2.6 Faraday's Law, Induction and Inductance

2.6.1 Faraday's law of induction

In the last section, we showed that a current produces a magnetic field. A more surprising physical discovery of Faraday is the reverse effect: *a changing magnetic field can induce an electric field that drives a current*. This link between a magnetic field and the electric field it induces is now called *Faraday's law of induction*.

Faraday originally stated his law of induction as "an emf is induced in a loop when the number of magnetic field lines that pass through the loop is changing". The number of magnetic field lines passing through the loop is quantitatively described by the magnetic flux. Suppose a loop enclosing an area A is placed in a magnetic field $\vec{B}$. Then, the magnetic flux through the loop is defined as

$$\Phi_B := \iint_A \vec{B} \cdot \mathrm{d}\vec{A}.$$

Since the magnetic field $\vec{B}$ is divergence-free, i.e., $\vec{\nabla} \cdot \vec{B} = 0$, by Corollary 2.2.2, its flux through a surface S is invariant under local continuous deformation of S. As a consequence, the flux Φ_B is well-defined and does not depend on the particular surface enclosed by the loop that we have chosen. The SI unit for magnetic flux is called the weber (Wb): $1\ \mathrm{Wb} = 1\ \mathrm{T} \cdot \mathrm{m}^2$.

Physics law 12 (Faraday's law of induction)**.** *The magnitude of the emf $\mathscr{E}$ induced in a conducting loop is equal to the rate at which the magnetic flux Φ_B through that loop changes with time. Moreover, the induced emf $\mathscr{E}$ tends to oppose the flux change. In formula, it writes that*

$$\mathscr{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}. \tag{2.6.1}$$

If we change the magnetic flux through a coil of n turns, an induced emf appears in every turn and the total emf induced in the coil is the sum of these individual induced emfs:

$$\mathscr{E} = -n\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}.$$

Here are some common means by which we can change the magnetic flux through a coil:

- Change the magnitude of the magnetic field within the coil.
- Change the area of the coil or the portion of that area that lies within the magnetic field.
- Change the angle between the direction of the magnetic field and the plane of the coil.

The rule of thumb to determine the directions of the induced currents is as follows:

An induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic flux that induces the current.

This is called Lenz's law. We illustrate it with the following figure.

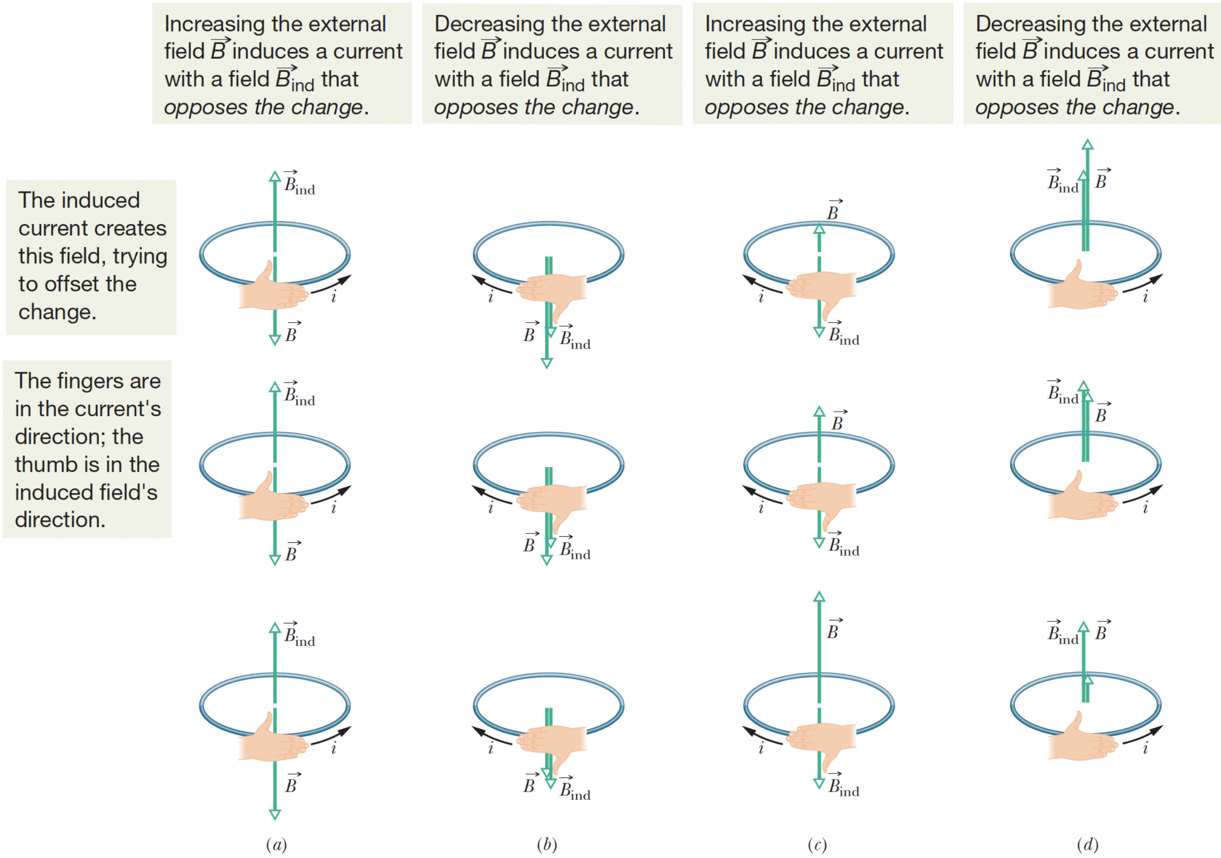


Figure 2.24: The direction of the current i induced in a loop is such that the current's magnetic field $\vec{B}_{\text{ind}}$ opposes the change in the magnetic field $\vec{B}$ inducing i. The field $\vec{B}_{\text{ind}}$ is always directed opposite an increasing field $\vec{B}$ (figures a, c) and in the same direction as a decreasing field $\vec{B}$ (figures b, d). The curled–straight right-hand rule gives the direction of the induced current based on the direction of the induced field.

Example. *Suppose we pull a closed conducting loop out of a uniform magnetic field $\vec{B}$ (pointing inside the paper) at constant velocity v as in Figure 2.25. Suppose the resistance of the loop is R. Find the induced current, the force we act on the loop, the rate of work we do on the loop, and the rate of energy dissipation in the loop.*

Solution: We choose the coordinate axes in Figure 2.25 such that $\vec{v} = v\vec{e}_x$ is along x direction, and $\vec{B}$ is along the $-z$ direction, i.e., $\vec{B} = -B\vec{e}_z$.

As we move the loop to the right, the portion of its area within the magnetic field decreases, so the flux through the loop also decreases. The magnetic flux is $\Phi_B = BLx$, which is changing with rate

$$\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = BL\frac{\mathrm{d}x}{\mathrm{d}t} = -BLv,$$

where the $-$ sign indicates the fact that x is decreasing. Then, by Faraday's law, an emf is induced in the loop with magnitude $\mathscr{E} = BLv$, which leads to an induced current

$$i = \frac{BLv}{R}.$$

Moreover, by Lenz's law, the induced current is in the clockwise direction.

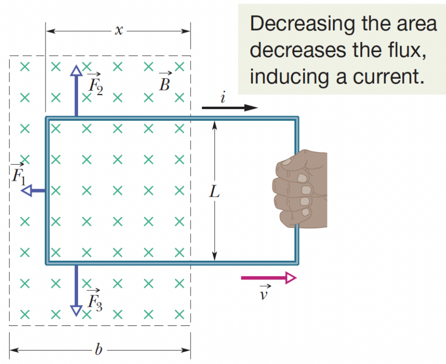


Figure 2.25

With the induced current, we can calculate the magnetic forces on the loop. For the left edge, we have

$$\vec{F}_1 = i(L\vec{e}_y) \times \vec{B} = -iLB\vec{e}_x = -\frac{B^2L^2v}{R}\vec{e}_x.$$

For the upper and lower edges, we have

$$\vec{F}_2 = i(x\vec{e}_x) \times \vec{B} = ixB\vec{e}_y, \quad \vec{F}_3 = -i(x\vec{e}_x) \times \vec{B} = -ixB\vec{e}_y.$$

Hence, to ensure that the loop moves at a constant velocity, the force we should act on the loop is

$$\vec{F} = -(\vec{F}_1 + \vec{F}_2 + \vec{F}_3) = \frac{B^2L^2v}{R}\vec{e}_x.$$

Then, the rate of work done on the loop is

$$\vec{F} \cdot \vec{v} = \frac{B^2L^2v^2}{R}.$$

Finally, the rate of energy dissipation is

$$P = i^2R = \frac{B^2L^2v^2}{R}.$$

Note that this is exactly equal to the rate at which we are doing work on the loop. In other words, this shows that the work that we have done in pulling the loop through the magnetic field finally transfers to the thermal energy in the loop.

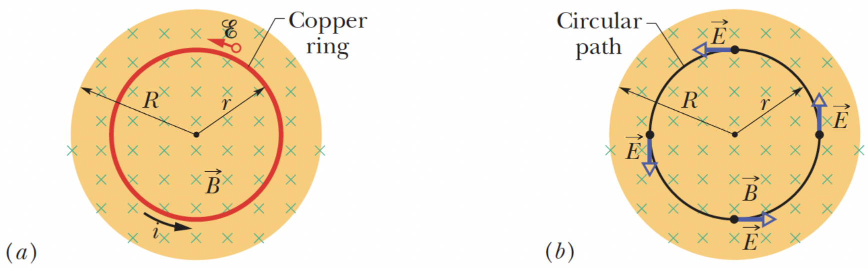
2.6.2 Induced electric fields

Faraday's law of induction tells us that a changing magnetic field induces an emf (and hence a current) in a copper ring. Then, an electric field must be present along the ring because an electric field is needed to move the conduction electrons. This induced electric field $\vec{E}$ by a changing magnetic field is just as real as an electric field produced by static charges—either field will exert a force $q_0\vec{E}$ on a particle of charge q_0. Now, Faraday's law can be equivalently stated as follows:

A changing magnetic field induces an electric field.

Note that this statement is more general in the sense that the electric field is induced even if there is no such copper ring, i.e., the electric field would appear even if the changing magnetic field were in a vacuum.

To derive the formula for the induced electric field, we look at the following figures. We assume that the magnetic field $\vec{B}$ is increasing in magnitude at a rate $\mathrm{d}\vec{B}/\mathrm{d}t$.



In (a), an emf $\mathscr{E}$ is induced along the copper ring as in (2.6.1). In (b), we replace the copper ring with a hypothetical circular path of radius r. Consider a particle of charge q_0 moving around this circular path, the work W done on it in one revolution by the induced electric field is

$$W = q_0\mathscr{E} = -q_0\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -q_0\iint_S \frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\cdot \mathrm{d}\vec{A},$$

where S is the disk enclosed by the circular path. On the other hand, the work done on a particle of charge q_0 along the circular path $\mathcal{C}$ can also be evaluated from the electric field as

$$W = q_0\oint_{\mathcal{C}} \vec{E}\cdot \mathrm{d}\vec{s} = q_0\iint_S (\vec{\nabla}\times\vec{E})\cdot \mathrm{d}\vec{A}.$$

where we used Theorem 2.2.3 in the second step. Comparing the above equations, we obtain that

$$-\iint_S \frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\cdot \mathrm{d}\vec{A} = \iint_S (\vec{\nabla}\times\vec{E})\cdot \mathrm{d}\vec{A}.$$

Note that this equation should hold for any surface S, so there should be

$$\vec{\nabla}\times\vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}. \tag{2.6.2}$$

This is Faraday's law in Maxwell's equations.

Now, we have two types of electric fields: static electric fields produced by static charges, and induced electric fields produced by changing magnetic flux. Although electric fields produced in either way exert forces on charged particles, there is an important difference between them. The static electric field is non-rotational, so there is an electric potential associated with it by Poincaré lemma, Theorem 2.2.4. But this is not the case for induced electric fields, which have non-vanishing curl: *electric potential has no meaning for electric fields that are produced by induction.* We can derive a contradiction assuming there is indeed an electric potential. Consider a charged q_0 that makes a loop around the circular path in the above Figure (b). It starts at a certain point and, on its return to that same point, has experienced an emf $\mathscr{E}$, that is, work of $q_0\mathscr{E}$ has been done on the particle by the electric field. However, that is impossible because the particle is back at the same point, which can have only one particular potential.

The difference between the two types of electric fields can also be told from their field lines. Field lines of induced electric fields form closed loops. But field lines produced by static charges never do so but must start on positive charges and end on negative charges. Thus, a field line from a charge can never loop around and back onto itself.

2.6.3 Inductors and inductance, self-induction

In the previous lecture, we discussed how capacitors can generate desired electric fields. Now we will shift our focus to inductors, which play a crucial role in generating desired magnetic fields. Specifically, we will examine the behavior of a long solenoid, with our attention drawn to a short length near the middle to mitigate any potential fringing effects.

When a current i flows through the windings of the solenoid, it generates a magnetic flux Φ_B in the central region of the inductor. The magnitude of the magnetic flux increases with the current. Similarly, increasing the number of turns N in the solenoid increases the magnetic flux. To quantify the ability of the solenoid to produce magnetic flux, we define the inductance L as:

$$L = \frac{N\Phi_B}{i}. \tag{2.6.3}$$

The inductance represents the ratio of the magnetic flux to the current and provides a measure of the solenoid's capability as an inductor. The SI unit of magnetic flux is the tesla-square meter ($\mathrm{T{\cdot}m^2}$), and as a result, the SI unit of inductance is the tesla-square meter per ampere ($\mathrm{T{\cdot}m^2/A}$). This unit is referred to as the henry (H) in honor of Joseph Henry, an American physicist who co-discovered the law of induction alongside Faraday. Therefore, we can express 1 henry as $1\ \mathrm{H} = 1\ \mathrm{T{\cdot}m^2/A}$.

To calculate the inductance of a solenoid, let's consider a solenoid with length l. The magnetic flux in this region can be expressed as

$$N\Phi_B = (nl)(BA), \tag{2.6.4}$$

where n represents the number of turns per unit length of the solenoid and B is the magnitude of the magnetic field inside the solenoid. As we calculated before, the magnetic field within the solenoid is given by $B = \mu_0 in$, where μ_0 is the permeability of free space and i is the current. Hence, the inductance of the solenoid per unit length can be determined as

$$\frac{L}{l} = \frac{N\Phi_B}{il} = \mu_0 n^2 A. \tag{2.6.5}$$

The previous equation for the inductance of a solenoid provides a good approximation when the solenoid is much longer than its radius. This approximation neglects the spreading of magnetic field lines near the ends of the solenoid. Similarly, the parallel-plate capacitor formula ($C = \epsilon_0 A/d$) neglects the fringing of electric field lines near the edges of the capacitor plates. These approximations hold under specific conditions but may not fully account for the behavior near the boundaries.

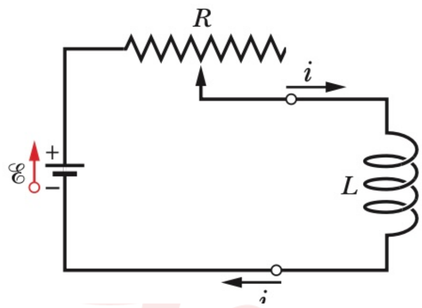


Figure 2.26

If the current in an inductor, such as a coil, is changing, it induces a changing magnetic flux, leading to a self-induced electric field according to Faraday's law. This phenomenon is known as self-induction. By the definition of inductance, $N\Phi_B = Li$, and applying Faraday's law, the electric potential is given by:

$$\mathscr{E} = -\frac{\mathrm{d}(N\Phi_B)}{\mathrm{d}t} = -L\frac{\mathrm{d}i}{\mathrm{d}t}. \tag{2.6.6}$$

Therefore, in any inductor, whether it be a coil, solenoid, or toroid, a self-induced electric field arises whenever the current changes with time, regardless of the current's magnitude. The direction of a self-induced potential is determined by Lenz's law. According to Lenz's law, the self-induced electric field has a direction that opposes the change in current. This is why we have a negative sign in above equation.

2.6.4 RL circuits

In section 2.4.7, we discussed RC circuits that involve both a resistor and a capacitor. Now, let's consider an RL circuit where we have a resistor and an inductor connected together in a circuit. We aim to derive and solve the differential equation that governs the behavior of this circuit.

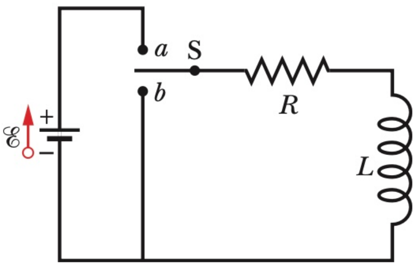


Figure 2.27

Let's first consider the stationary case when the switch is at a. Initially, due to the presence of the inductor, the increasing current gives rise to an induced electric potential that is opposite in direction to the battery potential, following Lenz's law. As time progresses, the current reaches its final maximum value, which is $i = \mathscr{E}/R$. This implies that after a long time, the inductor behaves like an ordinary connecting wire. And the voltage across the two ends of the inductor is zero. So, in the stationary state, the circuit can be simplified to a battery connected to a resistance.

In the stationary state, when the switch is on position b, the circuit consists of the inductor and the resistor without a battery. Without a battery to drive the current, the circuit eventually reaches an equilibrium state where no current flows and no voltage is present across the components.

Now let us analyze the time-dependent situations quantitatively. When the switch S in the circuit is closed on position a, the current in the resistor begins to increase. A self-induced electric potential $\mathscr{E}_L = -L\frac{\mathrm{d}i}{\mathrm{d}t}$ is induced in the resistor, opposing the rise of the current according to Lenz's law. This self-induced potential opposes the battery $\mathscr{E}$. Therefore, the voltage across the resistor is given by $\mathscr{E} + \mathscr{E}_L = \mathscr{E} - L\frac{\mathrm{d}i}{\mathrm{d}t}$. By Ohm's law, the voltage across the resistor should also be iR. Thus, the differential equation for the circuit is:

$$\mathscr{E} - L\frac{\mathrm{d}i}{\mathrm{d}t} - iR = 0. \tag{2.6.7}$$

With the initial conditions $i(t=0) = 0$ and the long-time limit $i(t=+\infty) = \frac{\mathscr{E}}{R}$, the solution to the above differential equation is

$$i(t) = \frac{\mathscr{E}}{R}(1 - e^{-\frac{t}{\tau_L}}), \tag{2.6.8}$$

where the inductive time constant τ_L is given by

$$\tau_L = \frac{L}{R}. \tag{2.6.9}$$

It is customary to verify that the dimension of τ_L defined above does indeed have the dimension of time. The potential difference across the resistor $V_R(t) = i(t)R$ exhibits a similar behavior as the current. The potential difference across the inductor is $|V_L(t)| = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = \mathscr{E}e^{-\frac{t}{\tau_L}}$, which decays exponentially to zero as expected.

On the other hand, if the switch is initially on position a for a long time and then switched to position b at $t = 0$, the initial current through the circuit is $i(0) = \mathscr{E}/R$. Initially, the inductor behaves like a wire. As time progresses without a battery in the circuit, the current will decrease. This time-varying current induces a potential $\mathscr{E}_L = -L\frac{\mathrm{d}i}{\mathrm{d}t}$ across the inductor. This potential should be equal to the potential difference across the resistor $V_R = iR$. Thus, we have the differential equation

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + iR = 0. \tag{2.6.10}$$

Using the initial condition $i(0) = \mathscr{E}/R$, the solution to the above differential equation is

$$i(t) = \frac{\mathscr{E}}{R}e^{-t/\tau_L}, \tag{2.6.11}$$

where the inductive time constant is again $\tau_L = \frac{L}{R}$. We observe that the current decays exponentially from its initial value $\mathscr{E}/R$ to zero.

2.6.5 Energy of a magnetic field

Let us consider the RL circuit again. By multiplying i to both sides of the RL differential equation, we obtain

$$\mathscr{E}i = Li\frac{\mathrm{d}i}{\mathrm{d}t} + i^2R. \tag{2.6.12}$$

The left-hand side $\mathscr{E}i = \mathscr{E}\frac{\mathrm{d}q}{\mathrm{d}t}$ can be interpreted as the rate at which the battery delivers energy to the rest of the circuit. On the right-hand side, the second term i^2R represents the rate at which energy appears as thermal energy in the resistor. Therefore, the first term $Li\frac{\mathrm{d}i}{\mathrm{d}t}$ should be understood as the rate of energy lost due to the inductor. Since an inductor stores energy in the form of a magnetic field, we can say that it represents the rate of energy stored in the magnetic field:

$$\frac{\mathrm{d}U_B}{\mathrm{d}t} = Li\frac{\mathrm{d}i}{\mathrm{d}t}. \tag{2.6.13}$$

By integrating this expression, we obtain the magnetic energy

$$U_B = \int \frac{\mathrm{d}U_B}{\mathrm{d}t}\mathrm{d}t = \int_0^i Li'\mathrm{d}i' = \frac{1}{2}Li^2. \tag{2.6.14}$$

The expression bears a similarity to the expression for electric energy stored in a capacitor, $U_E = \frac{q^2}{2C}$.

If the inductor is a long solenoid of cross-sectional area A and length l, the magnetic energy density of it is approximately given by

$$u_B = \frac{U_B}{Al} = \frac{L}{l}\frac{i^2}{2A} = \mu_0 n^2 A\frac{i^2}{2A} = \frac{B^2}{2\mu_0}. \tag{2.6.15}$$

In this equation, we used the relationship $\frac{L}{l} = \mu_0 n^2 A$ from Eq. (2.6.5) and $B = \mu_0 i n$ for a long solenoid. This final expression represents the density of stored magnetic energy at any point where the magnitude of the magnetic field is B. Although we derived it specifically for the case of a solenoid, it holds true for all magnetic fields, regardless of their generation. Notably, the magnetic energy density expression bears a resemblance to the expression for electric energy density, $u_E = \frac{1}{2}\epsilon_0 E^2$.

2.6.6 LC harmonic oscillations

We have discussed RC and RL circuits in previous lectures. In this section, we will consider an LC circuit consisting of an inductor with inductance L and a capacitor with capacitance C. Let's denote the charge on the capacitor as q, and the current in the circuit as i, which is equal to $\frac{\mathrm{d}q}{\mathrm{d}t}$. The electric potential differences across the inductor and the capacitor are $-L\frac{\mathrm{d}i}{\mathrm{d}t}$ and $\frac{q}{C}$, respectively. Therefore, we have the differential equation

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{q}{C} = 0, \tag{2.6.16}$$

or equivalently,

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -\omega^2 q. \tag{2.6.17}$$

The angular frequency is given by

$$\omega = \frac{1}{\sqrt{LC}} \tag{2.6.18}$$

This equation represents a harmonic oscillator. The solution to this differential equation is

$$q = Q\cos(\omega t + \phi), \tag{2.6.19}$$

where Q is the amplitude of the oscillation and ϕ is the phase angle. The corresponding current is given by

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = -\omega Q\sin(\omega t + \phi) \tag{2.6.20}$$

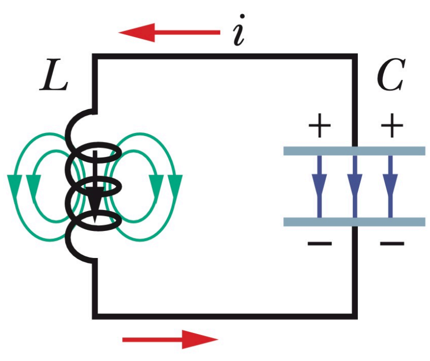


Figure 2.28

In a harmonic oscillator, such as a spring system, we know that the total energy is conserved. The energy is transformed between kinetic energy and the potential energy of the spring. Similarly, in the LC circuit, we expect total energy conservation. The total energy consists of the electric energy of the capacitor and the magnetic energy of the inductor. Using the energy formulas we derived earlier, the total energy is given by:

$$U = U_E + U_B = \frac{1}{2}Li^2 + \frac{q^2}{2C}. \tag{2.6.21}$$

The conservation of this total energy implies that:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = Li\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{q}{C}\frac{\mathrm{d}q}{\mathrm{d}t} = Li\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{q}{C}i = 0. \tag{2.6.22}$$

By eliminating i on both sides of the equation, this equation is precisely the harmonic oscillator differential equation that we derived earlier. It confirms that the total energy of the LC circuit is conserved, just like in other harmonic oscillator systems.

2.6.7 RLC damped oscillations

In this subsection, we will delve into the behavior of RLC circuits. RLC circuits are fundamental electrical circuits that incorporate resistors (R), inductors (L), and capacitors (C).

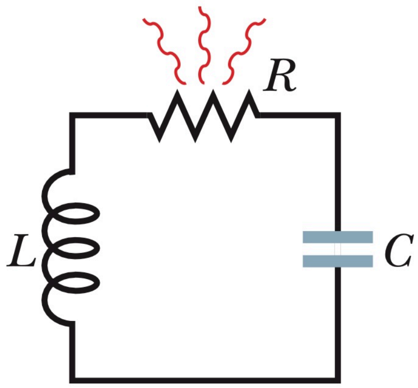


Figure 2.29

The differential equation for a series RLC circuit can be derived by applying Kirchhoff's law to the circuit. The voltage drop across the resistor is given by Ohm's Law as $V_R(t) = i(t)R$, where $i(t)$ is the current flowing through the circuit. According to Faraday's law, the voltage across the inductor is given by $V_L(t) = -L(\mathrm{d}i(t)/\mathrm{d}t)$. The voltage across the capacitor is given by $V_C(t) = (1/C)\int i(t)\mathrm{d}t$. Applying Kirchhoff's law to the series RLC circuit, we have:

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0. \tag{2.6.23}$$

where we used the relation $i(t) = \mathrm{d}q(t)/\mathrm{d}t$. This is the differential equation that describes the behavior of a series RLC circuit. Interestingly, it has the same form as the equation of motion for a damped harmonic oscillator, as given by Eq. (1.5.14):

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0. \tag{2.6.24}$$

By making the following correspondences between physical quantities and parameters:

$$\begin{cases} q & \leftrightarrow & x \\ i = \frac{\mathrm{d}q}{\mathrm{d}t} & \leftrightarrow & v = \frac{\mathrm{d}x}{\mathrm{d}t} \end{cases}, \qquad \begin{cases} L & \leftrightarrow & m \\ R & \leftrightarrow & b \\ C & \leftrightarrow & \frac{1}{k} \end{cases}, \tag{2.6.25}$$

we can establish a direct connection between the series RLC circuit and the damped harmonic oscillator.

Depending on the values of the parameters, the series RLC circuit can exhibit three different types of behavior:

- Underdamped: When $R < \sqrt{\frac{4L}{C}}$, the circuit is underdamped. In this case, the system exhibits oscillatory behavior, with the current and charge undergoing decaying oscillations.
- Overdamped: When $R > \sqrt{\frac{4L}{C}}$, the circuit is overdamped. In this scenario, the system does not exhibit oscillations. The current and charge approach their equilibrium values gradually without oscillatory behavior.
- Critically damped: When $R = \sqrt{\frac{4L}{C}}$, the circuit is critically damped. Here, the system reaches equilibrium in the shortest time possible without oscillating.

Understanding the behavior of the RLC circuit in terms of the damped harmonic oscillator helps us analyze and predict the response of the circuit under different parameter values.

Additionally, we can introduce an alternating current (AC) battery to the RLC circuit, which adds a time-varying term $\mathscr{E}(t)$ to the system. In this scenario, the differential equation has the same form as that of the forced damped harmonic oscillator, with the battery acting as an external driving force. By leveraging the results obtained for the forced damped harmonic oscillator, we can gain insights into the behavior of the RLC circuit with the battery. We can study, for example, the phenomena of resonance, where the circuit exhibits a maximum response to a specific frequency of the AC battery. By understanding the behavior of the forced damped harmonic oscillator, we can indeed explore its applications in various fields, including signal processing and communication systems. One of the key applications is signal amplification or magnification. The forced damped harmonic oscillator can be used to enhance or amplify weak signals by applying an AC battery as external force to the system.