

# Differential Geometry

LIN150117

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

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# 1 Smooth Manifold

**Definition 1.1** (Topological manifold). A space  $M$  is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

**Definition 1.2** (Smooth Manifold). A smooth structure is given by an equivalence class of smooth atlas  $\{(U_\alpha, \varphi_\alpha)\}$  s.t.  $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$  is smooth  $\forall \alpha, \beta$ .  $M = \cup U_\alpha$ .

A **smooth manifold** is a topological manifold with a smooth structure.

Define when a continuous map  $f : M_1 \rightarrow M_2$  is smooth if  $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$ , we have  $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$  is smooth.

**Definition 1.3.** Given  $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ . A homeomorphism  $f : M_1 \rightarrow M_2$  is called a diffeomorphism if  $f, f^{-1}$  is smooth.

In this case we say  $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$  are diffeomorphism.

**Theorem 1.4** (Kervaire).  $\exists$  1 10-dimensional topological manifold without smooth manifold.

**Theorem 1.5** (Milnor).  $\exists$  a smooth manifold  $M$  s.t.  $M \cong S^7$  but not in diffeomorphism meaning.

**Theorem 1.6** (Kervaire-Milnor).  $\exists$  28 smooth structures (up to orientation preserving diffeomorphism) on  $S^7$

**Theorem 1.7** (Morse-Birg). On  $S^7$ . If  $n \leq 3$ , then any  $n$ -dimensional topological manifold  $M$  has a unique smooth structure up to diffeomorphism.

**Theorem 1.8** (Stallings). *If  $n \neq 4$ , then  $\exists$  a unique smooth structure on  $\mathbb{R}^n$  up to diffeomorphism.*

**Theorem 1.9** (Donaldson-Freedman-Gompf-Faubes).  *$\exists$  uncountable smooth structures on  $\mathbb{R}^4$  up to diffeomorphism.*

**Definition 1.10** (topological manifold with boundary). A space  $M$  is called a topological manifold with boundary if

1.  $M$  is Hausdorff
2.  $M$  is second countable
3.  $\forall p \in M, \exists$  a neighbourhood  $U$  of  $p$  and a homeomorphism  $\varphi : U \rightarrow V$  where  $V$  is open in  $\mathbb{H}^n$

We say a manifold  $M$  is closed if  $M$  is compact and  $\partial M$  is empty.

Our motivation for studying manifold is to study the space of solution for equations.

**Question 1.** Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  smooth,  $q \in \mathbb{R}^n$ , when is  $f^{-1}(q)$  is a smooth manifold?

For  $f : U \rightarrow \mathbb{R}^n$  smooth,  $U$  open in  $\mathbb{R}^m$ , the differential of  $f$  at  $p \in U$  denoted as  $df(p)$ .

**Definition 1.11.** We say  $p \in U$  is a **regular point** of  $f$  if  $df(p)$  is surjective. Otherwise we say  $p \in U$  is a **critical point**.

A point  $q \in \mathbb{R}^n$  is called a **regular value** of  $f$  if  $\forall p \in f^{-1}(q)$ ,  $p$  is a regular point of  $f$ .

A point  $q \in \mathbb{R}^n$  is called a **critical value** of  $f$  if  $\forall p \in f^{-1}(q)$ ,  $p$  is a critical point of  $f$ .

**Theorem 1.12** (Implicit function theorem). *If  $p \in U$  is a regular point of  $f : U \rightarrow \mathbb{R}^n$ . Then there exists*

- *An open neighbourhood  $V$  of  $p$  in  $U$*
- *An open subset  $V'$  of  $\mathbb{R}^m$*
- *A diffeomorphism  $\varphi : V \rightarrow V'$  such that  $P \circ \varphi = f$  where  $P$  is the projection from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .*

*In other words, near a regular point, we can do local coordinate change to turn  $f$  into the projection.*

**Remark 1.13.** Inverse function theorem and Implicit function theorem gives a way to find the related from "a point" to "a beighbourhood"!

In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \dots, x_m) \in V' \mid (x_1, \dots, x_n) = f(p)\}$$

*i.e. if we set  $M = f^{-1}(f(p))$ , then  $(M \cap V, \varphi_p)$  is a chart that contains  $p$ .*

**Corollary 1.14.** *If  $q$  is a regular value of  $f : U \rightarrow \mathbb{R}^n$  then  $f^{-1}(q)$  is a smooth manifold.*

**Remark 1.15.** It suffices to show that the corresponding charts are compatible.

**Theorem 1.16** (Sard). *If  $f : U \rightarrow \mathbb{R}^n$  is a smooth map, then the set of critical values of  $f$  has measure 0.*

**Remark 1.17.** For a "generic"  $q$ ,  $f^{-1}(q)$  is a manifold of dimension  $m - n$ .

**Corollary 1.18.** *If  $f : U \rightarrow \mathbb{R}^n$  is smooth and  $m < n$  then  $f(U)$  has measure 0.*

## 1.1 Lie groups and homogeneous spaces

**Definition 1.19.** We say  $G$  is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map  $\cdot : G \times G \rightarrow G$  and the inverse map  $G \rightsquigarrow G$  is smooth.

**Example 1.20.**  $GL(n, \mathbb{R}) = \{n \times n \text{ matrices with non-zero determinant}\} \subset \mathbb{R}^{n \times n}$

$$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$$

$$SO(n) = \{A \in O(n) | \det A = 1\}$$

$$U(n) = \{A \in GL(n, \mathbb{C}) | A\bar{A}^T = I\}$$

$$SU(n) = \{A \in U(n) | \det A = 1\}$$

**Exercise 1.21.**

$$O(1) \cong S^2 \qquad SO(1) \cong * \qquad (1.1)$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \qquad (1.2)$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \qquad (1.3)$$

The last one is a diffeomorphism but do not preserve the multiplication, *i.e.* not an isomorphism of Lie group.

**Theorem 1.22** (Cartan). *Let  $H$  be a closed subgroup of Lie group  $G$ . Then  $H$  is a Lie group. More precisely,  $H$  is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also,  $G/H$  is a smooth manifold*

**Definition 1.23.** Let  $M$  be a smooth manifold. We say  $M$  is a **homogeneous space** if  $\exists$  a Lie group  $G$  with a smooth transitive action  $\rho : G \times M \rightarrow M$ .

**Definition 1.24.** For  $M$  be a homogeneous space. The **isotropy** group of  $x \in M$  is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of  $G$

Given any  $x, x' \in M$ ,  $Iso(x) \cong Iso(x')$  because the group action is transitive.

Hence, we have a well-defined map

$$p : G/Iso(x) \rightarrow M \tag{1.4}$$

$$g \mapsto gx \tag{1.5}$$

**Theorem 1.25.**  *$p$  is always a diffeomorphism.*

Therefore, we have this proposition

**Proposition 1.26.**  *$M$  is a homogeneous space  $\Leftrightarrow M = G/H$  for some closed subgroup  $H$ .*

**Example 1.27.** If  $M = S^n$ , let  $G = SO(n+1)$ .

Then  $Iso(1, 0, \dots, 0) \cong SO(n)$ .

So  $S^n \cong SO(n+1)/(SO(n))$ .

Similarly, we can prove  $\mathbb{RP}^n \cong SO(n+1)/(O(n))$ ,  $\mathbb{CP}^n \cong SO(n+1)/(U(n))$

The isotropy  $k$  dimensional linear subspaces of  $\mathbb{R}^n$  can be  $O(k) \times O(n-k)$  if  $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to  $\mathbb{RP}^2$ ,  $S^2$ ,  $T^2$  and Klein bottle.

**Theorem 1.28** (Whithead). *Any smooth manifold has a triangulation.*

**Theorem 1.29** (Poincare-Hopf).  *$G$  is compact Lie group  $\Rightarrow \chi(G) = 0$ .*

**Theorem 1.30** (Mostow2005).  *$M$  is a compact homogeneous space  $\Rightarrow \chi(M) \geq 0$ .*

## 1.2 Bump Function and Partition of Unity

**Theorem 1.31** (Urysohn smooth version). *Given  $M$ , closed disjoint  $A, B$ ,  $\exists$  smooth  $f : M \rightarrow [0, 1]$  s.t.  $f|_A = 0, f|_B = 1$ .*

**Theorem 1.32** (Tietze). *Given  $M$ , closed  $A$ , smooth  $f : A \rightarrow \mathbb{R}^n$ , there exists smooth  $\hat{f} : M \rightarrow \mathbb{R}^n$  s.t.  $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

**Lemma 1.33.** *Let  $U$  be a neighbourhood of  $p \in M$ . Then  $\exists$  smooth  $\sigma : M \rightarrow [0, 1]$  s.t.*

1.  $\sigma \equiv 1$  near  $p$
2.  $\text{Supp } \sigma \subset U$

*Such  $\sigma$  is called a **bump function** at  $p$ , supported in  $U$ .*

**Definition 1.34.** An open cover of a space  $X$  is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

**Proposition 1.35.** *Given compact  $K \subset U$  and open neighbourhood  $U$  of  $K$ ,  $\exists$  a smooth  $g : M \rightarrow [0, +\infty)$  s.t.  $g|_K \equiv 1$  and  $\text{Supp } g \subset U$ .*

**Definition 1.36.** An **exhaust** of a space  $X$  is a sequence of open sets  $\{U_i\}$  s.t.

1.  $X = \bigcup_{i=1}^{\infty} U_i$
2.  $\overline{U_i}$  is compact and contained in  $U_{i+1}$

**Theorem 1.37.** *Any topological manifold has an exhaust.*

Given two open covers  $\mathcal{U}, \mathcal{V}$ , we say  $\mathcal{V}$  is a **refinement** of  $\mathcal{U}$  if  $\forall U_\alpha \in \mathcal{U}, \exists V_\beta \in \mathcal{V}$  s.t.  $V_\beta \subset U_\alpha$ .



We say a space  $X$  is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact. (The proof is hard)

**Proposition 1.38.** *Let  $\mathcal{U} = \{U_\alpha\}$  be an open cover of a topological manifold  $M$ . Then there exists countable open covers  $\mathcal{W} = \{W_i\}$ ,  $\mathcal{V} = \{V_i\}$  s.t.*

- *For any  $i$ ,  $\overline{V_i}$  is compact and  $\overline{V_i} \subset W_i$*
- *$\mathcal{W}$  is locally finite.*
- *$\mathcal{W}$  is a refinement of  $\mathcal{U}$ .*

As a corollary, we have any topological manifold is paracompact.

**Definition 1.39.** Given open cover  $\mathcal{U}$  of a smooth  $M$ , a partition of unity subordinate to  $\mathcal{U}$  is a collection of smooth functions  $\{\rho_\alpha : M \rightarrow [0, 1]\}_{\alpha \in \mathcal{A}}$  s.t.

1.  $\forall p \in M, \exists$  only finitely many  $\alpha \in \mathcal{A}$  s.t.  $p \in \text{Supp } \rho_\alpha$
2.  $\sum_{\alpha \in \mathcal{A}} \rho_\alpha(p) = 1$
3.  $\text{Supp } \rho_\alpha \subset U_\alpha$

**Theorem 1.40** (Existence of P.O.U). *For any open cover  $\mathcal{U}$  of smooth  $M$ ,  $\exists$  a P.O.U subordinate to  $\mathcal{U}$*

**Theorem 1.41** (Whitney approximation theorem). *Given any smooth  $M$ , any closed  $A$  and any continuous  $f : M \rightarrow \mathbb{R}$ ,  $\delta : M \rightarrow (0, +\infty)$ . Suppose  $f$  is smooth on  $A$ . Then  $\exists g : M \rightarrow \mathbb{R}$  smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M, |g(p) - f(p)| < \delta(p)$ .

## 2 Tangent space and tangent vectors

### 2.1 Tangent Space

Given  $p \in M$ , consider the set  $C_p^\infty(M) = \{\text{smooth function } V \rightarrow \mathbb{R}\} / \sim$  where  $f_1 \sim f_2$  if and only if  $\exists$  neighbourhood  $U$  of  $p$ ,  $f_1|_U = f_2|_U$ .

$C_p^\infty(M)$  is the space of **germs of smooth function** near  $p$ .

A **partial-derivative** of  $p$  is a  $\mathbb{R}$ -linear map  $D : C_p^\infty(M) \rightarrow \mathbb{R}$  that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

**Definition 2.1.** A **tangent vector** of  $M$  at  $p$  is a partial-derivative at  $p$ .

Define the **tangent space**  $T_p M = \{\text{all partial-derivative at } p\}$ , which is a  $\mathbb{R}$ -vector space.

**Proposition 2.2.** For  $M = U \subset \mathbb{R}^n$  open. We have  $\{\frac{\partial}{\partial x_i}\}$  is a basis for  $T_p U$ .

**Proposition 2.3.**

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leq i \leq n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

$M, N$  smooth manifolds,  $C^\infty(N, M) = \{\text{smooth } F : N \rightarrow M\}$ .

Given  $F \in C^\infty(N, M)$ ,  $F$  induces  $F^* : C_{F(p)}^\infty(M) \rightarrow C_p^\infty(N)$ ,  $f \mapsto f \circ F$ .

By taking dual, we get

$$F_* : T_p N \rightarrow T_{F(p)} M$$

we also write  $F_*$  as  $F_{*,p}$ , call it the **differential** of  $F$  at  $p$ .

where

$$F_*\left(\frac{\partial}{\partial x^i}\right)|_p = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

**Proposition 2.4.** *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_p N \rightarrow T_{G \circ F(p)} W$$

**Definition 2.5.** A smooth **curve** is a smooth map  $\gamma : (a, b) \rightarrow M$ . We say  $\gamma$  starts at  $p$  if  $\gamma(0) = p$ . We define the **velocity** of  $\gamma$  at  $\gamma(0)$  as  $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)} M$

Take charts  $(U, x^1, \dots, x^n)$  about  $p$ , let  $\gamma^i = x^i \circ \gamma$ .

We say  $\gamma, \delta$  are **tangent** to each other at  $p$  if  $(\gamma^i)'(0) = (\delta^i)'(0)$ .

Now we can define

$$(T_p M)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\} / \sim$$

where  $\gamma \sim \delta$  iff they are tangent to each other.

Then these definition is more geometric.

**Lemma 2.6.** *Given  $F \in C^\infty(M, M)$ ,  $p \in N$ , the diagram commutes:*

$$\begin{array}{ccc} \gamma \in (T_p N)_{curve} & \xrightarrow{\cong} & T_p N \\ \downarrow & & \downarrow \\ F \circ \gamma \in (T_{F(p)} M)_{curve} & \xrightarrow{\cong} & T_{F(p)} M \end{array}$$

## 2.2 Tangent Bundle

Let  $(M, \mathcal{A})$  be a smooth manifold,  $TM = \bigcup_{p \in M} T_p M$ , called the **tangent bundle**

Now we want to define a natural topology and smooth structure on  $TM$ . Take any chart  $(U, \varphi) = (U, x^1, \dots, x^n) \in \mathcal{A}$ .

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \quad (2.1)$$

$$X \in T_p U \mapsto (\varphi(p), X^1, \dots, X^n) \quad (2.2)$$

where  $X = \sum X^i \frac{\partial}{\partial x^i} |_p$ .

Then pull back standard topology on  $\varphi(U) \times \mathbb{R}^n$  to a topology on  $TU$ .

$$\mathcal{B} = \{ \hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- $\mathcal{B}$  is a basis
- $\mathcal{B}$  generates a Hausdorff, second countable topology on  $TM$ .

So  $TM$  is a topological manifold covered by charts  $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$ .

Given  $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$ , the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi} \circ \hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \quad (2.3)$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \quad (2.4)$$

So  $\hat{\mathcal{A}}$  is a smooth atlas on  $TM$ , making  $TM$  into a smooth manifold.

**Definition 2.7** (vector bundle). Given a continuous map  $f : E \rightarrow B$ , we say  $f$  is a  $n$ -dimensional **vector bundle** if:  $\exists$  an open cover  $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$  of  $B$  and homeomorphisms  $\{f^{-1}(U_\alpha) \xrightarrow[\cong]{\rho_\alpha} U_\alpha \times \mathbb{R}^n\}$  s.t.

$$\begin{array}{ccc} f^{-1}(U_\alpha) & \xrightarrow{\rho_\alpha} & U_\alpha \times \mathbb{R}^n \\ \downarrow f & \swarrow \text{projection} & \\ U_\alpha & & \end{array} \quad \text{commutes for } \alpha \in I$$

- $\forall p \in U_\alpha \cap U_\beta$ , the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

is linear.

Call  $f^{-1}(p)$  the **fiber** over  $p$ .

**Proposition 2.8.** *Given vector bundle  $f : E \rightarrow B$ , the fiber  $f^{-1}(p)$  has a structure of a vector space.*

**Example 2.9** (Product bundle).  $E = \mathbb{R}^n \times B$

**Example 2.10** (Tautological bundle).

$$B = \mathbb{CP}^n = \{1\text{-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map  $(L, v) \mapsto L$

Given vector bundles  $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$ , a bundle map consists of  $(\hat{f}, f)$  s.t.

- $$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow \pi & & \downarrow \pi \\ B_1 & \xrightarrow{f} & B_2 \end{array} \text{ commutes.}$$
- $\forall b \in B, \hat{f} : \pi_1^{-1}(b) \rightarrow \pi_2^{-1}(f(b))$  is linear.

If  $\hat{f}, f$  are diffeomorphisms, then we call  $(\hat{f}, f)$  an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

**Example 2.11.**  $TS^1, TS^2$  are both trivial.

$$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$$

**Theorem 2.12.** *If  $G$  is a Lie group, then  $TG$  is trivial.*

*Proof.* For  $(x^1, x^2, \dots, x^n)$  is a basis of  $T_e G$  The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG, (g, c^1, \dots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \rightarrow G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism  $(l_g)_*$

□

**Proposition 2.13** (Adams, 1960s).  $TS^n$  is trivial if and only if  $n = 0, 1, 3, 7$ .

**Proposition 2.14.**

1. Given any  $F \in C^\infty(M, N)$ ,  $F_* : TM \rightarrow TN$  is a bundle map.
2.  $TS^n$  is isomorphic to the following bundle:

$$B = S^n \quad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

**Definition 2.15** (smooth section). Given a smooth vector bundle  $\pi : E \rightarrow B$ , a **smooth section** is a smooth map  $S : B \rightarrow E$  s.t.  $\pi \circ S = id_B$ .

$$s_0 : B \rightarrow E, b \mapsto 0 \in 0\text{-vector in } \pi^{-1}b.$$

## 2.3 Vector Field, Curves and Flows

**Definition 2.16.** A (tangent) **vector field** is a smooth section of  $TM$ . i.e. a smooth map  $M \xrightarrow{X} TM$  s.t.  $X(p) \in T_p M, \forall p \in M$

Given any  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

**Example 2.17.**  $X = f^1 \partial x^1 + f^2 \partial x^2$  is a gradient field if and only if  $\frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}$

**Theorem 2.18** (Poincare-Hopf). *For closed  $M$ ,  $M$  has a nowhere vanishing vector field if and only if  $\chi(M) = 0$ .*

So  $S^n$  has a nowhere vanishing vector field if and only if  $n$  is odd.

**Theorem 2.19** (MaoQiu).  *$S^2$  has no no-where vanishing vector field.*

So We cannot smooth out all the hairs on a ball.

Given a vector field  $X = \{X_p\}_{p \in M}$ , a curve  $\gamma : (a, b) \rightarrow M$  is called an **integral curve** of  $X$  if  $\gamma'(t) = X_{\gamma(t)}, \forall t \in (a, b)$ , where  $\gamma'(t) = \gamma_*\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M$ .

We say  $\gamma$  is maximal if the domain cannot be extended to a larger interval.

Denote the set of all smooth vector fields on  $M$  by  $\mathfrak{X}M$

Recall that  $\gamma$  is maximal if it's domain can not be extended to a large open interval.

In a local chart  $(U, x^1, \dots, x^n)$ ,  $X|_U = \sum_{i=1}^n a^i \partial x^i$ . Then  $\gamma$  is an integral curve if and only if  $(\gamma^i)'(t) = a^i(\gamma(t)), \forall 1 \leq i \leq n$ , where  $\gamma^i = x^i \circ \gamma : (a, b) \rightarrow \mathbb{R}$ .

And in this case the initial value condition:  $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$ .

Locally, solving integral curve starting at  $p$  is equivalent to solving ODE with initial value  $p^1, \dots, p^n$ . By existence and uniqueness of solutions of ODE, we have

**Theorem 2.20** (Fundamental theorem of integral curve). *Let  $X \in \mathfrak{X}M, p \in M$ , then:*

(1) (Uniqueness) *Given any two integral curves  $\gamma_1, \gamma_2 : (a, b) \rightarrow M$ , then we have:*

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a, b) \Rightarrow \gamma_1 = \gamma_2$$

(2) *there exists a unique max integral curve  $\gamma : (a(p), b(p)) \rightarrow M$  starting at  $p$ .*

(3) (integral curve smoothly depend on initial values)  $\exists$  Nbh  $U$  of  $p, \varepsilon > 0$ , and smooth

$\varphi : (-\varepsilon, \varepsilon) \times U \rightarrow M$  s.t.  $\forall q \in U, \varphi_\varepsilon := \varphi(-, q) : (-\varepsilon, \varepsilon) \rightarrow M$  is an integral

curve starting at  $q$ .

we call such  $\varphi$  a local **flow** generated by  $X$ .

**Definition 2.21.** Given  $X \in \mathfrak{X}M$ , a global **flow** generated by  $X$  is a smooth map  $\varphi : \mathbb{R} \times M \rightarrow M$  s.t.  $\forall q \in M$ ,  $\varphi_q := \varphi(-, q)$  is the maximal integral curve of  $X$  starting at  $q$ .

$$\Leftrightarrow \frac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s, p)}, \forall s \in \mathbb{R}, p \in M \text{ and } \varphi(0, p) = p, \forall p \in M.$$

If such global flow exists, then we say  $X$  is **complete**.

**Example 2.22.**

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$  is complete, where global flow  $\varphi : \mathbb{R} \times M \rightarrow M$ ,  $\varphi(t, p) = p \cdot e^t$ .
- $X = x^2 \partial x$  is not complete. Max integral curve starting at 1 is given by  $\gamma(t) = \frac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$ .

Given  $X \in \mathfrak{X}M$ , we define  $\text{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$ .

**Theorem 2.23.** If a vector field  $X$  is compactly supported, then  $X$  is complete.

**Corollary 2.24.** Any vector field on closed manifold is complete.

**Lemma 2.25** (Escaping lemma). Suppose  $\gamma : (a, b) \rightarrow M$  is a max integral curve, with  $(a, b) \neq \mathbb{R}$ . Then  $\nexists$  compact  $K \subset M$  s.t.  $\gamma(a, b) \subset K$

*Proof.* Otherwise, suppose  $\gamma(a, b) \subset K$ . WLOG, we may assume  $b < +\infty$ .

Take  $(t_i) \rightarrow b$  from left. Then  $\gamma(t_i) \in K$ . After passing to subsequence, we may assume  $(\gamma(t_i)) \rightarrow p \in K$ .

Then  $\exists U$  Nbh of  $p$ , local flow  $\varphi : (-\varepsilon, \varepsilon) \times U \rightarrow M$ . Take  $n$  large enough s.t.  $b - t_n < \varepsilon, \gamma(t_n) \in U$ . Then  $\gamma(- + t_n) : (a - t_n, b - t_n) \rightarrow M, \varphi(-, \gamma(t_n)) : (-\varepsilon, \varepsilon) \rightarrow M$  are both integral curves for  $X$  starting at  $\gamma(t_n)$ . By uniqueness, they coincide.



Let  $\hat{\gamma} : (a, t_n + \varepsilon) \rightarrow M$  be defined by  $\hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \varepsilon) \end{cases}$

Then  $\hat{\gamma}$  is an integral curve with larger domain, then  $\gamma$  contradiction with the maximality of  $\gamma$ . □

*Proof of 2.23.* Take any max integral curve  $\gamma : (a, b) \rightarrow M$ . Suppose  $(a, b) \neq \mathbb{R}$ . Then  $X_{\gamma(s)} \neq 0, \forall s$ . Otherwise, the constant map  $\mathbb{R} \rightarrow M, t \mapsto \gamma(s)$  is an integral curve with larger domain.

So  $\forall s, \gamma(s) \in \text{Supp} X \Rightarrow \gamma(a, b) \subset \text{Supp} X$  which is compact  $\Rightarrow (a, b) = \mathbb{R}$  by the lemma. This causes contradiction! □

A smooth  $\varphi : \mathbb{R} \times M \rightarrow M$  is called an **one-parameter transformation group** if

$$(1) \quad \varphi_0 := \varphi(0, -) = \text{id}_M$$

$$(2) \quad \varphi_s \circ \varphi_t = \varphi_{s+t} \text{ for all } s, t \in \mathbb{R}. \text{ In particular, } \varphi_s^{-1} = \varphi_{-s}.$$

**Theorem 2.26.**  $\varphi \in C^\infty(\mathbb{R} \times M, M)$ , then  $\varphi$  is an one-parameter transformation group if and only if  $\varphi$  is the global flow generated by some  $X \in \mathfrak{X}M$

**Lemma 2.27** (Translation lemma). If  $\gamma : (a, b) \rightarrow M$  is an integral curve for some  $X \in \mathfrak{X}M$ , then  $\forall s \in \mathbb{R}, \gamma(- + s) : (a - s, b - s) \rightarrow M$  is also an integral curve for  $X$ .

*Proof.* Let  $\iota = \gamma(- + s)$ . Then  $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$  □

**Lemma 2.28.** Let  $\varphi : (-\varepsilon, \varepsilon) \times U \rightarrow M$  be a local flow for some  $X \in \mathfrak{X}M$ . Then  $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$  provided that  $s, t, s + t \in (-\varepsilon, \varepsilon), p, \varphi_r(p) \in U$ .

*Proof.*  $\gamma_p = \varphi(-, p)$  is an integral curve for  $X$ .

$\Rightarrow \gamma_p(- + s)$  is an integral curve for  $X$  starting at  $\gamma_p(s) = \varphi_s(p)$ . But  $\gamma_{\varphi_s(p)}$  is also an integral curve starting at  $\varphi_s(p)$ . Thus  $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$  □

**Lemma 2.29.** Let  $\varphi : (-\varepsilon, \varepsilon) \times U \rightarrow M$  be a local flow for some  $X \in \mathfrak{X}M$ . Then  $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$  i.e. any vector field is invariant under its flow.

*Proof.* Take  $f \in C_{\varphi(p)}^\infty(M)$ .

$$\begin{aligned}\varphi_{s,*}(X_p)(f) &= X_p(f \circ \varphi_s) \\ &= \frac{d}{dt}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \\ &= \frac{d}{dt}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \\ &= X_{\varphi_s(p)}(f)\end{aligned}$$

□

*Proof of 2.26.* " $\Leftarrow$ " is because the lemma  $\varphi_s \circ \varphi_r = \varphi_{s+r}$

" $\Rightarrow$ " Let  $X = \{X_p\}$  where  $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$ .

Leave it as an exercise.

□

**Time dependent** vector field is a smooth map  $X : \mathbb{R} \times M \rightarrow TM$  s.t.  $X_{(t,p)} \in T_pM$ .

A smooth curve  $\gamma(a, b) \rightarrow M$  is the **integral curve** for  $X$  if  $\gamma'(t) = X_{(t,\gamma(t))}$ .

In local chart, solving  $\gamma$  is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness:  $\gamma_1, \gamma_2$  are both integral curves for  $X$ ,  $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define  $\text{Supp}X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$ .

Then  $X$  is compactly supported, then  $X$  is complete( i.e. a global flow  $\varphi : \mathbb{R} \times M \rightarrow M$ )

But something is not true for time dependent vector field:

- translation lemma is not true.
- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

## 2.4 Another definition of vector field

A derivation on  $M$  is a  $\mathbb{R}$ -linear map  $C^\infty(M) \xrightarrow{D} C^\infty(M)$  that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

**Theorem 2.30.** *We have a bijection:*

$$\begin{aligned} \rho : \mathfrak{X}M &\xrightarrow{1:1} \{\text{derivation on } M\} \\ X &\mapsto D_X : f \mapsto X(f) \end{aligned}$$

**Lemma 2.31.**  $D_p : \mathfrak{X}_p M \rightarrow \mathbb{R}$ -linear map  $C^\infty(M) \rightarrow \mathbb{R}$  s.t.  $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$  is an isomorphism of vector spaces.

*Proof.* Leave it as an exercise. □

**Lemma 2.32.** *Given a vector field(not necessarily smooth)  $X = \{X_p\}_{p \in M}$ ,  $X$  is smooth  $\Leftrightarrow \forall f \in C^\infty(M)$ ,  $X(f)$  is smooth.*

*Proof.* " $\Leftarrow$ "  $\forall p \in M$ , take chart  $(U, x^1, x^2, \dots, x^n)$  around  $p$ .  $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$  :  $U \rightarrow \mathbb{R}$ , where  $f^i = X|_U(x^i)$ . Take  $\varphi : M \rightarrow [0, 1]$  s.t.  $\varphi \equiv 1$  near  $p$ ,  $\text{Supp} \varphi \subset U$ ,  $\varphi \cdot x^i \in C^\infty(M)$ .

Then  $X(\varphi \cdot x^i) = f^i$  near  $p$ . By assumption,  $f^i$  is smooth near  $p$ . So  $f^i$  is smooth, so  $X$  is smooth.

" $\Rightarrow$ " Similar. □

**Theorem 2.33.** *The map  $\rho : \mathfrak{T}M \rightarrow \{\text{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$  is well-defined and bijective.*

*Proof.*  $\rho$  is well-defined:  $X(f) \in C^\infty(M)$  by Lemma 2.32, and  $D_x(fg) = D_x(f)g + fD_x(g)$  since  $X$  is a point-derivation.

$\rho$  is injective:  $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$  as maps  $C^\infty(M)$  to  $\mathbb{R}$ . By Lemma 2.31, we have  $X_p = Y_p, \forall p$ . So  $X = Y$ .

$\rho$  is surjective: Given  $D : C^\infty(M) \rightarrow C^\infty(M)$ . Define  $D_p : C^\infty(M) \rightarrow \mathbb{R}$  by  $D_p(f) := D(f)(p)$  satisfies the Leibniz rule. By Lemma 2.31,  $D_p = D_{X_p}$  for some  $X_p \in T_p M$ . Define  $X = \{X_p\}_{p \in M}$ . Then  $X(f) = D(f), \forall f \in C^\infty(M)$ . By Lemma 2.32,  $X$  is a smooth vector field. □

## 3 Lie group, Lie algebra and Lie bracket

### 3.1 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned} \{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \text{ s.t.} \\ &\quad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\} \end{aligned}$$

$$\begin{aligned} \{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\} \end{aligned}$$

**Notation 3.1.** We will identify  $X \in \mathfrak{X}M$  with its derivation  $D_x : C^\infty(M) \rightarrow C^\infty(M)$ . So a vector field is just a  $\mathbb{R}$ -linear map  $X : C^\infty(M) \rightarrow C^\infty(M)$  s.t.  $X(fg) = fX(g) + X(f)g$ .

**Definition 3.2** (Lie bracket). Given two (smooth) vector field  $X, Y : C^\infty(M) \rightarrow C^\infty(M)$ , we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \rightarrow C^\infty(M)$$

**Theorem 3.3.** For any  $X, Y \in \mathfrak{X}M$ ,  $[X, Y] \in \mathfrak{X}M$

*Proof.* Easy to check that  $[X, Y]$  is linear.

By Leibuniz rule,

$$\begin{aligned} [X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\ &= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\ &= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\ &= [X, Y](f) \cdot g - f \cdot [X, Y](g) \end{aligned}$$

□

So What is the geometric meaning of  $[X, Y]$ ? Non commutativity of flows.

**Fact 3.4.** Given  $X, Y \in \mathfrak{X}M$ , we say  $X, Y$  are commutative vector field if  $[X, Y] = 0$

$X, Y$  are commutative iff for any local flows  $\varphi^X : (-\varepsilon, \varepsilon) \times U \rightarrow M$ ,  $\varphi^Y : (-\varepsilon, \varepsilon) \times U \rightarrow M$  we have  $\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X$

**Proposition 3.5** (Calculation of  $[V, W]$  using local charts). *Chart  $(U, x^1, \dots, x^n)$ ,  $V, W \in \mathfrak{X}M$ ,  $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$ ,  $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$ . Then*

$$\begin{aligned} [V, W]|_U &= \sum_{i=1}^n (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n \left( \sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} \\ &= \sum_{1 \leq i, j \leq n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i} \end{aligned}$$

**Example 3.6.**  $V = x\partial x + y\partial y$ ,  $W = -y\partial x + x\partial y$  commutes.

**Proposition 3.7** (Properties of Lie bracket).

(a) *Natuality under push-forward.*

Given any  $F \in \text{Diff}(M, N)$ ,  $V \in \mathfrak{X}M$ ,  $W \in \mathfrak{X}M$ , we have  $[F_*V, F_*W] = F_*[V, W]$ .

(b)  *$\mathbb{R}$ -linearity  $\forall a, b \in \mathbb{R}$*

$$[aX + bV, W] = a[X, W] + b[V, W]$$

$$[W, aX + bV] = b[W, X] + a[W, V]$$

(c) *anti-symmetric  $[V, W] = -[W, V]$*

(d) *Jacobi identity*

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

(f) Leibniz rule

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

**Definition 3.8.** Given  $F \in C^\infty(M, N)$ ,  $V \in \mathfrak{X}M$ ,  $W \in \mathfrak{X}N$ . We say  $W$  is  **$F$ -related** to  $V$  if  $\forall p \in M$ ,  $F_{p,*}(V_p) = W_{F(p)}$  where  $F_{p,*} : T_pM \rightarrow T_{f(p)}N$

**Example 3.9.**  $F : S^1 \rightarrow \mathbb{R}^2, \theta \mapsto (\cos \theta, \sin \theta)$ ,  $V = \partial\theta$ ,  $W = -y\partial x + x\partial y$ .

*Note 1.* In general, given  $V \in \mathfrak{X}M$  and  $F \in C^\infty(M, N)$ . There may not exist  $W \in \mathfrak{X}N$  s.t.  $V, W$  are  $F$ -related. Even such  $W$  exists, it may not be unique.

However, if  $F$  is a diffeomorphism, given any  $V$ ,  $\exists$  unique  $W$  s.t.  $V$  and  $W$  are  $F$ -related. Actually,  $W_p = F_*V_{F^{-1}(p)}$ .

Such  $W$  is called **push forward** of  $V$  along  $F$ , denoted by  $F_*V$ , only defined when  $F$  is a diffeomorphism.

**Lemma 3.10.**  $\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N, F \in C^\infty(M, N)$ . Then  $W$  is  $F$ -related to  $V$  iff  $\forall f \in C^\infty(N)$ ,  $V(f \circ F) = W(f) \circ F \in C^\infty(M)$

*Proof.* Check that  $F_{p,*}(V_p)(f) = W_{F(p)}(f), \forall f \in C^\infty(N)$  □

**Proposition 3.11.** Given  $V_0, V_1 \in \mathfrak{X}M$ ,  $W_0, W_1 \in \mathfrak{X}N$ ,  $F \in C^\infty(M, N)$ ,  $W_i$  is  $F$ -related to  $V_i, i = 0, 1 \Rightarrow [W_0, W_1]$  is  $F$ -related to  $[V_0, V_1]$

**Corollary 3.12** (Naturality of Lie bracket). Given any  $F \in \text{Diff}(M, N)$ ,  $V \in \mathfrak{X}M, W \in \mathfrak{X}M$ , we have  $[F_*V, F_*W] = F_*[V, W]$

The rest of Proposition 3.7 is easy to check if it is viewed as a mapping  $C^\infty(M) \rightarrow C^\infty(M)$ .

## 3.2 Lie algebra of a Lie group

**Definition 3.13.** A **Lie algebra**  $\mathfrak{g}$  is  $\mathbb{R}$ -linear space  $\mathfrak{g}$  with map  $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  s.t. it is bilinear, anti-symmetric and satisfies the Jacobian identity.

Then  $(\mathfrak{TM}, [-, -])$  is an infinite dimensional Lie algebra.

For  $G$  Lie group,  $\forall g \in G$  we have diffeomorphism

$$l^g : G \rightarrow G, h \mapsto gh$$

$$r^g : G \rightarrow G, h \mapsto hg$$

We say  $X \in \mathfrak{X}G$  is **left invariant** if  $l_*^g(X) = X, \forall g \in G$ . Similarly,  $X$  is **right invariant** if  $r_*^g(X) = X$ .

**Proposition 3.14.**  $X, Y$  are left/right invariant  $\Rightarrow [X, Y]$  is left/right invariant.

*Proof.*  $l_*^g[X, Y] = [l_*^g X, l_*^g Y] = [X, Y]$  □

So we can find a natural Lie algebra of  $G$ :

$\text{Lie}(G) := \{\text{left invariant vector fields on } G\}$ , with  $[-, -]$  restricted from  $\mathfrak{X}G$

**Theorem 3.15.** Given any  $V \in T_e G, \exists$  unique left invariant  $\hat{V} \in \mathfrak{X}G$  s.t.  $\hat{V}_e = V$ .

**Corollary 3.16.**  $\text{Lie}(G) \cong T_e G$  as vector spaces.

*Proof of Theorem 3.15.*

**Uniqueness of  $\hat{V}$ :**  $\hat{V}_g = l_{e,*}^g(\hat{V}_e) = l_{e,*}^g(V)$ . So  $\hat{V}$  is determined by  $V$ .

**Existence of  $\hat{V}$ :** Let  $\hat{V} = \{\hat{V}_g\}_{g \in G}$  where  $\hat{V}_g = l_{e,*}^g(\hat{V}_e)$ .



$\hat{V}$  is left-invariant because

$$(l_*^h(\hat{V}))_g = l_{h^{-1}g,*}^h(\hat{V}_{h^{-1}g}) = l_{h^{-1}g,*}^h(l_{e,*}^{h^{-1}g}(V)) = l_{e,*}^g(V) = \hat{V}_g$$

$\hat{V}$  is smooth: Take any  $f \in C^\infty(G)$  suffices to show  $\hat{V}(f) \in C^\infty(G)$ .

Take any smooth  $\gamma : \mathbb{R} \rightarrow G$  s.t.  $\gamma(0) = e, \gamma'(0) = V$ . Then  $l^g \circ \gamma : \mathbb{R} \rightarrow G$  satisfies  $l^g \circ \gamma(0) = g, (l^g \circ \gamma)(0) = g, (l^g \circ \gamma)'(0) = l_{e,*}^g(V) = \hat{V}_g$

So

$$\hat{V}(f)(g) = \hat{V}_g(f) = \frac{d}{dt}f(l^g \circ \gamma(t))|_{t=0} = \frac{d}{dt}f(g \cdot \gamma(t))|_{t=0} \quad (3.1)$$

Consider the map

$$\begin{aligned} \hat{f} : G \times \mathbb{R} &\xrightarrow{\text{id} \times \gamma} G \times G \xrightarrow{f} \mathbb{R} \\ (g, t) &\mapsto (g, \gamma(t)) \mapsto g \cdot \gamma(t) \mapsto f(g \cdot \gamma(t)) \end{aligned}$$

Then  $\hat{f}$  is smooth,  $\frac{\partial \hat{f}}{\partial t}|_{t=0} : G \rightarrow \mathbb{R}$  is smooth, but  $\frac{\partial \hat{f}}{\partial t}|_{t=0}(g) = \hat{V}(f)(g)$  by 3.1. So  $\hat{V}(f) \in C^\infty(G)$ . □

**Example 3.17.**  $G = \text{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^2$ .

$$\mathfrak{gl}(n, \mathbb{R}) = \text{Lie}(\text{GL}(n, \mathbb{R})) = T_I \text{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$$

**Theorem 3.18.**  $\forall A, B \in \mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}), [A, B] = AB - BA$ .

**Remark 3.19.** This theorem shows that the Lie bracket viewed as the Lie algebra and matrix are the same. In some sense, it means the Lie bracket defined in three sets  $\mathfrak{gl}(n, \mathbb{R}) = T_I \text{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$  can commute with those corresponding, or equivalently, are just the same.

**Lemma 3.20.**  $\forall A \in \mathfrak{gl}(n, \mathbb{R})$ , the left invariant vector field  $\hat{A}$  is complete and generate the flow  $\varphi_t : \text{GL}(n, \mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2 t^2}{2!} + \dots)$

*Proof.*

$$\hat{A}_g = g \cdot A \in T_g G = M_n(\mathbb{R})$$

$$\frac{\partial}{\partial t} \varphi_t(g) = \frac{\partial}{\partial t} (g(e^{At})) = g e^{At} A = \hat{A}_{g \cdot e^{At}} = \hat{A}_{\varphi_t(g)}$$

□

**Remark 3.21.** This lemma tells how to compute  $A(f) = \hat{A}(f)(I)$  as a tangent vector or a vector field, as we will see in the next proof.

*Proof of Theorem 3.18.* Take  $A, B \in \mathfrak{gl}(n, \mathbb{R})$ . Want to show  $[\hat{A}, \hat{B}]_I = AB - BA$ .

Pick  $f \in C_I^\infty(G)$ , need to show  $A(\hat{B}(f)) - B(\hat{A}(f)) = (AB - BA)(f)$

Further Simplification: Just need to focus on  $f = x^{ij}$ , where  $x^{ij} : \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}, E \mapsto (E - I)_{ij}$ . Actually,  $\partial x^{ij}$  is what we choose as a basis of  $T_I \text{GL}(n, \mathbb{R})$ .

Such  $f$  satisfies  $f(I + -)$  is  $\mathbb{R}$ -linear.

Recall that Given  $W \in \mathfrak{Z}M$ ,  $W(f)(p) = \frac{d}{dt} f(\varphi_t^W(p))|_{t=0}$ .

So  $\hat{B}(f)(g) = \frac{d}{dt} f(g e^{Bt})|_{t=0}$ .

So since  $A(\hat{B}(f)) = \hat{A}((\hat{B}(f)))(I) = \frac{d}{ds} (\hat{B}(f)(e^{As}))|_{s=0}$ ,

$$A(\hat{B}(f)) = \frac{d}{ds} (\hat{B}(f)(e^{As}))|_{s=0} = \frac{d^2}{ds dt} f(I + sA + tB + \frac{s^2}{2} A^2 + stAB + \frac{t^2}{2} B^2 + \dots)|_{s=t=0}$$

Similarly,

$$B(\hat{A}(f)) = \frac{d^2}{ds dt} f(I + sA + tB + \frac{s^2}{2} A^2 + stBA + \frac{t^2}{2} B^2 + \dots)|_{s=t=0}$$

So  $A(\hat{B}(f)) - B(\hat{A}(f)) = f(I + (AB - BA)) = (AB - BA)(f)$  since  $f$  is  $\mathbb{R}$ -linear. □

Similarly, for  $G = \text{GL}(n, \mathbb{C})$ ,  $\text{Lie}(G) = \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$ , we have  $[A, B] = AB - BA$ .

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group  $G$ , its universal cover  $\hat{G}$  is simply-connected with  $\pi^{-1}(G) \subset Z(\hat{G})$ .

What is the meaning of Lie bracket. There is a fact about it:

**Fact 3.22.**  $G$  is connected Lie group.  $G$  is abelian iff  $[-, -] = 0$  on  $\text{Lie}(G)$

### 3.3 Morphisms between Lie group and Lie algebras

A smooth map  $F : G \rightarrow H$  between two Lie group is called a **morphism** if  $F(gh) = F(g)F(h)$ .

A linear map  $L : g \rightarrow h$  between Lie algebra is called a **morphism** if  $L[u, v] = [Lu, Lv]$ .

**Proposition 3.23.** Let  $F : G \rightarrow H$  be a morphism of Lie groups. Then  $F_{e,*} : \text{Lie}(G) \rightarrow \text{Lie}(H)$  is a morphism of Lie algebra.

*Proof.*  $V_0, V_1 \in \text{Lie}(G) = T_e G$ ,  $W_i = F_{e,*}(V_i) \in \text{Lie}(H) = T_e H$ . Let  $\hat{V}, \hat{W}$  be left-invariant vector fields.

*Claim.*  $\hat{W}_i$  is  $F$ -compatible with  $\hat{V}_i$  for  $i = 0, 1$ .

*Proof of Claim.*  $\forall g \in G$ ,  $F_*(\hat{V}_g) = F_*(l_g^*(V)) = (F \circ l_g)_*(V) = (l^{F(g)} \circ F)_*(V) = l^{F(g)}(W) = \hat{W}_{F(g)}$  □

So  $[\hat{W}_0, \hat{W}_1]$  is  $F$ -compatible with  $[\hat{V}_0, \hat{V}_1]$ . In particular,  $[W_0, W_1] = F_*([V_0, V_1])$ . □

## 4 Vector Field

### 4.1 Canonical form of a field

Recall that  $V \in \mathfrak{X}M$ ,  $p \in M$  is called a **regular point** if  $V_p \neq 0$ , and is called a **singular point** if  $V_p = 0$ .

**Theorem 4.1** (Canonical Form Theorem). *Let  $p$  be a regular point of  $V$ . Then  $\exists$  local chart  $(U, x^1, \dots, x^n)$  around  $p$  s.t.  $V|_U = \partial x^1$*

*Proof.* This is a local problem. We may assume  $M \subset \mathbb{R}^n$  open. We may also assume  $p = 0$ ,  $V_0 = \partial r^1|_0$  where  $r^i$  coordinate function.

Let  $\varphi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)^n \rightarrow M$  be the local flow of  $V$ .

Define  $\psi : (-\varepsilon, \varepsilon)^n \rightarrow M$  by  $\psi(t, r^2, \dots, r^n) = \varphi(t, (0, r^2, \dots, r^n))$ . Then  $\psi(-, r^2, \dots, r^n)$  is an integral curve for  $V$ . Therefore,  $\psi_*(\partial t) = V$ .

At  $\vec{0}$ , we have  $\psi_{\vec{0},*}(\partial t) = V_{\vec{0}} = \partial r^1$ ,  $\psi_{\vec{0},*}(\partial r^i) = \partial r^i$ .

So  $\psi_{*,\vec{0}} : T_{\vec{0}}(-\varepsilon, \varepsilon)^n \rightarrow T_{\vec{0}}M$  is an isomorphism.

By the inverse function theorem,  $\exists U' \subset (-\varepsilon, \varepsilon)^n$ ,  $U \subset M$  s.t.  $\psi|_{U'} : U' \rightarrow U$  is a diffeomorphism.

Then  $(U, (\psi|_{U'})^{-1})$  is the local chart what we need. □

**Remark 4.2.** Regular point in a vector field is simple, as we can view it in the standard chart locally. However, behavior of  $V$  at a singular point can be complicated. For example, for  $f(x, y) = x^2 - y^2$ ,  $\nabla f = 2x\partial x - 2y\partial y$ ,  $g : \mathbb{C} \rightarrow \mathbb{C}$ ,  $z \mapsto z^n$ , they behave differently at  $\vec{0}$ .

### 4.2 Lie derivative of vector field

$V, W \in \mathfrak{X}M$ ,  $\mathcal{L}_V W$  is the directional derivative of  $W$  in the direction of  $V$ .

**Definition 4.3.** The **Lie derivative**  $\mathcal{L}_V W \in \mathfrak{T}M$  is defined as follows:  $\forall p \in M$ , let  $\{\theta_t : U \rightarrow M\}_{t \in (-\varepsilon, \varepsilon)}$  be the local flow for  $V$ . Then

$$(\mathcal{L}_V W)_p = \lim_{t \rightarrow 0} \frac{(\theta_{-t})_* W_{\theta_t(p)} - W_p}{t}$$

**Remark 4.4.** This defintion is actually a difference between  $T_{\theta_t(p)}$  and  $T_p$ , which need pullback.

**Lemma 4.5.**  $\mathcal{L}_V W$  is well-defined and smooth.

*Proof.* For  $p \in M$ , take local chart  $(U, x^1, \dots, x^n)$ . Let  $\theta : (-\varepsilon, \varepsilon) \times U \rightarrow M$  be the flow of  $V$ . Take  $J_0 \subset (-\varepsilon, \varepsilon)$ ,  $U_0 \subset U$ . Let  $\theta^i = x^i \circ \theta : J_0 \times U_0 \rightarrow \mathbb{R}$ ,  $W|_U = \sum_{i=1}^n W^i \partial x^i$ .

Under the basis  $\{\partial x^i\}$ ,  $(\theta_{-t})_* : T_{\theta_t(p)} M \rightarrow T_p M$  is represented by

$$\left( \frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j} \right)_{i,j}$$

So  $(\theta_{-t})_* W_{\theta_t(x)} = \sum_{i,j} \frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j} W^j(\theta(t, x)) \cdot \partial x^i$  is smooth in  $t, x$ . So

$$(\mathcal{L}_V W)_x = \frac{\partial((\theta_{-t})_*(W_{\theta_t(x)}))}{\partial t} \Big|_{t=0}$$

is well-defined and smooth. □

**Theorem 4.6.** For all  $V, W \in \mathfrak{T}M$ ,  $\mathcal{L}_V W = [V, W]$ .

*Proof.* For  $p$  is a regular point of  $V$ . By canonical form theorem 4.1,  $\exists$  local chart  $(U, x^1, \dots, x^n)$  around  $p$  s.t.  $V|_U = \partial x^1$ . Let  $W|_U = \sum_{i=1}^n W^i \partial x^i$ .

Then  $\theta_t(x_1, \dots, x_n) = (x_1 + t, x_2, \dots, x_n)$ . So

$$\mathcal{L}_V W|_U = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

$$[V, W]|_U = \sum_i V(W^i) \partial x^i - \sum_i W(V^i) \partial x^i = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

Then  $[V, W]|_U = \mathcal{L}_V W$ .

For  $p$  is a singular point but  $p \in \text{Supp}(V)$ . Then  $\exists p_i \rightarrow p$  s.t.  $V_p \neq 0$ . By the previous case  $(\mathcal{L}_V W)_{p_i} = [V, W]|_{p_i}$ . By continuity, We have  $(\mathcal{L}_V W)_p = [V, W]_p$ .

For  $p \notin \text{Supp}(V)$ ,  $\exists$  Nbd  $U$  of  $p$  s.t.  $V|_U = 0$ . Then  $\theta_t(q) = q$ . So

$$(\mathcal{L}_V W)|_U = 0 = [V, W]|_U$$

□

### Corollary 4.7.

- $\mathcal{L}_V W$  is  $\mathbb{R}$ -linear with respect to  $V, W$ .
- $\mathcal{L}_V W = -\mathcal{L}_W V$ .
- $\mathcal{L}_V[W, X]$ .
- (Jacobian identity)  $\mathcal{L}_V[W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$ .
- (Jacobian identity)  $\mathcal{L}_{[V, W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$ .
- $\mathcal{L}_V(fW) = (Vf) \cdot W + f \mathcal{L}_V W$
- Let  $F : M \rightarrow N$  be a diffeomorphism. Then  $F_*(\mathcal{L}_V W) = \mathcal{L}_{F_*(V)} F_*(W)$ .

## 4.3 Commuting vector fields

**Definition 4.8.** We say  $V, W \in \mathfrak{X}M$  **commutes** if  $[V, W] = 0$ .

**Theorem 4.9.** TFAE:

1  $V, W$  commutes.

2  $W$  is invariant under the flow generated by  $V$ , i.e.  $\theta_{t,*}(W_p) = W_{\theta_t(p)}$

3 The flow for  $V, W$  commutes, i.e.  $\theta_t \circ \eta_s = \eta_s \circ \theta_t$  whenever either side is defined or equivalently, whose the domain is compatible.

**Lemma 4.10.** Given  $F \in C^\infty(M, N)$ ,  $V \in \mathfrak{X}M, W \in \mathfrak{X}N$ . Then  $W$  is  $F$ -related to  $V$  if and only if  $\forall t \in \mathbb{R}, \eta_t \circ F = F \circ \theta_t$  on the domain of  $\theta_t$ , which means

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow \theta_t & & \downarrow \eta_t \\ M & \xrightarrow{F} & N \end{array} \text{ commutes.}$$

*Proof.* " $\Rightarrow$ " Let  $\gamma = F \circ \theta^p : J \rightarrow N$  satisfies

$$\gamma'(t) = (F \circ \theta^p)'(t) = F_*((\theta^p)'(t)) = F_*(V_{\theta^p(t)}) = W_{F(\theta^p(t))} = W_{\gamma(t)}$$

So  $\gamma$  is an inetgral curve of  $W$  starting at  $\gamma(0) = F(p)$  i.e.  $F \circ \theta^p = \gamma(t) = \eta^{F(p)}(t)$   
i.e.  $F \circ \theta_t = \eta \circ F$ .

" $\Leftarrow$ " Suppose  $F \circ \theta_t = \eta \circ F$ . Then  $(F \circ \theta^p)(t) = \eta^{F(p)}(t)$ .

Then  $F_*V_p = F_*((\theta^p)'(0)) = (F \circ \theta^p)'(0) = (\eta^{F(p)})'(0) = W_{F(p)}$ . So  $W$  is  $F$ -related to  $V$ . □

*Proof of Theorem 4.9.*  $2 \Rightarrow 1$ :  $(\theta_{-t})_*(W_{\theta_t(p)}) = W_p$ . So

$$\mathcal{L}_V W = \lim_{t \rightarrow 0} \frac{(\theta_{-t})_*(W_{\theta_t(p)}) - W_p}{t} = 0$$

$1 \Rightarrow 2$ : Let  $X(t) = (\theta_{-t})_*(W_{\theta_t(p)})$ ,  $p \in M$ .

Want to show that  $X(t) = X_p$  for all  $t$ . Suffices to show  $\frac{d}{dt}|_{t=t_0} X(t) = 0$ .

For  $t_0 = 0$ ,  $\frac{d}{dt}|_{t=0} X(t) = (\mathcal{L}_V W)_p = 0$ .

In general, set  $s = t - t_0$ ,  $X(t) = (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))})$ . Then

$$\begin{aligned}
\frac{d}{dt}|_{t=t_0} X(t) &= \frac{d}{ds}|_s X(s + t_0) \\
&= \frac{d}{ds}|_s (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))}) \\
&= (\theta_{t_0})_* \frac{d}{ds}|_{s=0} (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))}) \\
&= (\theta_{t_0})_*(\mathcal{L}_V W)_{\theta_{t_0}(p)} \\
&= 0
\end{aligned}$$

2  $\Rightarrow$  3. For simplicity, assume  $V, W$  are complete.  $F = \theta_s : M \rightarrow M$ . By 2,  $W$  is  $F$ -related to  $W$ . So by the lemma,

$$\begin{array}{ccc}
M & \xrightarrow{F} & M \\
\downarrow \theta_t & & \downarrow \eta_t \\
M & \xrightarrow{F} & M
\end{array} \text{ commutes.}$$

$\eta_t$  is flow for  $W$ . i.e.  $\theta_s \circ \eta_t = \eta \circ \theta_s$

3  $\Rightarrow$  2 is similar. The diagram commutes, so  $W$  is  $F$ -related to  $W$ . □

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### 4.3.1 Canonical form of commuting vector field

**Theorem 4.11.** Given  $V_1, \dots, V_k \in \mathfrak{X}M$ , s.t.

1)  $[V_i, V_j] = 0, \forall i, j$ .

2)  $V_{1,p}, V_{2,p}, \dots, V_{k,p}$  linearly independent at some  $p \in M$

Then  $\exists$  local chart  $(U, x^1, \dots, x^n)$  around  $p$  s.t.  $V_i|_U = \frac{\partial}{\partial x^i}, \forall 1 \leq i \leq k$

We prove it using the inverse function theorem.



*Proof.* This is a local problem. So we may assume  $M \subset \mathbb{R}^m$  be open with coordinate function  $r^i : M \rightarrow \mathbb{R}, 1 \leq i \leq m$ .

After translation and linear transformation, we may assume  $p = \vec{0}, V_{i,\vec{0}} = \left. \frac{\partial}{\partial x^i} \right|_{\vec{0}}, 1 \leq i \leq k$ .

Take local flow  $\{\theta_t^i : (-\varepsilon, \varepsilon)^m \rightarrow M\}_{t \in (-\varepsilon, \varepsilon)}$  for  $V_i$ .

Define  $\psi : (-\varepsilon, \varepsilon)^k \times (-\varepsilon, \varepsilon)^{m-k} \rightarrow M, \psi(t^1, \dots, t^k, r^{k+1}, \dots, r^m) = \theta_{t_1}^1 \circ \theta_{t_2}^2 \cdots \circ \theta_{t_k}^k(0, 0, \dots, 0, r^{k+1}, \dots, r^m)$ , where  $\theta^i$  commutes with each other.

So if we fix  $t^j, j \neq i$  except  $t^i, \psi(t^1, \dots, t^{i-1}, -, t^{i+1}, \dots, t^k, r^{k+1}, \dots, r^m)$  is an integral curve for  $V^i$ . Then  $V^i$  is  $\psi$ -related to  $\partial t^i$ .

On the other hand.  $\psi(0, 0, \dots, 0, r^{k+1}, \dots, r^m) = (0, 0, \dots, 0, r^{k+1}, \dots, r^m)$ . So  $\psi_{\vec{0},*} : T_{\vec{0},*} : T_{\vec{0}}(-\varepsilon', \varepsilon')^m \rightarrow T_{\vec{0}}M, \partial t^i \mapsto V_{i,0} = \partial x^i|_0$  and  $\partial r^i \mapsto \partial r^i, k+1 \leq i \leq m$ .

So  $\psi_{\vec{0},*}$  is an isomorphism.

By the inverse function theorem, there exists Nbh  $U' \subset (-\varepsilon', \varepsilon')^m$  s.t.  $\psi : U' \rightarrow U$  is a diffeomorphism and  $U \subset M$  open.

Then  $(U, (\psi|_U)^{-1})$  is the local chart we need. □

## 4.4 The constant rank theorem

$F \in C^\infty(M, N), p \in M$ . The **rank** of  $F$  at  $p$  is

$$\begin{aligned} \text{rank}_p F &:= \text{rank}(F_{p,*} : T_p M \rightarrow T_{F(p)} N) \\ &= \text{rank} \left( \frac{\partial F^i(p)}{\partial x^j} \right)_{i,j} \end{aligned}$$

We say  $F$  has **constant rank**  $k$  near  $p$  if  $\exists$  Nbh  $U$  of  $p$  s.t.  $\text{rank}_q F = k, \forall q \in U$

**Proposition 4.12.**

$$\text{rank}_q(F) \leq \min(\dim(M), \dim(N))$$

**Theorem 4.13** (The constant rank theorem). *Suppose  $F : M \rightarrow N$  has constant rank  $k$  near  $p \in M$ , then  $\exists$  local charts  $U \xrightarrow[\cong]{\varphi} \mathbb{R}^m$  around  $p$ ,  $V \xrightarrow[\cong]{\psi} \mathbb{R}^n$  around  $F(p)$  s.t.*

$$\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^n \text{ is given by } (x^1, \dots, x^m) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$$

*Proof.* This is a local problem. So we may assume  $M = \mathbb{R}^m$ ,  $N = \mathbb{R}^n$  by restricting to local charts. And  $p = 0$ ,  $F(p) = 0$ . After changing orders of coordinates, may assume  $\left( \frac{\partial F^i}{\partial x^j}(0) \right)_{1 \leq i, j \leq k}$  is invertible. Write  $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ ,  $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ .

Then  $F(x, y) = (Q(x, y), R(x, y))$ . Consider  $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $(x, y) \mapsto (Q(x, y), y)$ .

Then

$$\varphi_{(0,0),*} = \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}(0) & 0 \\ \frac{\partial Q^i}{\partial y^j}(0) & I_{m-k} \end{bmatrix} \quad (4.1)$$

is invertible.

By inverse function theorem,  $\exists$  Nbh  $U_0 \subset \mathbb{R}^m$ ,  $\tilde{U}_0 \subset \mathbb{R}^m$  of 0 s.t.  $\varphi : U_0 \rightarrow \tilde{U}_0$  is a diffeomorphism.

$$\tilde{U}_0 \xrightleftharpoons[\varphi]{\varphi^{-1}} U_0 \xrightarrow{F} \mathbb{R}^n$$

$$(Q(x, y), y) \leftarrow (x, y) \mapsto (Q(x, y), R(x, y))$$

So  $F \circ \varphi^{-1} : \tilde{U}_i \rightarrow \mathbb{R}^n$ ,  $(x, y) \mapsto (x, A(x, y))$ . And

$$(F \circ \varphi^{-1})_{p,*} = \begin{bmatrix} I_k & 0 \\ \frac{\partial A}{\partial x}(p) & \frac{\partial A}{\partial y}(p) \end{bmatrix} \quad (4.2)$$

Since  $\text{rank}(F \circ \varphi^{-1})$  is  $k$ ,  $\frac{\partial A}{\partial y}(p) = 0$ . i.e.  $A(x, y) = A(x)$ .

We can find a map  $\psi : (x, y) \mapsto (x, y - A(x))$  in a smaller neighborhood of 0 by the inverse theorem similarly.

And  $\psi \circ F \circ \varphi$  maps  $(x, y)$  to  $(x, 0)$ . So we end the proof.  $\square$

**Definition 4.14.**  $F \in C^\infty(M, N)$ .

We say  $F$  is **submersion** if  $F_{p,*}$  is surjective  $\forall p \in M$ .

We say  $F$  is **immersion** if  $F_{p,*}$  is injective  $\forall p \in M$ .

We say  $F$  is **embedding** if  $F$  is immersion and  $F$  is a topological embedding. (i.e.  $F : M \rightarrow F(M)$  is a homeomorphism)

If  $F$  is embedding (immersion resp.), we say  $M$  or  $F(M)$  is an **embedded submanifold** (immersed submanifold, resp.) of  $N$ .

Denote  $M \looparrowright N$  be the immersion.  $M \hookrightarrow N$  be the embedding.

**Example 4.15.**

- There is an example  $F : S^1 \rightarrow \mathbb{R}^2$  where  $F$  is an immersion but not an embedding.
- Projection  $M \times N \rightarrow M$  is a submersion.
- $E \xrightarrow{p} B$  is a smooth vector bundle, then  $p$  is a submersion.
- $\gamma : \mathbb{R} \rightarrow M$  is an immersion  $\Leftrightarrow \gamma'(t) \neq 0, \forall t$ .
- There is an example  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  is injective immersion but not an embedding
- $\gamma : \mathbb{R} \rightarrow \mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}, x \mapsto (x, cx), c \notin \mathbb{Q}$  is injective immersion but not embedding.

**Definition 4.16.** For  $F : X \rightarrow Y$ , we say  $F$  is **proper** if for any compact set  $K \subset N$ ,  $F^{-1}(K)$  is compact.

**Lemma 4.17.**  $X$  is compact,  $Y$  Hausdorff, then  $F : X \rightarrow Y$  is proper.

**Proposition 4.18.**  $F \in C^\infty(M, N)$  is an injective immersion, and  $F$  is proper. Then  $F$  is an embedding.

*Proof.*  $F : M \rightarrow F(M)$  is a closed map. □

**Definition 4.19.** For  $F \in C^\infty(M, N)$ .

$p \in M$  is called **regular point** if  $F_{p,*} : T_p M \rightarrow T_{F(p)} N$  is surjective.

$p \in M$  is called **critical point** if  $F_{p,*} : T_p M \rightarrow T_{F(p)} N$  is not surjective.

$q \in N$  is called **regular value** if  $\forall p \in F^{-1}(q)$ ,  $p$  is a regular point.

$q \in N$  is called **critical value**(or **singular value**) if  $\exists p \in F^{-1}(q)$ ,  $p$  is a critical point.

**Theorem 4.20 (Sard).** Singular value has measure 0.

*Proof.* We will not prove it in this lecture. □

**Theorem 4.21.**  $M$  is an embedded submanifold of  $N$  if and only if  $\forall p \in M \subset N$ ,  $\exists$  local chart  $(U, x^1, \dots, x^n)$  around  $p$  of  $N$  s.t.  $M \cap U = \{(x^1, \dots, x^m, 0, \dots, 0)\}$

*Proof.* " $\Rightarrow$ ":  $F : M \rightarrow N$  is embedding  $\Rightarrow F$  has constant rank  $m$ . Apply constant rank theorem near  $p$ , and we finish the proof of " $\Rightarrow$ "

The converse is trivial. □

**Theorem 4.22.**  $F \in C^\infty(M, N)$ ,  $q$  is a regular value of  $F$ . Then  $F^{-1}(q)$  is an embedded submanifold of  $M$ . And

$$\forall p \in F^{-1}(q), T_p F^{-1}(q) = \ker(F_{p,*} : T_p M \rightarrow T_{F(p)} N)$$

*Proof.*  $q$  is regular value  $\Rightarrow \text{rank}_p F = n, \forall p \in F^{-1}(q)$ .

$\Rightarrow \text{rank}_{p'} F = n, \forall p' \text{ near } p$ , since we know the rank of  $p'$  near  $p$  should not be less than that of  $p$

So by the constant rank theorem,  $F^{-1}(q)$  is a submanifold near  $p$ .

□

Denote

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) | A + A^T = 0\}$$

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) | A + A^* = 0\}$$

$$\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) | \text{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \text{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) | \text{tr} A = 0\}$$

**Theorem 4.23.** *Those above sets are the Lie algebra of the corresponding Lie group. For instance,  $\mathfrak{su}(n) = \text{Lie}(\text{SU}(n))$ .*

## 5 Differential forms

### 5.1 Introduction

Our goal is to define the integration  $\int_M \alpha$  s.t.

- Works for any smooth manifold  $M$ , without embedding  $M$  into  $\mathbb{R}^n$
- Generalize two types of surface integral, i.e.  $\int_{\Sigma} f dS$  and  $\int_{\Sigma} f dx \wedge dy$

For Canton's idea,  $\alpha$  is a "differential  $k$ -form" on  $M$  s.t.

- $\forall F \in C^\infty(N, M), F^*(\alpha)$  is a  $k$ -form on  $N$

- If  $k = \dim M$ , then  $\int_M \alpha \in \mathbb{R}$

## 5.2 Alternating vector linear algebra

For  $V_1, \dots, V_n, W$  be  $\mathbb{R}$ -vector spaces,  $f : V_1 \times \dots \times V_n \rightarrow W$  is called **multi  $\mathbb{R}$ -linear** if

$$\begin{aligned} f(v_1, \dots, v_{i-1}, av_i + bv'_i, v_{i+1}, \dots, v_n) &= af(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n) \\ &+ bf(v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_n) \end{aligned} \quad (5.1)$$

**Example 5.1.**

- Inner product  $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\cdot} \mathbb{R}$ .
- Matrix multiplication  $M_{n \times m}(\mathbb{R}) \times M_{m \times k}(\mathbb{R}) \rightarrow M_{n \times k}(\mathbb{R})$ .
- Cross product  $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\times} \mathbb{R}^3$ .
- Bilinear form.

We hope that we can construct a vector space  $V_1 \otimes \dots \otimes V_n$  s.t. we have canonical isomorphism:

$$\{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \dots \times V_n \rightarrow W\} \cong \{\text{linear map } V_1 \otimes \dots \otimes V_n \rightarrow W\} \quad (5.2)$$

Then we can transform the study of multilinear algebra into the study of the normal linear algebra.

For any set  $S$ , let

$$\mathbb{R} \langle S \rangle = \left\{ \text{formal linear combination } \sum_{i=1}^n a_i s_i \mid a_i \in \mathbb{R}, s_i \in S, n < \infty \right\} \quad (5.3)$$

Consider  $\mathbb{R} \langle V_1 \times \cdots \times V_n \rangle = \left\{ \sum_{i=1}^k a^i(V_{i,1}, \dots, V_{i,n}) | a^i \in \mathbb{R}, v_{i,j} \in V_j \right\}$ . Denote

$$W = \text{Span}\{(\cdots, av_i + bv'_i, \cdots) - a(\cdots, v_i, \cdots) - b(\cdots, v'_i, \cdots) | a, b \in \mathbb{R}, v_i, v'_i \in V_i\} \quad (5.4)$$

Define  $V_1 \otimes \cdots \otimes V_n = \mathbb{R} \langle V_1 \times \cdots \times V_n \rangle / W$ , write  $[(v_1, \dots, v_n)]$  as  $v_1 \otimes \cdots \otimes v_n$ , called a  **$n$ -tensor**.

**Proposition 5.2** (Universal Property). *We have a multi  $\mathbb{R}$ -linear map  $q : V_1 \times \cdots \times V_n \rightarrow V_1 \otimes \cdots \otimes V_n$ ,  $(v_1, v_2, \dots, v_n) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$ . It satisfies the universal property:*

$\forall$  multi  $\mathbb{R}$ -linear map  $f : V_1 \times \cdots \times V_n \rightarrow W$ ,  $\exists$  unique linear map  $\tilde{f} : V_1 \otimes \cdots \otimes V_n \rightarrow W$  s.t.  $\tilde{f} \circ q = f$ . i.e. The diagram commutes:

$$\begin{array}{ccc} V_1 \otimes \cdots \otimes V_n & & \\ \uparrow \rho & \searrow \exists! \tilde{f} & \\ V_1 \times \cdots \times V_n & \xrightarrow{f} & W \end{array}$$

**Corollary 5.3.**

$$\{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V_n \rightarrow W\} \cong \{\text{linear map } V_1 \otimes \cdots \otimes V_n \rightarrow W\} \quad (5.5)$$

$$f \leftrightarrow \tilde{f}$$

**Proposition 5.4.**

- Any element in  $V_1 \otimes \cdots \otimes V_n$  can be written as  $\sum a_i v_i^1 \otimes \cdots \otimes v_i^n$  for some  $a_i \in \mathbb{R}$ .
- If  $(e_i^j)_{j \in \mathcal{A}_i}$  is a basis for  $V_i$ , then  $\{e_1^{j_1} \otimes e_2^{j_2} \otimes \cdots \otimes e_n^{j_n} | j_i \in \mathcal{A}_i\}$  is a basis of  $V_1 \otimes \cdots \otimes V_n$ .
- $\dim(V_1 \otimes \cdots \otimes V_n) = \prod_{i=1}^n \dim(V_i)$

**Proposition 5.5.** Denote  $W^* = \text{Hom}(W, \mathbb{R})$ , then we have an injection

$$\begin{aligned} V \otimes W^* &\xrightarrow{e} \text{Hom}(W, V) \\ v \otimes f &\mapsto (w \mapsto f(w) \cdot v) \end{aligned} \quad (5.6)$$

If  $\dim V$  or  $\dim W$  is finite, then  $e$  is an isomorphism.

Indeed, if  $\dim V = \infty$ , then  $\text{id}_V \notin e(V \otimes V^*)$

Given any  $l_i \in \text{Hom}(V_i, W_i)$ ,  $1 \leq i \leq n$ , we define

$$\begin{aligned} l_1 \otimes \cdots \otimes l_n &\in \text{Hom}(V_1 \otimes \cdots \otimes V_n, W_1 \otimes \cdots \otimes W_n) \\ (l_1 \otimes \cdots \otimes l_n)(v_1 \otimes \cdots \otimes v_n) &= l_1(v_1) \otimes \cdots \otimes l_n(v_n) \end{aligned} \quad (5.7)$$

**Proposition 5.6.** If  $\dim V_i < \infty$ ,  $\forall 1 \leq i \leq n$ , then we have isomorphism

$$\begin{aligned} V_1^* \otimes \cdots \otimes V_n^* &\xrightarrow{\cong} (V_1 \otimes \cdots \otimes V_n)^* \\ f_1 \otimes \cdots \otimes f_n &\mapsto \left( (v_1 \otimes \cdots \otimes v_n \mapsto \prod_{i=1}^n f_i(v_i)) \right) \end{aligned} \quad (5.8)$$

For  $\bigotimes_n V = \underbrace{V \otimes \cdots \otimes V}_n$ ,  $S_n = \{\text{bijection on } \{1, 2, \dots, n\}\}$  acts on  $\bigotimes_n V$ , where

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \quad (5.9)$$

A tensor  $T \in \bigotimes_n V$  is called **symmetric** if  $\sigma(T) = T$ ,  $\forall \sigma \in S_n$ .

$T$  is called **anti-symmetric** if  $\sigma(T) = \text{sgn}(\sigma) \cdot T$ ,  $\forall \sigma \in S_n$ .

Define

$$\begin{aligned} \text{Sym}^n(V) &= \{\text{symmetric tensors in } \bigotimes_n V\} \\ \bigwedge^n(V) &= \{\text{anti-symmetric tensors in } \bigotimes_n V\} \end{aligned} \quad (5.10)$$



which are both in  $\bigotimes_n V$ . And

$$\dim(\text{Sym}^n(V)) = \binom{\dim(V) + n - 1}{n} \quad \dim(\bigwedge^n V) = \binom{\dim(V)}{n} \quad (5.11)$$

From now on, **we may assume**  $\dim V < \infty$ . Define

$$L^n(V) = \left( \bigotimes_n V \right)^* \cong \bigotimes_n V^* \cong \{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V \rightarrow \mathbb{R}\} \quad (5.12)$$

And by the assumption we can obtain

$$\begin{aligned} \text{Sym}^n(V^*) &\cong \{\text{symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \rightarrow \mathbb{R}\} \\ \bigwedge^n(V^*) &\cong \{\text{anti-symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \rightarrow \mathbb{R}\} \end{aligned} \quad (5.13)$$

We will mainly focus on  $\bigwedge^n(V^*)$ , also denoted as  $\text{Alt}^k(V) = \bigwedge^n(V^*)$ . An element in  $\text{Alt}^k(V)$  is called a (linear)  **$k$ -form** on  $V$ . Now for  $V = \mathbb{R}\langle e_1, \dots, e_n \rangle$ ,  $V^* = \mathbb{R}\langle e_1^*, \dots, e_n^* \rangle$ . Then

$$\begin{aligned} L^2(V) &= \{\text{all bilinear forms on } V\} \\ L^2(V) &\cong \text{Sym}^2(V^*) \oplus \bigwedge^2(V^*) \end{aligned}$$

And  $\text{Sym}^2(V^*) = \mathbb{R}\langle e_i^* \otimes e_j^* + e_j^* \otimes e_i^* | 1 \leq i \leq i \leq n \rangle$  is symmetric bilinear form

$\text{Alt}^2(V) = \bigwedge^2(V^*) = \mathbb{R}\langle e_i^* \otimes e_j^* - e_j^* \otimes e_i^* | 1 \leq i \leq i \leq n \rangle$  is anti-symmetric bilinear form.

The determinant  $\det \in \text{Alt}^n(\mathbb{R}^n)$ .

**Definition 5.7** (Exterior product).

$$\bigwedge : \text{Alt}^k(V) \times \text{Alt}^l(V) \rightarrow \text{Alt}^{k+l}(V)$$

$$\begin{aligned}
\omega_1 \wedge \omega_2(v_1, \dots, v_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \omega_2(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\
&= \sum_{\sigma \in S_{k,l}} \text{sgn}(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \omega_2(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})
\end{aligned}$$

where  $S_{k,l} = \{\sigma \in S_{k+l} \mid \sigma(1) < \dots < \sigma(k), \sigma(k+1) < \dots < \sigma(k+l)\} \subset S_{k+l}$ .

Then we have those properties:

**Proposition 5.8.**

- (1)  $\omega_1 \wedge \omega_2 = (-1)^{|\omega_1| \cdot |\omega_2|} \omega_2 \wedge \omega_1$ ,  $|\omega| = k$  is  $\omega \in \text{Alt}^k(V)$ . In particular,  $\omega \wedge \omega = 0$  if  $|\omega|$  is odd.
- (2)  $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$
- (3) Given any  $\omega_1, \dots, \omega_k \in \text{Alt}^1(V) = V^*$ ,  $v_1, \dots, v_k \in V$ . Then

$$(\omega_1 \wedge \dots \wedge \omega_k)(v_1, \dots, v_k) = \det [w_i(v_j)]_{i,j} \quad (5.14)$$

Moreover,  $\omega_1 \wedge \dots \wedge \omega_n \neq 0$  iff  $\omega_i$  are linearly independent.

- (4)  $V = \mathbb{R} \langle e_1, \dots, e_n \rangle$ . Then

$$\text{Alt}^k(V) = \mathbb{R} \langle e_{i_1}^* \wedge \dots \wedge e_{i_k}^* \mid i_1 < \dots < i_k \rangle \quad (5.15)$$

In particular,  $\text{Alt}^n(V) = \mathbb{R} \langle e_1^* \wedge \dots \wedge e_n^* \rangle$ . And we denote  $\text{Alt}^0(V) = \mathbb{R}$ ,  $\text{Alt}^k(V) = 0$ ,  $k > n$ .

- (5) Any  $f \in \text{Hom}(V, W)$  induces  $\text{Alt}^k(f) \in \text{Hom}(\text{Alt}^k(V), \text{Alt}^k(W))$ , where

$$\text{Alt}^k(f)(\omega)(w_1, \dots, w_k) = \omega(f(w_1), \dots, f(w_k)) \quad (5.16)$$

We have  $\text{Alt}^k(f \circ g) = \text{Alt}^k(g) \circ \text{Alt}^k(f)$ ,  $\text{Alt}^k(\text{id}_V) = \text{id}_{\text{Alt}^k(V)}$ . Such  $\text{Alt}^k(-)$  is called a **contravariant functor**.

*Proof.*

(1) By definition,

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{k+l}) = \omega_2 \wedge \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k+l)})$$

$$\text{where } \sigma(i) = \begin{cases} i + k & 1 \leq i \leq l \\ i - l & l + 1 \leq i \leq k + l \end{cases} \cdot \text{sgn}(\sigma) = (-1)^{k+l}.$$

(2) By definition.

(3) By linearity, we assume  $\omega_i = e_{a(i)}^*$ ,  $v_j = e_{b(j)}$  for some  $a(i), b(j)$ . Further more, can assume  $\{a(i)\} = \{b(i)\}$ . (Otherwise,  $LHS = RHS = 0$ .)

Then  $e_{a(i)}^*(e_{b(j)}) = \delta_{a(i), b(j)}$ . After permutation, may assume  $a(i) = b(i), \forall i$ . It is direct to check  $LHS = 1 = RHS$ .

(4) If  $\omega_1, \dots, \omega_k$  are linear independent. Then  $\exists$  basis  $e_1^*, \dots, e_n^*$  of  $V^*$ , basis  $e_1, \dots, e_n$  of  $V$  s.t.  $\omega_i = e_i^*, \forall 1 \leq i \leq n$ .

$$(\omega_1 \wedge \dots \wedge \omega_n)(e_1, \dots, e_n) = \det(I) = 1 \neq 0 \Rightarrow \omega_1 \wedge \dots \wedge \omega_n \neq 0$$

If  $\omega_1, \dots, \omega_k$  are linearly dependent. WLOG, we assume  $\omega_k = \sum_{i=1}^{k-1} a_i \omega_i$ .

$$(\omega_1 \wedge \dots \wedge \omega_k)(e_1, \dots, e_n) = \sum_{i=1}^{k-1} a_i (\omega_1 \wedge \dots \wedge \omega_{k-1} \wedge \omega_i)(e_1, \dots, e_n) = 0$$

(5) For  $i_1 < \cdots < i_k, j_1 < \cdots < j_n$  we have

$$(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)(e_{j_1}, \cdots, e_{j_k}) = \begin{cases} 1 & j_t = i_t, \forall 1 \leq t \leq k \\ 0 & \text{otherwise} \end{cases} \quad (5.17)$$

Since  $\dim \text{Alt}(V) = \dim \bigwedge^k(V^*) = \binom{n}{k} = |\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1 < \cdots < i_k\}|$ .

(6) For  $\omega \in \text{Alt}^k(W), f \in \text{Hom}(V, W)$ , define  $\text{Alt}^k(f)(\omega) \in \text{Alt}^k(V)$  by

$$\text{Alt}^k(f)(\omega(V_1, \cdots, V_k)) = \omega(fV_1, \cdots, fV_k) \in \mathbb{R}$$

□

### Definition 5.9.

An  **$\mathbb{R}$ -algebra** consists of an  $\mathbb{R}$ -vector space  $A$  with a bilinear map  $\mu : A \times A \rightarrow A$  that is associate, i.e.  $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ .

Say  $A$  is **unitary** if  $\exists 1 \in A$  s.t.  $\mu(a, 1) = \mu(1, a) = a, \forall a \in A$

Say  $A$  is **graded** if  $A = \bigoplus_{k \in \mathbb{Z}} A_k$  as vector space, and  $\mu(A_k \times A_l) \subset A_{k+l}$ . Elements in  $A_k$  are called **homogeneous elements** of degree  $k$ .

If  $A$  is graded  $\mathbb{R}$ -algebra, we say  $A$  is **anticommutative** if  $\mu(a, b) = (-1)^{k+l} \mu(b, a), \forall a \in A_k, b \in A_l$ . And say  $A$  is **commutative** if  $\mu(a, b) = \mu(b, a), \forall a, b$ .

If  $A$  is graded  $\mathbb{R}$ -algebra, say  $A$  is **connected** if  $\exists$  unit  $1 \in A_0$  s.t. the map  $\varepsilon : \mathbb{R} \rightarrow A_0, r \mapsto r \cdot 1$  is an isomorphism.

Given vector space  $V$ , let

$$\begin{array}{ccc} \text{Alt}^k(V) & \xlongequal{\quad} & \bigoplus_{k \geq 0} \text{Alt}^k(V) \\ \parallel & & \parallel \\ \text{Alt}^*(V^*) & \xlongequal{\quad} & \bigoplus_{k \geq 0} \wedge^k(V^*) \end{array}$$

By Proposition 5.8, we have the theorem

**Theorem 5.10.**  $(\text{Alt}^*(V), \wedge)$  is a graded connected anticommutative  $\mathbb{R}$ -algebra, called the exterior algebra of  $V$  or exterior algebra of  $V$

### 5.3 Operation on vector bundles

Given  $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$ , meaning a vector bundle  $E \xrightarrow{\pi} M$  of dimension  $n$ , local trivialization  $\left\{ U_\alpha, \varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n \right\}_{\alpha \in \mathcal{A}}$ . By shrinking  $U_\alpha$ , we may assume we have an smooth atlas  $\left\{ \varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^m \right\}_{\alpha \in \mathcal{A}}$ .

For  $x \in M$ , use  $E_x$  to denote  $\pi^{-1}(x)$ , fiber over  $x$ , which is a vector space of dimension  $n$ .

Then **Dual bundle of a vector bundle**  $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$  is

$$E^* := \{(x, l) | x \in M, l \in (E_x)^*\}, \pi' : E^* \rightarrow M, (x, l) \mapsto x, (\pi')^{-1}(x) = (E_x)^* \quad (5.18)$$

Define topology or smooth structure on  $E^*$  s.t.  $\pi' : E^* \rightarrow M$  is a smooth vector bundle.

For  $\alpha \in \mathcal{A}$ , let  $E_\alpha^* = \pi'^{-1}(U_\alpha)$ , we have a bijection

$$\tilde{\varphi}_\alpha : E_\alpha^* \xrightarrow{\text{bijection}} \mathbb{R}^m \times (\mathbb{R}^n)^* \xrightarrow{\cong} \mathbb{R}^{m+n}$$

$$(x, l) \longmapsto (\psi_\alpha(x), (\varphi_{\alpha,x})^{-1}(l))$$

We can check that

(1)  $\{\tilde{\varphi}_\alpha^{-1} | \alpha \in \mathcal{A}, V \subset \mathbb{R}^{m+n} \text{ open}\}$  is a basis, we use it to generate a topology on  $E^*$ .

(2) Use  $\tilde{\varphi}_\alpha : E_\alpha^* \xrightarrow{\cong} \mathbb{R}^{m+n}, \alpha \in \mathcal{A}$  as an atlas to give  $E^*$  a smooth structure.

(3)  $E^* \xrightarrow{\pi'} M$  is a smooth vector bundle, called the **dual vector bundle** of  $E \xrightarrow{\pi} M$ , where

$$(E^*)_x = E_x^*$$

We can define other operations on vector bundles in similar way:

Given  $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$ ,  $\mathbb{R}^m \hookrightarrow F \xrightarrow{\pi} M$ , we can define

$$\mathbb{R}^{m+n} \hookrightarrow E \oplus F \xrightarrow{\pi} M \text{ with } (E \oplus F)_x = E_x \oplus F_x$$

$$\mathbb{R}^{mn} \hookrightarrow E \otimes F \xrightarrow{\pi} M \text{ with } (E \otimes F)_x = E_x \otimes F_x$$

$$\mathbb{R}^{mn} \hookrightarrow \text{Hom}(E, F) \xrightarrow{\pi} M \text{ with } \text{Hom}(E, F)_x = \text{Hom}(E_x, F_x)$$

$$\mathbb{R}^{\binom{n}{k}} \hookrightarrow \text{Alt}^k(E) \rightarrow M \text{ with}$$

$$\text{Alt}^k(E)_x = \text{Alt}^k(E_x) = \{\text{alternating } k\text{-linear } l : E_x \times \cdots \times E_x \rightarrow \mathbb{R}\}$$

$$\text{Then } \text{Alt}^k(TM) = \bigwedge^k(T^*M).$$

$$\text{Alt}^k(M)_x = \{\text{alternating } k\text{-linear } l : T_x M \times \cdots \times T_x M \rightarrow \mathbb{R}\} = \{\text{linear } k\text{-form on } T_x M\}$$

Define

$$\Gamma(E) := \{\text{smooth sections of } E\} = \{s \in C^\infty(M, E) : \pi \circ s = \text{id}_M\}$$

**Definition 5.11.** Given smooth  $M$ , define a differential  $k$ -form on  $M$  to be an element in  $\Gamma(\text{Alt}^k(TM))$  is a differential  $k$ -form  $\alpha$  assigns each  $x \in M$  a linear  $k$ -form  $\alpha(x) \in \text{Alt}^k(T_x M)$ .

Denote  $\Omega^k(M)$  be the set of all the differential  $k$ -forms.

Then  $\Omega^0(M) = C^\infty(M, \mathbb{R})$ .  $\text{Alt}^1(TM) = T^*M \Rightarrow$  a 1-form on  $M$  is just a "cotangent vector field" on  $M$ .

$$\Omega^k(M) = 0 \text{ if } k \geq \dim(M).$$

## 5.4 Differential forms using local chart

Given local chart  $(U, x^1, \dots, x^n)$  of  $M$ .

For any  $p \in U$ ,  $\{\frac{\partial}{\partial x^i}|_p\}_{1 \leq i \leq n}$  is a basis of  $T_x M$ .

We denote the dual basis of  $T_x^* M$  by  $\{dx^i|_p\}_{1 \leq i \leq n}$ .

For any  $\alpha \in \Omega^1(M)$ ,  $\alpha|_U$  can be written as  $\sum_{i=1}^n f_i dx^i$ , where  $f^i \in C^\infty(U, \mathbb{R})$ .

Similarly,  $\{dx^{i_1}|_p \wedge \dots \wedge dx^{i_k}|_p | i_1 < \dots < i_k\}$  is a basis for  $\bigwedge^k(T_x^* M)$ , so  $\forall \alpha \in \Omega^k(M)$ ,

$$\alpha|_U = \sum_{i_1 < \dots < i_k} f_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad f_{i_1, \dots, i_k} \in C^\infty(U, \mathbb{R})$$

We give the notation that  $I = (i_1, \dots, i_k)$ , write  $f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$  as  $f^I dx^I$ .

**Change of coordinate** If  $(U, x^1, \dots, x^n)$  and  $(V, y^1, \dots, y^n)$  two charts of  $M$  and  $p \in U \cap V$ , then

$$dy^i = \sum_{1 \leq j \leq n} \frac{\partial y_i}{\partial x^j} dx^j. \quad (5.19)$$

## 5.5 Exterior differential

For  $k = 0$ , define  $d : \Omega^0(M) \rightarrow \Omega^1(M)$  as follows:

$$\forall p \in M, X_p \in T_p M, df|_p(X_p) = X_p(f) \in \mathbb{R}. \text{ In local chart, } df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

**Theorem 5.12.**  $\exists$  linear operator  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  s.t. For  $\alpha \in \Omega^k(M)$ ,

$$\alpha|_U = \sum_I f^I dx^I \Rightarrow d\alpha|_U = \sum_I df^I \wedge dx^I \quad (5.20)$$

Called the *exterior differential*

*Proof.* It suffices to prove that (5.20) is compatible for two charts  $(U, x^1, \dots, x^n), (V, y^1, \dots, y^n)$ , i.e. the diagram is commutative.

$$\begin{array}{ccc}
 f dy^1 \wedge \dots \wedge dy^k & \xleftarrow{\quad} & \sum_{1 \leq i_1, i_2, \dots, i_k \leq n} f \frac{\partial y^{i_1}}{\partial x^1} \dots \frac{\partial y^{i_k}}{\partial x^k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \\
 \downarrow d & & \downarrow ?d \\
 \sum_{1 \leq i \leq n} \frac{\partial f}{\partial y^i} dy^i \wedge dy^{i_1} \wedge \dots \wedge dy^{i_k} & \xleftarrow{?} & \sum_{1 \leq i_1, i_2, \dots, i_k \leq n} \frac{\partial f}{\partial x^j} \cdot \frac{\partial y^{i_1}}{\partial x^1} \dots \frac{\partial y^{i_k}}{\partial x^k} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}
 \end{array}$$

□

**Theorem 5.13.**

(1)  $d^2 = 0$ .

(2)  $\forall \alpha \in \Omega^k(M), \beta \in \Omega^l(M), d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ .

*Proof.*

(1) If  $\alpha|_U = \sum_U f^I dx^I$ . By linearity suffices to check

$$\begin{aligned}
 d \circ d(f dx^I) &= d \left( \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) \\
 &= \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i \wedge dx^I \\
 &= 0
 \end{aligned}$$

(2) By linearity, suffices to assume  $\alpha = f dx^I, \beta = g dx^J$ .

$$\begin{aligned}
 d(\alpha \wedge \beta) &= d(f g dx^I \wedge dx^J) \\
 &= \sum_{1 \leq i \leq n} \frac{\partial(fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\
 &= \sum_{1 \leq i \leq n} \left( \frac{f}{\partial x^i} g + f \frac{\partial g}{\partial x^i} \right) dx^i \wedge dx^I \wedge dx^J
 \end{aligned}$$



And

$$d\alpha \wedge \beta = \sum_i \frac{\partial f}{\partial x^i} g dx^i \wedge dx^I \wedge dx^J$$

$$\alpha \wedge d\beta = \sum_i \frac{\partial g}{\partial x^i} f dx^I \wedge dx^i \wedge dx^J = \sum_i (-1)^k \frac{\partial g}{\partial x^i} f dx^i \wedge dx^I \wedge dx^J$$

□

**Example 5.14.** For  $M = \mathbb{R}^3$ ,

$$\begin{array}{ccc} \Omega^0(\mathbb{R}^3) & \xlongequal{\quad} & C^\infty(\mathbb{R}^3) \\ \downarrow d & & \downarrow \text{gradient} \\ \Omega^1(\mathbb{R}^3) & \longleftrightarrow & \mathfrak{T}\mathbb{R}^3, & f dx + g dy + h dz & \longleftrightarrow & f \partial x + g \partial y + h \partial z \\ \downarrow d & & \downarrow \text{curl} \\ \Omega^2(\mathbb{R}^3) & \longleftrightarrow & \mathfrak{T}\mathbb{R}^3, & f dx \wedge dy + g dx \wedge dz + h dy \wedge dz & \longleftrightarrow & f \partial z + g \partial x + h \partial y \\ \downarrow d & & \downarrow \text{divergent} \\ \Omega^3(\mathbb{R}^3) & \longleftrightarrow & C^\infty(\mathbb{R}^3), & f dx \wedge dy \wedge dz & \longleftrightarrow & f \end{array}$$

## 5.6 Pull back of differential forms

For  $F \in C^\infty(M, N)$ ,  $\alpha \in \Omega^k(N)$ , define the **pullback**  $F^*(\alpha) \in \Omega^k(M)$  as follows:

$$\forall p \in M, V_1 \cdots, V_k \in T_p M, F^*(\alpha)|_p(V_1, \cdots, V_k) = \alpha|_{F(p)}(F_{p,*}(V_1), \cdots, F_{p,*}(V_k)) \in \mathbb{R}$$

Actually,  $F^*(\alpha) = \text{Alt}^k(F_{p,*}) : \text{Alt}^k(T_{F(p)}N) \rightarrow \text{Alt}^k(T_p M)$ .

**Proposition 5.15.** For  $F : M \rightarrow N$ ,  $G : N \rightarrow L$ .

$$(1) \quad f \in \Omega^0(N), F^*(f) = f \circ F \in \Omega^0(M).$$

$$(2) \quad F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta).$$

$$(3) \quad F^*(d\alpha) = dF^*(\alpha).$$

$$(4) \quad (G \circ F)^* = F^* \circ G^*$$

*Proof.*

$$(1) \quad \begin{array}{ccc} \text{Alt}^0(T_{F(p)}N) & \xrightarrow{\text{Alt}^0(F_p,*)} & \text{Alt}^0(T_pM) \\ \parallel & & \parallel \\ \mathbb{R} & \xrightarrow{id} & \mathbb{R} \end{array} \quad \text{commutes.}$$

$$(2) \quad \begin{array}{ccc} \text{Alt}^k(T_{F(p)}N) \times \text{Alt}^l(T_{F(p)}N) & \xrightarrow{\wedge} & \text{Alt}^{k+l}(T_pM) \\ \downarrow \text{Alt}^k(F_p,*) \times \text{Alt}^l(F_p,*) & & \downarrow \text{Alt}^{k+l}(F_p,*) \\ \text{Alt}^k(T_pM) \times \text{Alt}^l(T_pM) & \xrightarrow{\wedge} & \text{Alt}^{k+l}(T_pM) \end{array} \quad \text{commutes.}$$

(3) By linearity it suffices to check

$$dF^*(f dx^I) = F^*d(f dx^I)$$

By Leibniz rule for  $d$  and (2), it suffices to show

$$(a) \quad dF^*(df) = F^*(df)$$

$$(b) \quad dF^*(dx^i) = F^*(d(dx^i))$$

Which leaves to the readers.

(4) By definition.

□

**Definition 5.16.** A  $k$ -form  $\omega$  is **closed** if  $\omega \in \ker \left( \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \right)$ .

A  $k$ -form  $\omega$  is **exact** if there exists a  $(k-1)$ -form  $\eta$  such that  $d\eta = \omega$ , or equivalently,  $\omega \in \text{Im} \left( \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \right)$ .

By Proposition 5.15 (1), exact  $k$ -form are all closed.

So we may define the  $k$ -th **de Rham cohomology** of  $M$

$$H_{\text{DR}}^k(M) := \frac{\ker \left( d : \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \right)}{\text{Im} \left( \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \right)} \quad (5.21)$$

By Proposition 5.15 (2), we have  $\forall F \in C^\infty(M, N), \omega \in \Omega^k(N)$ .

Then  $\omega$  closed  $\Rightarrow F^*\omega$  is closed.  $\omega$  exact  $\Rightarrow F^*\omega$  exact.

So  $F$  induces a linear map

$$F^* : H_{\text{DR}}^k(N) \rightarrow H_{\text{DR}}^k(M)$$

$$[\omega] \mapsto [F^*\omega]$$

**Proposition 5.17** (Key properties of  $H_{\text{DR}}^k(M)$ ).

$$(1) (F \circ G)^* = G^* \circ F^*$$

$$(2) (\text{id})^* = \text{id}.$$

$$(3) F, G \in C^\infty(M, N), F \text{ homotopic to } G \Rightarrow F^* = G^*$$

$$(4) \text{ If } F \text{ is a homotopy equivalence} \Rightarrow F^* : H_{\text{DR}}^k(M) \rightarrow H_{\text{DR}}^k(N) \text{ is an isomorphism.}$$

**Remark 5.18.** Properties (3),(4) are nontrivial, which is the essential part of the theory of de Rham cohomology

**Proposition 5.19.**  $H_{\text{DR}}^k(M) \cong \mathbb{R} \langle \pi_0(M) \rangle$ , where  $\pi_0(M) = \{\text{path component of } M\}$ .

It suffices to prove the lemma that

**Lemma 5.20.**  $\alpha \in \Omega^0(M) = C^\infty(M, \mathbb{R})$ . Then  $\alpha$  is closed iff  $\alpha$  is constant on each component of  $M$ .

*Proof.* The inverse part is trivial.

Assume  $\alpha$  is closed. Pick  $p, q \in M$  in some path component.  $\exists$  smooth path

$$\gamma : \mathbb{R} \rightarrow M, \gamma(0) = p, \gamma(1) = q.$$

$$d\alpha = 0 \Rightarrow d(\gamma^*\alpha) = 0 \Rightarrow d(\alpha \circ \gamma) = 0 \Rightarrow \frac{d(\alpha \circ \gamma)}{dt} = 0 \Rightarrow \alpha \circ \gamma(1) = \alpha \circ \gamma(0).$$

$$\text{So } \alpha(p) = \alpha(q)$$

□

We have  $H_{\text{DR}}^k(M) \cong \text{Ab}(\pi_1(M)) \otimes_{\mathbb{Z}} \mathbb{R}$ .  $\text{Ab}(\pi_1(M))$  is the Abelian group of  $\pi_1(M)$ . In particular,  $H_{\text{DR}}^1(\mathbb{R}^2) = 0$ ,  $H_{\text{DR}}^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$ .

Let us stop the discussion of de Rham cohomology for a moment, and move on to the next topic.

## 6 Integration of differential form

### 6.1 Orientation on manifold

An orientation on a finite dimensional vector space  $V$  is an equivalent class of ordered basis

$$\alpha = (\alpha_1, \dots, \alpha_n)^T \sim \beta = (\beta_1, \dots, \beta_n)^T \Leftrightarrow \det(\alpha\beta^T) > 0$$

Each vector space has exactly two orientations. And we actually have the 1-1 correspondence

$$\{\text{orientation on } V\} \leftrightarrow (\text{Alt}^n(V) \setminus \{0\}) / \mathbb{R}^+$$

$$[(e_1, \dots, e_n)] \leftrightarrow [e_1^* \wedge \dots \wedge e_n^*]$$

An **orientation form** on  $M$  of dimension  $n$  is a nowhere vanishing  $\omega \in \Omega^n(M)$  *i.e.* an orientation form is a nowhere vanishing section of  $\text{Alt}^n(TM)$ .

Two orientation forms  $\omega_1, \omega_2$  are equivalent if  $\exists f \in C^\infty(M, \mathbb{R}^+) \text{ s.t. } \omega_1 = f\omega_2$ .

An **orientation** on  $M$  is an equivalent class of orientation form.

An **orientation manifold** is a manifold that has an orientation.

An **oriented manifold** is a manifold equipped with an orientation.

**Example 6.1.**  $|\pi_0(M)| = k \Rightarrow M$  has  $2^k$  orientations or no orientations.

**Example 6.2.**

1.  $U \subset \mathbb{R}^n$  open.  $U$  has a standard orientation, represented by the form  $dx^1 \wedge \cdots \wedge dx^n$ . Denote this standard orientation as  $\mathcal{O}_{\text{std}}$
2.  $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$  oriented manifolds  $\Rightarrow (M \times N, \mathcal{O}_M \times \mathcal{O}_N)$ . If  $\mathcal{O}_M = [\omega_M], \mathcal{O}_N = [\omega_N]$ , then  $\mathcal{O}_M \times \mathcal{O}_N$  is defined by  $[\pi_M^*(\omega_M) \wedge \pi_N^*(\omega_N)]$ . ( $\pi_M, \pi_N$  is the pullback of the projection map)
3.  $T^n, S^n$  are orientable.
4.  $\mathbb{RP}^n$  orientable iff  $n$  is odd.

**Proposition 6.3.** Let  $\mathcal{U} = \{U_\alpha\}$  be an open cover of  $M$ . Suppose we have an orientation  $\mathcal{O}_\alpha$  on each  $U_\alpha$  s.t.  $\mathcal{O}_\alpha|_{U_\alpha \cap U_\beta} = \mathcal{O}_\beta|_{U_\alpha \cap U_\beta}, \forall \alpha, \beta$ . Then  $\exists$  unique orientation  $\mathcal{O}_M$  on  $M$  s.t.  $\mathcal{O}_M|_{U_\alpha} = \mathcal{O}_\alpha$ .

*Proof.* For each  $\alpha$ , we have  $\omega_\alpha \in \Omega^n(U_\alpha)$  nowhere-vanishing. And

$$\omega_\alpha|_{U_\alpha \cap U_\beta} = f_{\alpha\beta} \cdot \omega_\beta|_{U_\alpha \cap U_\beta}, f_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{R}^+ \quad (6.1)$$

Take partition of unity subordinate to  $\mathcal{U}, \{\varphi_\alpha\}$ .

Set  $\omega = \sum_{\alpha} \varphi_\alpha \cdot \omega_\alpha$ . Then  $\omega$  is nowhere-vanishing by (6.1).

The uniqueness follows from the fact that  $n$ -form is equivalent if and only if it is equivalent on each chart. □

**Definition 6.4.** Given  $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ .  $f \in \text{Diff}(M, N)$ . Say  $f$  is **orientation preserving** if  $f^*(\mathcal{O}_N) = \mathcal{O}_M$ .  $f$  is **orientation reversing** if  $f^*(\mathcal{O}_N) = -\mathcal{O}_M$ .

**Lemma 6.5.**  $U_1, U_2 \subset \mathbb{R}^n$  open. Then  $f : (U_1, \mathcal{O}_{\text{std}}) \rightarrow (U_2, \mathcal{O}_{\text{std}})$  is orientation preserving iff

$$\forall p \in U_1, \det(Df|_p) > 0 \quad Df = \left( \frac{\partial f^i}{\partial x^j} \right)_{1 \leq i, j \leq n}$$

*Proof.* For  $\mathcal{O}_{\text{std}} = [dx^1 \wedge \cdots \wedge dx^n]$ ,

$$\begin{aligned} f^*(dx^1 \wedge \cdots \wedge dx^n) &= df^1 \wedge \cdots \wedge df^n, \quad df^i = \sum_{j=1}^n \frac{\partial f^i}{\partial x^j} dx^j \\ &= \det(Df) dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

Then

$$\begin{aligned} \det(Df) > 0 &\Leftrightarrow f^*(dx^1 \wedge \cdots \wedge dx^n) \sim dx^1 \wedge \cdots \wedge dx^n \\ &\Leftrightarrow f^*\mathcal{O}_{\text{std}} = \mathcal{O}_{\text{std}} \\ &\Leftrightarrow f \text{ is orientation preserving} \end{aligned}$$

□

Given  $(M, \mathcal{O})$ ,  $p \in M$ , a basis  $e_1, \dots, e_n$  of  $T_p M$  is called **oriented** if  $\mathcal{O}_p = [(e_1, \dots, e_n)]$ .

A chart  $U \xrightarrow{\varphi} V \subset \mathbb{R}^n$  is **oriented** if  $\varphi^*(\mathcal{O}_{\text{std}}) = \mathcal{O}|_U$ .

A smooth atlas  $\left\{ U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \right\}_{\alpha \in \mathcal{A}}$  is **oriented** if each chart  $U_\alpha \xrightarrow{\varphi_\alpha}$  is oriented.

A smooth atlas  $\left\{ U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \right\}_{\alpha \in \mathcal{A}}$  is called **positive** if  $\forall \alpha, \beta \in \mathcal{A}$ ,

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi'_\alpha : \varphi(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta) \text{ is orientation preserving}$$

By Lemma 6.5, this is equivalent to  $\det(D\varphi_{\alpha\beta}|_p) > 0$  for any  $p \in \varphi_\alpha(U_\alpha \cap U_\beta)$ .

## 6.2 Integration on Oriented Manifold

**Goal:** Given  $M, \mathcal{O}$ ,  $\omega \in \Omega_c^n(M) = \{\text{compactly supported } n\text{-form on } M\}$ . Then  $\text{Supp}(\omega) = \overline{\{p \in M | \omega_p \neq 0 \in \text{Alt}^n(T_p M)\}}$  is compact.

We hope to define  $\int_M \omega \in \mathbb{R}$ .

For  $M \overset{\text{open}}{\subset} \mathbb{R}^n$ ,  $\mathcal{O} = \mathcal{O}_{\text{std}}$ . Then  $\forall \omega \in \Omega_c^n(M)$ ,

$$\omega = f dx^1 \wedge \cdots \wedge dx^n, f \in C_c^\infty(M)$$

Define  $\int_M \omega = \int_M f d\mu$  where  $\mu$  is the standart Lebesgue measure on  $\mathbb{R}^n$ .

**Lemma 6.6.**  $U, V \overset{\text{open}}{\subset} \mathbb{R}^n$ ,  $\varphi : U \xrightarrow{\cong} V$  is orientation preserving. Then  $\forall \omega \in \Omega_c^n(V)$ , we have  $\int_U \varphi^*(\omega) = \int_V \omega$ .

*Proof.* If  $\omega = f dx^1 \wedge \cdots \wedge dx^n$ , then

$$\begin{aligned} \varphi^*(\omega) &= \varphi^*(f) \wedge d\varphi^1 \wedge \cdots \wedge d\varphi^n \\ &= (f \circ \varphi) \det \left( \frac{\partial \varphi^i}{\partial x^j} \right)_{1 \leq i, j \leq n} dx^{1^1} \wedge \cdots \wedge dx^{1^n} \end{aligned} \tag{6.2}$$

$$\text{So } \int_U \varphi^*(\omega) = \int_U (f \circ \varphi) \det \left( \frac{\partial \varphi^i}{\partial x^j} \right)_{1 \leq i, j \leq n} d\mu = \int_V f d\mu = \int_V \omega$$

□

So we can define the integral over special  $\omega$  and general  $M$ .

**Definition 6.7.** If  $\omega \in \Omega_s^n(M) = \{n\text{-forms with "small" support}\}$   
 $= \{\omega \in \Omega_c^n(M) | \exists \text{ oriented chart } \varphi : U \xrightarrow{\cong} \text{ s.t. } \text{Supp}(\omega) \subset U\}.$

We define  $\int_M \omega := \int_V \varphi^{-1,*}(\omega)$

*Claim.* If  $\text{Supp}(\omega) \subset U_\alpha \cap U_\beta$ , then  $\int_{V_\alpha} \varphi_\alpha^{-1,*}(\omega) = \int_{V_\beta} \varphi_\beta^{-1,*}(\omega)$

*Proof.*

$$\begin{aligned} \varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) &\xrightarrow{\cong} \varphi_\beta(U_\alpha \cap U_\beta) \\ \varphi_\alpha^{-1,*}(\omega) &\mapsto \varphi_\beta^*(\omega) \end{aligned}$$

By Lemma 6.6, we have  $\int_{V_\alpha} \varphi_\alpha^{-1,*}(\omega) = \int_{V_\beta} \varphi_\beta^{-1,*}(\omega)$

□

**Theorem 6.8.** For any oriented  $(M, \mathcal{O})$ ,  $\exists$  unique linear map  $\int_M : \Omega_c^n(M) \rightarrow \mathbb{R}$  that extends  $\int_M : \Omega_s^n(M) \rightarrow \mathbb{R}$ .

*Proof. Step1:* There exists an oriented atlas  $\mathcal{U} = \{\varphi_\alpha : U_\alpha \rightarrow V_\alpha \overset{\text{open}}{\subset} \mathbb{R}^n\}_{\alpha \in \mathcal{A}}$ . Indeed, pick any smooth atlas  $\mathcal{U} = \{U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha\}_{\alpha \in \mathcal{A}}$ . By replacing  $\varphi_\alpha$  with  $r \circ \varphi_\alpha$  where  $r(x_1, \dots, x_n) = (-x_1, \dots, x_n)$ . We can get the oriented atlas  $\mathcal{U}'$ .

**Step2.** Pick a partition of unity subordinate to  $\mathcal{U}$ ,  $\{\varphi_\alpha : M \rightarrow [0, 1]\}$

Now we begin the main proof:

Let  $\omega_\alpha = \rho_\alpha \cdot \omega$ .  $\text{Supp}(\omega_\alpha) \subset \text{Supp}(\rho_\alpha) \cap \text{Supp}(\omega) \subset U_\alpha$ . And  $\omega_\alpha \in \Omega_s^n(M)$

*Claim.*  $\omega_\alpha \neq 0$  for only finite many  $\alpha \in \mathcal{A}$

*Proof.*  $\forall p \in \text{Supp}(\omega)$ ,  $\exists$  neighbourhood  $W_p$  only intersects  $\text{Supp}(\rho_\alpha)$  for finitely many  $\alpha$ .

Since  $\{W_p\}_{p \in \text{Supp}(\omega)}$  is an open cover of  $\text{Supp}(\omega)$ , by compactness,  $\text{Supp}(\omega)$  only intersects  $\text{Supp}(\rho_\alpha)$  for finitely many  $\alpha$ .

Therefore,  $\omega_\alpha \neq 0$  for only finitely many  $\alpha$  □

By this claim, since  $\sum_{\alpha \in \mathcal{A}} \rho_\alpha = 1 \Rightarrow \omega = \omega_{\alpha_1} + \dots + \omega_{\alpha_n}$  for some  $\alpha_1, \dots, \alpha_n \in \mathcal{A}$ .

We may define

$$\int_M \omega = \sum_{\alpha \in \mathcal{A}} \int_M \omega_\alpha = \int_M \omega_{\alpha_1} + \dots + \int_M \omega_{\alpha_n} \in \mathbb{R} \quad (6.3)$$

This proves existence  $\int_{(M, \mathcal{O}, \mathcal{U}, \{\rho_\alpha\})} : \Omega_c^n(M) \rightarrow \mathbb{R}$ .

**Uniqueness:**  $\forall \omega \in \Omega_c^n(M)$ ,  $\exists \omega_1, \dots, \omega_k \in \Omega_s^n(M)$ ,  $\omega = \omega_1 + \dots + \omega_k$  as the claim proved.

So  $\int_M \omega = \sum_{i=1}^n \int_M \omega_i$  is uniquely defined. □

**Proposition 6.9.**



1.  $f : (M, \mathcal{O}_M) \xrightarrow{\cong} (N, \mathcal{O}_N)$ . If  $f$  is orientation preserving, then  $\int_M f^*(\omega) = \int_N \omega$ . If  $f$  is orientation reversing, then  $\int_M f^*(\omega) = -\int_N \omega$ .
2.  $\int_{(M, \mathcal{I})} \omega = -\int_{(M, \overline{\mathcal{O}})} \omega$ .
3. If  $\text{Supp}(\omega) \subset U \stackrel{\text{open}}{\subset} M$ , then  $\int_M \omega = \int_U \omega$ .

*Proof.* Leave as exercise. □

### 6.3 Smooth Manifold with boundary

Now let  $M$  be the smooth manifold with boundary,  $\partial M = N$ . *i.e.*  $M$  has a smooth atlas  $\{\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha \stackrel{\text{open}}{\subset} \mathbb{R}_- \times \mathbb{R}^{n-1}\}$ .

We can define  $T_p M$ ,  $TM$ ,  $\text{Alt}^k(M)$ ,  $\Omega^k(M)$ ,  $\Omega_c^k(M)$ ,  $\Omega_s^k(M)$  and orientation similar as before.

For  $p \in \partial M$ ,  $X \in T_p M$  is called **outward** if  $\exists$  local chart  $\varphi : U \xrightarrow{\cong} V$  around  $p$  s.t.

$$\varphi_{p,*}(X) = a_1 \partial x^1 + \cdots + a_n \partial x^n \text{ with } a_1 > 0$$

Recall that if  $M$  is  $n$  dimensional manifold with boundary, then  $N = \partial M$  is a  $n - 1$  dimensional manifold without boundary.

**Proposition 6.10.** For any orientation  $\mathcal{O}_M$  on  $M$ ,  $\exists$  a unique induced orientation  $\mathcal{O}_N$  on  $N = \partial M$  s.t.  $\forall p \in N$ ,  $X = T_p M$  outward,  $e_2, \dots, e_n \in T_p N$  is an oriented basis. Moreover,  $(X, e_2, \dots, e_n)$  is oriented basis of  $T_p M$ .

*Proof.* Take oriented atlas  $\mathcal{U} = \{\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha\}_{\alpha \in \mathcal{A}}$ .

We have  $\varphi_\alpha(U_\alpha \cap U_\beta) \subset \{0\} \times \mathbb{R}^{n-1}$ .

Define  $\psi_\alpha : N \cap U_\alpha \xrightarrow{\cong} (\{0\} \times \mathbb{R}^{n-1}) \cap U_\alpha$ . Then  $\mathcal{U}' = \{\psi_\alpha\}$  is a smooth atlas for  $N$ .

$\mathcal{U}$  is oriented implies  $\mathcal{U}$  is positive, so is  $\mathcal{U}'$ . So there exists the unique  $\mathcal{O}_N$  s.t.  $\mathcal{U}'$  is oriented. □

**Theorem 6.11** (Stokes' Theorem).  *$M$  is  $n$  dimensional manifold with boundary, oriented by  $\mathcal{O}_M$ .  $N = \partial M$ , with induced orientation  $\mathcal{O}_N$ .  $\iota : N \hookrightarrow M$  is the inclusion map. Then  $\forall \omega \in \Omega_c^{n-1}(M)$ , we have*

$$\int_M d\omega = \int_N \iota^*(\omega) \quad (6.4)$$

*Proof.*  $\forall \omega \in \Omega_c^{n-1}(M)$ ,  $\omega = \omega_1 + \cdots + \omega_k$ ,  $\omega_j \in \Omega_s^n(M)$ .

By linearity, we may assume  $\omega \in \Omega_s^n(M)$ ,  $\text{Supp}(\omega) \subset U_\alpha$  for chart  $\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha$ .

$\varphi_\alpha^{-1,*}(\omega) \in \Omega_c^n(V_\alpha)$  induces  $\omega' \in \Omega_c^n(\mathbb{R}_- \times \mathbb{R}^{n-1})$  if we extend it by 0.

By considering  $\omega'$  instead of  $\omega$ , we may assume  $M = \mathbb{R}_- \times \mathbb{R}^{n-1}$  □

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