Tsinghua University

Numberical Analysis

Fall 2024

Homework 6

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- **Collaborators:** I finish this homework by myself. However, I ask matlab for help when I solve the Problem 5.

Problem 1.

$$\begin{aligned}\hat{r} - r &= Ax - A\hat{x} \\ &= A(A^TA)^{-1}A^Tb - A(A^TA+F)^{-1}A^Tb \\ &= A(A^TA+F)^{-1}\left((A^TA+F)-(A^TA)\right)(A^TA)^{-1}A^Tb \\ &= A(A^TA+F)^{-1}Fx\end{aligned}$$

We know from the lecture that

$$\mathcal{K}_2(A)^2 = \mathrm{cond}_2(A^TA) = \frac{|\rho(A^TA)|}{|\sigma_n(A)|^2} = \frac{||A||_2^2}{|\sigma_n(A)|^2}$$

Noticed that $||(A^TA)^{-1}||_2 = ||A^+||_2^2 = \dfrac{1}{|\sigma_n(A)|^2} \le \dfrac{1}{2||F||_2}$, hence

$$||(A^TA+F)^{-1}||_2 \le \frac{||(A^TA)^{-1}||_2}{1-||F||_2\cdot||(A^TA)^{-1}||_2} \le \frac{||A^+||_2^2}{\frac{1}{2}}$$

Therefore

$$\begin{aligned}||\hat{r} - \hat{r}||_2 &\le ||A||_2 \cdot ||(A^TA+F)^{-1}||_2 \cdot ||F||_2 \cdot ||x||_2 \\ &\le 2||A||_2 \cdot ||A^+||_2^2 \cdot ||F||_2 \cdot ||x||_2 \\ &= 2\mathcal{K}_2(A)^2\frac{||F||_2}{||A||_2}||x||_2\end{aligned}$$

Problem 2.

$$\begin{aligned}
||x-\hat{x}||_2 &= ||(A^TA)^{-1}f||_2 \\
&\le ||(A^TA)^{-1}||\cdot||f||_2 \\
&\le ||A^+||_2^2\cdot\epsilon||A||_2||b||_2 \\
&= \epsilon\mathcal{K}_2(A)^2\frac{||A||_2||b||_2}{||A||_2^2} \\
&= \epsilon\mathcal{K}_2(A)^2\frac{||A||_2||b||_2}{||A^TA||_2}
\end{aligned}$$

Then

$$\frac{||x-\hat{x}||_2}{||x||_2}\le\epsilon\mathcal{K}_2(A)^2\frac{||A||_2||b||_2}{||A^TA||_2||x||_2}\le\epsilon\mathcal{K}_2(A)^2\frac{||A||_2||b||_2}{||A^TAx||_2}=\epsilon\mathcal{K}_2(A)^2\frac{||A||_2||b||_2}{||A^Tb||_2}$$

Problem 3. We already have method tp compute A^TA, A^Tb, and we do Gauss elimination to it to get an equation:

$$Ux=b'$$

where U is upper triangular matrix, with

$$U=\begin{bmatrix}U' & B\\ 0 & 0\end{bmatrix}, b'=\begin{bmatrix}\hat{b}'\\ 0\end{bmatrix}$$

U' is the upper triangular matrix with all diagonal elements 1.
Then solutions of $A^TAx=A^Tb$ are

$$x=\begin{bmatrix}U^{-1}(\hat{b}'-Bx')\\ x'\end{bmatrix}=\begin{bmatrix}U^{-1}\hat{b}'\\ 0\end{bmatrix}-\begin{bmatrix}U^{-1}B\\ -I\end{bmatrix}x'$$

Therefore the solution of least norm is exactly the least square solution of

$$\begin{bmatrix}U^{-1}B\\ -I\end{bmatrix}x'=\begin{bmatrix}U^{-1}\hat{b}'\\ 0\end{bmatrix}$$