A TikTok Online Assignment Problem

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I solved this problem in an online-assignment of TikTok last year. Recently, a friend of mine talked about this problem when we had dinner together, and I decided to write up this interesting problem. This problem is essentially equivalent to the classical problem, counting the number of non-negative integer solution of linear equation, so it can be solved by the standard approach together with inclusion-exclusion principle.

Problem 1. *Real Programmer Game (RPG) is about having a hero swinging sticks at a monster. The monster has N Health Point(HP). It is killed when HP drops to 0 or negative. Each swing of the hero reduces monster's HP by a (evenly distributed) random integer in [0, M]. What's the probability for hero to kill the monster in K swings?*

sol: (1) For the simplest case that $M = 1$, i.e. each swing takes value either 0 or 1. As swings are independent, we need to find the probability that at least K swings take value 1. Denote event A_m=$\{$exactly m swings take value 1$\}$. Then, we want to find

$$P(\cup_{m=k}^{n} A_m) = \sum_{m=N}^{K} P(A_m) = \sum_{m=N}^{K} \frac{\binom{K}{m}}{2^K} = \frac{1}{2^K} \sum_{m=N}^{K} \binom{K}{m}$$

(2). For the general case, we have that each swing is the uniform distribution on the set $\{0, \cdots, M\}$ independently. This problem can be transformed to the classical combinatorical problem, the number of integer solution. Given K swings, $\sum_{i=1}^{K} x_i$ has $(M+1)^K$ possible combinations (the sum is not necessarily distinct though), i.e. $|\Omega| = (M+1)^K$. Now, we need to find the number of integer solutions among the all possible distinct combinations that satisfy

$$x_1 + \cdots x_K \geq N \quad s.t. \quad 0 \leq x_i \leq M \quad for \quad 1 \leq i \leq K$$

Let S be the set of all distinct solutions that satisfy the constrain above, the desired probability is just $\frac{|S|}{|\Omega|}$. We will compute $|S|$ via computing $|S^C|$, which is the number of integer solutions of

$$x_1 + \cdots x_K < N \quad s.t. \quad 0 \leq x_i \leq M \quad for \quad 1 \leq i \leq K$$

It equals the number of integer solutions of

$$x_1 + \cdots x_K + x_{K+1} = N + K \quad s.t. \;\; x_{K+1} \geq 1, \quad 1 \leq x_i \leq M \quad for \;\; 1 \leq i \leq K$$

If we ignore the upper bounds in the constrain, it is just the number of positive solutions, which is $\binom{N+K-1}{K}$*. Now, we need to substract all solutions where at least one* x_i *violates the upper bound. To do it, we need to use inclusion-exclusion principle.*

Define A_m=$\{$*solutions such that* x_i *violate the upper bound for* $1 \leq i \leq m\}$*. Then*

$$|S^C| = \binom{N+K-1}{K} + \sum_{m \geq 1} (-1)^m \binom{K}{m} \cdot |A_m|$$

For $m = 1$*, there are* $\binom{N+K-M-2}{K}$ *solutions such that* $x_1 > M$*. It is because to count the number of solutions with constrain* $x_1 \geq M + 1, \quad x_i \geq 1$ *for* $2 \leq i \leq M$ *is equivalent to count the number of solutions of*

$$x_1 + \cdots x_K + x_{K+1} = N + K - (M+1) \quad s.t. \;\; x_i \geq 1 \quad for \;\; 1 \leq i \leq K+1$$

Therefore, we want to subtract all solutions that $x_i > M$*, in total* $K * \binom{N+K-M-2}{K}$*. However, there are a lot of double counting in this way, so we need to keep computing* A_m *for* $m > 1$*.*

For general m*, the constrain is* $x_1, \cdots, x_m \geq M+1$*,* $x_{m+1}, \cdots, x_{K+1} \geq 1$*, which is equivalent to the number of solutions of*

$$x_1 + \cdots x_K + x_{K+1} = N + K - m(M+1) \quad s.t. \;\; x_i \geq 1 \quad for \;\; 1 \leq i \leq K+1$$

Then, number of solutions is $\binom{N+K-1-m(M+1)}{K} = |A_m|$*. Hence, we have*

$$|S^C| = \binom{N+K-1}{K} + \sum_{m \geq 1} (-1)^m \binom{K}{m} \binom{N+K-1-m(M+1)}{K}$$

The desired probability is $1 - \frac{|S^C|}{|\Omega|}$*. In the implementation, we can break the loop if* $N + K - 1 - m * (M+1) < K$*, as* $\binom{n}{k} = 0$ *for* $n < k$*.*