Tsinghua University

Numerical Analysis

Fall 2024

Homework 7

Lin Zejin 2025 年 5 月 27 日

Problem 1.

$$
\begin{aligned}
y(x+h)-y(x)-hf(x+h,y+hf(x,y)) &= hy'(x)+\frac{1}{2}h^2y''(x)-h(f(x,y)+h\frac{\partial f}{\partial x}+hf(x,y)\frac{\partial f}{\partial y})+O(h^3)\\
&= \frac{h^2}{2}y''(x)-h^2y''(x)+O(h^3)\\
&= -\frac{h^2}{2}y''(x)+O(h^3)
\end{aligned}
$$

Problem 2.

$$
k_1=\frac{\mathrm{d}y}{\mathrm{d}x}
$$

$$
\begin{aligned}
k_2 &= f(x+c_2h,y+ha_{21}k_1)\\
&= f(x,y)+hc_2\frac{\partial f}{\partial x}+k_1ha_{21}\frac{\partial f}{\partial y}+\left(c_2^2h^2\frac{\partial^2 f}{\partial x^2}+a_{21}^2k_1^2h^2\frac{\partial^2 f}{\partial y^2}+2c_2a_{21}h^2\frac{\partial^2 f}{\partial x\partial y}\right)+O(h^3)
\end{aligned}
$$

$$
\begin{aligned}
k_3 &= f(x+c_3h,y+ha_{31}k_1+ha_{32}k_2)\\
&= f(x+c_3h,y+ha_{31}k_1)+ha_{32}k_2\frac{\partial f}{\partial y}+\frac{h^2a_{32}^2k_2^2}{2}\frac{\partial^2 f}{\partial y^2}+O(h^3)\\
&= f(x,y)+hc_3\frac{\partial f}{\partial x}+k_1ha_{32}\frac{\partial f}{\partial y}+\left(c_3^2h^2\frac{\partial^2 f}{\partial x^2}+a_{32}^2k_1^2h^2\frac{\partial^2 f}{\partial y^2}+2c_3a_{32}h^2\frac{\partial^2 f}{\partial x\partial y}\right)\\
&+ha_{32}k_2\frac{\partial f}{\partial y}+\frac{h^2a_{32}^2k_2^2}{2}\frac{\partial^2 f}{\partial y^2}+O(h^3)
\end{aligned}
$$

where $\tilde{k}_2=k_2/h$

Noticed that

$$
y'''(x)=\left(\frac{\partial f}{\partial x}+y'(x)\frac{\partial f}{\partial y}\right)'=\frac{\partial^2 f}{\partial x^2}+2y'\frac{\partial^2 f}{\partial x\partial y}+(y')^2\frac{\partial^2 f}{\partial y^2}+y''\frac{\partial f}{\partial y}
$$

Then compared each term, $b_1k_1+b_2k_2+b_3k_3=y'+\frac{1}{2}hy''+\frac{1}{6}h^2y'''+O(h^3)$ if

$$
\begin{aligned}
b_1+b_2+b_3 &= 1\\
b_2c_2+b_3c_3 &= \frac{1}{2}\\
b_2c_2^2+b_3c_3^2 &= \frac{1}{3}\\
b_3c_2a_{32} &= \frac{1}{6}
\end{aligned}
$$

Problem 3.

$$
\begin{aligned}
T_{n+3} &= y(x_{n+3}) + \alpha(y(x_{n+2}) - y(x_{n+1})) - y(x_n) - \frac{1}{2}(3+\alpha)h[f(x_{n+2}, y(x_{n+2})) + f(x_{n+1}, y(x_{n+1}))] \\
&= \left[3hy'(x_{n+1}) + \frac{3}{2}h^2y^{(2)}(x_{n+1}) + \frac{3}{2}h^3y^{(3)}(x_{n+1}) + \frac{15}{24}y^{(4)}(X_{n+1})\right] \\
&+ \alpha\left[hy'(x_{n+1}) + \frac{1}{2}h^2y^{(2)}(x_{n+1}) + \frac{1}{6}h^3y^{(3)}(x_{n+1}) + \frac{1}{24}y^{(4)}(X_{n+1})\right] \\
&- \frac{3+\alpha}{h}\left[2y'(x_{n+1}) + hy^{(2)}(X_{n+1}) + \frac{h^2}{2}y^{(3)}(x_{n+1}) + \frac{h^3}{6}y^{(4)}(x_{n+1})\right] + O(h^5) \\
&= \frac{9-\alpha}{12}h^3y^{(3)}(x_{n+1}) + \frac{9-\alpha}{24}h^4y^{(4)}(x_{n+1}) + O(h^5)
\end{aligned}
$$

So take $\alpha = 9$ it is a convergent method of order 4.

Problem 4. 考虑常微分方程

$$
y' = f(x,y), \quad x \in [x_n,\, x_{n+1}], \quad h = x_{n+1} - x_n.
$$

为了导出四步显式 Adams–Bashforth 方法，我们在区间 $[x_n,\, x_{n+1}]$ 上引入配点

$$
x = x_n + s\,h, \qquad s \in [0,1].
$$

令

$$
F(s) \;=\; f\big(x_n + s\,h,\; y(x_n + s\,h)\big).
$$

我们希望用 $F(s)$ 在 $s = 0,\, -1,\, -2,\, -3$ 处的值来做三次 Lagrange 插值，然后对插值多项式在 $s \in [0,1]$ 上积分，得到

$$
\int_0^1 F(s)\,ds \;\approx\; \sum_{j=0}^{3} b_j\,F(-j),
$$

从而公式

$$
y_{n+1} \;=\; y_n \;+\; h\int_0^1 F(s)\,ds \;\approx\; y_n \;+\; h\sum_{j=0}^{3} b_j\,f\big(x_{n-j},\, y_{n-j}\big).
$$

下面计算各系数 $b_j = \displaystyle\int_0^1 \ell_j(s)\,ds$，其中 $\{\ell_j(s)\}$ 是通过配点 $s = 0, -1, -2, -3$ 构造的 Lagrange 基函数。

配点为

$$
s_0 = 0, \quad s_1 = -1, \quad s_2 = -2, \quad s_3 = -3.
$$

对应的 Lagrange 基函数 $\ell_j(s)$ 满足 $\ell_j(s_i) = \delta_{ij}$。具体地：

$$
\ell_0(s) = \frac{(s-s_1)(s-s_2)(s-s_3)}{(s_0-s_1)(s_0-s_2)(s_0-s_3)} = \frac{(s+1)(s+2)(s+3)}{(0+1)(0+2)(0+3)} = \frac{(s+1)(s+2)(s+3)}{6},
$$

$$
\ell_1(s) = \frac{(s-s_0)(s-s_2)(s-s_3)}{(s_1-s_0)(s_1-s_2)(s_1-s_3)} = \frac{s\,(s+2)(s+3)}{(-1-0)\,(-1+2)\,(-1+3)} = \frac{s\,(s+2)(s+3)}{(-1)\,(1)\,(2)} = -\frac{s\,(s+2)(s+3)}{2},
$$

$$
\ell_2(s) = \frac{(s-s_0)(s-s_1)(s-s_3)}{(s_2-s_0)(s_2-s_1)(s_2-s_3)} = \frac{s\,(s+1)\,(s+3)}{(-2-0)\,(-2+1)\,(-2+3)} = \frac{s\,(s+1)\,(s+3)}{(-2)\,(-1)\,(1)} = \frac{s\,(s+1)\,(s+3)}{2},
$$

$$
\ell_3(s) = \frac{(s-s_0)(s-s_1)(s-s_2)}{(s_3-s_0)(s_3-s_1)(s_3-s_2)} = \frac{s\,(s+1)\,(s+2)}{(-3-0)\,(-3+1)\,(-3+2)} = \frac{s\,(s+1)\,(s+2)}{(-3)\,(-2)\,(-1)} = -\frac{s\,(s+1)\,(s+2)}{6}.
$$

- **系数 $b_0 = \int_0^1 \ell_0(s)\,ds$.**

$$\ell_0(s) = \frac{(s+1)(s+2)(s+3)}{6} = \frac{1}{6}\big(s^3 + 6\,s^2 + 11\,s + 6\big).$$

因此

$$b_0 = \int_0^1 \ell_0(s)\,ds = \frac{1}{6}\int_0^1 \big(s^3 + 6\,s^2 + 11\,s + 6\big)\,ds = \frac{1}{6}\Big[\frac{s^4}{4} + 6\,\frac{s^3}{3} + 11\,\frac{s^2}{2} + 6\,s\Big]_0^1.$$

计算括号内:

$$\frac{s^4}{4}\Big|_0^1 = \frac{1}{4}, \quad 6\,\frac{s^3}{3}\Big|_0^1 = 2, \quad 11\,\frac{s^2}{2}\Big|_0^1 = \frac{11}{2}, \quad 6\,s\big|_0^1 = 6.$$

所以

$$b_0 = \frac{1}{6}\Big(\frac{1}{4} + 2 + \frac{11}{2} + 6\Big) = \frac{1}{6}\Big(\frac{1+8+22+24}{4}\Big) = \frac{1}{6}\cdot\frac{55}{4} = \frac{55}{24}.$$

- **系数 $b_1 = \int_0^1 \ell_1(s)\,ds$.**

$$\ell_1(s) \;=\; -\,\frac{s\,(s+2)\,(s+3)}{2} = -\,\frac{1}{2}\big(s^3 + 5\,s^2 + 6\,s\big).$$

因此

$$b_1 = \int_0^1 \ell_1(s)\,ds = -\,\frac{1}{2}\int_0^1 \big(s^3 + 5\,s^2 + 6\,s\big)\,ds = -\,\frac{1}{2}\Big[\frac{s^4}{4} + 5\,\frac{s^3}{3} + 6\,\frac{s^2}{2}\Big]_0^1.$$

计算括号内:

$$\frac{s^4}{4}\Big|_0^1 = \frac{1}{4}, \quad 5\,\frac{s^3}{3}\Big|_0^1 = \frac{5}{3}, \quad 6\,\frac{s^2}{2}\Big|_0^1 = 3.$$

因此

$$\int_0^1 \big(s^3 + 5\,s^2 + 6\,s\big)\,ds = \frac{1}{4} + \frac{5}{3} + 3 = \frac{3}{12} + \frac{20}{12} + \frac{36}{12} = \frac{59}{12}.$$

于是

$$b_1 = -\,\frac{1}{2}\cdot\frac{59}{12} = -\,\frac{59}{24}.$$

- **系数 $b_2 = \int_0^1 \ell_2(s)\,ds$.**

$$\ell_2(s) = \frac{s\,(s+1)\,(s+3)}{2} = \frac{1}{2}\big(s^3 + 4\,s^2 + 3\,s\big).$$

因此

$$b_2 = \int_0^1 \ell_2(s)\,ds = \frac{1}{2}\int_0^1 \big(s^3 + 4\,s^2 + 3\,s\big)\,ds = \frac{1}{2}\Big[\frac{s^4}{4} + 4\,\frac{s^3}{3} + 3\,\frac{s^2}{2}\Big]_0^1.$$

计算括号内:

$$\frac{s^4}{4}\Big|_0^1 = \frac{1}{4}, \quad 4\,\frac{s^3}{3}\Big|_0^1 = \frac{4}{3}, \quad 3\,\frac{s^2}{2}\Big|_0^1 = \frac{3}{2}.$$

因此

$$\int_0^1 \big(s^3 + 4\,s^2 + 3\,s\big)\,ds = \frac{1}{4} + \frac{4}{3} + \frac{3}{2} = \frac{3}{12} + \frac{16}{12} + \frac{18}{12} = \frac{37}{12}.$$

于是

$$b_2 = \frac{1}{2}\cdot\frac{37}{12} = \frac{37}{24}.$$

- **系数** $b_3 = \int_0^1 \ell_3(s)\,ds$**.**

$$\ell_3(s) = -\frac{s\,(s+1)\,(s+2)}{6} = -\frac{1}{6}\big(s^3 + 3\,s^2 + 2\,s\big).$$

因此

$$b_3 = \int_0^1 \ell_3(s)\,ds = -\frac{1}{6}\int_0^1 \big(s^3 + 3\,s^2 + 2\,s\big)\,ds = -\frac{1}{6}\Big[\frac{s^4}{4} + 3\,\frac{s^3}{3} + 2\,\frac{s^2}{2}\Big]_0^1.$$

计算括号内：

$$\frac{s^4}{4}\Big|_0^1 = \frac{1}{4}, \quad 3\,\frac{s^3}{3}\Big|_0^1 = 1, \quad 2\,\frac{s^2}{2}\Big|_0^1 = 1.$$

因此

$$\int_0^1 \big(s^3 + 3\,s^2 + 2\,s\big)\,ds = \frac{1}{4} + 1 + 1 = \frac{1+4+4}{4} = \frac{9}{4}.$$

于是

$$b_3 = -\frac{1}{6}\cdot\frac{9}{4} = -\frac{9}{24}.$$

综上可得四步显式 Adams–Bashforth 方法（AB4）为

$$\boxed{y_{n+1} \;=\; y_n \;+\; h\Big(\frac{55}{24}\,f(x_n, y_n) - \frac{59}{24}\,f(x_{n-1}, y_{n-1}) + \frac{37}{24}\,f(x_{n-2}, y_{n-2}) - \frac{9}{24}\,f(x_{n-3}, y_{n-3})\Big).}$$

考虑隐式 Adams–Moulton 的三步（k=3）情形，公式形式为

$$y_{n+1} \;=\; y_n \;+\; h\sum_{j=-1}^{2} a_j\,f\big(x_{n-j},\,y_{n-j}\big),$$

其中 a_{-1} 对应 $f(x_{n+1}, y_{n+1})$（隐式项），其余 a_0, a_1, a_2 对应前面三个点 x_n, x_{n-1}, x_{n-2}。
同样在区间 $[x_n,\ x_{n+1}]$ 上引入参数

$$x = x_n + s\,h, \qquad s\in[0,1], \qquad G(s) \;=\; f\big(x_n + s\,h,\ y(x_n + s\,h)\big).$$

我们用 $G(s)$ 在 $s = 1,\,0,\,-1,\,-2$ 处的值做三次 Lagrange 插值，然后对插值多项式在 $s\in[0,1]$ 上积分：

$$\int_0^1 G(s)\,ds \;\approx\; \sum_{j=-1}^{2} a_j\,G\big(s_j\big),$$

其中配点为

$$s_{-1} = 1, \quad s_0 = 0, \quad s_1 = -1, \quad s_2 = -2.$$

于是

$$y_{n+1} = y_n + h\int_0^1 G(s)\,ds \approx y_n + h\Big(a_{-1}f(x_{n+1}, y_{n+1}) + a_0 f(x_n, y_n) + a_1 f(x_{n-1}, y_{n-1}) + a_2 f(x_{n-2}, y_{n-2})\Big).$$

下面利用 Lagrange 基函数计算 $a_{-1},\,a_0,\,a_1,\,a_2$。

配点：

$$s_{-1} = 1, \quad s_0 = 0, \quad s_1 = -1, \quad s_2 = -2.$$

对应的 Lagrange 基函数 $\ell_j(s)$（为了标号方便，令下标与 s_j 同名）满足 $\ell_j(s_i)=\delta_{ij}$。

- 当 $j=-1$（节点 $s=1$）时，

$$\ell_{-1}(s)=\frac{(s-s_0)(s-s_1)(s-s_2)}{(s_{-1}-s_0)(s_{-1}-s_1)(s_{-1}-s_2)}=\frac{(s-0)\,(s+1)\,(s+2)}{(1-0)\,(1+1)\,(1+2)}=\frac{s\,(s+1)\,(s+2)}{1\cdot 2\cdot 3}=\frac{s\,(s+1)\,(s+2)}{6}.$$

- 当 $j=0$（节点 $s=0$）时，

$$\ell_0(s)=\frac{(s-s_{-1})(s-s_1)(s-s_2)}{(s_0-s_{-1})(s_0-s_1)(s_0-s_2)}=\frac{(s-1)\,(s+1)\,(s+2)}{(0-1)\,(0+1)\,(0+2)}=\frac{(s-1)\,(s+1)\,(s+2)}{(-1)\,(1)\,(2)}=-\frac{(s-1)\,(s+1)\,(s+2)}{2}.$$

- 当 $j=1$（节点 $s=-1$）时，

$$\ell_1(s)=\frac{(s-s_{-1})(s-s_0)(s-s_2)}{(s_1-s_{-1})(s_1-s_0)(s_1-s_2)}=\frac{(s-1)\,(s-0)\,(s+2)}{(-1-1)\,(-1-0)\,(-1+2)}=\frac{(s-1)\,s\,(s+2)}{(-2)\,(-1)\,(1)}=\frac{(s-1)\,s\,(s+2)}{2}.$$

- 当 $j=2$（节点 $s=-2$）时，

$$\ell_2(s)=\frac{(s-s_{-1})(s-s_0)(s-s_1)}{(s_2-s_{-1})(s_2-s_0)(s_2-s_1)}=\frac{(s-1)\,(s-0)\,(s+1)}{(-2-1)\,(-2-0)\,(-2+1)}=\frac{(s-1)\,s\,(s+1)}{(-3)\,(-2)\,(-1)}=-\frac{(s-1)\,s\,(s+1)}{6}.$$

- **系数 $a_{-1}=\int_0^1 \ell_{-1}(s)\,ds$.**

$$\ell_{-1}(s)=\frac{s\,(s+1)\,(s+2)}{6}=\frac{1}{6}\big(s^3+3\,s^2+2\,s\big).$$

因此

$$a_{-1}=\int_0^1 \ell_{-1}(s)\,ds=\frac{1}{6}\int_0^1\big(s^3+3\,s^2+2\,s\big)\,ds=\frac{1}{6}\Big[\frac{s^4}{4}+3\,\frac{s^3}{3}+2\,\frac{s^2}{2}\Big]_0^1.$$

计算括号内：

$$\frac{s^4}{4}\Big|_0^1=\frac{1}{4},\quad 3\,\frac{s^3}{3}\Big|_0^1=1,\quad 2\,\frac{s^2}{2}\Big|_0^1=1.$$

所以

$$\int_0^1\big(s^3+3\,s^2+2\,s\big)\,ds=\frac{1}{4}+1+1=\frac{9}{4},\quad a_{-1}=\frac{1}{6}\cdot\frac{9}{4}=\frac{9}{24}.$$

- **系数 $a_0=\int_0^1 \ell_0(s)\,ds$.**

$$\ell_0(s)=-\frac{(s-1)\,(s+1)\,(s+2)}{2}=-\frac{1}{2}\big(s^3+2\,s^2-s-2\big).$$

因此

$$a_0=\int_0^1 \ell_0(s)\,ds=-\frac{1}{2}\int_0^1\big(s^3+2\,s^2-s-2\big)\,ds=-\frac{1}{2}\Big[\frac{s^4}{4}+2\,\frac{s^3}{3}-\frac{s^2}{2}-2\,s\Big]_0^1.$$

计算括号内：

$$\frac{s^4}{4}\Big|_0^1=\frac{1}{4},\quad 2\,\frac{s^3}{3}\Big|_0^1=\frac{2}{3},\quad -\frac{s^2}{2}\Big|_0^1=-\frac{1}{2},\quad -2\,s\big|_0^1=-2.$$

所以

$$\int_0^1 \big(s^3 + 2\,s^2 - s - 2\big)\,ds = \frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2 = \frac{3}{12} + \frac{8}{12} - \frac{6}{12} - \frac{24}{12} = -\frac{19}{12},$$

$$a_0 = -\frac{1}{2}\cdot\Big(-\frac{19}{12}\Big) = \frac{19}{24}.$$

- **系数** $a_1 = \displaystyle\int_0^1 \ell_1(s)\,ds$**.**

$$\ell_1(s) = \frac{(s-1)\,s\,(s+2)}{2} = \frac{1}{2}\big(s^3 + s^2 - 2\,s\big).$$

因此

$$a_1 = \int_0^1 \ell_1(s)\,ds = \frac{1}{2}\int_0^1 \big(s^3 + s^2 - 2\,s\big)\,ds = \frac{1}{2}\Big[\frac{s^4}{4} + \frac{s^3}{3} - s^2\Big]_0^1.$$

计算括号内:

$$\frac{s^4}{4}\Big|_0^1 = \frac{1}{4},\quad \frac{s^3}{3}\Big|_0^1 = \frac{1}{3},\quad -s^2\big|_0^1 = -1.$$

所以

$$\int_0^1 \big(s^3 + s^2 - 2\,s\big)\,ds = \frac{1}{4} + \frac{1}{3} - 1 = \frac{3}{12} + \frac{4}{12} - \frac{12}{12} = -\frac{5}{12},$$

$$a_1 = \frac{1}{2}\cdot\Big(-\frac{5}{12}\Big) = -\frac{5}{24}.$$

- **系数** $a_2 = \displaystyle\int_0^1 \ell_2(s)\,ds$**.**

$$\ell_2(s) = -\frac{(s-1)\,s\,(s+1)}{6} = -\frac{1}{6}\big(s^3 - s\big).$$

因此

$$a_2 = \int_0^1 \ell_2(s)\,ds = -\frac{1}{6}\int_0^1 \big(s^3 - s\big)\,ds = -\frac{1}{6}\Big[\frac{s^4}{4} - \frac{s^2}{2}\Big]_0^1.$$

计算括号内:

$$\frac{s^4}{4}\Big|_0^1 = \frac{1}{4},\quad -\frac{s^2}{2}\Big|_0^1 = -\frac{1}{2}.$$

所以

$$\int_0^1 \big(s^3 - s\big)\,ds = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4},\quad a_2 = -\frac{1}{6}\cdot\Big(-\frac{1}{4}\Big) = \frac{1}{24}.$$

综上可得三步隐式 Adams–Moulton 方法（AM3）为

$$\boxed{y_{n+1} = y_n + h\Big(\frac{9}{24}\,f(x_{n+1}, y_{n+1}) + \frac{19}{24}\,f(x_n,\,y_n) - \frac{5}{24}\,f(x_{n-1},\,y_{n-1}) + \frac{1}{24}\,f(x_{n-2},\,y_{n-2})\Big).}$$

注意到若 $f(x,y)$ 关于 y 是线性的（例如 $f(x,y) = y - x^2 + 1$），则上述公式隐式项

$$\frac{9}{24}\,f(x_{n+1},\,y_{n+1})$$

可以展开成 $\frac{9}{24}\big(y_{n+1} - x_{n+1}^2 + 1\big)$，从而将 y_{n+1} 的系数收集后可显式解出 y_{n+1}。
根据 integral.m 的代码生成结果如下

步长 h	AB4 最大误差	AM3 最大误差
$h = 0.1250$	4.1240×10^{-4}	4.3885×10^{-5}
$h = 0.0625$	3.2025×10^{-5}	2.8674×10^{-6}
$h = 0.0312$	2.2203×10^{-6}	1.8273×10^{-7}

估计收敛阶	AB4	AM3
$h = 0.1250 \to 0.0625$	3.6868	3.9359
$h = 0.0625 \to 0.0312$	3.8503	3.9719

可见两者的收敛阶都在 4 附近，说明两者都是四阶方法。