

Differential Geometry

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1 Smooth Manifold

Definition 1.1 (Topological manifold). A space M is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

Definition 1.2 (Smooth Manifold). A smooth structure is given by an equivalence class of smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ s.t. $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth $\forall \alpha, \beta$. $M = \cup U_\alpha$.

A **smooth manifold** is a topological manifold with a smooth structure.

Define when a continuous map $f : M_1 \rightarrow M_2$ is smooth if $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$, we have $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$ is smooth.

Definition 1.3. Given $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$. A homeomorphism $f : M_1 \rightarrow M_2$ is called a diffeomorphism if f, f^{-1} is smooth.

In this case we say $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ are diffeomorphism.

Theorem 1.4 (Kervaire). \exists a 10-dimensional topological manifold without smooth manifold.

Theorem 1.5 (Milnor). \exists a smooth manifold M s.t. $M \cong S^7$ but not in diffeomorphism meaning.

Theorem 1.6 (Kervaire-Milnor). \exists 28 smooth structures (up to orientation preserving diffeomorphism) on S^7

Theorem 1.7 (Morse-Birg). On S^7 . If $n \leq 3$, then any n -dimensional topological manifold M has a unique smooth structure up to diffeomorphism.

Theorem 1.8 (Stallings). *If $n \neq 4$, then \exists a unique smooth structure on \mathbb{R}^n up to diffeomorphism.*

Theorem 1.9 (Donaldson-Freedman-Gompf-Faubes). *\exists uncountable smooth structures on \mathbb{R}^4 up to diffeomorphism.*

Definition 1.10 (topological manifold with boundary). A space M is called a topological manifold with boundary if

1. M is Hausdorff
2. M is second countable
3. $\forall p \in M, \exists$ a neighbourhood U of p and a homeomorphism $\varphi : U \rightarrow V$ where V is open in \mathbb{H}^n

We say a manifold M is closed if M is compact and ∂M is empty.

Our motivation for studying manifold is to study the space of solution for equations.

Question 1. Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth, $q \in \mathbb{R}^n$, when is $f^{-1}(q)$ is a smooth manifold?

For $f : U \rightarrow \mathbb{R}^n$ smooth, U open in \mathbb{R}^m , the differential of f at $p \in U$ denoted as $df(p)$.

Definition 1.11. We say $p \in U$ is a **regular point** of f if $df(p)$ is surjective. Otherwise we say $p \in U$ is a **critical point**.

A point $q \in \mathbb{R}^n$ is called a **regular value** of f if $\forall p \in f^{-1}(q)$, p is a regular point of f .

A point $q \in \mathbb{R}^n$ is called a **critical value** of f if $\forall p \in f^{-1}(q)$, p is a critical point of f .

Theorem 1.12 (Implicit function theorem). *If $p \in U$ is a regular point of $f : U \rightarrow \mathbb{R}^n$. Then there exists*

- *An open neighbourhood V of p in U*
- *An open subset V' of \mathbb{R}^m*
- *A diffeomorphism $\varphi : V \rightarrow V'$ such that $P \circ \varphi = f$ where P is the projection from \mathbb{R}^m to \mathbb{R}^n .*

In other words, near a regular point, we can do local coordinate change to turn f into the projection.

Remark 1.13. In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \dots, x_m) \in V' \mid (x_1, \dots, x_n) = f(p)\}$$

i.e. if we set $M = f^{-1}(f(p))$, then $(M \cap V, \varphi_p)$ is a chart that contains p .

Corollary 1.14. *If q is a regular value of $f : U \rightarrow \mathbb{R}^n$ then $f^{-1}(q)$ is a smooth manifold.*

Remark 1.15. It suffices to show that the corresponding charts are compatible.

Theorem 1.16 (Sard). *If $f : U \rightarrow \mathbb{R}^n$ is a smooth map, then the set of critical values of f has measure 0.*

Remark 1.17. For a "generic" q , $f^{-1}(q)$ is a manifold of dimension $m - n$.

Corollary 1.18. *If $f : U \rightarrow \mathbb{R}^n$ is smooth and $m < n$ then $f(U)$ has measure 0.*

1.1 Lie groups and homogeneous spaces

Definition 1.19. We say G is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map $\cdot : G \times G \rightarrow G$ and the inverse map $G \rightsquigarrow G$ is smooth.

Example 1.20. $GL(n, \mathbb{R}) = \{n \times n \text{ matrices with non-zero determinant}\} \subset \mathbb{R}^{n \times n}$

$$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$$

$$SO(n) = \{A \in O(n) | \det A = 1\}$$

$$U(n) = \{A \in GL(n, \mathbb{C}) | A\bar{A}^T = I\}$$

$$SU(n) = \{A \in U(n) | \det A = 1\}$$

Exercise 1.21.

$$O(1) \cong S^2 \qquad SO(1) \cong * \qquad (1.1)$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \qquad (1.2)$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \qquad (1.3)$$

The last one is a diffeomorphism but do not preserve the multiplication, *i.e.* not an isomorphism of Lie group.

Theorem 1.22 (Cartan). *Let H be a closed subgroup of Lie group G . Then H is a Lie group. More precisely, H is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also, G/H is a smooth manifold*

Definition 1.23. Let M be a smooth manifold. We say M is a **homogeneous space** if \exists a Lie group G with a smooth transitive action $\rho : G \times M \rightarrow M$.

Definition 1.24. For M be a homogeneous space. The **isotropy** group of $x \in M$ is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of G

Given any $x, x' \in M$, $Iso(x) \cong Iso(x')$ because the group action is transitive.

Hence, we have a well-defined map

$$p : G/Iso(x) \rightarrow M \qquad (1.4)$$

$$g \mapsto gx \tag{1.5}$$

Theorem 1.25. *p is always a diffeomorphism.*

Therefore, we have this proposition

Proposition 1.26. *M is a homogeneous space $\Leftrightarrow M = G/H$ for some closed subgroup H .*

Example 1.27. If $M = S^n$, let $G = SO(n + 1)$.

Then $ISO(1, 0, \dots, 0) \cong SO(n)$.

So $S^n \cong SO(n + 1)/(SO(n))$.

Similarly, we can prove $\mathbb{RP}^n \cong SO(n + 1)/(O(n))$, $\mathbb{CP}^n \cong SO(n + 1)/(U(n))$

The isotropy k dimensional linear subspaces of \mathbb{R}^n can be $O(k) \times O(n - k)$ if $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to \mathbb{RP}^2, S^2, T^2 and Klein bottle.

Theorem 1.28 (Whithead). *Any smooth manifold has a triangulation.*

Theorem 1.29 (Poincare-Hopf). *G is compact Lie group $\Rightarrow \chi(G) = 0$.*

Theorem 1.30 (Mostow2005). *M is a compact homogeneous space $\Rightarrow \chi(M) \geq 0$.*

1.2 Bump Function and Partition of Unity

Theorem 1.31 (Urysohn smooth version). *Given M , closed disjoint A, B , \exists smooth $f : M \rightarrow [0, 1]$ s.t. $f|_A = 0, f|_B = 1$.*

Theorem 1.32 (Tietze). *Given M , closed A , smooth $f : A \rightarrow \mathbb{R}^n$, there exists smooth $\hat{f} : M \rightarrow \mathbb{R}^n$ s.t. $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

Lemma 1.33. *Let U be a neighbourhood of $p \in M$. Then \exists smooth $\sigma : M \rightarrow [0, 1]$ s.t.*

1. $\sigma \equiv 1$ near p

2. $\text{Supp } \sigma \subset U$

Such σ is called a **bump function** at p , supported in U .

Definition 1.34. An open cover of a space X is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

Proposition 1.35. *Given compact $K \subset U$ and open neighbourhood U of k , $\exists \alpha$ a smooth $g : M \rightarrow [0, +\infty)$ s.t. $g|_K \equiv 1$ and $\text{Supp } g \subset U$.*

Definition 1.36. An **exhaust** of a space X is a sequence of open sets $\{U_i\}$ s.t.

1. $X = \bigcup_{i=1}^{\infty} U_i$

2. $\overline{U_i}$ is compact and contained in U_i

Theorem 1.37. *Any topological manifold has an exhaust.*

Given two open covers \mathcal{U}, \mathcal{V} , we say \mathcal{V} is a **refinement** of \mathcal{U} if $\forall U_\alpha \in \mathcal{U}, \exists V_\beta \in \mathcal{V}$ s.t. $V_\beta \subset U_\alpha$.

We say a space X is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact.(The proof is hard)

Proposition 1.38. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a topological manifold M . Then there exists countable open covers $\mathcal{W} = \{W_i\}, \mathcal{V} = \{V_i\}$ s.t.*

- For any i , $\overline{V_i}$ is compact and $\overline{V_i} \subset W_i$

- \mathcal{W} is locally finite.
- \mathcal{W} is a refinement of \mathcal{U} .

As a corollary, we have any topological manifold is paracompact.

Definition 1.39. Given open cover \mathcal{U} of a smooth M , a partition of unity subordinate to \mathcal{U} is a collection of smooth functions $\{\rho_\alpha : M \rightarrow [0, 1]\}_{\alpha \in \mathcal{A}}$ s.t.

1. $\forall p \in M, \exists$ only finitely many $\alpha \in \mathcal{A}$ s.t. $p \in \text{Supp } \rho_\alpha$
2. $\sum_{\alpha \in \mathcal{A}} \rho_\alpha(p) = 1$
3. $\text{Supp } \rho_\alpha \subset U_\alpha$

Theorem 1.40 (Existence of P.O.U). *For any open cover \mathcal{U} of smooth M , \exists a P.O.U subordinate to \mathcal{U}*

Theorem 1.41 (Whitney approximation theorem). *Given any smooth M , any closed A and any continuous $f : M \rightarrow \mathbb{R}$, $\delta : M \rightarrow (0, +\infty)$. Suppose f is smooth on A . Then $\exists g : M \rightarrow \mathbb{R}$ smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M, |g(p) - f(p)| < \delta(p)$.

2 Tangent space and tangent vectors

2.1 Tangent Space

Given $p \in M$, consider the set $C_p^\infty(M) = \{\text{smooth function } V \rightarrow \mathbb{R}\} / \sim$ where $f_1 \sim f_2$ if and only if \exists neighbourhood U of p , $f_1|_U = f_2|_U$.

$C_p^\infty(M)$ is the space of **germs of smooth function** near p .

A **partial-derivative** of p is a \mathbb{R} -linear map $D : C_p^\infty(M) \rightarrow \mathbb{R}$ that satisfies the Leibnize rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

Definition 2.1. A **tangent vector** of M at p is a partial-derivative at p .

Define the **tangent space** $T_p M = \{\text{all partial-derivative at } p\}$, which is a \mathbb{R} -vector space.

Proposition 2.2. For $M = U \subset \mathbb{R}^n$ open. We have $\{\frac{\partial}{\partial x_i}\}$ is a basis for $T_p U$.

Proposition 2.3.

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leq i \leq n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

M, N smooth manifolds, $C^\infty(N, M) = \{\text{smooth } F : N \rightarrow M\}$.

Given $F \in C^\infty(N, M)$, F induces $F^* : C_{F(p)}^\infty(M) \rightarrow C_p^\infty(N)$ $f \rightarrow f \circ F$.

By taking dual, we get

$$F_* : T_p N \rightarrow T_{F(p)} M$$

we also write F_* as $F_{*,p}$, call it the **differential** of F at p .

where

$$F_*\left(\frac{\partial}{\partial x^i}|_p\right) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

Proposition 2.4. The differential satisfies the composition law.

$$(G \circ F)_* = G_* \circ F_* : T_p N \rightarrow T_{G \circ F(p)} W$$

Definition 2.5. A smooth **curve** is a smooth map $\gamma : (a, b) \rightarrow M$. We say γ starts at p if $\gamma(0) = p$. We define the **velocity** of γ at $\gamma(0)$ as $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)}M$

Take charts (U, x^1, \dots, x^n) about p , let $\gamma^i = x^i \circ \gamma$.

We say γ, δ are **tangent** to each other at p if $(\gamma^i)'(0) = (\delta^i)'(0)$.

Now we can define

$$(T_p M)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\} / \sim$$

where $\gamma \sim \delta$ iff they are tangent to each other.

Then these definition is more geometric.

Lemma 2.6. Given $F \in C^\infty(M, M)$, $p \in N$, the diagram commutes:

$$\begin{array}{ccc} \gamma \in (T_p N)_{curve} & \xrightarrow{\cong} & T_p N \\ \downarrow & & \downarrow \\ F \circ \gamma \in (T_{F(p)} M)_{curve} & \xrightarrow{\cong} & T_{F(p)} M \end{array}$$

2.2 Tangent Bundle

Let (M, \mathcal{A}) be a smooth manifold, $TM = \bigcup_{p \in M} T_p M$, called the **tangent bundle**

Now we want to define a natural topology and smooth structure on TM . Take any chart $(U, \varphi) = (U, x^1, \dots, x^n) \in \mathcal{A}$.

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \quad (2.1)$$

$$X \in T_p U \mapsto (\varphi(p), X^1, \dots, X^n) \quad (2.2)$$

where $X = \sum X^i \frac{\partial}{\partial x^i} |_p$.

Then pull back standard topology on $\varphi(U) \times \mathbb{R}^n$ to a topology on TU .

$$\mathcal{B} = \{ \hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- \mathcal{B} is a basis
- \mathcal{B} generates a Hausdorff, second countable topology on TM .

So TM is a topological manifold covered by charts $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$.

Given $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$, the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi} \circ \hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \quad (2.3)$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \quad (2.4)$$

So $\hat{\mathcal{A}}$ is a smooth atlas on TM , making TM into a smooth manifold.

Definition 2.7 (vector bundle). Given a continuous map $f : E \rightarrow B$, we say f is a n -dimensional **vector bundle** if: \exists an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of B and homeomorphisms $\{f^{-1}(U_\alpha) \xrightarrow[\cong]{\rho_\alpha} U_\alpha \times \mathbb{R}^n\}$ s.t.

- $$\begin{array}{ccc} f^{-1}(U_\alpha) & \xrightarrow{\rho_\alpha} & U_\alpha \times \mathbb{R}^n \\ \downarrow f & \swarrow \text{projection} & \\ U_\alpha & & \end{array} \quad \text{commutes for } \alpha \in I$$
- $\forall p \in U_\alpha \cap U_\beta$, the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

is linear.

Call $f^{-1}(p)$ the **fiber** over p .

Proposition 2.8. *Given vector bundle $f : E \rightarrow B$, the fiber $f^{-1}(p)$ has a structure of a vector space.*

Example 2.9 (Product bundle). $E = \mathbb{R}^n \times B$

Example 2.10 (Tautological bundle).

$$B = \mathbb{CP}^n = \{1\text{-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map $(L, v) \mapsto L$

Given vector bundles $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$, a bundle map consists of (\hat{f}, f) s.t.

- $$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow \pi & & \downarrow \pi \\ B_1 & \xrightarrow{f} & B_2 \end{array} \text{ commutes.}$$
- $\forall b \in B, \hat{f} : \pi_1^{-1}(b) \rightarrow \pi_2^{-1}(f(b))$ is linear.

If \hat{f}, f are diffeomorphisms, then we call (\hat{f}, f) an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

Example 2.11. TS^1, TS^2 are both trivial.

$$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$$

Theorem 2.12. *If G is a Lie group, then TG is trivial.*

Proof. For (x^1, x^2, \dots, x^n) is a basis of $T_e G$ The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG, (g, c^1, \dots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \rightarrow G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism $(l_g)_*$

□

Proposition 2.13 (Adams, 1960s). TS^n is trivial if and only if $n = 0, 1, 3, 7$.

Proposition 2.14. 1. Given any $F \in C^\infty(M, N)$, $F_* : TM \rightarrow TN$ is a bundle map.

2. TS^n is isomorphic to the following bundle:

$$B = S^n \quad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

Definition 2.15 (smooth section). Given a smooth vector bundle $\pi : E \rightarrow B$, a **smooth section** is a smooth map $S : B \rightarrow E$ s.t. $\pi \circ S = id_B$.

$$s_0 : B \rightarrow E, b \mapsto 0 \in 0\text{-vector in } \pi^{-1}b.$$

2.3 Vector Field, Curves and Flows

Definition 2.16. A (tangent) **vector field** is a smooth section of TM . i.e. a smooth map $M \xrightarrow{X} TM$ s.t. $X(p) \in T_p M, \forall p \in M$

Given any $f : \mathbb{R}^n \rightarrow \mathbb{R}$, define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

Example 2.17. $X = f^1 \partial x^1 + f^2 \partial x^2$ is a gradient field if and only if $\frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}$

Theorem 2.18 (Poincare-Hopf). For closed M , M has a nowhere vanishing vector field if and only if $\chi(M) = 0$.

So S^n has a nowhere vanishing vector field if and only if n is odd.

Theorem 2.19 (MaoQiu). S^2 has no no-where vanishing vector field.

So We cannot smooth out all the hairs on a ball.

Given a vector field $X = \{X_p\}_{p \in M}$, a curve $\gamma : (a, b) \rightarrow M$ is called an **integral curve** of X if $\gamma'(t) = X_{\gamma(t)}, \forall t \in (a, b)$, where $\gamma'(t) = \gamma_*\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M$.

We say γ is maximal if the domain cannot be extended to a larger interval.

Denote the set of all smooth vector fields on M by $\mathfrak{T}M$

Recall that γ is maximal if it's domain can not be extended to a large open interval.

In a local chart (U, x^1, \dots, x^n) , $X|_U = \sum_{i=1}^n a^i \partial x^i$. Then γ is an integral curve if and only if $(\gamma^i)'(t) = a^i(\gamma(t)), \forall 1 \leq i \leq n$, where $\gamma^i = x^i \circ \gamma : (a, b) \rightarrow \mathbb{R}$.

And in this case the initial value condition: $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$.

Locally, solving integral curve starting at p is equivalent to solving ODE with initial value p^1, \dots, p^n . By existence and uniqueness of solutions of ODE, we have

Theorem 2.20 (Fundamental theorem of integral curve). Let $X \in \mathfrak{T}M, p \in M$, then:

(1) (Uniqueness) Given any two integral curves $\gamma_1, \gamma_2 : (a, b) \rightarrow M$, then we have:

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a, b) \Rightarrow \gamma_1 = \gamma_2$$

(2) there exists a unique max integral curve $\gamma : (a(p), b(p)) \rightarrow M$ starting at p .

(3) (integral curve smoothly depend on initial values) \exists Nbh U of $p, \epsilon > 0$, and smooth

$\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$ s.t. $\forall q \in U, \varphi_\epsilon := \varphi(-, q) : (-\epsilon, \epsilon) \rightarrow M$ is an integral curve starting at p .

we call such φ a local **flow** generated by X .

Definition 2.21. Given $X \in \mathfrak{X}M$, a global **flow** generated by X is a smooth map $\varphi : \mathbb{R} \times M \rightarrow M$ s.t. $\forall q \in M, \varphi_q := \varphi(-, q)$ is the maximal integral curve of X starting at q .

$$\Leftrightarrow \frac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s, p)}, \forall s \in \mathbb{R}, p \in M \text{ and } \varphi(0, p) = p, \forall p \in M.$$

If such global flow exists, then we say X is complete.

Example 2.22.

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$ is complete, where global flow $\varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \varphi(t, p) = p \cdot e^t$.
- $X = x^2 \partial x$ is not complete. Max integral curve starting at 1 is given by $\gamma(t) = \frac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$.

Given $X \in \mathfrak{X}M$, we define $\text{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$.

Theorem 2.23. If a vector field X is compactly supported, then X is complete.

Corollary 2.24. Any vector field on closed M is complete.

Lemma 2.25 (Escaping lemma). Suppose $\gamma : (a, b) \rightarrow M$ is a max integral curve, with $(a, b) \neq \mathbb{R}$. Then \nexists compact $K \subset M$ s.t. $\gamma(a, b) \subset K$

Proof. Otherwise, suppose $\gamma(a, b) \subset K$. WLOG, we may assume $b < +\infty$.

Take $(t_i) \rightarrow b$ from left. Then $\gamma(t_i) \in K$. After passing to subsequence, we may assume $(\gamma(t_i)) \rightarrow p \in K$.

Then $\exists U$ Nbh of p , local flow $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$. Take n large enough s.t. $b - t_n < \epsilon, \gamma(t_n) \in U$. Then $\gamma(- + t_n) : (a - t_n, b - t_n) \rightarrow M, \varphi(-, \gamma(t_n)) : (-\epsilon, \epsilon) \rightarrow M$ are both integral curves for X starting at $\gamma(t_n)$. By uniqueness, they coincide.

$$\text{Let } \hat{\gamma} : (a, t_n + \epsilon) \rightarrow M \text{ be defined by } \hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \epsilon) \end{cases}$$

Then $\hat{\gamma}$ is an integral curve with larger domain, then γ contradiction with the maximality of γ . □

Proof of 2.23. Take any max integral curve $\gamma : (a, b) \rightarrow M$. Suppose $(a, b) \neq \mathbb{R}$. Then $X_{\gamma(s)} \neq 0, \forall s$. Otherwise, the constant map $\mathbb{R} \rightarrow M, t \mapsto \gamma(s)$ is an integral curve with larger domain.

So $\forall s, \gamma(s) \in \text{Supp} X \Rightarrow \gamma(a, b) \subset \text{Supp} X$ which is compact $\Rightarrow (a, b) = \mathbb{R}$ by the lemma. This causes contradiction! \square

A smooth $\varphi : \mathbb{R} \times M \rightarrow M$ is called an **one-parameter transformation group** if

$$(1) \quad \varphi_0 := \varphi(0, -) = \text{id}_M$$

$$(2) \quad \varphi_s \circ \varphi_t = \varphi_{s+t} \text{ for all } s, t \in \mathbb{R}. \text{ In particular, } \varphi_s^{-1} = \varphi_{-s}.$$

Theorem 2.26. $\varphi \in C^\infty(\mathbb{R} \times M, M)$, then φ is an one-parameter transformation group if and only if φ is the global flow generated by some $X \in \mathfrak{X}M$

Lemma 2.27 (Translation lemma). If $\gamma : (a, b) \rightarrow M$ is an integral curve for some $X \in \mathfrak{X}M$, then $\forall s \in \mathbb{R}, \gamma(- + s) : (a - s, b - s) \rightarrow M$ is also an integral curve for X .

Proof. Let $\iota = \gamma(- + s)$. Then $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$ \square

Lemma 2.28. Let $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$ provided that $s, t, s + t \in (-\epsilon, \epsilon), p, \varphi_r(p) \in U$.

Proof. $\gamma_p = \varphi(-, p)$ is an integral curve for X .

$\Rightarrow \gamma_p(- + s)$ is an integral curve for X starting at $\gamma_p(s) = \varphi_s(p)$. But $\gamma_{\varphi_s(p)}$ is also an integral curve starting at $\varphi_s(p)$. Thus $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$ \square

Lemma 2.29. Let $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$ i.e. any vector field is invariant under its flow.

Proof. Take $f \in C_{\varphi(p)}^\infty(M)$.

$$\varphi(s, *) (X_p)(f) = X_p(f \circ \varphi_s) \quad (2.5)$$

$$= \frac{d}{dt}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \quad (2.6)$$

$$= \frac{d}{dt}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \quad (2.7)$$

$$= X_{\varphi_s(p)}(f) \quad (2.8)$$

□

Proof of 2.26. " \Leftarrow " is because the lemma $\varphi_s \circ \varphi_r = \varphi_{s+r}$

" \Rightarrow " Let $X = \{X_p\}$ where $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$.

Leave it as an exercise. □

Time dependent vector field is a smooth map $X : \mathbb{R} \times M \rightarrow TM$ s.t. $X_{(t,p)} \in T_p M$.

A smooth curve $\gamma(a, b) \rightarrow M$ is the **integral curve** for X if $\gamma'(t) = X_{(t, \gamma(t))}$.

In local chart, solving γ is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness: γ_1, γ_2 are both integral curves for X , $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define $\text{Supp} X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$.

Then X is compactly supported, then X is complete(i.e. a global flow $\varphi : \mathbb{R} \times M \rightarrow M$)

But something is not true for time dependent vector field:

- translation lemma is not true.

- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

2.4 Another definition of vector field

A derivation on M is a \mathbb{R} -linear map $C^\infty(M) \xrightarrow{D} C^\infty(M)$ that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

Theorem 2.30. *We have a bijection:*

$$\begin{aligned} \rho : \mathfrak{X}M &\xrightarrow{1:1} \{\text{derivation on } M\} \\ X &\mapsto D_X : f \mapsto X(f) \end{aligned}$$

Lemma 2.31. $\mathfrak{X}_p M \rightarrow \mathbb{R}$ -linear map $C^\infty(M) \rightarrow \mathbb{R}$ s.t. $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$ is an isomorphism of vector spaces.

Proof. Leave it as an exercise. □

Lemma 2.32. *Given a vector field(not necessarily smooth) $X = \{X_p\}_{p \in M}$, X is smooth $\Leftrightarrow \forall f \in C^\infty(M)$, $X(f)$ is smooth.*

Proof. " \Leftarrow " $\forall p \in M$, take chart $(U, x^1, x^2, \dots, x^n)$ around p . $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$: $U \rightarrow \mathbb{R}$, where $f^i = X|_U(x^i)$. Take $\varphi : M \rightarrow [0, 1]$ s.t. $\varphi \equiv 1$ near p , $\text{Supp} \varphi \subset U$, $\varphi \cdot x^i \in C^\infty(M)$.

Then $X(\varphi \cdot x^i) = f^i$ near p . By assumption, f^i is smooth near p . So f^i is smooth, so X is smooth.

" \Rightarrow " Similar. □

Theorem 2.33. *The map $\rho : \mathfrak{X}M \rightarrow \{\text{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$ is well-defined and bijective.*

Proof. ρ is well-defined: $X(f) \in C^\infty(M)$ by Lemma 2.32, and $D_x(fg) = D_x(f)g + fD_x(g)$ since X is a point-derivation.

ρ is injective: $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$ as maps $C^\infty(M)$ to \mathbb{R} . By Lemma 2.31, we have $X_p = Y_p, \forall p$. So $X = Y$.

ρ is surjective: Given $D : C^\infty(M) \rightarrow C^\infty(M)$. Define $D_p : C^\infty(M) \rightarrow \mathbb{R}$ by $D_p(f) := D(f)(p)$ satisfies the Leibniz rule. By Lemma 2.31, $D_p = D_{X_p}$ for some $X_p \in T_pM$. Define $X = \{X_p\}_{p \in M}$. Then $X(f) = D(f), \forall f \in C^\infty(M)$. By Lemma??, X is a smooth vector field. \square

2.5 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned} \{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \text{ s.t.} \\ &\quad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\} \end{aligned}$$

$$\begin{aligned} \{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\} \end{aligned}$$

Notation 2.34. We will identify $X \in \mathfrak{X}M$ with its derivation $D_x : C^\infty(M) \rightarrow C^\infty(M)$. So a vector field is just a \mathbb{R} -linear map $X : C^\infty(M) \rightarrow C^\infty(M)$ s.t. $X(fg) = fX(g) + X(f)g$.

Definition 2.35 (Lie bracket). Given two (smooth) vector field $X, Y : C^\infty(M) \rightarrow C^\infty(M)$, we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \rightarrow C^\infty(M)$$

Theorem 2.36. For any $X, Y \in \mathfrak{X}M$, $[X, Y] \in \mathfrak{X}M$

Proof. Easy to check that $[X, Y]$ is linear.

By Leibniz rule,

$$\begin{aligned} [X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\ &= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\ &= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\ &= [X, Y](f) \cdot f \cdot [X, Y](g) \end{aligned}$$

□

So What is the geometric meaning of $[X, Y]$? Non commutativity of flows.

Fact 2.37. Given $X, Y \in \mathfrak{X}M$, we say X, Y are commutative vector field if $[X, Y] = 0$

X, Y are commutative iff for any local flows $\varphi^X : (-\epsilon, \epsilon) \times U \rightarrow M$, $\varphi^Y : (-\epsilon, \epsilon) \times U \rightarrow M$ we have $\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X$

Proposition 2.38 (Calculation of $[V, W]$ using local charts). Chart (U, x^1, \dots, x^n) ,

$V, W \in \mathfrak{X}M$, $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$, $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned} [V, W]|_U &= \sum_{i=1}^n (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} \end{aligned}$$

$$= \sum_{1 \leq i, j \leq n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i}$$

Example 2.39. $V = x\partial x + y\partial y$, $W = -y\partial x + x\partial y$ commutes.

Proposition 2.40 (Properties of Lie bracket).

- *Natuality under push-forward.*

Given any $F \in \text{Diff}(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}M$, we have $[F_*V, F_*W] = F_*[V, W]$.

- \mathbb{R} -linearity $\forall a, b \in \mathbb{R}$

$$[aX + bV, W] = a[X, W] + b[V, W]$$

$$[W, aX + bV] = b[W, X] + a[W, V]$$

- *anti-symmetric* $[V, W] = -[W, V]$
- *Jacobi identity*

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

- *Leibniz rule*

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

Definition 2.41. Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}N$. We say W is **F -related** to V if $\forall p \in M$, $F_{p,*}(V_p) = W_{F(p)}$, $F_{p,X} : T_pM \rightarrow T_{f(p)}N$

Example 2.42. $F : S^1 \rightarrow \mathbb{R}^2$, $\theta \mapsto (\cos \theta, \sin \theta)$, $V = \partial \theta$, $W = -y\partial x + x\partial y$.

note 1. In general, given $V \in \mathfrak{T}M$ and $F \in C^\infty(M, N)$. There may not exist $W \in \mathfrak{T}M$ s.t. V, W are F -related. Even such W exists, it may not be unique.

However, if F is a diffeomorphism, given any V, \exists unique W s.t. V and W are F -related. Actually, $W_p = F_* V_{F^{-1}(p)}$.

Such W is called **push forward** of V along F , denoted by $F_* V$, only defined when F is a diffeomorphism.

Lemma 2.43. $\forall V \in \mathfrak{T}M, W \in \mathfrak{T}N, F \in C^\infty(M, N)$. Then W is F -related to V iff $\forall f \in C^\infty N, V(f \circ F) = W(f) \circ F \in C^\infty(M)$

Proof. Check that $F_{p,*}(V_p)(f) = W_{F(p)}(f), \forall f \in C^\infty(N)$ □

Proposition 2.44. Given $V_0, V_1 \in \mathfrak{T}M, W_0, W_1 \in \mathfrak{T}N, F \in C^\infty(M, N), W_i$ is F -related to $V_i, i = 0, 1 \Rightarrow [W_0, W_1]$ is F -related to $[V_0, V_1]$

Proof. □

Corollary 2.45 (Naturality of Lie bracket). Given any $F \in \text{Diff}(M, N), V \in \mathfrak{T}M, W \in \mathfrak{T}M$, we have $[F_* V, F_* W] = F_* [V, W]$

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