Tsinghua University

Numerical Analysis

Fall 2024

Homework 7

Lin Zejin 2025 年 5 月 21 日

Problem 1.

$$\begin{aligned}
T(x,y;h) &= y(x+h) - y(x) - hf(x_n + \frac{h}{2}, y_n + \frac{1}{2}hf(x_n, y_n)) \\
&= hy'(x) + \frac{1}{2}h^2 y''(x) + O(h^3) - h(f(x,y) + \frac{h}{2}\cdot\frac{\partial f}{\partial x} + \frac{h}{2}f(x,y)\frac{\partial f}{\partial y}) \\
&= \frac{h^2}{2}y''(x) - \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\partial f}{\partial y}\right) + O(h^3) \\
&= O(h^3)
\end{aligned}$$

Problem 2.

$$\begin{aligned}
T(x,y;h) &= y(x+h) - y(x) - hf(x+h, y+h) \\
&= hy'(x) - h(f(x,y) + h\frac{\partial f}{\partial x} + h\frac{\partial f}{\partial y}) + O(h^2) \\
&= O(h^2)
\end{aligned}$$

所以一阶相容

$$y_{n+1} = y_n + f(x_{n+1}, y_{n+1})h$$

$$y(x_{n+1}) = y(x_n) + f(x_{n+1}, y(x_{n+1}))h + O(h^2)$$

则

$$\begin{aligned}
e_{n+1} - e_n &= f(x_{n+1}, y_{n+1})h - f(x_{n+1}, y(x_{n+1}))h + O(h^2) \\
&\le hL|e_n| + hO(h)
\end{aligned}$$

由引理知

$$E(h) \le O(h)$$

故他 1 阶收敛

Problem 3.

$$k_1 = h\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\begin{aligned}
k_2 &= hf(x+\alpha h, y+\alpha k_1) \\
&= h\left[f(x,y) + h\alpha\frac{\partial f}{\partial x} + k_1\alpha\frac{\partial f}{\partial y} + \frac{h^2\alpha^2}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\tilde{k}_1^2 + 2\frac{\partial^2 f}{\partial x\partial y}\tilde{k}_1\right)\right] + O(h^4) \\
&= hy'(x) + h^2\alpha y''(x) + \frac{1}{2}h^3\alpha^2\left(\frac{\partial^2 f}{\partial x^2} + 2y'\frac{\partial^2 f}{\partial x\partial y} + (y')^2\frac{\partial^2 f}{\partial y^2}\right) + O(h^4)
\end{aligned}$$

where $\tilde{k}_1 = k_1/h$

$$\begin{aligned}
k_3 &= hf(x+(1-\alpha)h, y+(1-\alpha)k_2) \\
&= h\left[f(x,y) + h(1-\alpha)\frac{\partial f}{\partial x} + k_2(1-\alpha)\frac{\partial f}{\partial y} + \frac{h^2(1-\alpha)^2}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\tilde{k}_2^2 + 2\frac{\partial^2 f}{\partial x \partial y}\tilde{k}_2\right)\right] + O(h^4) \\
&= hy'(x) + h^2(1-\alpha)y''(x) + \frac{1}{2}h^3(1-\alpha)^2\left(\frac{\partial^2 f}{\partial x^2} + 2y'\frac{\partial^2 f}{\partial x \partial y} + (y')^2\frac{\partial^2 f}{\partial y^2}\right) \\
&+ h^3(1-\alpha)\alpha y''(x)\frac{\partial f}{\partial y} \\
&+ O(h^4)
\end{aligned}$$

where $\tilde{k}_2 = k_2/h$

注意到

$$y'''(x) = \left(\frac{\partial f}{\partial x} + y'(x)\frac{\partial f}{\partial y}\right)' = \frac{\partial^2 f}{\partial x^2} + 2y'\frac{\partial^2 f}{\partial x \partial y} + (y')^2\frac{\partial^2 f}{\partial y^2} + y''\frac{\partial f}{\partial y}$$

所以

$$\begin{aligned}
T(x,y;h) &= y(x+h) - y(x) - \frac{1}{2}(k_2+k_3) \\
&= y'(x)h + y''(x)\frac{h^2}{2} + y'''(x)\frac{h^3}{6} - \frac{1}{2}(k_2+k_3) + O(h^4) \\
&= \frac{h^3}{6}y'''(x) + O(h^4) - O(h^3)
\end{aligned}$$

故他是 2 阶相容

Problem 4.

$$\begin{aligned}
T(x,y;h) &= y(x+h) - y(x) - \frac{h^2}{2}g(x+\frac{1}{3}h, y+\frac{1}{3}hf(x,y)) - hf(x,y) \\
&= hy'(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y'''(x) - \frac{h^2}{2}\left[g(x,y) + \frac{1}{3}h\left(\frac{\partial^2 f}{\partial x^2} + 2y'\frac{\partial^2 f}{\partial x \partial y} + (y')^2\frac{\partial^2 f}{\partial y^2} + y''\frac{\partial f}{\partial y}\right)\right] - hy' + O(h^4) \\
&= O(h^4)
\end{aligned}$$

故他是 3 阶相容。

类似问题 2 的操作，我们很容易将定理 8.3.2 扩展到 3 阶相容。即由

$$|e_{n+1}| \le (1+hL)|e_n| + h\alpha(h)$$

结合引理知

$$|e_n| \le \frac{\alpha(h)}{L}(e^{L(b-a)} - 1) = O(h^3)$$

故他是 3 阶收敛