Analysis-1 Note

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Contents

1 Introduction

1.1 proposition,logic,simple set theory

There are some symbols we should to know:

$$P \vee Q : \text{or} \quad | \quad P \wedge Q : \text{and} \quad | \quad \neg P : \text{not} \quad |$$

xRy means $(x, y) \in R$, for $R \subset X \times Y$, which is called a relationship.
For $R \subset X \times Y$, **define** $R^{-1} := \{(y, x) \in Y \times X : xRy\}$.
For $S \subset Y \times Z$, **define** $S \circ R := \{(x, y) \in X \times Z : \exists y \in Y, s.t. xRy \wedge ySz\}$

Definition 1.1.1 (equivalent relationship)**.** $\sim \subset X \times X$ *is an equivalent relationship if*

(1) $\forall x \in X, x \sim x$

(2) $\forall x, y \in X, x \sim y \Rightarrow y \sim x$

(3) $x \sim y, y \sim z \Rightarrow x \sim z$

For an equivalent relationship, we can **define** $[x] := \{y \in X : y \sim x\}$ be the equivalent class of x. For that there is a map called **quotient mapping**

Definition 1.1.2 (partially ordered relation). *For* $\leqslant \subset X \times X$, *if*

(1) $x \leqslant x$

(2) $x \leqslant y, y \leqslant z \Rightarrow x \leqslant z$

(3) $x \leqslant y, y \leqslant x \Rightarrow y = x$

we call it ***partial ordered relation***
If $\forall x, y \in X$, $(x \leqslant y) \vee (y \leqslant x)$, *we call it* ***total order*** *or* ***linear order***.

Definition 1.1.3. $(X, \leqslant)$ *is a* ***partially ordered set*** *for* $A \subset X$, ***define***

1. $x < y$ *iff* $x \leqslant y$ *and* $x \neq y$

2. $s \in X$ *is an upper bound(lower bound) iff* $\forall a \in A, a \leqslant s (s \leqslant a)$

3. $m \in A$ *is a maximal(minimal) element iff* $\nexists a \in A s.t. m < a (a < m)$

4. $m \in A$ *is the greatest(least) element iff* m *is the upper(lower) bound and* $m \in A$

1.2 Metric spaces, convergence of sequences, and continuous functions

1.3 Introduction

1.4 Basic definitions and examples

Definition 1.4.1. *Let* X *be a set. A function* $d : X \times X \to \mathbb{R}_{\geq 0}$ *is called a* ***metric*** *if for all* $x, y, z \in X$ *we have*

(1) $d(x, y) = d(y, x)$

(2) $d(x, y) = 0$ *iff* $x = y$

(3) *(Triangle inequality)* $d(x, z) \leq d(x, y) + d(y, z)$

The pair (X,d), or simply X is called a ***metric space****. If $x \in X$ and $r \in (0,+\infty]$, the set*

$$B_X(x,r) = \{y \in X : d(x,y) < r\}$$

often abbreviated to $B(x,r)$, is called the open ball with center x and radius r. If $r \in [0,+\infty)$,

$$\overline{B}_X(x,r) = \{y \in X : d(x,y) \leq r\}$$

also abbreviated to $\overline{B}(x,r)$ is called the ***closed ball*** *with center x and radius r.*

Unless otherwise stated, the metric on $\mathbb{R}^n$ and $\mathbb{C}^n$(and their subsets) are assumed to be the **Euclidean metrics**.

Example 1. *Let $X = X_1 \times \cdots X_N$ where each X_i is a metric space with metris d_i. Write $x = (x_1, \cdots, x_N) \in X$ and $y = (y_1 \cdots, y_N) \in X$. Then the following are metrics on X:*

$$d(x,y) = d_1(x_1,y_1) + \cdots + d_N(x_N,y_N) \tag{1}$$

$$\delta(x,y) = \max\{d_1(x_1,y_1), \cdots, d_N(x_N,y_N)\} \tag{2}$$

$$\rho(x,y) = \sqrt{d_1(x_1,y_1)^2 + \cdots + d_N(x_n,y_N)^2} \tag{3}$$

With respect to the metris δ, the open balls of X are "polydisks"

$$B_X(x,r) = B_{X_1}(x_1,r) \times \cdots \times B_{X_N}(x_N,r)$$

There is no standard choice of metric on the product of metric spaces. d, δ, ρ are all good, and they are equivalent in the following sense:

Definition 1.4.2. *We say that two metrics d_1, d_2 on a set X are* ***equivalent*** *and write $d_1 \approx d_2$, if there exist $\alpha, \beta > 0$ such that for any $x, y \in X$ we have*

$$d_1(x,y) \leq \alpha d_2(x, \qquad d_2(x,y) \leq \beta d_1(x,y)$$

This is an equivalence relation. More generally, we may write $d_1 \lesssim d_2$ if $d_1 \leq \alpha d_2$ for some $\alpha > 0$. Then $d_1 \approx d_2$ iff $d_1 \lesssim d_2$ and $d_2 \lesssim d_1$.

Example 2. *We have $\delta \leq \rho \leq d \leq N\delta$. So $\delta \approx \rho \approx \rho \approx d$.*

Given finitely many metric spaces $X_1, \cdots, x_N$, the metric on the product space $X = X_1 \times \cdots \times X_N$ is chosen to be any one that is equivalent to the ones defined before. In the case that each X_i is a subset of $\mathbb{R}$ or $\mathbb{C}$, we choose the metric on X to be the **Euclidean metric**

Definition 1.4.3. *Let (X, d) be a metric space. Then a **metric subspace** is denotes an object $(Y, d|Y)$ where $Y \subset X$ and $d|_Y$ is the restriction of d to Y, nemly for all $y_1, y_2 \in Y$ we set*

$$d|_Y(y_1, y_2) = d(y_1, y_2)$$

1.5 Convergence of sequences

Definition 1.5.1. *Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in a metric space X. Let $x \in X$. We say that x is a **limit** of x_n and write $\lim_{n\to\infty} x_n = x$, if: For every real number $\epsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that for every $n \geq N$ we hace $d(x_n, x) < \epsilon$.*

Proposition 1. *Any sequence $(x_n)_{n\in\mathbb{Z}_+}$ in a metric space X has at most one limit*

Proposition 2 (Squeeze theorem)**.** *Suppose that (x_n) is a sequence in a metric space X ,Let $x \in X$. Suppose that there is a sequence (a_n) in $\mathbb{R}_{\geq} 0$ such that $\lim_{n\to\infty} a_n = 0$ and that $d(x_n, x) \leq a_n$ for all n. Then $\lim_{n\to\infty} x_n = x$*

Definition 1.5.2. *X is a metric space. We say X is **sequentially compact** if every sequence in X has a convergent subsequence.*

Lemma 1 (Extreme Value Theorem)**.** *If X is sequentially compact, $f : X \to \mathbb{R}$ continuous. Then f attains its max and min on X. In particular $f(x)$ is a bounded subset of $\mathbb{R}$.*

Example 3. *If $X = A_1 \cup \cdots A_n$, each A_i is sequentially compact, then X is sequentially compact.*

Example 4. *Finite set is sequentially compact.*

Proposition 3. *If X, Y are sequentially compact, then $X \times Y$ is sequentially compact.*

Proof. Pick (x_n, y_n) in $X \times Y$. Since X is sequentially compact, x_n has subsequence, $x_{n_k} \to x \in X$. Since Y is sequentially compact, y_{n_k} has subsequence, $y_{n_{k_i}} \to y \in Y$. Then $(x_{n_{k_i}}, y_{n_{k_i}}) \to (x, y) \in X \times Y$. $\square$

Proposition 4. *If $f : X \to Y$ continuous, X is sequentially compact, then $f(X)$ is sequentially compact.*

Example 5. *If A is sequentially compact subset of $\mathbb{R}$, then $\sup A, \inf A \in A$.*

Theorem 1.5.1. *$[a, b] \subset \overline{\mathbb{R}}$ is sequentially compact. Then $I_1 \times \cdots \times I_n$ is sequentially compact where $I_i = [a, b]$.*

Definition 1.5.3. *If (x_n) is a sequence in X. We say $x \in X$ is a **cluster/accumulation point** of (x_n) if x is a limit point of a subsequence of (x_n).*

Definition 1.5.4. *Let (x_n) in $\overline{\mathbb{R}}$. $\alpha_n = \inf(x_n, x_{n+1}, \cdots), \beta_n = \sup(x_n, x_{n+1}, \cdots)$. Then we have $\alpha_n \leq x_n \leq \beta_n$. **Define***

$$\liminf x_n = \lim_{n\to\infty} \alpha_n = \sup \alpha_n$$

$$\limsup x_n = \lim_{n\to\infty} \beta_n = \inf \beta_n$$

Theorem 1.5.2. *Let $S := \{$cluster point of x_n in $\overline{\mathbb{R}}\}$, $B := \limsup x_n$, $A := \liminf x_n$. Then $B = \max S$, $A = \min S$. In particular, $A, B \in S$, so $S \neq \emptyset$*

Theorem 1.5.3. *Let X be sequentially compact, (x_n) in X. The following are equivalent:*

(1) x_n converges.

(2) (x_n) has only one cluster point.

Corollary 1. *Let (x_n) in $\mathbb{R}^N$. The following is equivalent:*

(1) (x_n) converges.

(2) (x_n) is bounded and has at most one cluster point.

Corollary 2. *Let (x_n) in $\overline{\mathbb{R}}$, then (x_n) converges in $\overline{\mathbb{R}}$ iff* $\liminf x_n = \limsup x_n$.

Corollary 3. *If (x_n) in $\mathbb{R}$. Then (x_n) converges in $\mathbb{R}$ iff (x_n) is bounded and* $\liminf x_n = \limsup x_n$

Definition 1.5.5. *A sequence (x_n) in X is called a* ***Cauchy sequence*** *if*

· $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_+$, *s.t.* $\forall m, n \geq N$, $d(x + m, x_n) < \epsilon$.

· $\forall \epsilon > 0$, $\exists N$, *s.t.* $\forall n \geq N$, $d(x_n, x_N) < \epsilon$.

Cauchy sequence are bounded.

Proposition 5. *Every convergent sequence in X is Cauchy sequence.*

Definition 1.5.6. *X is called* ***complete*** *if every Cauchy sequence in X converges.*

Theorem 1.5.4. *If (x_n) is Cauchy with at least one cluster point. Then (x_n) converges. Therefore, <u>every sequentially compact space is complete</u>.*

Corollary 4. $\mathbb{R}^N, \mathbb{C}^N \cong \mathbb{R}^N$ *are complete under Euclidean metric.*

Definition 1.5.7. *$A \subset X$. We say A is closed if the following is true:*

$$\text{If } (x_n) \in A \text{ converges to } x \in X, \text{ then } x \in A$$

Proposition 6. *$A \subset X$, $d_A = d_X|A$.*

$$\text{(1) } A \text{ is complete; (2) } A \text{ is closed.}$$

Then (1) $\Rightarrow$ (2) and if X is complete then (2) $\Rightarrow$ (1)

Theorem 1.5.5. *If X is a metric space. Then X is compact $\Leftrightarrow$ X is sequentially compact.*

Definition 1.5.8. *X is a topo. space. X is called* ***net-compact*** *if every net in X has a convergent subnet.*
X is called ***countably compact*** *if every countable open cover of X has a finite subcover.*

Easy to see that:

compact $\Rightarrow$ countably compact.

Proposition 7. *For topology space, net-compact $\Leftrightarrow$ compact.*

For metric space, Four compactness are equivalent.

Example 6 (Extreme Value Theorem)**.** *If X is compact, $f : X \to \mathbb{R}$ continuous, then f attains it max(and min).*

It suffices to prove $f(X)$ is bounded.

<u>steps of proof</u>:

Step1: Prove finiteness locally;

Step2: Use compactness to go from local to global.

Proposition 8. *X is a topo. space. TFAE:*

(1) *X is compact.*

(2) *(Increasing chain property) If $(U_\mu)_{\mu \in I}$ is an increasing net of open subsets of X, s.t. $\cup_{\mu \in I} U_\mu = X$, then $\exists \mu \in I$ s.t. $U_\mu = X$.*

(3) *(Decreasing chain property) If $(U_\mu)_{\mu \in I}$ is an dncreasing net of closd subsets of X, s.t. $\forall \mu \in I, E_\mu \neq \emptyset$, then $\cap_{\mu \in I} E_\mu \neq \emptyset$*