Tsinghua University

Differential Geometry

Fall 2024

Homework 6

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- **Collaborators:** I finish this homework by myself.

Problem 1. Noticed that $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a global flow of vector field if and only if

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}(s,p) = X_{\varphi(s,p)},\ \varphi(0,p) = p$$

So it suffices to solve an differential equation, which is unqiue if we gives the initial contidion by the previous result.

Now I give the answer and omit the check.

(1) $\varphi(t,p_1,p_2) = (\frac{1}{2}t^2 + p_2t + p_1, t + p_2)$

(2) $\varphi(t,p_1,p_2) = (p_1e^t, p_2e^{2t})$

(3) $\varphi(t,p_1,p_2) = (p_1e^t, p_2e^{-t})$

(4) $\varphi(t,p_1,p_2) = (\frac{p_1+p_2}{2}e^t + \frac{p_1+p_2}{2}e^{-t}, \frac{p_1+p_2}{2}e^t - \frac{p_1+p_2}{2}e^{-t})$

Problem 2. For (U,φ) chart of p, since $\lim_{t\to\infty}\gamma(t) = p$, if we restrict M to U, then we obtain an integral curve on U, $\hat{\gamma} : \mathbb{R} \cong (M, +\infty) \to U \cong \mathbb{R}^n$.

By proper transformation, WLOG we may assume $M = \mathbb{R}^n$.

Then $\gamma'(t) = X_{\gamma(t)} \Rightarrow$

$$\gamma(n) = \gamma(m) + \int_m^n X_{\gamma(t)}\mathrm{d}t \tag{2.1}$$

Since $\gamma(n)$ converges to p, $\gamma(t)$ is Cauchy sequence $\Rightarrow$

$0 = \lim_{m,n\to\infty}\int_m^n X_{\gamma(t)} = \lim_{m,n\to\infty}\int_m^n X_p$.

Now if $X_p \neq 0$, then

$$0 = \lim_{m,n\to\infty} \int_m^n X_{\gamma(t)} = \lim_{m,n\to\infty} \int_m^n X_p = (n-m)X_p$$

Therefore, $X_p = 0$.

Problem 3. Dnote $M = \mathbb{R}^n$

Otherwise, suppose $\lim\limits_{t\to b} |\gamma(t)| = p < \infty$.

Take $(t_i) \to b$ from left. Then $(\gamma(t_i)) \to p$.

Then $\exists$ U Nbh of p, local flow $\varphi : (-\epsilon, \epsilon) \times U \to M$. Take n large enough *s.t.* $b - t_n < \epsilon$, $\gamma(t_n) \in U$. Then $\gamma(- + t_n) : (a - t_n, b - t_n) \to M$, $\varphi(-, \gamma(t_n)) : (-\epsilon, \epsilon) \to M$ are both integral curves for X starting at $\gamma(t_n)$. By uniqueness, they coincide.

Let $\hat{\gamma} : (a, t_n + \epsilon) \to M$ be defined by $\hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a,b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \epsilon) \end{cases}$

Then $\hat{\gamma}$ is an integral curve with larger domain, then γ contradiction with the maxity of γ.

Problem 4. The global flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ generated by X only need satisfy:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t, x_1, x_2) = X_{\varphi(t,x_1,x_2)},\ \varphi(0, x_1, x_2) = (x_1, x_2)$$

Denote $\varphi = (\varphi_1, \varphi_2)$.

Then equivalently, $\varphi_1' = f \circ \varphi, \varphi_2' = g \circ \varphi, \varphi(0, -) = \mathrm{id}$ Noticed that f, g is Lipschitz continuous by the condition, hence, by the Picard-Lindelöf theorem, φ exists.