1、解. $r = \frac{L}{\sqrt{3}}$

$|\vec{F}| = \frac{GM^2}{L^2} \cdot \sqrt{3}$

则 $|\vec{F}| = \frac{m\vec{v}^2}{r} \Rightarrow v = \sqrt{\frac{\frac{L}{\sqrt{3}} \times \frac{\sqrt{3}GM^2}{L^2}}{m}}$

$= \sqrt{\frac{GM}{L}}$

2、 $G\frac{Mm}{r^2} = m\omega^2 r \Rightarrow \omega = \sqrt{\frac{GM}{r^3}} \Rightarrow$ (a) $T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}}$

(b) $v_0 = \omega r = \sqrt{\frac{GM}{r}}$

(c) $v' = 0.99\, v_0 \Rightarrow K = \frac{1}{2} m v'^2 = \frac{1}{2} m (0.99 v_0)^2$

(d) $U = -\frac{GMm}{r}$

(e) $E = K + U = \frac{1}{2} m (0.99 v_0)^2 - \frac{GMm}{r}$

(f) 设椭圆分别在左右端点处速度为 v_1, v_2，半长轴为 a，焦距为 c

由 Kepler 2nd law, $v_1(a+c) = v_2(a-c)$ ∴ --①

又由机械能守恒定律 $\frac{1}{2} m v_1^2 - \frac{GMm}{a+c} = \frac{1}{2} m v_2^2 - \frac{GMm}{a-c}$

结合①得 $\frac{1}{2} m v_1^2 = \frac{a-c}{2a(a+c)} GMm$

$\Rightarrow E = K + U = \frac{a-c}{2a(a+c)} GMm - \frac{GMm}{a+c}$

$= -\frac{GMm}{2a}$

(f) 即 $a = -\dfrac{GMm}{2(K+U)}$. 由 Kepler 3rd law

$$\frac{T^2}{a^3} = \frac{T_0^2}{r^3} = \frac{\frac{4\pi^2 r^3}{GM}}{r^3} = \frac{4\pi^2}{GM}$$

(g) $\Rightarrow T = \sqrt{-\dfrac{4\pi^2}{GM} \cdot \dfrac{G^3M^3m^3}{8(K+U)^3}} = \sqrt{-\dfrac{4\pi^2 G^2 M^2 m^3}{8E^3}}$

$$E = K+U = \frac{1}{2}mv_1^2 - \frac{GMm}{r}$$

(h) $t_{earlier} = T - T_0 - 908$

$$= -\frac{4\pi^2 G^2 M^2 m^3}{8E^3} - \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}} - 908$$

3、环形物质量均匀，

(a) 故 $F = \dfrac{GmM}{x^2+R^2} \cdot \dfrac{x}{\sqrt{x^2+R^2}} = \dfrac{GmMx}{(x^2+R^2)^{\frac{3}{2}}}$

(b) $\frac{1}{2}mv^2 = \displaystyle\int_0^x \frac{GmMx'}{(x'^2+R^2)^{\frac{3}{2}}}\,dx'$

$$= \int_0^x \frac{GmM\,\frac{1}{2}\,dx'^2}{(x'^2+R^2)^{\frac{3}{2}}}, \qquad (x^2+r^2)^{-\frac{1}{2}}$$

$$= \frac{GMm}{r} - \frac{GMm}{\sqrt{x^2+r^2}}$$

$$\Rightarrow v = \sqrt{\frac{2GM}{r} - \frac{2GM}{\sqrt{x^2+r^2}}}$$

4. $\Delta K = -\Delta U = Gm^2\left(\frac{1}{\frac{1}{2}R} - \frac{1}{R}\right)$

$= \frac{Gm^2}{R}$

而系统开始动能为0

故 (a) $K_{总} = \frac{Gm^2}{R}$

(b) $K_{每个} = \frac{1}{2}K_{总} = \frac{Gm^2}{2R}$

(c) $\frac{1}{2}mv^2 = \frac{Gm^2}{2R}$

$\Rightarrow v = \sqrt{\frac{Gm}{R}}$

(d) $v_{B相对于A} = 2v = 2\sqrt{\frac{Gm}{R}}$

(e) $K_B = -\Delta U = Gm^2\left(\frac{1}{R} - \frac{1}{\frac{1}{2}R}\right) = \frac{Gm^2}{R}$

(f) $\frac{1}{2}mv_B^2 = K_B$

$\Rightarrow v_B = \sqrt{\frac{2Gm}{R}}$

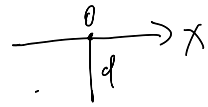
(g) 因为(f)基于非惯性参考系，故(f)是错误的

而(d)正确

5. 设圆心到轨道垂直距离为 d；以垂足为原点，轨道为 x 轴建立

$$F = -\frac{Gm}{d^2+x^2}\cdot\frac{\vec{x}}{\sqrt{d^2+x^2}}\cdot M\left(\frac{\sqrt{d^2+x^2}}{R}\right)^3$$

$$= \frac{GmM\vec{x}}{R^3}$$



故 $-\frac{GmMx}{R^3} = m\ddot{x}$

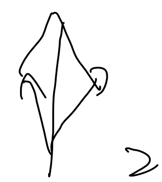
$$\Rightarrow \quad x = \sqrt{R^2-d^2}\cos\sqrt{\frac{GM}{R^3}}\,t$$

从而 $t = \frac{\pi}{\omega} = \pi\cdot\sqrt{\frac{R^3}{GM}}$.

6. 将圆环分为 $2k+1$ 份，则第 x 份和第 $2k+1-x$ 份引力的合力为

$-\frac{Gm}{R^2}\cdot\frac{2M}{2k+1}\sin\frac{x\pi}{2k+1}$，即总引力 $F = \sum_{x=0}^{2k+1}\frac{Gm}{R^2}\frac{M}{2k+1}\sin\frac{x\pi}{2k+1}$

故 $$\lim_{2k+1\to\infty} F = \frac{GmM}{\pi R^2}\int_0^{\pi}\sin x\,dx = \frac{2GmM}{\pi R^2}$$



即 (a) $F = \frac{-2GmM}{\pi R^2}$，方向向上。

(b) 对一个圆，对径二点对圆心引力方向相反，大小相等

故在圆心上没有引力作用！