Analysis2 Note

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Contents

25 Positive linear functionals and Radon measures

25.4 Regularity and Lusin's theorem

Corollary 25.29. Let $E \subset X$ such that E is contained in an open set with finite measure. Then the following are equivalent:

1. $E \in \mathfrak{M}$

2. For every $\epsilon > 0$ there exists a compact set $K \subset X$ and an open set $U \subset X$ such that $K \subset E \subset U$ and $\mu(U \backslash K) < \epsilon$

2' There exist a σ-compact set $A \subset X$ and a G_δ set $BA \subset X$ such that $A \subset E \subset B$ and $\mu(B \backslash A) < \epsilon$

25.5 Regularity beyond finite measures

Theorem 25.37. μ is σ-finite. Then $\forall E \subset X$, $E \in \mathfrak{M} \Leftrightarrow \forall \epsilon, \exists U \supset E$ open in X, $F \subset E$ closed, such that $\mu(U \backslash F) < \epsilon$

Theorem 25.38. Let X be second countable LCH. $\mu : \mathfrak{B}_X \to [0, +\infty]$ s.t. $\mu(K) < +\infty, \forall K$ compact. Then μ is a Radon measure.

Proof. We need to prove a lemma:

Lemma 25.39. μ is inner regular on open sets.

The first proof is given by RM theorem.
The second proof use the Cor 25.29
The third proof shows that the regular sets are regular. □

Theorem 25.40. If $f \in C([a, b], \mathbb{R})$. Then f is Stieltjes integrable. $I_p : C([a, b], \mathbb{R}) \to \mathbb{R}, I_p(f) = f(a)\rho(a) + \int_a^b f \mathrm{d}\rho$

Lemma 25.41. Let $\rho : [a, b] \to \mathbb{R}_{\geq 0}$ be increasing. Assume that $a \leq c < d \leq b$. Let $f \in C([a, b], [0, 1])$ s.t. $f|_{[a,c]} = 1, f|_{[d,b]} = 0$. Then $\rho(c) \leq I_p(f) \leq \rho(d)$

Theorem 25.42 (Riesz Representation Theorem)**.** We have a bijection $\rho \mapsto I_\rho$ between increasing right continuous $\rho : [a, b] \to \mathbb{R}_{\geq 0}$ and positive linear functionals $\Lambda : C([a, b], \mathbb{R}) \to \mathbb{R}$

26 Theorems of Fubini and Tonelli for Radon measures

26.1 Products of Radon measure

Lemma 26.1. X be LCH, $\Lambda : C_c(X) \to \mathbb{C}$ positive linear functional. For each precompact open $U \subset X$, $\Lambda|_{C_c(U)} : C_c(U) \to \mathbb{C}$ is bounded.

Theorem 26.2. $X_1, \cdots, X_N$ with positive linear function $\Lambda_i : C_c(X_i) \to \mathbb{C}$. Then there exists a unique positive linear function $\Lambda : C_c(X_1 \times \cdots \times X_N) \to \mathbb{C}$ such that $\forall f_i \in C_c(X_i)$, $\Lambda(f_1 \cdots f_N) = \Lambda_1(f_1) \cdots \Lambda_N(f_N)$
where $f_1 \cdots f_N : X_1 \times \cdots \times X_N \to \mathbb{C}, (x_1, \cdots x_N) \mapsto f_1(x_1) \cdots f_N(x_N)$

Definition 26.1.1. The completion of the Radon measure associated to $\Lambda_1 \otimes \cdots \otimes \Lambda_N$ is denoted by $\mu_1 \times \cdots \times \mu_N$($\mu_i$ is the completion of the Radon measure for Λ_n) called **Radon Product**

26.2 Theorems of Fubini and Tonelli

Theorem 26.3 (Tonelli's theorem)**.** $f \in LSC_+(X \times Y)$. Then $\int_Y f \mathrm{d}v : X \to [0, +\infty]$

Theorem 26.4 (Tonelli's theorem)**.** Assume μ, ν are σ-finite. Let $f \in \mathcal{L}(X \times Y)$ i.e. f is $(\mu \times \nu)$-measurable

(a) $f(x, \cdot) : Y \to [0, +\infty]$ is $\nu - measurable$ for $x \in X$ a.e.

(b) $x \mapsto \int_Y f(x, \cdot)\mathrm{d}v$ is measurable.

(c) $\int_{X \times Y} f \mathrm{d}(\mu \times \nu) = \int_X \int_Y f \mathrm{d}\nu \mathrm{d}\mu$

Proposition 26.5. Assume μ, ν are σ-finte. Then for measurable $A \subset X, B \subset Y$, $A \times B$ is $(\mu \times \nu)$-measurable, and $(\mu \times v)(A \times B) = \mu(A)\nu(B)$.

Example 26.6. μ is completion of Radon on X, $(Y, 2^Y, \nu)$ counting measure.
Then if $E \subset X \times Y$ is open $\Rightarrow \chi_E$ is lower semicontinuous.
Tonelli's theorem $\Rightarrow (\mu \times \nu)(E) = \sum\limits_{y \in y} \nu(E_y)$
Assume μ is σ-finite, $f : X \times Y \to [0, +\infty]$ is Borel. Apply Prop 26.5, we have

$$\sum_{y \in Y} \int_X f_y \mathrm{d}\mu = \int_X \sum_{y \in Y} f_y \mathrm{d}\mu \tag{a}$$

(a) is true if f is LCH or if Y is countable.
Let $I = fin(2^Y)$, $\forall \alpha \in I$, $g_\alpha = \sum\limits_{y \in \alpha} f_\alpha$. We have

$$\lim_\alpha \int_X g_\alpha \mathrm{d}\mu = \int_X \lim_\alpha g_\alpha \mathrm{d}\mu$$

if f is LSC or $Y = \mathbb{Z}$.

27 The marriage of Hilbert spaces and integral theory

27.1 The definition of L^p spaces

Definition 27.1.1. $f \in \mathcal{L}(X, \mathbb{C}), ||f||_{L^p} = ||f||_p = (\int_X |f|^p \mathrm{d}\mu)^{\frac{1}{p}}$

Theorem 27.1. Assume $f, g \in \mathcal{L}(X, \mathbb{C})$ or $f, g \in \mathcal{L}_+(X)$. We have **Minkowski's inequality**

$$||f + g||_p \leq ||f||_p + ||g||_p$$

And if $\frac{1}{p}+\frac{1}{q}$, $1<p,q<+\infty$. We have **Hölder's inequality**

$$|\int_X fg\mathrm{d}\mu| \leq ||f||_p||g||_q$$

$$||f+g||_p \leq ||\,|f|+|g|\,||_p$$

Definition 27.1.2. Let $\mathcal{L}^p(X,\mu)=\{f\in\mathcal{L}(X,\mathbb{C}):||f||_p<+\infty\}$. Then $||\cdot||_p$ is a semi-norm on $\mathcal{L}^p(X,\mu)$

$L^p(X,\mu)=\mathcal{L}^p(X,\mu)/_{\{f\in\mathcal{L}^p(X,\mu):||f||_p=0\}}$ is a NVS with norm $||\cdot||_p$.

$$||f||_p=0 \Leftrightarrow \int |f|^p=0 \Leftrightarrow f=0 \text{ a.e.}$$

$L^p(X,\mu)$ is the space of all $f\in\mathcal{L}(X,\mathbb{C})$ satisfying $||f||_p<+\infty$, but f,g are the same iff $f=g$ a.e.

Definition 27.1.3. $f\in\mathcal{L}(X,\mathbb{C})$. Define

$$||f||_{L^\infty}=||f||_\infty=\inf\{a\in\overline{\mathbb{R}}_{\geq 0}:\mu\{|f|>a\}=0\}$$

where $\{|f|>a\}=\{x\in X:||f(x)||>a\}$

Proposition 27.2. Let $f,g\in\mathcal{L}(X,\mathbb{C})$

(a) If $f=g$ a.e., then $||f||_\infty=||g||_\infty$

(b) $f=0$ a.e. iff $||f||_{L^\infty}=0$

Proposition 27.3. Let $f\in\mathcal{L}(X,\mathbb{C}),\lambda=||f||_{L^\infty}$. Then

$$\{a\in\overline{\mathbb{R}}\geq 0:\mu\{|f|>a\}=0\}=[\lambda,+\infty]$$

In particular, λ is the smallest number s.t. $\{|f|>\lambda\}$ is null.

Corollary 27.4. Let $A=\{|f|\leq\lambda\}$. Then $X\backslash A$ is null, and $||f\chi_A||_{l^\infty}=||f||_{L^\infty}$

Proposition 27.5. Let (f_n) be in $\mathcal{L}(X,\mathbb{C})$. TFAE

(1) $\lim\limits_{n\to\infty}||f_n||_{L^\infty}=0$

(2) $\exists$ measurable A with $\mu(A^c)=0$ s.t. $f_n|_A$ uniformly converge to 0.

Proposition 27.6.

$$||f+g||_{L^\infty}\leq||f||_{L^\infty}+||g||_{L^\infty}$$

$$||af||_{L^\infty}=|a|\cdot||f||_{L^\infty}$$

Definition 27.1.4.

$$\mathcal{L}^\infty(X) = \{f \in \mathcal{L}(X, \mathbb{C}) : ||f||_{l^\infty} < +\infty\}$$

And

$$\begin{aligned} L^\infty(X,\mu) &= \mathcal{L}^\infty(X)/_{\{f\in\mathcal{L}^\infty(X):||f||_{L^\infty}=0\}} \\ &= \mathcal{L}^\infty(X)/_{\{f\in\mathcal{L}^\infty(X):f=0 \text{ a.e.}\}} \end{aligned}$$

27.2 Approximation in L^p spaces

Theorem 27.7. Let X be LCH. Let μ be (the completion of) a Radon measure on X $1 \leq p < +\infty$ Then $C_c(X)$ is dense in $L^p(X, \mu)$

Hint. First we prove $||f||_{l^\infty} < +\infty$, second we approximate f by $f\chi_{E_n}$ □

Corollary 27.8. Let $e_n : x \mapsto e^{inx}$ in $C(S^1)$. If $1 \leq p < +\infty$, then e_n spans a dense subspace of $L^p(S^1, \frac{m}{2\pi})$ where m is the Lebesgue measure.

Theorem 27.9. Let X be second countable LCH. μ is (the completion of) a Radon measure on X. Let $1 \leq p < +\infty$. Then $L^p(X, \mu)$ is seperable.

Hint. First for X compact. We have proved that $C(X)$ is l^∞-separable. Easy to check that it is true for L^∞.
For arbitary X. Let $K_1 \subset K_2 \subset \cdots K_n \subset \cdots \subset X$, $\bigcup\limits_n K_n = X$. And $\lim\limits_n f\chi_{K_n} = f$. It suffices to prove that $\mu|_{K_n}$ is Radno measure. □

Theorem 27.10. Let (X, μ) be measurable, $1 \leq p \leq +\infty$. Then $L^p(X, \mu) \cap S(X, \mathbb{C})$ is dense in $L^p(X\mu)$.

Note. Elements in $L^p \cap S$ are exactly:

$$\begin{cases} S(X, \mathbb{C}), & p = +\infty \\ \sum a_n\chi_{E_n}, a_n \in \mathbb{C}, \mu(E_n) < \infty, & p < +\infty \end{cases}$$

And we only need to check that for $f \geq 0$. □

Proposition 27.11. $L^\infty(X, \mu)$ is complete.

In inner product space, we prove a similar theorem for completeness

Theorem 27.12. If V is NVS, then V is complete $\Leftrightarrow$ if (v_n) in V s.t. $\sum ||v_n|| < +\infty$, then $\sum v_n$ converges.

27.3 The Riesz-Fischer Theorem

Theorem 27.13 (Riesz-Fischer Theorem)**.** If $1 \leq p < +\infty$. Then $L^p(X, \mu)$ is complete(Banach) space.

Moreover, if (f_n) in $L^p(X, \mu)$, $f \in L^p(X, \mu)$ and $\lim_n ||f - f_n||_{L^p} = 0$, then (f_n) has a subsequence converging a.e. to f.

Corollary 27.14 (Riesz-Fischer)**.** We have a unitary $L^2([-\pi, \pi], \frac{m}{2\pi}) \to l^2(\mathbb{Z}), f \mapsto \hat{f}$.

27.4 Introduction to dualities in L^p spaces

We now assume $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq +\infty$. (X, μ) measurable space.

Proposition 27.15. Assume μ is σ-finte if $p = +\infty, q = 1$. Then $\exists$ linear isometry $\Psi : L^p(X, \mu) \to L^q(X, \mu)^*$ s.t. $\forall F \in L^p, g \in L^q$,

$$\langle \Psi(f), g \rangle = \int_X fg \mathrm{d}\mu$$

Proposition 27.16. $|\langle \Psi(f), g \rangle| \leq ||f||_p \cdot ||g||_q$

Example 27.17. (X, μ) measurable space, $f \in L^\infty(X, \mu)$. Define

$$M_f : L^2(X, \mu) \to L^2(X, \mu), g \mapsto fg$$

Called **multiplication operator**

M_x has no eigenvalue, when $X = [0, 1]$

Theorem 27.18. Let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint, $a \leq T \leq B$. There exists $(\mu_i)_I$ of Randon measures on $[a, b]$ and unitary $U : \mathcal{H} \to \bigoplus_{i \in I} L^2([a, b], \mu_i)$ s.t. $UTU^{-1} = \bigoplus_{i \in I} M_x$.

Definition 27.4.1. A **unitary representation** of $\mathcal{A}$ on $\mathcal{H}$ is a linear $\pi : \mathcal{A} \to \mathfrak{L}(\mathcal{H})$ s.t. $\pi(ab) = \pi(a)\pi(b)$, $\pi(1) = 1_{\mathcal{H}}$, $\pi(a^*) = \pi(a)^*$. pi is called a **unital *-homomorphism**.

If $\Omega \in \mathcal{H}$ s.t. $\pi(\mathcal{A})\Omega = \{\pi(a)\Omega : a \in \mathcal{A}\}$ is dense, we say π is a **cyclic representation**. Ω is called a **cyclic vector**.

If $\mathcal{K}$ is a closed linear subspace of $\mathcal{H}$. If $\mathcal{K}$ is $\mathcal{A}$-invariant, i.e. $\pi(a)\mathcal{K} \subset \mathcal{K}$ for all $a \in \mathcal{A}$, we call $\mathcal{K}$ is a **subrepresentation**.

Fact 27.19. If $\mathcal{K}$ is subrepresentation, then $\mathcal{H} \cong \mathcal{K} \oplus \mathcal{K}^\perp$.

Proposition 27.20. Let $\pi : \mathcal{A} \to \mathfrak{L}(\mathcal{H})$ be unitary representation. then $\exists\, (\mathcal{H}_i)_{i \in I}$ of unitary subrepresentation of $(\pi, \mathcal{H})$ s.t.

(a) Every $\mathcal{H}_i$ is a cyclic representation.

(b) $\mathcal{H}_i \perp \mathcal{H}_j$ if $i \neq j$.

(c) $\mathrm{Span}\{\mathcal{H}_i : i \in I\}$ is dense in $\mathcal{H}$.

Theorem 27.21. Let X be compact Hausdorff. $\pi : C(X) \to \mathfrak{L}(\mathcal{H})$ cyclic representation with cyclic vector Ω. Then $\exists$ Randon measure μ on X and an unitary equivalence $U : (\mathcal{H}, \pi) \to (L^2(X, \mu), M)$, $U\Omega = 1$

For $T \in \mathfrak{L}(\mathcal{H})$, define

$$\begin{aligned} \pi_T : \mathbb{C}[x] &\to \mathfrak{L}\mathcal{H} \\ f &\mapsto f(T) \end{aligned}$$

Theorem 27.22. Let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint. $a \leq T \leq b$. Then $\pi_T : \mathbb{C}[x] \to \mathfrak{L}(\mathcal{H})$ has operator norm $||\pi_T|| \leq 1$ if $\mathbb{C}[x]$ is equipped with $l^\infty[a, b]$.

Proposition 27.23. Let $f \in \mathbb{C}[x]$, T_α is a net in $\mathfrak{L}(\mathcal{H})$. If $T_\alpha \to T$, $\sup ||T_\alpha|| < +\infty$. Then $f(T_\alpha) \to f(T)$

Proposition 27.24. If $\sup ||T_\alpha|| < +\infty$, $T_\alpha \to T, \eta_\beta \to \eta$, then $\lim\limits_{\alpha,\beta} T_\alpha \eta_\beta = T\eta$

28 From the implicit function theorem to differential manifolds

28.1 The inverse function theorem

Definition 28.1.1. Let $U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^m$ open. $f : U \to V$ is called a C^r**-diffeomorphism** $(0 \leq r \leq \infty)$, if f is bijective and $f, f^{-1} \in C^r$.
In fact, $m = n$ since $\mathrm{Jac}(f) \cdot \mathrm{Jac}(f^{-1}) = I$

Theorem 28.1 (Inverse Function Theorem)**.** Let $\Omega \subset \mathbb{R}^n$ open. Let $\varphi : \Omega \to \mathbb{R}^n$ be C^r-map, $1 \leq r \leq +\infty$. Let $p \in \Omega$. Assume $\mathrm{d}\varphi|_p : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then $\exists$ a neighborhood $U \subset \Omega$ of p, neighborhood $V \subset \mathbb{R}^n$ of $q = \varphi(p)$ s.t. $\varphi : U \to V$ is C^r-diffeomorphism.

Lemma 28.2. First prove it is true for φ bijective.

Corollary 28.3. $r \geq 1$, $\Omega \subset \mathbb{R}^n$ open. $\varphi : \Omega \to \mathbb{R}^n$ injective C^r-map s.t. $\mathrm{Jac}(\varphi)$ is invertible everywhere. Then $\varphi(\Omega)$ is open in $\mathbb{R}^n$, $\varphi : \Omega \to \varphi(\Omega)$ is a C^r-diffemmorphism.

28.2 The Implicit Theorem

Corollary 28.4. $\Omega \subset \mathbb{R}^d \times \mathbb{R}^k$ open, $(x, y) = (x^1, \cdots, x^d, y^1, \cdots, y^k)$, $f = (f^1, \cdots, f^k) : \Omega \to \mathbb{R}^k$ is C^r function, $r \geq 1$. Assume $\mathrm{Jac}_y(f)$ is invertible at $p \in \Omega$. Then $\exists$ neighborhood $U \subset \Omega$ of p and open $V \subset \mathbb{R}^d \times \mathbb{R}^k$ s.t. we have C^r-diffemmorphism

$$(x^1, \cdots, x^d, f^1, \cdots, f^k) : U \to V$$

Corollary 28.5. $f = (f^1, \cdots, f^k) : \Omega \to \mathbb{R}^k$, Ω open in $\mathbb{R}^d \times \mathbb{R}^k$. If $\mathrm{Jac}_y(f)$ invertible at $p \in \Omega$. Then $\exists\, p \in U \subset \Omega$, $V \subset \mathbb{R}^d \times \mathbb{R}^k$ s.t.

$$(x^1, \cdots, x^d, f^1, \cdots, f^k) = (x, f) : U \cong V$$

is a diffeomorphism.

Definition 28.2.1. Let $M \subset \mathbb{R}$, $r \geq 1$. We say M is an **(embedding) C^r-submanifold** of $\mathbb{R}^n$ if for every $p \in M$, $\exists U \in Nbh_{\mathbb{R}^n}(p)$, $0 \leq d \leq n, k = n - d$, $\exists\, C^r$-functions

$$(\varphi^1, \cdots, \varphi^d, f^1, \cdots, f^k) : U \to \mathbb{R}^n$$

satisfying:

(a) $(\varphi, f) : U \cong V$ is a C^r-diffeomorphism.

(b) $(\varphi, f)^{-1}((\mathbb{R}^d \times 0) \cap V) = M \cap U$,(equivalently, $(\varphi, 0)$ is bijective)

Proposition 28.6. $\varphi(M \cap U)$ is an open subset of $\mathbb{R}^d$. Moreover, $\forall h \in C^r(U, \mathbb{R})$, $\exists!\, g : \varphi(M \cap U) \to \mathbb{R}$ s.t.

$$h|_{M\cap U} = g \circ \varphi|_{M\cap U} \tag{$\triangle$}$$

Moreover, any g satisfying $(\triangle)$ is C^r

Lemma 28.7. Let $\Omega \subset \mathbb{R}^n$ open, $\psi = (\psi^1, \cdots, \psi^n) : \Omega \to \mathbb{R}^d$ is a C^r function. Assume Ω, ψ satisfying a similar property as U, φ in Def 28.2.1. Then $(U, \varphi|_{M\cap U})$ and $(\Omega, \psi|_{M\cap\Omega})$ are C^r-compatible, i.e.

$$\Psi|_{M\cap, U, \Omega} \circ (\varphi|_{M\cap U\cap\Omega})^{-1} : \varphi(M \cap U \cap \Omega) \to \Psi(M \cap U \cap \Omega)$$

is a C^r-diffeomoephism of open subsets of $\mathbb{R}^d$.

Definition 28.2.2. Let M be a nonempty Hausdorff. A set $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)_{\alpha\in\mathcal{A}}\}$ is called a **C^r-atlas** of M if

- $M = \bigcup\limits_{\alpha} U_\alpha$, U_α is open,
- $\varphi_\alpha : U_\alpha \xrightarrow{\cong} \varphi(U_\alpha)$ is homeomorphism, $\varphi(U_\alpha)$ is open in $\mathbb{R}^{d_\alpha}$
- $\forall \alpha, \beta \in \mathcal{A}$, φ_α and φ_β are C^r compatible, i.e. $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\cong} \varphi_\beta(U_\alpha \cap U_\beta)$ is a C^r-diffemmorphism.

$(M, \mathcal{U})$ is called a **C^r-manifold**.

We assume M is second countable.

We call C^∞-manifold differential manifold or smooth manifold.

Definition 28.2.3. U_α is d_α-dimensional. If $p \in U_\alpha$, $\dim_p M = d_\alpha$.

If $\dim_p M = d$ independent of p, we say M is equidimensional.

Proposition 28.8. $\forall d \in \mathbb{N}$, $U_d = \{x \in M : \dim_x M = d\}$ is open and closed in M.

In particular, if M is connected, then M is equidimensional.

Definition 28.2.4. A C^r**-chart** on $(M,\mathcal{U})$ is (V,ψ) s.t.

- V is open in M, $\psi(V)$ open in $\mathbb{R}^d$.
- $\psi : V \to \psi(V)$ homeomorphism.
- (V,ψ) is C^r-compatible with any member of $\mathcal{U}$.

$\overline{\mathcal{U}} = \{C^r - \text{chart of } (M,\mathcal{U})\}$ is the maximal C^r atlas contain $\mathcal{U}$.

A maximal C^r-atlas on M is called a C^r-structure.

Definition 28.2.5. M, N are C^r-manifolds, $F : M \to N$ is a C^r-map if F is continuous and if for every chart (U,φ) of M and (V,ψ) of N, we have

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$$

is C^r function.

If F is bijective, F and F^{-1} are C^r, we sat F is a C^r-diffeomorphism.

Proposition 28.9. Let M be a C^r-submanifold of N, then the inclusion map $\iota : M \to N$ is C^r.

Proposition 28.10. Let X, N be C^r-submanifolds. M is C^r-submanifold of N. Let $\iota : M \to N$. Let $F : X \to M$. Then F is C^r iff $\iota \circ F$ is C^r.

Example 28.11. $P \subset M, Q \subset N$ submanifold, then $P \times Q \subset M \times N$ submanifold.

Theorem 28.12 (Implict Function Theorem)**.** Let $(x,y) = (x^1, \cdots, x^d, y^1, \cdots, y^k)$ be standard coordinates on $\mathbb{R}^d \times \mathbb{R}^k$. $\Omega \subset \mathbb{R}^d \times \mathbb{R}^k$ open. Let $M \subset \mathbb{R}^d \times \mathbb{R}^k$. Assume $\exists C^r$ $f = (f^1, \cdots, f^k) : \Omega \to \mathbb{R}^k$ *s.t.*

1. $M \cap \Omega = Z(f)$.
2. $\operatorname{Jac}_y f$ is invertible at $p \in M$

Then $\exists$ Neighborhood $p \in U \subset \Omega$ *s.t.* $M \cap U$ is a C^r-submanifold of $\mathbb{R}^d \times \mathbb{R}^k$ and $(M \cap U, x|_{M\cap U})$ is a chart on $M \cap U$.

29 Differential calculus on manifold

Recall that for V is $\mathbb{F}$-vector space, with $e_i, 1 \leq i \leq n$ basis. There is dual basis e^i in V^* *s.t.* for $\xi \in V$, $\xi = \sum_{i=1}^{n} \langle \xi, e^i \rangle e_i$.

29.1 Tangent Space and Cotangent Space

Definition 29.1.1. For C^∞-map $\gamma : (a, b) \to M$. $p \in M$. Define

$$T_pM = \{\text{smooth } \gamma : (-\epsilon, \epsilon) \to M, \gamma(0) = p\}/_\sim = \{\gamma'(t_0) : \gamma(t_0) = p, t_0 \in \mathbb{R}\}$$

where $\gamma_1 \sim \gamma_2$ if and only if $\exists$ chart (U, φ) *s.t.* $\operatorname{Jac}(\varphi \circ \gamma_1)|_0 = \operatorname{Jac}(\varphi \circ \gamma_2)|_0$
$\gamma'(t_0)$ is the equivalance class of $t \mapsto \gamma(t + t_0)$ in $T_{\gamma(t_0)}M$

Theorem 29.1. Let $p \in M$, for each chart $(U, \varphi^1, \cdots, \varphi^n)$ containing p, $\exists$ bijection $\mathrm{d}\varphi|_p$ defined by $\mathrm{d}\varphi|_p : T_pM \to \mathbb{R}^n$, $\mathrm{d}\varphi|_p \cdot \gamma'(0) = \operatorname{Jac}(\varphi \circ \gamma)|_0$ if γ is smooth path and $\gamma(0) = p$.

Remark 29.2. In this way, we can define a $\mathbb{R}$-vector space structure on TM. cf. Def 29.1.2

Definition 29.1.2. Tangent bundle

$$TM = \bigsqcup_{p \in M} T_pM$$

$X : M \to TM$ is called a vector field if $\forall p \in M, X|_p \in T_pM$

Definition 29.1.3. For (U, φ) is chart on M. Define

$$\partial_{\varphi^i} = \frac{\partial}{\partial \varphi^i} : U \to TM$$

$$\partial_{\varphi^i}|_p = (\mathrm{d}\varphi|_p)^{-1} e_i$$

Theorem 29.3. Let $F : M \to N$ be C^∞. Then $\forall p \in M$, let $q = F(p)$. Then $\exists$ unique linear map $\mathrm{d}F|_p : T_pM \to T_qN$, $\mathrm{d}F|_p \cdot \gamma'(0) = (F \circ \gamma)'(p)$ for $\gamma : (-\epsilon, \epsilon) \to M$ smooth, $\gamma(0) = p$.
Moreover, if $(U, \varphi^1, \cdots, \varphi^m)$ and $(V, \psi^1, \cdots, \psi^n)$ are charts of M, N containing p, q, then

$$\mathrm{d}F|_p \cdot (\frac{\partial}{\partial \varphi^1}, \cdots, \frac{\partial}{\partial \varphi^m})_p = (\frac{\partial}{\partial \psi^1}, \cdots, \frac{\partial}{\partial \psi^1})_q \cdot \operatorname{Jac}(\psi \cdot \varphi^{-1})|_{\varphi(p)}$$

Remark 29.4. It's hard to prove the existence of $\mathrm{d}F|_p$ but the chain rule.

Proposition 29.5 (Chain rule)**.**

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G|_{F(p)} \cdot \mathrm{d}F|_p$$

Definition 29.1.4. $X : M \to TM$ is called a **smooth vector field** if TFEC true:

1. $\forall$ chart $(U, \varphi^1, \cdots, \varphi^n)$ if $X|_U = \sum_{i=1}^{n} X^i \frac{\partial}{\partial \varphi^i}$, $X^i : U \to \mathbb{R}$ smooth.

2. $\exists$ atlas $\mathcal{U}$ *s.t.* $\forall (U, \varphi) \in \mathcal{U}$, X^i is smooth.

Definition 29.1.5. The **cotangent space** is

$$T^*M = \bigsqcup_{p \in M} T_p^* M$$

where T_p^*M is the (real) dual space of T_pM.

$\omega : M \to T^*M$ is called a **1-form** if $\forall p \in M, \omega|_p \in T_p^*M$

Definition 29.1.6. **1-form** $\omega : M \to T^*M$ is called smooth if $\forall$ open $U \subset M$, $\forall$ smooth $X : U \to TM$, $\langle \omega, X \rangle : p \in U \mapsto \langle \omega|_p, X|p \rangle$ is smooth.

Definition 29.1.7. $f \in C^\infty(M, \mathbb{R})$, $\forall p \in M$,

$$\mathrm{d}f|_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}, \mathrm{d}f|_p \in T_p^*M$$

$\mathrm{d}f : M \to T^*M$ is 1-form.

Proposition 29.6. Let $f \in C^\infty(M, \mathbb{R})$, $\mathrm{d}f$ is smooth 1-form. Moreover, if $(U, \varphi^1, \cdots, \varphi^n)$ is chart

$$\frac{\partial}{\partial \varphi^j} f|_p = \partial_j (f \circ \varphi^{-1})|_{\varphi(p)}, p \in M$$

$$\partial_{\varphi^j} f = (\partial_j (f \circ \varphi^{-1})) \circ \varphi$$

Corollary 29.7. $f \in C^\infty(M, \mathbb{R})$, $(U, \varphi^1, \cdots, \varphi^n)$ chart, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial \varphi^j} \mathrm{d}\varphi^j$$

Corollary 29.8. $f, g \in C^\infty(M, \mathbb{R})$, then $\mathrm{d}(fg) = \mathrm{d}f \cdot g + \mathrm{d}g \cdot f$

Definition 29.1.8. If $F : M \to N$ smooth, $F^*|_p : T^*_{F(p)}N \to T_p^*M$ is defined by transpose of $\mathrm{d}F|_p : T_pM \to T_{F(p)}N$.

F^* is called **cotangent map**

$F^* \cdot \omega'$ called **pullback** of ω' by F.

If $\omega : N \to T^*N$ is a 1-form, its **pullback** is $F^*\omega : M \to T^*M, p \mapsto F^*(\omega|_{F(p)}) \in T_p^*M$.

Proposition 29.9. Let $\omega : N \to T^*N$ be smooth 1-form, then $F^*\omega$ is smooth. Moreover, if $f \in C^\infty(N, \mathbb{R})$, $F^*\mathrm{d}f = \mathrm{d}(f \circ F)$

Definition 29.1.9. $F : M \to N$ is called a (smooth) **embedding** if $F(M)$ is a C^∞-submanifold of N, and F restricts to a diffeomorphism on M.

Proposition 29.10. Let $F : M \to N$ be smooth embedding, then $\forall p \in M$, $\mathrm{d}F|_p : T_pM \to T_{F(p)}N$ is injective.

Theorem 29.11. Let $F : M \to N$ be smooth, let $q \in N$. Assume $\forall p \in F^{-1}(q)$, $\mathrm{d}F|_p : T_pM \to T_qN$ is surjective(F is a submersion at q) Then $F^{-1}(q)$ is a smooth submanifold of M.
Moreover, $T_p(F^{-1}(q) = \ker(\mathrm{d}F|_p)$
In particular, $\dim_p F^{-1}(q) = \dim_p M - \dim_q N$

Theorem 29.12. Let $F : M \to N$ be smooth. Let $p \in M$, $q = F(p)$, $\mathrm{d}F|_p : T_pM \to T_qN$ linear isomorphism. Then $\exists U \in Nbh(p)$, $V \in Nbh(q)$ *s.t.* F restructs to a diffeomorphism $F : U \to V$

30 The change of variables formula

Definition 30.0.1. μ, ν are Borel measures of X. $\mu \leq \nu$ if one of the following equivalent holds.

1. $\forall E \in \mathcal{B}_X, \mu(E) \leq \nu(E)$
2. $\forall$ Borel $f : X \to [0, +\infty]$, $\int f \mathrm{d}f \leq \int f \mathrm{d}v$.
3. $\forall$ open $U \subset X$, $\mu(U) \leq \nu(U)$
4. $\forall$ compact $K \subset X$, $\mu(K) \leq \nu(K)$
5. $\forall f \in C_c(X, \mathbb{R}_{\geq 0}), \int f \mathrm{d}\mu \leq \int f \mathrm{d}v$

Proposition 30.1. Let μ, ν be Randon measures of LCH $X, \mathcal{U}$ is an open cover of X. Then $\mu \leq \nu$ iff $\forall u \in \mathcal{U}$ we have $\mu|_U \leq \nu|_U$

Proposition 30.2. Let $\Omega \subset \mathbb{R}^n$ open, μ, ν Radon on Ω. TFAE

1. $\mu \leq \nu$
2. $\forall$ cube $Q \subset \Omega$, $\mu(Q) \leq \nu(Q)$
3. $\forall$ open cube $Q \subset \Omega$, $\mu(Q) \leq \nu(Q)$

Theorem 30.3. $\Omega, \triangle \subset \mathbb{R}^n$ open, $\Phi : \Omega \xrightarrow[C^1]{\cong} \triangle$. Then

$$\Phi^* \mathrm{d}m_\triangle = |J(\Phi)| \mathrm{d}m_\Omega$$

Equivalently, $\forall$ Borel $f : \triangle \to [0, +\infty]$, we have

$$\int_\triangle f \mathrm{d}m = \int_\Omega (f \circ \Phi) \cdot |J\Phi| \mathrm{d}m$$

Definition 30.0.2. A (linear) **product** $\bigodot : V_1 \times \cdots \times V_N \to V_1 \odot \cdots \odot V_N$ is a N-linear map.

Definition 30.0.3. A tensor product is a surjective linear product s.t. for all linear product there exists a homomorphism to it. $\boxdot$

Index