1.2 Newton's Laws

1.2.1 Newtonian mechanics

Classical mechanics is a physical theory describing the motion, including accelerations, of macroscopic objects, and what can cause an object to accelerate. That *cause* is called a *force*, which is, loosely speaking, a push or pull on the object. The force is said to act on the object to change its velocity. In the next section, we will see some examples of forces in nature.

In classical mechanics, the relation between a force and the acceleration it causes was fully understood by the celebrated Newton's three laws of motion, which is the subject of this chapter. The study of that relation, as Newton presented it, is then called *Newtonian mechanics*. However, we remark that Newtonian mechanics does not apply to all situations. On the one hand, it is only a low-speed approximation of Einstein's special theory of relativity. If the speeds of the objects become an appreciable fraction of the speed of light, Newtonian mechanics fails. On the other hand, if the interacting bodies are on the scale of atomic structure, then Newtonian mechanics should be replaced with more sophisticated quantum mechanics. Although physicists now view Newtonian mechanics as a special case of these two deeper theories, it is a very useful and important special case because it applies to the motion of objects ranging in size from the very small (almost on the scale of atomic structure) to astronomical (galaxies and clusters of galaxies).

It is also worth mentioning that even though the underlying Newton's laws of classical mechanics are very simple, there are still sufficiently many phenomena in classical mechanics that remain mysterious to physicists and mathematicians. Studies of these phenomena have led to many profound mathematical theories. One famous example is the Millennium Prize about the Navier–Stokes equation, which is in the regime of classical mechanics and has been one of the central problems in PDE studies. Another example is the chaos phenomena in classical mechanics (you may know the three-body problem), which has had far-reaching impacts on the development of the dynamical systems theory.

1.2.2 Newton's first law

Before Newton, Aristotle proposed that some force is needed to keep a body moving at constant velocity. A body was thought to be in its "natural state" when it was at rest, and for a body to move with constant velocity, it seems that we had to push or pull it in some way. Otherwise, the body would "naturally" stop moving. However:

Example (Galileo's thought experiment)**.** *Galileo considered a sliding body on inclined planes in the absence of friction. The speed acquired by a body moving down a plane from a height, say h, was sufficient to enable it to reach the same height when climbing up another plane at a different inclination, say θ. As θ decreases, the body should travel a greater and greater distance. Galileo proposed that the body could travel indefinitely far as $\theta \to 0$, contrary to the Aristotelian notion of the natural tendency of an object to remain at rest unless acted upon by an external force.*

Due to his contribution, Galileo is credited with introducing the concept of *inertia*. Newton later exploited it as his first law of motion.

Physics law 1 (Newton's first law/the law of inertia)**.** *In an inertial frame, if no force acts on a body, the body's velocity cannot change; that is, the body cannot accelerate.*

In other words, if a free body is at rest, it stays at rest. If it is moving, it continues to move with the same velocity (same magnitude and same direction).

Note Newton's first law does not hold in non-inertial frames. For example, in the frame of a bus accelerating from rest, a ball will accelerate backward even if no force acts on it. Hence, we can give another definition of the inertial frame: *an inertial frame is a frame where Newton's first law holds.*

1.2.3 Newton's second law

Newton's second law is perhaps the greatest law in physics and has deeply affected physics and even human history since after Newton. Before introducing Newton's second law, we first need to make some preparations.

Force is a *vector quantity*, i.e., it has not only magnitude but also direction. So, if two or more forces act on a body, we find the *net force* by adding them as vectors. This leads to the *principle of superposition (or decomposition) for forces*, i.e., a single force that has the same magnitude and direction as the calculated net force would then have the same effect as all the individual forces. In this note, we will often use $\vec{F}$ to represent a force.

There may be multiple forces acting on a body, but if their net force is zero, the body cannot accelerate. So, a more precise statement of Newton's first law is:

Physics law 2 (Newton's first law)**.** *In an inertial frame, if no net force acts on a body, the body's velocity cannot change; that is, the body cannot accelerate.*

Mass is a quantitative measure of inertia. From everyday experience, we know that the object with the larger mass is accelerated less. With careful experiments, people find that the acceleration is actually inversely related to the mass (rather than, say, the square of the mass). In other words, applying the same force F to two bodies of masses m_1 and m_2, their accelerations a_1 and a_2 satisfy

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}.$$

This suggests that

$$m_1 a_1 = m_2 a_2 = CF$$

for a universal dimensionless C that does not depend on any physical quantities of the bodies. By choosing the proper physical units, we can let $C = 1$, which leads to Newton's second law.

Physics law 3 (Newton's second law)**.** *In an inertial frame, the net force on a body is equal to the product of the body's mass and its acceleration. In other words, suppose a net force $\vec{F}$ is acted on a body of mass m, then the generated acceleration satisfies*

$$\vec{F} = m\vec{a}.$$

Note Newton's first law is a special case of the second law with $\vec{F} = 0$ and $\vec{a} = 0$. We now check that Newton's second law is "invariant" under Galilean transformations:

$$\vec{x} \quad \text{in inertial frame } O \quad \rightarrow \quad \vec{x}' = \vec{x} - \vec{v}t \quad \text{in inertial frame } O',$$

where $\vec{v}$ is a constant velocity vector. Then, in O', we have that

$$\vec{a}' = \frac{\mathrm{d}^2\vec{x}'}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \vec{a}.$$

Hence, in the new frame O', we still have $\vec{F} = m\vec{a}'$.

The standard units of mass and acceleration are kg and m/s^2. Then, the unit of force is kg $\cdot$ m/s^2, which is called "Newton", denoted by N. Hence, a 1 N force acting on a body of 1 kg mass leads to an acceleration 1 m/s^2.

Remark. Mass is an intrinsic characteristic of a body. However, what, exactly, is mass? This turns out to be a much deeper question than it may look. In everyday language, it is often confused with weight, but this is wrong. In classical mechanics, by Newton's second law, we can only say that the mass of a body is *a characteristic that relates a force on the body to the resulting acceleration.* There is no more familiar definition—we can have a physical sensation of mass only when we try to accelerate a body. The weight of a body is actually the gravitational force acting on the body, from which we can measure the mass by observing how the body accelerates under gravitation. In more advanced physics (such as relativity and quantum field theory), people have a much deeper understanding of "mass".

1.2.4 Newton's third law

Two bodies are said to interact when they push or pull on each other, that is when a force acts on each body due to the other body. Newton's third law states that the action force is equal to the reaction force.

Physics law 4 (Newton's third law/Action-reaction law)**.** *In an inertial frame, when two bodies interact, the forces on the bodies from each other are always equal in magnitude and opposite in direction. In other words, let A and B be two interacting bodies. $\vec{F}_{AB}$ is the force acting on B from A, and $\vec{F}_{BA}$ is the force acting on A from B. Then, we have*

$$\vec{F}_{AB} = -\vec{F}_{BA}.$$

If we say Newton's first law is about setting up inertial frames and the second law gives how an object changes its motion if some force is acting on it, Newton's third law is purely a description of the nature of forces. In Feynman's lectures, he said:

In our discussion of Newton's laws, it was explained that these laws are a kind of program that says "Pay attention to the forces," and that Newton told us only two things about the nature of forces. In the case of gravitation, he gave us the complete law of the force. In the case of the very complicated forces between atoms, he was not aware of the right laws for the forces; however, he discovered one rule, one general property of forces, which is expressed in his Third Law, and that is the total knowledge that Newton had about the nature of forces—the law of gravitation and this principle, but no other details.

Nowadays, we know there are four fundamental interactions: *weak, strong, electromagnetic* and *gravitation* forces. We will discuss Newton's law of gravitation in Section 1.6, and the electromagnetic force in Chapter 2. In applications, we will also often consider *phenomenological forces* (i.e., forces due to fundamental interactions), such as frictions, pressure, buoyancy, elastic force, damping force, and so on.

1.2.5 Applying Newton's laws

In principle, with Newton's second law, we can predict the motion of a given object at any future time. Denote the position of the object by $\vec{x}$. In physics, most forces depend only on the position and velocity (but not the acceleration) of the object. Then, Newton's second law can be written as a second-order differential equation of $\vec{x}$:

$$m\ddot{\vec{x}} = \vec{F}(\vec{x}, \dot{\vec{x}}),$$

where we used Newton's notation of differentiation: $\dot{\vec{x}} = \mathrm{d}\vec{x}/\mathrm{d}t$. To get a solution, we need to solve the equation and fix the two integration constants using the initial conditions for $\vec{x}(0)$ and $\vec{v}(0) = \dot{\vec{x}}(0)$. In general, for a system of multiple objects, we can write down a differential equation for each object. Thus, to predict the evolution of the system, we need to solve the system of differential equations with the given initial condition. In this note, we will look at some examples such that solving the given differential equation is possible and simple. In reality, however, such simple scenarios are rare and very few problems can be solved exactly by pure analysis.

Everybody on earth is pulled toward the ground by a gravitational force that is proportional to the mass m:

$$\vec{F}_g = m\vec{g},$$

where $\vec{g}$ is the gravitational acceleration. Near the ground, $\vec{g}$ can be treated as a constant vector with a magnitude approximately 9.8 $\mathrm{m/s^2}$ and pointing downward.

Free fall is any motion of a body where gravity is the only force acting upon it. If we release a body from rest, then its free fall is a linear motion described by (1.1.30) with $a = -g$ (where the positive direction is upward) and $v = 0$. If the body has a horizontal velocity, then its free fall is described by the parabolic motion (1.1.42).

Example. *A box starts from rest at height h_0 and falls along a frictionless plane inclined at angle θ. How long does it take to reach the ground? Suppose the mass of the box is m. What is the normal force on the box?*

Solution: We can set up the axes such that the x-axis is pointing downward along the inclined plane, the y-axis is pointing upward along the normal direction of the inclined plane, and the origin is at the position of the box at $t = 0$.

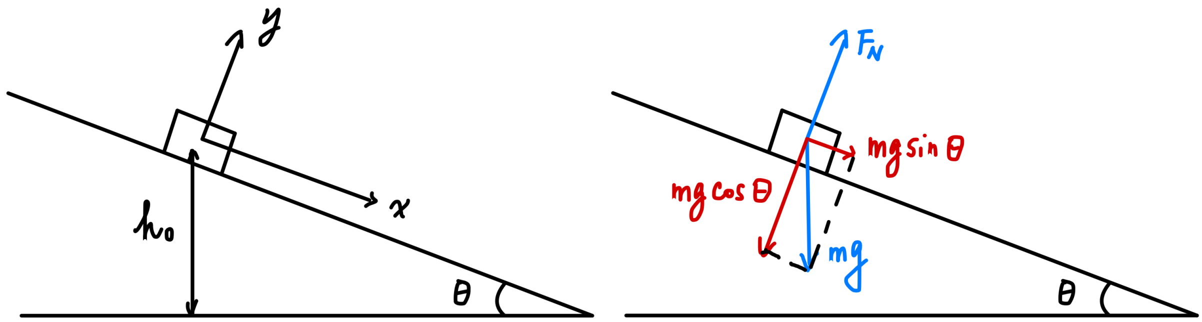

Figure 1.5: Forces on the box on the inclined plane.

Then, the gravitational force on the box can be written as

$$\vec{F}_g = (mg\sin\theta, -mg\cos\theta).$$

On the other hand, the normal force is $\vec{F}_N = (0, F_N)$. Since the box will slide along the inclined plane, its acceleration can be written as $\vec{a} = (a, 0)$. By Newton's second law,

$$\vec{F}_g + \vec{F}_N = m\vec{a}.$$

Along the normal direction, it gives the normal force $F_N = mg\cos\theta$. Along the x-axis, the above equation gives $a = g\sin\theta$. Then, by (1.1.30), we have that the position of the box $x(t)$ is given by

$$x(t) = \frac{1}{2}g\sin\theta \cdot t^2.$$

When the box reaches the ground, we have $x(t) = h_0/\sin\theta$. Solving for t, we get

$$t = \sqrt{\frac{2h_0}{g\sin^2\theta}}.$$

Recall that for a point particle doing circular motion, its acceleration point towards the rotation center and has a magnitude $v^2/r = \omega^2 r$, where r is the radius of the circular motion, v is the speed of the particle, and $\omega = v/r$ is the angular speed. We now look at the following example:

Example. *In a double-star system, two stars of masses m_1 and m_2 are rotating around a center with angular speed ω. The gravitation force between them is given by Gm_1m_2/r^2, where G is the gravitational constant and r is the distance between the two stars. Given ω, determine r and the position of the rotation center.*

Solution: Suppose the distances from the two stars to the rotation center are r_1 and $r_2 = r - r_1$, respectively. The acceleration of the two stars are towards the rotation center and equal to

$$a_1 = \omega^2 r_1, \quad a_2 = \omega^2 r_2,$$

in magnitude. Then, by Newton's second law, we have

$$\frac{Gm_1m_2}{r^2} = m_1\omega^2 r_1 = m_2\omega^2 r_2. \tag{1.2.1}$$

From this equation, we get $m_1r_1 = m_2r_2 = m_2(r - r_1)$, which gives

$$r_1 = \frac{m_2}{m_1 + m_2} r, \quad r_2 = \frac{m_1}{m_1 + m_2} r.$$

Plugging it into Equation (1.2.1), we can solve that

$$r = \left(\frac{G(m_1 + m_2)}{\omega^2}\right)^{1/3}.$$

Remark. Note that the rotation center is in fact the system's center of mass. In general, for a system of n bodies of masses $m_1, \ldots, m_n$ and at position $\vec{x}_1, \ldots, \vec{x}_n$, the center of mass of this system is a vector $\vec{x}$ such that

$$(m_1 + \cdots + m_n)\vec{x} = m_1\vec{x}_1 + \cdots + m_n\vec{x}_n.$$

Next, we see some examples where solving second-order differential equations is necessary.

Example. *Consider a ball falling in some liquid. The liquid exerts a damping force $F_d = bv$ that is proportional to the speed v of the ball and in the opposite direction of the velocity, where b is the damping constant. Suppose the ball is released from rest and has mass m. Determine its motion with respect to t and the velocity of the ball when $t \to \infty$.*

Solution: Let the positive x-axis be the downward direction and let the origin be the starting position of the ball. By Newton's second law, we need to solve the following equation

$$m\ddot{x} = mg - b\dot{x},$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. Denoting $y = \dot{x}$, the above equation is written as

$$m\dot{y} = mg - by,$$

which can be solved as

$$\dot{x}(t) = y(t) = Ce^{-\frac{b}{m}t} + \frac{mg}{b}.$$

With the initial condition, we can get that $C = -mg/b$. Integrating $\dot{x}$, we get

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)\mathrm{d}\tau = \frac{mg}{b}t + \frac{m^2 g}{b^2}\left(e^{-\frac{b}{m}t} - 1\right).$$

Note that when $t \to \infty$, $\dot{x}(t) \to mg/b$, in which case the $F_d = mg$. This means the gravitational force and the damping force will finally balance, and the velocity of the ball will not change anymore.

Example. *Consider a box of mass m lying on a frictionless plane. It is connected to a spring. The spring force satisfies Hooke's law $\vec{F} = -k\vec{x}$, where $\vec{x}$ is the displacement of the spring's free end from its position when the spring is in its relaxed state (which is chosen as the origin), and k is the spring constant. At the time $t = 0$, the box is released from rest at position $\vec{x}_0$. Determine the position $\vec{x}(t)$ of the box at any time t.*

Solution: Without loss of generality, suppose $\vec{x}_0$ is along the x-axis. By Newton's second law, we need to solve the following equation

$$m\ddot{x} = -kx,$$

with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. We can rewrite the above equation as

$$\ddot{x} = -\omega^2 x, \quad \omega := \sqrt{\frac{k}{m}}.$$

We know the above equation has a general solution

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t).$$

From the initial conditions, we can determine that $C_1 = x_0$ and $C_2 = 0$. Hence, the solution is $\vec{x}(t) = x_0\cos(\omega t)\vec{i}$, where $\vec{i}$ is the basis unit vector along the x-axis.

Remark. The above system is called a *harmonic oscillator*. It is one of the most important examples in almost every field of physics. We will discuss this system in the context of classical mechanics in more detail in Section 1.5. You will see it again and again in more advanced physics courses (in particular, it will be one of the first few examples when you study quantum mechanics).

Finally, we look at a two-body problem. We will show that it can be reduced to solving two one-body problems.

Example. *Consider two boxes of have masses m_1 and m_2. They are connected by a spring, which is of length ℓ when it is in a relaxed state. Moreover, the spring constant is k. At the time $t = 0$, the two boxes have the same velocity $\vec{v}$ along the direction of the spring and the spring is compressed to length $\ell/2$. Determine the positions $\vec{x}_1(t)$ and $\vec{x}_2(t)$ of the two boxes at any time t.*

Solution: Without loss of generality, we choose the x-axis such that the spring is along the x-direction, and

$$x_1(0) = -\frac{m_2}{m_1+m_2}\frac{\ell}{2}, \quad x_2(0) = \frac{m_1}{m_1+m_2}\frac{\ell}{2}.$$

(They are chosen such that the center of mass is at the origin.) By Newton's second law, we can write down the following system of equations:

$$m_1\ddot{x}_1 = k(x_2 - x_1 - \ell), \quad m_2\ddot{x}_2 = -k(x_2 - x_1 - \ell), \tag{1.2.2}$$

with initial conditions

$$x_1(0) = -\frac{m_2}{m_1+m_2}\frac{\ell}{2}, \quad \dot{x}_1(0) = v, \quad x_2 = \frac{m_1}{m_1+m_2}\frac{\ell}{2}, \quad \dot{x}_2(0) = v.$$

To solve (1.2.2), we introduce

$$y_1 := \frac{m_1x_1 + m_2x_2}{m_1+m_2}, \quad y_2 = x_2 - x_1.$$

Note that y_1 is the center of mass of the two boxes, while y_2 describes their relative position. Then, from the two equations in (1.2.2), we get

$$\ddot{y}_1 = 0, \quad \ddot{y}_2 = -\omega^2(y_2 - \ell), \quad \text{with} \quad \omega = \sqrt{\frac{k}{m_1} + \frac{k}{m_2}}, \tag{1.2.3}$$

with initial conditions

$$y_1(0) = 0, \quad \dot{y}_1(0) = v, \quad y_2(0) = \ell/2, \quad \dot{y}_2(0) = 0. \tag{1.2.4}$$

It is easy to see that the two differential equations in (1.2.2) have general solutions

$$y_1(t) = C_1t + C_2, \quad y_2(t) = \ell + C_3\cos(\omega t) + C_4\sin(\omega t).$$

From the initial conditions (1.2.4), we can determine that

$$C_1 = v, \quad C_2 = 0, \quad C_3 = -\ell/2, \quad C_4 = 0.$$

Hence, we get the solution

$$y_1(t) = vt, \quad y_2(t) = \ell - \frac{\ell}{2}\cos(\omega t),$$

from which we can solve $x_1(t)$ and $x_2(t)$.

Remark. In physics, the interactions between two bodies only depend on their relative positions (which is due to the translational symmetry of physics laws). Hence, the two-body problem in classical mechanics can always be reduced to two one-body problems, which are much easier to solve. Due to this reason, people say that two-body problems are exactly solvable. However, this is not true anymore for problems with three or more bodies. In particular, it is well known that for some initial conditions, the three-body motion can exhibit chaotic behaviors.

In general, with more and more bodies, the problem becomes harder and harder. However, when there are a significant amount of particles, the whole system will exhibit some *collective behaviors* that are predictable. This will be the focus of statistical mechanics.