac{\sigma^2}{\mu(\mu - 1)}.$$

sol: (a). Since Z_n is integer valued, we have $Z_n = \sum_{k=1}^{\infty} k \mathbf{1}_{\{Z_n = k\}}$.

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \frac{1}{\mu^{n+1}} \mathbb{E}[\sum_{k=1}^{\infty} X_{n+1} \mathbf{1}_{\{Z_n = k\}} | \mathcal{F}_n] = \frac{1}{\mu^{n+1}} \sum_{k=1}^{\infty} \mathbf{1}_{\{Z_n = k\}} \mathbb{E}[X_1^{n+1} + \cdots X_k^{n+1} | \mathcal{F}_n]$$

since $\{X_k^{n+1}\}$ is independent to $\mathcal{F}_n$, we have

$$= \frac{\mu \sum_{k=1}^{\infty} k \mathbf{1}_{\{Z_n=k\}}}{\mu^{n+1}} = \frac{Z_n}{\mu^n}$$

(b).

$$\mathbb{E}[(X_n + X_{n+1} - X_n)^2 | \mathcal{F}_n] = \mathbb{E}[X_n^2 + 2X_n(X_{n+1} - X_n) + (X_{n+1} - X_n)^2 | \mathcal{F}_n]$$

since X_n is $\mathcal{F}_n$ measureable, and $E[X_{n+1} - X_n | \mathcal{F}_n] = 0$ a.s.

$$= X_n^2 + 0 + \mathbb{E}[(X_{n+1} - X_n)^2 | \mathcal{F}_n] = X_n^2 + \frac{1}{\mu^{2n+2}} \mathbb{E}[(Z_{n+1} - \mu Z_n)^2 | \mathcal{F}_n]$$

note that $\sum_{k=1}^{\infty} \mathbf{1}_{\{Z_n=k\}} = 1$ and $Z_n = \sum_{k=1}^{\infty} k \mathbf{1}_{\{Z_n=k\}}$. We have

$$= X_n^2 + \frac{1}{\mu^{2n+2}} \mathbb{E}[(Z_{n+1} \sum_{k=1}^{\infty} \mathbf{1}_{\{Z_n=k\}} - \sum_{k=1}^{\infty} \mu k \mathbf{1}_{\{Z_n=k\}})^2 | \mathcal{F}_n]$$

since $\mathbf{1}_{\{Z_n=k_1\}} \mathbf{1}_{\{Z_n=k_2\}} = 0$ for $k_1 \neq k_2$, by Monotone convergence theorem,

$$= X_n^2 + \frac{1}{\mu^{2n+2}} \sum_{k=1}^{\infty} \mathbf{1}_{\{Z_n=k\}} \mathbb{E}[(\sum_{i=1}^{k} X_i^{n+1} - \mu k)^2 | \mathcal{F}_n]$$

since X_i^{n+1} is independent to $\mathcal{F}_n$, the sum inside expectation is $\mathbb{E}[\Big(\sum_{i=1}^{k} (X_i^{n+1} - \mu)\Big)^2] = \mathbb{E}[\sum_{i=1}^{k} (X_i^{n+1} - \mu)^2] = k\sigma^2$. Therefore,

$$\mathbb{E}[X_{n+1}^2 | \mathcal{F}_n] = X_n^2 + \frac{1}{\mu^{2n+2}} \sum_{k=1}^{\infty} \mathbf{1}_{\{Z_n=k\}} k\sigma^2] = X_n^2 + \frac{1}{\mu^{2n+2}} \sigma^2 Z_n$$

$$\Rightarrow \mathbb{E}[Z_{n+1}^2 | \mathcal{F}_n] = \mu^2 Z_n + \sigma^2 Z_n$$

(c)&(d). Since $\mathbb{E}[X_n] = \mathbb{E}[X_0] = 1 = \mathbb{E}[X_0^2], \forall n$, so $E[X_{n+1}^2] = \mathbb{E}[X_n^2] + \frac{1}{\mu^{n+2}} \sigma^2 \mathbb{E}[\frac{Z_n}{\mu^n}] = \mathbb{E}[X_n^2] + \frac{\sigma^2}{\mu^{n+2}}$. Hence,

$$\mathbb{E}[X_n^2] = 1 + \sum_{k=2}^{n} \frac{\sigma^2}{\mu^k}$$

Then, $\mathbb{E}[X_n^2]$ is uniformly bounded if and only if $\mu > 1$.
If $\mu > 1$, by Thm 5.4.5, we have X_n converges to X a.s. for some X and $\mathbb{E}[X^2] = \lim_{n\to\infty} \mathbb{E}[X_n^2] = 1 + \frac{\sigma^2}{\mu(1-\mu)}$. As $\mathbb{E}[X_n^2]$ are uniformly bounded, $\{X_n\}$ is uniformly integrable, so $\mathbb{E}[X] = \lim_{n\to\infty} E[X_n] = 1$. Then, we conclude that $\operatorname{Var} X = \frac{\sigma^2}{\mu(1-\mu)}$

Problem 8 (Unbounded martingale that converges a.s.)

Find an example of a martingale (X_n) that converges a.s. to some random variable X, but for which $\limsup_n \mathbb{E}\,|X_n| = \infty$.

(Hint: define the sequence $a_1 := 2$, $a_n := 4\sum_{i=1}^{n-1} a_i$. Consider independent random variables Z_n that take value $\pm a_n$ with probability $(2n)^{-2}$ and 0 with probability $1 - n^{-2}$. Define $X_n := \sum_{i=1}^{n} Z_i$.)

sol: Define $\{a_n\}$ as above. Let $\{Z_i\}$ be a sequence of random variables that $\mathbb{P}\{Z_i = -a_i\} =$

$\mathbb{P}\{Z_i = a_i\} = \frac{1}{2n^2}$ and $\mathbb{P}\{Z_i = 0\} = 1 - \frac{1}{n^2}$ Let $\mathcal{F}_n = \sigma(Z_1, \cdots, Z_n)$, we see that X_n is a martingale with filtration $\mathcal{F}_n$, as Z_n are zero mean. We note that $a_n > 2^n$, so $\lim_{n\to\infty} \frac{a_n}{n^2} = \infty \Rightarrow \lim_{n\to\infty} \frac{3}{4}\frac{a_n}{n^2} = \infty$. Then, $\mathbb{E}[|X_n|] \le \mathbb{E}|X_n| \cdot \mathbf{1}_{\{|Z_n|=a_i\}} \le \frac{3a_n}{4n^2} \to \infty$. However, by borel cantelli lemma, we see that for a.s. ω, $Z_i(\omega) = 0$ for all but finitely many i. Hence, $\lim_{n\to\infty} X_n(\omega)$ exists a.s. Then define $X = \begin{cases} \lim_{n\to\infty} X_n(\omega) & \text{if the limit exsits} \\ 0 & else \end{cases}$. Then, $X_n \to X$ a.s. and X is a random variable (measurable).