Tsinghua University

Numerical Analysis

Fall 2024

Homework 1

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- **Collaborators:** I finish this homework by myself.

Problem 1. It is exactly Theorem 2.4.7

Let $T_n(x) = \cos(n \arccos(X))$ be the Chebyshev polynomial of degree n.

Then

$$L_n f(x) = \frac{1}{n^2} T_n^2(x) \sum_{i=1}^{n} f(x_i) \frac{1 - x x_i}{(x - x_i)^2}$$

which impilies that $L_n f$ is a monotone linear operator.

By the monotone linear operator theorem, it suffices to prove it for $f = 1, \varphi_t$.

Since $L_n 1 = 1$,

$$|L_n \varphi_t(t)| = \left| \frac{1}{n^2} T_n^2(t) \sum_{i=1}^{n} (t - x_i)^2 \frac{1 - t x_i}{(t - x_i)^2} \right| \leq \frac{1}{n^2} \cdot 2n = \frac{2}{n} \to 0$$

Problem 2.

Problem 3. That's because, $\bar{Q}_i(x)$, $0 \leq i \leq n$ is an orthogonormal basis of polynomial space of degree n, $\mathbb{P}_n$. So

$$p(x) = \sum_{i=0}^{n} \left\langle p, \bar{Q}_i \right\rangle \bar{Q}_i(x) = \int_a^b p(t) \bar{Q}_i(t) \omega(t) \, \mathrm{d}t \bar{Q}_i(x) = \int_a^b p(t) K_n(t, x) \omega(t) \, \mathrm{d}t$$

Problem 4. If $\lim\limits_{n \to \infty} \|L_n f - f\|_\infty = 0$, obviously it holds for $f = 1, \sin x, \cos x$.

Conversely, if it holds for $f = 1, \sin x, \cos x$, then for any $f \in \mathrm{Span}\{1, \sin x, \cos x\}$, $\lim\limits_{n \to \infty} \|L_n f - f\|_\infty = 0$. Then let $\varphi_t(x) = 1 - \cos(x - t) = 1 - \cos x \cos t - \sin t \sin x$

$$\begin{aligned}
|(L_n \varphi_t)(t)| &= |(L_n \cdot 1)(t) - \sin t (L_n \sin x)(t) - \cos t (L_n \cos x)(t) - (1 - \cos(t - t))| \\
&\leq \|(L_n \cdot 1) - 1\|_\infty + |\sin t| \cdot \|(L_n \sin x) - \sin x\|_\infty + |cost| \cdot \|(L_n \cos x) - \cos x\|_\infty \\
&\rightrightarrows 0
\end{aligned}$$

uniformly converges.

Now $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall |x - y| \leq \arcsin \delta$, $|f(x) - f(y)| < \epsilon$.

Then if $|x - t| < \arccos \delta$, $|f(x) - f(t)| < \delta$.

If $|x - t| \geq \arccos \delta$, then $|f(x) - f(t)| \leq 2\|f\|_\infty \leq 2\|f\|_\infty \cdot \frac{\varphi_t(x)}{1 - \delta} \leq \alpha \varphi_t(x)$ with $\alpha = \dfrac{2\|f\|_\infty}{1 - \delta}$.

Then $\forall x \in [-\pi, \pi]$,

$$-(\epsilon \cdot 1 + \alpha\varphi_t(x)) \leq f(x) - f(t) \leq \epsilon \cdot 1 + \alpha\varphi_t(x)$$

By monotonity,

$$-(\epsilon \cdot (L_n 1) + \alpha(L_n\varphi_t)(t)) \leq (L_n f)(t) - f(t)(L_n \cdot 1)(t) \leq \epsilon \cdot (L_n 1) + \alpha(L_n\varphi_t)(t)$$

Therefore

$$\|L_n f - (L_n \cdot 1)f\|_\infty \leq \epsilon\|L_n 1\| + \alpha\|L_n\varphi_t(t)\|$$

Since $L_n\varphi_t(t) \rightrightarrows 0$ for all t, letting $n \to \infty, \epsilon \to 0$, we have

$$\lim_{x\to\infty} \|L_n f - (L_n \cdot 1)f\|_\infty = 0$$

Noticed that $\lim\limits_{n\to\infty} \|L_n 1 - 1\|_\infty = 0$, this causes

$$\lim_{n\to\infty} \|L_n f - f\|_\infty = 0$$

Problem 5. $s_n \sin x = \sin x, s_n \cos x = \cos x$. So $G_n \sin x = \sin x, G_n \cos x = \cos x, G_n \cdot 1 = 1$ remains. By problem 4,

$$\lim_{n\to\infty} \|G_n f - f\|_\infty = 0$$