For group G of order 2275

4. Let n_p denote the number of Sylow p-subgroup. By Third Sylow Theorem, we have

$$n_7 \equiv 1 \pmod 7), \quad n_7 \mid 5^2 \cdot 13 \quad \Rightarrow n_7 = 1$$
$$n_{13} \equiv 1 \pmod{13} \quad n_{13} \mid 5^2 \cdot 7 \quad \Rightarrow n_{13} = 1.$$

Let K_7, K_{13} be the unique Sylow 7-subgroup, 13-subgroup respectively.

Then $K_7 \trianglelefteq G$, $K_{13} \trianglelefteq G$ by Second Sylow thm.

Since $K_7 \cap K_{13} < K_7, K_{13} \Rightarrow |K_7 \cap K_{13}| \mid |K_7|, |K_{13}|$
$\Rightarrow K_7 \cap K_{13} = \{1\}$

Then by Prop 7.3.3, $K_7 K_{13} \cong K_7 \times K_{13}$.

Since K_7, K_{13} cyclic, hence abelian, $K_7K_{13} \cong K_7 \times K_{13}$ is abelian.

$\forall g \in G$, $gK_7K_{13}g^{-1} = gK_7g^{-1}gK_{13}g^{-1} = K_7K_{13} \Rightarrow K_7K_{13} \trianglelefteq G$

Let K_5 be the Sylow 5-subgroup.

Choose $g \neq 1$ in K_5.

Let $H = \langle g \rangle$

Let $S \subset K_7K_{13}$ denote all elements of order 91.

For $h \in H$, $s \in S$, $hsh^{-1} \in K_7K_{13}$ since $K_7K_{13} \trianglelefteq G$.

$(hsh^{-1})^n = hs^nh^{-1} = 1$ if and only if $s^n = 1 \Rightarrow hsh^{-1} \in S$ has order of 91

Consider action $H \curvearrowright S$, $h * k = hkh^{-1} \in S$

Since $K_7K_{13} \cong K_7 \times K_{13}$, $|S| = |\{(e_1, e_2) \in K_7 \times K_{13} : e_1 \neq 1 \text{ or } e_2 \neq 1\}|$
$= 6 \times 12 = 72$.

order of orbit in S should divide $|H| \mid 25$

$|S| = 72$
$\Rightarrow \exists$ orbit of order 1. i.e. $\exists k \in S$, $gkg^{-1} = k$.

$\Rightarrow k^n = (g k g^{-1})^n = g k^n g^{-1}$

since k has order of 91 $\Rightarrow g$ commutes with all elements in $K_7 K_{13}$

Thus $\forall g \in K_5$, g commutes with all elements in $K_7 K_{13}$

By 2nd isom thm $K_5(K_7 K_{13}) < G$, $|K_5(K_7K_{13})| = \dfrac{|K_5||K_7K_{13}|}{|K_5 \cap K_7K_{13}|} = 2275 = |G|$

$\Rightarrow K_5(K_7K_{13}) = G$

$\forall h_1, h_2 \in K_5$, $k_1, k_2 \in K_7K_{13}$

$$h_1 k_1 h_2 k_2 = h_1 h_2 k_1 k_2 = h_2 h_1 k_2 k_1 \quad (K_5 \text{ has order of } 5^2 \text{, hence abelian})$$
$$= h_2 k_2 h_1 k_1$$

$\Rightarrow G = K_5(K_7K_{13})$ commutes