数学中的问题介绍

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Pb 1. *Noticed that*

$$\begin{aligned} ||y - As||_2 &= ||Es||_2 \\ &\le ||E||_2| \cdot |s||_2 \end{aligned}$$

Then

$$||E||_2 \ge \frac{||y - As||_2}{||s||_2}$$

Then we try to compute $||E^*||_2$.

$$\begin{aligned} ||s^T s \cdot E^*||_2^2 &= ||(y - As)s^T||_2^2 \\ &= \max_{|x|=1} |x^T s(y - As)^T (y - As)s^T x| \\ &= |y - As|_2^2 \cdot \max_{|x|=1} |x^T s s^T x| \\ &= |y - As|_2^2 \cdot \max_{|x|=1} |x^T s|^2 \\ &= |y - As|_2^2 \cdot |s|_2^2 \end{aligned}$$

The final equation is inferred from the Cauchy-Schwarz inequality.

Thus,

$$||E^*||_2 = \frac{||y - As||_2}{||s||_2}$$

Noticed that

$$E^* \cdot s = \frac{(y - As)s^T s}{s^T s} = y - As$$

Therefore, E^* *is a solution of the question.*

Pb 2. *For* $|x| = 1$, $y = A^{-1}x$.

Then $Ay = x$ *with norm 1.*

$\Rightarrow |\sum_{j=1}^{n} a_{ij} y_j| \le 1 \ \forall 1 \le j \le n.$

If $|y_i| = \max |y_1|, \cdots, |y_n|$, *then we have*

$$1 \ge \sum_{j=1}^{n} |a_{ij} y_j| \ge |a_{ii} y_i| - \sum_{j \ne i} |y_i| \cdot |a_{ij}|$$

$\Rightarrow$

$$|y|_\infty = |y_i| \le \frac{1}{|a_{ii}| - \sum_{j \ne i} |a_{ij}|}$$

Therefore

$$|A^{-1}x| =_\infty |y|_\infty \le \max_{1 \le i \le n} \frac{1}{|a_{ii}| - \sum_{j \ne i} |a_{ij}|} = (\min_{1 \le i \le n} |a_{ii}| - \sum_{j \ne i} |a_{ij}|)^{-1}$$

So

$$||A^{-1}||_\infty \le (\min_{1 \le i \le n} |a_{ii}| - \sum_{j \ne i} |a_{ij}|)^{-1}$$

Pb 3. $||L||_2^2 = |\rho(L^T L)|$.

$||A||_2^2 = |\rho(A^T A)| = |\rho(A^2)| = |\rho(A)|^2 = |\rho(LL^T)| = ||L^T||_2^2.$

Noticed that if $L^T L\vec{x} = \lambda \vec{x}$, *then* $LL^T(L\vec{x}) = \lambda L\vec{x}$. *The converse is also true. Hence,*

$$\rho(L^T L) = \rho(LL^T)$$

Therefore, $||L||_2^2 = ||L^T||_2^2 = ||A||_2^2$.

Pb 4. *Since by the definition, for each step k, the matrix should be like:*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2k} & \cdots & a_{2n} \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & a_{kk} & \cdots & a_{kn} \\ & & & \vdots & \ddots & \vdots \\ & & & a_{nk} & \cdots & a_{nn} \end{pmatrix}$$

where me will make sure that $|a_{kk} = \max_{i\geq k}\{|a_{ik}|, |a_{ki}|\}$.
Since the following steps do not change the k^{th} *row, so the obtained upper triangular matrix U satisfies the condition:*

$$u_{ii} = \max_{i\geq k} |u_{ik}|$$

Pb 5. *Let* $e_i = (0, 0, \cdots, 1, \cdots, 0)$ *with 1 in the* i^{th} *term.*
Now $Ae_i = a_i$ *and* $||e_i||_p \leq 1$.
So $||A||_p \geq ||a_i||_p$.
Let $x = \frac{Ae_i}{||a_i||_p}$, *then* $||x||_p = 1$, $A^{-1}x = \frac{e_i}{||a_i||_p}$.
$\Rightarrow ||A^{-1}||_p \geq \frac{1}{||a_i||_p}$.
So

$$||A||_p \cdot ||A^{-1}||_p \geq \frac{||a_i||_p}{||a_j||_p}$$

Pb 6. *For* $n = 1$, *let* $L = I, D = A$, *then* $A = LDL^T$ *satisfies the condition.*
If $n-1$ *there exists the representation, the for* n, *denote*

$$A = \begin{pmatrix} A' & b \\ b^T & c \end{pmatrix}$$

Let $A' = L'D'L'^T$, *where* L' *lower triangular matrix and* D' *diagonal matrix.*
Then let $y = (L'D')^{-1}b$

$$A = \begin{pmatrix} L' & \\ y^T & 1 \end{pmatrix} \cdot \begin{pmatrix} D' & 0 \\ 0 & c - yD'y^T \end{pmatrix} \cdot \begin{pmatrix} L'^T & y \\ & 1 \end{pmatrix}$$

So A *can be represented by* LDL^T.

Pb 7. *The above problem actually gives a possible coomputation method.*
Let $L_1 = I_1$, $D_1 = [a_{11}]$.
We obtain L_i, D_i *by the equation*

$$L_i = \begin{pmatrix} L_{i-1} & \\ y_i^T & 1 \end{pmatrix}$$

$$D_i = \begin{pmatrix} D_{i-1} & 0 \\ 0 & a_{ii} - y_i D_{i-1} y_i^T \end{pmatrix}$$

where $b = (a_{1i}, a_{2i}, \cdots, a_{ni})^T$, $y = (L_{i-1}D_{i-1})^{-1}b$

Pb 8. 代码附件在压缩包中；

以下报告顺序按 *1*、高斯消元法；*2*、高斯选主元消元法；*3*、平方根法进行，分别对应 *Gauss.m*, *Gauss_choose.m*, *Cholesky.m*

通过时间测试（*sys1.m*，*sys2.m*），第一题、第二题三种方法平均用时时间从小到大为 *1*，*2*，*3*。原因应该是 *1* 循环计算数最少，而 *2* 相比于 *1* 多了选主元的循环。*3* 则在解出 *LU* 分解的基础上又调用了一遍高斯算法，故时间最长。

在误差测试中，第一题三者没有明显区别，平方根法输出上略大，但三者相对误差量级均在 *e-16*，可忽略不计；

而第二题三者相对误差均超过 *400*，第三题偏低。从数据表现上看，前两种方法仅四项在 *1* 附近，而第三种方法为 *6* 项，说明第三种方法精度比前两者更好。

误差的精度差距是由于计算顺序决定的，第三种方法多乘法和减法，除法较少，因此解决各项接近的 *Hilbert* 矩阵更有优势。

数据附件 *data.docx* 在压缩包中；