Differential Geometry

LIN150117

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

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Contents

1 Smooth Manifold

Definition 1.1 (Topological manifold)**.** A space M is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

Definition 1.2 (Smooth Manifold)**.** A smooth structure is given by an equivalence class of smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ *s.t.* $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth $\forall \alpha, \beta$. $M = \cup U_\alpha$.
A **smooth manifold** is a topological manifold with a smooth structure.
Define when a continuous map $f : M_1 \to M_2$ is smooth if $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$, we have $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is smooth.

Definition 1.3. Given $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$. A homeomorphism $f : M_1 \to M_2$ is called a diffeomorphism if f, f^{-1} is smooth.
In this case we say $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ are diffeomorphism.

Theorem 1.4 (Kervaire)**.** *$\exists$ 1 10-dimensional topological manifold without smooth manifold.*

Theorem 1.5 (Milnor)**.** *$\exists$ a smooth manifold M s.t. $M \cong S^7$ but not in diffeomorphism meaning.*

Theorem 1.6 (Kervaire-Milnor)**.** *$\exists$ 28 smooth structures (up to orientation preserving diffeomorphism) on S^7*

Theorem 1.7 (Morse-Birg)**.** *On S^7. If $n \leqslant 3$, then any n-dimensional topological manifold M has a unique smooth structure up to diffeomorphism.*

Theorem 1.8 (Stallings). *If $n \neq 4$, then $\exists$ a unique smooth structure on $\mathbb{R}^n$ up to diffeomorphism.*

Theorem 1.9 (Donaldson-Freedom-Gompf-Faubes). *$\exists$ uncountable smooth structures on $\mathbb{R}^4$ up to diffeomorphism.*

Definition 1.10 (topological manifold with boundary). A space M is called a topological manifold with boundary if

1. M is Hausdorff

2. M is second countable

3. $\forall p \in M$, $\exists$ a neighbourhood U of p and a homeomorphism $\varphi : U \to V$ where V is open in $\mathbb{H}^n$

We say a manifold M is closed if M is compact and ∂M is empty.

Our motivation for studying manifold is to study the space of solution for equations.

Question 1. Given $f : \mathbb{R}^n \to \mathbb{R}$ smooth, $q \in \mathbb{R}^n$, when is $f^{-1}(q)$ is a smooth manifold?

For $f : U \to \mathbb{R}^n$ smooth, U open in $\mathbb{R}^m$, the differential of f at $p \in U$ denoted as $\mathrm{d}f(p)$.

Definition 1.11. We say $p \in U$ is a **regular point** of f if $\mathrm{d}f(p)$ is surjective. Otherwise we say $p \in U$ is a **critical point**.
A point $q \in \mathbb{R}^n$ is called a **regular value** of f if $\forall p \in f^{-1}(q)$, p is a regular point of f.
A point $q \in \mathbb{R}^n$ is called a **critical value** of f if $\forall p \in f^{-1}(q)$, p is a critical point of f.

Theorem 1.12 (Implicit function theorem). *If $p \in U$ is a regular point of $f : U \to \mathbb{R}^n$. Then there exists*

- *An open neighbourhood V of p in U*
- *An open subset V' of $\mathbb{R}^m$*
- *A diffeomorphism $\varphi : V \to V'$ such that $P \circ \varphi = f$ where P is the projection from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

In other words, near a regular point, we can do local coordinate change to turn f into the projection.

Remark 1.13. Inverse function theorem and Implicit function theorem gives a way to find the related from "a point" to "a beibourhood"!

In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \ldots, x_m) \in V' | (x_1, \cdots .x_n) = f(p)\}$$

i.e. if we set $M = f^{-1}(f(p))$, then $(M \cap V, \varphi_p)$ is a chart that contains p.

Corollary 1.14. *If q is a regular value of $f : U \to \mathbb{R}^n$ then $f^{-1}(q)$ is a smooth manifold.*

Remark 1.15. It suffices to show that the corresponding charts are compatible.

Theorem 1.16 (Sard). *If $f : U \to \mathbb{R}^n$ is a smooth map, then the set of critical values of f has measure 0.*

Remark 1.17. For a "generic" q, $f^{-1}(q)$ is a manifold of dimension $m - n$.

Corollary 1.18. *If $f : U \to \mathbb{R}^n$ is smooth and $m < n$ then $f(U)$ has measure 0.*

1.1 Lie groups and homogeneous spaces

Definition 1.19. We say G is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map $\cdot : G \times G \to G$ and the inverse map $G \rightsquigarrow G$ is smooth.

Example 1.20. $GL(n, \mathbb{R}) = \{n \times n$ matrices with non-zero determinant$\} \subset \mathbb{R}^{n\times n}$
$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$
$SO(n) = \{A \in O(n) | \det A = 1\}$
$U(n) = \{A \in GL(n, \mathbb{C}) | A\overline{A}^T = I\}$
$SU(n) = \{A \in U(n) | \det A = 1\}$

Exercise 1.21.

$$O(1) \cong S^2 \qquad SO(1) \cong * \tag{1.1}$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \tag{1.2}$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \tag{1.3}$$

The last one is a diffeomorphism but do not preserve the multiplicatioin, *i.e.* not an isomorphism of Lie group.

Theorem 1.22 (Carton)**.** *Let H be a closed subgroup of Lie group G. Then H is a Lie group. More precisely, H is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also, G/H is a smooth manifold*

Definition 1.23. Let M be a smooth manifold. We say M is a **homogeneous space** if $\exists$ a Lie group G with a smooth transitive action $\rho : G \times M \to M$.

Definition 1.24. For M be a homogeneous space. The **isotropy** group of $x \in M$ is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of G

Given any $x, x' \in M$, $Iso(x) \cong Iso(x')$ because the group action is transitive.

Hence, we have a well-defined map

$$p : G/_{Iso(x)} \to M \tag{1.4}$$

$$g \mapsto gx \tag{1.5}$$

Theorem 1.25. *p is always a diffeomorphism.*

Therefore, we have this proposition

Proposition 1.26. *M is a homogeneous space $\Leftrightarrow$ $M = G/H$ for some closed subgroup H.*

Example 1.27. If $M = S^n$, let $G = SO(n+1)$.

Then $Iso(1, 0, \cdots, 0) \cong SO(n)$.

So $S^n \cong SO(n+1)/(SO(n))$.

Similarly, we can prove $\mathbb{RP}^n \cong SO(n+1)/(O(n))$, $\mathbb{CP}^n \cong SO(n+1)/(U(n))$

The isotropy k dimensional linear subspaces of $\mathbb{R}^n$ can be $O(k) \times O(n-k)$ if $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to $\mathbb{RP}^2, S^2, T^2$ and Klein bottle.

Theorem 1.28 (Whithead)**.** *Any smooth manifold has a triangulation.*

Theorem 1.29 (Poincare-Hopf)**.** *G is compact Lie group $\Rightarrow$ $\chi(G) = 0$.*

Theorem 1.30 (Mostow2005)**.** *M is a compact homogeneous space $\Rightarrow$ $\chi(M) \geqslant 0$.*

1.2 Bump Function and Partition of Unity

Theorem 1.31 (Urysohn smooth version)**.** *Given M, closed disjoint A, B, $\exists$ smooth $f : M \to [0,1]$ s.t. $f|_A = 0, f|_B = 1$.*

Theorem 1.32 (Tietze)**.** *Given M, closed A, smooth $f : A \to \mathbb{R}^n$, there exists smooth $\hat{f} : M \to \mathbb{R}^n$ s.t. $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

Lemma 1.33. *Let U be a neighbourhood of $p \in M$. Then $\exists$ smooth $\sigma : M \to [0,1]$ s.t.*

1. *$\sigma \equiv 1$ near p*
2. *Supp $\sigma \subset U$*

*Such σ is called a **bump function** at p, supported in U.*

Definition 1.34. An open cover of a space X is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

Proposition 1.35. *Given compact $K \subset U$ and open neighbourhood U of K, $\exists$ a smooth $g : M \to [0, +\infty)$ s.t. $g|_K \equiv 1$ and Supp $g \subset U$.*

Definition 1.36. An **exhaust** of a space X is a sequence of open sets $\{U_i\}$ *s.t.*

1. $X = \bigcup_{i=1}^{\infty} U_i$
2. $\overline{U_i}$ is compact and contained in U_{i+1}

Theorem 1.37. *Any topological manifold has an exhaust.*

Given two open covers $\mathcal{U}, \mathcal{V}$, we say $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if $\forall U_\alpha \in U$, $\exists V_\beta \in \mathcal{V}$ *s.t.* $V_\beta \subset U_\alpha$.

We say a space X is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact.(The proof is hard)

Proposition 1.38. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a topological manifold M. Then there exists countable open covers $\mathcal{W} = \{W_i\}$, $\mathcal{V} = \{V_i\}$ s.t.*

- *For any i, $\overline{V_i}$ is compact and $\overline{V_i} \subset W_i$*
- *$\mathcal{W}$ is locally finite.*
- *$\mathcal{W}$ is a refinement of $\mathcal{U}$.*

As a corollary, we have any topological manifold is paracompact.

Definition 1.39. Given open cover $\mathcal{U}$ of a smooth M, a partition of unity subordinate to $\mathcal{U}$ is a collection of smooth functions $\{\rho_\alpha : M \to [0,1]\}_{\alpha \in \mathcal{A}}$ *s.t.*

1. $\forall p \in M$, $\exists$ only finitely many $\alpha \in \mathcal{A}$ *s.t.* $p \in Supp\, \rho_\alpha$
2. $\sum\limits_{\alpha \in \mathcal{A}} \rho_\alpha(p) = 1$
3. $Supp\, \rho_\alpha \subset U_\alpha$

Theorem 1.40 (Existence of P.O.U)**.** *For any open cover $\mathcal{U}$ of smooth M, $\exists$ a P.O.U subordinate to $\mathcal{U}$*

Theorem 1.41 (Whitney approximation theorem)**.** *Given any smooth M, any closed A and any continuous $f : M \to \mathbb{R}$, $\delta : M \to (0, +\infty)$. Suppose f is smooth on A. Then $\exists g : M \to \mathbb{R}$ smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M$, $|g(p) - f(p)| < \delta(p)$.

2 Tangent space and tangent vectors

2.1 Tangent Space

Given $p \in M$, consider the set $C_p^\infty(M) = \{\text{smooth function } V \to \mathbb{R}\}/_\sim$ where $f_1 \sim f_2$ if and only if $\exists$ neighbourhood U of p, $f_1|_U = f_2|_U$.

$C_p^\infty(M)$ is the space of **genus of smooth function** near p.

A **partial-derivative** of p is a $\mathbb{R}$-linear map $D : C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

Definition 2.1. A **tangent vector** of M at p is a partial-derivative at p.

Define the **tangent space** $T_pM = \{\text{all partial-derivative at } p\}$, which is a $\mathbb{R}$-vector space.

Proposition 2.2. *For $M = U \subset \mathbb{R}^n$ open. We have $\{\frac{\partial}{\partial x_i}\}$ is a basis for T_pU.*

Proposition 2.3.

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leqslant i \leqslant n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

M, N smooth manifolds, $C^\infty(N, M) = \{\text{smooth } F : N \to M\}$.

Given $F \in C^\infty(N, M)$, F induces $F^* : C^\infty_{F(p)}(M) \to C_p^\infty(N)$, $f \mapsto f \circ F$.

By taking dual, we get

$$F_* : T_pN \to T_{F(p)}M$$

we also write F_* as $F_{*,p}$, call it the **differential** of F at p.

where

$$F_*(\frac{\partial}{\partial x^i}|_p) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

Proposition 2.4. *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_pN \to T_{G\circ F(p)}W$$

Definition 2.5. A smooth **curve** is a smooth map $\gamma : (a, b) \to M$. We say γ starts at p if $\gamma(0) = p$. We define the **velocity** of γ at $\gamma(0)$ as $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)}M$

Take charts $(U, x^1, \cdots, x^n)$ about p, let $\gamma^i = x^i \circ \gamma$.

We say γ, δ are **tangent** to each other at p if $(\gamma^i)'(0) = (\delta^i)'(0)$.

Now we can define

$$(T_pM)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\ \}/_{\sim}$$

where $\gamma \sim \delta$ iff they are tangent to each other.

Then these definition is more geometric.

Lemma 2.6. *Given $F \in C^\infty(M, M)$, $p \in N$, the diagram commutes:*

$$\begin{array}{ccc} \gamma \in (T_pN)_{curve} & \xrightarrow{\cong} & T_pN \\ \downarrow & & \downarrow \\ F \circ \gamma \in (T_{F(p)}M)_{curve} & \xrightarrow{\cong} & T_{F(p)}M \end{array}$$

2.2 Tangent Bundle

Let $(M, \mathcal{A})$ be a smooth manifold, $TM = \bigcup_{p\in M} T_pM$, called the **tangent bundle**

Now we want to define a natural topology and smooth structure on TM. Take any chart $(U, \varphi) = (U, x^1, \cdots, x^n) \in \mathcal{A}$.

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \tag{2.1}$$

$$X \in T_pU \mapsto (\varphi(p), X^1, \cdots, X^n) \tag{2.2}$$

where $X = \sum X^i \frac{\partial}{\partial x^i}|_p$.

Then pull back standard topology on $\varphi(U) \times \mathbb{R}^n$ to a topology on TU.

$$\mathcal{B} = \{\hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- $\mathcal{B}$ is a basis
- $\mathcal{B}$ generates a Hausdorff, second countable topology on TM.

So TM is a topological manifold covered by charts $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$.

Given $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$, the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi} \circ \hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \tag{2.3}$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \tag{2.4}$$

So $\hat{\mathcal{A}}$ is a smooth atlas on TM, making TM into a smooth manifold.

Definition 2.7 (vector bundle)**.** Given a continuous map $f : E \to B$, we say f is a n-dimensional **vector bundle** if: $\exists$ an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of B and homeomorphisms $\{f^{-1}(U_\alpha) \xrightarrow[\cong]{\rho_\alpha} U_\alpha \times \mathbb{R}\}$ *s.t.*

- $\begin{tikzcd} f^{-1}(U_\alpha) \arrow[r, "\rho_\alpha"] \arrow[d, "f"] & U_\alpha \times \mathbb{R}^n \arrow[dl, "projection"] \\ U_\alpha & \end{tikzcd}$ commutes for $\alpha \in I$
- $\forall\, p \in U_\alpha \cap U_\beta$, the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

is linear.

Call $f^{-1}(p)$ the **fiber** over p.

Proposition 2.8. *Given vector bundle $f : E \to B$, the fiber $f^{-1}(p)$ has a structure of a vector space.*

Example 2.9 (Product bundle)**.** $E = \mathbb{R}^n \times B$

Example 2.10 (Tautological bundle)**.**

$$B = \mathbb{CP}^n = \{\text{1-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map $(L, v) \mapsto L$

Given vector bundles $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$, a bundle map consists of $(\hat{f}, f)$ *s.t.*

- $\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$ commutes.
- $\forall b \in B$, $\hat{f} : \pi_1^{-1}(b) \to \pi_2^{-1}(f(b))$ is linear.

If $\hat{f}, f$ are diffeomorphisms, then we call $(\hat{f}, f)$ an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

Example 2.11. TS^1, TS^2 are both trivial.

$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$

Theorem 2.12. *If G is a Lie group, then TG is trivial.*

Proof. For $(x^1, x^2, \cdots, x^n)$ is a basis of T_eG The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG,\ (g, c^1, \cdots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \to G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism $(l_g)_*$

$\square$

Proposition 2.13 (Adams, 1960s). *TS^n is trivial if and only if $n = 0, 1, 3, 7$.*

Proposition 2.14.

1. *Given any $F \in C^\infty(M, N)$, $F_* : TM \to TN$ is a bundle map.*

2. *TS^n is isomorphic to the following bundle:*

$$B = s^n \qquad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

Definition 2.15 (smooth section). Given a smooth vector bundle $\pi : E \to B$, a **smooth section** is a smooth map $S : B \to E \quad s.t.\ \pi \circ S = id_b$.

$s_0 : B \to E, b \mapsto 0 \in$ 0-vector in $\pi^{-1}b$.

2.3 Vector Field, Curves and Flows

Definition 2.16. A (tangent) **vector field** is a smooth section of TM. *i.e.* a smooth map $M \xrightarrow{X} TM \quad s.t.\ X(p) \in T_pM, \forall p \in M$

Given any $f : \mathbb{R}^n \to \mathbb{R}$, define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leqslant i \leqslant n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

Example 2.17. $X = f^1 \partial x^1 + f^2 \partial x^2$ is a gradient field if and only if $\dfrac{\partial f^1}{\partial x^2} = \dfrac{\partial f^2}{\partial x^1}$

Theorem 2.18 (Poincare-Hopf). *For closed M, M has a nowhere vanishing vector field if and only if $\chi(M) = 0$.*

So S^n has a nowhere vanishing vector field if and only if n is odd.

Theorem 2.19 (MaoQiu). *S^2 has no no-where vanishing vector field.*

So We cannot smooth out all the hairs on a ball.

Given a vector field $X = \{X_p\}_{p\in M}$, a curve $\gamma : (a, b) \to M$ is called an **integral curve** of X if $\gamma'(t) = X_{\gamma(t)}$, $\forall t \in (a, b)$, where $\gamma'(t) = \gamma_*(\dfrac{\partial}{\partial t}) \in T_{\gamma(t)}M$.

We say γ is maximal if the domain cannot be extended to a larger interval. Denote the set of all smooth vector fields on M by $\mathfrak{X}M$

Recall that γ is maximal if it's domain can not be extended to a large open interval.

In a local chart $(U, x^1, \cdots, x^n)$, $X|_U = \sum_{i=1}^{n} a^i \partial x^i$. Then γ is an integral curve if and only if $(\gamma^i)'(t) = a^i(\gamma(t))$, $\forall 1 \leqslant i \leqslant n$, where $\gamma^i = x^i \circ \gamma : (a, b) \to \mathbb{R}$.

And in this case the initial value condition: $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$.

Locally, solving integral curve starting at p is equivalent to solving ODE with initial value $p^1, \cdots, p^n$. By existence and uniqueness of solutions of ODE, we have

Theorem 2.20 (Fundamental theorem of integral curve). *Let $X \in \mathfrak{X}M$, $p \in M$, then:*

(1) (Uniqueness) Given any two integral curves $\gamma_1, \gamma_2 : (a, b) \to M$, then we have:

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a, b) \ \Rightarrow \gamma_1 = \gamma_2$$

(2) there exists a unique max integral curve $\gamma : (a(p), b(p)) \to M$ starting at p.

(3) (integral curve smoothly depend on initial values) $\exists$ Nbh U of p, $\varepsilon > 0$, and smooth $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ s.t. $\forall q \in U$, $\varphi_\varepsilon := \varphi(-, q) : (-\varepsilon, \varepsilon) \to M$ is an integral

curve starting at q.

*we call such φ a local **flow** generated by X.*

Definition 2.21. Given $X \in \mathfrak{X}M$, a global **flow** generated by X is a smooth map $\varphi : \mathbb{R} \times M \to M \quad s.t.\ \forall q \in M$, $\varphi_q := \varphi(-, q)$ is the maximal integral curve of X starting at q.
$\Leftrightarrow \dfrac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s,p)}, \forall s \in \mathbb{R}, p \in M$ and $\varphi(0, p) = p, \forall p \in M$.

If such global flow exists, then we say X is **complete**.

Example 2.22.

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$ is complete, where global flow $\varphi : \mathbb{R} \times M \to M$, $\varphi(t, p) = p \cdot e^t$.
- $X = x^2 \partial x$ is not complete. Max integral curve starting at 1 is given by $\gamma(t) = \dfrac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$.

Given $X \in \mathfrak{X}M$, we define $\mathrm{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$.

Theorem 2.23. *If a vector field X is compactly supported, then X is complete.*

Corollary 2.24. *Any vector field on closed manifold is complete.*

Lemma 2.25 (Escaping lemma). *Suppose $\gamma : (a, b) \to M$ is a max integral curve, with $(a, b) \neq \mathbb{R}$. Then $\nexists$ compact $K \subset M \quad s.t.\ \gamma(a, b) \subset K$*

Proof. Otherwise, suppose $\gamma(a, b) \subset K$. WLOG, we may assume $b < +\infty$.

Take $(t_i) \to b$ from left. Then $\gamma(t_i) \in K$. After passing to subsequence, we may assume $(\gamma(t_i)) \to p \in K$.

Then $\exists\ U$ Nbh of p, local flow $\varphi : (-\varepsilon, \varepsilon) \times U \to M$. Take n large enough $\quad s.t.$ $b - t_n < \varepsilon, \gamma(t_n) \in U$. Then $\gamma(- + t_n) : (a - t_n, b - t_n) \to M$, $\varphi(-, \gamma(t_n)) : (-\varepsilon, \varepsilon) \to M$ are both integral curves for X starting at $\gamma(t_n)$. By uniqueness, they coincide.

Let $\hat{\gamma} : (a, t_n + \varepsilon) \to M$ be defined by $\hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \varepsilon) \end{cases}$

Then $\hat{\gamma}$ is an integral curve with larger domain, then γ contradiction with the maxity of γ. □

Proof of 2.23. Take any max integral curve $\gamma : (a, b) \to M$. Suppose $(a, b) \neq \mathbb{R}$. Then $X_{\gamma(s)} \neq 0$, $\forall s$. Otherwise, the constant map $\mathbb{R} \to M, t \mapsto \gamma(s)$ is an integral curve with lager domain.

So $\forall s$, $\gamma(s) \in \mathrm{Supp}X \Rightarrow \gamma(a, b) \subset \mathrm{Supp}X$ which is compact $\Rightarrow (a, b) = \mathbb{R}$ by the lemma. This causes contradiction! □

A smooth $\varphi : \mathbb{R} \times M \to M$ is called an **one-parameter transformation group** if

(1) $\varphi_0 := \varphi(0, -) = \mathrm{id}_M$

(2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}$. In particular, $\varphi_s^{-1} = \varphi_{-s}$.

Theorem 2.26. *$\varphi \in C^\infty(\mathbb{R} \times M, M)$, then φ is an one-parameter transformation group if and only if φ is the global flow generated by some $X \in \mathfrak{X}M$*

Lemma 2.27 (Translation lemma). *If $\gamma : (a, b) \to M$ is an integral curve for some $X \in \mathfrak{X}M$, then $\forall s \in \mathbb{R}$, $\gamma(- + s) : (a - s, b - s) \to M$ is also an integral curve for X.*

Proof. Let $\iota = \gamma(- + s)$. Then $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$ □

Lemma 2.28. *Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$ provided that $s, t, s + t \in (-\varepsilon, \varepsilon), p, \varphi_r(p) \in U$.*

Proof. $\gamma_p = \varphi(-, p)$ is an integral curve for X.

$\Rightarrow \gamma_p(- + s)$ is an integral curve for X starting at $\gamma_p(s) = \varphi_s(p)$. But $\gamma_{\varphi_s(p)}$ is also an integral curve starting at $\varphi_s(p)$. Thus $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$ □

Lemma 2.29. *Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$ i.e. any vector field is invariant under its flow.*

Proof. Take $f \in C^{\infty}_{\varphi(p)}(M)$.

$$\begin{aligned} \varphi_{s,*}(X_p)(f) &= X_p(f \circ \varphi_s) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \\ &= X_{\varphi_s(p)}(f) \end{aligned}$$

□

Proof of 2.26. "$\Leftarrow$" is because the lemma $\varphi_s \circ \varphi_r = \varphi_{s+r}$

"$\Rightarrow$" Let $X = \{X_p\}$ where $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$.

Leave it as an exercise. □

Time dependent vector field is a smooth map $X : \mathbb{R} \times M \to TM$ *s.t.* $X_{(t,p)} \in T_pM$.

A smooth curve $\gamma(a, b) \to M$ is the **integral curve** for X if $\gamma'(t) = X_{(t,\gamma(t))}$.

In local chart, solving γ is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness: γ_1, γ_2 are both integral curves for X, $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define $\mathrm{Supp}X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$.

Then X is compactly supported, then X is complete(*i.e.* a global flow $\varphi : \mathbb{R} \times M \to M$)

But something is not true for time dependent vector field:

- translation lemma is not true.
- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

2.4 Another definition of vector field

A derivation on M is a $\mathbb{R}$-linear map $C^\infty(M) \xrightarrow{D} C^\infty(M)$ that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

Theorem 2.30. *We have a bijection:*

$$\rho : \mathfrak{T}M \xrightarrow{1:1} \{\textit{derivation on } M\}$$

$$X \mapsto D_X : f \mapsto X(f)$$

Lemma 2.31. $D_p : \mathfrak{T}_pM \to$ *$\mathbb{R}$-linear map $C^\infty(M) \to \mathbb{R}$ s.t. $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$ is an isomorphism of vector spaces.*

Proof. Leave it as an exercise. □

Lemma 2.32. *Given a vector field(not necessarily smooth) $X = \{X_p\}_{p\in M}$, X is smooth $\Leftrightarrow \forall f \in C^\infty(M), X(f)$ is smooth.*

Proof. "$\Leftarrow$" $\forall p \in M$, take chart $(U, x^1, x^2, \cdots, x^n)$ around p. $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$ $f^i : U \to \mathbb{R}$, where $f^i = X|_U(x^i)$. Take $\varphi : M \to [0, 1]$ *s.t.* $\varphi \equiv 1$ near p, Supp$\varphi \subset U$,$\varphi \cdot x^i \in C^\infty(M)$.

Then $X(\varphi \cdot x^i) = f^i$ near p. By assumption, f^i is smooth near p. So f^i is smooth, so X is smooth.

"$\Rightarrow$" Similar. $\square$

Theorem 2.33. *The map $\rho : \mathfrak{X}M \to \{$derivation on $M\}, X \mapsto (D_x : f \mapsto X(f))$ is well-defined and bijective.*

Proof. ρ is well-defined: $X(f) \in C^\infty(M)$ by Lemma 2.32, and $D_x(fg) = D_x(f)g + fD_x(g)$ since X is a point-derivation.

ρ is injective: $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$ as maps $C^\infty(M)$ to $\mathbb{R}$. By Lemma 2.31, we have $X_p = Y_p$, $\forall p$. So $X = Y$.

ρ is surjective: Given $D : C^\infty(M) \to C^\infty(M)$. Define $D_p : C^\infty(M) \to \mathbb{R}$ by $D_p(f) := D(f)(p)$ satisfies the Leibniz rule. By Lemma 2.31, $D_p = D_{X_p}$ for some $X_p \in T_pM$. Define $X = \{X_p\}_{p\in M}$. Then $X(f) = D(f), \forall f \in C^\infty(M)$. By Lemma2.32, X is a smooth vector field. $\square$

3 Lie group, Lie algebra and Lie bracket

3.1 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned}\{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \quad s.t. \\ &\qquad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\}\end{aligned}$$

$$\begin{aligned}\{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\}\end{aligned}$$

Notation 3.1. We will identify $X \in \mathfrak{X}M$ with its derivation $D_x : C^\infty(M) \to C^\infty(M)$. So a vector field is just a $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M) \quad s.t.\ X(fg) = fX(g) + X(f)g$.

Definition 3.2 (Lie bracket)**.** Given two (smooth) vector field $X, Y : C^\infty(M) \to C^\infty(M)$, we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

Theorem 3.3. *For any $X, Y \in \mathfrak{X}M$, $[X, Y] \in \mathfrak{X}M$*

Proof. Easy to check that $[X, Y]$ is linear.

By Leibuniz rule,

$$\begin{aligned}
[X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\
&= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\
&= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\
&= [X, Y](f) \cdot g - f \cdot [X, Y](g)
\end{aligned}$$

□

So What is the geometric meaning of $[X, Y]$? Non commutatiy of flows.

Fact 3.4. *Given $X, Y \in \mathfrak{X}M$, we say X, Y are commutative vector field if $[X, Y] = 0$*

X, Y are commuative iff for any local flows $\varphi^X : (-\varepsilon, \varepsilon) \times U \to M$, $\varphi^Y : (-\varepsilon, \varepsilon) \times U \to M$ we have $\varphi^X_s \circ \varphi^T_t = \varphi^Y_t \circ \varphi^X_s$

Proposition 3.5 (Calculation of $[V, W]$ using local charts)**.** *Chart* $(U, x^1, \cdots, x^n)$, $V, W \in \mathfrak{X}M$, $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$, $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$. *Then*

$$\begin{aligned} [V, W]|_U &= \sum_{i=1}^n (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n (\sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i} \\ &= \sum_{1 \leqslant i,j \leqslant n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i} \end{aligned}$$

Example 3.6. $V = x\partial x + y\partial y$, $W = -y\partial x + x\partial y$ commutes.

Proposition 3.7 (Properties of Lie bracket)**.**

(a) Natuality under push-forword.

Given any $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, *we have* $[F_*V, F_*W] = F_*[V, W]$.

(b) $\mathbb{R}$*-linearity* $\forall a, b \in \mathbb{R}$

$$[aX + bV, W] = a[X, W] + b[V, W]$$

$$[W, aX + bV] = b[W, X] + a[W, V]$$

(c) anti-symmetric $[V, W] = -[W, V]$

(d) Jacobi identity

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

(f) *Leibuniz rule*

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

Definition 3.8. Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}N$. We say W is F**-related** to V if $\forall p \in M$, $F_{p,*}(V_p) = W_{F(p)}$ where $F_{p,*} : T_pM \to T_{f(p)}N$

Example 3.9. $F : S^1 \to \mathbb{R}^2, \theta \mapsto (\cos\theta, \sin\theta)$, $V = \partial\theta, W = -y\partial x + x\partial y$.

Note 1. In general, given $V \in \mathfrak{X}M$ and $F \in C^\infty(M, N)$. There may not exist $W \in \mathfrak{X}M$ *s.t.* V, W are F-related. Even such W exists, it may not be unique.

However, if F is a diffeomorphism, given any V, $\exists$ unique W *s.t.* V and W are F-related. Actually, $W_p = F_* V_{F^{-1}(p)}$.

Such W is called **push forward** of V along F, denoted by F_*V, only defined when F is a diffeomorphism.

Lemma 3.10. $\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N$, $F \in C^\infty(M, N)$. *Then W is F-related to V iff* $\forall f \in C^\infty(N), V(f \circ F) = W(f) \circ F \in C^\infty(M)$

Proof. Check that $F_{p,*}(V_p)(f) = W_{F(p)}(f), \forall f \in C^\infty(N)$ □

Proposition 3.11. *Given $V_0, V_1 \in \mathfrak{X}M$, $W_0, W_1 \in \mathfrak{X}N$, $F \in C^\infty(M, N)$, W_i is F-related to V_i, $i = 0, 1 \Rightarrow [W_0, W_1]$ is F-related to $[V_0, V_1]$*

Corollary 3.12 (Naturality of Lie bracket)**.** *Given any $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, we have $[F_*V, F_*W] = F_*[V, W]$*

The rest of Proposition 3.7 is easy to check if it is viewed as a mapping $C^\infty(M) \to C^\infty(M)$.

3.2 Lie algebra of a Lie group

Definition 3.13. A **Lie algebra** g is $\mathbb{R}$-linear space g with map $[-,-] : g \times g \to g$ *s.t.* it is bilinear, anti-symmetric and satisfies the Jacobian identity.

Then $(\mathfrak{X}M, [-,-])$ is an infinite dimensional Lie algebra.

For G Lie group, $\forall g \in G$ we have diffeomorphism

$$l^g : G \to G, h \mapsto gh$$

$$r^g : G \to G, h \mapsto hg$$

We say $X \in \mathfrak{X}G$ is **left invariant** if $l^g_*(X) = X$, $\forall g \in G$. Similarly, X is **right invariant** if $r^g_*(X) = X$.

Proposition 3.14. *X, Y are left/right invariant $\Rightarrow [X, Y]$ is left/right invariant.*

Proof. $l^g_*[X, Y] = [l^g_* X, l^g_* Y] = [X, Y]$ □

So we can find a natural Lie algebra of G:

$$\mathrm{Lie}(G) := \{\text{left invariant vector fields on } G\}, \text{with } [-,-] \text{ restricted from } \mathfrak{X}G$$

Theorem 3.15. *Given any $V \in T_eG$, $\exists$ unique left invariant $\hat{V} \in \mathfrak{X}G$ s.t. $\hat{V}_e = V$.*

Corollary 3.16. $\mathrm{Lie}(G) \cong T_eG$ *as vector spaces.*

Proof of Theorem 3.15.

Uniqueness of $\hat{V}$: $\hat{V}_g = l^g_{e,*}(\hat{V}_e) = l^g_{e,*}(V)$. So $\hat{V}$ is determined by V.

Existence of $\hat{V}$: Let $\hat{V} = \{\hat{V}_g\}_{g \in G}$ where $\hat{V}_g = l^g_{e,*}(\hat{V}_e)$.

$\hat{V}$ is left-invariant because

$$(l^h_*(\hat{V}))_g = l^h_{h^{-1}g,*}(\hat{V}_{h^{-1}g}) = l^h_{h^{-1}g,*}(l^{h^{-1}g}_{e,*}(V)) = l^g_{e,*}(V) = \hat{V}_g$$

$\hat{V}$ is smooth: Take any $f \in C^\infty(G)$ suffices to show $\hat{V}(f) \in C^\infty(G)$.

Take any smooth $\gamma : \mathbb{R} \to G \quad s.t.\ \gamma(0) = e, \gamma'(0) = V$. Then $l^g \circ \gamma : \mathbb{R} \to G$ satisfies $l^g \circ \gamma(0) = g, (l^g \circ \gamma)(0) = g, (l^g \circ \gamma)'(0) = l^g_{e,*}(V) = \hat{V}_g$

So

$$\hat{V}(f)(g) = \hat{V}_g(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(l^g \circ \gamma(t))|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(g \cdot \gamma(t))|_{t=0} \tag{3.1}$$

Consider the map

$$\hat{f} : G \times \mathbb{R} \xrightarrow{\mathrm{id}\times\gamma} G \times G \xrightarrow{\cdot} G \xrightarrow{f} \mathbb{R}$$

$$(g, t) \mapsto (g, \gamma(t)) \mapsto g \cdot \gamma(t) \mapsto f(g \cdot \gamma(t))$$

Then $\hat{f}$ is smooth, $\frac{\partial \hat{f}}{\partial t}|_{t=0} : G \to \mathbb{R}$ is smooth, but $\frac{\partial f}{\partial t}|_{t=0}(g) = \hat{V}(f)(g)$ by 3.1. So $\hat{V}(f) \in C^\infty(G)$. $\square$

Example 3.17. $G = \mathrm{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det A \neq 0\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^2$.

$\mathrm{gl}(n, \mathbb{R})$= $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) = T_I\mathrm{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$

Theorem 3.18. *$\forall A, B \in \mathrm{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$, $[A, B] = AB - BA$.*

Remark 3.19. This theorem shows that the Lie bracket viewed as the Lie algebra and matrix are the same. In some sense, it means the Lie bracket defined in three sets $\mathrm{gl}(n, \mathbb{R}) = T_I\mathrm{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$ can commute with those corresponding, or equivalently, are just the same.

Lemma 3.20. *$\forall A \in \mathrm{gl}(n, \mathbb{R})$, the left invariant vector field $\hat{A}$ is complete and generate the flow $\varphi_t : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2t^2}{2!} + \cdots)$*

Proof.

$$\hat{A}_g = g \cdot A \in T_g G = M_n(\mathbb{R})$$

$$\frac{\partial}{\partial t}\varphi_t(g) = \frac{\partial}{\partial t}(g(e^{At})) = ge^{At}A = \hat{A}_{g \cdot e^{At}} = \hat{A}_{\varphi_t(g)}$$

□

Remark 3.21. This lemma tells how to compute $A(f) = \hat{A}(f)(I)$ as a tangent vector or a vector field, as we will see in the next proof.

Proof of Theorem 3.18. Take $A, B \in \mathrm{gl}(n, \mathbb{R})$. Want to show $[\hat{A}, \hat{B}]_I = AB - BA$.

Pick $f \in C_I^\infty(G)$, need to show $A(\hat{B}(f)) - B(\hat{A}(f)) = (AB - BA)(f)$

Further Simplification: Just need to focus on $f = x^{ij}$, where $x^{ij} : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}, E \mapsto (E - I)_{ij}$. Actually, ∂x^{ij} is what we choose as a basis of $T_I \mathrm{GL}(n, \mathbb{R})$.

Such f satisfies $f(I + -)$ is $\mathbb{R}$-linear.

Recall that Given $W \in \mathfrak{X}M$, $W(f)(p) = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t^W(p))|_{t=0}$.

So $\hat{B}(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(ge^{Bt})|_{t=0}$.

So since $A(\hat{B}(f)) = \hat{A}((\hat{B})(f))(I) = \frac{\mathrm{d}}{\mathrm{d}s}(\hat{B}(f)(e^{As}))|_{s=0}$,

$$A(\hat{B}(f)) = \frac{\mathrm{d}}{\mathrm{d}s}(\hat{B}(f)(e^{As}))|_{s=0} = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t} f(I + sA + tB + \frac{s^2}{2}A^2 + stAB + \frac{t^2}{2}B^2 + \cdots)|_{s=t=0}$$

Similarly,

$$B(\hat{A}(f)) = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t} f(I + sA + tB + \frac{s^2}{2}A^2 + stBA + \frac{t^2}{2}B^2 + \cdots)|_{s=t=0}$$

So $A(\hat{B}(f)) - B(\hat{A}(f)) = f(I + (AB - BA)) = (AB - BA)(f)$ since f is $\mathbb{R}$-linear. □

Similarly, for $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{Lie}(G) = \mathrm{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$, we have $[A, B] = AB - BA$.

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group G, its universal cover $\hat{G}$ is simply-connected with $\pi^{-1}(G) \subset Z(\hat{G})$.

What is the meaning of Lie bracket. There is a fact about it:

Fact 3.22. *G is connected Lie group. G is abelian iff $[-,-] = 0$ on $\mathrm{Lie}(G)$*

3.3 Morphisms between Lie group and Lie algebras

A smooth map $F : G \to H$ between two Lie group is called a **morphism** if $F(gh) = F(g)F(h)$.

A linear map $L : g \to h$ between Lie algebra is called a **morphism** if $L[u, v] = [Lu, Lv]$.

Proposition 3.23. *Let $F : G \to H$ be a morphism of Lie groups. Then $F_{e,*} : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a morphism of Lie algebra.*

Proof. $V_0, V_1 \in \mathrm{Lie}(G) = T_eG$, $W_i = F_{e,*}(V_i) \in \mathrm{Lie}(H) = T_eH$. Let $\hat{V}, \hat{W}$ be left-invariant vector fields.

Claim. $\hat{W}_i$ is F-compatible with $\hat{V}_i$ for $i = 0, 1$.

Proof of Claim. $\forall g \in G$, $F_*(\hat{V}_g) = F_*(l^g_*(V)) = (F \circ l^g)_*(V) = (l^{F(g)} \circ F)_*(V) = l^{F(g)}(W) = \hat{W}_{F(g)}$ □

So $[\hat{W}_0, \hat{W}_1]$ is F-compatible with $[\hat{V}_0, \hat{V}_1]$. In particular, $[W_0, W_1] = F_*([V_0, V_1])$. □

4 Vector Field

4.1 Canonical form of a field

Recall that $V \in \mathfrak{X}M$, $p \in M$ is called a **regular point** if $V_p \neq 0$, and is called a **singular point** if $V_p = 0$.

Theorem 4.1 (Canonical Form Theorem). *Let p be a regular point of V. Then $\exists$ local chart $(U, x^1, \cdots, x^n)$ around p s.t. $V|_U = \partial x^1$*

Proof. This is a local problem. We may assume $M \subset \mathbb{R}^n$ open. We may also assume $p = 0, V_0 = \partial r^1|_0$ where r^i coordinate function.

Let $\varphi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)^n \to M$ be the local flow of V.

Define $\psi : (-\varepsilon, \varepsilon)^n \to M$ by $\psi(t, r^2, \cdots, r^n) = \varphi(t, (0, r^2, \cdots, r^n))$. Then $\psi(-, r^2, \cdots, r^n)$ is an integral curve for V. Therefore, $\psi_*(\partial t) = V$.

At $\vec{0}$, we have $\psi_{\vec{0},*}(\partial t) = V_{\vec{0}} = \partial r^1$, $\psi_{\vec{0},*}(\partial r^i) = \partial r^i$.

So $\psi_{*,\vec{0}} : T_{\vec{0}}(-\varepsilon, \varepsilon)^n \to T_{\vec{0}}M$ is an isomorphism.

By the inverse function theorem, $\exists U' \subset (-\varepsilon, \varepsilon)^n$, $U \subset M$ s.t. $\psi|_{U'} : U' \to U$ is a diffeomorphism.

Then $(U, (\psi|_{U'})^{-1})$ is the local chart what we need. $\square$

Remark 4.2. Regular point in a vector field is simple, as we can view it in the standard chart locally. However, behavior of V art a singular point can be complicated. For example, for $f(x, y) = x^2 - y^2$, $\nabla f = 2x\partial x - 2y\partial y$, $g : \mathbb{C} \to C, z \mapsto z^n$, they behave differently at $\vec{0}$.

4.2 Lie derivative of vector field

$V, W \in \mathfrak{X}M$, $\mathcal{L}_V W$ is the directional derivative of W in the direction of V.

Definition 4.3. The **Lie derivative** $\mathcal{L}_V W \in \mathfrak{X}M$ is defined as follows: $\forall p \in M$, let $\{\theta_t : U \to M\}_{t\in(-\varepsilon,\varepsilon)}$ be the local flow for V. Then

$$(\mathcal{L}_V W)_p = \lim_{t\to 0} \frac{(\theta_{-t})_* W_{\theta_t(p)} - W_p}{t}$$

Remark 4.4. This defintion is actually a difference between $T_{\theta_t(p)}$ and T_p, which need pullback.

Lemma 4.5. *$\mathcal{L}_V W$ is well-defined and smooth.*

Proof. For $p \in M$, take local chart $(U, x^1, \cdots, x^n)$. Let $\theta : (-\varepsilon, \varepsilon) \times U \to M$ be the flow of V. Take $J_0 \subset (-\varepsilon, \varepsilon), U_0 \subset U$. Let $\theta^i = x^i \circ \theta : J_0 \times U_0 \to \mathbb{R}, W|_U = \sum_{i=1}^n W^i \partial x^i$.

Under the basis $\{\partial x^i\}$, $(\theta_{-t})_* : T_{\theta_t(p)}M \to T_pM$ is represented by

$$\left(\frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j}\right)_{i,j}$$

So $(\theta_{-t})_* W_{\theta_t(x)} = \sum_{i,j} \frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j} W^j(\theta(t, x)) \cdot \partial x^i$ is smooth in t, x. So

$$(\mathcal{L}_V W)_x = \frac{\partial((\theta_{-t})_*(W_{\theta_t(x)}))}{\partial t}|_{t=0}$$

is well-defined and smooth. □

Theorem 4.6. *For all $V, W \in \mathfrak{X}M$, $\mathcal{L}_V W = [V, W]$.*

Proof. For p is a regular point of V. By canonical form theorem 4.1, $\exists$ local chart $(U, x^1, \cdots, x^n)$ around p *s.t.* $V|_U = \partial x^1$. Let $W|_U = \sum_{i=1}^n W^i \partial x^i$.

Then $\theta_t(x_1, \cdots, x_n) = (x_1 + t, x_2, \cdots, x_n)$. So

$$\mathcal{L}_V W|_U = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

.

$$[V, W]|_U = \sum_i V(W^i)\partial x^i - \sum_i W(V^i)\partial x^i = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

Then $[V, W]|_U = \mathcal{L}_V W$.

For p is a singular point but $p \in \mathrm{Supp}(V)$. Then $\exists\ p_i \to p \quad s.t.\ V_p \neq 0$. By the previous case $(\mathcal{L}_V W)_{p_i} = [V, W]|_{p_i}$. By continuity, We have $(\mathcal{L}_V W)_p = [V, W]_p$.

For $p \notin \mathrm{Supp}(V)$, $\exists$ Nbd U of $p \quad s.t.\ V|_U = 0$. Then $\theta_t(q) = q$. So

$$(\mathcal{L}_V W)|_U = 0 = [V, W]|_U$$

$\square$

Corollary 4.7.

- *$\mathcal{L}_V W$ is $\mathbb{R}$-linear with respect to V, W.*
- *$\mathcal{V}W = -\mathcal{L}_W V$.*
- *$\mathcal{L}_V[W, X]$.*
- *(Jacobian identity) $\mathcal{L}_V[W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.*
- *(Jacobian identity) $\mathcal{L}_{[V,W]}X = \mathcal{L}_V\mathcal{L}_W X - \mathcal{L}_W\mathcal{L}_V X$.*
- *$\mathcal{L}_V(fW) = (Vf) \cdot W + f\mathcal{L}_V W$*
- *Let $F : M \to N$ be a diffeomorphism. Then $F_*(\mathcal{L}_V W) = \mathcal{L}_{F_*(V)}F_*(W)$.*

4.3 Commuting vector fields

Definition 4.8. We say $V, W \in \mathfrak{T}M$ **commutes** if $[V, W] = 0$.

Theorem 4.9. *TFAE:*

1 *V, W commutes.*

2 *W is invariant under the flow generated by V, i.e. $\theta_{t,*}(W_p) = W_{\theta_t(p)}$*

3 *The flow for V, W commutes, i.e. $\theta_t \circ \eta_s = \eta_s \circ \theta_t$ whenever either side is defined or equivalently, whose the domain is compatible.*

Lemma 4.10. *Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}N$. Then W is F-related to V if and only if $\forall t \in \mathbb{R}$, $\eta_t \circ F = F \circ \theta_t$ on the domain of θ_t, which means*

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\theta_t"] & N \arrow[d, "\eta_t"] \\ M \arrow[r, "F"] & N \end{tikzcd}$$ *commutes.*

Proof. "$\Rightarrow$" Let $\gamma = F \circ \theta^p : J \to N$ satisfies

$$\gamma'(t) = (F \circ \theta^p)'(t) = F_*((\theta^p)'(t)) = F_*(V_{\theta^p(t)}) = W_{F(\theta^p(t))} = W_{\gamma(t)}$$

So γ is an inetgral curve of W starting at $\gamma(0) = F(p)$ *i.e.* $F \circ \theta^p = \gamma(t) = \eta^{F(p)}(t)$ *i.e.* $F \circ \theta_t = \eta \circ F$.

"$\Leftarrow$" Suppose $F \circ \theta_t = \eta \circ F$. Then $(F \circ \theta^p)(t) = \eta^{F(p)}(t)$.

Then $F_* V_p = F_*((\theta^p)'(0)) = (F \circ \theta^p)'(0) = (\eta^{F(p)})'(0) = W_{F(p)}$. So W is F-related to V. □

Proof of Theorem 4.9. $2 \Rightarrow 1$: $(\theta_{-t})_*(W_{\theta_t(p)}) = W_p$. So

$$\mathcal{L}_V W = \lim_{t \to 0} \frac{(\theta_{-t})_*(W_{\theta_t(p)}) - W_p}{t} = 0$$

$1 \Rightarrow 2$: Let $X(t) = (\theta_{-t})_*(W_{\theta_t(p)})$, $p \in M$.

Want to show that $X(t) = X_p$ for all t. Suffices to show $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} X(t) = 0$.

For $t_0 = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} X(t) = (\mathcal{L}_V W)_p = 0$.

In general, set $s = t - t_0$, $X(t) = (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))})$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} X(t) &= \frac{\mathrm{d}}{\mathrm{d}s}|_s X(s+t_0) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}|_s (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))}) \\
&= (\theta_{t_0})_* \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))}) \\
&= (\theta_{t_0})_* (\mathcal{L}_V W)_{\theta_{t_0}(p)} \\
&= 0
\end{aligned}$$

$2 \Rightarrow 3$. For simplicity, assume V, W are complete. $F = \theta_s : M \to M$. By 2, W is F-related to W. So by the lemma,

$$\begin{array}{ccc}
M & \xrightarrow{F} & M \\
\Big\downarrow{\scriptstyle \theta_t} & & \Big\downarrow{\scriptstyle \eta_t} \\
M & \xrightarrow{F} & M
\end{array} \quad \text{commutes.}$$

η_t is flow for W. *i.e.* $\theta_s \circ \eta_t = \eta \circ \theta_s$

$3 \Rightarrow 2$ is similar. The diagram commutes, so W is F-related to W. □

""""""

4.3.1 Canonical form of commuting vector field

Theorem 4.11. *Given $V_1, \cdots, V_k \in \mathfrak{X}M$, s.t.*

1) $[V_i, V_j] = 0, \forall i, j$.

2) $V_{1,p}, V_{2,p}, \cdots, V_{k,p}$ linearly independent at some $p \in M$

Then $\exists$ local chart $(U, x^1, \cdots, x^n)$ around p s.t. $V_i|_U = \frac{\partial}{\partial x^i}, \forall 1 \leqslant i \leqslant k$

We prove it using the inverse function theorem.

Proof. This is a local problem. So we may assume $M \subset \mathbb{R}^m$ be open with coordinate function $r^i : M \to \mathbb{R}, 1 \leqslant i \leqslant m$.

After translation and linear transformation, we may assume $p = \vec{0}$, $V_{i,\vec{0}} = \left.\frac{\partial}{\partial x^i}\right|_{\vec{0}}, 1 \leqslant i \leqslant k$.

Take local flow $\{\theta_t^i : (-\varepsilon, \varepsilon)^m \to M\}_{t \in (-\varepsilon,\varepsilon)}$ for V_i.

Define $\psi : (-\varepsilon, \varepsilon)^k \times (-\varepsilon, \varepsilon)^{m-k} \to M$, $\psi(t^1, \cdots, t^k, r^{k+1}, \cdots, r^m) = \theta^1_{t_1} \circ \theta^2_{t_2} \cdots \circ \theta^k_{t_k}(0, 0, \cdots, 0, r^{k+1}, \cdots, r^m)$, where θ^i commutes with each other.

So if we fix $t^j, j \neq i$ except t^i, $\psi(t^1, \cdots, t^{i-1}, -, t^{i+1}, \cdots, t^k, r^{k+1}, \cdots, r^m)$ is an integral curve for V^i. Then V^i is ψ-related to ∂t^i.

On the other hand. $\psi(0, 0, \cdots, 0, r^{k+1}, \cdots, r^m) = (0, 0, \cdots, 0, r^{k+1}, \cdots, r^m)$. So $\psi_{\vec{0},*} : T_{\vec{0},*} : T_{\vec{0}}(-\varepsilon', \varepsilon')^m \to T_{\vec{0}}M, \partial t^i \mapsto V_{i,0} = \partial x^i|_0$ and $\partial r^i \mapsto \partial r^i, k+1 \leqslant i \leqslant m$.

So $\psi_{\vec{0},*}$ is an isomorphism.

By the inverse function theorem, there exists Nbh $U' \subset (-\varepsilon', \varepsilon')^m$ *s.t.* $\psi : U' \to U$ is a diffeomorphism and $U \subset M$ open.

Then $(U, (\psi|_U)^{-1})$ is the local chart we need. □

4.4 The constant rank theorem

$F \in C^\infty(M, N)$, $p \in M$. The **rank** of F at p is

$$\begin{aligned} \operatorname{rank}_p F &:= \operatorname{rank}(F_{p,*} : T_pM \to T_{F(p)}N) \\ &= \operatorname{rank}\left(\frac{\partial F^i(p)}{\partial x^j}\right)_{i,j} \end{aligned}$$

We say F has **constant rank** k near p if $\exists$ Nbh U of p *s.t.* $\operatorname{rank}_q F = k$, $\forall q \in U$

Proposition 4.12.

$$\operatorname{rank}_q(F) \leqslant \min(\dim(M), \dim(N))$$

Theorem 4.13 (The constant rank theorem). *Suppose $F : M \to N$ has constant rank k near $p \in M$, then $\exists$ local charts $U \underset{\cong}{\xrightarrow{\varphi}} \mathbb{R}^m$ around p, $V \underset{\cong}{\xrightarrow{\psi}} \mathbb{R}^n$ around $F(p)$ s.t.*

$$\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \textit{ is given by } (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^k, 0, \cdots, 0)$$

Proof. This is a local problem. So we may assume $M = \mathbb{R}^m, N = \mathbb{R}^n$ by restricting to local charts. And $p = 0, F(p) = 0$. After changing orders of coordinates, may assume $\left(\frac{\partial F^i}{\partial x^j}(0)\right)_{1 \leqslant i,j \leqslant k}$ is invertible. Write $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}, \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Then $F(x,y) = (Q(x,y), R(x,y))$. Consider $\varphi : \mathbb{R}^m \to \mathbb{R}^m, (x,y) \mapsto (Q(x,y), y)$. Then

$$\varphi_{(0,0),*} = \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}(0) & 0 \\ & \\ \frac{\partial Q^i}{\partial y^j}(0) & I_{m-k} \end{bmatrix} \tag{4.1}$$

is invertible.

By inverse function theorem, $\exists$ Nbh $U_0 \subset \mathbb{R}^m$, $\tilde{U}_0 \subset \mathbb{R}^m$ of 0 *s.t.* $\varphi : U_0 \to \tilde{U}_0$ is a diffeomorphism.

$$\tilde{U}_0 \underset{\varphi}{\overset{\varphi^{-1}}{\rightleftarrows}} U_0 \xrightarrow{F} \mathbb{R}^n$$

$$(Q(x,y), y) \leftarrow\!\shortmid (x,y) \mapsto (Q(x,y), R(x,y))$$

So $F \circ \varphi^{-1} : \tilde{U}_i \to \mathbb{R}^n, (x,y) \mapsto (x, A(x,y))$. And

$$(F \circ \varphi^{-1})_{p,*} = \begin{bmatrix} I_k & 0 \\ & \\ \frac{\partial A}{\partial x}(p) & \frac{\partial A}{\partial y}(p) \end{bmatrix} \tag{4.2}$$

Since $\operatorname{rank}(F \circ \varphi^{-1})$ is k, $\dfrac{\partial A}{\partial y}(p) = 0$. *i.e.* $A(x, y) = A(x)$.

We can find a map $\psi : (x, y) \mapsto (x, y - A(x))$ in a smaller neighborhood of 0 by the inverse theorem similarly.

And $\psi \circ F \circ \varphi$ maps (x, y) to $(x, 0)$. So we end the proof. $\square$

Definition 4.14. $F \in C^\infty(M, N)$.

We say F is **submersion** if $F_{p,*}$ is surjective $\forall p \in M$.

We say F is **immersion** if $F_{p,*}$ is injective $\forall p \in M$.

We say F is **embedding** if F is immersion and F is a topological embedding.(i.e. $F : M \to F(M)$ is a homeomorphism)

If F is embedding(immersion resp.), we say M or $F(M)$ is an **embedded submanifold(immersed submanifold**, resp.) of N.

Denote $M \looparrowright N$ be the immersion. $M \hookrightarrow N$ be the embedding.

Example 4.15.

- There is an example $F : S^1 \to \mathbb{R}^2$ where F is an immersion but not an embedding.
- Projection $M \times N \to M$ is a submersion.
- $E \xrightarrow{p} B$ is a smooth vector bundle, then p is a submersion.
- $\gamma : \mathbb{R} \to M$ is an immersion $\Leftrightarrow \gamma'(t) \neq 0$, $\forall t$.
- There is an example $\gamma : \mathbb{R} \to \mathbb{R}^2$ is injective immersion but not an embedding
- $\gamma : \mathbb{R} \to \mathrm{T}^2 = \mathbb{R}/_{\mathbb{Z}} \times \mathbb{R}/_{\mathbb{Z}}$, $x \mapsto (x, cx)$, $c \notin \mathbb{Q}$ is injective immersion but not embedding.

Definition 4.16. For $F : X \to Y$, we say F is **proper** if for any compact set $K \subset N$, $F^{-1}(K)$ is compact.

Lemma 4.17. *X is compact, Y Hausdorff, then $F : X \to Y$ is proper.*

Proposition 4.18. *$F \in C^\infty(M, N)$ is an injective immersion, and F is proper. Then F is an embedding.*

Proof. $F : M \to F(M)$ is a closed map. □

Definition 4.19. For $F \in C^\infty(M, N)$.

$p \in M$ is called **regular point** if $F_{p,*} : T_pM \to T_{F(p)}N$ is surjective.

$p \in M$ is called **critical point** if $F_{p,*} : T_pM \to T_{F(p)}N$ is not surjective.

$q \in N$ is called **regular value** if $\forall p \in F^{-1}(q)$, p is a regular point.

$q \in N$ is called **critical value**(or **singular value**) if $\exists p \in F^{-1}(q)$, p is a critical point.

Theorem 4.20 (Sard). *Singular value has measure 0.*

Proof. We will not prove it in this lecture. □

Theorem 4.21. *M is an embedded submanifold of N if and only if $\forall p \in M \subset N$, $\exists$ local chart $(U, x^1, \cdots, x^n)$ around p of N s.t. $M \cap U = \{(x^1, \cdots, x^m, 0, \cdots, 0)\}$*

Proof. "$\Rightarrow$": $F : M \to N$ is embedding $\Rightarrow$ F has constant rank m. Apply constant rank theorem near p, and we finish the proof of "$\Rightarrow$"

The converse is trivial.

□

Theorem 4.22. *$F \in C^\infty(M, N)$, q is a regular value of F. Then $F^{-1}(q)$ is an embedded submanifold of M. And*

$$\forall p \in F^{-1}(q), T_pF^{-1}(q) = \ker(F_{p,*} : T_pM \to T_{F(p)}N)$$

Proof. q is regular value $\Rightarrow$ $\mathrm{rank}_p F = n$, $\forall p \in F^{-1}(q)$.

$\Rightarrow \text{rank}_{p'} F = n$, $\forall p'$ near p, since we know the rank of p' near p should not be less than that of p

So by the constant rank theorem, $F^{-1}(q)$ is a submanifold near p.

$\square$

Denote

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) | A + A^T = 0\}$$

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) | A + A^* = 0\}$$

$$\mathfrak{su}(n) = \{A \in u(n) | \text{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \text{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) | \text{tr} A = 0\}$$

Theorem 4.23. *Those above sets are the Lie algebra of the corresponding Lie group. For instance,* $\mathfrak{su}(n) = \text{Lie}(\text{SU}(n))$.

5 Differential forms

5.1 Introduction

Our goal is to define the integration $\displaystyle\int_M \alpha$ *s.t.*

- Works for any smooth manifold M, without embedding M into $\mathbb{R}^n$
- Generalize two types of surface integral, *i.e.* $\int_\Sigma f \mathrm{d}S$ and $\int_\Sigma f \mathrm{d}x \wedge \mathrm{d}y$

For Canton's idea, α is a "differential k-form" on M *s.t.*

- $\forall F \in C^\infty(N, M)$, $F^*(\alpha)$ is a k-form on N

- If $k = \dim M$, then $\int_M \alpha \in \mathbb{R}$

5.2 Alternating vector linear algebra

For $V_1, \cdots, V_n, W$ be $\mathbb{R}$-vector spaces, $f : V_1 \times \cdots \times V_n \to W$ is called **multi $\mathbb{R}$-linear** if

$$\begin{aligned} f(v_1, \cdots, v_{i-1}, av_i + bv_i', v_{i+1}, \cdots, v_n) = & af(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_n) \\ & + bf(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_n) \end{aligned} \tag{5.1}$$

Example 5.1.

- Inner product $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\cdot} \mathbb{R}$.
- Matrix multiplication $M_{n\times m}(\mathbb{R}) \times M_{m\times k}(\mathbb{R}) \to M_{n\times k}(\mathbb{R})$.
- Cross product $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\times} \mathbb{R}^3$.
- Bilinear form.

We hope that we can construct a vector space $V_1 \otimes \cdots \otimes V_n$ *s.t.* we have canonical isomorphism:

$$\{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V_n \to W\} \cong \{\text{linear map } V_1 \otimes \cdots \otimes V_n \to W\} \tag{5.2}$$

Then we can transform the study of multilinear algebra into the study of the normal linear algebra.

For any set S, let

$$\mathbb{R}\langle S\rangle = \left\{\text{formal linear combination } \sum_{i=1}^{n} a_i s_i | a_i \in \mathbb{R}, s_i \in S, n < \infty\right\} \tag{5.3}$$

Consider $\mathbb{R}\langle V_1 \times \cdots \times V_n \rangle = \left\{ \sum_{i=1}^{k} a^i (V_{i,1}, \cdots, V_{i,n}) | a^i \in \mathbb{R}, v_{i,j} \in V_j \right\}$. Denote

$$W = \mathrm{Span}\{(\cdots, av_i + bv_i', \cdots) - a(\cdots, v_i, \cdots) - b(\cdots, v_i', \cdots) | a, b \in \mathbb{R}, v_i, v_i' \in V_i\} \tag{5.4}$$

Define $V_1 \otimes \cdots \otimes V_n = \mathbb{R}\langle V_1 \times \cdots \times V_n \rangle /_W$, write $[(v_1, \cdots, v_n)]$ as $v_1 \otimes \cdots \otimes v_n$, called a n-**tensor**.

Proposition 5.2 (Universal Property). *We have a multi $\mathbb{R}$-linear map $q : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$, $(v_1, v_2, \cdots, v_n) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$. It satisfies the universal property:*

$\forall$ multi $\mathbb{R}$-linear map $f : V_1 \times \cdots \times V_n \to W$, $\exists$ unique linear map $\tilde{f} : V_1 \otimes \cdots \otimes V_n \to W$ s.t. $\tilde{f} \circ q = f$. i.e. The diagram commutes:

$$\begin{tikzcd}
V_1 \otimes \cdots \otimes V_n \arrow[dr, "\exists ! \tilde{f}"] & \\
V_1 \times \cdots \times V_n \arrow[u, "\rho"] \arrow[r, "f"] & W
\end{tikzcd}$$

Corollary 5.3.

$$\begin{gathered}
\{\textit{multi } \mathbb{R}\textit{-linear maps } V_1 \times \cdots \times V_n \to W\} \cong \{\textit{linear map } V_1 \otimes \cdots \otimes V_n \to W\} \\
f \leftrightarrow \tilde{f}
\end{gathered} \tag{5.5}$$

Proposition 5.4.

- *Any element in $V_1 \otimes \cdots \otimes V_n$ can be written as $\sum a_i v_i^1 \otimes \cdots \otimes v_i^n$ for some $a_i \in \mathbb{R}$.*
- *If $(e_i^j)_{j \in \mathcal{A}_i}$ is a basis for V_i, then $\{e_1^{j_1} \otimes e_2^{j_2} \otimes \cdots \otimes e_n^{j_n} | j_i \in \mathcal{A}_i\}$ is a basis of $V_1 \otimes \cdots \otimes V_n$.*
- $\dim(V_1 \otimes \cdots \otimes V_n) = \prod_{i=1}^{n} \dim(V_i)$

Proposition 5.5. *Denote $W^* = \mathrm{Hom}(W, \mathbb{R})$, then we have an injection*

$$\begin{aligned} V \otimes W^* &\xrightarrow{e} \mathrm{Hom}(W, V) \\ v \otimes f &\mapsto (w \mapsto f(w) \cdot v) \end{aligned} \tag{5.6}$$

If $\dim V$ or $\dim W$ is finite, then e is an isomorphism.

Indeed, if $\dim V = \infty$, then $\mathrm{id}_V \notin e(V \otimes V^*)$

Given any $l_i \in \mathrm{Hom}(V_i, W_i)$, $1 \leqslant i \leqslant n$, we define

$$\begin{aligned} l_1 \otimes \cdots \otimes l_n &\in \mathrm{Hom}(V_1 \otimes \cdots \otimes V_n, W_1 \otimes \cdots W_n) \\ (l_1 \otimes \cdots \otimes l_n)(v_1 \otimes \cdots \otimes v_n) &= l_1(v_1) \otimes \cdots \otimes l_n(v_n) \end{aligned} \tag{5.7}$$

Proposition 5.6. *If $\dim V_i < \infty$, $\forall 1 \leqslant i \leqslant n$, then we have isomorphism*

$$\begin{aligned} V_1^* \otimes \cdots \otimes V_n^* &\xrightarrow{\cong} (V_1 \otimes \cdots \otimes V_n)^* \\ f_1 \otimes \cdots \otimes f_n &\mapsto \left((v_1 \otimes \cdots \otimes v_n \mapsto \prod_{i=1}^n f_i(v_i)) \right) \end{aligned} \tag{5.8}$$

For $\bigotimes_n V = \underbrace{V \otimes \cdots \otimes V}_{n}$, $S_n = \{\text{bijection on } \{1, 2, \cdots, n\}\}$ acts on $\bigotimes_n V$, where

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \tag{5.9}$$

A tensor $T \in \bigotimes_n V$ is called **symmetric** if $\sigma(T) = T$, $\forall \sigma \in S_n$.
T is called **anti-symmetric** if $\sigma(T) = \mathrm{sgn}(\sigma) \cdot T$, $\forall \sigma \in S_n$.

Define

$$\begin{aligned} \mathrm{Sym}^n(V) &= \{\text{symmetric tensors in } \bigotimes_n V\} \\ \bigwedge{}^n(V) &= \{\text{anti-symmetric tensors in } \bigotimes_n V\} \end{aligned} \tag{5.10}$$

which are both in $\bigotimes_n V$. And

$$\dim(\mathrm{Sym}^n(V)) = \binom{\dim(V)+n-1}{n} \quad \dim(\bigwedge{}^n V) = \binom{\dim(V)}{n} \tag{5.11}$$

From now on, **we may assume** $\dim V < \infty$. Define

$$L^n(V) = \left(\bigotimes_n V\right)^* \cong \bigotimes_n V^* \cong \{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V \to \mathbb{R}\} \tag{5.12}$$

And by the assumption we can obtain

$$\begin{aligned} \mathrm{Sym}^n(V^*) &\cong \{\text{symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \to \mathbb{R}\} \\ \bigwedge{}^n(V^*) &\cong \{\text{anti-symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \to \mathbb{R}\} \end{aligned} \tag{5.13}$$

We will mainly focus on $\bigwedge^n(V^*)$, also denoted as $\mathrm{Alt}^k(V)$= $\bigwedge^n(V^*)$. An element in $\mathrm{Alt}^k(V)$ is called a (linear) k**-form** on V Now for $V = \mathbb{R}\langle e_1, \cdots, e_n\rangle$, $V^* = \mathbb{R}\langle e_1^*, \cdots, e_n^*\rangle$. Then

$$L^2(V) = \{\text{all bilinear forms on } V\}$$
$$L^2(V) \cong \mathrm{Sym}^2(V^*) \oplus \bigwedge{}^2(V^*)$$

And $\mathrm{Sym}^2(V^*) = \mathbb{R}\left\langle e_i^* \otimes e_j^* + e_j^* \otimes e_i^* | 1 \leqslant i \leqslant i \leqslant n\right\rangle$ is symmetric bilinear form

$\mathrm{Alt}^2(V) = \bigwedge^2(V^*) = \mathbb{R}\left\langle e_i^* \otimes e_j^* - e_j^* \otimes e_i^* | 1 \leqslant i \leqslant i \leqslant n\right\rangle$ is anti-symmetric bilinear form.

The determinant $\det \in \mathrm{Alt}^n(\mathbb{R}^n)$.

Definition 5.7 (Exterior product)**.**

$$\bigwedge : \mathrm{Alt}^k(V) \times \mathrm{Alt}^l(V) \to \mathrm{Alt}^{k+l}(V)$$

$$\omega_1 \wedge \omega_2(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\omega_2(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$
$$= \sum_{\sigma \in S_{k,l}} \mathrm{sgn}(\sigma)\omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\omega_2(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

where $S_{k,l} = \{\sigma \in S_{k+l} | \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\} \subset S_{k+l}$.

Then we have those properties:

Proposition 5.8.

(1) $\omega_1 \wedge \omega_2 = (-1)^{|\omega_1| \cdot |\omega_2|}\omega_2 \wedge \omega_1$, $|\omega| = k$ *is* $\omega \in \mathrm{Alt}^k(V)$. *In particular,* $\omega \wedge \omega = 0$ *if* $|\omega|$ *is odd.*

(2) $(\omega_1 \wedge \omega_2) \wedge w_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$

(3) *Given any* $\omega_1, \cdots, \omega_k \in \mathrm{Alt}^1(V) = V^*$, $v_1, \cdots, v_k \in V$. *Then*

$$(\omega_1 \wedge \cdots \wedge \omega_k)(v_1, \cdots, v_k) = \det \left[w_i(v_j)\right]_{i,j} \tag{5.14}$$

Moreover, $\omega_1 \wedge \cdots \wedge_n \neq 0$ *iff* ω_i *are linearly independent.*

(4) $V = \mathbb{R}\langle e_1, \cdots, e_n \rangle$. *Then*

$$\mathrm{Alt}^k(V) = \mathbb{R}\left\langle e^*_{i_1} \wedge \cdots \wedge e^*_{i_k} | i_1 < \cdots < i_k \right\rangle \tag{5.15}$$

In particular, $\mathrm{Alt}^n(V) = \mathbb{R}\langle e^*_1 \wedge \cdots \wedge e^*_n \rangle$. *And we denote* $\mathrm{Alt}^0(V) = \mathbb{R}$, $\mathrm{Alt}^k(V) = 0$, $k > n$.

(5) *Any* $f \in \mathrm{Hom}(V, W)$ *induces* $\mathrm{Alt}^k(f) \in \mathrm{Hom}(\mathrm{Alt}^k(V), \mathrm{Alt}^k(W))$, *where*

$$\mathrm{Alt}^k(f)(\omega)(w_1, \cdots, w_k) = \omega(f(w_1), \cdots, f(w_k)) \tag{5.16}$$

We have $\mathrm{Alt}^k(f \circ g) = \mathrm{Alt}^k(g) \circ \mathrm{Alt}^k(f)$, $\mathrm{Alt}^k(\mathrm{id}_V) = \mathrm{id}_{\mathrm{Alt}^k(V)}$. *Such* $\mathrm{Alt}^k(-)$ *is called a* ***contravariant functor****.*

Proof.

(1) By definition,

$$\omega_1 \wedge \omega_2(v_1, \cdots, v_{k+l}) = \omega_2 \wedge \omega_1(v_{\sigma(1)}), \cdots, v_{\sigma(k+l)}$$

where $\sigma(i) = \begin{cases} i+k & 1 \leqslant i \leqslant l \\ i-l & l+1 \leqslant i \leqslant k+l \end{cases}$. $\mathrm{sgn}(\sigma) = (-1)^{k+l}$.

(2) By definition.

(3) By linearity, we assume $\omega_i = e^*_{a(i)}, v_j = e_{b(j)}$ for some $a(i), b(j)$. Further more, can assume $\{a(i)\} = \{b(i)\}$. (Otherwise, $LHS = RHS = 0$.)

Then $e^*_{a(i)}(e_{b(j)}) = \delta_{a(i),b(j)}$. After permutation, may assume $a(i) = b(i), \forall i$. It is direct to check $LHS = 1 = RHS$.

(4) If $\omega_1, \cdots, \omega_k$ are linear independent. Then $\exists$ basis $e^*_1, \cdots, e^*_n$ of V^*, basis $e_1, \cdots, e_n$ of V *s.t.* $\omega_i = e^*_i$, $\forall 1 \leqslant i \leqslant n$.

$$(\omega_1 \wedge \cdots \wedge \omega_n)(e_1, \cdots, e_n) = \det(I) = 1 \neq 0 \Rightarrow \omega_1 \wedge \cdots \wedge \omega_n \neq 0$$

If $\omega_1, \cdots, \omega_k$ are linearly dependent. WLOG, we assume $\omega_k = \sum_{i=1}^{k-1} a_i \omega_i$.

$$(\omega_1 \wedge \cdots \wedge \omega_k)(e_1, \cdots, e_n) = \sum_{i=1}^{k-1} a_i (\omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega_i)(e_1, \cdots, e_n) = 0$$

(5) For $i_1 < \cdots < i_k, j_1 < \cdots < j_n$ we have

$$(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)(e_{j_1}, \cdots, e_{j_k}) = \begin{cases} 1 & j_t = i_t, \forall 1 \leqslant t \leqslant k \\ 0 & \text{otherwise} \end{cases} \tag{5.17}$$

Since $\dim \mathrm{Alt}(V) = \dim \bigwedge^k(V^*) = \binom{n}{k} = |\{e_{i_1} \wedge \cdots e_{i_k} | i_1 < \cdots < i_k\}|$.

(6) For $\omega \in \mathrm{Alt}^k(W), f \in \mathrm{Hom}(V, W)$, define $\mathrm{Alt}^k(f)(\omega) \in \mathrm{Alt}^k(V)$ by

$$\mathrm{Alt}^k(f)(\omega(V_1, \cdots, V_k)) = \omega(fV_1, \cdots, fV_k) \in \mathbb{R}$$

□

Definition 5.9.

An $\mathbb{R}$**-algebra** consists of an $\mathbb{R}$-vector space A with a bilinear map $\mu : A \times A \to A$ that is associate, *i.e.* $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$.

Say A is **unitary** if $\exists 1 \in A$ *s.t.* $\mu(a, 1) = \mu(1, a) = a, \forall a \in A$

Say A is **graded** if $A = \bigoplus\limits_{k \in \mathbb{Z}} A_k$ as vector space, and $\mu(A_k \times A_l) \subset A_{k+l}$. Elements in A_k are called **homogeneous elements** of degree k.

If A is graded $\mathbb{R}$-algebra, we say A is **anticommutative** if $\mu(a, b) = (-1)^{k+l}\mu(b, a), \forall a \in A_k, b \in A_l$. And say A is **commutative** if $\mu(a, b) = \mu(b, a), \forall a, b$.

If A is graded $\mathbb{R}$-algebra, say A is **connected** if $\exists$ unit $1 \in A_0$ *s.t.* the map $\varepsilon : \mathbb{R} \to A_0, r \mapsto r \cdot 1$ is an isomorphism.

Given vector space V, let

$$\begin{tikzcd} \mathrm{Alt}^k(V) \arrow[r, equal] \arrow[d, equal] & \bigoplus_{k \geqslant 0} \mathrm{Alt}^k(V) \arrow[d, equal] \\ \mathrm{Alt}^*(V^*) \arrow[r, equal] & \bigoplus_{k \geqslant 0} \wedge^k(V^*) \end{tikzcd}$$

By Proposition 5.8, we have the theorem

Theorem 5.10. $(\mathrm{Alt}^*(V), \wedge)$ *is a graded connected anticommutative $\mathbb{R}$-algebra, called the **exterior algebra** of V or **exterior algebra** of V*

5.3 Operation on vector bundles

Given $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$, meaning a vector bundle of dimension n, local trivialization $\left\{U_\alpha, \varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n\right\}_{\alpha\in\mathcal{A}}$. By shrinking U_α, we may assume we have an smooth atlas $\{\varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^m\}_{\alpha\in\mathcal{A}}$.

For $x \in M$, use E_x to denote $\pi^{-1}(x)$, fiber over x, which is a vector space of dimension n.

Then **Dual bundle of a vector bundle** $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$ is

$$E^* := \{(x,l)|x \in M, l \in (E_x)^*\}, \pi' : E^* \to M, (x,l) \mapsto x, (\pi')^{-1}(x) = (E_x)^* \quad (5.18)$$

Define topology or smooth structure on E^* *s.t.* $\pi' : E^* \to M$ is a smooth vector bundle.

For $\alpha \in \mathcal{A}$, let $E^*_\alpha = \pi'^{-1}(U_\alpha)$, we have a bijection

$$\tilde{\varphi_\alpha} : E^*_\alpha \xrightarrow{bijection} \mathbb{R}^m \times (\mathbb{R}^n)^* \xrightarrow{\quad\cong\quad} \mathbb{R}^{m+n}$$

$$(x,l) \longmapsto (\psi_\alpha(x), (\varphi_{\alpha,x})^{-1}(l))$$

We can check that

(1) $\{\tilde{\varphi_\alpha}^{-1}|\alpha \in \mathcal{A}, V \subset \mathbb{R}^{m+n} \text{ open}\}$ is a basis, we use it to generate a topology on E^*.

(2) Use $\tilde{\varphi_\alpha} : E^*_\alpha \xrightarrow{\cong} \mathbb{R}^{m+n}, \alpha \in \mathcal{A}$ as an atlas to give E^* a smooth structure.

(3) $E^* \xrightarrow{\pi'} M$ is a smooth vector bundle, called the **dual vector bundle** of $E \xrightarrow{\pi}$

M, where

$$(E^*)_x = E_x^*$$

We can define other operations on vector bundles in similar way:

Given $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$, $\mathbb{R}^m \hookrightarrow F \xrightarrow{\pi} M$, we can define

$$\mathbb{R}^{m+n} \hookrightarrow E \oplus F \xrightarrow{\pi} M \text{ with } (E \oplus F)_x = E_x \oplus F_x$$

$$\mathbb{R}^{mn} \hookrightarrow E \otimes F \xrightarrow{\pi} M \text{ with } (E \otimes F)_x = E_x \otimes F_x$$

$$\mathbb{R}^{mn} \hookrightarrow \mathrm{Hom}(E, F) \xrightarrow{\pi} M \text{ with} \mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x)$$

$$\mathbb{R}^{\binom{n}{k}} \hookrightarrow \mathrm{Alt}^k(E) \to M \text{ with}$$

$$\mathrm{Alt}^k(E)_x = \mathrm{Alt}^k(E_l) = \{\text{alternating } k\text{-linear } l : E_x \times \cdots \times E_x \to \mathbb{R}\}$$

Then $\mathrm{Alt}^k(TM) = \bigwedge^k(T^*M)$.

$\mathrm{Alt}^k(M)_x = \{\text{alternating } k\text{-linear } l : T_xM \times \cdots \times T_xM \to \mathbb{R}\} = \{\text{linear } k\text{-form on } T_xM\}$

Define

$$\Gamma(E) := \{\text{smooth sections of } E\} = \{s \in C^\infty(M, E) : \pi \circ s = \mathrm{id}_M\}$$

Definition 5.11. Given smooth M, define a differential k-form on M to be an element in $\Gamma(\mathrm{Alt}^k(TM))$ is a differential k-form α assigns each $x \in M$ a linear k-form $\alpha(x) \in \mathrm{Alt}^k(T_xM)$.

Denote $\Omega^k(M)$ be the set of all the differential k-forms.

Then $\Omega^0(M) = C^\infty(M, \mathbb{R})$. $\mathrm{Alt}^1(TM) = T^*M \Rightarrow$ a 1-form on M is just a "cotangent vector field" on M.

$\Omega^k(M) = 0$ if $k \geqslant \dim(M)$.

5.4 Differential forms using local chart

Given local chart $(U, x^1, \cdots, x^n)$ of M.

For any $p \in U$, $\{\frac{\partial}{\partial x^i}|_p\}_{1 \leqslant i \leqslant n}$ is a basis of $T_x M$.

We denote the dual basis of $T_x^* M$ by $\{\mathrm{d}x^i|_p\}_{1 \leqslant i \leqslant n}$.

For any $\alpha \in \Omega^1(M)$, $\alpha|_U$ can be written as $\sum\limits_{i=1}^{n} f_1 \mathrm{d}x^i$, where $f^i \in C^\infty(U, \mathbb{R})$.

Similarly, $\{\mathrm{d}x^{i_1}|_1 \wedge \cdots \wedge \mathrm{d}x^{i_k}|_p | i_1 < \cdots < i_k\}$ is a basis for $\bigwedge^k(T_x^* M)$, so $\forall \alpha \in \Omega^k(M)$,

$$\alpha|_U = \sum_{i_1 < \cdots < i_k} f_{i_1, \cdots, i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},\ f_{i_1, \cdots, i_k} \in C^\infty(U, \mathbb{R})$$

We give the notation that $I = (i_1, \cdots, i_k)$, write $f_{i_1, \cdots, i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ as $f^I \mathrm{d}x^I$.

Change of coordinate If $(U, x^1, \cdots, x^n)$ and $(V, y^1, \cdots, y^n)$ two charts of M and $p \in U \cap V$, then

$$\mathrm{d}y^i = \sum_{1 \leqslant i \leqslant n} \frac{\partial y_i}{\partial x^i} \mathrm{d}x^i. \tag{5.19}$$

5.5 Exterior differential

For $k = 0$, define $d : \Omega^0(M) \to \Omega^1(M)$ as follows:

$\forall p \in M$, $X_p \in T_p M$, $\mathrm{d}f|_p(X_p) = X_p(f) \in \mathbb{R}$. In local chart, $\mathrm{d}f = \sum\limits_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x^i$.

Theorem 5.12. *$\exists$ linear operator $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ s.t. For $\alpha \in \Omega^k(M)$,*

$$\alpha|_U = \sum_I f^I \mathrm{d}x^I \Rightarrow \mathrm{d}\alpha|_U = \sum_I \mathrm{d}f^I \wedge \mathrm{d}x^I \tag{5.20}$$

Called the ***exterior differential***

Proof. It suffices to prove that (5.20) is compatible for two charts $(U, x^1, \cdots, x^n), (V, y^1, \cdots, y^n)$, *i.e.* the diagram is commutative.

$$\begin{tikzcd}
f\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^k \arrow[r, leftrightarrow] \arrow[d, "\mathrm{d}"] & \sum\limits_{1\leqslant i_1, i_2, \cdots, i_k \leqslant n} f\frac{\partial y^{i_1}}{\partial x^1} \cdots \frac{\partial y^{i_k}}{\partial x^k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}, \arrow[d, "d", "?"'] \\
\sum\limits_{1\leqslant i \leqslant n} \frac{\partial f}{\partial y^i}\mathrm{d}y^i \wedge \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k} \arrow[r, leftrightarrow, "?"] & \sum\limits_{1\leqslant i_1, i_2, \cdots, i_k \leqslant n} \frac{\partial f}{\partial x^j} \cdot \frac{\partial y^{i_1}}{\partial x^1} \cdots \frac{\partial y^{i_k}}{\partial x^k} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}
\end{tikzcd}$$

□

Theorem 5.13.

(1) $\mathrm{d}^2 = 0$.

(2) $\forall \alpha \in \Omega^k(M), \beta \in \Omega^l(M)$, $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$.

Proof.

(1) If $\alpha|_U = \sum\limits_U f^I \mathrm{d}x^I$. By linearity suffices to check

$$\begin{aligned}
\mathrm{d} \circ \mathrm{d}(f\mathrm{d}x^I) &= \mathrm{d}\left(\sum_{1\leqslant i\leqslant n} \frac{\partial f}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^I\right) \\
&= \sum_{\substack{1\leqslant i\leqslant n \\ 1\leqslant j\leqslant n}} \frac{\partial^2 f}{\partial x^i}\mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I \\
&= 0
\end{aligned}$$

(2) By linearity, suffices to assume $\alpha = f\mathrm{d}x^I$, $\beta = g\mathrm{d}x^I$.

$$\begin{aligned}
\mathrm{d}(\alpha \wedge \beta) &= \mathrm{d}(fg\mathrm{d}x^I \wedge x^J) \\
&= \sum_{1\leqslant i\leqslant n} \frac{\partial(fg)}{\partial x^i}\mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{1\leqslant i\leqslant n} \left(\frac{f}{\partial x^i}g + f\frac{\partial g}{\partial x^i}\right)\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J
\end{aligned}$$

And

$$\mathrm{d}\alpha \wedge \beta = \sum_i \frac{\partial f}{\partial x^i} g \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J$$

$$\alpha \wedge \mathrm{d}\beta = \sum_i \frac{\partial g}{\partial x^i} f \mathrm{d}x^I \wedge \mathrm{d}x^i \wedge \mathrm{d}x^J = \sum_i (-1)^k \frac{\partial g}{\partial x^i} f \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J$$

□

Example 5.14. For $M = \mathbb{R}^3$,

$$\begin{tikzcd}
\Omega^0(\mathbb{R}^3) \arrow[r, equal] \arrow[d, "d"] & C^\infty(\mathbb{R}^3) & & \\
\Omega^1(\mathbb{R}^3) \arrow[r, leftrightarrow] \arrow[d, "d"] & \mathfrak{X}\mathbb{R}^3, \arrow[d, "\text{curl}"] & f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z \arrow[r, leftrightarrow] & f\partial x + g\partial y + h\partial z \\
\Omega^2(\mathbb{R}^3) \arrow[r, leftrightarrow] \arrow[d, "d"] & \mathfrak{X}\mathbb{R}^3, \arrow[d, "\text{divergent}"] & f\mathrm{d}x \wedge \mathrm{d}y + g\mathrm{d}x \wedge \mathrm{d}z + h\mathrm{d}y \wedge \mathrm{d}z \arrow[r, leftrightarrow] & f\partial z + g\partial x + h\partial y \\
\Omega^3(\mathbb{R}^3) \arrow[r, leftrightarrow] & C^\infty(\mathbb{R}^3), & f\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \arrow[r, leftrightarrow] & f
\end{tikzcd}$$

5.6 Pull back of differential forms

For $F \in C^\infty(M, N)$, $\alpha \in \Omega^k(N)$, define the **pullback** $F^*(\alpha) \in \Omega^k(M)$ as follows:

$$\forall p \in M, V_1 \cdots, V_k \in T_pM, F^*(\alpha)|_p(V_1, \cdots, V_k) = \alpha|_{F(p)}(F_{p,*}(V_1), \cdots, F_{p,*}(V_k)) \in \mathbb{R}$$

Actually, $F^*(\alpha) = \mathrm{Alt}^k(F_{p,*}) : \mathrm{Alt}^k(T_{F(p)N}) \to \mathrm{Alt}^k(T_pM)$.

Proposition 5.15. *For $F : M \to N, G : N \to L$.*

(1) $f \in \Omega^0(N)$, $F^*(f) = f \circ F \in \Omega^0(M)$.

(2) $F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$.

(3) $F^*(\mathrm{d}\alpha) = \mathrm{d}F^*(\alpha)$.

(4) $(G \circ F)^* = F^* \circ G^*$

Proof.

(1)
$$\begin{array}{ccc} \mathrm{Alt}^0(T_{F(p)}N) & \xrightarrow{\mathrm{Alt}^0(F_{p,*})} & \mathrm{Alt}^0(T_pM) \\ \| & & \| \\ \mathbb{R} & \xrightarrow{id} & \mathbb{R} \end{array}$$
commutes.

(2)
$$\begin{array}{ccc} \mathrm{Alt}^k(T_{F(p)N}) \times \mathrm{Alt}^l(T_{F(p)}N) & \xrightarrow{\wedge} & \mathrm{Alt}^{k+l}(T_pM) \\ \downarrow{\scriptstyle \mathrm{Alt}^k(F_{p,*})\times\mathrm{Alt}^l(F_{p,*})} & & \downarrow{\scriptstyle \mathrm{Alt}^{k+l}(F_{p,*})} \\ \mathrm{Alt}^k(T_pM) \times \mathrm{Alt}^l(T_pM) & \xrightarrow{\wedge} & \mathrm{Alt}^{k+l}(T_pM) \end{array}$$
commutes.

(3) By linearity it suffices to check

$$\mathrm{d}F^*(f\mathrm{d}x^I) = F^*\mathrm{d}(f\mathrm{d}x^I)$$

By Leibniz rule for d and (2), it suffices to show

(a) $\mathrm{d}F^*(\mathrm{d}f) = F^*(\mathrm{d}f)$

(b) $\mathrm{d}F^*(\mathrm{d}x^i) = F^*(\mathrm{d}(\mathrm{d}x^i))$

Which leaves to the readers.

(4) By definition.

□

Definition 5.16. A k-form ω is **closed** if $\omega \in \ker\left(\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)\right)$.

A k-form ω is **exact** if there exists a $(k-1)$-form η such that $d\eta = \omega$, or equivalently, $\omega \in \mathrm{Im}\left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M)\right)$.

By Proposition 5.15 (1), exact k-form are all closed.

So we may define the k-th **de Rham cohomology** of M

$$H^k_{\mathrm{DR}}(M) := \frac{\ker\left(d : \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)\right)}{\mathrm{Im}\left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M)\right)} \tag{5.21}$$

By Proposition 5.15 (2), we have $\forall F \in C^\infty(M,N)$, $\omega \in \Omega^k(N)$.

Then ω closed $\Rightarrow F^*\omega$ is closed. ω exact $\Rightarrow F^*\omega$ exact.

So F induces a linear map

$$\begin{aligned} F^* : H^k_{\mathrm{DR}}(N) &\to H^k_{\mathrm{DR}}(M) \\ [\omega] &\mapsto [F^*\omega] \end{aligned}$$

Proposition 5.17 (Key properties of $H^k_{\mathrm{DR}}(M)$)**.**

(1) $(F \circ G)^* = G^* \circ F^*$

(2) $(\mathrm{id})^* = \mathrm{id}.$

(3) $F, G \in C^\infty(M,N)$, *F homotopic to G* $\Rightarrow F^* = G^*$

(4) If F is a homotopy equivalence $\Rightarrow F^* : H^k_{\mathrm{DR}}(M) \to H^k_{\mathrm{DR}}(N)$ *is an isomorphism.*

Remark 5.18. Properties (3),(4) are nontrivial, which is the essential part of the theory of de Rham cohomology

Proposition 5.19. $H^k_{\mathrm{DR}}(M) \cong \mathbb{R}\langle \pi_0(M)\rangle$, *where* $\pi_0(M) = \{$*path component of* $M\}$.

It suffices to prove the lemma that

Lemma 5.20. $\alpha \in \Omega^0(M) = C^\infty(M, n\mathbb{R})$. *Then α is closed iff α is constant on each component of M.*

Proof. The inverse part is trivial.

Assume α is closed. Pick $p, q \in M$ in some path component. $\exists$ smooth path $\gamma : \mathbb{R} \to M$, $\gamma(0) = p, \gamma(1) = q$.

$\mathrm{d}\,\alpha = 0 \Rightarrow \mathrm{d}(\gamma^*\alpha) = 0 \Rightarrow \mathrm{d}(\alpha\circ\gamma) = 0 \Rightarrow \dfrac{\mathrm{d}(\alpha\circ\gamma)}{\mathrm{d}\,t} = 0 \Rightarrow \alpha\circ\gamma(1) = \alpha\circ\gamma(0).$

So $\alpha(p) = \alpha(q)$ $\square$

We have $H^k_{\mathrm{DR}}(M) \cong \mathrm{Ab}(\pi_1(M)) \otimes_{\mathbb{Z}} \mathbb{R}$. $\mathrm{Ab}(\pi_1(M))$ is the Abelian group of $\pi_1(M)$. In particular, $H^1_{\mathrm{DR}}(\mathbb{R}^2) = 0$, $H^1_{\mathrm{DR}}(\mathbb{R}^2\backslash\{0\}) \neq 0$.

Let us stop the discussion of de Rham cohomology for a moment, and move on to the next topic.

6 Integration of differential form

6.1 Orientation on manifold

An orientation on a finite dimensional vector space V is an equivalent class of ordered basis

$$\alpha = (\alpha_1, \cdots, \alpha_n)^T \sim \beta = (\beta_1, \cdots, \beta_n)^T \Leftrightarrow \det(\alpha\beta^T)$$

Each vector space has exactly two orientations. And we actually have the 1-1 correspondence

$$\{\text{orientation on } V\} \leftrightarrow (\mathrm{Alt}^n(V)\backslash\{0\})/_{\mathbb{R}^+}$$

$$[(e_!, \cdots, e_n)] \leftrightarrow [e_1^* \wedge \cdots \wedge e_n^*]$$

An **orientation form** on M of dimension n is a nowhere vanishing $\omega \in \Omega^n(M)$ *i.e.* an orientation form is a nowhere vanishing section of $\mathrm{Alt}^n(TM)$.

Two orientation forms ω_1, ω_2 are equivalent if $\exists f \in C^\infty(M, \mathbb{R}^+)$ *s.t.* $\omega_1 = f\omega_2$. An **orientation** on M is an equivalent class of orientation form.

An **orientation manifold** is a manifold that has an orientation.

An **oriented manifold** is a manifold equipped with an orientation.

Example 6.1. $|\pi_0(M)| = k \Rightarrow M$ has 2^k orientations or no orientations.

Example 6.2.

1. $U \subset \mathbb{R}^n$ open. U has a standard orientation, represented by the form $\mathrm{d}\,x^1 \wedge \cdots \wedge \mathrm{d}\,x^n$.

2. $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ oriented manifolds $\Rightarrow (M \times N, \mathcal{O}_M \times \mathcal{O}_N)$. If $\mathcal{O}_M = [\omega_M], \mathcal{O}_N = [\omega_N]$, then $\mathcal{O}_M \times \mathcal{O}_N$ is defined by $[\pi_M^*(\omega_M) \wedge \pi_N^*(\omega_N)]$. ($\pi_M, \pi_N$ is the pullback of the projection map)

3. T^n, S^n are orientable.

4. $\mathbb{RP}^n$ orientable iff n is odd.

Proposition 6.3. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M. Suppose we have an orientation $\mathcal{O}_\alpha$ on each U_α s.t. $\mathcal{O}_\alpha|_{U_\alpha \cap U_\beta} = \mathcal{O}_\beta|_{U_\alpha \cap U_\beta}$, $\forall \alpha, \beta$. Then $\exists$ unique orientation $\mathcal{O}_M$ on M s.t. $\mathcal{O}_M|_{U_\alpha} = \mathcal{O}_\alpha$.*

Proof. For each α, we have $\omega_\alpha \in \Omega^n(U_\alpha)$ nowhere-vanishing. And

$$\omega_\alpha|_{U_\alpha \cap U_\beta} = f_{\alpha\beta} \cdot \omega_\beta|_{U_\alpha \cap U_\beta},\ f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}^+ \tag{6.1}$$

Take partition of unity subordinate to $\mathcal{U}$, $\{\varphi_\alpha\}$.

Set $\omega = \sum_\alpha \varphi_\alpha \cdot \omega_\alpha$. Then ω is nowhere-vanishing by (6.1).

The uniqueness follows from the fact that n-form is equivalent if and only if it is equivalent on each chart. □

Definition 6.4. Given $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$. $f \in \mathrm{Diff}(M, N)$. Say f is **orientation preserving** if $f^*(\mathcal{O}_N) = \mathcal{O}_M$. f is **orientation reversing** if $f^*(\mathcal{O}_N) = -\mathcal{O}_M$.

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List of Theorems