1. A planet with mass m rotates around a star with mass M ($M \gg m$). There is dust in the universe and the density is ρ

① If the planet move around the star in a circle and the angular momentum is L. Get the relation between L and r_0

② If the planet get a small impact in the radial direction Get the frequency of the small oscillation.

The force of the dust :

$$F = -\frac{G\rho\frac{4}{3}\pi r^3 m}{r^2} = -\frac{4}{3}G\rho\pi m r = -mkr$$

$$k = \frac{4}{3}\pi G\rho$$

$$L = mr_0^2\dot{\varphi}$$

$$-\frac{GmM}{r_0^2} - mkr_0 = -mr\dot{\varphi}^2$$

$$= -\frac{L^2}{m^2r_0^3}$$

$$\Rightarrow \frac{L^2}{m^2r_0^3} - \frac{GM}{r_0^2} - kr_0 = 0$$

(2) $r = r_0 + \epsilon$

$$\ddot{r} = -\frac{GM}{r^2} - kr + \frac{L^2}{m^2 r^3}$$

$$\ddot{\epsilon} = -\frac{GM}{r_0^2}\left(1 - \frac{2\epsilon}{r_0}\right) - k(r_0 + \epsilon) + \frac{L^2}{m^2 r_0^3}\left(1 - \frac{3\epsilon}{r_0}\right)$$

$$= -\left(\frac{L^2}{m^2 r_0^4} + 3k\right)\epsilon$$

$$\Rightarrow \omega = \sqrt{\frac{L^2}{m^2 r_0^4} + 3k}$$

2. A particle moves in the potential $V(r)$

Define $h = r^2\dot\theta$, $u = \frac{1}{r}$. Prove $-\frac{dV}{dr} = -m h^2 u^2 \left(\frac{d^2u}{d\theta^2} + u\right)$

If the path f

$$\vec{F} = -\nabla V(r) = \left(-\frac{\partial}{\partial r} V(r)\right)\hat{r}$$

$$\Rightarrow -\frac{\partial V(r)}{\partial r} = m(\ddot{r} - r\dot\theta^2)$$

$$r^2\dot\theta = \text{const} = h \Rightarrow \qquad 2\dot r\dot\theta + r\ddot\theta = 0$$

$$\dot\theta = \frac{h}{r^2} = hu^2$$

$$\dot r = \frac{d(\frac{1}{u})}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\dot\theta = -h\frac{du}{d\theta}$$

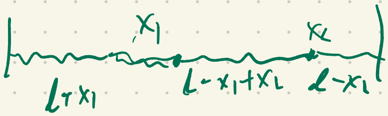
$$\ddot r = -h^2\frac{d^2u}{d\theta^2}u^2 \qquad r\dot\theta^2 = h^2u^3$$

$$\Rightarrow -\frac{\partial V(r)}{\partial r} = -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

3.

Consider three springs, the length of them are all l

Get the oscillation mode of the system



$$m_1 \ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -kx_2 - k(x_2 - x_1)$$

Solve the linear homogeneous equation

$$x_1 = Ae^{i\omega t}$$

$$x_2 = Be^{i\omega t}$$

$$(m_1\omega^2 - 2k)A + kB = 0$$

$$kA + (m_2\omega^2 - 2k)B = 0$$

Non-trivial solution

$$\begin{vmatrix} m_1\omega^2 - 2k & k \\ k & m_2\omega^2 - 2k \end{vmatrix} = 0$$

$$(m_1\omega^2 - 2k)(m_2\omega^2 - 2k) - k^2 = 0$$

$$m_1 m_2 w^4 - 2k(m_1+m_2)w^2 + 3k^2 = 0$$

$$w^2 = \frac{2k(m_1+m_2) \pm \sqrt{4(m_1^2+m_2^2+2m_1m_2) - 12m_1m_2}}{2m_1m_2}$$

$$= \frac{2k(m_1+m_2) \pm k\sqrt{4(m_1^2+m_2^2-m_1m_2)}}{2m_1m_2}$$

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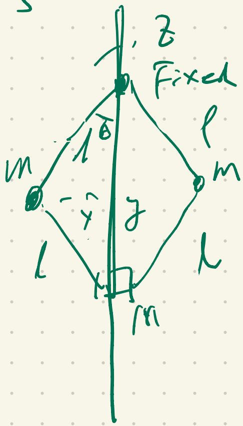

The system rotates in constant angular velocity ω_0.

M can move along side smoothly

No friction.

① Get the height of M when the system is stable

② Get the frequency of small

③

Centrifugal force

Solution:

$$x = \ell \sin\theta$$
$$y_m = -\ell\cos\theta$$
$$y_M = -2\ell\cos\theta$$

$$T = 2 \times \frac{1}{2} m (\dot{x}^2 + \dot{y}_m^2) \qquad + \frac{1}{2} M \dot{z}_M^2$$

$$= m\ell^2\dot{\theta}^2 + 2M\ell^2\dot{\theta}^2\sin^2\theta$$

$$V = (-2mg\ell - 2Mg\ell)\cos\theta - \underbrace{m\ell^2\omega_0^2\sin^2\theta}_{\text{centrifugal force}}$$

$$T+V = \text{const} = E$$

$$E = m\ell^2\dot{\theta}^2 + 2M\ell^2\dot{\theta}^2\sin^2\theta - 2(m+M)g\ell\cos\theta$$
$$- m\ell^2\omega_0^2\sin^2\theta$$

$$\frac{dU}{d\theta} = 2(m+M)\, g l \sin\theta \quad - 2ml^2\omega_0^2 \sin\theta\cos\theta$$

$$\Rightarrow \cos\theta_0 = \frac{(m+M)}{ml^2\omega_0^2}$$

$$0 = 2ml^2\dot\theta\ddot\theta + 4Ml^2\dot\theta\ddot\theta\sin^2\theta + 4Ml^2\dot\theta^3\sin\theta\cos\theta$$

$$+ 2(m+M)\, g l \sin\theta\,\dot\theta - 2ml^2\omega_0^2\sin\theta\cos\theta\,\dot\theta$$

$$\left(2ml^2 + 4Ml^2\sin^2\theta_0\right)\ddot\theta + \Big[2(m+M)\, g l\cos\theta_0(\Delta\theta)$$

$$-2ml^2\omega_0^2(2\cos^2\theta_0 - 1)\Delta\theta\Big]$$

$$(m + 2M\sin^2\theta_0)\, l^2\ddot\theta + m\omega_0^2 l^2\sin^2\theta_0\,\Delta\theta = 0$$

$$\Rightarrow \Omega = \sqrt{\frac{m\omega_0^2\sin^2\theta_0}{m + 2M\sin^2\theta_0}}$$