Homework 4

Math 270B, Winter 2020, Prof. Roman Vershynin

Problem 1 (Eigenvectors of transition matrices)

Let (X_n) be an irreducible and recurrent Markov chain with (doubly-infinite) transition matrix P. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a bounded function satisfying

$$\sum_{j=1}^{\infty} P_{ij}\psi(j) = \psi(i) \quad \text{for all } i \in \mathbb{N}.$$

Show that ψ is a constant function.

(Hint: check that $\psi(X_n)$ is a bounded martingale, and apply the martingale convergence theorem.)

sol: $X_n = \sum_{k=0}^{\infty} k \cdot \mathbf{1}_{\{X_n=k\}}$ and $\psi(X_n) = \sum_{k=0}^{\infty} \psi(k) \cdot \mathbf{1}_{\{X_n=k\}}$
We have

$$\mathbb{E}[\psi(X_{n+1})|\mathcal{F}_n] = \mathbb{E}[\psi(X_{n+1})(\sum_{k=1}^{\infty} \mathbf{1}_{\{X_n=k\}})|\mathcal{F}_n]$$

$$= \sum_{k=0}^{\infty} \mathbb{E}[\psi(X_{n+1})|\mathcal{F}_n] \cdot \mathbf{1}_{\{X_n=k\}} = \sum_{k=0}^{\infty} \mathbb{E}[\mathbb{E}[\psi(X_n+1)|X_n=k]] \cdot \mathbf{1}_{\{X_n=k\}}$$

$$= \sum_{k=0}^{\infty} (\sum_{j=1}^{\infty} P_{kj}\psi(j)) \cdot \mathbf{1}_{\{X_n=k\}}) = \sum_{k=0}^{\infty} \psi(k)\mathbf{1}_{\{X_n=k\}} = \psi(X_n)$$

Hence, $\psi(X_n)$ is a martingale. Since ψ is bounded, so $\psi(X_n)$ is a bounded martingale. Suppose $|\psi| \le M$, then we have $\mathbb{E}|\psi(X_n)| \le M$. By Martingale convergence theorem, we have for a.s. ω, $\psi(X_n(\omega))$ converges.
Now, suppose ψ is not a constant, then $\exists i, j$ s.t. $\psi(i) \neq \psi(i)$. Since the Markov chain is irreducible, we show that $\mathbb{P}\{A\} > 0$, where A={ X_n visits both i and j} (from our office hour discussion, $\mathbb{P}\{A\}$ is actually 1, but we only need it to be positive here). For any i_0, $\exists$ $n_1^{i_0}$, n_2 s.t. $P_{i_0 i}^{n_1} > 0$ and $P_{ij}^{n_2} > 0$ since it is irreducible. Then, $\mathbb{P}\left\{X_{n_1^{i_0}} = i, X_{n_1^{i_0}+n_2} = j | X_0 = i_0\right\} = P_{i_0 i}^{n_1^{i_0}} P_{ij}^{n_2} > 0$. Therefore,

$$\mathbb{P}\{A\} \ge \sum_{i_0=0}^{\infty} P_{i_0 i}^{n_1^{i_0}} P_{ij}^{n_2} \cdot \mathbb{P}\{X_0 = i_0\} > 0$$

Since both i and j are recurrent states, we know that

$$\mathbb{P}\{X_n \text{visits i infinitly many times} | X_N = i \text{ for some } N\} = 1$$

By formula of conditional probability, we have $\mathbb{P}\{X_n \text{visits i infinitely many times}\} = \mathbb{P}\{X_n \text{ visits } i\}$. Then, $\mathbb{P}\{X_n \text{visits both i and j infinitely many times}\} = \mathbb{P}\{A\} > 0$. However, this implies that $\psi(X_n)$ does not converge on A, as for $\omega \in A$, there are two subsequence for X_n to be i and j respectively. This contradicts to martingale convergence theorem. Hence, ψ is a constant.

Problem 2 (Stopped σ-algebra)

Let S and T be stopping times with respect to a filtration $(\mathcal{F}_n)$. Denote by $\mathcal{F}_T$ the collection of events F such that $F \cap \{T \leq n\} \in \mathcal{F}_n$ for all n.

(a) Show that $\mathcal{F}_T$ is a σ-algebra.

(b) Show that T is measurable with respect to $\mathcal{F}_T$.

(c) If $E \in \mathcal{F}_S$, show that $E \cap \{S \leq T\} \in \mathcal{F}_T$.

(d) Show that if $S \leq T$ a.s. then $\mathcal{F}_S \subset \mathcal{F}_T$.

sol: (a). We show $\mathcal{F}_T \neq \emptyset$ by showing that $\{T \leq m\} \in \mathcal{F}_T, \forall m$:

$$\forall n, m, \{T \leq m\} \cap \{T \leq n\} = \{T \leq min\{n, m\}\} \in \mathcal{F}_n$$

since the filtration $\mathcal{F}_n$ is an ascending chain of σ-algebra.
Now, it suffices to show that $\mathcal{F}_T$ is closed under countable union and intersection.
Let $\{F_i\}_{i=1}^{\infty} \subset \mathcal{F}_T$, we have

$$\forall n, \quad \Big(\cup_{i=1}^{\infty} F_i \Big) \cap \{T \leq n\} = \cup_{i=1}^{\infty} \Big(F_i \cap \{T \leq n\} \Big)$$

Since $F_i \cap \{T \leq n\} \in \mathcal{F}_n \ \forall i$, and $\mathcal{F}_n$ is a σ-algebra, so $\mathcal{F}_T$ is a σ-algebra.
(b). In (a), we showed that $\forall n$, $\{T \leq n\} \in \mathcal{F}_T$, so T is measurable.
(c).

$$\forall n, \{S \leq T\} \cap \{T \leq n\} = \cup_{k=1}^{n} \Big(\{S \leq k\} \cap \{T = k\} \Big)$$

Then,

$$E \cap \{S \leq T\} \cap \{T \leq n\} = \cup_{k=1}^{n} \Big(\{S \leq k\} \cap \{T = k\} \cap E \Big)$$

since $\{S \leq k\} \cap E \cap \{T = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$ for $1 \leq k \leq n$. Therefore, $E \cap \{S \leq T\} \cap \{T \leq n\} \in \mathcal{F}_n \ \forall n$.
(d). Since T and S are stopping times, $\{T \leq n\}, \{S \leq n\} \in \mathcal{F}_n, \forall n$. Also, as $S \leq T$, $\{T \leq n\} \subset \{S \leq n\}, \forall n$. For any $F \in \mathcal{F}_S$, we have

$$F \cap \{S \leq n\} \in \mathcal{F}_n, \forall n \ \text{ and } \{T \leq n\} \in \mathcal{F}_n, \forall n$$
$$\Rightarrow F \cap \{T \leq n\} = F \cap \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n$$

Hence, $\mathcal{F}_S \subset \mathcal{F}_T$

Problem 3 (Stopped σ-algebra: reconstruction from the limit)

Let (X_n) be a uniformly bounded martingale with respect to the filtration $(\mathcal{F}_n)$. Let S and T be two stopping times satisfying $S \leq T$ a.s. Prove that

$$X_T = \mathbb{E}[X|\mathcal{F}_T] \quad \text{and} \quad X_S = \mathbb{E}[X_T|\mathcal{F}_S]$$

where X is the almost sure limit of X_n.

sol: Since X_n is uniformly bounded, so is uniformly integrable. We have that $\mathbb{E}[X \Big| \mathcal{F}_n] = X_n \ \forall n$.

Firstly, we have that $X_T = \sum_{k=0}^{\infty} X_k \cdot \mathbf{1}_{\{T=k\}}$. We show that $\mathbb{E}[X_T \cdot \mathbf{1}_A] = \mathbb{E}[X \cdot \mathbf{1}_A]$ for any $A \in \mathcal{F}_T$:

$$\mathbb{E}[X \cdot \mathbf{1}_A] = \mathbb{E}[\sum_{k=0}^{\infty} X \cdot \mathbf{1}_{\{T=k\}\cap A}] = \sum_{k=0}^{\infty} \mathbb{E}[X \cdot \mathbf{1}_{\{T=k\}\cap A}]$$

since $\{T = k\} \cap A \in \mathcal{F}_k \ \forall k$ by definition of $\mathcal{F}_T$, we condition $\mathcal{F}_k$ on each term:

$$= \sum_{k=0}^{\infty} \mathbb{E}[\mathbb{E}[X \cdot \mathbf{1}_{\{T=k\}\cap A}\Big|\mathcal{F}_k]] = \sum_{k=0}^{\infty} \mathbb{E}[X_k \cdot \mathbf{1}_{\{T=k\}\cap A}] = \mathbb{E}[\sum_{k=0}^{\infty} X_k \cdot \mathbf{1}_{\{T=k\}\cap A}] = \mathbb{E}[X_T \cdot \mathbf{1}_A]$$

Then, we show that X_T is $\mathcal{F}_T$ measurable. Note that $\forall a \in \mathbb{R}$, we have that

$$\{X_T \le a\} = \cup_{k=0}^{\infty}\Big(\{X_k \le a\} \cap \{T = k\}\Big)$$

so $\forall n$, $\{X_T \le a\} \cap \{T \le n\}$ is

$$\{X_T \le a\} = \cup_{k=0}^{\infty}\Big(\{X_k \le a\} \cap \{T = k\} \cap \{T \le n\}\Big)$$

but each set is either empty if $n < k$, or $\{X_k \le a\} \cap \{T = k\} \in \mathcal{F}_n$ if $n \ge k$. Therefore, we have that $\{X_T \le a\} \cap \{T \le n\} \in \mathcal{F}_n$. We conclude X_T is $\mathcal{F}_T$ measurable.
Since X_n is uniformly bounded, we have that $\mathbb{E}|X| \le \lim_{n\to\infty} \mathbb{E}|X_n| < \infty$. Therefore, $\mathbb{E}[X] \le \infty$. Apply $A = \Omega$ to result of first part of this problem, we have $\mathbb{E}[X_T] \le \infty$, so $X_T \in L^1$.
Now, X_T satisfies the definition of conditional expectation, by uniqueness of conditional expectation, we have the desired result.
(b). Since $S \le T$, we have that $\mathcal{F}_S \subset \mathcal{F}_T$. Then,

$$\mathbb{E}[X_T|\mathcal{F}_S] = \mathbb{E}[\mathbb{E}[X\Big|\mathcal{F}_T]\Big|\mathcal{F}_S] = \mathbb{E}[X|\mathcal{F}_S] = X_S$$

by property of conditional expectation.

Problem 4

A die is rolled repeatedly. Which of the following are Markov chains? For those that are, compute the transition matrix.

(a) The largest number X_n shown up to the n-th roll.

(b) The number N_n of sixes in n rolls.

(c) At time r, the time C_r since the most recent six.

(d) At time r, the time B_r until the next six.

sol: (a). Yes, since X_{n+1} takes only two possible values X_n and $X_n + 1$, and the probabilities are completely determined by X_n. Transition matrix:

$$\begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where the states is $\{0, 1, 2, 3, 4, 5, 6\}$.
(b). Yes, as N_{n+1} only takes two values X_n and $X_n + 1$ with probabilities $\frac{5}{6}$ and $\frac{1}{6}$ respectively. The transition matrix:

$$\begin{bmatrix} \frac{5}{6} & \frac{1}{6} & 0 & \cdots & \cdots & \cdots \\ 0 & \frac{5}{6} & \frac{1}{6} & 0 & \cdots & \cdots \\ 0 & 0 & \frac{5}{6} & \frac{1}{6} & 0 & \cdots \\ \vdots & & & \ddots & \ddots & \end{bmatrix}$$

where the states are $\{0, 1, 2, \cdots\}$.
(c). Yes, since X_{n+1} takes two possible values $X_n + 1$ and 0, and the probabilities are completely determined by the value of X_n. $\mathbb{P}\{X_{n+1} = 0\} = \frac{1}{6}$ and $\mathbb{P}\{X_{n+1} = i + 1 | X_n = i\} = \frac{1}{6}$. The transition matrix:

$$\begin{bmatrix} \frac{1}{6} & \frac{5}{6} & 0 & \cdots & \cdots \\ \frac{1}{6} & 0 & \frac{5}{6} & \cdots & \cdots \\ \frac{1}{6} & 0 & 0 & \frac{5}{6} & \cdots \\ \vdots & & & & \ddots \end{bmatrix}$$

(d) Yes? Since when we get a six, the time we need to get the next six is a geometric distribution. Then we have that $\mathbb{P}\{X_{n+1} = i | X_n = 0\} = (\frac{5}{6})^{i-1}\frac{1}{6}$. If X_n is not 0, then X_{n+1} is $X_n - 1$ deterministically. Then, we have the transition matrix:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & \end{bmatrix}$$

where $a_i = (\frac{5}{6})^i \frac{1}{6}$ and states are $\{0, 1, 2, \cdots\}$

Problem 5 (Reflected random walk)

Let (S_n) be a simple random walk starting at $S_0 = 0$. Show that $X_n = |S_n|$ is a Markov chain.

sol: Since the walk is symmetric, $P_R = P_L = \frac{1}{2}$, we have that X_{n+1} takes two possible values $X_n - 1$ and $X_n + 1$ and the probabilities are completely determined by the value of X_n. Hence, we have X_n is a Markov chain with transition matrix (states are 0,1,2,3,$\cdots$):

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & & \ddots & \ddots & & \end{bmatrix}$$

where $\mathbb{P}\{X_{n+1} = 1 | X_n = 0\} = 1$ and $\mathbb{P}\{X_{n+1} = i+1 | X_n = i\} = \mathbb{P}\{X_{n+1} = i-1 | X_n = i\} = \frac{1}{2}$ for all $i \geq 1$.

Problem 6 (Markov property for stopping times)

Let (X_n) me a Markov chain, and let T be a stopping time with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that

$$\mathbb{P}\{X_{T+1} = j \mid X_k = x_k \text{ for } 0 \leq k < T,\ X_T = i\} = \mathbb{P}\{X_{T+1} = j \mid X_T = i\}$$

for all $m \geq 0$, i, j, and x_k.

sol: I think we have

$$\mathbb{P}\Big((A|B)|C\Big) = \frac{\mathbb{P}(A \cap B|C)}{\mathbb{P}(B|C)} = \mathbb{P}(A|B \cap C)$$

and consequently

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B|C)$$

Hence, we have that

$$\mathbb{P}(X_{T+1} = j|X_T = i) = \sum_{m=0}^{\infty} \mathbb{P}(X_{T+1} = j, T = m|X_T = i)$$

$$= \sum_{m=0}^{\infty} \mathbb{P}(X_{T+1} = j|X_T = i, T = m)\mathbb{P}(T = m|X_T = i) = P_{ij} \sum_{m=0}^{\infty} \mathbb{P}(T = m|X_T = i) = P_{ij}$$

On the other hand,

$$\mathbb{P}(X_{T+1} = j|X_k = x_k, X_T = i) = \sum_{m=0}^{\infty} \mathbb{P}(X_{T+1}, T = m|X_k = x_k, X_T = i)$$

$$= \mathbb{P}(X_{T+1} = j|X_T = i, X_k = x_k, T = m)\mathbb{P}(T = m|X_T = i)$$

$$= \mathbb{P}(X_{m+1} = j|X_m = i, X_k = x_k)\mathbb{P}(T = m|X_T = i)$$

by Markov property of X_n, we have that

$$= \sum_{m=0}^{\infty} P_{ij}\mathbb{P}(T = m|X_T = i) = P_{ij}$$

Problem 7 (Markov property for stopping times)

Find an example of two Markov chains (X_n) and (Y_n) such that $X_n + Y_n$ is not a Markov chain.

sol: Let S_n and S_n^* be two independent simple random walk, and X_n, Y_n be $|S_n|$ and $|S_n^*|$ respectively. We show that the probabilities of $X_{n+1} + Y_{n+1}$ cannot be completely determined by the value of $X_n + Y_n$. Suppose $X_n + Y_n = 2$. If $X_n = Y_n = 1$, $\mathbb{P}\{X_{n+1} + Y_{n+1} = 0\} = \mathbb{P}\{X_{n+1} + Y_{n+1} = 4\} = \frac{1}{4}$ and $\mathbb{P}\{X_{n=+1} + Y_{n+1} = 2\} = \frac{1}{2}$. If $X_n = 0 = 2 - Y_n$, then $\mathbb{P}\{X_{n+1} + Y_{n+1} = 2\} = \frac{1}{2} = \mathbb{P}\{X_{n+1} + Y_{n+1} = 4\}$. Therefore, $\mathbb{P}\{X_{n+1} + Y_{n+1} = i|X_n + Y_n = 2\}$ is undefined. $X_n + Y_n$ is not a Markov Chain.

Problem 8 (Random walk on a cube)

A particle performs a random walk on the vertices of a three-dimensional cube. At each step it remains where it is with probability 1/4, or moves to one of its neighboring vertices each having probability 1/4. Compute the mean number of steps until the particle returns to the vertex from which the walk started.

(Hint: Let $v \to s \to t \to w$ be a path from the original vertex v to the diametrically opposite vertex w. Conditioning on the first step and using a symmetry argument, write down a system of linear equations for μ_v, μ_s, μ_t and μ_w, the mean number of steps to reach v from v, s, t, w respectively.)

sol: I interpret the problem as find $\mathbb{E}[N]$ where $N = \{n|X_n = v, X_0 = v, X_i \neq v, 0 < i < n\}$. In particular, N can be 1.
Start from w, by the law of total expectation, we have $\mu_w = \mathbb{E}[W|\text{move}]\mathbb{P}\{\text{move}\} + \mathbb{E}[W|\text{stay}]\mathbb{P}\{\text{stay}\}$

$$\mu_w = \frac{3}{4}(\mu_t + 1) + \frac{1}{4}(1 + \mu_w)$$

since all vertices adjacent to w has the same expectation as t by symmetry. Similarly,

$$\mu_t = \frac{1}{2}(\mu_s + 1) + \frac{1}{4}(\mu_t + 1) + \frac{1}{4}(\mu_w + 1)$$

$$\mu_s = \frac{1}{4} \cdot 1 + \frac{1}{2}(\mu_t + 1) + \frac{1}{4}(1 + \mu_s)$$

$$\mu_v = \frac{1}{4} \cdot 1 + \frac{3}{4}(\mu_s + 1)$$

$\mu_v = 4.125$, which is the expected number of steps to return.

Problem 9 (Recurrence of a symmetric random walk)

Prove that the symmetric random walk on $\mathbb{Z}^2$ is recurrent, the symmetric random walk on $\mathbb{Z}^3$ is transient.

sol: For a random walk in $\mathbb{Z}^2$, observe that each step, X_n has possible moves $(1,0),(0,1),(-1,0),(0,-1)$ with probability $\frac{1}{4}$ each. In order to return, we need both vertical and horizontal displacement to be zero. Therefore, n needs to be even (so we use $2n$), and there are $2k$ of $2n$ move vertically with exactly k steps forward and k steps backward. Therefore:

$$P_{ii}^{2n} = \sum_{k=0}^{n} \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{4^{2n}}$$

claim:

$$\sum_{k=0}^{n} \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k} = \binom{2n}{n}^2$$

We give a combinatorial proof. Each step, the Markov Chain has four possible movements, and there are $2n$ movements in total. We construct a bijective map ψ that maps movements $(0,1),(0,-1),(1,0),(-1,0)$ to $(1,1),(-1,-1),(1,-1),(-1,1)$ respectively. That is, $\psi(m_i) = (x_i, y_i)$ for movements m_i, $i = 1, 2, \cdots, 2n$. Now, consider $\{x_i\}_{i=1}^{2n}$ and $\{y_i\}_{i=1}^{2n}$. Their preimages form a valid return tour if and only if each sequence contains n many 1 and n many -1. There are $\binom{2n}{n}\binom{2n}{n}$ pairs of such sequences, so we proved our claim. Now we see that

$$P_{ii}^{n} \sim (\frac{2^{2n}}{\sqrt{\pi n}})^2 \frac{1}{4^{2n}} = \frac{1}{\pi n}$$

Hence, we have

$$\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$$

By the recurrent criterion, the Markov chain is recurrent.

Problem 10 (Reversible Markov chains)

Which of the following are reversible Markov chains?

(a) Move from 0 to 1 with probability p and stay at 0 with probability $1-p$; move from 1 to 0 with probability q and stay at 1 with probability $1-q$.

(b) A random walk on the vertices of a given triangle: at each time, move to the next vertex counterclockwise with probability p and clockwise with probability $1-p$.

sol: Recall the reversible criterion, a M.C. is reversible if and only if $\pi_i P_{ij} = \pi_j P_{ji}, \forall i, \forall j$. Note that we only need to check $i \neq j$.
(a). The transition matrix is

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

A computation for the stationary distribution $\pi = [\pi_1, \pi_2]$, $\pi P = \pi$, leads to that $\frac{\pi_1}{\pi_2} = \frac{q}{p}$, then $\pi_1 P_{12} = \pi_2 P_{21}$. Hence, the Markov Chain is reversible.
(b) Label the vertices clockwise, we have that the transition matrix

$$P = \begin{bmatrix} 0 & 1-p & p \\ p & 0 & 1-p \\ 1-p & p & 0 \end{bmatrix}$$

Let $\pi = [\pi_1, \pi_2, \pi_3]$. From $\pi P = \pi$, we get $p\pi_i + (1-p)\pi_j = \pi_k$ for $(i,j,k) \in \{(2,3,1),(1,3,2),(1,2,3)\}$. By this symmetric system of equations, we have $\frac{\pi_i}{\pi_j} = 1$. Then, we need $\frac{P_{ij}}{P_{ji}} = 1$ for all $i \neq j$. However, this is true only if $p = \frac{1}{2}$. Hence, it is reversible if and only if $p = \frac{1}{2}$.

Problem 11 (Chess)

A king performs a random walk on a chessboard; at each step it is equally likely to make any one of the available moves (and never stays put). What is the mean recurrence time of a comer square?

sol: According to Wikipedia, a chessboard is 8×8 with 64 states we label them

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \vdots & \cdots & & \ddots & & & & \end{bmatrix}$$

In class, we showed that there is a unique stationary distrubution π s.t. that $\pi P = \pi$. Moreover, $\pi_i = \frac{d_i}{\sum_{j=1}^n d_j}$, where d_i is the degree (in degree) of state i. Then,

$$(\pi P)_i = \frac{\sum_{j=1}^n P_{ij} d_j}{\sum_{j=1}^n d_j} = \frac{d_i}{\sum_{j=1}^n d_j} = \pi_i$$

because P_{ij} is either 0 or $\frac{1}{d_j}$ and the sum in the numerator is the in degree of state i.
Now lets consider the in degree of each state. There are 4 corner states, 24 edge states (exclude corners), and 36 interior states. Their degrees are 3, 5, and 8 respectively.

$$\sum_{j=1}^n d_j = 4 \cdot 3 + 24 \cdot 5 + 36 \cdot 8 = 440$$

Hence, for a corner state, $\pi_1 = \frac{3}{440} = \frac{1}{140}$. By Egrodic Theorem, we have

$$\mathbb{E}[T_1] = \frac{1}{\pi_1} = 140$$