Complex Analysis HW8

1. (a) If $z_1 \neq z_2$ s.t. $z_1^2+z_1 = z_2^2+z_2$, then $(z_1-z_2)(z_1+z_2+1)=0$. So if $|z| \le \frac{1}{2}$, then w is injective.

If $|z| > \frac{1}{2}$, take $z_1 = \frac{1}{2}+i\varepsilon$, $z_2 = \frac{1}{2}-i\varepsilon$ $(\varepsilon>0)$ $\Rightarrow$ w injective on $B(0,\frac{1}{2})$. which is largest.

(b) $e^{z_1} = e^{z_2} \Leftrightarrow z_1 - z_2 \in 2\pi i\mathbb{Z}$. If $|z|<\pi$ then $|z_1-z_2| < 2\pi \Rightarrow w$ is injective on $B(0,\pi)$.

Note that $e^{i\pi} = e^{-i\pi}$ $\Rightarrow$ $B(0,\pi)$ is largest.

2. (a) let $T_w = \frac{z-w}{1-\bar{w}z}$, then $T_w \in \mathrm{Aut}(\mathbb{D})$ $(\forall |w|<1)$. Define $g(z) = T_{f(w)} \circ f \circ T_w^{-1}(z)$, then

$g(0)=0$, $g: \mathbb{D} \to \mathbb{D}$. Schwarz lemma $\begin{cases} |g(z)| \le z \\ |g'(0)| \le 1 \end{cases}$ $\Rightarrow$ $\begin{cases} \left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| \le \left|\frac{z-w}{1-\bar{w}z}\right| \\ \frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2} \end{cases}$

(b) If "=" holds, then $g(z) = cz$ for some $|c|=1$ $\Rightarrow$ f is Möbius transformation.

3. (a) let $S_w = \frac{z-w}{z-\bar{w}}$ $(w \in \mathbb{H})$, $g(z) = S_{f(w)} \circ f \circ S_w^{-1}$, then $g: \mathbb{D} \to \mathbb{D}$ and $g(0)=0$.

Schwarz lemma $\Rightarrow$ $\left|\frac{f(z)-f(w)}{f(z)-\overline{f(w)}}\right| \le \left|\frac{z-w}{z-\bar{w}}\right|$.

(b) If "=" holds, then $g(z) = cz$ for some $|c|=1$ $\Rightarrow$ f is Möbius transformation.

4. let $T_w = \frac{z-w}{1-\bar{w}z}$, $S_u = \frac{z-u}{z-\bar{u}}$ $(w \in \mathbb{D}, u \in \mathbb{H})$, $g(z) = S_{f(w)} \circ f \circ T_w^{-1}(z)$, then $g(0)=0$

Schwarz lemma $\Rightarrow$ $\left|\frac{f(z)-f(w)}{f(z)-\overline{f(w)}}\right| \le \left|\frac{z-w}{1-\bar{w}z}\right|$

5. let $T = \frac{z-w}{1-\bar{w}z}$, $w \in \mathbb{D}$, $g(z) = T_{f(w)} \circ f \circ T_w^{-1}$, then $g: \mathbb{D} \to \mathbb{D}$ is 1-1 conformal mapping.

$g(0)=0 \Rightarrow |g(z)| \le |z|$, $|g^{-1}(z)| \le |z|$ $\Rightarrow$ $|g(z)| = |z|$, i.e. $g(z) = cz$ $|c|=1 \Rightarrow f$ is Möbius transformation.

6. (a) Assume $\gamma: [a,b] \to \mathbb{C}$, then $\int_{f(\gamma)} \frac{1}{1-|z|^2}|dz| = \int_a^b \frac{|f'(\gamma(t))\cdot\gamma'(t)|}{1-|f(\gamma(t))|^2}|dt| = \int_\gamma \frac{|f'(z)|}{1-|f(z)|^2}|dz| \le \int_\gamma \frac{1}{1-|z|^2}|dz|$

(b) In this case, $\frac{|f'(z)|}{1-|f(z)|^2} = \frac{1}{1-|z|^2}$ $\Rightarrow$ $\int_\gamma \frac{1}{1-|z|^2}|dz| = \int_{f(\gamma)} \frac{1}{1-|z|^2}|dz|$.

(c) By HW4.4, $f(z) = c\frac{z-\alpha}{1-\bar{\alpha}z}$ for some $\alpha \in \mathbb{D}$.

$$\int_{f(\gamma)} \frac{1}{1-|z|^2}|dz| = \int_\gamma \frac{|f'(z)|}{1-|f(z)|^2}|dz| = \int_\gamma \frac{\frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}}{1-\frac{|z-\alpha|^2}{|1-\bar{\alpha}z|^2}}|dz| = \int_\gamma \frac{1}{1-|z|^2}|dz|.$$