Differential Geometry

LIN150117

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

October 17, 2024

Contents

1 Smooth Manifold

Definition 1.1 (Topological manifold)**.** A space M is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

Definition 1.2 (Smooth Manifold)**.** A smooth structure is given by an equivalence class of smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ *s.t.* $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth $\forall \alpha, \beta$. $M = \cup U_\alpha$.
A **smooth manifold** is a topological manifold with a smooth structure.
Define when a continuous map $f : M_1 \to M_2$ is smooth if $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$, we have $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is smooth.

Definition 1.3. Given $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$. A homeomorphism $f : M_1 \to M_2$ is called a diffeomorphism if f, f^{-1} is smooth.
In this case we say $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ are diffeomorphism.

Theorem 1.4 (Kervaire)**.** *$\exists$ 1 10-dimensional topological manifold without smooth manifold.*

Theorem 1.5 (Milnor)**.** *$\exists$ a smooth manifold M s.t. $M \cong S^7$ but not in diffeomorphism meaning.*

Theorem 1.6 (Kervaire-Milnor)**.** *$\exists$ 28 smooth structures (up to orientation preserving diffeomorphism) on S^7*

Theorem 1.7 (Morse-Birg)**.** *On S^7. If $n \leqslant 3$, then any n-dimensional topological manifold M has a unique smooth structure up to diffeomorphism.*

Theorem 1.8 (Stallings). *If $n \neq 4$, then $\exists$ a unique smooth structure on $\mathbb{R}^n$ up to diffeomorphism.*

Theorem 1.9 (Donaldson-Freedom-Gompf-Faubes). *$\exists$ uncountable smooth structures on $\mathbb{R}^4$ up to diffeomorphism.*

Definition 1.10 (topological manifold with boundary). A space M is called a topological manifold with boundary if

1. M is Hausdorff
2. M is second countable
3. $\forall p \in M$, $\exists$ a neighbourhood U of p and a homeomorphism $\varphi : U \to V$ where V is open in $\mathbb{H}^n$

We say a manifold M is closed if M is compact and ∂M is empty.

Our motivation for studying manifold is to study the space of solution for equations.

Question 1. Given $f : \mathbb{R}^n \to \mathbb{R}$ smooth, $q \in \mathbb{R}^n$, when is $f^{-1}(q)$ is a smooth manifold?

For $f : U \to \mathbb{R}^n$ smooth, U open in $\mathbb{R}^m$, the differential of f at $p \in U$ denoted as $\mathrm{d}f(p)$.

Definition 1.11. We say $p \in U$ is a **regular point** of f if $\mathrm{d}f(p)$ is surjective. Otherwise we say $p \in U$ is a **critical point**.
A point $q \in \mathbb{R}^n$ is called a **regular value** of f if $\forall p \in f^{-1}(q)$, p is a regular point of f.
A point $q \in \mathbb{R}^n$ is called a **critical value** of f if $\forall p \in f^{-1}(q)$, p is a critical point of f.

Theorem 1.12 (Implicit function theorem). *If $p \in U$ is a regular point of $f : U \to \mathbb{R}^n$. Then there exists*

- *An open neighbourhood V of p in U*
- *An open subset V' of $\mathbb{R}^m$*
- *A diffeomorphism $\varphi : V \to V'$ such that $P \circ \varphi = f$ where P is the projection from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

In other words, near a regular point, we can do local coordinate change to turn f into the projection.

Remark 1.13. In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \ldots, x_m) \in V' | (x_1, \cdots .x_n) = f(p)\}$$

i.e. if we set $M = f^{-1}(f(p))$, then $(M \cap V, \varphi_p)$ is a chart that contains p.

Corollary 1.14. *If q is a regular value of $f : U \to \mathbb{R}^n$ then $f^{-1}(q)$ is a smooth manifold.*

Remark 1.15. It suffices to show that the corresponding charts are compatible.

Theorem 1.16 (Sard). *If $f : U \to \mathbb{R}^n$ is a smooth map, then the set of critical values of f has measure 0.*

Remark 1.17. For a "generic" q, $f^{-1}(q)$ is a manifold of dimension $m - n$.

Corollary 1.18. *If $f : U \to \mathbb{R}^n$ is smooth and $m < n$ then $f(U)$ has measure 0.*

1.1 Lie groups and homogeneous spaces

Definition 1.19. We say G is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map $\cdot : G \times G \to G$ and the inverse map $G \rightsquigarrow G$ is smooth.

Example 1.20. $GL(n, \mathbb{R}) = \{n \times n \text{ matrices with non-zero determinant}\} \subset \mathbb{R}^{n\times n}$

$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$

$SO(n) = \{A \in O(n) | \det A = 1\}$

$U(n) = \{A \in GL(n, \mathbb{C}) | A\overline{A}^T = 1\}$

$SU(n) = \{A \in U(n) | \det A = 1\}$

Exercise 1.21.

$$O(1) \cong S^2 \qquad SO(1) \cong * \tag{1.1}$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \tag{1.2}$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \tag{1.3}$$

The last one is a diffeomorphism but do not preserve the multiplicatioin, *i.e.* not an isomorphism of Lie group.

Theorem 1.22 (Carton)**.** *Let H be a closed subgroup of Lie group G. Then H is a Lie group. More precisely, H is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also, G/H is a smooth manifold*

Definition 1.23. Let M be a smooth manifold. We say M is a **homogeneous space** if $\exists$ a Lie group G with a smooth transitive action $\rho : G \times M \to M$.

Definition 1.24. For M be a homogeneous space. The **isotropy** group of $x \in M$ is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of G

Given any $x, x' \in M$, $Iso(x) \cong Iso(x')$ because the group action is transitive.

Hence, we have a well-defined map

$$p : G/_{Iso(x)} \to M \tag{1.4}$$

$$g \mapsto gx \tag{1.5}$$

Theorem 1.25. *p is always a diffeomorphism.*

Therefore, we have this proposition

Proposition 1.26. *M is a homogeneous space $\Leftrightarrow$ $M = G/H$ for some closed subgroup H.*

Example 1.27. If $M = S^n$, let $G = SO(n+1)$.
Then $Iso(1, 0, \cdots, 0) \cong SO(n)$.
So $S^n \cong SO(n+1)/(SO(n))$.
Similarly, we can prove $\mathbb{RP}^n \cong SO(n+1)/(O(n))$, $\mathbb{CP}^n \cong SO(n+1)/(U(n))$
The isotropy k dimensional linear subspaces of $\mathbb{R}^n$ can be $O(k) \times O(n-k)$ if $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to $\mathbb{RP}^2, S^2, T^2$ and Klein bottle.

Theorem 1.28 (Whithead)**.** *Any smooth manifold has a triangulation.*

Theorem 1.29 (Poincare-Hopf)**.** *G is compact Lie group $\Rightarrow \chi(G) = 0$.*

Theorem 1.30 (Mostow2005)**.** *M is a compact homogeneous space $\Rightarrow \chi(M) \geqslant 0$.*

1.2 Bump Function and Partition of Unity

Theorem 1.31 (Urysohn smooth version)**.** *Given M, closed disjoint A, B, $\exists$ smooth $f : M \to [0,1]$ s.t. $f|_A = 0, f|_B = 1$.*

Theorem 1.32 (Tietze)**.** *Given M, closed A, smooth $f : A \to \mathbb{R}^n$, there exists smooth $\hat{f} : M \to \mathbb{R}^n$ s.t. $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

Lemma 1.33. *Let U be a neighbourhood of $p \in M$. Then $\exists$ smooth $\sigma : M \to [0,1]$ s.t.*

1. $\sigma \equiv 1$ *near* p
2. *Supp* $\sigma \subset U$

Such σ is called a ***bump function*** *at p, supported in U.*

Definition 1.34. An open cover of a space X is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

Proposition 1.35. *Given compact $K \subset U$ and open neighbourhood U of K, $\exists$ a smooth $g : M \to [0, +\infty)$ s.t. $g|_K \equiv 1$ and $Supp\, g \subset U$.*

Definition 1.36. An **exhaust** of a space X is a sequence of open sets $\{U_i\}$ *s.t.*

1. $X = \bigcup\limits_{i=1}^{\infty} U_i$
2. $\overline{U_i}$ is compact and contained in U_{i+1}

Theorem 1.37. *Any topological manifold has an exhaust.*

Given two open covers $\mathcal{U}, \mathcal{V}$, we say $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if $\forall U_\alpha \in U$, $\exists V_\beta \in \mathcal{V}$ *s.t.* $V_\beta \subset U_\alpha$.

We say a space X is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact.(The proof is hard)

Proposition 1.38. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a topological manifold M. Then there exists countable open covers $\mathcal{W} = \{W_i\}$, $\mathcal{V} = \{V_i\}$ s.t.*

- *For any i, $\overline{V_i}$ is compact and $\overline{V_i} \subset W_i$*

- *$\mathcal{W}$ is locally finite.*

- *$\mathcal{W}$ is a refinement of $\mathcal{U}$.*

As a corollary, we have any topological manifold is paracompact.

Definition 1.39. Given open cover $\mathcal{U}$ of a smooth M, a partition of unity subordinate to $\mathcal{U}$ is a collection of smooth functions $\{\rho_\alpha : M \to [0,1]\}_{\alpha\in\mathcal{A}}$ *s.t.*

1. $\forall p \in M$, $\exists$ only finitely many $\alpha \in \mathcal{A}$ *s.t.* $p \in Supp\,\rho_\alpha$

2. $\sum\limits_{\alpha\in\mathcal{A}} \rho_\alpha(p) = 1$

3. $Supp\,\rho_\alpha \subset U_\alpha$

Theorem 1.40 (Existence of P.O.U)**.** *For any open cover $\mathcal{U}$ of smooth M, $\exists$ a P.O.U subordinate to $\mathcal{U}$*

Theorem 1.41 (Whitney approximation theorem)**.** *Given any smooth M, any closed A and any continuous $f : M \to \mathbb{R}$, $\delta : M \to (0, +\infty)$. Suppose f is smooth on A. Then $\exists g : M \to \mathbb{R}$ smooth s.t.*

- $g|_A = f|_A$

- $\forall p \in M$, $|g(p) - f(p)| < \delta(p)$.

2 Tangent space and tangent vectors

2.1 Tangent Space

Given $p \in M$, consider the set $C_p^\infty(M) = \{\text{smooth function } V \to \mathbb{R}\}/_\sim$ where $f_1 \sim f_2$ if and only if $\exists$ neighbourhood U of p, $f_1|_U = f_2|_U$.

$C_p^\infty(M)$ is the space of **genus of smooth function** near p.

A **partial-derivative** of p is a $\mathbb{R}$-linear map $D : C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

Definition 2.1. A **tangent vector** of M at p is a partial-derivative at p.

Define the **tangent space** $T_pM = \{$all partial-derivative at p $\}$, which is a $\mathbb{R}$-vector space.

Proposition 2.2. *For $M = U \subset \mathbb{R}^n$ open. We have $\{\frac{\partial}{\partial x_i}\}$ is a basis for T_pU.*

Proposition 2.3.

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leqslant i \leqslant n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

M, N smooth manifolds, $C^\infty(N, M) = \{$smooth $F : N \to M\}$.

Given $F \in C^\infty(N, M)$, F induces $F^* : C^\infty_{F(p)}(M) \to C_p^\infty(N)$, $f \mapsto f \circ F$.

By taking dual, we get

$$F_* : T_pN \to T_{F(p)}M$$

we also write F_* as $F_{*,p}$, call it the **differential** of F at p.

where

$$F_*(\frac{\partial}{\partial x^i}|_p) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

Proposition 2.4. *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_pN \to T_{G \circ F(p)}W$$

Definition 2.5. A smooth **curve** is a smooth map $\gamma : (a, b) \to M$. We say γ starts at p if $\gamma(0) = p$. We define the **velocity** of γ at $\gamma(0)$ as $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)}M$

Take charts $(U, x^1, \cdots, x^n)$ about p, let $\gamma^i = x^i \circ \gamma$.

We say γ, δ are **tangent** to each other at p if $(\gamma^i)'(0) = (\delta^i)'(0)$.

Now we can define

$$(T_pM)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\ \}/_\sim$$

where $\gamma \sim \delta$ iff they are tangent to each other.

Then these definition is more geometric.

Lemma 2.6. *Given $F \in C^\infty(M, M)$, $p \in N$, the diagram commutes:*

$$\begin{array}{ccc} \gamma \in (T_pN)_{curve} & \xrightarrow{\cong} & T_pN \\ \downarrow & & \downarrow \\ F \circ \gamma \in (T_{F(p)}M)_{curve} & \xrightarrow{\cong} & T_{F(p)}M \end{array}$$

2.2 Tangent Bundle

Let $(M, \mathcal{A})$ be a smooth manifold, $TM = \bigcup\limits_{p \in M} T_pM$, called the **tangent bundle**

Now we want to define a natural topology and smooth structure on TM. Take any chart $(U, \varphi) = (U, x^1, \cdots, x^n) \in \mathcal{A}$.

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \tag{2.1}$$

$$X \in T_pU \mapsto (\varphi(p), X^1, \cdots, X^n) \tag{2.2}$$

where $X = \sum X^i \frac{\partial}{\partial x^i}|_p$.

Then pull back standard topology on $\varphi(U) \times \mathbb{R}^n$ to a topology on TU.

$$\mathcal{B} = \{\hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- $\mathcal{B}$ is a basis
- $\mathcal{B}$ generates a Hausdorff, second countable topology on TM.

So TM is a topological manifold covered by charts $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$.

Given $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$, the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi} \circ \hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \tag{2.3}$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \tag{2.4}$$

So $\hat{\mathcal{A}}$ is a smooth atlas on TM, making TM into a smooth manifold.

Definition 2.7 (vector bundle). Given a continuous map $f : E \to B$, we say f is a n-dimensional **vector bundle** if: $\exists$ an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of B and homeomorphisms $\{f^{-1}(U_\alpha) \underset{\cong}{\xrightarrow{\rho_\alpha}} U_\alpha \times \mathbb{R}\}$ *s.t.*

- $\begin{tikzcd} f^{-1}(U_\alpha) \arrow[r, "\rho_\alpha"] \arrow[d, "f"] & U_\alpha \times \mathbb{R}^n \arrow[dl, "projection"] \\ U_\alpha & \end{tikzcd}$ commutes for $\alpha \in I$
- $\forall\, p \in U_\alpha \cap U_\beta$, the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

  is linear.

Call $f^{-1}(p)$ the **fiber** over p.

Proposition 2.8. *Given vector bundle $f : E \to B$, the fiber $f^{-1}(p)$ has a structure of a vector space.*

Example 2.9 (Product bundle). $E = \mathbb{R}^n \times B$

Example 2.10 (Tautological bundle).

$$B = \mathbb{CP}^n = \{\text{1-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map $(L, v) \mapsto L$

Given vector bundles $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$, a bundle map consists of $(\hat{f}, f)$ *s.t.*

- $\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$ commutes.
- $\forall b \in B$, $\hat{f} : \pi_1^{-1}(b) \to \pi_2^{-1}(f(b))$ is linear.

If $\hat{f}, f$ are diffeomorphisms, then we call $(\hat{f}, f)$ an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

Example 2.11. TS^1, TS^2 are both trivial.

$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$

Theorem 2.12. *If G is a Lie group, then TG is trivial.*

Proof. For $(x^1, x^2, \cdots, x^n)$ is a basis of T_eG The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG,\ (g, c^1, \cdots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \to G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism $(l_g)_*$

□

Proposition 2.13 (Adams, 1960s). *TS^n is trivial if and only if $n = 0, 1, 3, 7$.*

Proposition 2.14. *1. Given any $F \in C^\infty(M, N)$, $F_* : TM \to TN$ is a bundle map.*

2. TS^n is isomorphic to the following bundle:

$$B = s^n \qquad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

Definition 2.15 (smooth section). Given a smooth vector bundle $\pi : E \to B$, a **smooth section** is a smooth map $S : B \to E \quad s.t. \ \pi \circ S = id_b$.

$s_0 : B \to E, b \mapsto 0 \in$ 0-vector in $\pi^{-1}b$.

2.3 Vector Field, Curves and Flows

Definition 2.16. A (tangent) **vector field** is a smooth section of TM. *i.e.* a smooth map $M \xrightarrow{X} TM \quad s.t. \ X(p) \in T_pM, \forall p \in M$

Given any $f : \mathbb{R}^n \to \mathbb{R}$, define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leqslant i \leqslant n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

Example 2.17. $X = f^1 \partial x^1 + f^2 \partial x^2$ is a gradient field if and only if $\dfrac{\partial f^1}{\partial x^2} = \dfrac{\partial f^2}{\partial x^1}$

Theorem 2.18 (Poincare-Hopf). *For closed M, M has a nowhere vanishing vector field if and only if $\chi(M) = 0$.*

So S^n has a nowhere vanishing vector field if and only if n is odd.

Theorem 2.19 (MaoQiu). *S^2 has no no-where vanishing vector field.*

So We cannot smooth out all the hairs on a ball.

Given a vector field $X = \{X_p\}_{p \in M}$, a curve $\gamma : (a,b) \to M$ is called an **integral curve** of X if $\gamma'(t) = X_{\gamma(t)}$, $\forall t \in (a,b)$, where $\gamma'(t) = \gamma_*(\frac{\partial}{\partial t}) \in T_{\gamma(t)}M$.

We say γ is maximal if the domain cannot be extended to a larger interval. Denote the set of all smooth vector fields on M by $\mathfrak{X}M$

Recall that γ is maximal if it's domain can not be extended to a large open interval.

In a local chart $(U, x^1, \cdots, x^n)$, $X|_U = \sum_{i=1}^{n} a^i \partial x^i$. Then γ is an integral curve if and only if $(\gamma^i)'(t) = a^i(\gamma(t))$, $\forall 1 \leqslant i \leqslant n$, where $\gamma^i = x^i \circ \gamma : (a,b) \to \mathbb{R}$.

And in this case the initial value condition: $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$.

Locally, solving integral curve starting at p is equivalent to solving ODE with initial value $p^1, \cdots, p^n$. By existence and uniqueness of solutions of ODE, we have

Theorem 2.20 (Fundamental theorem of integral curve). *Let $X \in \mathfrak{X}M$, $p \in M$, then:*

(1) (Uniqueness) Given any two integral curves $\gamma_1, \gamma_2 : (a,b) \to M$, then we have:

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a,b) \Rightarrow \gamma_1 = \gamma_2$$

(2) there exists a unique max integral curve $\gamma : (a(p), b(p)) \to M$ starting at p.

(3) (integral curve smoothly depend on initial values) $\exists$ Nbh U of p, $\epsilon > 0$, and smooth $\varphi : (-\epsilon, \epsilon) \times U \to M$ s.t. $\forall q \in U$, $\varphi_\epsilon := \varphi(-, q) : (-\epsilon, \epsilon) \to M$ is an integral curve starting at p.

*we call such φ a local **flow** generated by X.*

Definition 2.21. Given $X \in \mathfrak{X}M$, a global **flow** generated by X is a smooth map $\varphi : \mathbb{R} \times M \to M$ *s.t.* $\forall q \in M$, $\varphi_q := \varphi(-, q)$ is the maximal integral curve of X starting at q.

$\Leftrightarrow \dfrac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s,p)}, \forall s \in \mathbb{R}, p \in M$ and $\varphi(0, p) = p, \forall p \in M$.

If such global flow exists, then we say X is **complete**.

Example 2.22.

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$ is complete, where global flow $\varphi : \mathbb{R} \times M \to M$, $\varphi(t, p) = p \cdot e^t$.
- $X = x^2 \partial x$ is not complete. Max integral curve starting at 1 is given by $\gamma(t) = \dfrac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$.

Given $X \in \mathfrak{X}M$, we define $\mathrm{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$.

Theorem 2.23. *If a vector field X is compactly supported, then X is complete.*

Corollary 2.24. *Any vector field on closed manifold is complete.*

Lemma 2.25 (Escaping lemma). *Suppose $\gamma : (a, b) \to M$ is a max integral curve, with $(a, b) \neq \mathbb{R}$. Then $\nexists$ compact $K \subset M$ s.t. $\gamma(a, b) \subset K$*

Proof. Otherwise, suppose $\gamma(a, b) \subset K$. WLOG, we may assume $b < +\infty$.

Take $(t_i) \to b$ from left. Then $\gamma(t_i) \in K$. After passing to subsequence, we may assume $(\gamma(t_i)) \to p \in K$.

Then $\exists$ U Nbh of p, local flow $\varphi : (-\epsilon, \epsilon) \times U \to M$. Take n large enough *s.t.* $b - t_n < \epsilon, \gamma(t_n) \in U$. Then $\gamma(- + t_n) : (a - t_n, b - t_n) \to M$, $\varphi(-, \gamma(t_n)) : (-\epsilon, \epsilon) \to M$ are both integral curves for X starting at $\gamma(t_n)$. By uniqueness, they coincide.

Let $\hat{\gamma} : (a, t_n + \epsilon) \to M$ be defined by $\hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \epsilon) \end{cases}$

Then $\hat{\gamma}$ is an integral curve with larger domain, then γ contradiction with the maxity of γ. $\square$

Proof of 2.23. Take any max integral curve $\gamma : (a, b) \to M$. Suppose $(a, b) \neq \mathbb{R}$. Then $X_{\gamma(s)} \neq 0$, $\forall s$. Otherwise, the constant map $\mathbb{R} \to M, t \mapsto \gamma(s)$ is an integral curve with lager domain.

So $\forall s$, $\gamma(s) \in \mathrm{Supp}X \Rightarrow \gamma(a, b) \subset \mathrm{Supp}X$ which is compact $\Rightarrow (a, b) = \mathbb{R}$ by the lemma. This causes contradiction! $\square$

A smooth $\varphi : \mathbb{R} \times M \to M$ is called an **one-parameter transformation group** if

(1) $\varphi_0 := \varphi(0, -) = \mathrm{id}_M$

(2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}$. In particular, $\varphi_s^{-1} = \varphi_{-s}$.

Theorem 2.26. *$\varphi \in C^\infty(\mathbb{R} \times M, M)$, then φ is an one-parameter transformation group if and only if φ is the global flow generated by some $X \in \mathfrak{X}M$*

Lemma 2.27 (Translation lemma)**.** *If $\gamma : (a, b) \to M$ is an integral curve for some $X \in \mathfrak{X}M$, then $\forall s \in \mathbb{R}$, $\gamma(- + s) : (a - s, b - s) \to M$ is also an integral curve for X.*

Proof. Let $\iota = \gamma(- + s)$. Then $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$ $\square$

Lemma 2.28. *Let $\varphi : (-\epsilon, \epsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$ provided that $s, t, s + t \in (-\epsilon, \epsilon), p, \varphi_r(p) \in U$.*

Proof. $\gamma_p = \varphi(-, p)$ is an integral curve for X.

$\Rightarrow \gamma_p(- + s)$ is an integral curve for X starting at $\gamma_p(s) = \varphi_s(p)$. But $\gamma_{\varphi_s(p)}$ is also an integral curve starting at $\varphi_s(p)$. Thus $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$ $\square$

Lemma 2.29. *Let $\varphi : (-\epsilon, \epsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$ i.e. any vector field is invariant under its flow.*

Proof. Take $f \in C^\infty_{\varphi(p)}(M)$.

$$\varphi(s,*)(X_p)(f) = X_p(f \circ \varphi_s) \tag{2.5}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \tag{2.6}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \tag{2.7}$$
$$= X_{\varphi_s(p)}(f) \tag{2.8}$$

□

Proof of 2.26. "$\Leftarrow$" is because the lemma $\varphi_s \circ \varphi_r = \varphi_{s+r}$

"$\Rightarrow$" Let $X = \{X_p\}$ where $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$.

Leave it as an exercise. □

Time dependent vector field is a smooth map $X : \mathbb{R} \times M \to TM \quad s.t.\ X_{(t,p)} \in T_pM$.

A smooth curve $\gamma(a,b) \to M$ is the **integral curve** for X if $\gamma'(t) = X_{(t,\gamma(t))}$.

In local chart, solving γ is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness: γ_1, γ_2 are both integral curves for X, $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define $\mathrm{Supp} X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$.

Then X is compactly supported, then X is complete(*i.e.* a global flow $\varphi : \mathbb{R} \times M \to M$)

But something is not true for time dependent vector field:

- translation lemma is not true.

- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

2.4 Another definition of vector field

A derivation on M is a $\mathbb{R}$-linear map $C^\infty(M) \xrightarrow{D} C^\infty(M)$ that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

Theorem 2.30. *We have a bijection:*

$$\rho : \mathfrak{T}M \xrightarrow{1:1} \{\textit{derivation on } M\}$$

$$X \mapsto D_X : f \mapsto X(f)$$

Lemma 2.31. *$D_p : \mathfrak{T}_pM \to \mathbb{R}$-linear map $\mathbb{C}^\infty(M) \to \mathbb{R}$ s.t. $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$ is an isomorphism of vector spaces.*

Proof. Leave it as an exercise. □

Lemma 2.32. *Given a vector field(not necessarily smooth) $X = \{X_p\}_{p \in M}$, X is smooth $\Leftrightarrow \forall f \in C^\infty(M), X(f)$ is smooth.*

Proof. "$\Leftarrow$" $\forall p \in M$, take chart $(U, x^1, x^2, \cdots, x^n)$ around p. $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$ $f^i : U \to \mathbb{R}$, where $f^i = X|_U(x^i)$. Take $\varphi : M \to [0,1]$ *s.t.* $\varphi \equiv 1$ near p, Supp$\varphi \subset U$,$\varphi \cdot x^i \in C^\infty(M)$.

Then $X(\varphi \cdot x^i) = f^i$ near p. By assumption, f^i is smooth near p. So f^i is smooth, so X is smooth.

"$\Rightarrow$" Similar. □

Theorem 2.33. *The map $\rho : \mathfrak{X}M \to \{derivation\ on\ M\}, X \mapsto (D_x : f \mapsto X(f))$ is well-defined and bijective.*

Proof. ρ is well-defined: $X(f) \in C^\infty(M)$ by Lemma 2.32, and $D_x(fg) = D_x(f)g + fD_x(g)$ since X is a point-derivation.

ρ is injective: $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$ as maps $C^\infty(M)$ to $\mathbb{R}$. By Lemma 2.31, we have $X_p = Y_p$, $\forall p$. So $X = Y$.

ρ is surjective: Given $D : C^\infty(M) \to C^\infty(M)$. Define $D_p : C^\infty(M) \to \mathbb{R}$ by $D_p(f) := D(f)(p)$ satisfies the Leibniz rule. By Lemma 2.31, $D_p = D_{X_p}$ for some $X_p \in T_pM$. Define $X = \{X_p\}_{p\in M}$. Then $X(f) = D(f)$, $\forall f \in C^\infty(M)$. By Lemma**??**, X is a smooth vector field. $\square$

2.5 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned}\{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\}\\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \quad s.t.\\ &\qquad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\}\end{aligned}$$

$$\begin{aligned}\{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\}\\ &= \{\text{derivation on } M\}\end{aligned}$$

Notation 2.34. We will identify $X \in \mathfrak{X}M$ with its derivation $D_x : C^\infty(M) \to C^\infty(M)$. So a vector field is just a $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M)$ *s.t.* $X(fg) = fX(g) + X(f)g$.

Definition 2.35 (Lie bracket)**.** Given two (smooth) vector field $X, Y : C^\infty(M) \to C^\infty(M)$, we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

Theorem 2.36. *For any $X, y \in \mathfrak{T}M$, $[X, Y] \in \mathfrak{T}M$*

Proof. Easy to check that $[X, Y]$ is linear.

By Leibuniz rule,

$$\begin{aligned}
[X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\
&= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\
&= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\
&= [X, Y](f) \cdot f \cdot [X, Y](g)
\end{aligned}$$

□

So What is the geometric meaning of $[X, Y]$? Non commutatiy of flows.

Fact 2.37. *Given $X, Y \in \mathfrak{T}M$, we say X, Y are commutative vector field if $[X, Y] = 0$*

X, Y are commuative iff for any local flows $\varphi^X : (-\epsilon, \epsilon) \times U \to M$, $\varphi^Y : (-\epsilon, \epsilon) \times U \to M$ we have $\varphi^X_s \circ \varphi^T_t = \varphi^Y_t \circ \varphi^X_s$

Proposition 2.38 (Calculation of $[V, W]$ using local charts)**.** *Chart $(U, x^1, \cdots, x^n)$, $V, W \in \mathfrak{T}M$, $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$, $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$. Then*

$$\begin{aligned}
[V, W]|_U &= \sum_{i=1}^n (V(W^i - W(V^i))) \frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^n (\sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial X^j}) \frac{\partial}{\partial x^j}
\end{aligned}$$

$$= \sum_{1\leqslant i,j\leqslant n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^j}$$

Example 2.39. $V = x\partial x + y\partial y$, $W = -y\partial x + x\partial y$ commutes.

Proposition 2.40 (Properties of Lie bracket)**.**

(a) Natuality under push-forword.

Given any $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{T}M, W \in \mathfrak{T}M$, we have $[F_* V, F_* W] = F_*[V, W]$.

(b) $\mathbb{R}$-linearity $\forall a, b \in \mathbb{R}$

$$[aX + bV, W] = a[X, W] + b[V, W]$$
$$[W, aX + bV] = b[W, X] + a[W, V]$$

(c) anti-symmetric $[V, W] = -[W, V]$

(d) Jacobi identity

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

(f) Leibuniz rule

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

Definition 2.41. Given $F \in C^\infty(M, N)$, $V \in \mathfrak{T}M$, $W \in \mathfrak{T}N$. We say W is F**-related** to V if $\forall p \in M$, $F_{p,*}(V_p) = W_{F(p)}, F_{p,X} : T_pM \to T_{f(p)}N$

Example 2.42. $F : S^1 \to \mathbb{R}^2, \theta \mapsto (\cos\theta, \sin\theta)$, $V = \partial\theta, W = -y\partial x + x\partial y$.

note 1. In general, given $V \in \mathfrak{X}M$ and $F \in C^\infty(M, N)$. There may not exist $W \in \mathfrak{X}M$ *s.t.* V, W are F-related. Even such W exists, it may not be unique.

However, if F is a diffeomorphism, given any V, $\exists$ unique W *s.t.* V and W are F-related. Actually, $W_p = F_* V_{F^{-1}(p)}$.

Such W is called **push forward** of V along F, denoted by $F_* V$, only defined when F is a diffeomorphism.

Lemma 2.43. $\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N$, $F \in C^\infty(M, N)$. *Then* W *is* F*-related to* V *iff* $\forall f \in C^\infty N, V(f \circ F) = W(f) \circ F \in C^\infty(M)$

Proof. Check that $F_{p,*}(V_p)(f) = W_{F(p)}(f), \forall f \in C^\infty(N)$ □

Proposition 2.44. *Given* $V_0, V_1 \in \mathfrak{X}M$, $W_0, W_1 \in \mathfrak{X}N$, $F \in C^\infty(M, N)$, W_i *is* F*-related to* V_i, $i = 0, 1 \Rightarrow [W_0, W_1]$ *is* F*-related to* $[V_0, V_1]$

Corollary 2.45 (Natuality of Lie bracket)**.** *Given any* $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, *we have* $[F_* V, F_* W] = F_*[V, W]$

The rest of Proposition 2.40 is easy to check if it is viewed as a mapping $C^\infty(M) \to C^\infty(M)$.

2.6 Lie algebra of a Lie group

Definition 2.46. A **Lie algebra** g is $\mathbb{R}$-linear space g with map $[-, -] : g \times g \to g$ *s.t.* it is bilinear, anti-symmetric and satisfies the Jacobian identity.

Then $(\mathfrak{X}M, [-, -])$ is an infinite dimensional Lie algebra.

For G Lie group, $\forall g \in G$ we have diffeomorphism

$$l^g : G \to G, h \mapsto gh$$
$$r^g : G \to G, h \mapsto hg$$

We say $X \in \mathfrak{X}G$ is **left invariant** if $l^g_*(X) = X$, $\forall g \in G$. Similarly, X is **right invariant** if $r^g_*(X) = X$.

Proposition 2.47. *X, Y are left/right invariant $\Rightarrow [X, Y]$ is left/right invariant.*

Proof. $l^g_*[X, Y] = [l^g_* X, l^g_* Y] = [X, Y]$ $\square$

So we can find a natural Lie algebra of G:

$$\mathrm{Lie}(G) := \{\text{left invariant vector fields on } G\}, \text{with } [-,-] \text{ restricted from } \mathfrak{X}G$$

Theorem 2.48. *Given any $V \in T_eG$, $\exists$ unique left invariant $\hat{V} \in \mathfrak{X}G$ s.t. $\hat{V}_e = V$.*

Corollary 2.49. $\mathrm{Lie}(G) \cong T_eG$ *as vector spaces.*

Proof of Theorem 2.48. **Uniqueness of $\hat{V}$:** $\hat{V}_g = l^g_{e,*}(\hat{V}_e) = l^g_{e,*}(V)$. So $\hat{V}$ is determined by V.

Existence of $\hat{V}$: Let $\hat{V} = \{\hat{V}_g\}_{g \in G}$ where $\hat{V}_g = l^g_{e,*}(\hat{V}_e)$.

$\hat{V}$ is left-invariant because

$$(l^h_*(\hat{V}))_g = l^h_{h^{-1}g,*}(\widehat{h^{-1}g}) = l^h_{h^{-1}g,*}(l^{h^{-1}g}_{e,*}(V)) = l^g_{e,*}(V) = \hat{V}_g$$

$\hat{V}$ is smooth: Take any $f \in C^\infty(G)$ suffices to show $\hat{V}(f) \in C^\infty(G)$.

Take any smooth $\gamma : \mathbb{R} \to G$ s.t. $\gamma(0) = e, \gamma'(0) = V$. Then $l^g \circ \gamma : \mathbb{R} \to G$ satisfies $l^g \circ \gamma(0) = g, (l^g \circ \gamma)(0) = g, (l^g \circ \gamma)'(0) = l^g_{e,*}(V) = \hat{V}_g$

So

$$\hat{V}(f)(g) = \hat{V}_g(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(l^g \circ \gamma(t))|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(g \cdot \gamma(t))|_{t=0} \tag{2.9}$$

Consider the map

$$\hat{f} : G \times \mathbb{R} \xrightarrow{\mathrm{id} \times \gamma} G \times G \qquad\qquad \xrightarrow{\cdot} G \xrightarrow{f} \mathbb{R}$$

$$(g,t) \mapsto (g,\gamma(t)) \qquad\qquad \mapsto g\cdot\gamma(t) \mapsto f(g\cdot\gamma(t))$$

Then $\hat{f}$ s smooth, $\frac{\partial \hat{f}}{\partial t}|_{t=0} : G \to \mathbb{R}$ is smooth, but $\frac{\partial f}{\partial t}|_{t=0}(g) = \hat{V}(f)(g)$ by 2.9. So $\hat{V}(f) \in C^\infty(G)$. □

Example 2.50. $G = \mathrm{GL}(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) |\det A \neq 0\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^2$.

$\mathrm{gl}(n,\mathbb{R})$= $\mathrm{Lie}(\mathrm{GL}(n,\mathbb{R})) = T_I\mathrm{GL}(n,\mathbb{R}) = M_n(\mathbb{R})$

Theorem 2.51. $\forall A, B \in \mathrm{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$, $[A,B] = AB - BA$.

Remark 2.52. This theorem shows that the Lie bracket viewed as the Lie algebra and

Lemma 2.53. $\forall A \in \mathrm{gl}(n,\mathbb{R})$, *the left invariant vector field* $\hat{A}$ *is complete and generated the flow* $\varphi_t : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2t^2}{2!} + \cdots)$

Proof.

$$\hat{A}_g = g\cdot A \in T_gG = M_n(\mathbb{R})$$

$$\frac{\partial}{\partial t}\varphi_t(g) = \frac{\partial}{\partial t}(g(e^{At})) = ge^{At}A = \hat{A_{g\cdot e^{At}}} = \hat{A}_{\varphi_t(g)}$$

□

Proof of Theorem 2.51. Take $A, B \in \mathrm{gl}(n,\mathbb{R})$. Want to show $[\hat{A},\hat{B}]_I = AB - BA$.

Pick $f \in C_I^\infty(G)$, need to show $A(\hat{B}(f)) - B(\hat{A}(f)) = (AB - BA)(f)$

Further Simplification: Just need to focus on $f = x^{ij}$, where $x^ij : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}, E \mapsto (E-I)_{ij}$.

Such f satisfies $f(I + -)$ is $\mathbb{R}$-linear.

Recall that Given $W \in \mathfrak{X}M$, $W(f)(p) = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t^W(p))|_{t=0}$.

So $\hat{B}(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(ge^{tB})|_{t=0}$.

So

$$A(\hat{B}(f)) = \frac{\mathrm{d}}{\mathrm{d}t}(\hat{B}(f)(e^{As}))|_{s=0} = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t}f(I+sA+tB+\frac{s^2}{2}A^2+stAB+\frac{t^2}{2}B^2+\cdots)|_{s=t=0}$$

Similarly,

$$B(\hat{A}(f)) = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t}f(I + sA + tB + \frac{s^2}{2}A^2 + stBA + \frac{t^2}{2}B^2 + \cdots)|_{s=t=0}$$

So $A(\hat{B}(f)) - B(\hat{A}(f)) = f(I + (AB - BA)) = (AB - BA)(f)$ since f is $\mathbb{R}$-linear. $\square$

Similarly, for $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{Lie}(G) = \mathrm{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$, we have $[A, B] = AB - BA$.

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group G, its universal cover $\hat{G}$ is simply-connected with $\pi^{-1}(G) \subset Z(\hat{G})$.

What is the meaning of Lie bracket. There is a fact about it:

Fact 2.54. *G is connected Lie group. G is abelian iff $[-,-] = 0$ on $\mathrm{Lie}(G)$*

2.7 Morphisms between Lie group and Lie algebras

A smooth map $F : G \to H$ between two Lie group is called a **morphism** if $F(gh) = F(g)F(h)$.

A linear map $L : g \to h$ between Lie algebra is called a **morphism** if $L[u, v] = [Lu, Lv]$.

Claim 2.55. *$\hat{W}_i$ is F-compatible with $\hat{V}_i$ for $i = 0, 1$.*

Proof. $\forall g \in G$, $F_*(\hat{V}_g) = F_*(l^g_*(V)) = (F \circ l^g)_*(V) = (l^{F(g)} \circ F)_*(V) = l^{F(g)}(W) = \hat{W}_{F(g)}$ □

In particular, $[W_0, W_1] = F_*([V_0, V_1])$.

Index

List of Theorems