Complex Analysis

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2 Complex Functions

2.1 Analytic functions and rational functions

2.1.1 Harmonic function

Definition 2.1 (Cauchy-Riemann equation).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 2.2 (Harmonic function). A function u is **harmonic** if it satisfied **Laplace equation** $\triangle u = 0$.

If two harmonic function u and v satisfies Cauchy-Riemann equations, then we say that v is **conjugate harmonic function of** $u \Rightarrow u$ is conjugate harmonic of $-v$.

2.1.2 Polynomials and rational function

The polynomial $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ is analytic in $\mathbb{C}$.

We will prove the fundamental theorem of algebra

Theorem 2.3 (Fundamental Theorem of Algebra). *Every polynomial with degree $n > 0$ has at least one point.*

Theorem 2.4 (Gauss-Lucus theorem). *The smallest convex polygon that contain the zeros of P also contains the zeros of P'.*

Proof. Only need to check.

We can get this equation.

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{j=1}^{n} \frac{1}{\alpha - \alpha_j} = 0 \Rightarrow \sum_{j=1}^{n} \frac{\overline{\alpha - \alpha_j}}{|\alpha - \alpha_j|^2}$$

Hence α is linearly represented by α_j. $\square$

Proposition 2.5. *Let P and Q be two polynomial with no common zeros. Then the rational function $R(z) = \dfrac{P(z)}{Q(z)}$ is analytic away from the zeros of Q.*
The zeros of Q are called ***poles*** *of R, and the* ***order of a pole*** *is equal to the order of the corresponding zero of Q.*

We often view R as a function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. $R_1(z) := R(\frac{1}{z})$.

If $R_1(0) = 0$, the order of the zero at ∞ (of R) is the order of the zero of $R_1(z)$ at $z = 0$.

If $R_1(0) = \infty$, the order of the pole at ∞ (of R) is the order of the pole of $R_1(z)$ at $z = 0$.

Suppose

$$R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, a_n \neq 0, b_m \neq 0$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$$

By discussing m and n, we can infer the situation of $R(z)$ at ∞.
By adding the order of poles and zeros at ∞, we can get the following theorem.

Theorem 2.6. *The total number of zeros and poles of a rational function are the same.*

Remark 2.7. This common number is called the **order of the rational function**.

Corollary 2.8. *Suppose a rational function R has order p. Then every equation $R(z) = a$ has exactly p roots.*

Proof. $\hat{R}(z) = R(z) - a$ has the same poles as R. $\square$

A rational function of order 1 is a **linear fraction** $R(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$
Such fraction is often called **Möbius transformation**

Every rational function has a representation by **partial fractions**.

- If R has a pole at ∞. Then we can write

$$R(z) = G(z) + H(z) \tag{$*$}$$

where G is a polynomial without constant term, and H is finite at ∞.

The degree of G is the order of the pole of R at ∞. G is called the **singular part** of R at ∞.

- Let the distinct finite poles of R be $\beta_1, \cdots, \beta_k$. Let $R_j(\psi) = R(\beta_j + \frac{1}{\psi})$. Then R_j is a rational function with a pole at ∞. As in $(*)$, we can write

$$R_j = G_j + H_j$$

with H_j finite at ∞. Then

$$R(z) = G_j(\frac{1}{z - \beta_j}) + H(\frac{1}{z - \beta_j})$$

with G_j is a polynomial in $\frac{1}{z-\beta_j}$ without constant term called the **singular point** of R at β_j.

- Let $F(z) = R(z) - G(z) - \sum\limits_{j=1}^{k} G_j(\frac{1}{z-\beta_j})$.
Then F is a rational function which can only have poles among β_j, ∞

Since by our construction, F is finite at every $\beta_j, 1 \leqslant j \leqslant k$ and ∞.

So F is a constant.

In particular, $R(z) = G(z) + \sum\limits_{j=1}^{k} G_j(\frac{1}{z-\beta_j}) + c$.

2.2 Power Series

2.2.1 Power series

Theorem 2.9 (Abel's theorem). *If* $\sum a_n$ *converges, then* $f(z) = \sum a_n z^n \to f(1)$ *as* $z \to 1$ *in such a way that* $\dfrac{|1-z|}{1-|z|}$ *remains bounded.*

2.3 Exponential, Trigonometric and Logorithmic Functions

2.3.1 Exponential and Trigonometric function

The **exponential function** is defined as the solution if the differential equation

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1 \end{cases}$$

We denote $e^z = \exp z = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$.

The **trigonometric function** are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

2.3.2 Logorithmic Functions

The **logorithmic function** $\ln$ is defined by $z = \ln w$ is a root of the equation $e^z = w$.

For $w \neq 0$, we write $z = x + iy$, then

$$e^{x+iy} = w \Leftrightarrow \begin{cases} e^x = |w| \\ e^{iy} = \dfrac{w}{|w|} \end{cases}$$

The first equation has a unique solution $x = \ln|w|$.

The second equation $e^{iy} = \dfrac{w}{|w|}$ has a unique solution $y_0 \in [0, 2\pi)$.

If we write $w = re^{i\theta}$, then $x = \ln w, y = \theta = \arg w$.

Thus, for $w \neq 0$, we have

$$\ln w = \ln|w| + i \arg w$$

The function $\ln$ is actually not single-valued. But we can define a single-valued function Ln

We define

$$a^b = \exp(b \ln a)$$

We will prove Ln is analytic in $\mathbb{C} - (-\infty, 0]$ but not continuous in $(-\infty, 0]$.

Ln is the principal branch of the logithm.

3 Conformal Mappings

3.1 Basic topology

3.1.1 Connectedness

Theorem 3.1. *A nonempty open set in $\mathbb{C}$ is connected iff any two of its points can be joined by a polygon which lies in the set, i.e. Connectedness is equivalent to Path Connectedness*

An nonempty connected subset is called a **region**

3.1.2 Compactness

Definition 3.2. A set X is **totally bounded** if $\forall\varepsilon > 0$, X can be covered by finitely many balls of radius ε

Theorem 3.3. *A set is compact iff it is complete and totally bounded.*

Theorem 3.4. *A subset $X \subset$ is compact iff every infinite sequence of X has a limit point in X.*

3.1.3 Continuous Functions

Theorem 3.5. *Continous function maps connected space to connected space.*

Theorem 3.6. *Continous function maps compact space to compact space.*

3.2 Conformality, geometric consequences of the existence of a derivative

3.2.1 Arcs and closed curves

The equation of an **arc** r in $\mathbb{C}$ can be represented by one of the terms

- $x = x(t), y = y(t)$, $\alpha \leqslant t \leqslant \beta$, x, y are continuous at t
- $z(t) = x(t) + iy(t)$, $\alpha \leqslant t \leqslant \beta$.
- The continuous mapping $\gamma : [\alpha, \beta] \to \mathbb{C}$.

For a non-decreasing function $\varphi : [\alpha, \beta] \to [\alpha, \beta]$, $z = z(\varphi(t)), \alpha' \leqslant \tau \leqslant \beta'$ is **change of parameter** of $z(t)$.

The change is **reversible** iff φ is strictly increasing.

If γ is differentiable, then call γ a **curve**.

γ is **simple** , or a **Jordan curve**, if γ is injective.

γ is **closed curve** if $\gamma(0) = \gamma(1)$.

3.2.2 Analytic Functions in Regions

A function f is analytic on an arbitrary set A if it is the restriction to A of a function which is analytic in some open set containing A.

Theorem 3.7. *An analytic function in a region(i.e. open and connected) Ω whose derivative is 0 must reduce to a constant. The same hold if the real part, the imaginary part, the modulus, or the argument is constant.*

3.2.3 Conformal Mappings

Suppose $f : \Omega \to \mathbb{C}$ is analytic in Ω. $r_1 = z_1(t), r_2 = z_2(t)$, $\alpha \leqslant t \leqslant \beta$.
$z_0 = z_1(t_0) = z_2(t_0'), z_1'(t_0) \neq 0, z_2'(\hat{t_0}) \neq 0, \alpha < t_0, \hat{t_0} < \beta$.
$f'(z_0) \neq 0, w_1(t) = f(z_1(t_0)), w_2 = f(z_2(\hat{(t_0)}))$
$\Gamma_1 = \{w_1(t)|\alpha \leqslant t \leqslant \beta\}$, $\Gamma_2 = \{w_2(t)|\alpha \leqslant t \leqslant \beta\}$
Then

$$w_1'(t) = f'(z_1(t))z_1'(t)$$
$$w_2'(t) = f'(z_2(\hat{t}))z_2'(\hat{t})$$

$\Rightarrow$

$$w_1'(t_0) \neq 0, w_2'(t_0) \neq 0$$
$$\arg w_1'(t_0) = \arg f'(z_1(t_0))z_1'(t_0)$$
$$\arg w_2'(t_0) = \arg f'(z_2(\hat{t_0}))z_2'(\hat{t_0})$$

So the "angle" $\arg w_1'(t_0) - \arg w_2'(\hat{t_0} = \arg z_1(t_0) - \arg z_2(\hat{t_0})$ remains the same. Now we give the definition.

Definition 3.8. $w = f(z)$ is said to be **conformal** in Ω if f is analytic in Ω and $f'(z) \neq 0$ for $\forall z \in \Omega$.

Easy to prove that linear change of scale at z_0 is independent of the direction. *i.e.* $|f'(z_0)| = \lim\limits_{z \to z_0} \frac{\delta\sigma}{\delta s}$

3.2.4 Length and Area

The **length** of a differentiable arc γ with the equation $z(t) = x(t) + iy(t)$, $a \leqslant t \leqslant b$

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t = \int_a^b |z'(t)|\mathrm{d}t$$

For $\Gamma = f(\gamma)$ where f conformal mapping.

Then

$$L(\Gamma) = \int_a^b |f'(z(t))| \cdot |z'(t)|\mathrm{d}t$$

The **area** of $E \subset \mathbb{R}$ is $A(E) = \int\int_E \mathrm{d}x\mathrm{d}y$

Then by the differentiable functional transformation, the area $\hat{E} = f(E)$ is

$$A(\hat{E}) = \int\int_E |u_x v_y - u_y v_x|\mathrm{d}x\mathrm{d}y$$

If f is the conformal mapping of an open set containing E, then by Caucht-Riemann equation

$$A(\hat{E}) = \int\int_E |f'(z)|^2\mathrm{d}x\mathrm{d}y$$

3.3 Möbius Transformation

Recall that a **Möbius transformation** is a function of the form

$$w = s(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Then it has an inverse $z = S^{-1}(w) = \dfrac{dw - b}{-cw + a}$.

We may define $S(\infty) = \lim_{z\to\infty} S(z) = \frac{a}{c}$, $S(\frac{-d}{c}) = \infty$

With these definition, $S : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a topological mapping. Here one may use the chordal metric to define the topology.

$$S'(z) = \frac{ad - bc}{(cz + d)^2}$$

Then S is conformal in $\hat{\mathbb{C}} - \{-\frac{d}{c}, \infty\}$.

$w = z + \alpha$ is called a **parallel translation**.

$w = kz$ with $|k| = 1$ is a **rotation**.

$w = kz$ with $k > 0$ is a **homothetic transformation**.

$x = \frac{1}{z}$ is called an **inversion**.

Proposition 3.9. *Every Möbius transformation is a composition of the above four operations.*

3.3.1 Cross ratio

For three distinct points $z_2, z_3, z_4 \in \hat{\mathbb{C}}$,we can find a Möbius transformation S such that $S(z_2) = 0, S(z_3) = 1, S(z_4) = \infty$.

Lemma 3.10. *The Möbius transformation satisfying the above conditions is unique.*

The **cross ratio** (z_1, z_2, z_3, z_4) is the image z_1 under the Möbius transformation which maps z_2 to 1, z_3 to 0 and z_4 to ∞.

Theorem 3.11. *If $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ are distinct, and T is any Möbius transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Lemma 3.12. *Let T be a Möbius transformation, $T(\mathbb{R})$ is either a circle or a straight line.*

Theorem 3.13. *The cross ratio (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle or a straight line.*

Remark 3.14. One may prove the theorem by elementary geometry

Theorem 3.15. *A Möbius transformation maps circles into circles.*

3.3.2 Symmetry

Suppose T is a Möbius transformation which maps $\hat{\mathbb{R}}$ onto a circle C. We say that $w = Tz$ and $w^* = T\bar{z}$ are **symmetric** w.r.t. C.

Remark 3.16. This definition is independent of T. Suppose S is another Möbius transformation which maps $\hat{\mathbb{R}}$ onto C, then $S^{-1}T$ maps $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$, and this $S^{-1}w = S^{-1}Tz$ and $S^{-1}w^* = S^{-1}T\bar{z}$ are conjugate.

The points z and z^* are **symmetric w.r.t C through** z_1, z_2, z_3 iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.

This can be another definition.

Note that only the points on C are symmetric to themselves.

The mapping $z \mapsto z^*$ is 1-1 and is called **reflection** w.r.t. C.

Geometric Meaning of Symmetry

Case1: C is a straight line. We may assume $z_3 = \infty$.

z, z^* are symmetric w.r.t. C if and only if

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Then

$$|z^* - z_2| = |z - z_2|, \quad \forall z_2 \in C \text{ and } z_2 \neq \infty$$

$$\operatorname{Im} \frac{z^* - z_2}{z_1 - z_2} = \operatorname{Im} \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

So C is the bisecting normal of the segment between z and z^*.

Case2: C is the circle $|z - a| = R$.

Then for $\forall$ distinct $z_1, z_2, z_3 \in \mathbb{C}$, $\overline{(z, z_1, z_2, z_3)} = \overline{(z-a, z_1-a, z_2-a, z_3-a)}$
$= (\bar{z}-\bar{a}, \bar{z}_1-\bar{a}, \bar{z}_2-\bar{a}, \bar{z}_3-\bar{a}) = (\bar{z}-\bar{a}, \dfrac{R^2}{z_1-a}, \dfrac{R^2}{z_2-a}, \dfrac{R^2}{z_3-a}) = (\dfrac{R^2}{\bar{z}-\bar{a}}, z_1-a, z_2-a, z_3-a)$
$= (\dfrac{R^2}{\bar{z}-\bar{a}}, z_1, z_2, z_3)$.
Then the symmetric point of z w.r.t. C is

$$z^* = \frac{R^2}{\bar{z}-\bar{a}} + a$$

or

$$(z^*-a)(\bar{z}-\bar{a}) = R^2$$

$\Rightarrow$

$$\begin{cases} |z^*-a| \cdot |z-a| = R^2 \\ \dfrac{z^*-a}{z-a} = \dfrac{(z^*-a)(\bar{z}-\bar{a})}{|z-a|^2} > 0 \end{cases}$$

Theorem 3.17 (The Symmetric principle)**.** *If a Möbius transformation maps a circle C_1 onto a circle C_2, then it transforms any pair of symmetric points w.r.t. C_1 into a pair of symmetric points w.r.t. C_2.*

Proof. Case1: $C_1 = \hat{\mathbb{R}}$. Let T be the Möbius transformation which maps $\hat{\mathbb{R}}$ onto C_2. $\forall z \in \mathbb{C}$, by definition, $w = Tz$ and $w^* = T\bar{z}$ are symmetric w.r.t. C_2.
Case2: C_1 is a general circle. Let $T : C_1 \to C_2$ and $S : \mathbb{R} \to C_2$ be Möbius transformation.

Suppose w, w^* are symmetric w.r.t. C_1. Then there exists z *s.t.* $w = Sz, w^* = S\bar{z}$.
Then we can find $Tw = TSz, Tw^* = TS\bar{z}$ are symmetric w.r.t. C_2 since $TS : \hat{\mathbb{R}} \to C_2$ ∎

Remark 3.18. (1). The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w, z_2 \mapsto w_2, z_3 \mapsto w_3$ where $z_1, z_2, z_3 \in C_1, w_1, w_2, w_3 \in C_2$ is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3,)$$

(2). The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w_1$, $z_2 \mapsto w_2$ where $z_1 \in C_1, z_2 \notin C_1$, $w_1 \in C_2, w_2 \notin C_2$ is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

3.3.3 Steiner Circles, circular net

For $S(z) = \dfrac{az+b}{cz+d}$, $S'(z) = \dfrac{ad-bc}{(cz+d)^2}$.

A point $z \notin$ a circle C is said to on the **right(left, resp.)** of C if $\mathrm{Im}(z, z_1, z_2, z_3) > 0 (\mathrm{Im}(z, z_1, z_2, z_3) < 0)$

Remark 3.19.

(1). This agrees with everyday use since $(i, 1, 0, \infty) = i$

(2). This distinct between left and right is the same for all triples, while the meaning may be reversed.

(If $C = \hat{\mathbb{R}}$, then $(z, z_1, z_2, z_3) = \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R} \Rightarrow \mathrm{Im}(z, z_1, z_2, z_3) = \dfrac{ad-bc}{|cz+d|^2}\,\mathrm{Im}(z)$)

(3). We can define an absolute positive orientation of all finite circles by requiring that ∞ should be lie to the right of the oriented circles.

Consider a Möbius transformation of the form

$$w = k \cdot \frac{z-a}{z-b}$$

Here, $z = a \mapsto w = 0, z = b \mapsto w = \infty$.

Then circles through a, b maps to straight line through $0, \infty$.

The concentric circle about the origin, $|w| = \rho$, correspond to circles with the equation

$$\left|\frac{z-a}{z-b}\right| = \frac{\rho}{|k|}$$

These are the circles of **Apollonius** with limit points a and b.

Denote by C_1 the circles through a, b and C_2 the circles of Apollonius with these limit points. The configuration formed by all the circles C_1 and C_2 is called the **Steiner circles**(or **circular net**)

Theorem 3.20.

(a) There is exactly one C_1 and one C_2 through each point in $\hat{\mathbb{C}}\backslash\{a, b\}$

(b) Every C_1 meets every C_2 under right angle.

(c) Reflection in a C_1 transforms every C_2 into itself and every C_1 into another C_1.

(d) The limit points a, b are symmetric w.r.t. each C_2, but not w.r.t. other circles.

Proof. If the limit points are $0, \infty$, those properties are trivial in the w-plane. The general case follows since all properties are invariant under Möbius transformations. □

3.4 Elementary Conformal mapping

Example 3.21. $w = z^\alpha$ where $\alpha > 0$.

Let $S(u_1, u_2)$ with $0 < \varphi_2 - \varphi_1 \leqslant 2\pi$ be $\{z \in \mathbb{C} : z \neq 0, \varphi_1 < \arg(z) < \varphi_2\}$ where $\arg(z)$ can be chosen as any value of it.

Then $S(\varphi_1, \varphi_2)$ is a region.

In this region, a unique value of $w = z^\alpha$ is defined by $\arg w = \alpha \arg z$.

This function is analytic with $\dfrac{\mathrm{d}w}{\mathrm{d}z} = \alpha \dfrac{w}{z}$.

This function is $1-1$ only if $\alpha(\varphi_2 - \varphi_1) \leqslant 2\pi$.

Example 3.22. $w = e^z$ maps $\{z \in \mathbb{C} : -\dfrac{\pi}{2} < \mathrm{Im}(z) < \dfrac{\pi}{2}\}$ onto $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$

Example 3.23. $w = \dfrac{z-1}{z+1}$ maps $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ onto $\{ww \in \mathbb{C} : |w| < 1\}$

Example 3.24.

$$\mathbb{C}\backslash[-1,1] \xrightarrow{z_1=\frac{z+1}{z-1}} \mathbb{C}\backslash(-\infty,0] \xrightarrow{z_2=\sqrt{z_1}} \{\mathrm{Re}(z_2) > 0\} \xrightarrow{w=\frac{z_2-1}{z_2+1}} \{w \in \mathbb{C} : |w| < 1\} \quad (3.1)$$

3.4.1 Elementary Riemann surfaces

Example 3.25. $w = z^n$, $n \in \mathbb{Z}_+$ and $n > 1$.

There is a 1-1 correspondence between each angle $\dfrac{(k-1)2\pi}{n} < \arg z < \dfrac{k \cdot 2\pi}{n}, k = 1, 2, \cdots, n$ and while w-plane except for the positive real axis.

Example 3.26. $w = e^z$. This function maps each parallel strip $(k-1)2\pi < \mathrm{Im}\, z < k \cdot 2\pi, k \in \mathbb{Z}$ onto a sheet with a cut along the positive axis.

4 Complex Integration

4.1 Fundamental Theorems

4.1.1 Line integral and rectifiable arcs

Let $f(t) = u(t) + iv(t)$ be a complex-valued defined on $t \in [a,b] \subset \mathbb{R}$ where u, v are real-valued functions. If f is continuous on $[a,b]$, we may define the **integral**

$$\int_a^b f(t)\mathrm{d}t := \int_a^b u(t)\mathrm{d}t + i\int_a^b v(t)\mathrm{d}t$$

Let γ be a piecewise differential arc in $\mathbb{C}$ with the equation $z = z(t), a \leqslant t \leqslant b$. If f is continuous on γ, then $f(z(t))$ is continuous on $[a, b]$, and we define

$$\int_{\gamma} f(z)\mathrm{d}z = \int_{a}^{b} f(z(t))z'(t)\mathrm{d}t \tag{4.1}$$

The integral defined in 4.1 is independent of the parametrization of γ. Suppose that anther parametrization of γ is $\gamma : (\alpha, \beta) \to \mathbb{C}, \tau \mapsto z(t(\tau))$, where $t : (\alpha, \beta) \to (a, b), \tau \mapsto t(\tau)$ is piecewise differentiable. Then we have

$$\int_{a}^{b} f(z(t))z'(t)\mathrm{d}t = \int_{\alpha}^{\beta} f(z(t(\tau)))z'(t(\tau))t'(\tau)\mathrm{d}t = \int_{\alpha}^{\beta} f(z(t(\tau)))\frac{\mathrm{d}z(t(\tau))}{\mathrm{d}\tau}\mathrm{d}\tau \tag{4.2}$$

For an arc γ with equation $z = z(t), a \leqslant t \leqslant b$, we define $-\gamma$ by $z = z(-t), -b \leqslant t \leqslant a$.

Then we have

$$\begin{aligned}
\int_{-\gamma} f(z)\mathrm{d}z &= \int_{-b}^{-a} f(z(-t))\frac{\mathrm{d}z(-t)}{\mathrm{d}t}\mathrm{d}t \\
&= -\int_{-a}^{-b} f(z(-t))z'(-t)\mathrm{d}t \\
&= -\int_{a}^{b} f(z(\tau))z'(\tau)\mathrm{d}\tau \\
&= -\int_{\gamma} f(z)\mathrm{d}z
\end{aligned}$$

So we have those properties:

Proposition 4.1.

(a) $\displaystyle\int_{-\gamma} f(z)\mathrm{d}z = -\int_{\gamma} \mathrm{d}z$

(b) *Let f and g be two continuous functions on the piecewise differentiable arc γ, then*

$$\int_\gamma (\lambda_1 f + \lambda_2 g)\mathrm{d}z = \lambda_1 \int_\gamma f\mathrm{d}z + \lambda_2 \int_\gamma g\mathrm{d}z, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

(c) *If γ can be subdivided into two pieces differentiable arcs γ_1 and γ_2, and f is continuous on γ_1 , then*

$$\int_\gamma f\mathrm{d}z = \int_{\gamma_1} f\mathrm{d}z + \int_{\gamma_2} f\mathrm{d}z$$

(d) (c) *implies that the integral of a closed curve doesn't depend on the starting point on the curve*

Example 4.2. Evaluate $\displaystyle\int_\gamma \frac{1}{z-a}\mathrm{d}z$ where γ is the circle centered at $a \in \mathbb{C}$ with radius R.

Let $z = z(t) = a + Re^{it}$. Then the integral is $2\pi i$

4.1.2 The fundamental theorem of Calculus for integrals in $\mathbb{C}$

The line integral w.r.t. $\bar{z}$ is defined by

$$\int_\gamma f(z)\overline{\mathrm{d}z} = \overline{\int_\gamma \overline{f(z)}\mathrm{d}z}$$

With this notation, line integrals w.r.t. $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ can be defined by

$$\int_\gamma f(z)\mathrm{d}x = \frac{1}{2}[\int_\gamma f(z)\mathrm{d}z + \int_\gamma f(z)\overline{\mathrm{d}z}]$$

$$\int_\gamma f(z)\mathrm{d}y = \frac{1}{2i}[\int_\gamma f(z)\mathrm{d}z - \int_\gamma f(z)\overline{\mathrm{d}z}]$$

if we write $f(z) = \mu + i\nu$, we have

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}x + i\int_\gamma f(z)\mathrm{d}y = \int_\gamma (\mu\mathrm{d}x - \nu\mathrm{d}y) + i\int_\gamma (\nu\mathrm{d}x + \mu\mathrm{d}y)$$

Remark 4.3. It is followed by the intuition. We can view the integration as the multiplication between f and $\mathrm{d}z$.

The integral w.r.t. **arc length** is defined by

$$\int_\gamma f(z)|\mathrm{d}z| = \int_a^b f(z(t))|z'(t)|\mathrm{d}t$$

This integral is again independent of the parametrization. It is easy to check

$$\int_{-\gamma} f(z)|\mathrm{d}z| = \int_\gamma f(z)|\mathrm{d}z|$$

Now we define **length** of a curve γ: $L(\gamma) = \int_\gamma |\mathrm{d}z|$

We have the inequality:

$$\left|\int_\gamma f\mathrm{d}z\right| \leqslant \int_\gamma |f| \cdot |\mathrm{d}z| \leqslant L(\gamma) \cdot \sup_{z \leqslant \gamma} |f(z)|$$

The length of an arc γ ($z = z(t)$) can also be defined as the least upper bound of all sums

$$\sum_{i=1}^n |z(t_i) - z(t_{i-1})|$$

where $a = t_0 < t_1 < \cdots < t_n = b$ If this least upper bound is finite, we say that the arc is **rectifiable**

It is easy to show that piecewise differentiable arcs are rectifiable.

The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_\gamma f(z)\mathrm{d}z = \lim \sum_{k=1}^n f(z(\psi_k))[z(t_k) - z(t_{k-1})]$$

Theorem 4.4. *Let $\Omega \subset \mathbb{C}$ be a region, and P, Q two (possibly complex-valued) functions that are continuous on Ω, γ closed curve. The integral $\int_\gamma P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$ depends only on the end point of γ iff there exists a function $U(x,y)$ on Ω with $\dfrac{\partial U}{\partial x} = P$, $\dfrac{\partial U}{\partial y} = Q$.*

Proof. "$\Leftarrow$": If such a U exists, then

$$\int_\gamma P\mathrm{d}x + Q\mathrm{d}y = \int_\gamma \frac{\partial U}{\partial x}\mathrm{d}x + \frac{\partial U}{\partial y}\mathrm{d}y = \int_\gamma \frac{\mathrm{d}U}{\mathrm{d}t}\mathrm{d}t = U(\gamma(b)) - U(\gamma(a))$$

"$\Rightarrow$": Fix a point $(x_0, y_0) \in \Omega$. We define $U(x,y) = \int_\gamma P\mathrm{d}x + Q\mathrm{d}y$ where γ is any curve between (x_0, y_0) and (x,y). Easy to check that it is true. $\square$

Theorem 4.5 (Fundamental theorem of Calculus for integrals on $\mathbb{C}$)**.** *Let f be continuous on a region Ω containing γ. $\int_\gamma f\mathrm{d}z$ depends on the endpoints iff f is the derivative of an analytic function F in Ω.*

Remark 4.6. We will prove $\displaystyle\int_\gamma f\mathrm{d}z = F(\omega_2) - F(\omega_1)$ where γ begins at ω_1 and ends at ω_2.

Proof. Transform the line integration into the composition of two real integration.

$\square$

Corollary 4.7. *If F is analytic on Ω with $F' = f$, and γ is a closed curve in Ω, then $\int_\gamma f\mathrm{d}z = 0$. Conversely if f is continuous on Ω and $\int_\gamma f\mathrm{d}z = 0$ for any closed curve in Ω, then f is the derivative of an analytic function F in Ω.*

4.1.3 Cauchy's theorem for a rectangle

There are some notes in this section:

R is the rectangle in $\mathbb{C}$, $R = \{x + iy \in \mathbb{C} : a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$. And ∂R is boundary curve oriented in the counterclockwise direction.

Theorem 4.8 (Cauchy's theorem for a rectangle). *If f is analytic on an open set which contains R, then* $\int_{\partial R} f(z)\mathrm{d}z = 0$

Proof. For $\forall$ rectangle $\tilde{R}$ inside R, we define $Z(\tilde{R}) = \int_{\partial \tilde{R}} f(z)\mathrm{d}z$. Then $Z(R) = Z(R_1) + Z(R_2)$ if R is divided into Z_1, Z_2.

Since we can divide R into four equal rectangles, and find a rectangle with $|Z(R^{(1)})| \geqslant \frac{1}{4}|Z(R)|$. Then repeat the above steps and we obtain a sequence of nested rectangles $R \supset R^{(1)} \supset \cdots$ with the property

$$Z(R^{(n)}) \geqslant \frac{1}{4}|Z(R^{(n-1)})| \geqslant \cdots \geqslant \frac{1}{4^n} Z(R) \tag{4.3}$$

$\forall \delta > 0, \exists n \in \mathbb{N} \quad s.t. \ R^{(n)} \subset \{z \in \mathbb{C} : |z - z_0| < \delta\}, \forall n \geqslant N$, where z_0 is the limit of $R^{(n)}$ as $n \to \infty$.

f is analytic in $R \Rightarrow \forall \varepsilon, \exists \delta > 0 \quad s.t.$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon, \forall z \text{ with } |z - z_0| < \delta \tag{4.4}$$

We assume that δ satisfies both conditions. We have

$$Z(R^{(n)}) = \int_{\partial R^{(n)}} f(z)\mathrm{d}z = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)]\mathrm{d}z$$

$$\Rightarrow |Z(R^{(n)})| \leqslant \varepsilon \int_{\partial R^{(n)}} |z - z_0|\mathrm{d}z \text{ by 4.4}$$

Let d_n be the length of diagonal of $R^{(n)}$, L_n be the length of its perimeter. Then $|z - z_0| \leqslant d_n$, $\forall z \in \partial R^{(n)}$.

$\Rightarrow |Z(R^{(n)})| \leqslant \varepsilon d_n L_n = \varepsilon \frac{D}{2^n} \cdot \frac{L}{2^n}$ where D, L are the diameter and perimeter of R.

$\Rightarrow |Z(R)| \overset{4.3}{\leqslant} 4^n |Z(R^{(n)})| \leqslant \varepsilon DL \Rightarrow Z(R) = 0$ since ε is arbitrary. □

We will next prove the following stronger theorem:

Theorem 4.9 (stronger version of Cauchy's theorem for a rectangle)**.** *Let f be analytic on $R' = R\backslash\{\psi_1, \cdots, \psi_m\}, m \in \mathbb{N}$. If $\lim_{z\to\psi_j}(z - \psi_j)f(z) = 0, \forall 1 \leqslant j \leqslant m$, then $\displaystyle\int_{\partial R} f(z)\mathrm{d}z = 0$.*

Proof. WLOG, we may assume f is not analytic at only one point $\psi \in R$. If we put psi into a small rectangle S_0, then the previous theorem tells us $\int_{\partial R} f(z)\mathrm{d}z = \int_{\partial S_0} f(z)\mathrm{d}z$.

$\forall \varepsilon > 0$, we may choose S_0 small enough such that $|f(z)| \leqslant \dfrac{\varepsilon}{|z-\varepsilon}$, $\forall z \in \partial S_0$

$\Rightarrow |\displaystyle\int_{\partial R} f(z)\mathrm{d}z \leqslant \varepsilon \int_{\partial S_0} \frac{|\mathrm{d}z|}{|z-\psi|} \leqslant \varepsilon \frac{1}{\frac{l}{2}} \cdot 4l = 8\varepsilon$

$\Rightarrow \int_{\partial R} f(z)\mathrm{d}z = 0$ since ε is arbitrary. $\square$

4.1.4 Cauchy's Theorem for a disk

$\Delta := \{z \in \mathbb{C} : |z - z_0| < R\}$ where $R > 0$.

Theorem 4.10 (Cauchy's Theorem for a disk)**.** *If f is analytic in an open disk Δ, then $\int_\gamma f(z)\mathrm{d}z = 0$ for closed curve γ in Δ.*

Proof. Suppose the center of Δ is $z_0 = x_0 + iy_0$, $z = x + iy$. We define

$$F(z) = \int_\gamma f(z)\mathrm{d}z$$

where γ is the horizontal line segment from z_0 to (x, y_0) added with vertical line segment from (x, y_0) to z. We have

$$\frac{\partial F}{\partial y} = \lim_{\delta y\to 0} \frac{F(x, y+\delta y) - F(x,y)}{\delta y} = \lim_{\delta y \to 0} \frac{1}{\delta y}\int_{\delta\gamma} f(z)\mathrm{d}z = if(z) \tag{4.5}$$

By Cauchy' theorem on rectangles, one has $F(z) = -\int_{\tilde{\gamma}} f(z)\mathrm{d}z$, where $\tilde{\gamma}$ is the vertical line segment from z_0 to (x_0, y) added with horizontal line segment from

(x_0, y) to z.

Similarly, $\dfrac{\partial F}{\partial x} = f(z)$.

$\Rightarrow \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} \Rightarrow F$ is analytic in Δ with derivative f. By Fundamental Theorem 4.5 of Calulus $\Rightarrow \int_\gamma f(z)\mathrm{d}z = 0$ for $\forall$ closed curve in Δ. □

Here is a stronger version.

Theorem 4.11 (stronger version of Cauchy's Theorem for a disk)**.** *Let f be analytic in a region $\Delta' = \Delta\backslash\{\psi_1, \cdots, \psi_m\}$ with $m \in \mathbb{N}$. If f satisfies $\lim\limits_{z\to\psi_j}(z - \psi_j)f(z) = 0, \forall 1 \leqslant j \leqslant m$, then $\displaystyle\int_\gamma f(z)\mathrm{d}z = 0, \forall\gamma$ closed in Δ'*

Proof. It is similar to the above proof.

For the case no ψ_j lies on $x = x_0$ and $y = y_0$, we can find a similar curve γ with last segment is a vertical one. Let $F(z) = \int_\gamma f(z)\mathrm{d}z$. And continue the process of proof of the previous theorem.

For the case that $\exists\ \psi_j$ lies on the lines $x = x_0, y = y_0$, we actually can move the center to another point *s.t.* no ψ_j lies on the lines $x = x_0', y = y_0'$. □

4.2 Cauchy's integral formula

4.2.1 Index of a point with resect to a closed curve

Lemma 4.12. *If the piecewise differentiable closed curve γ does not pass through $z \in \mathbb{C}$, then the value of the integral $\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}$ is a multiple of $2\pi i$.*

Proof. $\gamma : \zeta = \zeta(t), \alpha \leqslant t \leqslant \beta$. $h(t) = \int_\alpha^t \frac{\zeta'(s)}{\zeta(s) - z}\mathrm{d}s$.

$z \in \gamma \Rightarrow h$ is defined and continuous on $[\alpha, \beta]$. For all t *s.t.* $\zeta'(t)$ is continuous, we have

$$h'(t) = \frac{\zeta'(t)}{\zeta(t) - z} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-h(t)}(\zeta(t) - z)\right] = 0$$

So $e^{-h(t)}(\zeta(t) - z)$ is constant on $[\alpha, \beta]$.

Then $e^{h(t)} = \dfrac{\zeta(t) - z}{\zeta(\alpha) - z} \Rightarrow e^{h(\beta)} = 1 \Rightarrow h(\beta) \in \{2k\pi i : k \in \mathbb{Z}\}$. □

The **index of the point** z w.r.t. the closed curve γ is the number

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}$$

n is also called the **winding number**.

Theorem 4.13. *Let γ be a piecewise differentiable closed curve. The function $z \mapsto n(\gamma, z)$ is constant on each connected set of $\mathbb{C} \backslash \gamma$, and zero if this set is unbounded.*

Proof. Define $f : \mathbb{C} \backslash \gamma \to \gamma, z \mapsto n(\gamma, z) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}$.

Then

$$|f(z) - f(z_0)| = \frac{1}{2\pi} \left| \int_\gamma \frac{z - z_0}{(\zeta - z)(\zeta - z_0)} \mathrm{d}\zeta \right| \leqslant \frac{|z - z_0|}{2\pi} \int_\gamma \frac{1}{|\zeta - z| \cdot |\zeta - z_0|} |\mathrm{d}\zeta|$$

$\Rightarrow f$ is continuous on each open connected set of $\mathbb{C} \backslash \gamma$. Let Ω be any open connected set of $\mathbb{C} \backslash \gamma$. We have $f(\Omega)$ is connected $\xRightarrow{f(\Omega) \subset \mathbb{Z}}$ $f(\Omega)$ contains at most one point $\Rightarrow f$ is constant on Ω.

If $|z|$ is sufficient large, $\exists$ a disk of radius R, $B(0, R)$, *s.t.* $\gamma \subset B(0, R)$ but $z \notin B(0, R)$. Cauchy's theorem for a disk 4.10 tells us that $f(z) = n(\gamma, z) = 0$. So it is zero if this set is unbounded. □

Lemma 4.14. *Let z_1, z_2 be two points on a closed curve γ and $0 \notin \gamma$.*

Suppose z_1 in the lower half space and z_2 in upper half space. If $\gamma_1 \cap \{(x, 0) : x \leqslant 0\} = \varnothing$, and $\gamma_2 \cap \{(x, 0) : x \geqslant 0\} = \varnothing$, then $n(\gamma, 0) = 1$.

Remark 4.15. One method to prove this lemma is to create two segment from z_i to the point in the unit circle. By divide the curve into two parts, we can easily remove the part of previous curve by using the theorem 4.13, since 0 is in the unbounded set.

In this proof, we can find that Theorem 4.13 is such powerful that we can change any curve to a more simple curve easily!

4.2.2 Cauchy's integral formula

Theorem 4.16 (Cauch's integral formula). *Suppose that f is analytic in an open disk $\triangle$, and let γ be a closed curve in $\triangle$. For $\forall z \notin \gamma$,*

$$n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

where $n(\gamma, z)$ is the index of z w.r.t. γ.

Proof. If $z \notin \triangle$, The both sides of the equation is 0.

So we may assume $z \in \triangle$ and $z \notin \gamma$. Define $F : \triangle\backslash\{z\} \to \mathbb{C}, \zeta \mapsto \dfrac{f(\zeta) - f(z)}{\zeta - z}$.

Then F is analytic in $\triangle\backslash\{z\}$, and $\lim\limits_{\zeta\to z}(\zeta - z)F(\zeta)$.

By Cauchy's Theorem 4.9 $\Rightarrow \int_\gamma F(\zeta)\mathrm{d}\zeta = 0 \Rightarrow \displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = f(z)\int_\gamma \frac{1}{\zeta - z}\mathrm{d}\zeta =$ $f(z)\cdot 2\pi i\cdot n(\gamma, z)$ □

Remark 4.17. This proof let us find that for a good-enough function, its integral over a closed curve is a constant.

The theorem still holds if f is analytic except at a finite number of ζ_j *s.t.*

$$\lim_{\zeta\to\zeta_j}(\zeta - \zeta_j)f(\zeta) = 0$$

and $z \neq \zeta_j$ for each j, since Cauchy's theorem is still applicable.

Theorem 4.18 (The mean value property for analytic functions). *f is analytic in a region Ω which contain $\overline{B(z, R)}$. Then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + Re^{i\theta})\mathrm{d}t$$

Proof. The previous theorem 4.16 $\Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta \xlongequal{\zeta = z + Re^{it}} \frac{1}{2\pi}\int_0^{2\pi} f(z + Re^{it})\mathrm{d}t$$

□

If f is analytic in an open disk $\triangle$, and γ is a closed curve in $\triangle$. And $n(\gamma, z) = 1$. Then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

This is usually referred to as **Cauchy's integral formula**

4.2.3 Higher derivatives

Lemma 4.19. *Let $\Omega \subset \mathbb{C}$ be a region and γ be an arc in Ω. If φ is continuous on γ, then the function*

$$F_n(z) := \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n}\mathrm{d}\zeta$$

is analytic in each of the regions $\Omega\backslash\gamma$, and its derivative is $F_n'(z) = nF_{n+1}(z)$

Proof. We prove it by induction.

The lemma is true if $n = 0$: $F_0(z) = \int_\gamma \varphi(\zeta)\mathrm{d}\zeta$ and $F_0'(z) = 0 = 0 \cdot F_1(z)$.

We suppose that the lemma holds for $n - 1$ with $n \in \mathbb{N}$: $\forall$ continuous φ on γ, F_{n-1} is analytic in $\Omega\backslash\gamma$ and $F_{n-1}'(z) = (n-1)F_n(z), \forall z \in \Omega\backslash\gamma$.

Fix $z_0 \in \Omega\backslash\gamma$. For $\forall z \in B(z_0, \frac{\delta}{2})$, with $B(z_0, \delta) \subset \Omega\backslash\gamma$, we have $|\zeta - z| > \frac{\delta}{2}$, $\forall \zeta \in \gamma$.

For $\forall$ continuous φ on γ,

$$\begin{aligned}
F_n(z) - F_n(z_0) &= \int_\gamma \frac{\varphi(\zeta)(\zeta - z + z - z_0)}{(\zeta - z)^n(\zeta - z_0)}\mathrm{d}\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n}\mathrm{d}\zeta \\
&= \left[\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)}\mathrm{d}\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^{n-1}(\zeta - z)}\mathrm{d}\zeta\right]
\end{aligned}$$

$$+ (z - z_0) \int_\gamma \frac{\varphi(\zeta)\mathrm{d}\zeta}{(\zeta - z)^n(\zeta - z_0)}$$

Let $\psi(\zeta) = \dfrac{\psi(\zeta)}{\zeta - z_0}$, which is continuous except γ.

Using the induction condition to ψ, we can finish the proof. □

Theorem 4.20. *An analytic function on a region Ω has derivatives of all orders which are analytic in Ω. More precisely, $\forall z_0 \in \Omega$, choose $B(z, \delta) \subset \Omega$ and a circle $C \subset B(z_0, \delta)$ with center z_0. For $\forall$ z in the interior of C, Cauchy's integral formula gives*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

Then the previous lemma implies $f'(z) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta$ is analytic in the interior of C. More generally, for $\forall n \in \mathbb{N}$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta \tag{4.6}$$

4.2.4 Consequences of Cauchy

Theorem 4.21 (Morera's Theorem). *If f is continuous in a region Ω, and if $\int_\gamma f(z)\mathrm{d}z = 0$ for $\forall$ closed curve γ in Ω. Then f is analytic in Ω.*

Proof. We proved in Corollary 4.7 that under the hypothesis of theorem, $f = F'$ where F is analytic in Ω. The last theorem $\Rightarrow$ f is analytic. □

Suppose f is analytic in a disk, $\overline{B(z_0, R)}$, and bounded on the circle γ given by $|z - z_0| = R$. Then $\forall z \in \gamma$, $|f(z)| \leqslant M$ for some $M \geqslant 0$. By (4.6),

$$|f^{(n)}(z)| \leqslant \frac{n!}{2\pi} \int_C \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}}|\mathrm{d}\zeta| \leqslant \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = MR^{-n}n! \tag{4.7}$$

This inequality is known as **Cauchy's estimate**.

Theorem 4.22 (Liouville's Theorem). *A bounded entire function (i.e. analytic in $\mathbb{C}$) is constant.*

Proof. Suppose $|f(z)| \leqslant M$, $\forall z \in \mathbb{C}$. Cauchy's estimate $\Rightarrow$

$$b|f'(z)| \leqslant \frac{M}{R}, \forall z \in \mathbb{C}, \forall R > 0 \tag{4.8}$$

□

$\xRightarrow{R\to\infty} f'(z) = 0$ for $z \in \mathbb{C} \Rightarrow f = 0$.

Theorem 4.23 (Fundamental Theorem for Algebra). *Every polynomial of degree $n \geqslant 1$ has n roots.*

Proof. It suffices to prove it has at least one root.

Suppose $P(z) = a_n z^n + \cdots a_1 z + a_0$ with $a_0 \neq 0$ does not have a root.

Then $f(z) := \frac{1}{P(z)}$ is an entire function. As $z \to \infty$, $\lim\limits_{|z|\to\infty} \dfrac{|P(z)|}{|z|^n} = |a_n| \Rightarrow \lim\limits_{|z|\to\infty} \dfrac{1}{|P(z)|} = 0$.

So f is bounded. By Liouville's Theorem, f is a constant. Where $f = f(\infty) = 0$. That causes contradiction. □

Theorem 4.24 (Power series). *If f is analytic in a region Ω which contains a closed disk $\overline{B(z_0, R)}$, then f has a power series expansion at z_0,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad \forall z \in B(z_0, R) \tag{4.9}$$

Proof. $\forall z \in B(z_0, R)$, $\forall \zeta$ with $|\zeta - z_0| = R$.

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\
&= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \\
&= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}
\end{aligned} \tag{4.10}$$

This series converges uniformly in ζ with $|\zeta - z_0| = R$.

For $\forall z \in B(z, R)$,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_{|\zeta - z| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \\
&\overset{\text{uniformly}}{=\!=\!=} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \cdot (z - z_0)^n \\
&\overset{(4.6)}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\end{aligned} \tag{4.11}$$

□

4.3 Local properties of analytic functions

4.3.1 Removable Singularities and Taylor's Theorem

We remarked that Cauchy's integral formula holds if f is analytic except at a finite number of point ζ_j *s.t.* $\lim_{\zeta \to \zeta_j} (\zeta - \zeta_j) f(\zeta) = 0$. We will prove f can be extended to an analytic function in $\triangle$. In other word, ζ_j are **removable singularities**.

Theorem 4.25 (Riemann's Removable Singularities Theorem). *Suppose that f is* *analytic in the region $\Omega' = \Omega \backslash \{\zeta_0\}$ where Ω is also a region. Then there exists an analytic function in Ω which coincides with f in Ω' if and only if $\lim_{z \to \zeta_0} (z - \zeta_0) f(z) = 0$.*

Proof. The uniqueness and "$\Rightarrow$" part is trivial since the extended function is continuous at ψ_0.

"$\Leftarrow$": Cauchy's integral formula $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta, \ \forall z \in \triangle \text{ and } z \neq \zeta_0 \tag{4.12}$$

Lemma 4.19 $\Rightarrow$ the RHS of the last equation 4.12 is analytic in $z \in \triangle$. Then

$$\hat{f}(z) = \begin{cases} f(z), & z \neq \zeta_0 \\ \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - \zeta_0} \mathrm{d}\zeta, z = \zeta_0 \end{cases} \tag{4.13}$$

is analytic in Ω. □

We apply Theorem 4.25 to the function $F(z) = \dfrac{f(z) - f(\zeta)}{z - \zeta}$, where f is analytic in a region Ω. Note that

$$\lim_{z \to \zeta} (z - \zeta) F(z) = 0, \ \lim_{z \to \zeta} F(z) = f'(\zeta) \tag{4.14}$$

Theorem 4.25 $\Rightarrow$ $\exists$ analytic function f_1 on Ω *s.t.*

$$f_1(z) = \begin{cases} F(z), & z \neq \zeta_0 \\ f'(\zeta), z = \zeta_0 \end{cases} \tag{4.15}$$

we may thus write $f(z) = f(\zeta) + (z - \zeta) f_1(z)$.

Repeating this process for f_1, we get an analytic function f_2 on Ω *s.t.*

$$f_1(z) = f_1(\zeta) + (z - \zeta) f_2(z) \tag{4.16}$$

where

$$f_2(z) = \begin{cases} \dfrac{f_1(z) - f_1(\zeta)}{z - \zeta}, & z \neq \zeta \\ f_2'(\zeta), & z = \zeta \end{cases} \tag{4.17}$$

Continuing the recursion, we have the general form

$$f_{n-1}(z) = f_{n-1}(\zeta) + (z - \zeta) f_n(z) \tag{4.18}$$

$\Rightarrow$

$$f(z) = f(\zeta) + (z - \zeta) f_1(\zeta) + \cdots + (z - \zeta)^{n-1} f_n(\zeta) + (z - \zeta)^n f_n(z) \tag{4.19}$$

Differentiating n times and setting $z = \zeta \Rightarrow f^{(n)}(\zeta) = n! f_n(\zeta)$

We just prove **Taylor's Theorem**

Theorem 4.26 (Taylor's Theorem). *If f is analytic in a region Ω, $\zeta \in \Omega$, then we have*

$$f(z) = f(\zeta) + (z - \zeta) f'(\zeta) + \cdot + \frac{f^{(n-1)}(\zeta)}{(n-1)!}(z - \zeta)^{n-1} + f_n(z)(z - \zeta)^n \tag{4.20}$$

where f_n is analytic in Ω. Moreover,

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - \zeta)^n (\omega - z)} \mathrm{d}\omega \tag{4.21}$$

where C is a circle in Ω s.t. its interior $\triangle$ is also in Ω and $\zeta, z \in \triangle$

Proof. It suffices to prove the second part.

Cauchy's integral formula $\Rightarrow f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\omega)}{\omega - z} \mathrm{d}\omega$, $\forall z \in \triangle$.

For $f_n(z)$, we substitute the expression from (4.20). The first term is

$$\frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega-\zeta)^n(\omega-z)}\mathrm{d}\omega \tag{4.22}$$

The remaining terms have the following form, except for constant factors:

$$g_k(\zeta)=\int_C \frac{1}{(\omega-\zeta)^n(\omega-z)}\mathrm{d}\omega,\ 1\leqslant k\leqslant n \tag{4.23}$$

The lemma 4.19 applies to $\varphi(\omega)=\dfrac{1}{\omega-z}$, $g_k'(\zeta)=kg_{k-1}(\zeta), k\in\mathbb{N}, \forall\zeta\in\triangle$. So

$$\begin{aligned}
g_1(\zeta)&=\int_C \frac{1}{(\omega-\zeta)(\omega-z)}\mathrm{d}\omega\\
&=\frac{1}{\zeta-z}\left[\int_C\frac{1}{\omega-\zeta}\mathrm{d}\omega-\int_C\frac{1}{\omega-z}\mathrm{d}\omega\right]\\
&=\frac{1}{\omega-z}[2\pi i-2\pi i]=0
\end{aligned} \tag{4.24}$$

So $g_k(z)=0,\ \forall k\in\mathbb{N},\forall z\in\triangle$. □

4.3.2 Zeros and poles

Theorem 4.27. *If f is analytic in a region Ω and $\exists a\in\Omega$ s.t. $f^{(n)}(a)=0$ for $\forall n\in\mathbb{N}\cup\{0\}$, then $f\equiv 0$ in Ω.*

Proof. Let $B(a,R)$ be the disk s.t. $\overline{B(a,R)}\subset\Omega$. Let $C=\partial B(0,R)$.

Taylor's theorem $\Rightarrow f(z)=(z-a)^n f_n(z)$ with

$$f_n(z)=\frac{1}{2\pi i}\int_C\frac{f(\omega)}{(\omega-a)^n(\omega-z)}\mathrm{d}\omega,\ \forall n\in\mathbb{N}\cup\{0\},\forall z\in B(a,R) \tag{4.25}$$

Let $M = \max_{z \in C} |f(z)|$.

$$\begin{aligned}
\Rightarrow & |f_n(z)| \leqslant \frac{1}{2\pi} \cdot \frac{M}{R^n(R - |z-a|)} \cdot 2\pi R \\
\Rightarrow & |f(z)| \leqslant \frac{|z-a|^n}{R^n} \cdot \frac{MR}{R - |z-a|} \to 0 \text{ as } n \to \infty, \ \forall z \in B(0,R) \\
\Rightarrow & f(z) = 0, \ \forall z \in B(0,R)
\end{aligned}$$

Now define

$$\begin{aligned}
E_1 &= \left\{ z \in \Omega | f^{(n)}(z) = 0, \forall n \in \mathbb{N} \cup \{0\} \right\} \\
E_2 = \Omega \backslash E_1 &= \left\{ z \in \Omega | f^{(n)}(z) \neq 0, \text{ for some } n \in \mathbb{N} \cup \{0\} \right\}
\end{aligned}$$

We just proved E_1 is open. E_2 is open because $f^{(n)}$ is continuous in Ω for $\forall n \in \mathbb{N} \cup \{0\}$. Ω is a region $\Rightarrow$ either $R_1 = \varnothing$ or $R_2 = \varnothing$.

The assumption of the theorem $\Rightarrow E_1 \neq \varnothing \Rightarrow E_1 = \Omega$. □

Let f be analytic in Ω which is not identically zero, $f(a) = 0$ for some $a \in \Omega$. The previous theorem implies $\exists$ first $N \in \mathbb{N}$ *s.t.* $f^{(N)}(a) \neq 0$. Taylor's theorem implies that $f(a) = (z-a)^N f_N(z)$ where f_N is analytic and $f_N(a) \neq 0$. We say that a is a **zero of order** N of f.

f_N is continuous $\Rightarrow \exists \delta > 0$ *s.t.* $f(z) \neq 0$ for $\forall z \in B(a, \delta) \backslash \{0\}$.

So we have just proved an important result: Zeros of analytic functions are isolated, or equivalently, we have a famous theorem:

Theorem 4.28 (Identity Theorem)**.** *If f and g are analytic in a region ω, and $f = g$ on a set which has an accumulation point in Ω, then $f(z) = g(z)$.*

Corollary 4.29.

(1) If $f \equiv 0$ in a subregion of Ω and f is analytic in Ω, then $f \equiv 0$ in Ω.

(2) *If f is analytic in Ω and vanishes on an arc in Ω which doesn't reduce to a point, then $f \equiv 0$ in Ω.*

If f is analytic in a neighborhood of a, but perhaps not at a itself, then a is called an **isolated singularity** of f.

If $\lim_{z \to a} f(z) = \infty$, then a is said to be a **pole** of f, and we set $f(a) = \infty$. Continuity implies $\exists \delta > 0 \quad s.t. \ f(z) \neq 0$ for $\forall z \in B(0, \delta) \backslash \{a\}$. Thus, $g(z) = \dfrac{1}{f(z)}$ is analytic in $B(a, \delta) \backslash \{a\}$. $\lim_{z \to a} (z - a) g(z) = 0 \Rightarrow a$ is a removable singularity of g, and g has an analytic extension with $g(a) = 0$. $g \not\equiv 0 \Rightarrow a$ is a zero of g with finite order. The **order of the pole** of f at a is the order N of the zero of g at a.

We can write

$$f(z) = (z - a)^{-N} f_N(z), \ \forall z \in B(a, \delta) \backslash \{a\} \tag{4.26}$$

where f_N is analytic and nonzero in a neighborhood of a.

Definition 4.30. A function which is analytic in a region Ω except for (isolated) poles is called a **meromorphic function**.

Example 4.31. If f and g are analytic in Ω and $g \not\equiv 0$, then $\frac{f}{g}$ is a meromorphic function in Ω. (See the Identity Theorem 4.28)

Remark 4.32. The sum, the product and quotient (if denominator is not always zero) of two meromorphic functions are meromorphic.

If f has a pole of order N at a, then $(z - a)^N f(z)$ is analytic at a, and Taylor's theorem 4.26 implies

$$(z - a)^N f(z) = b_N + b_{N-1}(z - a) + \cdots + b_1 (z - a)^{N-1} + \varphi(z) \cdot (z - a)^N \tag{4.27}$$

where φ is analytic at a.

$$\Rightarrow f(z) = b_N(z-a)^{-N} + b_{N-1}(z-a)^{-(N-1)} + \cdots + b_1(z-a)^{-1} + \varphi(z),\ \forall z \neq a. \quad (4.28)$$

Theorem 4.33. *If f is analytic in a neighborhood of a, but perhaps not at a itself, then exactly one of the following 3 cases occurs:*

(i) $f \equiv 0$ *in this neighborhood.*

(ii) $\exists$ *integer* $N \in \mathbb{Z}$ *s.t.* $\lim\limits_{z\to a} |z-a|^{\alpha} \cdot |f(z)| = \begin{cases} 0, & \alpha > N \\ \infty, & \alpha < N \end{cases}$

(iii) neither $\lim\limits_{z\to a} |z-a|^{\alpha} \cdot |f(z)| = 0$ *for any* $\alpha \in \mathbb{R}$ *nor* $\lim\limits_{z\to a} |z-a|^{\alpha} \cdot |f(z)| = \infty$ *for any* $\alpha \in \mathbb{R}$

Proof.

① If $\lim\limits_{z\to a} |z-a|^{\alpha} \cdot |f(z)| = 0$ for $\forall \alpha \in \mathbb{R}$, then $\lim\limits_{z\to a} |z-a|^{m} \cdot |f(z)| = 0$ for $\forall$ integer $m > \alpha$.

$\Rightarrow (z-a)^m f(z)$ has a removable singularity at a and vanishes at $z = a$

$\Rightarrow$ Either $f \equiv 0$ in $B(a,\delta)\backslash\{a\}$, which is case (i), or $(z-a)^m f(\alpha)$ has a zero of finite order k at $a \Rightarrow \lim\limits_{z\to a} |z-a|^{\alpha} \cdot |f(z)| = \begin{cases} 0, & \alpha > m-k \\ \infty, & \alpha < m-k \end{cases}$

② If $\lim\limits_{z\to a} |z-a|^{\alpha}|f(z)| = \infty$ for some $\alpha \in \mathbb{R}$, then $\lim\limits_{z\to a} |z-a|^{n} \cdot |f(z)| = \infty$ for $\forall$ integer $n < \alpha$.

$\Rightarrow (z-a)^n f(z)$ has a pole of finite order l at a

$\Rightarrow \lim\limits_{z\to a} |z-a|^{\alpha} \cdot |f(z)| = \begin{cases} 0, & \alpha > n+l \\ \infty, & \alpha < n+l \end{cases}$ □

Remark 4.34. In case (ii), N may be called the **algebraic order** of f at a. $N > 0$ if a is a pole, $N < 0$ if a is a zero, and $N = 0$ if f is analytic at a and $f(a) \neq 0$. The order is always an integer, there is no analytic function which tends to 0 or ∞, like a fractional power of $|z - a|$.

In some sense, three cases depends on whether $\lim\limits_{z\to a}(z-a)^N f(z)$ converges for some N.

In case (iii), the point a is an **essential isolated singularity**.

Example 4.35. $f(z) = \exp(\frac{1}{z})$ has an essential isolated singularity $z = 0$.

Theorem 4.36 (Weierstrass). *An analytic function comes arbitarily close to any complex value in every neighborhood of an essential singularity. Or equivalently, the codomain of f on every neighborhood of an essential singularity is dense in $\mathbb{C}$.*

Proof. Suppose the statement is false.

$\exists A \in \mathbb{C}$, $\delta > 0$ and $\varepsilon > 0$ *s.t.*

$$|f(z) - A| > \delta, \ \forall z \text{ with } 0 < |z - a| < \varepsilon \tag{4.29}$$

$\Rightarrow \lim\limits_{z\to a} |z-a|^{\alpha} \cdot |f(z) - A| = \infty$ for $\forall \alpha < 0$. $\Rightarrow a$ is not an essential singularity of $f(z) - A$.

The previous theorem $\Rightarrow \exists\, \beta \in \mathbb{R}$ *s.t.* $\lim\limits_{z\to a} |z-a|^{\beta} \cdot |f(z) - A| = 0$, and we may choose $\beta > 0$.

Then $\lim\limits_{z\to a} |z-a|^{\beta} \cdot |A| = 0 \Rightarrow \lim\limits_{z\to a} |z-a|^{\beta} \cdot |f(z)| = 0$ by the triangular inequality.

So a is not an essential singularity of f, which causes contradiction!

So the statement has to be true. □

Remark 4.37. If f is analytic in $|z| > R$. We treat ∞ as an isolated singularity. Removable singularity, pole or essential singularity of f at ∞ is defined according to $g(z) = f(\frac{1}{z})$ at $z = 0$.

4.3.3 The Local Mappings

Theorem 4.38 (The Argument Principle)**.** *Let f be analytic in a disk $\triangle$ s.t. f does not vanish identically. Let z_j be the zeros of f, each zero being counted as many times as* ***its order indicates****. For every closed curve γ in $\triangle$ which does not pass through a zero, we have*

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z \tag{4.30}$$

where the sum has only a finite number of terms with nonzero value.

Proof.

Case I: f has exactly n zeros $z_1, \cdots, z_n$.

By repeated application of Taylor' Theorem 4.26, we can write

$$f(z) = (z - z_1)(z - z_2)\cdots(z - z_n)g(z),\ z \in \triangle \tag{4.31}$$

where g is analytic in $\triangle$ and $g(z) \neq 0$ for $\forall z \in \triangle$. $\Rightarrow$

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)},\ \forall z \in \triangle \text{ and } z \neq z_j \tag{4.32}$$

Cauchy' Theorem 4.10 $\Rightarrow$

$$\int_\gamma \frac{g'(z)}{g(z)}\mathrm{d}z = 0 \Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_{j=1}^{n} n(\gamma, z_j) \tag{4.33}$$

Case II: f has infinitely many zeros in $\triangle$. Then γ is inside a concentric disk $\triangle'$ smaller than $\triangle$.

$f \not\equiv 0 \Rightarrow$ There is only a finite number of zeros in $\triangle'$.

So we can apply (4.33) to the disk $\triangle' \Rightarrow$ (4.30) holds since $n(\gamma, z_j) = 0$ if $z \notin \triangle'$. $\square$

Remark 4.39.

- The function $\omega = f(z)$ maps γ onto a closed curve Γ in the ω-plane, and we have

$$\int_{\Gamma} \frac{\mathrm{d}\omega}{\omega} = \int_{\gamma} \frac{f'(z)}{f(z)} \mathrm{d}z \tag{4.34}$$

  Then (4.30) can be interpreted as $n(\Gamma, 0) = \sum_j n(\gamma, z_j)$.

- The most useful application of the theorem is to the case when γ is a circle (or more generally a simple closed curve). So that $n(\gamma, z) = \begin{cases} 1, & z \text{ is inside } \gamma \\ 0, & z \text{ is outside } \gamma \end{cases}$ Then (4.30) yields a formula for the total number of zeros enclosed by γ.

Let $a \in \mathbb{C}$. Apply the previous theorem to $f(z) - a$

$$\sum_j n(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - a} \mathrm{d}z$$

where $z_j(a)$ are zeros of $f - a$ (or roots of $f(z) = a$), and γ is a closed curve in $\triangle$ which doesn't pass $z_j(a) \Rightarrow$

$$n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$$

If a and b are in the same region determined by Γ, then $n(\Gamma, a) = n(\Gamma, b) \Rightarrow$

$$\sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b)) \tag{4.35}$$

If γ is a circle, then f takes the values a and b equally many times inside γ, counted as many times as their orders indicate.

We have the equation that

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}\mathrm{d}z = n(\Gamma,a) = n(\Gamma,b)\\ = \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{d}\omega}{\omega-b} = \frac{1}{2\pi i}\frac{f'(z)\mathrm{d}z}{f(z)-b}\\ = \mathrm{card}\{z \text{ inside } \gamma : f(z)=b\}\end{aligned} \tag{4.36}$$

Theorem 4.40. *Suppose f is analytic at z_0, and $f(z)-\omega_0$ has a zero of order $N \in \mathbb{N}$ at z_0. Then for $\forall \varepsilon > 0$ sufficiently small, $\exists \delta > 0$ s.t. for $\forall a$ with $|a-\omega_0| < \delta$, the equation $f(z) = a$ has exactly N roots in the disk $|z-z_0| < \varepsilon$*

Proof. We choose $\varepsilon > 0$ *s.t.*

(1) f is analytic in $|z-z_0| \leqslant \varepsilon$

(2) z_0 is the only zero of $f(z)-\omega_0$ in this disk.

(3) $f'(z) \neq 0$ for $\forall z$ with $0 < |z-z_0| < \varepsilon$

Let γ be the circle $|z-z_0| < \varepsilon$ and $\Gamma = f \circ \gamma$.

$\omega_0 \notin \Gamma \Rightarrow \exists \delta > 0$ *s.t.* $B(\omega_0,\delta) \cap \Gamma = \varnothing$.

The consequence of the argument principle 4.38, *i.e.* (4.36) $\Rightarrow f$ takes all values $a \in B(\omega_0,\delta)$ the same number of times N inside γ, since $f(z) = \omega_0$ has exactly N coiciding roots inside γ.

(3) $\Rightarrow$ all roots $f(z) = a$ with $a \in B(\omega_0,\delta)\backslash\{\omega_0\}$ are simple ☐

Corollary 4.41 (open mapping theorem)**.** *A nonconstant analytic function maps open sets onto open sets.*

Proof. The previous theorem $\Rightarrow \forall \varepsilon > 0$, $f(B(z_0,\varepsilon)) \supset B(\omega_0,\delta)$ ☐

Corollary 4.42. *If f is analytic at z_0 with $f'(z_0) \neq 0$. It maps a neighborhood of z_0 conformally and topologically onto a region.*

Proof. This is the case $N = 0$. The previous theorem $\Rightarrow$ There is 1-1 corresponding between the disk $|\omega - \omega_0| < \delta$ and an open subset of $|z - z_0| < \varepsilon$. The open mapping theorem 4.41 $\Rightarrow f^{-1}$ is continuous $\Rightarrow f$ is a topological map. And f is conformal on $|z - z_0| < \varepsilon$ □

Remark 4.43. Under the assumption of Corollary 4.42, f^{-1} is continuous $\Rightarrow f^{-1}$ is analytic $\Rightarrow f^{-1}$ is conformal map.

If $f : \Omega \to \mathbb{C}$ is 1-1 and analytic, Theorem 4.40 can hold only with $N = 1 \Rightarrow f'(z) \neq 0$ for $\forall z \in \mathbb{C}$. So this condition is stronger than the conformal condition.

4.3.4 The Maximum Principle

Theorem 4.44 (The maximum principle)**.** *If f is analytic and nonconstant in a region Ω, then its modules $|f|$ has no maximum in Ω.*

Proof. $\forall z_0 \in \Omega$, the open mapping theorem 4.41 $\Rightarrow \exists$ an open disk $|\omega - f(z_0)| < \delta$ contained in $F(\Omega)$. In this disk, $\exists\, \omega$ *s.t.* $|\omega| > |f(z_0)| \Rightarrow |f(z_0)|$ is not the maximum of $|f|$. □

Theorem 4.45 (The maximum principle)**.** *If f is defined and continuous on a closed bounded set E and analytic in the interior of E, then the maximum of $|f|$ on E is assumed on the boundary of E.*

Remark 4.46. The maximum principle can also be proved by the mean value theorem 4.18 for analytic functions.

Theorem 4.47 (Schwarz Lemma)**.** *If f is analytic in the disk $|z| < 1$ and satisfies $f(0) = 0$, $|f(z)| \leqslant 1$, $\forall z \in B(0,1)$, then $|f(z)| \leqslant |z|$ and $|f'(0)| \leqslant 1$. Furthermore, if $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ where $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. We define $g(z) = \begin{cases} \dfrac{f(z)}{z}, & z \neq 0, z \in B(0,1) \\ f'(0), & z = 0 \end{cases}$.

Then g is analytic with $g'(0) = \dfrac{f'(0)}{2}$ using Taylor series (4.20).

The maximum principle implies that $|g(z)| \leqslant \dfrac{1}{r}$, $\forall z \in \overline{B(0,r)}$ where $0 < r < 1$.

Setting $r \to 1$, we get $|g(z)| \leqslant 1$, $\forall |z| < 1$.

If $|f(z)| = |z|$ for some $z \neq 0$, or $|f'(0)| = 1$, then $|g| = 1$ attains its maximum at some interior points. By maximum principle, g has to be a constant. □

Remark 4.48. For a general analytic function $f : B(0,R) \to B(0,M)$, $z_0 \mapsto w_0$.

Let $T(z) = \dfrac{\frac{z}{R} - \frac{z_0}{R}}{1 - \frac{\bar{z_0}}{R} \cdot \frac{z}{R}}$

$S(\omega) = \dfrac{\frac{\omega}{M} - \frac{\omega_0}{M}}{1 - \frac{\bar{\omega_0}}{M} \cdot \frac{\omega}{M}}$.

Then $S \circ f \circ T^{-1}$ satisfies $S \circ f \circ T^{-1}(0) = 0$ and $|S \circ f \circ T^{-1}(z)| \leqslant 1 \overset{Schwarz\,lemma}{\Longrightarrow}$ $|S \circ f \circ T^{-1}(\zeta)| \leqslant |\zeta|$.

$\Rightarrow |S \circ f(z)| \leqslant |T(z)| \Rightarrow$

$$\left| \frac{M(f(z) - \omega_0)}{M^2 - \bar{\omega_0} f(z)} \right| \leqslant \left| \frac{R(z - z_0)}{R^2 - \bar{z_0} z} \right|, \forall z \in B(0,R)$$

4.4 The General Form of Cauchy's Theorem

4.4.1 Chains and Cycles

Let $\Omega \subset \mathbb{C}$ be open. Let $\gamma_j : [\alpha_j, \beta_j] \to \Omega$ be piecewise continuously differentiable curves in Ω. The sum $\gamma_1 + \gamma_2 + \cdots + \gamma_N$, which need not be a curve is called a **chain**. The **integral** of a continuous f in Ω along this chain is defined by

$$\int_{\gamma_1 + \gamma_2 + \cdots + \gamma_N} f = \sum_{j=1}^{N} \int_{\gamma_j} f. \tag{4.37}$$

Two chains are **identical** if they yield the same line integrals for all function f.

A chain is a **cycle** if it can be represented as a finite sum of closed curves.

4.4.2 Simple connectivity and homology

A region is **simply connected** if its complement w.r.t. $\hat{\mathbb{C}}$ is connected.

Example 4.49. A disk, a half plane, a parallel strip are simply connected.

$\mathbb{C}\backslash\overline{B(0,1)}$ is not simply connected since its complement w.r.t. $\hat{\mathbb{C}}$ consists of $\overline{B(0,1)}$ and ∞.

Theorem 4.50. *A region $\Omega \subset \mathbb{C}$ is simply connected iff $n(\gamma, z) = 0$ for all cycles γ in Ω and all points $z \notin \Omega$.*

Proof. "$\Rightarrow$": $\forall$ cycle $\gamma \subset \Omega$, $\hat{\mathbb{C}}\backslash\Omega$ must be in one of the regions in $\hat{\mathbb{C}}\backslash\gamma$ since $\hat{\mathbb{C}}\backslash\Omega$ is connected.

$\infty \in \hat{\mathbb{C}}\backslash\Omega \Rightarrow \mathbb{C}\backslash\Omega$ is in the unbounded region of $\mathbb{C}\backslash\gamma$. By theorem 4.13 $n(\gamma, z) = 0, \forall z \in \mathbb{C}\backslash\Omega$.

"$\Leftarrow$": Suppose Ω is not simply connected, *i.e.* , $\hat{C}\backslash\Omega$ is not connected. Let $\hat{\mathbb{C}}\backslash\Omega = A \sqcup B$ with A, B disjoint closed sets.

Suppose that $\infty \in B$. Then A is the bounded set. δ is defined to be the distance between A and B. The $\delta > 0$. Cover A with a net of squares Ω of side less than $\frac{\delta}{\sqrt{2}}$.

Suppose $z_0 \in A$ lies at the center of a square cycle $\gamma := \sum_{Q: Q\cap A \neq 0} \partial\Omega$.

z_0 is only in one of these squares $\Rightarrow n(\gamma, z_0) = 1$.

Since sides of squares are less than $\frac{\delta}{\sqrt{2}}$, $\gamma \cap B \neq \varnothing$.

$\gamma \cap A = \varnothing$ after cancellations of the multiple sides.

$\Rightarrow \gamma \in \Omega$ with $n(\gamma, z_0) = 1$. That's a contradiction. $\square$

A cycle γ in an open set Ω is said to be **homologous to zero** w.r.t. Ω if $n(\gamma, z) = 0$ for $\forall z \in \mathbb{C}\backslash\Omega$.

In symbols, we write $\gamma \sim 0(\mathrm{mod}\,\Omega)$. So $\gamma_1 \sim \gamma_2$ means $\gamma_1 - \gamma_2 \sim 0(\mathrm{mod}\,\Omega)$.

4.4.3 The general form of Cauchy's theorem

Theorem 4.51 (General form of Cauchy's theorem)**.** *If f is analytic in an open set Ω, then $\int_\gamma f(z)\mathrm{d}z = 0$ for $\forall$ cycle γ which is homologous to zero in Ω.*

In combination with the theorem 4.50 in the previous section, we have

Corollary 4.52. *If f is analytic in a simply connected region Ω, then $\int_\gamma f(z)\mathrm{d}z = 0$ for all cycles γ in Ω.*

In combination with the fundamental theorem 4.5 of Calculus for integrals in $\mathbb{C}$, we have

Corollary 4.53. *If f is analytic in a simply connected region Ω, then $\exists$ an analytic function F in Ω s.t. $F'(z) = f(z)$ for $\forall z \in \Omega$.*

Corollary 4.54. *If f is analytic in a simply connected region Ω and $f(z) \neq 0$ for $\forall z \in \Omega$, then it is possible to define single-valued analytic branches of $\ln f(z)$ and $\sqrt[n]{f(z)}$ in Ω*

Proof. $\dfrac{f'(z)}{f(z)}$ is analytic in $\Omega \overset{Corollary 4.53}{\Longrightarrow} \exists$ an analytic function F s.t. $F'(z) = \dfrac{f'(z)}{f(z)}$, $\forall z \in \Omega$.

$\Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}z}\left[f(z)e^{-F(z)}\right] = 0, \forall z \in \Omega \Rightarrow f(z) = C \cdot e^{F(z)}$ for some $C \in \mathbb{C}\backslash\{0\}$.

Choose $z_0 \in \Omega$ and one of the infinite values of $\ln f(z_0)$.

$\Rightarrow \exp\left[F(z) - F(z_0) + \ln f(z_0)\right] = \dfrac{f(z)}{C} \cdot e^{-F(z_0)} = f(z), \forall z \in \Omega.$

We may define $\ln f(z) = F(z) - F(z_0) + \ln f(z_0)$, $\sqrt[n]{f(z)} = \exp\left[\dfrac{1}{n}\ln f(z)\right]$. □

Proof of Cauchy's Theorem 4.51. Let γ be a cycle in Ω satisfying $\gamma \sim 0 \mod \Omega$. The theorem 4.13 implies that

$$E := \{z \in \mathbb{C}\backslash\gamma : n(\gamma, z) = 0\} \text{ is open}$$

We define $g : \Omega \times \Omega \to \mathbb{C}$ by

$$g(z,\zeta) := \begin{cases} \dfrac{f(z)-f(\zeta)}{z-\zeta}, & z \neq \zeta \\ f'(z), & z = \zeta \end{cases} \tag{4.38}$$

Taylor's theorem implies g is continuous in $(z,\zeta) \in \Omega \times \Omega$. For $\forall \zeta_0 \in \Omega$, $g(z,\zeta_0)$ is analytic in Ω since $\lim\limits_{z \to \zeta_0} (z-\zeta_0) g(z,\zeta_0) = 0$.

Define $h(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\gamma g(z,\zeta)\mathrm{d}\zeta, & z \in \Omega \\ \dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{f(z)}{z-\zeta}\mathrm{d}\zeta, & z \in E \end{cases}$. $\gamma \sim 0 \Rightarrow n(\gamma, z) = 0, \forall z \in \mathbb{C}\backslash\Omega \Rightarrow$ $C\backslash\Omega \subset E \Rightarrow \Omega \cup E = \mathbb{C}$. So h is defined on $\mathbb{C}$.

These two expressions are equal on $\Omega \cap E$ since $n(\gamma, z) = 0$, $\forall z \in \Omega \cap E$.

Lemma 4.19 implies that h is analytic in E.

The last exercise in Homework 6 $\Rightarrow$ h is analytic on $\Omega \Rightarrow h$ is entire.

$n(\gamma, z) = 0$ if $|z|$ is sufficiently large $\Rightarrow z \in E$ if $|z|$ large enough.

f is bounded on $\gamma \Rightarrow h(z) \to 0$ as $|z| \to \infty \Rightarrow h$ is bounded and thus $h \equiv 0$. By Liouville's Theorem 4.22,

$\dfrac{1}{2\pi i}\displaystyle\int_\gamma g(z,\zeta)\mathrm{d}\zeta = 0$, $\forall z \in \Omega\backslash\gamma$. Then

$$n(\gamma, z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta,\ \forall z \in \Omega\backslash\gamma \tag{4.39}$$

Equation 4.39 is the **generalized version of Cauchy's integral formula**.

Let $z_0 \in \Omega\backslash\gamma$. Define $h_1(z) = (z-z_0)f(z)$. Then h_1 analytic and

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma \frac{h_1(z)}{z-z_0}\mathrm{d}z \overset{(4.39)}{=} 2\pi i \cdot n(\gamma, z_0) \cdot h_1(z_0) = 0 \tag{4.40}$$

□

4.5 The Calculus of Residues

4.5.1 The Residue Theorem

Suppose f is analytic in a region Ω except for the isolated singularity at a. Consider a circle C centered at a and contained in Ω. The **residue** of f at a is defined by

$$\mathrm{Res}_{z=a} f(z) := \frac{1}{2\pi i}\int_C f(z)\mathrm{d}z \tag{4.41}$$

It is independent of choice of circle followed from the general Cauchy's theorem 4.51.

Now suppose f is analytic in a region Ω except for finitely many singularities a_j. Let γ be cycle in $\Omega' = \Omega\backslash\{a_1, \cdots, a_n\}$ which is homologous to zero w.r.t. Ω. Then

$$\gamma \sim \sum_{j=1}^{N} n(\gamma, a_j)C_j \mod \Omega' \tag{4.42}$$

where C_j is any circle centered at a_j and contained in Ω'.

The general Cauchy's theorem 4.51 implies

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{N} n(\gamma, a_j)\int_{C_j} f(z)\mathrm{d}z \tag{4.43}$$

So $\frac{1}{2\pi i}\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{N} n(\gamma, a_j)\mathrm{Res}_{z=a_j} f(z)$.

We just proved the residue theorem under the assumption that there are only a finite number of singularities

Theorem 4.55 (The Residue Theorem)**.** *Let f be analytic except for countably many* *isolated singularities a_j in a region Ω. Then*

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\mathrm{d}z = \sum_{j=1}^{N} n(\gamma, a_j)\mathrm{Res}_{z=a_j} f(z) \tag{4.44}$$

for any circle γ which is homologous to zero in Ω and does not pass through any of a_j.

Proof. We already proved the case when number of singularities is finite. For the general case,

it is enough to prove that $n(\gamma, a_j) = 0$ except for a finite number of a_j.

Let $E := \{z \in \mathbb{C}\backslash\gamma : n(\gamma, z) = 0\}$.

Then E is open by theorem 4.13 and contains all points outside of a large circle. $\Rightarrow E^c$ is compact. So E^c contains a finite number of the isolated points $a_j \Rightarrow n(\gamma, a_j) \neq 0$ only for a finite number of a_j. □

Remark 4.56.

(1) In the applications it is often the case that each $n(\gamma, a_j) \in \{0, 1\}$.

(2) When f has essential singularity, there is usually no simple method to compute residues.

(3) If f has a pole of order N at a, we proved in §3.2 that

$$(z-a)^N f(z) = b_N + b_{N-1}(z-a) + \cdots + b_1(z-a)^{N-1} + \varphi(z)(z-a)^N,\ z \neq a \tag{4.45}$$

where $\varphi(z)$ is analytic at a and $b_N \neq 0$. So we have

$$\mathrm{Res}_{z=a} f(z) = b_1 = \frac{1}{(N-1)!} \cdot \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\left[(z-a)^N f(z)\right] \tag{4.46}$$

This is because when the term $b_1(z-a)^{-1}$ is omitted, the remainder of the RHS of (4.46) is a derivative.

In particular, if $f(z) = \frac{g(z)}{h(z)}$, h has a simple zero at a and $g(a) \neq 0$, then

$$\mathrm{Res}_{z=a} f(z) = \lim_{z \to a} \left[\frac{g(z)}{h(z)} (z-a) \right] = \lim_{z \to a} \frac{g(z)}{\frac{h(z)-h(a)}{h-a}} = \frac{g(a)}{h'(a)} \tag{4.47}$$

Example 4.57. Compute $\int_{|z|=1} \frac{e^{iz}}{z^3} \mathrm{d}z$.

Solution. The only pole is at $z = 0$ with order 3. The residue theorem 4.55 implies:

$$\int_{|z|=1} \frac{e^{iz}}{z^3} \mathrm{d}z = 2\pi i \mathrm{Res}_{z=0} f(z) = 2\pi i \frac{1}{2!} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \left[z^3 \cdot \frac{e^{iz}}{z^3} \right] |_{z=0} = -\pi i \tag{4.48}$$

Or one can use Taylor's series (4.20)

$$\int_{|z|=1} \frac{e^{iz}}{z^3} \mathrm{d}z = \int_{|z|=1} \frac{1 + iz + + \frac{(iz)^2}{2} + \cdots}{z^3} \mathrm{d}z = -\pi i \tag{4.49}$$

□

4.5.2 The Argument Principle

Theorem 4.58 (The Argument Principle)**.** *If f is meromorphic in a region Ω with zeros a_j and poles b_k. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k) \tag{4.50}$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of zeros and poles. The sums in (4.50) *are finite, and multiple zeros and poles have to be repeated as many times as their order indicates.*

Proof. We assume that f has a finite number of zeros and poles, and denote that number by K.

Let N_j be the order of the zero or pole of f at $z_j \in \{a_1, a_2, \cdots, b_1, b_2, \cdots\}$.

Define $\tilde{N}_j := \begin{cases} N_j, & z_j \text{ is a zero} \\ -N_j, & z_j \text{ is a pole} \end{cases}$

Let $g(z) = f(z) \cdot \prod_{j=1}^{K} (z - z_j)^{-\tilde{N}_j}$. Then g only has removable singularities in Ω, and we can view it as analytic in Ω. Moreover, $g(z) \neq 0$ for $\forall z \in \Omega$.

$f(z) = g(z) \cdot \prod_{j=1}^{K} (z - z_j)^{\tilde{N}_j}$ implies that

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{j=1}^{N} \frac{\tilde{N}_j}{(z - z_j)}, \forall z \neq z_j \tag{4.51}$$

Then

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z &= \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} \mathrm{d}z + \sum_{j=1}^{K} \frac{1}{2\pi i} \int_\gamma \frac{\tilde{N}_j}{z - z_j} \mathrm{d}z \\ &= \sum_{j=1}^{K} \tilde{N}_j \cdot n(\gamma, z_j) \\ &= \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k) \end{aligned}$$

If f has infinite number of zeros or poles, the proof is the same as that of the residue theorem. *i.e.* $n(\gamma, z) \neq= 0$ for finite many z zeros or poles. □

Theorem 4.59 (Rouchē's Theorem). *Let γ be a cycle which is homologous to zero in a region Ω s.t. $n(\gamma, z) \in \{0, 1\}, \forall z \in \Omega \backslash \gamma$.*

Suppose f, g are analytic in Ω, $|f(z) - g(z)| < |f(z)|, \forall z \in \gamma$. Then f and g have the same number of zeros enclosed by γ.

Proof. First we have $f(z) \neq 0, g(z) \neq 0$ for $z \in \gamma$.

Let $\psi(z) = \frac{g(z)}{f(z)}, z \in \gamma$. Then $|\psi(z) - 1| < 1$, $\forall z \in \gamma$. For $\Gamma = \psi(\gamma)$

$$\int_\gamma \frac{\psi'(z)}{\psi(z)} \mathrm{d}z = \int_\Gamma \frac{\mathrm{d}\omega}{\omega} = 2\pi i \cdot n(\Gamma, 0) = 0$$

since 0 is in the unbounded connected component of $\mathbb{C}\backslash\Gamma$.

The argument principle implies that $0 = \frac{1}{2\pi i}\int_\gamma \frac{\psi'(z)}{\psi(z)}\mathrm{d}z$ is equal to the difference of number of zeros of g and f. □

The argument principle can be generalized to

Theorem 4.60 (The Argument Principle). *Under the hypothesis of the argument principle 4.58, and if h is analytic in Ω, then we have*

$$\frac{1}{2\pi i}\int_\gamma h(z)\frac{f'(z)}{f(z)}\mathrm{d}z = \sum_j n(\gamma, a_j)h(a_j) - \sum_k n(\gamma, b_k)h(b_k) \tag{4.52}$$

Remark 4.61. In §5.3.3, we proved Theorem 4.40 that if f is analytic at z_0, and $f(z) - \omega_0$ has zero of order N at z_0, then for ε small enough, there exists $\delta > 0$ *s.t.* $\forall\omega$ with $|\omega - \omega_0| < \delta$, $f(z) = \omega$ has exactly N roots $z_j(\omega)$ in the disk $|z - z_0| < \varepsilon$. If we apply (4.52) with $h(z) = z$, we get

$$\sum_{j=1}^N z_j(\omega) = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} z\frac{f'(z)}{f(z)-\omega}\mathrm{d}z,\ \forall\omega \in B(\omega_0, \delta) \tag{4.53}$$

For $N = 1$, the inverse function $f^{-1}(\omega)$ can thus be represented by

$$f^{-1}(\omega) = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} z\frac{f'(z)}{f(z)-\omega}\mathrm{d}z,\ \forall\omega \in B(\omega_0, \delta) \tag{4.54}$$

If we apply (4.52) with $h(z) = z^m$, we get

$$\sum_{j=1}^N z_j^m(\omega) = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} \frac{z^m f'(z)}{f(z)-\omega}\mathrm{d}z,\ \forall\omega \in B(\omega_0, \delta) \tag{4.55}$$

4.5.3 Evaluation of Definite integrals

① All integrals of the form $\int_0^{2\pi} R(\cos\theta, \sin\theta)\mathrm{d}\theta$, where the integrand is a rational function of $\cos\theta$ and $\sin\theta$. The substitution $z = e^{i\theta}$ transform it into the line integral

$$\int_{|z|=1} R(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i})\frac{\mathrm{d}z}{iz}$$

It remains to determine the residues which correspond to the poles of the integrand inside $\{z : |z| < 1\}$.

Example 4.62. Compute $\displaystyle\int_0^{\pi} \frac{\mathrm{d}\theta}{a+\cos\theta}, a > 1.$

$$\begin{aligned}\int_0^{\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} &= \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} \\ &\overset{z=e^{i\theta}}{=} \frac{1}{a+\frac{z+z^{-1}}{2}} \cdot \frac{\mathrm{d}z}{iz} = \frac{1}{i}\int_{|z|=1} \frac{1}{z^2+2az+z}\mathrm{d}z \\ &= \frac{1}{i}\int_{|z|=1} \frac{1}{\left[z-(-a+\sqrt{a^2-1})\right]\cdot\left[z-(-a-\sqrt{a^2-1})\right]}\mathrm{d}z\end{aligned}$$

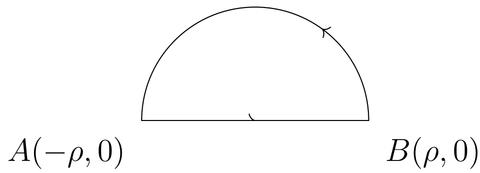
Note that $|-a+\sqrt{a^2-1}| = \frac{1}{|a+\sqrt{a^2-1}|} < 1$ and $|-a-\sqrt{a^2-1}| > 1$.

Residue Theorem 4.55 implies that

$$\begin{aligned}\int_0^{\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} &= \frac{1}{i}\cdot 2\pi i \mathrm{Res}_{z=-a+\sqrt{a^2-1}} f(z) \\ &= 2\pi \cdot \frac{1}{-a+\sqrt{a^2-1}-(-a-\sqrt{a^2-1})} \\ &= \frac{\pi}{\sqrt{a^2-1}}\end{aligned}$$

② An integral of the form $\int_{-\infty}^{\infty} R(x)\mathrm{d}x$ converges if and only if in the rational function R, the degree of denominator $\geqslant$ the degree of numerator+2 and has no pole lies in $\mathbb{R}$.

Consider this semicircle γ. If ρ is large enough, γ encloses all poles of R in the



upper half-plane. It is easy to see that

$$\lim_{\rho \to \infty} \int_{z=\rho e^{it}, 0 \leqslant t \leqslant \pi} R(z) \mathrm{d}z = 0$$

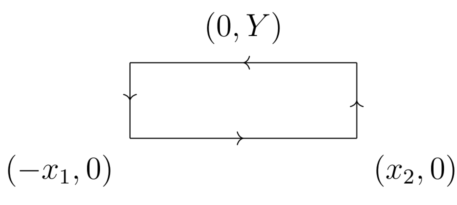
So we have $\displaystyle\int_{-\infty}^{+\infty} R(x)\mathrm{d}x = 2\pi i \sum_{y>0} \mathrm{Res}_{x+iy} R(z)$.

③

(a) The same method can be applied to $\displaystyle\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x$, where the rational function has a zero of at least two at ∞. Then $|e^{iz}| = e^{-y} \geqslant 1$ in the upper-half plane. So

$$\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x = 2\pi i \sum_{y>0} \mathrm{Res}_{x+iy} R(z)e^{iz}$$

(b) We now consider the case that R has only a simple zero at ∞ and no pole on $\mathbb{R}$.



There exists $M > 0$ and $C > 0$ *s.t.* this rectangle all poles of R in the upper half-plane if $x_1 > M, x_2 > M$, and $Y > M$. $|zR(z)| \leqslant C$ if $|z| \geqslant M$.

$$\left| \int_{\text{right vertical line}} R(z)e^{iz}\mathrm{d}z \right| \leqslant \int_0^Y \frac{C}{|z|} e^{-y}\mathrm{d}y \leqslant \frac{C}{x_2} \int_0^Y e^{-y}\mathrm{d}y \leqslant \frac{C}{x_2}$$

Similarly,

$$\left|\int_{\text{left vertical line}} R(z)e^{iz}\mathrm{d}z\right| \leqslant \frac{C}{x_1}$$

$$\left|\int_{\text{upper horizontal line}} R(z)e^{iz}\mathrm{d}z\right| \leqslant \int_{-x_1}^{x_2} \frac{C}{|z|}e^{-Y}\mathrm{d}x \leqslant \frac{Ce^{-Y}}{Y}\int_{-x_1}^{x_2}\mathrm{d}x = \frac{Ce^{-Y}(x_1+x_2)}{Y}$$

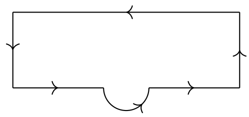
Fix x_1 and x_2, setting $Y \to \infty$. Then

$$\left|\int_{-x_1}^{x_2} R(x)e^{ix}\mathrm{d}x - 2\pi i\sum_{y>0}\mathrm{Res}_{x+iy}R(z)e^{iz}\right| \leqslant C\left(\frac{1}{x_1}+\frac{1}{x_2}\right)$$

So

$$\int_{-x_1}^{x_2} R(x)e^{ix}\mathrm{d}x = 2\pi i\sum_{y>0}\mathrm{Res}_{x+iy}R(z)e^{iz}$$

(c) R has only a single zero at ∞ and a simple pole at 0. Suppose that $R(z)e^{iz} = \frac{B}{z} + \varphi(z)$ where φ is analytic at 0.



Then it is easy to use this curve to prove that

$$\lim_{\delta\to 0^+}\left[\int_{-\infty}^{-\delta} R(x)e^{ix}\mathrm{d}x + \int_{\delta}^{\infty} R(x)e^{ix}\mathrm{d}x\right] = 2\pi i\left[\sum_{y>0}\mathrm{Res}_{x+iy}R(z)e^{iz} + \frac{B}{2}\right]$$

Denote this integral as $\mathrm{P.V.}\left[\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x\right]$, called **Cauchy principle value of the integral**.

Example 4.63.

$$\begin{aligned}\mathrm{P.V.}\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\mathrm{d}x\right) &= 2\pi i\cdot\frac{1}{2} = \pi i\\ &= \mathrm{P.V.}\left(\int_{-\infty}^{\infty}\frac{\cos x}{x}\mathrm{d}x + i\int_{-\infty}^{\infty}\frac{\sin x}{x}\mathrm{d}x\right)\end{aligned}$$

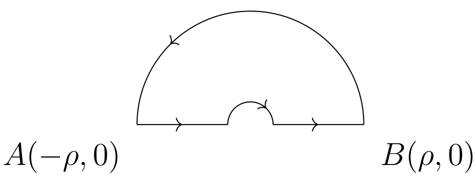
$$\begin{aligned}&= \text{P.V.}\left(\int_{-\infty}^{\infty}\frac{\cos x}{x}\mathrm{d}x\right) + i\int_{-\infty}^{\infty}\frac{\sin x}{x}\mathrm{d}x\\&= i\cdot 2\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x\end{aligned}$$

So we obtain $\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$

④ Calculate $\int_0^{\infty} x^{\alpha}R(x)\mathrm{d}x$, where $\alpha\in(0,1)$, $R(z)$ has a zero of order larger than 2 at ∞, and at most a simple pole at 0. Then

$$\int_0^{\infty} x^{\alpha}R(x)\mathrm{d}x \overset{x=t^2}{=} 2\int_0^{\infty} t^{2\alpha+1}R(t^2)\mathrm{d}t \tag{4.56}$$

$f(x) = z^{2\alpha}$ is analytic in $\mathbb{C}\backslash\{iy : y\leqslant 0\}$ if we require $\arg f(x)\in(-\pi\alpha, 3\pi\alpha)$.



Applying residue theorem 4.55 to $z^{2\alpha+1}R(z^2)$ and take limits we have

$$\int_{-\infty}^{\infty} z^{2\alpha+1}R(z^2)\mathrm{d}z = 2\pi i\sum_{y>0}\mathrm{Res}_{x+iy}z^{2\alpha+1}R(z^2)$$

And

$$\begin{aligned}\int_{-\infty}^{\infty} z^{2\alpha+1}R(z^2)\mathrm{d}z &= \int_0^{\infty} z^{2\alpha+1}R(z^2)\mathrm{d}z + \int_0^{\infty}(-z)^{2\alpha+1}R(z^2)\mathrm{d}z\\&= (1-e^{2\alpha\pi i})(1-e^{2\alpha\pi i})\int_0^{\infty} z^{2\alpha+1}R(z^2)\mathrm{d}z\end{aligned}$$

So $\int_0^{\infty} x^{\alpha}R(x) = \frac{2}{1-e^{2\alpha\pi i}}\cdot 2\pi i\cdot\sum_{y>0}\mathrm{Res}_{x+iy}z^{2\alpha+1}R(z^2)$.

Example 4.64. Compute $\int_0^{\infty}\frac{x^{\frac{1}{2}}}{1+x^2}\mathrm{d}x$.

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\mathrm{d}x = 2\int_0^\infty \frac{t^2}{1+t^4} = \int_{-\infty}^\infty \frac{t^2}{1+t^4}\mathrm{d}t$$

Take $f(z) = \dfrac{z^2}{1+z^4}$ and apply Residue Theorem 4.55 to f, we have

$$\int_{-\infty}^\infty \frac{t^2}{1+t^4} = \int_{-\infty}^\infty f(z)\mathrm{d}z = 2\pi i\sum_{y>0}\mathrm{Res}_{x+yi}f(z) = 2\pi i[\mathrm{Res}_{\exp(\frac{i\pi}{4})}f + \mathrm{Res}_{\exp(\frac{3i\pi}{4})}f] = \frac{\sqrt{2}\pi}{2}$$

4.6 Harmonic Functions

4.6.1 Definition and basis properties

A real-valued function $u(z) = u(x,y)$ in a region Ω is **harmonic** if it is in C^2 and satisfying the Laplace's equation

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.57}$$

We already know that if $f(z) = u(x,y) + iv(x,y)$ is analytic in Ω, then u and v satisfy the Cauchy-Riemann equations, and are therefore harmonic in Ω.

If u is harmonic in Ω, then $f(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ is analytic in Ω. This is because, for $U := \dfrac{\partial u}{\partial x}, V = -\dfrac{\partial u}{\partial y}$.

$$\begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial V}{\partial y} \\ \\ \dfrac{\partial U}{\partial y} = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial V}{\partial x} \end{cases} \tag{4.58}$$

We may write the differential

$$f\mathrm{d}z = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(\mathrm{d}x + i\mathrm{d}z) = \left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + i\left(\frac{\partial u}{\partial x}\mathrm{d}y - \frac{\partial u}{\partial y}\mathrm{d}x\right) \tag{4.59}$$

In this expression, the real part is $\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$. And if u has a conjugate harmonic function v, then the imaginary part is

$$\mathrm{d}v = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y = -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y$$

In general, however, there is no (single-valued) conjugate function. We thus define

$$^*\mathrm{d}u := -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y \tag{4.60}$$

and call $^*\mathrm{d}u$ the **conjugate differential of** $\mathrm{d}u$. We may write (4.59) as

$$f\mathrm{d}z = \mathrm{d}u + i^*\mathrm{d}u \tag{4.61}$$

Lemma 4.65. *Let γ be a cycle in a region Ω s.t. $\gamma \sim 0 \mod \Omega$. Then*

$$\int_\gamma {}^*\mathrm{d}u = 0 \tag{4.62}$$

Proof. (4.61) implies $\displaystyle\int_\gamma f(z)\mathrm{d}z = \int_\gamma \mathrm{d}u + i\int_\gamma {}^*\mathrm{d}u$.

Cauchy's Theorem 4.51 implies $\int_\gamma f(z)\mathrm{d}z = 0$. And $\int_\gamma \mathrm{d}u = 0$ since $\mathrm{d}u$ is an exact diffenrential.

Hence, $\int_\gamma {}^*\mathrm{d}u = 0$. □

Theorem 4.66. *If Ω is simply connected and u is harmonic in Ω, then u has a (single-valued) conjugate function v which uniquely determined up to additive constant.*

Proof. The last lemma 4.65 and theorem 4.4 imply that there is a (single-valued) function v *s.t.* $^*\mathrm{d}u = \mathrm{d}v$ *i.e.*

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

So v is a conjugate function of u. (Notice that we use the property of simply connection that every cycle in Ω is homologous to zero)

If v_1 and v_2 are two such harmonic functions, then $f_1 = u + iv_1$, $f_2 = u + iv_2$ are both analytic in Ω. So $f_1 - f_2 = i(v_1 - v_2)$ is analytic in Ω. The open mapping theorem 4.41 implies $f_1 - f_2$ is a constant. □

Remark 4.67. We see that the open mapping theorem 4.41 has such power that it gives a way to prove an analytic function with some closed property is constant.

Remark 4.68. The condition on simply connectness can not be removed. For instance, $u(z) = \ln|z|$ is harmonic in $\mathbb{C}\backslash\{0\}$, but it cannot be written as the real part of an analytic function since $\ln|z| = \operatorname{Re}\ln z$

4.6.2 The Mean-value Property

Theorem 4.69 (Mean-value Property)**.** *Let u be harmonic in a region Ω. If $\overline{B(z,R)} \subset \Omega$, then*

$$u(z) = \int_0^{2\pi} u(z_0 + Re^{i\theta})\mathrm{d}\theta \tag{4.63}$$

Proof. The previous theorem 4.66 implies that u has a conjugate function v on $\overline{B(z_0,R)}$. Consider the analytic function $f = u + iv$. The Cauchy integral formula 4.39 shows

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{z-z_0}\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\mathrm{d}\theta \tag{4.64}$$

This theorem follows by taking the real part of the equation. □

Theorem 4.70. *If u is harmonic in Ω, and $\{z \in \mathbb{C} : 0 < R_1 \leqslant |z - z_0| \leqslant R_2\} \subset \Omega$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\mathrm{d}\theta = \alpha \ln r + \beta \, r \in [R_1, R_2] \tag{4.65}$$

where α and β are constants

Proof. In polar coordinate (r, θ),

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2} = r^{-1}\frac{\partial}{\partial r}(r\cdot\frac{\partial}{\partial r}) + r^{-2}\frac{\partial^2}{\partial\theta^2}$$

Let $U(r) = \int_0^{2\pi} u(z_0 + re^{i\theta})\mathrm{d}\theta$. $z \mapsto u(z_0 + z)$ is harmonic. Then

$$\triangle U(r) = \frac{1}{2\pi}\int_0^{2\pi} \triangle u(z_0 + re^{i\theta})\mathrm{d}\theta = 0 \tag{4.66}$$

Therefore, $\dfrac{\partial}{\partial r}\left(r\dfrac{\partial U}{\partial r}\right) = 0$. Therefore, $U(r) = \alpha \ln\gamma + \beta$. □

Theorem 4.71 (Maximal Principle of Harmonic Function)**.** *A nonconstant harmonic function has neither a maximum nor a minimum in its region of definition.*

Proof. Suppose u attains a maximum at $z_0 \in \Omega$. $\exists R > 0 \;\; s.t. \; B(z_0, R) \subset \Omega$. Suppose $\exists a \in B(z_0, R) \;\; s.t. \; u(a) < u(z_0) = M$.

The mean-value property implies

$$M = u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\mathrm{d}\theta < M$$

by continuity. This causes a contradiction.

So u is a constant in $B(z_0, R)$.

Then for every z_1 in the region, since we can find a series of disk such that the center of the disk is in the previous disk and z_0 is the center of the first disk, z_1 is in the last disk.

Then by the property above, $u(z_0) = u(z_1)$. So u is a constant, which causes a contradiction! □

Theorem 4.72. *u is harmonic in the interior of E and continuous on $\overline{E}$, which is bounded, then the maximum and minimum of u are taken on ∂E.*

Proof. It is followed from Theorem 4.71 □

It follows that the maximal norm of harmonic function u is taken on ∂E, which implies a corollary

Corollary 4.73. *If u_1 and u_2 are continuous on a closed bounded set E which are harmonic in the interior of E and $u_1 = u_2$ on the boundary of E, then $u_1 = u_2$ on E.*

Proof. Apply the maximum and minimum principle to $u_1 - u_2$ □

4.6.3 Poisson's Formula

Theorem 4.74 (Poisson's formula)**.** *Suppose that u is harmonic on $B(0,R)$ and continuous on $\overline{B(0,R)}$. Then*

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z) \mathrm{d}\theta \tag{4.67}$$

for $\forall a \in B(0,R)$.

Proof. The idea is to use Möbius transformation and apply mean-value property.

Let $\zeta = S^{-1}(z) = \dfrac{\frac{z}{R} - \frac{a}{R}}{1 - \frac{\bar{a}}{R} \cdot \frac{z}{R}} = \dfrac{R(z-a)}{R^2 - \bar{a} \cdot z}$. So $z = S(\zeta) = \dfrac{R(R\zeta + a)}{R + \bar{a}\zeta}$ is a Möbius transformation mapping the unit circle into $B(0,R)$ in which $0 \mapsto a$.

Suppose $u(S(\zeta))$ is harmonic on $|\zeta| \leqslant 1$(See Remark 4.76). The mean-value property implies

$$u(S(0)) = u(a) = \frac{-i}{2\pi} \int_{|\zeta|=1} u(\zeta) \frac{\mathrm{d}\zeta}{\zeta} \tag{4.68}$$

where

$$\begin{aligned}
\frac{\mathrm{d}\zeta}{\zeta} &= \frac{R^2-\bar{a}z}{R(z-a)}\cdot\frac{R(R^2-|a|^2)}{(R^2-\bar{a}z)^2}\mathrm{d}z \\
&= \left[\frac{1}{z-a}+\frac{\bar{a}}{R^2-\bar{a}z}\right]\mathrm{d}z \\
&\overset{z=Re^{i\theta}}{=} \left[\frac{iz}{z-a}+\frac{i\bar{a}z}{R^2-\bar{a}z}\right]\mathrm{d}\theta \\
&\overset{R^2=z\bar{z}}{=} \left[\frac{iz}{z-a}+\frac{i\bar{a}}{\bar{z}-\bar{a}}\right]\mathrm{d}\theta \\
&= i\frac{R^2-a^2}{|z-a|^2}\mathrm{d}\theta
\end{aligned} \tag{4.69}$$

Combined with (4.68) and (4.69), we obtain (4.67) in a stronger assumption.

$$u(a) = \frac{1}{2\pi}\int_{|z|=R}\frac{R^2-|a|^2}{|z-a|^2}u(z)\mathrm{d}\theta$$

$\square$

Remark 4.75. Note that

$$\begin{aligned}
\frac{R^2-|a|^2}{|z-a|^2} &= \frac{z}{z-a}+\frac{\bar{a}}{\bar{z}-\bar{a}} \\
&= \frac{1}{2}\left[\frac{z}{z-a}+\frac{\bar{a}}{\bar{z}-\bar{a}}+\frac{\bar{z}}{\bar{z}-\bar{a}}+\frac{a}{z-a}\right] \\
&= \frac{1}{2}\left(\frac{z+a}{z-a}+\frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}\right) \\
&= \mathrm{Re}\left(\frac{z+a}{z-a}\right)
\end{aligned} \tag{4.70}$$

So the Poisson's formula can also be written as

$$u(a) = \frac{1}{2\pi}\int_{|z|=R}\mathrm{Re}\left(\frac{z+a}{z-a}\right)u(z)\mathrm{d}\theta,\ \forall a\in B(0,R) \tag{4.71}$$

or

$$u(a) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_{|z|=R}\frac{z+a}{z-a}\cdot\frac{u(z)}{z}\mathrm{d}z\right],\ \forall a\in B(0,R) \tag{4.72}$$

By Lemma 4.19, u is the real part of the analytic function

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{\zeta+z}{\zeta-z}\cdot\frac{u(\zeta)}{\zeta}\mathrm{d}\zeta + iC$$

where $C \in \mathbb{R}$. (4.72) is called the **Schwarz Formula**.

Remark 4.76. For the general assumption in the theorem 4.74, note that if $r \in (0,1)$, then $u(rz)$ is harmonic in $\overline{B(0,R)}$. The above proof implies

$$u(ra) = \frac{1}{2\pi}\int_{|z|=R}\frac{R^2-|a|^2}{|z-a|^2}u(rz)\mathrm{d}\theta \tag{4.73}$$

Since u is continuous on a compact set $\overline{B(0,R)}$, it is uniformly continuous. Then $u(rz) \rightrightarrows u(z)$ uniformly for $|z| = R$ as $r \to 1$.

Then take $r \to 0$ in (4.73) and we obtain Poisson's formula holds under the assumption of the theorem.

4.6.4 Schwarz's Theorem

We can easily define harmonic function u in the interior if u is piecewise continuous on the boundary by (4.72). However, it is not always continuous at the boundary. The next theorem gives a condition that such an extended function u exists if u is continuous on the boundary.

Theorem 4.77 (Schwarz's theorem)**.** *Given a piecewise continuous function u on $[0,2\pi]$, the* ***Poisson integral***

$$P_u(z) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)u(\theta)\mathrm{d}\theta \tag{4.74}$$

is harmonic for $|z| < 1$. Moreover, $\lim\limits_{\substack{z\to e^{i\theta} \\ |z|<1}} P_u(z) = u(\theta_0)$ if u is continuous t θ_0.

Proof. Lemma 4.19 implies P_u is harmonic in $|z| < 1$.

Note that P is a **linear functional** which maps piecewise continuous function u on $[0, 2\pi]$ to harmonic function P_u on the unit disk. Explicitly,

$$\begin{cases} P_{u_1+u_2} = P_{u_1} + P_{u_2} \\ P_{\lambda u} = \lambda P_u \end{cases} \tag{4.75}$$

Applying Poisson's formula 4.74 to $u \equiv 1$, we get $P_1 = 1$, and thus $P_c = c, \forall c \in \mathbb{R}$.

If $u \geqslant 0$ on $[0, 2\pi]$, then $P_u \geqslant 0$. (4.75) follows that if $-\infty < m \leqslant u(\theta) \leqslant M < \infty$ for $\forall \theta \in [0, 2\pi]$, then $m \leqslant P_u \leqslant M$.

By replacing u with $u - u(\theta_0)$, WLOG, we may assume $u(\theta_0) = 0$.

If u is continuous at θ_0, then $\forall \varepsilon > 0$, one can choose $C_2 \subset \partial B(0, 1)$ *s.t.* $e^{i\theta_0} \in \operatorname{int}(C_2)$ and $|u(\theta)| < \frac{\varepsilon}{2}$ for $\forall e^{i\theta} \in C_2$. Let $C_1 = \partial B(0, 1) \backslash C_2$. Define

$$u_1(\theta) = \begin{cases} u(\theta), & e^{i\theta} \in C_1 \\ 0, & \text{otherwise} \end{cases} \qquad u_2(\theta) = \begin{cases} u(\theta), & e^{i\theta} \in C_2 \\ 0, & \text{otherwise} \end{cases} \tag{4.76}$$

Linearity of P implies $P_u = P_{u_1} + P_{u_2}$. $|u_2| < \dfrac{\varepsilon}{2} \Rightarrow |P_{u_1}(z)| < \frac{\varepsilon}{2}, \forall z \in B(0, 1)$. So $\lim\limits_{\substack{z \to e^{i\theta} \\ |z|<1}} P_{u_2}(z) = 0$

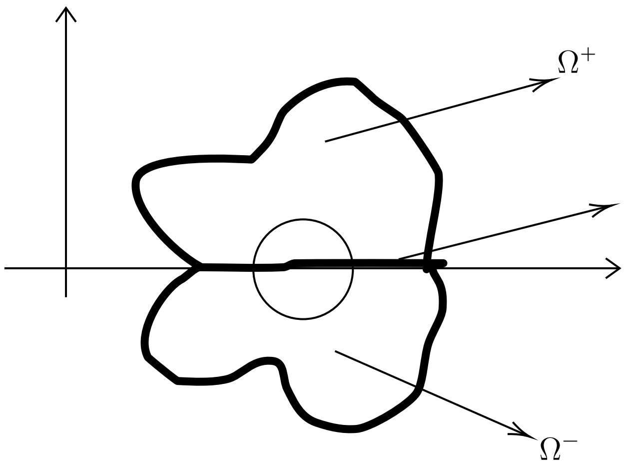
P_{u_1} can be viewed as a line integral over $C_1 \Rightarrow P_{u_1}$ is harmonic in $\mathbb{C} \backslash C_1$. So P_{u_1} is harmonic in $\mathbb{C} \backslash C_1$ by lemma 4.19.

$\operatorname{Re}\left(\dfrac{e^{i\theta} + z}{e^{i\theta} - z}\right) = \dfrac{1 - |z|^2}{|z - e^{i\theta}|^2} \Rightarrow P_{u_1}(z) = 0$ for $z \in C_2$. Continuity implies $\lim\limits_{\substack{z \to e^{i\theta} \\ |z|<1}} P_{u_1}(z) = 0$.

Therefore, $\lim\limits_{\substack{z \to e^{i\theta} \\ |z|<1}} P_u(z) = 0 = u(\theta_0)$ $\square$

4.6.5 The Reflection Principle

Theorem 4.78 (The reflection principle). *Let Ω be a region which is symmetric w.r.t. the x-axis, and $\Omega^+ := \Omega \cap \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, $\sigma = \Omega \cap \{z \in \mathbb{C} : \mathrm{Im} z = 0\}$. Suppose that v is continuous in $\Omega^+ \cup \sigma$, harmonic on Ω^+, and zero on σ. Them v has a harmonic extension to Ω, which satisfies $v(\bar{z}) = -v(z)$. In the same situation, if v is the imaginary part of an analytic function $f(z)$ in Ω^+, then $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\bar{z})}$.*



Proof. $h(z) = \begin{cases} v(z), & z \in \Omega^+ \\ 0, & z \in \sigma \\ v(\bar{z}), & z \in \Omega^- \end{cases}$.

We need to prove that h is harmonic in Ω. It suffices to prove h is harmonic on σ. Choose δ small *s.t.* $\overline{B(x,\delta)} \subset \Omega$. Let P_h be the Poisson integral w.r.t. $\partial B(x_0, \delta)$ with the boundary values h.

Schwarz' theorem 4.77 implies P_h is harmonic in $\overline{B(x,\delta)}$ and continuous on $\overline{B(x_0,\delta)}$. It follows that,

(1) $v - P_h$ is harmonic in the upper half disk $B(x_0, \delta) \cap \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$.

(2) $v - P_h = 0$ on $\partial B(x, \delta) \cap \{z \in \mathbb{C} : \mathrm{Im} z \geqslant 0\}$.

$\forall x \in B(x_0, \delta) \cap \sigma$, apply Poisson formula 4.67

$$P_h(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta^2 - |x|^2}{|\delta e^{i\theta} - x|^2} h(\delta e^{i\theta}) \mathrm{d}\theta = 0 \tag{4.77}$$

by symmetry.

Apply the maximum and minimum principle 4.73 to $h - P_h$, we get $h = P_h$ in $B(x, \delta) \cap \{z \in \mathbb{C} : \mathrm{Im} z \geqslant 0\}$.

The same argument works for the lower half disk.

So $h = P_h$ in $B(x_0, \delta) \Rightarrow h$ is harmonic at x_0.

For the second part of the theorem, it is enough to prove $\tilde{f}(z) = \begin{cases} f(z), & z \in \Omega^+ \\ \overline{f(\bar{z})}, & z \in \Omega^- \end{cases}$ is analytic on σ.

For $\forall x_0 \in \sigma$, let $B(x_0, \delta)$ be as before. We already proved v can be extended to a harmonic function in $B(x_0, \delta)$. v has a conjugate function $-u_0$ iin the same disk. We may normalize u_0 *s.t.* $u_0 = \mathrm{Re} f(z)$ in $B(x_0, \delta \cap \{z \in \mathbb{C}, \mathrm{Im} z > 0\})$. Define $g(z) := u_0(z) - u_0(\bar{z})$.

Then $g(x) = 0$ for $x \in B(x_0, \delta) \cap \sigma \Rightarrow$

$$\frac{\partial g}{\partial x}(z) = 0, \ \forall z \in B(x_0, \delta) \cap \sigma$$
$$\frac{\partial g}{\partial y}(x) = 2\frac{\partial u_0}{\partial y}(z) = -2\frac{\partial v}{\partial x}(z) = 0$$

So the analytic function $\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y} \equiv 0$ on $B(x, \delta) \cap \sigma$.(It is analytic because of (4.58)) Then $\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y} \equiv 0$ in $B(x_0, \delta) \Rightarrow g \equiv 0$.

So $u_0(z) = u_0(\bar{z})$, $\forall z \in B(x_0, \delta) \Rightarrow f(z) = u_0(z) + iv(z)$ is analytic in $B(x_0, \delta)$ and $f(z) = \overline{f(\bar{z})}$ for $\forall z \in B(x_0, \delta)$.

$\square$

Remark 4.79. The reflection principle can be applied to any circles with symmetric points by using Möbius transformation. However, the condition of $f(\mathbb{R}) \subset \mathbb{R}$ (*i.e.* $v(\mathbb{R}) = 0$) transforms to $f(C) \subset C$.

5 Series and Product Representations

5.1 Power Series Expansions

5.1.1 Weierstrass's Theorem

Theorem 5.1 (Weierstrass's Theorem). *Suppose f_n is analytic in the region Ω_n for each $n \in \mathbb{N}$, and $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. If f_n converges to f in Ω, uniformly on every compact subset of Ω, then f is analytic in Ω.*

Moreover, f_n' converges uniformly to f' on every compact subset of Ω.

Proof. $\forall$ compact subset $K \subset \Omega$, $K \subset \bigcup_{n=1}^{\infty} \Omega_n \Rightarrow \exists N \in \mathbb{N}$ such that $K \subset \bigcup_{n=1}^{N} \Omega_n$.

$\forall z_0 \in \Omega$, $\exists R > 0$ *s.t.* $\overline{B(z_0, R)} \subset \Omega$. Choose $N \in \mathbb{N}$ *s.t.* $\overline{B(z_0, R)} \subset \Omega_n$ for $\forall n \geqslant N$.

Cauchy's integral formula 4.51 implies

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,R)} \frac{f_n(\zeta)}{\zeta - z} \mathrm{d}\zeta, \ \forall z \in B(z_0, R) \tag{5.1}$$

$f_n \rightrightarrows f$ uniformly on $\overline{B(z_0, R)} \Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,R)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \tag{5.2}$$

Then f is analytic in $B(z_0, R)$ by lemma 4.19.

$$\begin{cases} f_n'(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial B(z_0,R)} \dfrac{f_n(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta & \forall z \in B(z_0, R) \\ f'(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial B(z_0,R)} \dfrac{f(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta & \forall z \in B(z_0, R) \end{cases} \tag{5.3}$$

Then $|f_n'(z) - f'(z)| \leqslant \dfrac{1}{2\pi}\displaystyle\int_{\partial B(z_0,R)} \dfrac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|^2}|\mathrm{d}\zeta|$. Therefore, f_n' uniformly converges to f' in $B(z_0, \rho)$ for $0 < \rho < R$.

Since any compact subset of Ω can be covered by a finite number of such closed disks, $f_n \rightrightarrows f$ uniformly on every compact subset of Ω. $\square$

Corollary 5.2. *If f_n is analytic in a region Ω for $n \in \mathbb{N}$, and $\sum\limits_{j=1}^{n} f_j \rightrightarrows f$ on every compact subset of Ω, then f is analytic in Ω and $f'(z) = \sum\limits_{j=1}^{\infty} f_j'(z)$, $\forall z \in \Omega$ uniformly on every compact subset of Ω.*

Theorem 5.3 (Hurwitz's Theorem)**.** *If the functions f_n are analytic and nowhere zero in a region Ω, and if $f_n \rightrightarrows f$ on every compact subset of Ω, then f is either identically zero or never equal to 0 in Ω.*

Proof. Suppose $f \not\equiv 0$. The zeros of f are isolated.

$\forall z_0 \in \Omega, \exists \delta > 0 \quad s.t.\ f(z) \neq 0, \forall z \in B(z_0, \delta)\backslash\{z_0\} \subset \Omega.$

Then $|f|$ has a positive minimum on $\partial B(z_0, \delta)$. Thus, $\dfrac{1}{f_n} \rightrightarrows \dfrac{1}{f}$ on $\partial B(z_0, \delta)$.

Combined with $f_n' \rightrightarrows f'$ on $\partial B(z_0, \delta) \Rightarrow$

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B(z_0,\delta)} \frac{f_n'(z)}{f_n(z)}\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial B(z_0,\delta)} \frac{f'(z)}{f(z)}\mathrm{d}z \tag{5.4}$$

By argument principle 4.38, this equation equals to 0. So f has no zeros on $\partial B(z_0, \delta)$, so is on Ω. $\square$

5.1.2 The Taylor Series

Theorem 5.4. *If f is analytic in the region Ω, and $z_0 \in \Omega$, then the expression*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{5.5}$$

is valid in the largest open disk of z_0 contained in Ω.

Proof. Taylor's theorem 4.26 implies

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}, \tag{5.6}$$

for $\forall z \in B(z_0, R) \subset \overline{B(z_0, R)} \subset \Omega$, where

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_{\partial B(z_0,R)} \frac{f(\zeta)}{(\zeta - z)^{n+1}(\zeta - z)}\mathrm{d}\zeta \tag{5.7}$$

Let $M := \max_{z \in \partial B(z_0,R)} |f(z)|$. Then $|f_{n+1}(z)(z - z_0)^{n+1}| < \dfrac{M}{R^n(R - |z - z_0|)} \cdot |z - z_0|^{n+1} \rightrightarrows 0$ in every disk $|z - z_0| \leqslant \rho < R$, from which we derive this theorem. ☐

Some known Taylor series:

$$\begin{aligned}
e^z &= 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots, \quad z \in \mathbb{C} \\
\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + \frac{(-1)^n z^{2n}}{(2n)!} + \cdots, \quad z \in \mathbb{C} \\
\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots, \quad z \in \mathbb{C} \\
\ln(1 + z) &= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^{n+1}\frac{z^n}{n} + \cdots, \quad \forall |z| < 1 \\
\forall \mu \in \mathbb{R}\backslash\mathbb{Z}_{\geqslant 0}, (1 + z)^\mu &= 1 + \mu z + \binom{\mu}{2} z^2 + \cdots + \binom{\mu}{n} z^n + \cdots, \quad \forall |z| < 1
\end{aligned} \tag{5.8}$$

where $\dbinom{\mu}{n} = \dfrac{\mu(\mu - 1)\cdots(\mu - n + 1)}{n!}$, and pick the branch with $\ln 1 = 0$.

5.1.3 Laurent Series

Lemma 5.5. *Let $A := \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$ be an annulus. For each analytic function $f : A \to \mathbb{C}$, there are analytic functions $f_1 : \{z \in \mathbb{C} : |z-a| < R_2\} \to \mathbb{C}$, $f_2 : \{z \in \mathbb{C} : |z-a| > R_1\} \to \mathbb{C}$ s.t. $f(z) = f_1(z) + f_2(z)$, $\forall z \in A$*

Proof. For $\forall z \in A$, $f_1(z) = \dfrac{1}{2\pi i}\displaystyle\int_{|\zeta-a|=r_1} \dfrac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$, $r_1 \in (|z-a|, R_2)$.

Cauchy's theorem 4.51 implies the integral is independent of the choice of r_1.

If we fix such r, then $f_1(z)$ is analytic for $\forall |z-a| < r_1$. $\overset{r_1 \to R_2}{\Rightarrow}$ $f_1(z)$ is well-defined and analytic on $B(a, R_2)$.

Let

$$f_2(z) = -\frac{1}{2\pi i}\int_{|\zeta-a|=r_2} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta,\ r_2 \in (R_1, |z-a|) \tag{5.9}$$

Then f_2 is well-defined and analytic in $\{z \in \mathbb{C} : |z-a| > R_1\}$.

Denote $\gamma_1 = \{z : |z-a| = r_1\}$, $\gamma_2 = \{z : |z-a| = r_2\}$, $R_1 < r_2 < |z-a| < r_1 < R_1$.

Cauchy's integral formula 4.39 implies

$$f(z) = n(\gamma_1 - \gamma_2, z)f(z) = \frac{1}{2\pi i}\int_{\gamma_1-\gamma_2} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = f_1(z) + f_2(z),\ \forall z \in A \tag{5.10}$$

□

Theorem 5.6 (Laurent Theorem)**.** *Any analytic function f on $A = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$ has a power series of the form*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{5.11}$$

This series, called ***Laurent series****, converges uniformly on each compact subset of A. Moreover,*

$$c_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\mathrm{d}\zeta,\ \forall n \in \mathbb{Z},\ \forall r \in (R_1, R_2) \tag{5.12}$$

Proof. The previous lemma implies $f(z) = f_1(z) + f_2(z)$, $\forall z \in A$, where f_1 is analytic in $|z-a| < R_2$ and f_2 is analytic in $|z-a| > R_1$. Then Taylor series for f_1 is

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \tag{5.13}$$

which converges uniformly on each compact subset of $|z-a| < R_2$.

Let $g(z) = f_2(a + \frac{1}{z})$, $|z| < \frac{1}{R_1}$. (5.9) tells us $\lim_{z\to\infty} f_2(z) = 0$. Then $\lim_{z\to 0} g(z) = 0 \Rightarrow$ g can be viewed as an analytic function in $B(0, \frac{1}{R_1})$.

The Taylor's series for g is $g(z) = \sum_{n=1}^{\infty} b_n z^n$, which converges uniformly on each compact subset of $B(0, \frac{1}{R_1})$. Now let $\zeta = a + \frac{1}{z}$. Then

$$f_2(\zeta) = g(z) = g(\frac{1}{\zeta - a}) = \sum_{n=1}^{\infty} b_n (\zeta - a)^n \tag{5.14}$$

which converges uniformly on each compact subset $|z-a| > R_1$.

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{5.15}$$

which converges uniformly on each compact subset of A. Then

$$\frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z = \frac{1}{2\pi i}\sum_{k=-\infty}^{\infty} c_k \sum_{|z-a|=r} (z-a)^{k-(m+1)} \mathrm{d}z,\ \forall z \in (R_1, R_2) \tag{5.16}$$

where $\int_{|z-a|=r} (z-a)^{k-(n+1)} \mathrm{d}z \neq 0$ iff $k = n$. So

$$c_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta-a)^{n+1}} \mathrm{d}\zeta,\ \forall n \in \mathbb{Z},\ \forall r \in (R_1, R_2) \tag{5.17}$$

$\square$

Theorem 5.7. *Let f be analytic in $\Omega\backslash\{a\}$, where Ω is a region and a is an isolated singularity. Its Laurent series is given by $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$, $\forall z \in B(a,R)\backslash\{a\} \subset \Omega\backslash\{a\}$. Then*

(a) f has a removable singularity at a iff $c_n = 0$ for $n < 0$

(b) f has a pole of order N at a iff $c_n = 0$ for $n < -N$ and $c_{-N} \neq 0$.

(c) f has an essential singularity at a iff $c_n \neq 0$ for infinitely many negative n.

Proof. (a) and (b) can be derived from the explicit expression of f_1, f_2.

For (c), "$\Rightarrow$" follows from (b) and (a).

"$\Leftarrow$" follows from the fact that isolated singularities belong to one of three categories: removable singularities, poles, and essential singularities, *i.e.* theorem 4.33. □

5.2 Partial Fractions and Factorization

5.2.1 Partial fractions

Theorem 5.8 (Mittag-Leffler Theorem). *Let $\{\zeta_k : k \in \mathbb{N}\}$ be a sequence in $\mathbb{C}$, $\lim_{k\to\infty} \zeta_k = \infty$, and let P_k be polynomials without constant term. Then there are functions which are meromorphic in $\mathbb{C}$ with poles at just the points ζ_k and the corresponding singular part $P_k\left(\frac{1}{z-\zeta_k}\right)$. Moreover, the most general meromorphic function of this kind can be written as*

$$f(z) = \sum_k \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right] + g(z) \tag{5.18}$$

where p_k are polynomials and g is entire.

Proof. WLOG, we assume $\zeta_k \neq 0$ for each k. Consider the Taylor expansion for

$P_k(\dfrac{1}{z-\zeta_k})$ around $z=0$:

$$\Psi(z) = P_k(\frac{1}{z-\zeta_k}) = \Psi(0) + \Psi'(0)z + \frac{\Psi''(0)}{2!}z^2 + \cdots + \frac{\Psi^{(N_k)(0)}}{N_k!}z^{N_k} + \Psi_{N_k+1}z^{N_k+1} \tag{5.19}$$

where N_k is to be specified later, and

$$\Psi_{N_k+1}(z) = \frac{1}{2\pi i}\int_C \frac{\Psi(\zeta)}{\zeta^{N_k+1}(\zeta - z)}\mathrm{d}\zeta \tag{5.20}$$

where C is the circle centered at 0 with radius $\dfrac{|\zeta_k|}{2}$. Let $M_k : \max\limits_{z\in C}|\Psi(z)|$. Then

$$|\Psi_{N_k+1}(z)| \leqslant \frac{1}{2\pi}\frac{M_k}{\left(\frac{|\zeta_k|}{2}\right)^{N_k+1}\cdot\frac{|\zeta_k|}{4}}\cdot 2\pi\cdot\frac{|\zeta_k|}{4} = 2M_k\left(\frac{2}{|\zeta_k|}\right)^{N_k+1}, \forall z \text{ with } |z| \leqslant \frac{|\zeta_k|}{4} \tag{5.21}$$

Let p_k be the partial sum of Ψ up to z^{N_k}. *i.e.* $p_k = \Psi(0) + \Psi'(0)z + \dfrac{\Psi''(0)}{2!}z^2 + \cdots + \dfrac{\Psi^{(N_k)(0)}}{N_k!}z^{N_k}$. Then

$$|\Psi(z) - p_k(z)| \leqslant 2M_k\left(\frac{2|z|}{|\zeta_k|}\right)^{N_k+1}, \forall z \text{ with } |z| \leqslant \frac{|\zeta_k|}{4} \tag{5.22}$$

Pick N_k large enough *s.t.* $M_k \cdot 2^k \leqslant 2^{N_k}$. Then

$$|\Psi(z) - p_k(z)| \leqslant 2^{-k} \Rightarrow |P_k(\frac{1}{z-\zeta_k}) - p_k(z)| \leqslant 2^{-k}, \forall z \text{ with } |z| \leqslant \frac{|\zeta_k|}{4} \tag{5.23}$$

Note that

$$\sum_k\left[P_k(\frac{1}{z-\zeta_k}) - p_k(z)\right] = \sum_{|\frac{\zeta_k}{4}|\leqslant R}\left[P_k(\frac{1}{z-\zeta_k}) - p_k(z)\right] + \sum_{|\frac{\zeta_k}{4}|>R}\left[P_k(\frac{1}{z-\zeta_k}) - p_k(z)\right]$$

where the first part is a finite sum and has $P_k(\frac{1}{z-\zeta_k})$ as the singular part at the pole ζ_k, and the second part is analytic in $z \in \overline{B(0,R)}$ by Weierstrass's theorem 4.36 and (5.23)

Therefore, $h(z) = \sum_k \left[P_k(\frac{1}{z-\zeta_k}) - p_k(z) \right]$ is the desired meromorphic function.

For the second part, if f is meromorphic in $\mathbb{C}$ with the some poles ζ_k and singular parts as h, then $g = f - h$ is analytic in $\mathbb{C}$. $\square$

Remark 5.9. We have given p_k as the partial sum of $P_k(\frac{1}{z-\zeta_k})$ up to some N_k

Example 5.10. Prove that

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \tag{5.24}$$

Proof. The singular part of $\frac{\pi^2}{\sin^2(\pi z)}$ at the pole $z = 0$ is $\frac{1}{z^2} \Rightarrow$ The singular part of $\frac{\pi^2}{\sin^2(\pi z)}$ at $z = n \in \mathbb{Z}$ is $\frac{1}{(z-n)^2}$. We know $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ converges uniformly on each compact set in $\mathbb{C}$ if we omit the terms which become infinite (*i.e.* $p_k = 0$ in the previous theorem)

The Mittag-Leffler Theorem 5.8 implies $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z)$ where g is analytic in $\mathbb{C}$.

It is easy to see that g has period 1 and $\lim_{|y|\to\infty} g(x+iy) = 0$ uniformly in $x \in \mathbb{R}$.

Then $|g(z)|$ is bounded in $\{z \in \mathbb{C} : 0 \leqslant \Re z \leqslant 1\} \Rightarrow |g(z)$ is bounded in $\mathbb{C}$ by its periodicty.

Then Liouville's theorem 4.22 implies g is a constant, hence of 0 since $\lim_{y\to\infty} g(x+iy) = 0$. $\square$

Similarly, one can prove

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\neq 0} \frac{1}{z-n} + \frac{1}{n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2},\ z \in \mathbb{C} \tag{5.25}$$

From (5.24) and (5.25), one can derive

$$\frac{\pi}{\sin(\pi z)} = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n},\ z \in \mathbb{C} \tag{5.26}$$

5.2.2 Infinite Products

An **infinite product** of complec numbers $\prod_{n=1}^{\infty} a_n$ converges if and only if at most a finite number of the factors are zero, and if the partial products formed by the non-vanishing factors tend to a finite limit which is different from zero.

Remark 5.11. $\prod_{n=1}^{\infty}$ converges $\Rightarrow a_n = \dfrac{\prod_{j=1}^{n} a_j}{\prod_{j=1}^{n-1} a_j} \to 1$ as $n \to \infty$ (if the zero factors are omitted)

Theorem 5.12. *The infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ *with* $1+a_n \neq 0$ *converges if and only if* $\sum_{n=1}^{\infty} \mathrm{Ln}(1+a_n)$ *converges, where* Ln *is the principal branch of the logarithm.*

Proof. "$\Leftarrow$": Let $S_n = \sum_{k=1}^{n} \mathrm{Ln}(1+a_k)$. Then $P_n = \prod_{k=1}^{n}(1+a_k) = e^{S_n}$.

So $s_n \to s$ as $n \to \infty \Rightarrow P_n \to P = e^s \neq 0$ as $n \to \infty$

"$\Rightarrow$" Suppose $P_n \to P \neq 0$ as $n \to \infty$.

There exists $M_n \in \mathbb{Z}$ *s.t.* $\mathrm{Ln}(\frac{P_n}{P}) = S_n - \mathrm{Ln}P + 2\pi i \cdot M_n$, $n \in \mathbb{N}$.

Then $2\pi(M_{n+1} - M_n) = \arg(\frac{P_{n+1}}{P}) - \arg(\frac{P_n}{P}) - \arg(1+a_{n+1})$. From $\lim_{k\to\infty} \frac{P_n}{P} = 1$ we can derive $\arg(\frac{P_{n+1}}{P}) - \arg(\frac{P_n}{P}) \to 0$ as $n \to \infty$.

$|\arg(1+a_{n+1})| \leqslant \pi \Rightarrow M_{n+1} - M_n = 0$ for n large enough. So $M_n = M \in \mathbb{Z}$ for all large n.

Then $\mathrm{Ln}(\frac{P_n}{P}) = S_n - \mathrm{Ln}P + 2\pi i \cdot M$, $n \in \mathbb{N} \Rightarrow S_n \to \mathrm{Ln}P - 2\pi i \cdot M$ as $n \to \infty$ $\square$

The infinite product $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$ is said to be **absolutely convergent** if the infinite sum $\displaystyle\sum_{n=1}^{\infty} \mathrm{Ln}(1+a_n)$ is absolutely convergent.

Theorem 5.13. *The product* $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$ *is absolutely convergent iff* $\displaystyle\sum_{n=1}^{\infty}|a_n|$ *converges.*

Proof. Convergence of either $\displaystyle\sum_{n=1}^{\infty} \mathrm{Ln}(1+a_n)$ or $\sum_{n=1}^{\infty}|a_n|$ implies $a_n \to 0$ as $n \to \infty$

$\displaystyle\lim_{z\to 0}\frac{\mathrm{Ln}(1+z)}{z} = 0 \Rightarrow \frac{1}{2}|a_n| < |\mathrm{Ln}(1+a_n)| < \frac{3}{2}|a_n|$ for all large n. $\square$

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