We have proved $C(A) = C(A^T), N(A) = N(A^T)$, so there exsts a invertible $P \in GL_n(K)$, such that $A = PA^T$, then $A^T = (PA^T)^T = AP^T$.

Then $A^2 = AA^T$ implies $A^2 = AAP^T \Rightarrow A^2(I_n - P^T) = 0$ which implies $A(I_n - P^T) = 0$ (for $N(A^2) = N(AA^T) = N(A^T) = N(A)$).

i.e. $A = AP^T = A^T$

Similar for $A^2 = -AA^T$(replace $I_n - P^T$ to $I_n + P^T$)