Homework 1
Math 270B, Winter 2020, Prof. Roman Vershynin

Problem 1 (Poisson central limit theorem)

Let X_i be i.i.d. random variables each having the Poisson distribution with mean 1, and consider $S_n = X_1 + \cdots + X_n$. Let $x \in \mathbb{R}$. Show that if $k = k(n)$ is such that $(k-n)/\sqrt{n} \to x$ as $n \to \infty$, we have

$$\sqrt{2\pi n}\, \mathbb{P}\{S_n = k\} \to \exp(-x^2/2).$$

(Hint: first show that S_n has Poisson distribution with mean n. Then use Stirling's formula to analyze the limiting behavior of the probability mass function of S_n.)

sol: the Ch.f for a poisson random variable is

$$\psi(t) = \mathbb{E}[e^{iXt}] = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} e^{ikt} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} = e^{\lambda(e^{it}-1)}$$

Then, by independence, we have

$$\psi_{S_n}(n) = \prod_{i=1}^{n} \psi(t) = e^{n\lambda(e^{it}-1)}$$

which is the Ch.f of poisson distribution with mean $n\lambda$. Hence, we have S_n is a poisson distribution with mean n. By Central Limit Theorem, we have

$$\frac{S_n - n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1)$$

We note that $\mathbb{P}\{S_n = k\} = \mathbb{P}\left\{k - \frac{1}{2} < S_n < k + \frac{1}{2}\right\} = \mathbb{P}\left\{\frac{k-n-\frac{1}{2}}{\sqrt{n}} < \frac{S_n - n}{\sqrt{n}} < \frac{k-n+\frac{1}{2}}{\sqrt{n}}\right\}$ and $\frac{k-n}{\sqrt{n}} \to x$.
Then, $\forall \epsilon > 0$, $|\sqrt{2\pi n}\mathbb{P}\left\{\frac{k-n-\frac{1}{2}}{\sqrt{n}} < \frac{S_n-n}{\sqrt{n}} < \frac{k-n+\frac{1}{2}}{\sqrt{n}}\right\} - \sqrt{n}\int_{x-\frac{1}{2\sqrt{n}}}^{x+\frac{1}{2\sqrt{n}}} e^{\frac{-t^2}{2}}dt| \leq \epsilon$ for large enough n. Let $n \to \infty$, by Lebesgue differentiation theorem, we have $\lim_{n\to\infty} \sqrt{n}\int_{x-\frac{1}{2\sqrt{n}}}^{x+\frac{1}{2\sqrt{n}}} e^{\frac{-t^2}{2}}dt = e^{\frac{-x^2}{2}}$. Then, we have

$$\sqrt{2\pi n}\mathbb{P}\{S_n = k\} \to e^{\frac{-x^2}{2}}$$

Problem 2 (Weak convergence without convergence of densities)

Find an example of random variables X_n with densities f_n so that X_n converge weakly to the uniform distribution on $[0,1]$ but $f_n(x)$ does not converge to 1 for any $x \in [0,1]$.

sol: Let $X = U([0,1])$ with C.D.F $F(x) = x$.

Define $f_n(x) = \begin{cases} 0 & x \in [\frac{2k}{2^n}, \frac{2k+1}{2^n}) \\ 2 & x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}] \end{cases}$ for $k = 0, 1, 2, \cdots 2^{n-1} - 1$. Let X_n and μ_n be the associated random variables and probability measures. Then, we that $\forall (a,b) \subseteq [0,1]$, $\lim_{n\to\infty} \mu_n\Big((a,b]\Big) \to b-a$ as $n \to \infty$. Consequently, we have $\forall x$, $F_n(x) = \mu_n\Big((0,x]\Big) \to x = F(x)$ as $n \to \infty$. Hence, X_n converges to X weakly, but $f_n(x)$ does not converges to 1 for any x.

Problem 3 (Extreme values)

Let X_i be i.i.d. random variables each having exponential distribution with mean 1, and consider $M_n := \max_{i\leq n} X_i$. Show that $M_n - \log n$ converges weakly to the standard Gubmel distribution, i.e. the distribution with cumulative distribution function $F(x) = \exp(-e^{-x})$.

sol: fix any x, $\{M_n - \ln n \leq x\} = \{\cap_{i=1}^n X_i \leq x + \ln x\}$. Since they are independent,

$$\mathbb{P}\left\{M_n - \ln x \leq x\right\} = \prod_{i=1}^n \mathbb{P}\left\{X_i \leq x + \ln x\right\} = (1 - e^{-(x+\ln n)})^n = (1 - \frac{e^{-x}}{n})^n \to e^{e^{-x}}$$

Since the C.D.F converges pointwisely, we conclude the converrgence in distribution.

Problem 4 (Convergence to a constant)

Let X_n be random variables and c be a constant. Prove that weak convergence of X_n to c is equivalent to convergence of X_n to c in probability.

sol: Let $X = c$, the $F_X(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$

If $X_n \xrightarrow{\mathcal{D}} X$, then $F_n(x) \to F_X(x)$ $\forall x \neq c$.
Fix any $\delta > 0$, then

$$\mathbb{P}\left\{|X_n - X| > \delta\right\} \leq \mathbb{P}\left\{X_n > c + \delta\right\} + \mathbb{P}\left\{X_n \leq c - \delta\right\} = 1 - F_n(c+\delta) + F_n(c-\delta)$$

Since $\delta > 0$, $\lim_{n\to\infty} F_n(c+\delta) = F_X(c+\delta) = 1$ and $\lim_{n\to\infty} F_n(c-\delta) = 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\left\{|X_n - X| > \delta\right\} = 0$$

Hence, $X_n \to X$ in probability.
Conversely, we know that convergent in probability implied convergence in distribution.

Problem 5 (Convergence together)

Consider the following statement:

$$\text{if } X_n \to X \text{ weakly and } Y_n \to Y \text{ weakly then } X_n + Y_n \to X + Y \text{ weakly.} \tag{1}$$

(a) Find an example showing that that implication (1) is false in general.

(b) Prove that if Y is a constant, then implication (1) is true.

(c) Prove that if X_n and Y_n are independent, then implication (1) is true.
sol:
(a). Let $\Omega = [0,1]^2 = \{(\omega_1, \omega_2)|(\omega_1, \omega_2) \in [0,1]^2\}$ with Borel measure. Let $X(\omega) = Y(\omega) = \begin{cases} 0 & \omega_1 < \frac{1}{2} \\ 1 & else \end{cases}$. Define $X_n = X, \forall n$, and $Y_n = \begin{cases} 0 & \omega_2 < \frac{1}{2} \\ 1 & else \end{cases}$, $\forall n$. Then, we see $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{\mathcal{D}} Y$. However, $X_n + Y_n \not\xrightarrow{\mathcal{D}} X + Y$, as $X + Y = \begin{cases} 0 & \omega_1 < \frac{1}{2} \\ 2 & else \end{cases}$ and $X_n + Y_n =$
(b). Suppose $Y = c$. Since we know that $Y_n \xrightarrow{\mathcal{D}} Y \Rightarrow Y_n \to c$ in probability. Let $F_X(x)$ be the C.D.F. of X. Then, $F_X(x) = F_{X+c}(x+c)$ and F_X continuous at x if and only if F_{X+c} continuous at $x+c$. Fix any x s.t. $F(x)$ is continuous at x, then $\forall \epsilon > 0$, we can find a neighborhood $B(x)$ such that $|F_X(y) - F_X(x)| < \epsilon \quad \forall y \in B(x)$. Since F_X can only have countable many discontinuities, we can find $x - \delta_1$, $x + \delta_2 \in B(x)$ s.t F_X is continuous at these two points. Then, we have the set relations:

$\{\omega|X_n(\omega) \le x - \delta_1\} \setminus \{\omega||Y_n(\omega) - c| \ge \delta_1\} \subseteq \{\omega|X_n(\omega) + Y_n(\omega) \le x + c\} \subseteq \{\omega||Y_n(\omega) \le c - \delta_2\} \cup \{\omega|X_n(\omega) \le x + \delta_2\}$
Take probability of the sets with a few operations:

$$\mathbb{P}\{X_n \le x - \delta_1\} - \mathbb{P}\{|Y_n - c| > \delta_1\} \le \mathbb{P}\{X_n + Y_n \le x + c\} \le \mathbb{P}\{Y_n \le c - \delta_2\} + \mathbb{P}\{X_n \le x + \delta_2\}$$

which is:

$$F_{X_n}(x - \delta_1) - \mathbb{P}\{|Y_n - c| > \delta_1\} \le F_{X_n+Y_n}(x + c) \le F_{Y_n}(c - \delta_2) + F_{X_n}(x + \delta_2)$$

Since F_X is continuous at $x - \delta_1$ and $x + \delta_2$ (so the pointwise convergence holds), F_Y is continous at $c - \delta_2$, also $Y_n \to c$ in probability, so $\exists N$ s.t. for all $n > N$:

$$F_X(x - \delta_1) - \epsilon - \epsilon \le F_{X_n+Y_n}(x + c) \le \epsilon + F_X(x + \delta_2) + \epsilon$$

Finally, by continuity of F_X at x, we have:

$$F_X(x) - 3\epsilon \le F_{X_n+Y_n} \le F_X(x) + 3\epsilon$$

Then we conclude that $F_{X_n+Y_n}(x + c)$ converges to $F_X(x) = F_{X+c}(x + c)$ pointwisely when F_{X+c} is continuous at $x + c$. Hence, it converges weakly.
(c).If X_n, Y_n converges to X and Y, where X and Y are independent. Then, we have $\forall t$, $\psi_{X_n}(t) \to \psi_X(t)$ and $\psi_{Y_n}(t) \to \psi_Y(t)$, then we have $\psi_{X_n+Y_n}(t) = \psi_{X_n}(t)\psi_{Y_n}(t) \to \psi_X(t)\psi_Y(t) = \psi_{X+Y}(t)$, both equalities from independence. Since both $\psi_X(t)$ and

$\psi_Y(t)$ as characteristics functions are continuous at 0, ψ_{X+Y} is continuous at 0. Then, by Levy's continuity theorem, we have $X_n + Y_n \xrightarrow{\mathcal{D}} X + Y$.

Problem 6 (Projection of the sphere is Gaussian)

(a) Prove the following implication: if $X_n \to X$ weakly, $Y_n \geq 0$ and $Y_n \to c$ weakly where c is a constant, then $X_n Y_n \to cX$.

(b) Let Z_n be a random vector uniformly distributed on the unit Euclidean sphere of radius $\sqrt{n}$ in $\mathbb{R}^n$. Prove that the distribution of the first coordinate of Z_n (and actually, of any given coordinate) converges weakly to the standard normal distribution.

(Hint: let X_n be standard normal random vector, and consider $Z_n = X_n \cdot \sqrt{n}/\|X_n\|_2$.)

(a). Let $C > 0$, then we first note that $Y_n \xrightarrow{\mathcal{D}} Y \Rightarrow CY_n \xrightarrow{\mathcal{D}} CY$, because $\psi_{Y_n}(t) \to \psi_Y(t)$ pointwisely $\Rightarrow \psi_{CY_n}(t) = \psi_{Y_n}(Ct) \to \psi_Y(Ct) = \psi_{CY}(t)$ pointwisely.
Then, assume $c > 0$, we have $\frac{Y_n}{c} \xrightarrow{\mathcal{D}} 1$ (so $\frac{Y_n}{c} \to 1$ in probability). If F_X continuous at x, then $\forall \epsilon > 0$, we can find a neighborhood $B(x)$ s.t. $|F_X(y) - F_X(x)| < \epsilon \;\; \forall y \in B(x)$. Since F_X can only have countable many discontinuities, we can find $x + \delta_2 \in B(x)$ ($\delta_2 > 0$) s.t. F_X continuous at $x+\delta_2$, then we pick $\delta_1 > 0$ very small s.t. $x\delta_1 + \delta_1\delta_2 < \delta_2$. Now, we have $(1-\delta_1)(x+\delta_2) > x$, so

$$\mathbb{P}\left\{X_n \frac{Y_n}{c} \leq x\right\} \leq \mathbb{P}\left\{\frac{Y_n}{c} \leq 1 - \delta_1\right\} + \mathbb{P}\left\{X_n \leq x + \delta_2\right\}$$

For large enough n, we have

$$\mathbb{P}\left\{\frac{X_n Y_n}{c} \leq x\right\} \leq \epsilon + F_X(x+\delta_2) + \epsilon$$

By continuity, we have

$$\mathbb{P}\left\{\frac{X_n Y_n}{c} \leq x\right\} \leq F_X(x) + 3\epsilon$$

Now, pick $\delta_3 > 0$ s.t. $x - \delta_3 \in B(x)$ and F_X is continuous at $x - \delta_3$. Pick $\delta_4 > 0$ small s.t. $\delta_3 \geq \delta_4(x - \delta_3)$. Now, we have that $(x-\delta_3)(1+\delta_4) \leq x$, so we have

$$\{X_n \leq x - \delta_3\} \setminus \{\frac{Y_n}{c} > 1 + \delta_4\} \subseteq \{\frac{X_n Y_n}{c} \leq x\}$$

Then, we have

$$\mathbb{P}\left\{X_n \leq x - \delta_3\right\} - \mathbb{P}\left\{Y_n > 1 + \delta_4\right\} \leq \mathbb{P}\left\{\frac{X_n Y_n}{c} \leq x\right\}$$

Similarly, by pointwise convergence of F_{X_n}, continuity of F_X at x, and convergence of $\frac{Y_n}{c}$ in probability, we have

$$F_X(x) - 3\epsilon \leq \mathbb{P}\left\{\frac{X_n Y_n}{c} \leq x\right\}$$

so $\mathbb{P}\left\{\frac{X_n Y_n}{c} \leq x\right\} \to F_X(x)$ where F_X is continuous, which implies the weak convergence.

(b). Define $X_n = (X_{n1}, \cdots, X_{nn})$ and Z_n as above. Then let $S_n = \frac{X_{n1}+\cdots X_{nn}}{n}$. By strong law of large number, $S_n \xrightarrow{a.s.} 1$. Then by continuity of $\sqrt{\frac{1}{x}}$ around $x = 1$, we have $\sqrt{\frac{1}{S_n}} = \frac{\sqrt{n}}{\|X_n\|_2} \xrightarrow{a.s.} 1 \Rightarrow \frac{\sqrt{n}}{\|X_n\|_2} \xrightarrow{\mathcal{D}} 1$. Clearly, $X_{n1} \xrightarrow{\mathcal{D}} N(0,1)$ since they all have distribution $N(0,1)$. Then, by part (a) we have $Z_{n1} = X_{n1}\frac{\sqrt{n}}{\|X_n\|_2} \xrightarrow{\mathcal{D}} N(0,1)$

Problem 8 (Operations on characteristic functions)

Prove that if ϕ is a characteristic function of some random variable, then $\mathrm{Re}\phi$ and $|\phi|^2$ are, too.

sol: Let $\phi(t)$ be Ch.f of random variable X. Let Y be identical and independent to X, the $\phi_{X-Y} = \phi_X(t)\phi_{-X}(t) = \phi(t)\overline{\phi(t)} = |\phi(t)|^2$
Now, suppose that we have a sample space Ω that X lives in. Let Z be a random variable independent to X s.t. $\mathbb{P}\{Z = 0\} = \mathbb{P}\{Z = 1\} = \frac{1}{2}$. Then consider random variable $ZX + (Z-1)X$

$\tilde{\phi}(t) = \mathbb{E}[e^{it(ZX+(Z-1)X)} \cdot \mathbf{1}_{\{Z=1\}} + e^{it(ZX+(Z-1)X)} \cdot \mathbf{1}_{\{Z=0\}}] = \mathbb{E}[e^{itX} \cdot \mathbf{1}_{\{Z=1\}} + e^{-itX} \cdot \mathbf{1}_{\{Z=0\}}]$

By independence of X and Z, we have

$$= \mathbb{E}[\mathbf{1}_{Z=1}]\mathbb{E}[e^{itX}] + \mathbb{E}[\mathbf{1}_{Z=0}]\mathbb{E}[e^{-itX}] = \frac{1}{2}(\phi(t) + \phi(-t)) = Re\phi(t)$$

Problem 9 (Point masses from characteristic function)

Let X be a random variable with characteristic function ϕ. Prove that for any $a \in \mathbb{R}$, we have

$$\mathbb{P}\{X = a\} = \lim_{T\to\infty} \frac{1}{2T} \int_{T}^{-T} e^{-ita}\phi(t)\, dt.$$

(Hint: imitate the proof of the inversion formula.)

sol:

$$\frac{1}{2T}\int_{-T}^{T} e^{-ita}\phi(t)dt = \frac{1}{2T}\int_{-T}^{T}\int e^{-ita}e^{itx}d\mu dt$$

since $\forall\theta|e^{i\theta}| \leq 1$, by Fubini's Theorem,

$$= \frac{1}{2T}\int\int_{-T}^{T} e^{it(x-a)}dtd\mu = \frac{1}{2T}\int\int_{-T}^{T} \cos[(x-a)t] + i\sin[(x-a)t]dtd\mu$$

Since $i\int_{-T}^{T}\sin[t(x-a)] = 0, \forall x$

$$= \int\int_{-T}^{T} \frac{\cos[(x-a)t]}{2T}dtd\mu$$

define $f_T(x) = \int_{-T}^{T} \frac{cos[(x-a)t]}{2T} dt$, then we have $|f_T(x)| \leq 1, \forall x, \forall T$, and $f_T(a) = 1, \forall T$. Also, $\forall x \neq a$, $\lim_{T\to\infty} f_T(x) = \lim_{T\to\infty} \frac{\sin[T(x-a)]}{T(x-a)} \to 0$ as $T \to \infty$. Hence, $f_T(x) \to \delta_a(x)$ as $T \to \infty$. By bounded convergence theorem,

$$\lim_{T\to\infty} \int f_T(x) d\mu = \int \delta_a(x) d\mu = \mu(\{a\})$$

Problem 10 (CLT for a random number of terms)

Let X_i be i.i.d. random variables with mean zero and unit variance. and let $S_n := X_1 + \cdots + X_n$. Let N_n be a sequence of nonnegative integer-valued random variables and a_n be a sequence of nonnegative integers such that $a_n \to \infty$ and $N_n/a_n \to 1$ in probability. Show that

$$S_{N_n}/\sqrt{a_n} \to N(0,1)$$

weakly.

(Hint: use Kolmogorov's maximal inequality to conclude that if $Y_n = S_{N_n}/\sqrt{a_n}$ and $Z_n = S_{a_n}/\sqrt{a_n}$, than $Y_n - Z_n \to 0$ in probability.)

sol: Define Y_n and Z_n as above. Let $b_n = \max\{N_n, a_n\}$ and $c_n = \min\{N_n, a_n\}$. we have that $\frac{b_n}{a_n} \to 1$ in probability, since fixed $\delta > 0$, $\mathbb{P}\left\{\frac{b_n}{a_n} - 1 > \delta\right\} \leq \mathbb{P}\left\{|\frac{N_n}{a_n} - 1| > \delta\right\} \to 0$ and, similarly, $\frac{c_n}{a_n} \to 1$ in probability as $\mathbb{P}\left\{1 - \frac{c_n}{a_n} > \delta\right\} \leq \mathbb{P}\left\{|\frac{N_n}{a_n} - 1| > \delta\right\} \to 0$.
Now, again fix any $\delta > 0$, by Kolmogorov's maximal inequality, we have $\mathbb{P}\left\{|Y_n - Z_n| > \delta\right\} \leq \frac{1}{\delta^2}\sum_{k=c_n}^{b_n} \mathbb{E}[(\frac{X_k}{\sqrt{a_n}})^2] = \frac{b_n - c_n}{\delta^2 a_n}$. We have that $\frac{b_n - c_n}{\delta^2 a_n} \to 0$ in probability.

Now, if we have N_n is independent to X_i. Let the space of $\{X_i\}$ be Ω_1 with probability $\mathbb{P}_1$ and the space of N_n be Ω_2 with $\mathbb{P}_2$ (so $\Omega = \Omega_1 \times \Omega_2$ and $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$). Then, fix $\omega_2 \in \Omega_2$, we have $\mathbb{P}_1\{\omega_1 ||Y_n - Z_n| > \delta\} \leq \frac{b_n(\omega_2) - c_n(\omega_2)}{\delta^2 a_n}$, then $\mathbb{P}\left\{|Y_n - Z_n| > \delta\right\}] = \int_{\Omega_2} \mathbb{P}_1(|Y_n - Z_n| > \delta) d\omega_2 \leq \mathbb{E}[\frac{b_n - c_n}{\delta^2 a_n}] \to 0$, so $Y_n - Z_n \to 0$ in probability.
By Central Limit theorem, we have $Z_n \xrightarrow{\mathcal{D}} N(0,1)$, then by part, then by problem 5(b) we have $Y_n = Z_n + (Y_n - Z_n) \xrightarrow{\mathcal{D}} N(0,1)$

Problem 11 (a non-example for Lindeberg-Feller CLT)

Consider independent random variables X_k such that X_k takes values $\pm k$ with probability $k^{-2}/2$ each and values ± 1 with probability $(1-k^{-2})/2$ each. Show that, although $\mathrm{Var}(S_n)/n \to 2$, $S_n/\sqrt{n}$ does not converge to $N(0,1)$ weakly. Why does this example not contradict Lindeberg-Feller central limit theorem?

I think the Lindeberg-Feller theorem states that $\frac{S_n}{\sqrt{2n}} \xrightarrow{\mathcal{D}} N(0,1)$
$\forall n$, $\mathrm{Var}(X_n) = 2$, Hence, $\frac{\mathrm{Var}(S_n)}{n} = \frac{\sum_{k=1}^{n}\mathrm{Var}(X_k)}{n} = 2$ by independent of X_k.
Let $s_n = \sqrt{\sum_{k=1}^{n}\mathrm{Var}(X_k)} = \sqrt{2n}$ and $\mu_k = \mathbb{E}[X_k]$. The condition for Lindeberg-Feller

theorem is

$$\forall \epsilon > 0, \quad \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}[(X_k - \mu_k)^2 \cdot \mathbf{1}_{\{|X_k-\mu_k|>\epsilon s_n\}}] = 0$$

Now take $\epsilon = \frac{1}{\sqrt{2}}$. Then,

$\mathbb{E}[(X_k - \mu_k)^2 \cdot \mathbf{1}_{\{|X_k-\mu_k|>\epsilon s_n\}}] = \begin{cases} 1 & \sqrt{n} < k \le n \\ 0 & k \le \sqrt{n} \end{cases}$ Then,

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}[(X_k - \mu_k)^2 \cdot \mathbf{1}_{\{|X_k-\mu_k|>\epsilon s_n\}}] = \lim_{n\to\infty} \frac{1}{2n}(n - \sqrt{n}) = \frac{1}{2} \neq 0$$

Now, we define $\tilde{X}_k(\omega) = \begin{cases} 1 & X_k(\omega) > 0 \\ -1 & X_k(\omega) < 0 \end{cases}$, and let $\tilde{S}_n = \sum_{k=1}^{n} \tilde{X}_k$. By borel-cantelli lemma, we see that for a.s. every ω, $X_k(w) \neq \tilde{X}_k(\omega)$ for finitely many k. Then, for some $N(\omega)$, we have that $\frac{S_n(\omega) - \tilde{S}_n(\omega)}{\sqrt{n}} = \frac{\sum_{k=1}^{N(\omega)} X_k(\omega) - \tilde{X}_k(\omega)}{\sqrt{n}} \to 0$ a.s. Note that $\tilde{X}_k$ are i.i.d, so by CLT we have $\frac{\tilde{S}_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1)$. However, we have that $\frac{S_n}{\sqrt{n}} - \frac{\tilde{S}_n}{\sqrt{n}} \xrightarrow{a.s.} 0$, so $\frac{S_n}{\sqrt{n}} - \frac{\tilde{S}_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} 0$. Then, by Problem 5 part(2), we have

$$\frac{S_n}{\sqrt{n}} = \frac{\tilde{S}_n}{\sqrt{n}} + \Big(\frac{S_n}{\sqrt{n}} - \frac{\tilde{S}_n}{\sqrt{n}}\Big) \xrightarrow{\mathcal{D}} 0 + N(0,1) = N(0,1)$$

Hence, we see that $\frac{S_n}{\sqrt{2n}} \xrightarrow{\mathcal{D}} N(0, \frac{1}{2})$.
I think for this problem, since the condition fails, the theorem gives a wrong normalization.