Homework 5

Lin Zejin June 13, 2025

- **Collaborators:** I finish this homework by myself.

Problem 1. (a)

$$\lambda(G) = \lambda_{\min}(I + \frac{1}{d}A) = \min_{\vec{x}} \frac{\vec{x}^T(I + \frac{1}{d}A)\vec{x}}{\vec{x}^T\vec{x}} = \min_{\vec{x}} \frac{\frac{1}{d}\sum_{((i,j)\in E)}(x_i + x_j)^2}{\sum_{i=1}^n x_i^2} \le \min_{\vec{y}\in\{0,1,-1\}^n} \frac{\frac{1}{d}\sum_{((i,j)\in E)}(x_i + x_j)^2}{\sum_{i=1}^n x_i^2} = \beta(G)$$

For $\vec{x}$ such that

$$\lambda(G) = \frac{1}{d} \cdot \frac{\sum_{(i,j)\in E}(x_i + x_j)^2}{\sum_{i=1}^n x_i^2}$$

The algorithm $\mathcal{A}$ is defined as follows:

5.1. Randomly sample $t \in [0, 1]$.

5.2. Normalize $\|\vec{x}\| = 1$.

5.3. Let $y_i = \begin{cases} 1 & x_i > t \\ -1 & x_i < -t \\ 0 & \text{otherwise} \end{cases}$.

Then

$$\mathop{\mathbb{E}}_t \beta(G) = \mathop{\mathbb{E}}_t \frac{1}{d} \cdot \frac{\sum_{(i,j)\in E}(y_i + y_j)^2}{\sum_{i=1}^n y_i^2} = \frac{1}{d} \cdot \frac{\sum_{(i,j)\in E}(x_i + x_j)^2}{\sum_{i=1}^n x_i^2} = \lambda(G) \le \sqrt{2\lambda(G)}$$

So the rounding algorithm will return a feasible solution efficiently.

As a result,

$$\beta(G) \le \sqrt{2\lambda(G)}$$

(b)

Problem 2. (a) In a connected component with diameter less than $\frac{1}{10}d$, each pair of elements in the component should differ at most $\frac{1}{10}d$.

Then the total size of the component is less than

$$2^{\frac{d}{10}} \cdot \binom{\frac{d}{10}}{d} \overset{Stirling}{<} 2^{0.8d}$$

Therefore, by isoperimetric inequality,

$$\text{the number of edges cut} = \frac{1}{2}\sum_{A:\text{connected component}} |\text{edges}(A,\bar{A})| \geq \sum_{A} |A|(d - \log_2|A|) \geq \sum_{A} \frac{1}{10} d|A| = O(n\log n)$$

Problem 3.

Problem 4.

Problem 5.

Problem 6.

Problem 7. (a) They are actually all Fourier basis $\pm\chi_S$, including the constant function.

(b)

$$\{f : \{\pm 1\}^n \to \{\pm 1\} : |\mathcal{S}| = k\} = \{\sum_{i=1}^{k} \pm\chi_{S_i} : S_i \subset [n] \text{ different}\}$$

(c) For all S odd, χ_S satisfies $\chi_S(\vec{x}) = -\chi_S(\vec{x})$. As a result, those functions f are odd. Conversely, if a function is odd, then

$$f(-\vec{x}) = \sum_{S\subset[n]} \hat{f}(S)\chi_S(-\vec{x}) = -f(\vec{x}) = -\sum_{S\subset[n]} \hat{f}(S)\chi_S(\vec{x})$$

Therefore, its support $\mathcal{S}$ should contain only S with odd elements.

Problem 8. (a)

WLOG we assume g is the subfunction of f gotten by fixing $x_1 = x_2 = \cdots = x_c = 1$.

$$\mathop{\mathbb{E}}_{\vec{x}, x_1=x_2=\cdots=x_c=1} f(\vec{x}) = \mathop{\mathbb{E}}_{\vec{x}} f(\vec{x}) \prod_{i=1}^{c}(x_i + 1) = \sum_{S\subset[c]} \mathop{\mathbb{E}}_{\vec{x}} f(\vec{x})\chi_S(\vec{x}) = \sum_{S\subset[c]} \hat{f}(S)$$

is within an additive $\pm 2^c\sqrt{\epsilon}$ of the bias $\mathop{\mathbb{E}}_{\vec{x}} f(\vec{x}) = \hat{f}(\emptyset)$, since $\hat{f}(S) \leq \sqrt{\epsilon}$ for all $S \subset [c]$.

(b) The hyperthesis implies that for all $\vec{y} \in \{\pm 1\}^c$,

$$\mathop{\mathbb{E}}_{\vec{x}} f(\vec{x}) \prod_{i=1}^{c}(x_i + y_i) = \sum_{S\subset[c]} \hat{f}(S)\chi_S(\vec{y})$$

is within an additive $\pm\sqrt{\epsilon}$ of the bias $\hat{f}(\emptyset)$. Then

$$h(\vec{y}) = \sum_{S\subset[c], S\neq\emptyset} \hat{f}(S)\chi_S(\vec{y})$$

satisfies that

$$|h(\vec{y})| \leq \sqrt{\epsilon}, \forall\vec{y}$$

So

$$\epsilon \geq \mathop{\mathbb{E}}_{\vec{y}} h(\vec{y})^2 = \sum_{S\subset[c]} \hat{h}(S)^2 = \sum_{S\subset[c], S\neq\emptyset} \hat{f}(S)^2$$

Therefore $\forall S \subset [c], |S| > 0$, $\hat{f}(S)^2 \leq \epsilon$. Similarly, one can prove it for all $|S| \leq c < \frac{1}{\delta}$.

So f is (ϵ, δ)-quasirandom.

(c) The bias of f is 0. By fixing c bits, the bias of g is at most

$$\frac{\sum_{-c \leq t-(n-t) \leq c} \binom{t}{n}}{2^n}$$

In particular, if $c = 1$, $\quad$ *i.e.* $\;1 > \delta > \frac{1}{2}$, we have the bias of g is at most

$$\frac{\binom{\frac{n-1}{2}}{n}}{2^n} \geq \sqrt{\frac{2}{\pi n}}$$

So it is $(\sqrt{\frac{2}{\pi n}}, \frac{1}{2})$-quasirandom

Problem 9. (a) In the lecture we have proved that

$$\mathrm{Inf}_i(f) = \sum_{S \ni i} \hat{f}(S)^2$$

Then

$$\mathrm{Inf}(f) = \sum_{i=1}^{n} \sum\nolimits_{,S \ni i} \hat{f}(S)^2 = \sum_{S \subset [n]} |S| \hat{f}(S)^2$$

(b) $\mathbb{E}_{\vec{x}} f(\vec{x}) = \hat{f}(\emptyset)$. Therefore,

$$\mathrm{Var}(f) = \mathop{\mathbb{E}}_{\vec{x}} f(\vec{x})^2 - \left(\mathop{\mathbb{E}}_{\vec{x}} f(\vec{x})\right)^2 = \sum_{S \subset [n]} \hat{f}(S)^2 - \hat{f}(\emptyset)^2 = \sum_{S \neq \emptyset} \hat{f}(S)^2$$

(c) Define $f(x) = \begin{cases} 1 & x \in A \\ -1 & x \in \bar{A} \end{cases}$.

Then

$$\mathrm{Var}(f) = \mathop{\mathbb{E}}_{\vec{x}} f(\vec{x})^2 - \left(\mathop{\mathbb{E}}_{\vec{x}} f(\vec{x})\right)^2 = 1 - \left(1 - \frac{2|A|}{2^n}\right)^2 = \frac{|A|}{2^{n-2}} - \frac{|A|^2}{2^{2n-2}}$$

$$|\mathrm{edges}(A, \bar{A})| = \sum_{i=1}^{n} 2^{n-1} \mathrm{Inf}_i(f) = 2^{n-1} \mathrm{Inf}(f) \geq 2^{n-1} \mathrm{Var}(f) = 2|A|(1 - \frac{|A|}{2^n})$$

Problem 10. Let

$$S = \sum_{j=1}^{n} a_j x_j$$

where x_j are independent uniform random signs. Then

$$\mathrm{Inf}_i(f) = \Pr\left[f(x) \neq f(x^{\oplus i})\right] = \Pr\left[S \cdot (S - 2a_i x_i) < 0\right].$$

This event occurs if and only if $|S| \leq 2|a_i|$. Therefore,

$$\mathrm{Inf}_i(f) = \Pr\left[|S| \leq 2|a_i|\right].$$

Since $\mathrm{Var}(S) = \sum_j a_j^2 = 1$, S is a sum with variance 1. By the Berry-Essen inequality,

$$\Pr\left[|S| \leq t\right] \leq O\left(t + \tau\right).$$

Setting $t = 2|a_i|$, we obtain

$$\mathrm{Inf}_i(f) \leq O\left(|a_i| + \tau\right).$$

Since $|a_i| \leq \tau$, this gives

$$\mathrm{Inf}_i(f) \leq O(\tau).$$

□