

Complex Analysis

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Contents

1	Harmonic function	3
1.1	Polynomial rational function	3
2	Power Series	6
3	Exponential, Trigonometric and logarithmic functions	6
3.1	Exponential and Trigonometric function	6
3.2	Logarithmic Functions	6
4	Conformal Mappings	7
4.1	Connectedness	7
4.2	Compactness	7
4.3	Continuous Functions	8
4.4	short	8
4.5	Analytic Functions in Regions	8

4.6	Conformal Mappings	9
4.7	Length and Area	10
5	Möbius Transformation	10
5.1	Cross ratio	11
5.2	Symmetry	12
5.3	Steiner Circles	14
6	Elementary Conformal mapping	15
6.1	Elementary Riemann surfaces	16
7	Complex Integration	16
7.1	Fundamental Theorems	16
7.2	Cauchy's integral formula	23
	Index	25
	List of Theorems	26

1 Harmonic function

Definition 1.1 (Cauchy-Riemann equation).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 1.2 (Harmonic function). A function u is **harmonic** if it satisfied **Laplace equation** $\Delta u = 0$.

If two harmonic function u and v satisfies Cauchy-Riemann equations, then we say that v is **conjugate harmonic function of u** $\Rightarrow u$ is conjugate harmonic of $-v$.

1.1 Polynomial rational function

The polynomial $P(z) = \sum_{j=0}^n a_j z^j$ is analytic in \mathbb{C} .

We will prove the fundamental theorem of algebra

Theorem 1.3 (Fundamental Theorem of Algebra). *Every polynomial with degree $n > 0$ has at least one point.*

Theorem 1.4 (Gauss-Lucus theorem). *The smallest convex polygon that contain the zeros of P also contains the zeros of P' .*

Proof. Only need to check.

We can get this equation.

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{j=1}^n \frac{1}{\alpha - \alpha_j} = 0 \Rightarrow \sum_{j=1}^n \frac{\overline{\alpha - \alpha_j}}{|\alpha - \alpha_j|^2}$$

Hence α is linearly represented by α_j . □

Proposition 1.5. Let P and Q be two polynomials with no common zeros. Then the rational function $R(z) = \frac{P(z)}{Q(z)}$ is analytic away from the zeros of Q .

The zeros of Q are called **poles** of R , and the **order of a pole** is equal to the order of the corresponding zero of Q .

We often view R as a function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. $R_1(z) := R(\frac{1}{z})$.

If $R_1(0) = 0$, the order of the zero at ∞ (of R) is the order of the zero of $R_1(z)$ at $z = 0$.

If $R_1(0) = \infty$, the order of the pole at ∞ (of R) is the order of the pole of $R_1(z)$ at $z = 0$.

Suppose

$$R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, a_n \neq 0, b_m \neq 0$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$$

By discussing m and n , we can infer the situation of $R(z)$ at ∞ .

By adding the order of poles and zeros at ∞ , we can get the following theorem.

Theorem 1.6. The total number of zeros and poles of a rational function are the same.

Remark 1.7. This common number is called the **order of the rational function**.

Corollary 1.8. Suppose a rational function R has order p . Then every equation $R(z) = a$ has exactly p roots.

Proof. $\hat{R}(z) = R(z) - a$ has the same poles as R . □

A rational function of order 1 is a **linear fraction** $R(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$

Such fraction is often called **Möbius transformation**

Every rational function has a representation by **partial fractions**.

- If R has a pole at ∞ . Then we can write

$$R(z) = G(z) + H(z) \quad (*)$$

where G is a polynomial without constant term, and H is finite at ∞ .

The degree of G is the order of the pole of R at ∞ . G is called the **singular part** of R at ∞ .

- Let the distinct finite poles of R be β_1, \dots, β_k . Let $R_j(\psi) = R(\beta_j + \frac{1}{\psi})$. Then R_j is a rational function with a pole at ∞ . As in $(*)$, we can write

$$R_j = G_j + H_j$$

with H_j finite at ∞ . Then

$$R(z) = G_j\left(\frac{1}{z - \beta_j}\right) + H\left(\frac{1}{z - \beta_j}\right)$$

with G_j is a polynomial in $\frac{1}{z - \beta_j}$ without constant term called the **singular point** of R at β_j .

- Let $F(z) = R(z) - G(z) - \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right)$.

Then F is a rational function which can only have poles among β_j, ∞

Since by our construction, F is finite at every $\beta_j, 1 \leq j \leq k$ and ∞ .

So F is a constant.

In particular, $R(z) = G(z) + \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right) + c$.

2 Power Series

Theorem 2.1 (Abel's theorem). *If $\sum a_n$ converges, then $f(z) = \sum a_n z^n \rightarrow f(1)$ as $z \rightarrow 1$ in such a way that $\frac{|1-z|}{1-|z|}$ remains bounded.*

3 Exponential, Trigonometric and logarithmic functions

3.1 Exponential and Trigonometric function

The **exponential function** is defined as the solution of the differential equation

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1 \end{cases}$$

We denote $e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

The **trigonometric functions** are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

3.2 Logarithmic Functions

The **logarithmic function** \ln is defined by $z = \ln w$ is a root of the equation $e^z = w$.

For $w \neq 0$, we write $z = x + iy$, then

$$e^{x+iy} = w \Leftrightarrow \begin{cases} e^x = |w| \\ e^{iy} = \frac{w}{|w|} \end{cases}$$

The first equation has a unique solution $x = \ln |w|$.

The second equation $e^{iy} = \frac{w}{|w|}$ has a unique solution $y_0 \in [0, 2\pi)$.

If we write $w = re^{i\theta}$, then $x = \ln w, y = \theta = \arg w$.

Thus, for $w \neq 0$, we have

$$\ln w = \ln |w| + i \arg w$$

The function \ln is actually not single-valued. But we can define a single-valued function Ln

We define

$$a^b = \exp(b \ln a)$$

We will prove Ln is analytic in $\mathbb{C} - (-\infty, 0]$ but not continuous in $(-\infty, 0]$.

Ln is the principle branch of the logarithm.

4 Conformal Mappings

4.1 Connectedness

Theorem 4.1. *A nonempty open set in \mathbb{C} is connected iff any two of its points can be joined by a polygon which lies in the set. (i.e. Connectedness is equivalent to Path Connectedness)*

An nonempty connected subset is called a **region**

4.2 Compactness

Definition 4.2. A set X is **totally bounded** if $\forall \epsilon > 0$, X can be covered by finitely many balls of radius ϵ

Theorem 4.3. *A set is compact iff it is complete and totally bounded.*

Theorem 4.4. *A subset $X \subset \mathbb{C}$ is compact iff every infinite sequence of X has a limit point in X .*

4.3 Continuous Functions

Theorem 4.5. *Continuous function maps connected space to connected space.*

Theorem 4.6. *Continuous function maps compact space to compact space.*

4.4 Arcs and closed curves

The equation of an **arc** r in \mathbb{C} can be represented by one of the terms

- $x = x(t), y = y(t), \alpha \leq t \leq \beta, x, y$ are continuous at t
- $z(t) = x(t) + iy(t), \alpha \leq t \leq \beta.$
- the continuous mapping $\gamma : [\alpha, \beta] \rightarrow \mathbb{C}.$

For a nondecreasing function $\varphi : [\alpha, \beta] \rightarrow [\alpha, \beta], z = z(\varphi(t)), \alpha' \leq \tau \leq \beta'$ is **change of parameter** of $z(t).$

The change is **reversible** iff φ is strictly increasing.

If γ is differentiable, then call γ a **curve**.

γ is **simple**, or a **Jordan curve**, if γ is injective.

γ is **closed curve** if $\gamma(0) = \gamma(1).$

4.5 Analytic Functions in Regions

A function f is analytic on an arbitrary set A if it is the restriction to A of a function which is analytic in some open set containing A .

Theorem 4.7. *An analytic function in a region(i.e. open and connected) Ω whose derivative is 0 must reduce to a constant. The same hold if the real part, the imaginary part, the modulus, or the argument is constant.*

4.6 Conformal Mappings

Suppose $f : \Omega \rightarrow \mathbb{C}$ is analytic in Ω . $r_1 = z_1(t), r_2 = z_2(t), \alpha \leq t \leq \beta$.

$$z_0 = z_1(t_0) = z_2(t'_0), z'_1(t_0) \neq 0, z'_2(\hat{t}_0) \neq 0, \alpha < t_0, \hat{t}_0 < \beta.$$

$$f'(z_0) \neq 0, w_1(t) = f(z_1(t)), w_2 = f(z_2(\hat{t}_0))$$

$$\Gamma_1 = \{w_1(t) | \alpha \leq t \leq \beta\}, \Gamma_2 = \{w_2(t) | \alpha \leq t \leq \beta\}$$

Then

$$w'_1(t) = f'(z_1(t))z'_1(t)$$

$$w'_2(t) = f'(z_2(\hat{t}))z'_2(\hat{t})$$

\Rightarrow

$$w'_1(t_0) \neq 0, w'_2(t_0) \neq 0$$

$$\arg w'_1(t_0) = \arg f'(z_1(t_0))z'_1(t_0)$$

$$\arg w'_2(t_0) = \arg f'(z_2(\hat{t}_0))z'_2(\hat{t}_0)$$

So the "angle" $\arg w'_1(t_0) - \arg w'_2(\hat{t}_0) = \arg z_1(t_0) - \arg z_2(\hat{t}_0)$ remains the same.

Now we give the definition.

Definition 4.8. $w = f(z)$ is said to be **conformal** in Ω if f is analytic in Ω and $f'(z) \neq 0$ for $\forall z \in \Omega$.

Easy to prove that linear change of scale at z_0 is independent of the direction.

$$i.e. |f'(z_0)| = \lim_{z \rightarrow z_0} \frac{\delta \sigma}{\delta s}$$

4.7 Length and Area

The **length** of a differentiable arc γ with the equation $z(t) = x(t) + iy(t)$, $a \leq t \leq b$

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b |z'(t)| dt$$

For $\Gamma = f(\gamma)$ where f conformal mapping.

Then

$$L(\Gamma) = \int_a^b |f'(z(t))| \cdot |z'(t)| dt$$

The **area** of $E \subset \mathbb{R}$ is $A(E) = \iint_E dx dy$

Then by the differentiable functional transformation, the area $\hat{E} = f(E)$ is

$$A(\hat{E}) = \int \int_E |u_x v_y - u_y v_x| dx dy$$

If f is the conformal mapping of an open set containing E , then by Cauchy-Riemann equation

$$A(\hat{E}) = \int \int_E |f'(z)|^2 dx dy$$

5 Möbius Transformation

Recall that a **Möbius transformation** is a function of the form

$$w = s(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Then it has an inverse $z = S^{-1}(w) = \frac{dw - b}{-cw + a}$.

We may define $S(\infty) = \lim_{z \rightarrow \infty} S(z) = \frac{a}{c}$, $S(-\frac{d}{c}) = \infty$

With these definition, $S : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a topological mapping. Here one may use the chordal metric to define the topology.

$$S'(z) = \frac{ad - bc}{(cz + d)^2}$$

Then S is conformal in $\hat{\mathbb{C}} - \{-\frac{d}{c}, \infty\}$.

$w = z + \alpha$ is called a **parallel translation**.

$w = kz$ with $|k| = 1$ is a **rotation**.

$w = kz$ with $k > 0$ is a **homothetic transformation**.

$x = \frac{1}{z}$ is called an **inversion**.

Proposition 5.1. *Every Möbius transformation is a composition of the above four operations.*

5.1 Cross ratio

For three distinct points $z_2, z_3, z_4 \in \hat{\mathbb{C}}$, we can find a Möbius transformation S such that $S(z_2) = 0, S(z_3) = 1, S(z_4) = \infty$.

Lemma 5.2. *The Möbius transformation satisfying the above conditions is unique.*

The **cross ratio** (z_1, z_2, z_3, z_4) is the image z_1 under the Möbius transformation which maps z_2 to 1, z_3 to 0 and z_4 to ∞ .

Theorem 5.3. *If $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ are distinct, and T is any Möbius transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Lemma 5.4. *Let T be a Möbius transformation, $T(\mathbb{R})$ is either a circle or a straight line.*

Theorem 5.5. *The cross ratio (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle or a straight line.*

Remark 5.6. One may prove the theorem by elementary geometry

Theorem 5.7. *A Möbius transformation maps circles into circles.*

5.2 Symmetry

Suppose T is a Möbius transformation which maps $\hat{\mathbb{R}}$ onto a circle C .

We say that $w = Tz$ and $w^* = T\bar{z}$ are **symmetric** w.r.t. C .

Remark 5.8. This definition is independent of T . Suppose S is another Möbius transformation which maps $\hat{\mathbb{R}}$ onto C , then $S^{-1}T$ maps $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$, and this $S^{-1}w = S^{-1}Tz$ and $S^{-1}w^* = S^{-1}T\bar{z}$ are conjugate.

The points z and z^* are **symmetric w.r.t C through** z_1, z_2, z_3 iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.

This can be another definition.

Note that only the points on C are symmetric to themselves.

The mapping $z \mapsto z^*$ is 1-1 and is called **reflection** w.r.t. C .

5.2.1 geometric meaning of symmetry

Case1: C is a straight line. We may assume $z_3 = \infty$.

z, z^* are symmetric w.r.t. C if and only if

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Then

$$|z^* - z_2| = |z - z_2|, \quad \forall z_2 \in C \text{ and } z_2 \neq \infty$$

$$\operatorname{Im} \frac{z^* - z_2}{z_1 - z_2} = \operatorname{Im} \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

So C is the bisecting normal of the segment between z and z^* .

Case2: C is the circle $|z - a| = R$.

$$\begin{aligned} \text{Then for } \forall \text{ distinct } z_1, z_2, z_3 \in \mathbb{C}, \quad & \overline{(z, z_1, z_2, z_3)} = \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)} \\ & = (\bar{z} - \bar{a}, \bar{z}_1 - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}) = (\bar{z} - \bar{a}, \frac{R^2}{\bar{z}_1 - a}, \frac{R^2}{\bar{z}_2 - a}, \frac{R^2}{\bar{z}_3 - a}) = (\frac{R^2}{\bar{z} - \bar{a}}, z_1 - a, z_2 - \end{aligned}$$

$$a, z_3 - a) \\ = \left(\frac{R^2}{\bar{z} - \bar{a}}, z_1, z_2, z_3 \right).$$

Then the symmetric point of z w.r.t. C is

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

or

$$(z^* - a)(\bar{z} - \bar{a}) = R^2$$

\Rightarrow

$$\begin{cases} |z^* - a| \cdot |z - a| = R^2 \\ \frac{z^* - a}{z - a} = \frac{(z^* - a)(\bar{z} - \bar{a})}{|z - a|^2} > 0 \end{cases}$$

Theorem 5.9 (The Symmetric principle). *If a Möbius transformation maps a circle C_1 onto a circle C_2 , then it transforms any pair of symmetric points w.r.t. C_1 into a pair of symmetric points w.r.t. C_2 .*

Proof. Case1: $C_1 = \hat{\mathbb{R}}$. Let T be the Möbius transformation which maps $\hat{\mathbb{R}}$ onto C_2 . $\forall z \in \mathbb{C}$, by definition, $w = Tz$ and $w^* = T\bar{z}$ are symmetric w.r.t. C_2 .

Case2: C_1 is a general circle. Let $T : C_1 \rightarrow C_2$ and $S : \mathbb{R} \rightarrow C_2$ be Möbius transformation.

Suppose w, w^* are symmetric w.r.t. C_1 . Then there exists z s.t. $w = Sz, w^* = S\bar{z}$.

Then we can find $Tw = TSz, Tw^* = TS\bar{z}$ are symmetric w.r.t. C_2 since $TS : \hat{\mathbb{R}} \rightarrow C_2$ □

Remark 5.10. 1. The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto$

$w, z_2 \mapsto w_2, z_3 \mapsto w_3$ where $z_1, z_2, z_3 \in C_1, w_1, w_2, w_3 \in C_2$ is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

2. The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w_1, z_2 \mapsto w_2$ where $z_1 \in C_1, z_2 \notin C_1, w_1 \in C_2, w_2 \notin C_2$ is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

5.3 Steiner Circles

For $S(z) = \frac{az+b}{cz+d}, S'(z) = \frac{ad-bc}{(cz+d)^2}$.

A point $z \notin$ a circle C is said to on the **right(left,resp.)** of C if $\text{Im}(z, z_1, z_2, z_3) > 0(\text{Im}(z, z_1, z_2, z_3) < 0)$

Remark 5.11.

1. This agrees with everyday use since $(i, 1, 0, \infty) = i$
2. This distinct between left and right is the same for all triples, while the meaning may be reversed.

(If $C = \hat{\mathbb{R}}$, then $(z, z_1, z_2, z_3) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R} \Rightarrow \text{Im}(z, z_1, z_2, z_3) = \frac{ad-bc}{|cz+d|^2} \text{Im}(z)$)

3. We can define an absolute positive orientation of all finite circles by requiring that ∞ should be lie to the right of the oriented circles.

Consider a Möbius transformation of the form

$$w = k \cdot \frac{z-a}{z-b}$$

Here, $z = a \mapsto w = 0, z = b \mapsto w = \infty$.

Then circles through a, b maps to straight line through $0, \infty$.

The concentric circle about the origin, $|w| = \rho$, correspond to circles with the equation

$$\left| \frac{z - a}{z - b} \right| = \frac{\rho}{|k|}$$

These are the circles of **Apollonius** with limit points a and b .

Denote by C_1 the circles through a, b and C_2 the circles of Apollonius with these limit points. The configuration formed by all the circles C_1 and C_2 is called the **Steiner circles**(or **circular net**)

Theorem 5.12.

- (a) *There is exactly one C_1 and one C_2 through each point in $\hat{\mathbb{C}} \setminus \{a, b\}$*
- (b) *Every C_1 meets every C_2 under right angle.*
- (c) *Reflection in a C_1 transforms every C_2 into itself and every C_1 into another C_1 .*
- (d) *The limit points a, b are symmetric w.r.t. each C_2 , but not w.r.t. other circles.*

Proof. If the limit points are $0, \infty$, those properties are trivial in the w -plane. The general case follows since all properties are invariant under Möbius transformations. □

6 Elementary Conformal mapping

Example 6.1. $w = z^\alpha$ where $\alpha > 0$.

Let $S(u_1, u_2)$ with $0 < \varphi_2 - \varphi_1 \leq 2\pi$ be $\{z \in \mathbb{C} : z \neq 0, \varphi_1 < \arg(z) < \varphi_2\}$ where $\arg(z)$ can be chosen as any value of it.

Then $S(\varphi_1, \varphi_2)$ is a region.

In this region, a unique value of $w = z^\alpha$ is defined by $\arg w = \alpha \arg z$.

This function is analytic with $\frac{dw}{dz} = \alpha \frac{w}{z}$.

This function is 1-1 only if $\alpha(\varphi_2 - \varphi_1) \leq 2\pi$.

Example 6.2. $w = e^z$ maps $\{z \in \mathbb{C} : -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2}\}$ onto $\{w \in \mathbb{C} : \text{Re}(w) > 0\}$

Example 6.3. $w = \frac{z-1}{z+1}$ maps $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$ onto $\{w \in \mathbb{C} : |w| < 1\}$

Example 6.4.

$$\mathbb{C} \setminus [-1, 1] \xrightarrow{z_1 = \frac{z+1}{z-1}} \mathbb{C} \setminus (-\infty, 0] \xrightarrow{z_2 = \sqrt{z_1}} \{\text{Re}(z_2) > 0\} \xrightarrow{w = \frac{z_2-1}{z_2+1}} \{w \in \mathbb{C} : |w| < 1\} \quad (6.1)$$

6.1 Elementary Riemann surfaces

Example 6.5. $w = z^n, n \in \mathbb{Z}_+$ and $n > 1$.

There is a 1-1 correspondence between each angle $\frac{(k-1)2\pi}{n} < \arg z < \frac{k \cdot 2\pi}{n}, k = 1, 2, \dots, n$ and while w -plane except for the positive real axis.

Example 6.6. $w = e^z$. This function maps each parallel strip $(k-1)2\pi < \text{Im } z < k \cdot 2\pi, k \in \mathbb{Z}$ onto a sheet with a cut along the positive axis.

7 Complex Integration

7.1 Fundamental Theorems

7.1.1 Line integral and rectifiable arcs

Let $f(t) = u(t) + iv(t)$ be a complex-valued defined on $t \in [a, b] \subset \mathbb{R}$ where u, v are real-valued functions. If f is continuous on $[a, b]$, we may define the **integral**

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt$$

Let γ be a piecewise differential arc in \mathbb{C} with the equation $z = z(t)$, $a \leq t \leq b$. If f is continuous on γ , then $f(z(t))$ is continuous on $[a, b]$, and we define

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt \quad (7.1)$$

The integral defined in 7.1 is independent of the parametrization of γ . Suppose that another parametrization of γ is $\gamma : (\alpha, \beta) \rightarrow \mathbb{C}, \tau \mapsto z(t(\tau))$, where $t : (\alpha, \beta) \rightarrow (a, b)$, $\tau \mapsto t(\tau)$ is piecewise differentiable. Then we have

$$\int_a^b f(z(t))z'(t)dt = \int_{\alpha}^{\beta} f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau = \int_{\alpha}^{\beta} f(z(t(\tau)))\frac{dz(t(\tau))}{d\tau}d\tau \quad (7.2)$$

For an arc γ with equation $z = z(t)$, $a \leq t \leq b$, we define $-\gamma$ by $z = z(-t)$, $-b \leq t \leq a$.

Then we have

$$\begin{aligned} \int_{-\gamma} f(z)dz &= \int_{-b}^{-a} f(z(-t))\frac{dz(-t)}{dt}dt \\ &= - \int_{-a}^{-b} f(z(-t))z'(-t)dt \\ &= - \int_a^b f(z(\tau))z'(\tau)d\tau \\ &= - \int_{\gamma} f(z)dz \end{aligned}$$

So we have those properties:

Proposition 7.1.

$$(a) \int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$$

(b) Let f and g be two continuous functions on the piecewise differentiable arc γ , then

$$\int_{\gamma} (\lambda_1 f + \lambda_2 g) dz = \lambda_1 \int_{\gamma} f dz + \lambda_2 \int_{\gamma} g dz, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

(c) If γ can be subdivided into two pieces differentiable arcs γ_1 and γ_2 , and f is continuous on γ_1 , then

$$\int_{\gamma} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz$$

(d) (c) implies that the integral of a closed curve doesn't depend on the starting point on the curve

Example 7.2. Evaluate $\int_{\gamma} \frac{1}{z-a} dz$ where γ is the circle centered at $a \in \mathbb{C}$ with radius R .

Let $z = z(t) = a + Re^{it}$. Then the integral is $2\pi i$

7.1.2 The fundamental theorem of Calculus for integrals in \mathbb{C}

The line integral w.r.t. \bar{z} is defined by

$$\int_{\gamma} f(z) d\bar{z} = \overline{\int_{\gamma} \overline{f(z)} dz}$$

With this notation, line integrals w.r.t. $x = \text{Re}(z)$ and $y = \text{Im}(z)$ can be defined by

$$\int_{\gamma} f(z) dx = \frac{1}{2} \left[\int_{\gamma} f(z) dz + \int_{\gamma} f(z) d\bar{z} \right]$$

$$\int_{\gamma} f(z) dy = \frac{1}{2i} \left[\int_{\gamma} f(z) dz - \int_{\gamma} f(z) d\bar{z} \right]$$

if we write $f(z) = \mu + i\nu$, we have

$$\int_{\gamma} f(z) dz = \int_{\gamma} f(z) dx + i \int_{\gamma} f(z) dy = \int_{\gamma} (\mu dx - \nu dy) + i \int_{\gamma} (\nu dx + \mu dy)$$

Remark 7.3. It is followed by the intuition. We can view the integration as the multiplication between f and dz .

The integral w.r.t. **arc length** is defined by

$$\int_{\gamma} f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt$$

This integral is again independent of the parametrization. It is easy to check

$$\int_{-\gamma} f(z)|dz| = \int_{\gamma} f(z)|dz|$$

Now we define **length** of a curve γ : $L(\gamma) = \int_{\gamma} |dz|$

We have the inequality:

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| \cdot |dz| \leq L(\gamma) \cdot \sup_{z \in \gamma} |f(z)|$$

The length of an arc γ ($z = z(t)$) can also be defined as the least upper bound of all sums

$$\sum_{i=1}^n |z(t_i) - z(t_{i-1})|$$

where $a = t_0 < t_1 < \dots < t_n = b$ If this least upper bound is finite, we say that the arc is **rectifiable**

It is easy to show that piecewise differentiable arcs are rectifiable.

The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_{\gamma} f(z)dz = \lim \sum_{k=1}^n f(z(\psi_k))[z(t_k) - z(t_{k-1})]$$

Theorem 7.4. Let $\Omega \subset \mathbb{C}$ be a region, and P, Q two (possibly complex-valued) functions that are continuous on Ω , γ closed curve. The integral $\int_{\gamma} p(x, y)dx + Q(x, y)dy$ depends only on the end point of γ iff there exists a function $U(x, y)$ on Ω with $\frac{\partial U}{\partial x} = P, \frac{\partial U}{\partial y} = Q$.

Proof. " \Leftarrow ": If such a U exists, then

$$\int_{\gamma} Pdx + Qdy = \int_{\gamma} \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = \int_{\gamma} \frac{dU}{dt}dt = U(\gamma(b)) - U(\gamma(a))$$

" \Rightarrow ": Fix a point $(x_0, y_0) \in \Omega$. We define $U(x, y) = \int_{\gamma} Pdx + Qdy$ where γ is any curve between (x_0, y_0) and (x, y) . Easy to check that it is true. \square

Theorem 7.5 (Fundamental theorem of Calculus for integrals on \mathbb{C}). Let f be continuous on a region Ω containing γ . $\int_{\gamma} f dz$ depends on the endpoints iff f is the derivative of an analytic function F in Ω .

Remark 7.6. We will prove $\int_{\gamma} f dz = F(\omega_2) - F(\omega_1)$ where γ begins at ω_1 and ends at ω_2 .

Proof. Transform the line integration into the composition of two real integration. \square

Corollary 7.7. If F is analytic on Ω with $F' = f$, and γ is a closed curve in Ω , then $\int_{\gamma} f dz = 0$. Conversely if f is continuous on Ω and $\int_{\gamma} f dz = 0$ for any closed curve in Ω , then f is the derivative of an analytic function F in Ω .

7.1.3 Cauchy's theorem for a rectangle

There is some notes in this section:

R is the rectangle in \mathbb{C} , $R = \{x + iy \in \mathbb{C} : a \leq x \leq b, c \leq y \leq d\}$. And ∂R is boundary curve oriented in the counterclockwise direction.

Theorem 7.8 (Cauchy's theorem for a rectangle). *If f is analytic on an open set which contains R , then $\int_{\partial R} f(z)dz = 0$*

Proof. For \forall rectangle \tilde{R} inside R , we define $Z(\tilde{R}) = \int_{\partial \tilde{R}} f(z)dz$. Then $Z(R) = Z(R_1) + Z(R_2)$ if R is divided into Z_1, Z_2 .

Since we can divide R into four equal rectangles, and find a rectangle with $|Z(R^{(1)})| \geq \frac{1}{4}|Z(R)|$. Then repeat the above steps and we obtain a sequence of nested rectangles $R \supset R^{(1)} \supset \dots$ with the property

$$|Z(R^{(n)})| \geq \frac{1}{4^n} |Z(R)| \geq \dots \geq \frac{1}{4^n} |Z(R)| \quad (7.3)$$

$\forall \delta > 0, \exists n \in \mathbb{N}$ s.t. $R^{(n)} \subset \{z \in \mathbb{C} : |z - z_0| < \delta\}$, $\forall n \geq N$, where z_0 is the limit of $R^{(n)}$ as $n \rightarrow \infty$.

f is analytic in $R \Rightarrow \forall \epsilon, \exists \delta > 0$ s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon, \forall z \text{ with } |z - z_0| < \delta \quad (7.4)$$

We assume that δ satisfies both conditions. We have

$$\begin{aligned} Z(R^{(n)}) &= \int_{\partial R^{(n)}} f(z)dz = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)]dz \\ &\Rightarrow |Z(R^{(n)})| \leq \epsilon \int_{\partial R^{(n)}} |z - z_0|dz \text{ by 7.4} \end{aligned}$$

Let d_n be the length of diagonal of $R^{(n)}$, L_n be the length of its perimeter. Then $|z - z_0| \leq d_n, \forall z \in \partial R^{(n)}$.

$\Rightarrow |Z(R^{(n)})| \leq \epsilon d_n L_n = \epsilon \frac{D}{2^n} \cdot \frac{L}{2^n}$ where D, L are the diameter and perimeter of R .

$\Rightarrow |Z(R)| \stackrel{7.3}{\leq} 4^n |Z(R^{(n)})| \leq \epsilon DL \Rightarrow Z(R) = 0$ since ϵ is arbitrary. \square

We will next prove the following stronger theorem:

Theorem 7.9 (stronger version of Cauchy's theorem for a rectangle). *Let f be analytic on $R' = R \setminus \{\psi_1, \dots, \psi_m\}$, $m \in \mathbb{N}$. If $\lim_{z \rightarrow \psi_j} (z - \psi_j)f(z) = 0$, $\forall 1 \leq j \leq m$, then*

$$\int_{\partial R} f(z)dz = 0.$$

Proof. WLOG, we may assume f is not analytic at only one point $\psi \in R$. If we put ψ into a small rectangle S_0 , then the previous theorem tells us $\int_{\partial R} f(z)dz = \int_{\partial S_0} f(z)dz$.

$\forall \epsilon > 0$, we may choose S_0 small enough such that $|f(z)| \leq \frac{\epsilon}{|z - \psi|}$, $\forall z \in \partial S_0$

$$\Rightarrow \left| \int_{\partial R} f(z)dz \right| \leq \epsilon \int_{\partial S_0} \frac{|dz|}{|z - \psi|} \leq \epsilon \frac{1}{\frac{l}{2}} \cdot 4l = 8\epsilon$$

$\Rightarrow \int_{\partial R} f(z)dz = 0$ since ϵ is arbitrary. □

7.1.4 Cauchy's Theorem for a disk

$\Delta := \{z \in \mathbb{C} : |z - z_0| < R\}$ where $R > 0$.

Theorem 7.10 (Cauchy's Theorem for a disk). *If f is analytic in an open disk Δ , then $\int_{\gamma} f(z)dz$ for closed curve γ in Δ .*

Proof. Suppose the center of Δ is $z_0 = x_0 + iy_0$, $z = x + iy$. We define

$$F(z) = \int_{\gamma} f(z)dz$$

where γ is the horizontal line segment from z_0 to (x, y_0) added with vertical line segment from (x, y_0) to z . We have

$$\frac{\partial F}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{F(x, y + \delta y) - F(x, y)}{\delta y} = \lim_{\delta y \rightarrow 0} \frac{1}{\delta y} \int_{\delta \gamma} f(z)dz = if(z) \quad (7.5)$$

By Cauchy's thm on rectangles, one has $F(z) = -\int_{\tilde{\gamma}} f(z)dz$, where $\tilde{\gamma}$ is the vertical line segment from z_0 to (x_0, y) added with horizontal line segment from (x_0, y) to

z .

Similarly, $\frac{\partial F}{\partial x} = f(z)$.

$\Rightarrow \frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y} \Rightarrow F$ is analytic in Δ with derivative f .

By Fundamental Theorem 7.5 of Calculus $\Rightarrow \int_{\gamma} f(z)dz = 0$ for \forall closed curve in

Δ . □

Here is a stronger version.

Theorem 7.11 (stronger version of Cauchy's Theorem for a disk). *Let f be analytic in a region $\Delta' = \Delta \setminus \{\psi_1, \dots, \psi_m\}$ with $m \in \mathbb{N}$. If f satisfies $\lim_{z \rightarrow \psi_j} (z - \psi_j)f(z) = 0, \forall 1 \leq j \leq m$, then $\int_{\gamma} f(z)dz = 0, \forall \gamma$ closed in Δ'*

Proof. It is similar to the above proof.

For the case no ψ_j lies on $x = x_0$ and $y = y_0$, we can find a similar curve γ with last segment is a vertical one. Let $F(z) = \int_{\gamma} f(z)dz$. And continue the process of proof of the previous theorem.

For the case that $\exists \psi_j$ lies on the lines $x = x_0, y = y_0$, we actually can move the center to another point s.t. no ψ_j lies on the lines $x = x'_0, y = y'_0$. □

7.2 Cauchy's integral formula

7.2.1 Index of a point with respect to a closed curve

Lemma 7.12. *If the piecewise differentiable closed curve γ does not pass through $z \in \mathbb{C}$, then the value of the integral $\int_{\gamma} \frac{d\zeta}{\zeta - z}$ is a multiple of $2\pi i$.*

Proof. $\gamma : \zeta = \zeta(t), \alpha \leq t \leq \beta$. $h(t) = \int_{\alpha}^t \frac{\zeta'(s)}{\zeta(s) - z} ds$.

$z \in \gamma \Rightarrow h$ is defined and continuous on $[\alpha, \beta]$. For all t s.t. $\zeta'(t)$ is continuous, we have

$$h'(t) = \frac{\zeta'(t)}{\zeta(t) - z} \Rightarrow \frac{d}{dt} [e^{-h(t)}(\zeta(t) - z)] = 0$$



Index

Apollonius, 15

arc, 8

area, 10

change of parameter, 8

circular net, 15

conformal, 9

conjugate harmonic function of u , 3

cross ratio, 11

curve, 8

 closed curve, 8

 Jordan curve, 8

 simple , 8

exponential function , 6

harmonic, 3

homothetic transformation, 11

integral, 16

\bar{z} , 18

 arc length, 19

inversion, 11

Laplace equation, 3

left, 14

length, 10, 19

linear fraction, 4

logarithmic function, 6

Möbius transformation, 4, 10

order of a pole, 4

order of the rational function, 4

parallel translation, 11

partial fractions, 4

poles, 4

rectifiable, 19

reflection, 12

region, 7

reversible , 8

right, 14

rotation, 11

singular part, 5

singular point, 5

Steiner circles, 15

symmetric, 12

symmetric w.r.t C through z_1, z_2, z_3 , 12

totally bounded , 7

trigonometric function, 6

List of Theorems

1.3	Theorem (Fundamental Theorem of Algebra)	3
1.4	Theorem (Gauss-Lucus theorem)	3
1.5	Proposition	3
1.6	Theorem	4
2.1	Theorem (Abel’s theorem)	6
4.1	Theorem	7
4.3	Theorem	8
4.4	Theorem	8
4.5	Theorem	8
4.6	Theorem	8
4.7	Theorem	9
5.1	Proposition	11
5.3	Theorem	11
5.5	Theorem	11
5.7	Theorem	11
5.9	Theorem (The Symmetric principle)	13
5.12	Theorem	15
7.1	Proposition	17
7.4	Theorem	20
7.5	Theorem (Fundamental theorem of Calculus for integrals on \mathbb{C})	20
7.8	Theorem (Cauch’s theorem for a rectangle)	21
7.9	Theorem (stronger version of Cauchy’s theorem for a rectangle)	22
7.10	Theorem (Cauchy’s Theorem for a disk)	22
7.11	Theorem (stronger version of Cauchy’s Theorem for a disk)	23