Tsinghua University

Numerical Analysis

Fall 2024

Homework 1

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- **Collaborators:** I finish this homework by myself.

Problem 1. *Proof of Lemma 2.5.3.* $B_1 = \begin{cases} 1 & x \leq 0 \\ 0 & x \geq 1 \\ 1-x & 0 < x < 1 \end{cases}$.

Then $B_2 = \begin{cases} 1 & x \leq -\frac{1}{2} \\ 0 & x \geq \frac{3}{2} \\ -\frac{x^2}{2} - \frac{1}{2}x + \frac{7}{8} & -\frac{1}{2} < x < \frac{1}{2} \\ \frac{1}{2}(\frac{3}{2} - x)^2 & \frac{1}{2} < x < \frac{3}{2} \end{cases}$

So $B_3 = \begin{cases} 0 & x \leq -1 \\ -\dfrac{t^3 + 3\,t^2 - 3\,t - 5}{6} & -1 < x < 0 \\ \dfrac{t^3}{3} - \dfrac{1}{2}t^2 - \dfrac{1}{2}t + \dfrac{5}{6} & 0 < x < 1 \\ \dfrac{(2-t)^3}{6} & 1 < x < 2 \\ 1 & x \geq 2 \\ 3 & -1 < x < 2 \end{cases}$ with B_2

So $B_3(x), B_3(x-1), B_3(x-2)$ linearly independent. □

Proof of Lemma 2.5.4. If

$$\sum_{k=-1}^{n+1} \lambda_k B_3(\frac{x-a-kh}{h}) = 0$$

Noticed that

$$B_3(\frac{x-a-kh}{h}) = B_3(\frac{x-a}{h} - k)$$

So on each interval $(a+jh, a+(j+1)h)$, it will be

$$\sum_{k=-1}^{n+1} \lambda_k B_3(j-k+\frac{x-jh-a}{h}) = \lambda_{j-1}B_3(-1+\frac{x-jh-a}{h}) + \lambda_j B_3(0+\frac{x-jh-a}{h}) + \lambda_{j+1}B_3(1+\frac{x-jh-a}{h}) = 0$$

where $\dfrac{x-jh-a}{h} \in (0,1)$

So we have matrix equation

$$\begin{pmatrix} B_3(-1+t) & B_3(t) & B_3(1+t)\cdots & \cdots & \cdots & 0 & \\ 0 & B_3(-1+t) & B_3(t) & B_3(1+t)\cdots & \cdots & 0 & \\ 0 & 0 & B_3(-1+t) & B_3(t) & B_3(1+t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & B_3(-1+t) & B_3(t) & B_3(1+t) \\ 0 & 0 & 0 & 0 & \cdots & B_3(-1+t) & B_3(t) \\ 0 & 0 & 0 & 0 & \cdots & 0 & B_3(-1+t) \end{pmatrix} \begin{pmatrix} \lambda_{-1} \\ \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \\ \lambda_n \end{pmatrix} = 0$$

on $(0,1)$ By lemma 2.5.3, we have $\lambda_{-1} = \lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. So they are linearly independent. □

Proof of lemma 2.6.2. We have proved in lemma 2.6.1 that

$$\sum |c_i| \leq \frac{\|P\|_Y}{\theta - \Omega(\delta)}$$

Then

$$\begin{aligned} \|f-P\|_X &\leq \max_{y\in Y, x\in X} |f(x)-f(y)| + |f(y)-P(y)| + |P(y)-P(x)| \\ &\leq \omega(\delta, f) + \|f-P\|_Y + \sum |c_i|\Omega(\delta) \\ &\leq \omega(\delta, f) + \|f-P\|_Y + \frac{\|P\|_Y \Omega(\delta)}{\theta - \Omega(\delta)} \end{aligned}$$

As $\Omega(\delta) \to 0$, $\dfrac{2}{\theta}$ is what we need. $\theta > 0$ since g_i linearly independent. □

Problem 2.

$$N_{i,0}(u) = \begin{cases} 1 & \text{if } u_i \leq u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

Easy to check the facts that

- $N_{i,p}(u)$ is non-zero on $[u_i, u_{i+p+1})$.
- On each interval $[u_i, u_{i+1})$, there is at most $p+1$ $N_{i,p}(u)$ for some i.

Therefore, $N_{i,3}$ is the basis of the B-spine space.

Problem 3. For f odd.

Consider the optimal approximation polynomial p on $[-1,1]$. Since $f(x)=-f(-x)$, so $-p(-x)$ is also the optimal approximation polynomial of $-f(-x)=f(x)$ on $[-1,0]$ $\Rightarrow$ By the uniqueness, $p(x)=-p(-x)$ *i.e.* $p(x)$ is odd.
For f is even
Consider the optimal approximation polynomial p on $[-1,1]$. Since $f(x)=f(-x)$, so $p(-x)$ is also the optimal approximation polynomial of $f(-x)=f(x)$ on $[-1,0]$ $\Rightarrow$ By the uniqueness, $p(x)=p(-x)$ *i.e.* $p(x)$ is even.

Problem 4. For $s(x_j)=m_j$, $s(x)$ can be expressed as

$$s(x)=m_j\alpha_j(x)+m_{j+1}\alpha_{j+1}(x)+y_j\beta_j(x)+y_{j+1}\beta_{j+1}(x)$$

on (x_j,x_{j+1}), where α_i,β_j is the Hermite polynomial of degree 3.
Use $s''(x_j^+)=s''(x_j^-)$ and we have

$$\frac{\mu_j}{h_j}m_{j+1}+(\frac{\lambda_j}{h_{j-1}}-\frac{\mu_j}{h_j})m_j-\frac{\lambda_j}{h_{j-1}}m_{j-1}=d_j$$

where

$$d_j=\frac{1}{3}\left(\lambda_j y_{j-1}+2y_j+\mu_j y_{j+1}\right),\lambda_j=\frac{h_j}{h_{j-1}+h_j},\mu_j=\frac{h_{j-1}}{h_{j-1}+h_j},h_j=x_{j+1}-x_j$$

The boundary condition $s(x_0)=m_0=a, s(x_n)=m_n=b$ contributes to

$$\begin{pmatrix}
\frac{\lambda_1}{h_0}-\frac{\mu_1}{h_1} & \frac{\mu_1}{h_1} & 0 & \cdots & 0 & 0\\
-\frac{\lambda_2}{h_1} & \frac{\lambda_2}{h_1}-\frac{\mu_2}{h_2} & \frac{\mu_2}{h_2} & \cdots & 0 & 0\\
0 & -\frac{\lambda_3}{h2} & \frac{\lambda_3}{h_2}-\frac{\mu_3}{h_3} & \frac{\mu_3}{h_3} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & -\frac{\lambda_{n-1}}{h_{n-2}} & \frac{\lambda_{n-1}}{h_{n-2}}-\frac{\mu_{n-1}}{h_{n-1}}
\end{pmatrix}\cdot\begin{pmatrix}m_1\\m_2\\\vdots\\m_{n-1}\end{pmatrix}=\begin{pmatrix}d_1+\frac{\lambda_1}{h_0}a\\d_2\\\vdots\\d_{n-1}-\frac{\mu_{n-1}}{h_{n-1}}b\end{pmatrix}$$

The boundary condition $s''(x_0)=y_0', s''(x_n)=y_n'$ contributes to

$$-2y_0-y_1+\frac{3}{h_0}m_1-\frac{3}{h_0}m_0=\frac{1}{2}y_0'h_0$$

So we have

$$\begin{pmatrix}
-\frac{3}{h_0} & \frac{3}{h_0} & 0 & \cdots & 0 & 0 & 0\\
-\frac{\lambda_1}{h_0} & \frac{\lambda_1}{h_0}-\frac{\mu_1}{h_1} & \frac{\mu_1}{h_1} & 0 & \cdots & 0 & 0\\
0 & -\frac{\lambda_2}{h_1} & \frac{\lambda_2}{h_1}-\frac{\mu_2}{h_2} & \frac{\mu_2}{h_2} & \cdots & 0 & 0\\
0 & 0 & -\frac{\lambda_3}{h_2} & \frac{\lambda_3}{h_2}-\frac{\mu_3}{h_2} & \frac{\mu_3}{h_3} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & -\frac{\lambda_{n-1}}{h_{n-2}} & \frac{\lambda_{n-1}}{h_{n-2}}-\frac{\mu_{n-1}}{h_{n-1}}\\
0 & 0 & \cdots & 0 & 0 & -\frac{3}{h_{n-1}} & \frac{3}{h_{n-1}}
\end{pmatrix}\cdot\begin{pmatrix}m_0\\m_1\\m_2\\\vdots\\m_{n-1}\\m_n\end{pmatrix}=\begin{pmatrix}2y_0+y_1+\frac{1}{2}y_0'h_0\\d_1\\d_2\\\vdots\\d_{n-1}\\2y_n+y_{n-1}+\frac{1}{2}y_n'h_{n-1}\end{pmatrix}$$

And the period boudary condtion $s(x_0) = s(x_n), s'(x_0) = s'(x_n), s''(x_0) = s''(x_n)$ contributes to

$$m_0 = m_n, \frac{\mu_j}{h_j} m_{j+1} + (\frac{\lambda_j}{h_{j-1}} - \frac{\mu_j}{h_j}) m_j - \frac{\lambda_j}{h_{j-1}} m_{j-1} = d_j$$

for $j = 0$, $\mu_{-1} = \mu_n$.

So

$$\begin{pmatrix} \frac{\lambda_1}{h_0} - \frac{\mu_1}{h_1} & \frac{\mu_1}{h_1} & 0 & \cdots & 0 & -\frac{\lambda_1}{h_0} \\ -\frac{\lambda_2}{h_1} & \frac{\lambda_2}{h_1} - \frac{\mu_2}{h_2} & \frac{\mu_2}{h_2} & \cdots & 0 & 0 \\ 0 & -\frac{\lambda_3}{h2} & \frac{\lambda_3}{h_2} - \frac{\mu_3}{h_3} & \frac{\mu_3}{h_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\mu_{n-1}}{h_{n-1}} & 0 & \cdots & 0 & -\frac{\lambda_{n-1}}{h_{n-2}} & \frac{\lambda_{n-1}}{h_{n-2}} - \frac{\mu_{n-1}}{h_{n-1}} \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{n-1} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{pmatrix}$$

In short, we discuss about three different boundary condition in this problem.

Problem 5. The boundary condition is

$$2m_0 + m_1 = 3f[x_0, x_1] - \frac{1}{2} f''(x_0) h$$

So combined with the equation in Lemma 2.5.2, we have

$$Am = d$$

where

$$A = \begin{pmatrix} 2 & 1 & & & & \\ \lambda_1 & 2 & \mu_1 & & & \\ 0 & \lambda_2 & 2 & \mu_2 & & \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \lambda_{n-1} & 2 & \mu_{n-1} \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$d = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{pmatrix}$$

$d_j = 3\lambda_j f[x_{j-1}, x_j] + 3\mu_j f[x_j, x_{j+1}]$, $1 \le j \le n-1$. $d_0 = 3f[x_0, x_1] - \frac{1}{2} f''(x_0) h$, $d_n = 3f[x_{n-1}, x_n] - \frac{1}{2} f''(x_n) h$.

Let $q = [m_0 - f_0', m_1 - f_1', \cdots, m_{n-1} - f_{n-1}']^T$, then

$$Aq = c$$

where $c = [c_j]^T$,

$$c_j = d_j - \lambda_j f'(x_{j-1}) - 2f'(x_j) - \mu_j f'(x_{j+1}), 1 \le j \le n-1$$

$$c_0 = d_0 - 2f'(x_0) - f'(x_1), c_n = d_n - f'(x_{n-1}) - 2f'(x_n)$$

Similar to Lemma 2.5.2, we can prove that

$$\|q\|_\infty \le \|A^{-1}\| \cdot \|c\| \le \|c\|_\infty$$

And we have proved in Lemma 2.5.2 that

$$|c_j| \le \frac{1}{24} h^3 \|f^{(4)}\|_\infty, 1 \le j \le n-1$$

Now suffices to prove it for c_0, c_n.

By Talor's equation

$$\begin{aligned}
c_0 &= 3 \cdot \frac{f(x_1) - f(x_0)}{h_0} - 2f'(x_0) - f'(x_1) - \frac{1}{2} f''(x_0) h_0 \\
&= 3 \left(f'(x_0) + \frac{1}{2} f''(x_0) h_0 + \frac{1}{6} f'''(x_0) h_0^2 + \frac{1}{6h_0} \int_{x_0}^{x_1} (x_1 - v)^3 f^{(4)}(v)\, \mathrm{d}v \right) \\
&- 2f'(x_0) - \left(f'(x_0) + f''(x_0) h_0 + \frac{1}{2} f'''(x_0) h_0^2 + \frac{1}{2} \int_{x_0}^{x_1} (x_1 - v)^2 f^{(4)}(v)\, \mathrm{d}v \right) - \frac{1}{2} f''(x_0) h_0 \\
&= \frac{1}{2} \int_{x_0}^{x_1} \left[\frac{1}{h_0} (x_1 - v)^3 - (x_1 - v)^2 \right] f^{(4)}(v)\, \mathrm{d}v \\
&\le \frac{1}{2} \|f^{(4)}(v)\|_\infty \cdot \frac{1}{h_0} \left| \int_0^{h_0} \tau^3 - h_0 \tau^2 \, \mathrm{d}\tau \right| \\
&= \frac{1}{24} \|f^{(4)}(v)\|_\infty
\end{aligned}$$

By symmetry, c_n also satisfy the same bound.

So Lemma 2.5.2 still holds. Hence Theorem 2.5.1 still holds. *i.e.*

$$\|s - f\|_\infty \le \frac{5}{384} h^4 \|f^{(4)}\|_\infty$$

Problem 6. Let $\varphi(x)$ be the Hermite polynomial of degree 3 w.r.t f, *i.e.*

$$\varphi(x) = f(x_j)\alpha_j(x) + f(x_{j+1})\alpha_{j+1}(x) + f'(x_j)\beta_j(x) + f'(x_{j+1})\beta_{j+1}(x)$$

Then we have $\varphi(x) - f(x) = \dfrac{f^{(4)}(\xi)}{24} \omega^2(x)$, where $\omega(x) = (x - x_j)(x - x_{j+1})$.

Note that Lemma 2.5.2 tells us $|m_j - f_j'| \le \dfrac{1}{24} h^3 \|f^{(4)}\|_\infty$ Since

$$s(x) - \varphi(x) = (m_j - f_j')\beta_j(x) + (m_{j+1} - f_{j+1}')\beta_{j+1}(x)$$

we have

$$\begin{aligned}|s'(x)-f'(x)| &\le |s'(x)-\varphi'(x)|+|\varphi'(x)-f'(x)|\\ &\le \frac{1}{24}h^3\|f^{(4)}\|_\infty A+\frac{1}{24}\|f^{(4)}\|_\infty B\\ &\le \frac{1}{6}h^3\|f^{(4)}\|_\infty\end{aligned}$$

$$\begin{aligned}|s''(x)-f''(x)| &\le |s''(x)-\varphi''(x)|+|\varphi''(x)-f''(x)|\\ &\le \frac{1}{24}h^3\|f^{(4)}\|_\infty A'+\frac{1}{24}\|f^{(4)}\|_\infty B'\\ &\le \frac{2}{3}h^3\|f^{(4)}\|_\infty\end{aligned}$$

where

$$A=\max|\beta_j'(x)|+|\beta_{j+1}'(x)|\le 1,\ B=\max|\omega(x)\omega'(x)|\le 3h^3$$

$$A'=\max|\beta_j''(x)|+|\beta_{j+1}''(x)|\le \frac{4}{h}\ B'=\max|(\omega^2(x))''|\le 12h^2$$

So $\|s'-f'\|\le \frac{1}{6}h^3\|f^{(4)}\|_\infty$, $\|s''-f''\|\le \frac{2}{3}h^3\|f^{(4)}\|_\infty$