

Complex Analysis

LIN150117

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

November 11, 2024

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2 Complex Functions

2.1 Analytic functions and rational functions

2.1.1 Harmonic function

Definition 2.1 (Cauchy-Riemann equation).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 2.2 (Harmonic function). A function u is **harmonic** if it satisfied **Laplace equation** $\triangle u = 0$.

If two harmonic function u and v satisfies Cauchy-Riemann equations, then we say that v is **conjugate harmonic function of u** $\Rightarrow u$ is conjugate harmonic of $-v$.

2.1.2 Polynomials and rational function

The polynomial $P(z) = \sum_{j=0}^n a_j z^j$ is analytic in \mathbb{C} .

We will prove the fundamental theorem of algebra

Theorem 2.3 (Fundamental Theorem of Algebra). *Every polynomial with degree $n > 0$ has at least one point.*

Theorem 2.4 (Gauss-Lucus theorem). *The smallest convex polygon that contain the zeros of P also contains the zeros of P' .*

Proof. Only need to check.

We can get this equation.

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{j=1}^n \frac{1}{\alpha - \alpha_j} = 0 \Rightarrow \sum_{j=1}^n \frac{\overline{\alpha - \alpha_j}}{|\alpha - \alpha_j|^2}$$

Hence α is linearly represented by α_j . □

Proposition 2.5. *Let P and Q be two polynomial with no common zeros. Then the rational function $R(z) = \frac{P(z)}{Q(z)}$ is analytic away from the zeros of Q .*

*The zeros of Q are called **poles** of R , and the **order of a pole** is equal to the order of the corresponding zero of Q .*

We often view R as a function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. $R_1(z) := R(\frac{1}{z})$.

If $R_1(0) = 0$, the order of the zero at ∞ (of R) is the order of the zero of $R_1(z)$ at $z = 0$.

If $R_1(0) = \infty$, the order of the pole at ∞ (of R) is the order of the pole of $R_1(z)$ at $z = 0$.

Suppose

$$R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, a_n \neq 0, b_m \neq 0$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$$

By discussing m and n , we can infer the situation of $R(z)$ at ∞ .

By adding the order of poles and zeros at ∞ , we can get the following theorem.

Theorem 2.6. *The total number of zeros and poles of a rational function are the same.*

Remark 2.7. This common number is called the **order of the rational function**.

Corollary 2.8. *Suppose a rational function R has order p . Then every equation $R(z) = a$ has exactly p roots.*

Proof. $\hat{R}(z) = R(z) - a$ has the same poles as R . □

A rational function of order 1 is a **linear fraction** $R(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$

Such fraction is often called **Möbius transformation**

Every rational function has a representation by **partial fractions**.

- If R has a pole at ∞ . Then we can write

$$R(z) = G(z) + H(z) \quad (*)$$

where G is a polynomial without constant term, and H is finite at ∞ .

The degree of G is the order of the pole of R at ∞ . G is called the **singular part** of R at ∞ .

- Let the distinct finite poles of R be β_1, \dots, β_k . Let $R_j(\psi) = R(\beta_j + \frac{1}{\psi})$. Then R_j is a rational function with a pole at ∞ . As in $(*)$, we can write

$$R_j = G_j + H_j$$

with H_j finite at ∞ . Then

$$R(z) = G_j\left(\frac{1}{z - \beta_j}\right) + H\left(\frac{1}{z - \beta_j}\right)$$

with G_j is a polynomial in $\frac{1}{z - \beta_j}$ without constant term called the **singular point** of R at β_j .

- Let $F(z) = R(z) - G(z) - \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right)$.

Then F is a rational function which can only have poles among β_j, ∞

Since by our construction, F is finite at every $\beta_j, 1 \leq j \leq k$ and ∞ .

So F is a constant.

In particular, $R(z) = G(z) + \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right) + c$.

2.2 Power Series

2.2.1 Power series

Theorem 2.9 (Abel's theorem). *If $\sum a_n$ converges, then $f(z) = \sum a_n z^n \rightarrow f(1)$ as $z \rightarrow 1$ in such a way that $\frac{|1-z|}{1-|z|}$ remains bounded.*

2.3 Exponential, Trigonometric and logarithmic functions

2.3.1 Exponential and Trigonometric function

The **exponential function** is defined as the solution of the differential equation

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1 \end{cases}$$

We denote $e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

The **trigonometric function** are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

2.3.2 Logarithmic Functions

The **logarithmic function** \ln is defined by $z = \ln w$ is a root of the equation $e^z = w$.

For $w \neq 0$, we write $z = x + iy$, then

$$e^{x+iy} = w \Leftrightarrow \begin{cases} e^x = |w| \\ e^{iy} = \frac{w}{|w|} \end{cases}$$

The first equation has a unique solution $x = \ln |w|$.

The second equation $e^{iy} = \frac{w}{|w|}$ has a unique solution $y_0 \in [0, 2\pi)$.

If we write $w = re^{i\theta}$, then $x = \ln w$, $y = \theta = \arg w$.

Thus, for $w \neq 0$, we have

$$\ln w = \ln |w| + i \arg w$$

The function \ln is actually not single-valued. But we can define a single-valued function Ln

We define

$$a^b = \exp(b \ln a)$$

We will prove Ln is analytic in $\mathbb{C} - (-\infty, 0]$ but not continuous in $(-\infty, 0]$.

Ln is the principal branch of the logarithm.

3 Conformal Mappings

3.1 Basic topology

3.1.1 Connectedness

Theorem 3.1. *A nonempty open set in \mathbb{C} is connected iff any two of its points can be joined by a polygon which lies in the set, i.e. Connectedness is equivalent to Path Connectedness*

An nonempty connected subset is called a **region**

3.1.2 Compactness

Definition 3.2. A set X is **totally bounded** if $\forall \varepsilon > 0$, X can be covered by finitely many balls of radius ε

Theorem 3.3. *A set is compact iff it is complete and totally bounded.*

Theorem 3.4. *A subset $X \subset \mathbb{C}$ is compact iff every infinite sequence of X has a limit point in X .*

3.1.3 Continuous Functions

Theorem 3.5. *Continuous function maps connected space to connected space.*

Theorem 3.6. *Continuous function maps compact space to compact space.*

3.2 Conformality, geometric consequences of the existence of a derivative

3.2.1 Arcs and closed curves

The equation of an **arc** r in \mathbb{C} can be represented by one of the terms

- $x = x(t), y = y(t), \alpha \leq t \leq \beta, x, y$ are continuous at t
- $z(t) = x(t) + iy(t), \alpha \leq t \leq \beta.$
- The continuous mapping $\gamma : [\alpha, \beta] \rightarrow \mathbb{C}.$

For a non-decreasing function $\varphi : [\alpha, \beta] \rightarrow [\alpha', \beta'], z = z(\varphi(t)), \alpha' \leq \tau \leq \beta'$ is **change of parameter** of $z(t).$

The change is **reversible** iff φ is strictly increasing.

If γ is differentiable, then call γ a **curve**.

γ is **simple**, or a **Jordan curve**, if γ is injective.

γ is **closed curve** if $\gamma(0) = \gamma(1).$

3.2.2 Analytic Functions in Regions

A function f is analytic on an arbitrary set A if it is the restriction to A of a function which is analytic in some open set containing A .

Theorem 3.7. *An analytic function in a region (i.e. open and connected) Ω whose derivative is 0 must reduce to a constant. The same hold if the real part, the imaginary part, the modulus, or the argument is constant.*

3.2.3 Conformal Mappings

Suppose $f : \Omega \rightarrow \mathbb{C}$ is analytic in Ω . $r_1 = z_1(t), r_2 = z_2(t), \alpha \leq t \leq \beta$.

$$z_0 = z_1(t_0) = z_2(t'_0), z'_1(t_0) \neq 0, z'_2(\hat{t}_0) \neq 0, \alpha < t_0, \hat{t}_0 < \beta.$$

$$f'(z_0) \neq 0, w_1(t) = f(z_1(t)), w_2 = f(z_2(\hat{t}_0))$$

$$\Gamma_1 = \{w_1(t) | \alpha \leq t \leq \beta\}, \Gamma_2 = \{w_2(t) | \alpha \leq t \leq \beta\}$$

Then

$$w'_1(t) = f'(z_1(t))z'_1(t)$$

$$w'_2(t) = f'(z_2(\hat{t}))z'_2(\hat{t})$$

\Rightarrow

$$w'_1(t_0) \neq 0, w'_2(t_0) \neq 0$$

$$\arg w'_1(t_0) = \arg f'(z_1(t_0))z'_1(t_0)$$

$$\arg w'_2(t_0) = \arg f'(z_2(\hat{t}_0))z'_2(\hat{t}_0)$$

So the "angle" $\arg w'_1(t_0) - \arg w'_2(t_0) = \arg z'_1(t_0) - \arg z'_2(\hat{t}_0)$ remains the same.

Now we give the definition.

Definition 3.8. $w = f(z)$ is said to be **conformal** in Ω if f is analytic in Ω and $f'(z) \neq 0$ for $\forall z \in \Omega$.

Easy to prove that linear change of scale at z_0 is independent of the direction.

$$i.e. |f'(z_0)| = \lim_{z \rightarrow z_0} \frac{\delta\sigma}{\delta s}$$

3.2.4 Length and Area

The **length** of a differentiable arc γ with the equation $z(t) = x(t) + iy(t)$, $a \leq t \leq b$

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b |z'(t)| dt$$

For $\Gamma = f(\gamma)$ where f conformal mapping.

Then

$$L(\Gamma) = \int_a^b |f'(z(t))| \cdot |z'(t)| dt$$

The **area** of $E \subset \mathbb{R}$ is $A(E) = \iint_E dx dy$

Then by the differentiable functional transformation, the area $\hat{E} = f(E)$ is

$$A(\hat{E}) = \iint_E |u_x v_y - u_y v_x| dx dy$$

If f is the conformal mapping of an open set containing E , then by Cauchy-Riemann equation

$$A(\hat{E}) = \iint_E |f'(z)|^2 dx dy$$

3.3 Möbius Transformation

Recall that a **Möbius transformation** is a function of the form

$$w = s(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Then it has an inverse $z = S^{-1}(w) = \frac{dw - b}{-cw + a}$.

We may define $S(\infty) = \lim_{z \rightarrow \infty} S(z) = \frac{a}{c}$, $S(\frac{-d}{c}) = \infty$

With these definition, $S : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a topological mapping. Here one may use the chordal metric to define the topology.

$$S'(z) = \frac{ad - bc}{(cz + d)^2}$$

Then S is conformal in $\hat{\mathbb{C}} - \{-\frac{d}{c}, \infty\}$.

$w = z + \alpha$ is called a **parallel translation**.

$w = kz$ with $|k| = 1$ is a **rotation**.

$w = kz$ with $k > 0$ is a **homothetic transformation**.

$x = \frac{1}{z}$ is called an **inversion**.

Proposition 3.9. *Every Möbius transformation is a composition of the above four operations.*

3.3.1 Cross ratio

For three distinct points $z_2, z_3, z_4 \in \hat{\mathbb{C}}$, we can find a Möbius transformation S such that $S(z_2) = 0$, $S(z_3) = 1$, $S(z_4) = \infty$.

Lemma 3.10. *The Möbius transformation satisfying the above conditions is unique.*

The **cross ratio** (z_1, z_2, z_3, z_4) is the image z_1 under the Möbius transformation which maps z_2 to 1, z_3 to 0 and z_4 to ∞ .

Theorem 3.11. *If $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ are distinct, and T is any Möbius transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Lemma 3.12. *Let T be a Möbius transformation, $T(\mathbb{R})$ is either a circle or a straight line.*

Theorem 3.13. *The cross ratio (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle or a straight line.*

Remark 3.14. One may prove the theorem by elementary geometry

Theorem 3.15. *A Möbius transformation maps circles into circles.*

3.3.2 Symmetry

Suppose T is a Möbius transformation which maps $\hat{\mathbb{R}}$ onto a circle C .

We say that $w = Tz$ and $w^* = T\bar{z}$ are **symmetric** w.r.t. C .

Remark 3.16. This definition is independent of T . Suppose S is another Möbius transformation which maps $\hat{\mathbb{R}}$ onto C , then $S^{-1}T$ maps $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$, and this $S^{-1}w = S^{-1}Tz$ and $S^{-1}w^* = S^{-1}T\bar{z}$ are conjugate.

The points z and z^* are **symmetric w.r.t C through z_1, z_2, z_3** iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.

This can be another definition.

Note that only the points on C are symmetric to themselves.

The mapping $z \mapsto z^*$ is 1-1 and is called **reflection** w.r.t. C .

Geometric Meaning of Symmetry

Case1: C is a straight line. We may assume $z_3 = \infty$.

z, z^* are symmetric w.r.t. C if and only if

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Then

$$|z^* - z_2| = |z - z_2|, \quad \forall z_2 \in C \text{ and } z_2 \neq \infty$$

$$\operatorname{Im} \frac{z^* - z_2}{z_1 - z_2} = \operatorname{Im} \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

So C is the bisecting normal of the segment between z and z^* .

Case2: C is the circle $|z - a| = R$.

$$\begin{aligned} \text{Then for } \forall \text{ distinct } z_1, z_2, z_3 \in \mathbb{C}, \overline{(z, z_1, z_2, z_3)} &= \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)} \\ &= (\bar{z} - \bar{a}, \bar{z}_1 - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}) = (\bar{z} - \bar{a}, \frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}) = (\frac{R^2}{\bar{z} - \bar{a}}, z_1 - a, z_2 - a, z_3 - a) \\ &= (\frac{R^2}{\bar{z} - \bar{a}}, z_1, z_2, z_3). \end{aligned}$$

Then the symmetric point of z w.r.t. C is

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

or

$$(z^* - a)(\bar{z} - \bar{a}) = R^2$$

\Rightarrow

$$\begin{cases} |z^* - a| \cdot |z - a| = R^2 \\ \frac{z^* - a}{z - a} = \frac{(z^* - a)(\bar{z} - \bar{a})}{|z - a|^2} > 0 \end{cases}$$

Theorem 3.17 (The Symmetric principle). *If a Möbius transformation maps a circle C_1 onto a circle C_2 , then it transforms any pair of symmetric points w.r.t. C_1 into a pair of symmetric points w.r.t. C_2 .*

Proof. Case1: $C_1 = \hat{\mathbb{R}}$. Let T be the Möbius transformation which maps $\hat{\mathbb{R}}$ onto C_2 . $\forall z \in \mathbb{C}$, by definition, $w = Tz$ and $w^* = T\bar{z}$ are symmetric w.r.t. C_2 .

Case2: C_1 is a general circle. Let $T : C_1 \rightarrow C_2$ and $S : \mathbb{R} \rightarrow C_2$ be Möbius transformation.

Suppose w, w^* are symmetric w.r.t. C_1 . Then there exists z s.t. $w = Sz, w^* =$

$S\bar{z}$.

Then we can find $Tw = TSz, Tw^* = TS\bar{z}$ are symmetric w.r.t. C_2 since $TS : \hat{\mathbb{R}} \rightarrow C_2$ □

Remark 3.18. (1). The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w, z_2 \mapsto w_2, z_3 \mapsto w_3$ where $z_1, z_2, z_3 \in C_1, w_1, w_2, w_3 \in C_2$ is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

(2). The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w_1, z_2 \mapsto w_2$ where $z_1 \in C_1, z_2 \notin C_1, w_1 \in C_2, w_2 \notin C_2$ is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

3.3.3 Steiner Circles, circular net

For $S(z) = \frac{az + b}{cz + d}, S'(z) = \frac{ad - bc}{(cz + d)^2}$.

A point $z \notin$ a circle C is said to on the **right(left, resp.)** of C if $\text{Im}(z, z_1, z_2, z_3) > 0(\text{Im}(z, z_1, z_2, z_3) < 0)$

Remark 3.19.

- (1). This agrees with everyday use since $(i, 1, 0, \infty) = i$
- (2). This distinct between left and right is the same for all triples, while the meaning may be reversed.

(If $C = \hat{\mathbb{R}}$, then $(z, z_1, z_2, z_3) = \frac{az + b}{cz + d}$ with $a, b, c, d \in \mathbb{R} \Rightarrow \text{Im}(z, z_1, z_2, z_3) = \frac{ad - bc}{|cz + d|^2} \text{Im}(z)$)

- (3). We can define an absolute positive orientation of all finite circles by requiring that ∞ should be lie to the right of the oriented circles.

Consider a Möbius transformation of the form

$$w = k \cdot \frac{z - a}{z - b}$$

Here, $z = a \mapsto w = 0, z = b \mapsto w = \infty$.

Then circles through a, b maps to straight line through $0, \infty$.

The concentric circle about the origin, $|w| = \rho$, correspond to circles with the equation

$$\left| \frac{z - a}{z - b} \right| = \frac{\rho}{|k|}$$

These are the circles of **Apollonius** with limit points a and b .

Denote by C_1 the circles through a, b and C_2 the circles of Apollonius with these limit points. The configuration formed by all the circles C_1 and C_2 is called the **Steiner circles**(or **circular net**)

Theorem 3.20.

- (a) *There is exactly one C_1 and one C_2 through each point in $\hat{\mathbb{C}} \setminus \{a, b\}$*
- (b) *Every C_1 meets every C_2 under right angle.*
- (c) *Reflection in a C_1 transforms every C_2 into itself and every C_1 into another C_1 .*
- (d) *The limit points a, b are symmetric w.r.t. each C_2 , but not w.r.t. other circles.*

Proof. If the limit points are $0, \infty$, those properties are trivial in the w -plane. The general case follows since all properties are invariant under Möbius transformations. □

4 Elementary Conformal mapping

Example 4.1. $w = z^\alpha$ where $\alpha > 0$.

Let $S(u_1, u_2)$ with $0 < \varphi_2 - \varphi_1 \leq 2\pi$ be $\{z \in \mathbb{C} : z \neq 0, \varphi_1 < \arg(z) < \varphi_2\}$ where $\arg(z)$ can be chosen as any value of it.

Then $S(\varphi_1, \varphi_2)$ is a region.

In this region, a unique value of $w = z^\alpha$ is defined by $\arg w = \alpha \arg z$.

This function is analytic with $\frac{dw}{dz} = \alpha \frac{w}{z}$.

This function is 1-1 only if $\alpha(\varphi_2 - \varphi_1) \leq 2\pi$.

Example 4.2. $w = e^z$ maps $\{z \in \mathbb{C} : -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2}\}$ onto $\{w \in \mathbb{C} : \text{Re}(w) > 0\}$

Example 4.3. $w = \frac{z-1}{z+1}$ maps $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$ onto $\{w \in \mathbb{C} : |w| < 1\}$

Example 4.4.

$$\mathbb{C} \setminus [-1, 1] \xrightarrow{z_1 = \frac{z+1}{z-1}} \mathbb{C} \setminus (-\infty, 0] \xrightarrow{z_2 = \sqrt{z_1}} \{\text{Re}(z_2) > 0\} \xrightarrow{w = \frac{z_2-1}{z_2+1}} \{w \in \mathbb{C} : |w| < 1\} \quad (4.1)$$

4.1 Elementary Riemann surfaces

Example 4.5. $w = z^n, n \in \mathbb{Z}_+$ and $n > 1$.

There is a 1-1 correspondence between each angle $\frac{(k-1)2\pi}{n} < \arg z < \frac{k \cdot 2\pi}{n}, k = 1, 2, \dots, n$ and while w -plane except for the positive real axis.

Example 4.6. $w = e^z$. This function maps each parallel strip $(k-1)2\pi < \text{Im } z < k \cdot 2\pi, k \in \mathbb{Z}$ onto a sheet with a cut along the positive axis.

5 Complex Integration

5.1 Fundamental Theorems

5.1.1 Line integral and rectifiable arcs

Let $f(t) = u(t) + iv(t)$ be a complex-valued defined on $t \in [a, b] \subset \mathbb{R}$ where u, v are real-valued functions. If f is continuous on $[a, b]$, we may define the **integral**

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt$$

Let γ be a piecewise differential arc in \mathbb{C} with the equation $z = z(t), a \leq t \leq b$. If f is continuous on γ , then $f(z(t))$ is continuous on $[a, b]$, and we define

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt \quad (5.1)$$

The integral defined in 5.1 is independent of the parametrization of γ . Suppose that another parametrization of γ is $\gamma : (\alpha, \beta) \rightarrow \mathbb{C}, \tau \mapsto z(t(\tau))$, where $t : (\alpha, \beta) \rightarrow (a, b), \tau \mapsto t(\tau)$ is piecewise differentiable. Then we have

$$\int_a^b f(z(t))z'(t)dt = \int_{\alpha}^{\beta} f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau = \int_{\alpha}^{\beta} f(z(t(\tau)))\frac{dz(t(\tau))}{d\tau}d\tau \quad (5.2)$$

For an arc γ with equation $z = z(t), a \leq t \leq b$, we define $-\gamma$ by $z = z(-t), -b \leq t \leq a$.

Then we have

$$\int_{-\gamma} f(z)dz = \int_{-b}^{-a} f(z(-t))\frac{dz(-t)}{dt}dt$$

$$\begin{aligned}
&= - \int_{-a}^{-b} f(z(-t))z'(-t)dt \\
&= - \int_a^b f(z(\tau))z'(\tau)d\tau \\
&= - \int_{\gamma} f(z)dz
\end{aligned}$$

So we have those properties:

Proposition 5.1.

(a) $\int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$

(b) Let f and g be two continuous functions on the piecewise differentiable arc γ , then

$$\int_{\gamma} (\lambda_1 f + \lambda_2 g)dz = \lambda_1 \int_{\gamma} f dz + \lambda_2 \int_{\gamma} g dz, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

(c) If γ can be subdivided into two pieces differentiable arcs γ_1 and γ_2 , and f is continuous on γ_1 , then

$$\int_{\gamma} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz$$

(d) (c) implies that the integral of a closed curve doesn't depend on the starting point on the curve

Example 5.2. Evaluate $\int_{\gamma} \frac{1}{z-a} dz$ where γ is the circle centered at $a \in \mathbb{C}$ with radius R .

Let $z = z(t) = a + Re^{it}$. Then the integral is $2\pi i$

5.1.2 The fundamental theorem of Calculus for integrals in \mathbb{C}

The line integral w.r.t. \bar{z} is defined by

$$\int_{\gamma} f(z) d\bar{z} = \int_{\gamma} \overline{f(\bar{z})} dz$$

With this notation, line integrals w.r.t. $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$ can be defined by

$$\int_{\gamma} f(z) dx = \frac{1}{2} \left[\int_{\gamma} f(z) dz + \int_{\gamma} f(z) \overline{dz} \right]$$

$$\int_{\gamma} f(z) dy = \frac{1}{2i} \left[\int_{\gamma} f(z) dz - \int_{\gamma} f(z) \overline{dz} \right]$$

if we write $f(z) = \mu + i\nu$, we have

$$\int_{\gamma} f(z) dz = \int_{\gamma} f(z) dx + i \int_{\gamma} f(z) dy = \int_{\gamma} (\mu dx - \nu dy) + i \int_{\gamma} (\nu dx + \mu dy)$$

Remark 5.3. It is followed by the intuition. We can view the integration as the multiplication between f and dz .

The integral w.r.t. **arc length** is defined by

$$\int_{\gamma} f(z) |dz| = \int_a^b f(z(t)) |z'(t)| dt$$

This integral is again independent of the parametrization. It is easy to check

$$\int_{-\gamma} f(z) |dz| = \int_{\gamma} f(z) |dz|$$

Now we define **length** of a curve γ : $L(\gamma) = \int_{\gamma} |dz|$

We have the inequality:

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| \cdot |dz| \leq L(\gamma) \cdot \sup_{z \in \gamma} |f(z)|$$

The length of an arc γ ($z = z(t)$) can also be defined as the least upper bound of

all sums

$$\sum_{i=1}^n |z(t_i) - z(t_{i-1})|$$

where $a = t_0 < t_1 < \dots < t_n = b$. If this least upper bound is finite, we say that the arc is **rectifiable**.

It is easy to show that piecewise differentiable arcs are rectifiable.

The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_{\gamma} f(z) dz = \lim \sum_{k=1}^n f(z(\psi_k)) [z(t_k) - z(t_{k-1})]$$

Theorem 5.4. *Let $\Omega \subset \mathbb{C}$ be a region, and P, Q two (possibly complex-valued) functions that are continuous on Ω , γ closed curve. The integral $\int_{\gamma} p(x, y) dx + Q(x, y) dy$ depends only on the end point of γ iff there exists a function $U(x, y)$ on Ω with $\frac{\partial U}{\partial x} = P, \frac{\partial U}{\partial y} = Q$.*

Proof. " \Leftarrow ": If such a U exists, then

$$\int_{\gamma} P dx + Q dy = \int_{\gamma} \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = \int_{\gamma} \frac{dU}{dt} dt = U(\gamma(b)) - U(\gamma(a))$$

" \Rightarrow ": Fix a point $(x_0, y_0) \in \Omega$. We define $U(x, y) = \int_{\gamma} P dx + Q dy$ where γ is any curve between (x_0, y_0) and (x, y) . Easy to check that it is true. \square

Theorem 5.5 (Fundamental theorem of Calculus for integrals on \mathbb{C}). *Let f be continuous on a region Ω containing γ . $\int_{\gamma} f dz$ depends on the endpoints iff f is the derivative of an analytic function F in Ω .*

Remark 5.6. We will prove $\int_{\gamma} f dz = F(\omega_2) - F(\omega_1)$ where γ begins at ω_1 and ends at ω_2 .

Proof. Transform the line integration into the composition of two real integration. \square

Corollary 5.7. *If F is analytic on Ω with $F' = f$, and γ is a closed curve in Ω , then $\int_{\gamma} f dz = 0$. Conversely if f is continuous on Ω and $\int_{\gamma} f dz = 0$ for any closed curve in Ω , then f is the derivative of an analytic function F in Ω .*

5.1.3 Cauchy's theorem for a rectangle

There are some notes in this section:

R is the rectangle in \mathbb{C} , $R = \{x + iy \in \mathbb{C} : a \leq x \leq b, c \leq y \leq d\}$. And ∂R is boundary curve oriented in the counterclockwise direction.

Theorem 5.8 (Cauchy's theorem for a rectangle). *If f is analytic on an open set which contains R , then $\int_{\partial R} f(z) dz = 0$*

Proof. For \forall rectangle \tilde{R} inside R , we define $Z(\tilde{R}) = \int_{\partial \tilde{R}} f(z) dz$. Then $Z(R) = Z(R_1) + Z(R_2)$ if R is divided into Z_1, Z_2 .

Since we can divide R into four equal rectangles, and find a rectangle with $|Z(R^{(1)})| \geq \frac{1}{4}|Z(R)|$. Then repeat the above steps and we obtain a sequence of nested rectangles $R \supset R^{(1)} \supset \dots$ with the property

$$|Z(R^{(n)})| \geq \frac{1}{4}|Z(R^{(n-1)})| \geq \dots \geq \frac{1}{4^n}|Z(R)| \quad (5.3)$$

$\forall \delta > 0, \exists n \in \mathbb{N}$ s.t. $R^{(n)} \subset \{z \in \mathbb{C} : |z - z_0| < \delta\}$, $\forall n \geq N$, where z_0 is the limit of $R^{(n)}$ as $n \rightarrow \infty$.

f is analytic in $R \Rightarrow \forall \varepsilon, \exists \delta > 0$ s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon, \forall z \text{ with } |z - z_0| < \delta \quad (5.4)$$

We assume that δ satisfies both conditions. We have

$$Z(R^{(n)}) = \int_{\partial R^{(n)}} f(z) dz = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)] dz$$

$$\Rightarrow |Z(R^{(n)})| \leq \varepsilon \int_{\partial R^{(n)}} |z - z_0| dz \text{ by 5.4}$$

Let d_n be the length of diagonal of $R^{(n)}$, L_n be the length of its perimeter. Then

$$|z - z_0| \leq d_n, \forall z \in \partial R^{(n)}.$$

$\Rightarrow |Z(R^{(n)})| \leq \varepsilon d_n L_n = \varepsilon \frac{D}{2^n} \cdot \frac{L}{2^n}$ where D, L are the diameter and perimeter of R .

$$\Rightarrow |Z(R)| \stackrel{5.3}{\leq} 4^n |Z(R^{(n)})| \leq \varepsilon DL \Rightarrow Z(R) = 0 \text{ since } \varepsilon \text{ is arbitrary.} \quad \square$$

We will next prove the following stronger theorem:

Theorem 5.9 (stronger version of Cauchy's theorem for a rectangle). *Let f be analytic on $R' = R \setminus \{\psi_1, \dots, \psi_m\}$, $m \in \mathbb{N}$. If $\lim_{z \rightarrow \psi_j} (z - \psi_j)f(z) = 0, \forall 1 \leq j \leq m$, then*

$$\int_{\partial R} f(z) dz = 0.$$

Proof. WLOG, we may assume f is not analytic at only one point $\psi \in R$. If we put ψ into a small rectangle S_0 , then the previous theorem tells us $\int_{\partial R} f(z) dz = \int_{\partial S_0} f(z) dz$.

$$\forall \varepsilon > 0, \text{ we may choose } S_0 \text{ small enough such that } |f(z)| \leq \frac{\varepsilon}{|z - \psi|}, \forall z \in \partial S_0$$

$$\Rightarrow \left| \int_{\partial R} f(z) dz \right| \leq \varepsilon \int_{\partial S_0} \frac{|dz|}{|z - \psi|} \leq \varepsilon \frac{1}{\frac{l}{2}} \cdot 4l = 8\varepsilon$$

$$\Rightarrow \int_{\partial R} f(z) dz = 0 \text{ since } \varepsilon \text{ is arbitrary.} \quad \square$$

5.1.4 Cauchy's Theorem for a disk

$$\Delta := \{z \in \mathbb{C} : |z - z_0| < R\} \text{ where } R > 0.$$

Theorem 5.10 (Cauchy's Theorem for a disk). *If f is analytic in an open disk Δ , then $\int_{\gamma} f(z) dz = 0$ for closed curve γ in Δ .*

Proof. Suppose the center of Δ is $z_0 = x_0 + iy_0$, $z = x + iy$. We define

$$F(z) = \int_{\gamma} f(z) dz$$

where γ is the horizontal line segment from z_0 to (x, y_0) added with vertical line segment from (x, y_0) to z . We have

$$\frac{\partial F}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{F(x, y + \delta y) - F(x, y)}{\delta y} = \lim_{\delta y \rightarrow 0} \frac{1}{\delta y} \int_{\delta \gamma} f(z) dz = if(z) \quad (5.5)$$

By Cauchy's theorem on rectangles, one has $F(z) = -\int_{\tilde{\gamma}} f(z) dz$, where $\tilde{\gamma}$ is the vertical line segment from z_0 to (x_0, y) added with horizontal line segment from (x_0, y) to z .

Similarly, $\frac{\partial F}{\partial x} = f(z)$.

$\Rightarrow \frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y} \Rightarrow F$ is analytic in Δ with derivative f . By Fundamental Theorem 5.5 of Calculus $\Rightarrow \int_{\gamma} f(z) dz = 0$ for \forall closed curve in Δ . \square

Here is a stronger version.

Theorem 5.11 (stronger version of Cauchy's Theorem for a disk). *Let f be analytic in a region $\Delta' = \Delta \setminus \{\psi_1, \dots, \psi_m\}$ with $m \in \mathbb{N}$. If f satisfies $\lim_{z \rightarrow \psi_j} (z - \psi_j) f(z) = 0, \forall 1 \leq j \leq m$, then $\int_{\gamma} f(z) dz = 0, \forall \gamma$ closed in Δ'*

Proof. It is similar to the above proof.

For the case no ψ_j lies on $x = x_0$ and $y = y_0$, we can find a similar curve γ with last segment is a vertical one. Let $F(z) = \int_{\gamma} f(z) dz$. And continue the process of proof of the previous theorem.

For the case that $\exists \psi_j$ lies on the lines $x = x_0, y = y_0$, we actually can move the center to another point s.t. no ψ_j lies on the lines $x = x'_0, y = y'_0$. \square

5.2 Cauchy's integral formula

5.2.1 Index of a point with respect to a closed curve

Lemma 5.12. *If the piecewise differentiable closed curve γ does not pass through $z \in \mathbb{C}$, then the value of the integral $\int_{\gamma} \frac{d\zeta}{\zeta - z}$ is a multiple of $2\pi i$.*

Proof. $\gamma : \zeta = \zeta(t), \alpha \leq t \leq \beta$. $h(t) = \int_{\alpha}^t \frac{\zeta'(s)}{\zeta(s) - z} ds$.

$z \in \gamma \Rightarrow h$ is defined and continuous on $[\alpha, \beta]$. For all t s.t. $\zeta'(t)$ is continuous, we have

$$h'(t) = \frac{\zeta'(t)}{\zeta(t) - z} \Rightarrow \frac{d}{dt} [e^{-h(t)}(\zeta(t) - z)] = 0$$

So $e^{-h(t)}(\zeta(t) - z)$ is constant on $[\alpha, \beta]$.

$$\text{Then } e^{h(t)} = \frac{\zeta(t) - z}{\zeta(\alpha) - z} \Rightarrow e^{h(\beta)} = 1 \Rightarrow h(\beta) \in \{2k\pi i : k \in \mathbb{Z}\}.$$

□

The **index of the point** z w.r.t. the closed curve γ is the number

$$n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z}$$

n is also called the **winding number**.

Theorem 5.13. *Let γ be a piecewise differentiable closed curve. The function $z \mapsto n(\gamma, z)$ is constant on each connected set of $\mathbb{C} \setminus \gamma$, and zero if this set is unbounded.*

Proof. Define $f : \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}, z \mapsto n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z}$.

Then

$$|f(z) - f(z_0)| = \frac{1}{2\pi} \left| \int_{\gamma} \frac{z - z_0}{(\zeta - z)(\zeta - z_0)} d\zeta \right| \leq \frac{|z - z_0|}{2\pi} \int_{\gamma} \frac{1}{|\zeta - z| \cdot |\zeta - z_0|} |d\zeta|$$

$\Rightarrow f$ is continuous on each open connected set of $\mathbb{C} \setminus \gamma$. Let Ω be any open connected set of $\mathbb{C} \setminus \gamma$. We have $f(\Omega)$ is connected $\xrightarrow{f(\Omega) \subset \mathbb{Z}} f(\Omega)$ contains at most one point $\Rightarrow f$ is constant on Ω .

If $|z|$ is sufficient large, \exists a disk of radius R , $B(0, R)$, s.t. $\gamma \subset B(0, R)$ but $z \notin B(0, R)$. Cauchy's theorem for a disk 5.10 tells us that $f(z) = n(\gamma, z) = 0$. So it is zero if this set is unbounded. □

Lemma 5.14. *Let z_1, z_2 be two points on a closed curve γ and $0 \notin \gamma$.*

Suppose z_1 in the lower half space and z_2 in upper half space. If $\gamma_1 \cap \{(x, 0) : x \leq 0\} = \emptyset$, and $\gamma_2 \cap \{(x, 0) : x \geq 0\} = \emptyset$, then $n(\gamma, 0) = 1$.

Remark 5.15. One method to prove this lemma is to create two segment from z_i to the point in the unit circle. By divide the curve into two parts, we can easily remove the part of previous curve by using the theorem 5.13, since 0 is in the unbounded set.

In this proof, we can find that Theorem 5.13 is such powerful that we can change any curve to a more simple curve easily!

5.2.2 Cauchy's integral formula

Theorem 5.16 (Cauchy's integral formula). Suppose that f is analytic in an open disk Δ , and let γ be a closed curve in Δ . For $\forall z \notin \gamma$,

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $n(\gamma, z)$ is the index of z w.r.t. γ .

Proof. If $z \notin \Delta$, The both sides of the equation is 0.

So we may assume $z \in \Delta$ and $z \notin \gamma$. Define $F : \Delta \setminus \{z\} \rightarrow \mathbb{C}, \zeta \mapsto \frac{f(\zeta) - f(z)}{\zeta - z}$.

Then F is analytic in $\Delta \setminus \{z\}$, and $\lim_{\zeta \rightarrow z} (\zeta - z)F(\zeta)$.

By Cauchy's Theorem 5.9 $\Rightarrow \int_{\gamma} F(\zeta) d\zeta = 0 \Rightarrow \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \int_{\gamma} \frac{1}{\zeta - z} d\zeta = f(z) \cdot 2\pi i \cdot n(\gamma, z)$ □

Remark 5.17. This proof let us find that for a good-enough function, its integral over a closed curve is a constant.

The theorem still holds if f is analytic except at a finite number of ζ_j s.t.

$$\lim_{\zeta \rightarrow \zeta_j} (\zeta - \zeta_j)f(\zeta) = 0$$

and $z \neq \zeta_j$ for each j , since Cauchy's theorem is still applicable.

Theorem 5.18 (The mean value property for analytic functions). *f is analytic in a region Ω which contain $\overline{B(z, R)}$. Then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\theta}) dt$$

Proof. The previous theorem 5.16 \Rightarrow

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta-z|=R} \frac{f(\zeta)}{\zeta - z} d\zeta \stackrel{\zeta=z+Re^{it}}{=} \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{it}) dt$$

□

If f is analytic in an open disk Δ , and γ is a closed curve in Δ . And $n(\gamma, z) = 1$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

This is usually referred to as **Cauchy's integral formula**

5.2.3 Higher derivatives

Lemma 5.19. *Let $\Omega \subset \mathbb{C}$ be a region and γ be an arc in Ω . If φ is continuous on γ , then the function*

$$F_n(z) := \int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z)^n} d\zeta$$

is analytic in each of the regions $\Omega \setminus \gamma$, and its derivative is $F'_n(z) = nF_{n+1}(z)$

Proof. We prove it by induction.

The lemma is true if $n = 0$: $F_0(z) = \int_{\gamma} \varphi(\zeta) d\zeta$ and $F'_0(z) = 0 = 0 \cdot F_1(z)$.

We suppose that the lemma holds for $n - 1$ with $n \in \mathbb{N}$: \forall continuous φ on γ , F_{n-1} is analytic in $\Omega \setminus \gamma$ and $F'_{n-1}(z) = (n - 1)F_n(z)$, $\forall z \in \Omega \setminus \gamma$.

Fix $z_0 \in \Omega \setminus \gamma$. For $\forall z \in B(z_0, \frac{\delta}{2})$, with $B(z_0, \delta) \subset \Omega \setminus \gamma$, we have $|\zeta - z| > \frac{\delta}{2}$, $\forall \zeta \in \gamma$.

For \forall continuous φ on γ ,

$$\begin{aligned} F_n(z) - F_n(z_0) &= \int_{\gamma} \frac{\varphi(\zeta)(\zeta - z + z - z_0)}{(\zeta - z)^n(\zeta - z_0)} d\zeta - \int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta \\ &= \left[\int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z_0)^{n-1}(\zeta - z)} d\zeta \right] \\ &\quad + (z - z_0) \int_{\gamma} \frac{\varphi(\zeta) d\zeta}{(\zeta - z)^n(\zeta - z_0)} \end{aligned}$$

Let $\psi(\zeta) = \frac{\varphi(\zeta)}{\zeta - z_0}$, which is continuous except γ .

Using the induction condition to ψ , we can finish the proof. \square

Theorem 5.20. *An analytic function on a region Ω has derivatives of all orders which are analytic in Ω . More precisely, $\forall z_0 \in \Omega$, choose $B(z, \delta) \subset \Omega$ and a circle $C \subset B(z_0, \delta)$ with center z_0 . For $\forall z$ in the interior of C , Cauchy's integral formula gives*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Then the previous lemma implies $f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$ is analytic in the interior of C . More generally, for $\forall n \in \mathbb{N}$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad (5.6)$$

5.2.4 Consequences of Cauchy

Theorem 5.21 (Morera's Theorem). *If f is continuous in a region Ω , and if $\int_{\gamma} f(z) dz = 0$ for \forall closed curve γ in Ω . Then f is analytic in Ω .*

Proof. We proved in Corollary 5.7 that under the hypothesis of theorem, $f = F'$ where F is analytic in Ω . The last theorem $\Rightarrow f$ is analytic. \square

Suppose f is analytic in a disk, $\overline{B(z_0, R)}$, and bounded on the circle γ given by $|z - z_0| = R$. Then $\forall z \in \gamma, |f(z)| \leq M$ for some $M \geq 0$. By (5.6),

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_C \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} |d\zeta| \leq \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = MR^{-n}n! \quad (5.7)$$

This inequality is known as **Cauchy's estimate**.

Theorem 5.22 (Liouville's Theorem). *A bounded entire function (i.e. analytic in \mathbb{C}) is constant.*

Proof. Suppose $|f(z)| \leq M, \forall z \in \mathbb{C}$. Cauchy's estimate \Rightarrow

$$n|f'(z)| \leq \frac{M}{R}, \forall z \in \mathbb{C}, \forall R > 0 \quad (5.8)$$

□

$$\xrightarrow{R \rightarrow \infty} f'(z) = 0 \text{ for } z \in \mathbb{C} \Rightarrow f = 0.$$

Theorem 5.23 (Fundamental Theorem for Algebra). *Every polynomial of degree $n \geq 1$ has n roots.*

Proof. It suffices to prove it has at least one root.

Suppose $P(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_0 \neq 0$ does not have a root.

Then $f(z) := \frac{1}{P(z)}$ is an entire function. As $z \rightarrow \infty, \lim_{|z| \rightarrow \infty} \frac{|P(z)|}{|z|^n} = |a_n| \Rightarrow$
 $\lim_{|z| \rightarrow \infty} \frac{1}{|P(z)|} = 0.$

So f is bounded. By Liouville's Theorem, f is a constant. Where $f = f(\infty) = 0$.

That causes contradiction. □

Theorem 5.24 (Power series). *If f is analytic in a region Ω which contains a closed disk $\overline{B(z_0, R)}$, then f has a power series expansion at z_0 ,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \quad \forall z \in B(z_0, R) \quad (5.9)$$

Proof. $\forall z \in B(z_0, R), \forall \zeta$ with $|\zeta - z_0| = R$.

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\
&= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \\
&= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}
\end{aligned} \tag{5.10}$$

This series converges uniformly in ζ with $|\zeta - z_0| = R$.

For $\forall z \in B(z, R)$,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi i} \int_{|\zeta - z| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta \\
&\stackrel{\text{uniformly}}{=} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \cdot (z - z_0)^n \\
&\stackrel{(5.6)}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\end{aligned} \tag{5.11}$$

□

5.3 Local properties of analytic functions

5.3.1 Removable Singularities and Taylor's Theorem

We remarked that Cauchy's integral formula holds if f is analytic except at a finite number of point ζ_j s.t. $\lim_{\zeta \rightarrow \zeta_j} (\zeta - \zeta_j) f(\zeta) = 0$. We will prove f can be extended to an analytic function in Δ . In other word, ζ_j are **removable singularities**.

Theorem 5.25 (Riemann's Removable Singularities Theorem). *Suppose that f is*

analytic in the region $\Omega' = \Omega \setminus \{\zeta_0\}$ where Ω is also a region. Then there exists an analytic function in Ω which coincides with f in Ω' if and only if $\lim_{z \rightarrow \zeta_0} (z - \zeta_0)f(z) = 0$.

Proof. The uniqueness and " \Rightarrow " part is trivial since the extended function is continuous at ψ_0 .

" \Leftarrow ": Cauchy's integral formula \Rightarrow

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta, \quad \forall z \in \Delta \text{ and } z \neq \zeta_0 \quad (5.12)$$

Lemma 5.19 \Rightarrow the RHS of the last equation 5.12 is analytic in $z \in \Delta$. Then

$$\hat{f}(z) = \begin{cases} f(z), & z \neq \zeta_0 \\ \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - \zeta_0} d\zeta, & z = \zeta_0 \end{cases} \quad (5.13)$$

is analytic in Ω . □

We apply Theorem 5.25 to the function $F(z) = \frac{f(z) - f(\zeta)}{z - \zeta}$, where f is analytic in a region Ω . Note that

$$\lim_{z \rightarrow \zeta} (z - \zeta)F(z) = 0, \quad \lim_{z \rightarrow \zeta} F(z) = f'(\zeta) \quad (5.14)$$

Theorem 5.25 $\Rightarrow \exists$ analytic function f_1 on Ω s.t.

$$f_1(z) = \begin{cases} F(z), & z \neq \zeta_0 \\ f'(\zeta), & z = \zeta_0 \end{cases} \quad (5.15)$$

we may thus write $f(z) = f(\zeta) + (z - \zeta)f_1(z)$.

Repeating this process for f_1 , we get an analytic function f_2 on Ω s.t.

$$f_1(z) = f_1(\zeta) + (z - \zeta)f_2(z) \quad (5.16)$$

where

$$f_2(z) = \begin{cases} \frac{f_1(z) - f_1(\zeta)}{z - \zeta}, & z \neq \zeta \\ f_2'(\zeta), & z = \zeta \end{cases} \quad (5.17)$$

Continuing the recursion, we have the general form

$$f_{n-1}(z) = f_{n-1}(\zeta) + (z - \zeta)f_n(z) \quad (5.18)$$

\Rightarrow

$$f(z) = f(\zeta) + (z - \zeta)f_1(\zeta) + \cdots + (z - \zeta)^{n-1}f_n(\zeta) + (z - \zeta)^n f_n(z) \quad (5.19)$$

Differentiating n times and setting $z = \zeta \Rightarrow f^{(n)}(\zeta) = n!f_n(\zeta)$

We just prove **Taylor's Theorem**

Theorem 5.26 (Taylor's Theorem). *If f is analytic in a region Ω , $\zeta \in \Omega$, then we have*

$$f(z) = f(\zeta) + (z - \zeta)f'(\zeta) + \cdots + \frac{f^{(n-1)}(\zeta)}{(n-1)!}(z - \zeta)^{n-1} + f_n(z)(z - \zeta)^n \quad (5.20)$$

where f_n is analytic in Ω . Moreover,

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - \zeta)^n(\omega - z)} d\omega \quad (5.21)$$

where C is a circle in Ω s.t. its interior \triangle is also in Ω and $\zeta, z \in \triangle$

Proof. It suffices to prove the second part.

Cauchy's integral formula $\Rightarrow f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\omega)}{\omega - z} d\omega, \forall z \in \triangle$.

For $f_n(z)$, we substitute the expression from (5.20). The first term is

$$\frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - \zeta)^n(\omega - z)} d\omega \quad (5.22)$$

The remaining terms have the following form, except for constant factors:

$$g_k(\zeta) = \int_C \frac{1}{(\omega - \zeta)^n(\omega - z)} d\omega, \quad 1 \leq k \leq n \quad (5.23)$$

The lemma 5.19 applies to $\varphi(\omega) = \frac{1}{\omega - z}$, $g'_k(\zeta) = kg_{k-1}(\zeta)$, $k \in \mathbb{N}$, $\forall \zeta \in \Delta$. So

$$\begin{aligned} g_1(\zeta) &= \int_C \frac{1}{(\omega - \zeta)(\omega - z)} d\omega \\ &= \frac{1}{\zeta - z} \left[\int_C \frac{1}{\omega - \zeta} d\omega - \int_C \frac{1}{\omega - z} d\omega \right] \\ &= \frac{1}{\omega - z} [2\pi i - 2\pi i] = 0 \end{aligned} \quad (5.24)$$

So $g_k(z) = 0$, $\forall k \in \mathbb{N}$, $\forall z \in \Delta$. □

5.3.2 Zeros and poles

Theorem 5.27. *If f is analytic in a region Ω and $\exists a \in \Omega$ s.t. $f^{(n)}(a) = 0$ for $\forall n \in \mathbb{N} \cup \{0\}$, then $f \equiv 0$ in Ω .*

Proof. Let $B(a, R)$ be the disk s.t. $\overline{B(a, R)} \subset \Omega$. Let $C = \partial B(0, R)$.

Taylor's theorem $\Rightarrow f(z) = (z - a)^n f_n(z)$ with

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - a)^n(\omega - z)} d\omega, \quad \forall n \in \mathbb{N} \cup \{0\}, \forall z \in B(a, R) \quad (5.25)$$

Let $M = \max_{z \in C} |f(z)|$.

$$\begin{aligned} \Rightarrow |f_n(z)| &\leq \frac{1}{2\pi} \cdot \frac{M}{R^n(R - |z - a|)} \cdot 2\pi R \\ \Rightarrow |f(z)| &\leq \frac{|z - a|^n}{R^n} \cdot \frac{MR}{R - |z - a|} \rightarrow 0 \text{ as } n \rightarrow \infty, \forall z \in B(0, R) \\ \Rightarrow f(z) &= 0, \forall z \in B(0, R) \end{aligned}$$

Now define

$$\begin{aligned} E_1 &= \{z \in \Omega \mid f^{(n)}(z) = 0, \forall n \in \mathbb{N} \cup \{0\}\} \\ E_2 &= \Omega \setminus E_1 = \{z \in \Omega \mid f^{(n)}(z) \neq 0, \text{ for some } n \in \mathbb{N} \cup \{0\}\} \end{aligned}$$

We just proved E_1 is open. E_2 is open because $f^{(n)}$ is continuous in Ω for $\forall n \in \mathbb{N} \cup \{0\}$. Ω is a region \Rightarrow either $R_1 = \emptyset$ or $R_2 = \emptyset$.

The assumption of the theorem $\Rightarrow E_1 \neq \emptyset \Rightarrow E_1 = \Omega$. □

Let f be analytic in Ω which is not identically zero, $f(a) = 0$ for some $a \in \Omega$. The previous theorem implies \exists first $N \in \mathbb{N}$ s.t. $f^{(N)}(a) \neq 0$. Taylor's theorem implies that $f(z) = (z - a)^N f_N(z)$ where f_N is analytic and $f_N(a) \neq 0$. We say that a is a **zero of order N** of f .

f_N is continuous $\Rightarrow \exists \delta > 0$ s.t. $f(z) \neq 0$ for $\forall z \in B(a, \delta) \setminus \{0\}$.

So we have just proved an important result: Zeros of analytic functions are isolated, or equivalently, we have a famous theorem:

Theorem 5.28 (Identity Theorem). *If f and g are analytic in a region ω , and $f = g$ on a set which has an accumulation point in Ω , then $f(z) = g(z)$.*

Corollary 5.29.

(1) *If $f \equiv 0$ in a subregion of Ω and f is analytic in Ω , then $f \equiv 0$ in Ω .*

(2) If f is analytic in Ω and vanishes on an arc in Ω which doesn't reduce to a point, then $f \equiv 0$ in Ω .

If f is analytic in a neighborhood of a , but perhaps not at a itself, then a is called an **isolated singularity** of f .

If $\lim_{z \rightarrow a} f(z) = \infty$, then a is said to be a **pole** of f , and we set $f(a) = \infty$. Continuity implies $\exists \delta > 0$ s.t. $f(z) \neq 0$ for $\forall z \in B(0, \delta) \setminus \{a\}$. Thus, $g(z) = \frac{1}{f(z)}$ is analytic in $B(a, \delta) \setminus \{a\}$. $\lim_{z \rightarrow a} (z - a)g(z) = 0 \Rightarrow a$ is a removable singularity of g , and g has an analytic extension with $g(a) = 0$. $g \not\equiv 0 \Rightarrow a$ is a zero of g with finite order. The **order of the pole** of f at a is the order N of the zero of g at a .

We can write

$$f(z) = (z - a)^{-N} f_N(z), \forall z \in B(a, \delta) \setminus \{a\} \quad (5.26)$$

where f_N is analytic and nonzero in a neighborhood of a .

Definition 5.30. A function which is analytic in a region Ω except for (isolated) poles is called a **meromorphic function**.

Example 5.31. If f and g are analytic in Ω and $g \not\equiv 0$, then $\frac{f}{g}$ is a meromorphic function in Ω . (See the Identity Theorem 5.28)

Remark 5.32. The sum, the product and quotient (if denominator is not always zero) of two meromorphic functions are meromorphic.

If f has a pole of order N at a , then $(z - a)^N f(z)$ is analytic at a , and Taylor's theorem 5.26 implies

$$(z - a)^N f(z) = b_N + b_{N-1}(z - a) + \cdots + b_1(z - a)^{N-1} + \varphi(z) \cdot (z - a)^N \quad (5.27)$$

where φ is analytic at a .

$$\Rightarrow f(z) = b_N(z-a)^{-N} + b_{N-1}(z-a)^{-(N-1)} + \cdots + b_1(z-a)^{-1} + \varphi(z), \quad \forall z \neq a. \quad (5.28)$$

Theorem 5.33. *If f is analytic in a neighborhood of a , but perhaps not at a itself, then exactly one of the following 3 cases occurs:*

(i) $f \equiv 0$ in this neighborhood.

$$(ii) \exists \text{ integer } N \in \mathbb{Z} \text{ s.t. } \lim_{z \rightarrow a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > N \\ \infty, & \alpha < N \end{cases}$$

(iii) neither $\lim_{z \rightarrow a} |z-a|^\alpha \cdot |f(z)| = 0$ for any $\alpha \in \mathbb{R}$ nor $\lim_{z \rightarrow a} |z-a|^\alpha \cdot |f(z)| = \infty$ for any $\alpha \in \mathbb{R}$

Proof.

① If $\lim_{z \rightarrow a} |z-a|^\alpha \cdot |f(z)| = 0$ for $\forall \alpha \in \mathbb{R}$, then $\lim_{z \rightarrow a} |z-a|^m \cdot |f(z)| = 0$ for \forall integer $m > \alpha$.

$\Rightarrow (z-a)^m f(z)$ has a removable singularity at a and vanishes at $z = a$

\Rightarrow Either $f \equiv 0$ in $B(a, \delta) \setminus \{a\}$, which is case (i), or $(z-a)^m f(z)$ has a zero of

finite order k at $a \Rightarrow \lim_{z \rightarrow a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > m-k \\ \infty, & \alpha < m-k \end{cases}$

② If $\lim_{z \rightarrow a} |z-a|^\alpha |f(z)| = \infty$ for some $\alpha \in \mathbb{R}$, then $\lim_{z \rightarrow a} |z-a|^n \cdot |f(z)| = \infty$ for \forall integer $n < \alpha$.

$\Rightarrow (z-a)^n f(z)$ has a pole of finite order l at a

$$\Rightarrow \lim_{z \rightarrow a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > n+l \\ \infty, & \alpha < n+l \end{cases}$$

□

Remark 5.34. In case (ii), N may be called the **algebraic order** of f at a . $N > 0$ if a is a pole, $N < 0$ if a is a zero, and $N = 0$ if f is analytic at a and $f(a) \neq 0$. The order is always an integer, there is no analytic function which tends to 0 or ∞ , like a fractional power of $|z - a|$.

In some sense, three cases depends on whether $\lim_{z \rightarrow a} (z - a)^N f(z)$ converges for some N .

In case (iii), the point a is an **essential isolated singularity**.

Example 5.35. $f(z) = \exp(\frac{1}{z})$ has an essential isolated singularity $z = 0$.

Theorem 5.36 (Weierstrass). *An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity. Or equivalently, the codomain of f on every neighborhood of an essential singularity is dense in \mathbb{C} .*

Proof. Suppose the statement is false.

$\exists A \in \mathbb{C}, \delta > 0$ and $\varepsilon > 0$ s.t.

$$|f(z) - A| > \delta, \forall z \text{ with } 0 < |z - a| < \varepsilon \quad (5.29)$$

$\Rightarrow \lim_{z \rightarrow a} |z - a|^\alpha \cdot |f(z) - A| = \infty$ for $\forall \alpha < 0$. $\Rightarrow a$ is not an essential singularity of $f(z) - A$.

The previous theorem $\Rightarrow \exists \beta \in \mathbb{R}$ s.t. $\lim_{z \rightarrow a} |z - a|^\beta \cdot |f(z) - A| = 0$, and we may choose $\beta > 0$.

Then $\lim_{z \rightarrow a} |z - a|^\beta \cdot |A| = 0 \Rightarrow \lim_{z \rightarrow a} |z - a|^\beta \cdot |f(z)| = 0$ by the triangular inequality.

So a is not an essential singularity of f , which causes contradiction!

So the statement has to be true. □

Remark 5.37. If f is analytic in $|z| > R$. We treat ∞ as an isolated singularity. Removable singularity, pole or essential singularity of f at ∞ is defined according to $g(z) = f(\frac{1}{z})$ at $z = 0$.

5.3.3 The Local Mappings

Theorem 5.38 (The Argument Principle). *Let f be analytic in a disk Δ s.t. f does not vanish identically. Let z_j be the zeros of f , each zero being counted as many times as its order indicates. For every closed curve γ in Δ which does not pass through a zero, we have*

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz \quad (5.30)$$

where the sum has only a finite number of terms with nonzero value.

Proof.

Case I: f has exactly n zeros z_1, \dots, z_n .

By repeated application of Taylor' Theorem 5.26, we can write

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)g(z), \quad z \in \Delta \quad (5.31)$$

where g is analytic in Δ and $g(z) \neq 0$ for $\forall z \in \Delta$. \Rightarrow

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}, \quad \forall z \in \Delta \text{ and } z \neq z_j \quad (5.32)$$

Cauchy' Theorem 5.10 \Rightarrow

$$\int_{\gamma} \frac{g'(z)}{g(z)} dz = 0 \Rightarrow \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^n n(\gamma, z_j) \quad (5.33)$$

Case II: f has infinitely many zeros in Δ . Then γ is inside a concentric disk Δ' smaller than Δ .

$f \not\equiv 0 \Rightarrow$ There is only a finite number of zeros in Δ' .

So we can apply (5.33) to the disk $\Delta' \Rightarrow$ (5.30) holds since $n(\gamma, z_j) = 0$ if $z \notin \Delta'$. □

Remark 5.39.

- The function $\omega = f(z)$ maps γ onto a closed curve Γ in the ω -plane, and we have

$$\int_{\Gamma} \frac{d\omega}{\omega} = \int_{\gamma} \frac{f'(z)}{f(z)} dz \quad (5.34)$$

Then (5.30) can be interpreted as $n(\Gamma, 0) = \sum_j n(\gamma, z_j)$.

- The most useful application of the theorem is to the case when γ is a circle (or more generally a simple closed curve). So that

$$n(\gamma, z) = \begin{cases} 1, & z \text{ is inside } \gamma \\ 0, & z \text{ is outside } \gamma \end{cases} \quad \text{Then (5.30) yields a formula for the total number of zeros enclosed by } \gamma.$$

Let $a \in \mathbb{C}$. Apply the previous theorem to $f(z) - a$

$$\sum_j n(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - a} dz$$

where $z_j(a)$ are zeros of $f - a$ (or roots of $f(z) = a$), and γ is a closed curve in Δ which doesn't pass $z_j(a) \Rightarrow$

$$n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$$

If a and b are in the same region determined by Γ , then $n(\Gamma, a) = n(\Gamma, b) \Rightarrow$

$$\sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b)) \quad (5.35)$$

If γ is a circle, then f takes the values a and b equally many times inside γ , counted as many times as their orders indicate.

We have the equation that

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - a} dz &= n(\Gamma, a) = n(\Gamma, b) \\
&= \frac{1}{2\pi i} \int_{\Gamma} \frac{d\omega}{\omega - b} = \frac{1}{2\pi i} \frac{f'(z) dz}{f(z) - b} \\
&= \text{card}\{z \text{ inside } \gamma : f(z) = b\}
\end{aligned} \tag{5.36}$$

Theorem 5.40. Suppose f is analytic at z_0 , and $f(z) - \omega_0$ has a zero of order $N \in \mathbb{N}$ at z_0 . Then for $\forall \varepsilon > 0$ sufficiently small, $\exists \delta > 0$ s.t. for $\forall a$ with $|a - \omega_0| < \delta$, the equation $f(z) = a$ has exactly N roots in the disk $|z - z_0| < \varepsilon$

Proof. We choose $\varepsilon > 0$ s.t.

- (1) f is analytic in $|z - z_0| \leq \varepsilon$
- (2) z_0 is the only zero of $f(z) - \omega_0$ in this disk.
- (3) $f'(z) \neq 0$ for $\forall z$ with $0 < |z - z_0| < \varepsilon$

Let γ be the circle $|z - z_0| < \varepsilon$ and $\Gamma = f \circ \gamma$.

$$\omega_0 \notin \Gamma \Rightarrow \exists \delta > 0 \text{ s.t. } B(\omega_0, \delta) \cap \Gamma = \emptyset.$$

The consequence of the argument principle 5.38, i.e. (5.36) $\Rightarrow f$ takes all values $a \in B(\omega_0, \delta)$ the same number of times N inside γ , since $f(z) = \omega_0$ has exactly N coinciding roots inside γ .

$$(3) \Rightarrow \text{all roots } f(z) = a \text{ with } a \in B(\omega_0, \delta) \setminus \{\omega_0\} \text{ are simple} \quad \square$$

Corollary 5.41 (open mapping theorem). A nonconstant analytic function maps open sets onto open sets.

Proof. The previous theorem $\Rightarrow \forall \varepsilon > 0, f(B(z_0, \varepsilon)) \supset B(\omega_0, \delta)$ \square

Corollary 5.42. *If f is analytic at z_0 with $f'(z_0) \neq 0$. It maps a neighborhood of z_0 conformally and topologically onto a region.*

Proof. This is the case $N = 0$. The previous theorem \Rightarrow There is 1-1 corresponding between the disk $|\omega - \omega_0| < \delta$ and an open subset of $|z - z_0| < \varepsilon$. The open mapping theorem 5.41 $\Rightarrow f^{-1}$ is continuous $\Rightarrow f$ is a topological map. And f is conformal on $|z - z_0| < \varepsilon$ \square

Remark 5.43. Under the assumption of Corollary 5.42, f^{-1} is continuous $\Rightarrow f^{-1}$ is analytic $\Rightarrow f^{-1}$ is conformal map.

If $f : \Omega \rightarrow \mathbb{C}$ is 1-1 and analytic, Theorem 5.40 can hold only with $N = 1 \Rightarrow f'(z) \neq 0$ for $\forall z \in \mathbb{C}$. So this condition is stronger than the conformal condition.

5.3.4 The Maximum Principle

Theorem 5.44 (The maximum principle). *If f is analytic and nonconstant in a region Ω , then its modulus $|f|$ has no maximum in Ω .*

Proof. $\forall z_0 \in \Omega$, the open mapping theorem 5.41 $\Rightarrow \exists$ an open disk $|\omega - f(z_0)| < \delta$ contained in $F(\Omega)$. In this disk, $\exists \omega$ s.t. $|\omega| > |f(z_0)| \Rightarrow |f(z_0)|$ is not the maximum of $|f|$. \square

Theorem 5.45 (The maximum principle). *If f is defined and continuous on a closed bounded set E and analytic in the interior of E , then the maximum of $|f|$ on E is assumed on the boundary of E .*

Remark 5.46. The maximum principle can also be proved by the mean value theorem 5.18 for analytic functions.

Theorem 5.47 (Schwarz Lemma). *If f is analytic in the disk $|z| < 1$ and satisfies $f(0) = 0$, $|f(z)| \leq 1$, $\forall z \in B(0, 1)$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Furthermore, if $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ where $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. We define $g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0, z \in B(0, 1) \\ f'(0), & z = 0 \end{cases}$.

Then g is analytic with $g'(0) = \frac{f'(0)}{2}$ using Taylor series (5.20).

The maximum principle implies that $|g(z)| \leq \frac{1}{r}, \forall z \in \overline{B(0, r)}$ where $0 < r < 1$.

Setting $r \rightarrow 1$, we get $|g(z)| \leq 1, \forall |z| < 1$.

If $|f(z)| = |z|$ for some $z \neq 0$, or $|f'(0)| = 1$, then $|g| = 1$ attains its maximum at some interior points. By maximum principle, g has to be a constant. \square

Remark 5.48. For a general analytic function $f : B(0, R) \rightarrow B(0, M), z_0 \mapsto w_0$.

$$\text{Let } T(z) = \frac{\frac{z}{R} - \frac{z_0}{R}}{1 - \frac{\bar{z}_0}{R} \cdot \frac{z}{R}}$$

$$S(\omega) = \frac{\frac{\omega}{M} - \frac{\omega_0}{M}}{1 - \frac{\bar{\omega}_0}{M} \cdot \frac{\omega}{M}}.$$

Then $S \circ f \circ T^{-1}$ satisfies $S \circ f \circ T^{-1}(0) = 0$ and $|S \circ f \circ T^{-1}(z)| \leq 1 \xrightarrow{\text{Schwarz lemma}}$

$$|S \circ f \circ T^{-1}(\zeta)| \leq |\zeta|.$$

$$\Rightarrow |S \circ f(z)| \leq |T(z)| \Rightarrow$$

$$\left| \frac{M(f(z) - \omega_0)}{M^2 - \bar{\omega}_0 f(z)} \right| \leq \left| \frac{R(z - z_0)}{R^2 - \bar{z}_0 z} \right|, \forall z \in B(0, R)$$

5.4 The General Form of Cauchy's Theorem

5.4.1 Chains and Cycles

Let $\Omega \subset \mathbb{C}$ be open. Let $\gamma_j : [\alpha_j, \beta_j] \rightarrow \Omega$ be piecewise continuously differentiable curves in Ω . The sum $\gamma_1 + \gamma_2 + \cdots + \gamma_N$, which need not be a curve is called a **chain**. The **integral** of a continuous f in Ω along this chain is defined by

$$\int_{\gamma_1 + \gamma_2 + \cdots + \gamma_N} f = \sum_{j=1}^N \int_{\gamma_j} f. \quad (5.37)$$

Two chains are **identical** if they yield the same line integrals for all function f .

A chain is a **cycle** if it can be represented as a finite sum of closed curves.

5.4.2 Simple connectivity and homology

A region is **simply connected** if its complement w.r.t. $\hat{\mathbb{C}}$ is connected.

Example 5.49. A disk, a half plane, a parallel strip are simply connected.

$\mathbb{C} \setminus \overline{B(0,1)}$ is not simply connected since its complement w.r.t. $\hat{\mathbb{C}}$ consists of $\overline{B(0,1)}$ and ∞ .

Theorem 5.50. A region $\Omega \subset \mathbb{C}$ is simply connected iff $n(\gamma, z) = 0$ for all cycles γ in Ω and all points $z \notin \Omega$.

Proof. " \Rightarrow ": \forall cycle $\gamma \subset \Omega$, $\hat{\mathbb{C}} \setminus \Omega$ must be in one of the regions in $\hat{\mathbb{C}} \setminus \gamma$ since $\hat{\mathbb{C}} \setminus \Omega$ is connected.

$\infty \in \hat{\mathbb{C}} \setminus \Omega \Rightarrow \mathbb{C} \setminus \Omega$ is in the unbounded region of $\mathbb{C} \setminus \gamma$. By theorem 5.13 $n(\gamma, z) = 0, \forall z \in \mathbb{C} \setminus \Omega$.

" \Leftarrow ": Suppose Ω is not simply connected, i.e., $\hat{\mathbb{C}} \setminus \Omega$ is not connected. Let $\hat{\mathbb{C}} \setminus \Omega = A \sqcup B$ with A, B disjoint closed sets.

Suppose that $\infty \in B$. Then A is the bounded set. δ is defined to be the distance between A and B . The $\delta > 0$. Cover A with a net of squares Ω of side less than $\frac{\delta}{\sqrt{2}}$.

Suppose $z_0 \in A$ lies at the center of a square cycle $\gamma := \sum_{Q: Q \cap A \neq \emptyset} \partial\Omega$.

z_0 is only in one of these squares $\Rightarrow n(\gamma, z_0) = 1$.

Since sides of squares are less than $\frac{\delta}{\sqrt{2}}, \gamma \cap B \neq \emptyset$.

$\gamma \cap A = \emptyset$ after cancellations of the multiple sides.

$\Rightarrow \gamma \in \Omega$ with $n(\gamma, z_0) = 1$. That's a contradiction. \square

A cycle γ in an open set Ω is said to be **homologous to zero** w.r.t. Ω if $n(\gamma, z) = 0$ for $\forall z \in \mathbb{C} \setminus \Omega$.

In symbols, we write $\gamma \sim 0(\text{mod } \Omega)$. So $\gamma_1 \sim \gamma_2$ means $\gamma_1 - \gamma_2 \sim 0(\text{mod } \Omega)$.

5.4.3 The general form of Cauchy's theorem

Theorem 5.51 (General form of Cauchy's theorem). *If f is analytic in an open set Ω , then $\int_{\gamma} f(z)dz = 0$ for \forall cycle γ which is homologous to zero in Ω .*

In combination with the theorem 5.50 in the previous section, we have

Corollary 5.52. *If f is analytic in a simply connected region Ω , then $\int_{\gamma} f(z)dz = 0$ for all cycles γ in Ω .*

In combination with the fundamental theorem 5.5 of Calculus for integrals in \mathbb{C} , we have

Corollary 5.53. *If f is analytic in a simply connected region Ω , then \exists an analytic function F in Ω s.t. $F'(z) = f(z)$ for $\forall z \in \Omega$.*

Corollary 5.54. *If f is analytic in a simply connected region Ω and $f(z) \neq 0$ for $\forall z \in \Omega$, then it is possible to define single-valued analytic branches of $\ln f(z)$ and $\sqrt[n]{f(z)}$ in Ω*

Proof. $\frac{f'(z)}{f(z)}$ is analytic in Ω $\xrightarrow{\text{Corollary 5.53}}$ \exists an analytic function F s.t. $F'(z) = \frac{f'(z)}{f(z)}$, $\forall z \in \Omega$.

$$\Rightarrow \frac{d}{dz} [f(z)e^{-F(z)}] = 0, \forall z \in \Omega \Rightarrow f(z) = C \cdot e^{F(z)} \text{ for some } C \in \mathbb{C} \setminus \{0\}.$$

Choose $z_0 \in \Omega$ and one of the infinite values of $\ln f(z_0)$.

$$\Rightarrow \exp [F(z) - F(z_0) + \ln f(z_0)] = \frac{f(z)}{C} \cdot e^{-F(z_0)} = f(z), \forall z \in \Omega.$$

We may define $\ln f(z) = F(z) - F(z_0) + \ln f(z_0)$, $\sqrt[n]{f(z)} = \exp \left[\frac{1}{n} \ln f(z) \right]$.

□

Proof of Cauchy's Theorem 5.51. Let γ be a cycle in Ω satisfying $\gamma \sim 0 \pmod{\Omega}$. The theorem in 5.13 implies that

$$E := \{z \in \mathbb{C} \setminus \gamma : n(\gamma, z) = 0\} \text{ is open}$$

We define $g : \Omega \times \Omega \rightarrow \mathbb{C}$ by

$$g(z, \zeta) := \begin{cases} \frac{f(z) - f(\zeta)}{z - \zeta}, & z \neq \zeta \\ f'(z), & z = \zeta \end{cases} \quad (5.38)$$

Taylor's theorem implies g is continuous in $(z, \zeta) \in \Omega \times \Omega$. For $\forall \zeta_0 \in \Omega$, $g(z, \zeta_0)$ is analytic in Ω since $\lim_{z \rightarrow \zeta_0} (z - \zeta_0)g(z, \zeta_0) = 0$.

Define $h(z) = \begin{cases} \frac{1}{2\pi i} \int_{\gamma} g(z, \zeta) d\zeta, & z \in \Omega \\ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - \zeta} d\zeta, & z \in E \end{cases}$. $\gamma \sim 0 \Rightarrow n(\gamma, z) = 0, \forall z \in \mathbb{C} \setminus \Omega \Rightarrow C \setminus \Omega \subset E \Rightarrow \Omega \cup E = \mathbb{C}$. So h is defined on \mathbb{C} .

These two expressions are equal on $\Omega \cap E$ since $n(\gamma, z) = 0, \forall z \in \Omega \cap E$.

Lemma 5.19 implies that h is analytic in E .

The last exercise in Homework 6 $\Rightarrow h$ is analytic on $\Omega \Rightarrow h$ is entire.

$n(\gamma, z) = 0$ if $|z|$ is sufficiently large $\Rightarrow z \in E$ if $|z|$ large enough.

f is bounded on $\gamma \Rightarrow h(z) \rightarrow 0$ as $|z| \rightarrow \infty \Rightarrow h$ is bounded and thus $h \equiv 0$. By

Liouville's Theorem 5.22,

$\frac{1}{2\pi i} \int_{\gamma} g(z, \zeta) d\zeta = 0, \forall z \in \Omega \setminus \gamma$. Then

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \forall z \in \Omega \setminus \gamma \quad (5.39)$$

Equation 5.39 is the **generalized version of Cauchy's integral formula**.

Let $z_0 \in \Omega \setminus \gamma$. Define $h_1(z) = (z - z_0)f(z)$. Then h_1 analytic and

$$\int_{\gamma} f(z) dz = \int_{\gamma} \frac{h_1(z)}{z - z_0} dz \stackrel{(5.39)}{=} 2\pi i \cdot n(\gamma, z_0) \cdot h_1(z_0) = 0 \quad (5.40)$$

□

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