Complex Analysis HW6

1.(a) By Cauchy's formula, $\int_{|z|=1} \frac{e^z}{z}\,dz = 2\pi i\, e^0 = 2\pi i$

(b) $\int_{|z|=2}\frac{1}{z^2+1}dz = \frac{1}{2i}\int_{|z|=2}\frac{1}{z-i}dz - \frac{1}{2i}\int_{|z|=2}\frac{1}{z+i}dz = \frac{1}{2i}2\pi i - \frac{1}{2i}2\pi i = 0$

2.(a) $\int_{|z|=\rho}\frac{|dz|}{|z-a|^2} = \int_{|z|=\rho}\frac{-i\rho\frac{dz}{z}}{(z-a)(\bar z-\bar a)} = -i\rho\int_{|z|=\rho}\frac{dz}{(z-a)(\rho^2-\bar a z)} =: I$

If $a=0$, then $I = -i\rho\int_{|z|=\rho}\frac{dz}{z\rho^2} = \frac{-i}{\rho}\cdot 2\pi i = \frac{2\pi}{\rho}$

If $0<|a|<\rho$, then $I = -i\rho\int_{|z|=\rho} \frac{\frac{1}{\rho^2-\bar a z}dz}{z-a} = \frac{-i\rho}{\rho^2-|a|^2}\cdot 2\pi i = \frac{2\pi\rho}{\rho^2-|a|^2}$

If $|a|>\rho$, then $I = -i\rho\int_{|z|=\rho} \frac{\frac{1}{z-a}dz}{-\bar a(z-\frac{\rho^2}{\bar a})} = \frac{-i\rho}{\frac{\rho^2}{\bar a}-a}\cdot\frac{1}{-\bar a}\cdot 2\pi i = \frac{2\pi\rho}{|a|^2-\rho^2}$

$\Rightarrow I = \frac{2\pi\rho}{||a|^2-\rho^2|}$

(b) $\int_{|z|=\rho} \frac{|dz|}{|z-a|^4} = \int_{|z|=\rho}\frac{-i\rho\frac{dz}{z}}{(z-a)^2(\bar z-\bar a)^2} = -i\rho\int_{|z|=\rho} \frac{z\, dz}{(z-a)^2(\rho^2-\bar a z)^2} =: J$

If $a=0$, then $J = \frac{-i\rho}{\rho^4}\int_{|z|=\rho}\frac{dz}{z} = \frac{-i}{\rho^3}\cdot 2\pi i = \frac{2\pi}{\rho^3}$

If $0<|a|<\rho$, then $J = -i\rho\int_{|z|=\rho} \frac{\frac{z}{(\rho^2-\bar a z)^2}dz}{(z-a)^2} = -i\rho\cdot 2\pi i\cdot \left(\frac{z}{(\rho^2-\bar a z)^2}\right)'\Big|_{z=a} = \frac{2\pi\rho(\rho^2+|a|^2)}{(\rho^2-|a|^2)^3}$

If $|a|>\rho$, then $J = -i\rho\int_{|z|=\rho} \frac{\frac{z}{(z-a)^2}dz}{\bar a^2(z-\frac{\rho^2}{\bar a})^2} = \frac{-i\rho}{\bar a^2}\cdot 2\pi i\left(\frac{z}{(z-a)^2}\right)'\Big|_{z=\frac{\rho^2}{\bar a}} = \frac{2\pi\rho(\rho^2+|a|^2)}{(|a|^2-\rho^2)^3}$

$J = \frac{2\pi\rho(\rho^2+|a|^2)}{||a|^2-\rho^2|^3}$

Note that $J = \frac{-i\rho}{\bar a^2}\int_{|z|=\rho} \frac{z\, dz}{(z-a)^2(z-b)^2}$ where $b=\frac{\rho^2}{\bar a}$ if $a\neq 0$, and

$\frac{z}{(z-a)^2(z-b)^2} = -\frac{a+b}{(a-b)^3}\frac{1}{z-a} + \frac{a+b}{(a-b)^3}\frac{1}{z-b} + \frac{a}{(a-b)^2}\frac{1}{(z-a)^2} + \frac{b}{(a-b)^2}\frac{1}{(z-b)^2}$

we can also use the Laurent expension to calculate J.

3. $\forall z_0\in\mathbb{C}$, $R>0$, $C=\{|z-z_0|=R\}$, $\|f\|_C = \sup_{z\in C}|f(z)|$, we have the Cauchy inequality $|f^{(n)}(z_0)|\le \frac{n!\,\|f\|_C}{R^n}$.

$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta\right| \le \frac{n!}{2\pi}\int_0^{2\pi}\frac{|f(z_0+Re^{i\theta})|}{R^{n+1}}R\, d\theta \le \frac{n!\,\|f\|_C}{R^n}$.

If $|f(z)|\le|z|^n$, then $|f^{(n+1)}(z_0)| \le \frac{(n+1)!\,\|f\|_C}{R^{n+1}} \le \frac{(n+1)!\,(|z_0|+R)^n}{R^{n+1}} \to 0$ as $R\to\infty$.

So $f^{(n+1)}(z_0)=0$ $\forall z_0\in\mathbb{C} \Rightarrow f$ is a polynomial with $\deg\le n$.

4. (a) By Cauchy's estimate, $|f^{(n)}(0)| \le \frac{n!\,\|f\|_C}{R^n} \le \frac{n!}{R^n(1-R)}$. If $n=0$, then $|f(0)| \le 1$.

If $n \ge 1$, then $|f^{(n)}(0)| \le \frac{n!}{R^n(1-R)} \ge \frac{n!\,(n+1)^{n+1}}{n^n}$ by AM-GM inequality ($R=\frac{n}{n+1}$).

It's the best estimate: If $n=0$, take $f \equiv 1$. If $n \ge 1$, take $f=\frac{(n+1)^{n+1}}{n^n} z^n$.

(b) ① The radius of convergence satisfies $\frac{1}{R} = \limsup \left|\frac{f^{(n)}(z_0)}{n!}\right|^{\frac{1}{n}} > \limsup |n| = \infty \Rightarrow R=0$, contradiction.

② Suppose f is analytic on a region containing $\{|z-z_0| \le R\}$. Then by Cauchy's estimate,

$|f^{(n)}(z_0)| \le \frac{n!\,\|f\|_C}{R^n}$ $(\forall n \ge 0)$. If $|f^{(n)}(z_0)| > n!\,n^n$, then $\|f\|_C > (Rn)^n$ for all $n>0$, contradiction.

5. (a) $\varphi(z,t)$ is analytic for fixed $t \Rightarrow \varphi(z,t) = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta,t)}{\zeta - z}\,d\zeta$ by Cauchy's formula.

(b) $F(z) = \int_\alpha^\beta \varphi(z,t)\,dt \overset{(a)}{=} \int_\alpha^\beta \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta,t)}{\zeta-z}\,d\zeta\,dt \overset{\text{Fubini}}{=} \int_C \frac{1}{2\pi i}\int_\alpha^\beta \varphi(\zeta,t)\,dt\,\frac{d\zeta}{\zeta-z}$.

(c) $F'(z) \overset{\text{lemma}}{=} \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{(\zeta-z)^2}\,d\zeta = \frac{1}{2\pi i}\int_C \int_\alpha^\beta \varphi(\zeta,t)\,dt\,\frac{d\zeta}{(\zeta-z)^2} \overset{\text{Fubini}}{=} \int_\alpha^\beta \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta,t)}{(\zeta-z)^2}\,d\zeta\,dt$

$\underset{(a)}{\overset{\text{lemma}}{=}} \int_\alpha^\beta \frac{\partial\varphi(z,t)}{\partial z}\,dt$.

F is holomorphic: For any triangle T in Δ, $\int_T F(z)\,dz = \int_T \int_\alpha^\beta \varphi(z,t)\,dt\,dz$

$\overset{\text{Fubini}}{=} \int_\alpha^\beta \int_T \varphi(z,t)\,dz\,dt = \int_\alpha^\beta 0\,dt = 0$

By Morera's theorem, F is analytic on Δ.

Morera's theorem: If f is a continuous function in an open disc D such that for any triangle T in D, $\int_T f(z)\,dz = 0$, then f is holomorphic in D.