In this section we compute the ordinary cohomology and the compactly supported cohomology of $\mathbb{R}^n$.

4.1 The Poincaré Lemma for de Rham Cohomology

Let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the projection on the first factor and $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}, x \mapsto (x, 0)$ be the zero section.
Trivially, $s^* \circ \pi^* = 1$. We now to prove $\pi^* \circ s^*$ is the identity in cohomology $H^*(\mathbb{R}^n \times \mathbb{R})$. It is enough to find a map K on $\Omega^*(\mathbb{R}^n \times \mathbb{R})$ such that

$$1 - \pi^* \circ s^* = \pm(d \circ K \pm K \circ d)$$

Easy to find that $dK \pm Kd$ maps closed forms to exact forms and therefore induces zero in cohomology. Such a K is called a <u>homotopy operator</u>. if it exists, we say that $\pi^* \circ s^*$ is <u>chain homotopic</u> to the identity.
Note that the homotopy operator K decreases the degree by 1.
We will use $\mathrm{d}f$ as we define before ($\sum\limits_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i$) and $\int g$ for $\int g(x,t)\,\mathrm{d}t$.
Every form on $\mathbb{R}^n \times \mathbb{R}$ is uniquely a linear combination of the folowing two types of forms:

$$\begin{aligned}&(1)\,(\pi^*\phi)f(x,t),\\&(2)\,(\pi^*\phi)f(x,t)\mathrm{d}t\end{aligned}$$

where ϕ is a form on the base $\mathbb{R}^n$.
We define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$ by

$$\begin{aligned}&(1)\,(\pi^*\phi)f(x,t) \mapsto 0,\\&(2)\,(\pi^*\phi)f(x,t)\mathrm{d}t \mapsto (\pi^*\phi)\int_0^t f.\end{aligned}$$

Let's check K is indeed a homotopy operator.

On the forms of type (1):

$$\omega = (\pi^*\phi)\cdot f(x,t), \qquad \deg\omega = q$$
$$(1-\pi^*\circ s^*)\omega = (\pi^*\phi)\cdot f(x,t) - \pi^*\phi\cdot f(x,0),$$
$$(dK-Kd)\omega = -Kd\omega = -K\left((d\pi^*\phi)f + (-1)^q\pi^*\phi(\mathrm{d}f + \frac{\partial f}{\partial t}\mathrm{d}t)\right)$$
$$= (-1)^{q-1}\pi^*\phi\int_0^t \frac{\partial f}{\partial t} = (-1)^{q-1}\pi^*\phi(f(x,t)-f(x,0))$$

Thus,

$$(1-\pi^*\circ s^*)\omega = (-1)^{q-1}(dK-Kd)\omega$$

On forms of type (2),

$$\omega = (\pi^*\phi)f(x,t)dt, \qquad \deg\omega = q$$
$$\mathrm{d}\omega = (\pi^*\mathrm{d}\phi)fdt + (-1)^{q-1}(\pi^*\phi)\mathrm{d}f\mathrm{d}t.$$
$$(1-\pi^*s^*)\omega = \omega \text{ because } s^*(\mathrm{d}t) = 0$$
$$K\mathrm{d}\omega = (\pi^*\mathrm{d}\phi)\int_0^t f + (-1)^{q-1}(\pi^*\phi)\int_0^t \mathrm{d}f$$
$$\mathrm{d}K\omega = (\pi^*\mathrm{d}\phi)\int_0^t f + (-1)^{q-1}(\pi^*\phi)\left[\mathrm{d}\int_0^t f + f\mathrm{d}t\right].$$

Thus

$$(1-\pi^*\circ s^*)\omega = (-1)^{q-1}(dK-Kd)\omega$$

In this case,

$$1-\pi^*\circ s^* = (-1)^{q-1}(\mathrm{d}K - K\mathrm{d}) \qquad \text{on} \quad \Omega^q(\mathbb{R}^n\times\mathbb{R})$$

This proves

Proposition 4.1. *The maps $H^*(\mathbb{R}^n\times\mathbb{R}) \leftrightarrows H^*(\mathbb{R}^n)$ are isomorphisms.*

Corollary 4.1.1 (Poincará Lemma)**.**

$$H^*(\mathbb{R}^n) = H^*(point) = \begin{cases}\mathbb{R} & \text{in dimension } 0\\ 0 & \text{elsewhere}\end{cases}$$

Similarly, we can show that $H^*(\mathbb{R}^n\times\mathbb{R}) \simeq H^*(\mathbb{R}^n)$ is an isomorphism via π^* and s^* defined before.

Corollary 4.1.2. *(Homotopy Axiom for de Rham cohomology) Homotopic maps induce the same map in cohomology.*

证明. Let $f = F \circ s_1, g = F \circ s_0$, where s_0 and $s_1 : M \to M \times \mathbb{R}$ are the 0-section and 1-section.
Then $f^* = (F \circ s_1)^* = s_1^* \circ F^*, g^* = (F \circ s_0)^* = s_0^* \circ F^*$.
Since s_1^* and S_0^* both invert π^* in $H^*(\mathbb{R}^n \times \mathbb{R})$, they are equal. Hence

$$f^* = g^*$$

. □

Two manifolds M and N are said to have the same homotopy type in the C^∞ sense if there are C^∞ maps $f : M \to N$ and $g : N \to M$ such that $g \circ f$ and $f \circ g$ are C^∞ homotopic to the identity on M and N respectively. A manifold having the homotopy type of a point is said to be contratible.

Corollary 4.1.3. *Two manifolds with the same homotopy type have the same de Rham cohomology.*

If $i : A \subset M$ is the inclusion $r : M \to A$ is a map which restrict to the identity on A, then r is called a retraction of M onto A. Equivalently, $r \circ i : A \to A$ is the identity. If in addition $i \circ r : M \to M$ is homotopic to the identity on M, then r is said to be a deformation retraction of M onto A. In this case we have:

Corollary 4.1.4. *If A is a deformation retract of M, then A and M have the same homotopy type.*

Exercise 4.2. Show that $r : \mathbb{R}^2 - \{0\} \to S^1$ given by $r(x) = \frac{x}{||x||}$ is a deformation retraction.

Exercise 4.3. The cohomology of the n-sphere S^n. Cover S^n by two open sets U and V where $U \cap V$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$. Using the Mayer Vietoris sequence, show that

$$H^*(S^n) = \begin{cases} \mathbb{R} & \text{in dimensions 0,n} \\ 0 & \text{otherwise} \end{cases}$$

Exercise 4.3.1 Volume form on a sphere. Let $S^n(r)$ be the spere of radius r

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2$$

in $\mathbb{R}^{n+1}$, and let

$$\omega = \frac{1}{r}\sum_{i=1}^{n+1}(-1)^{i-1}dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}$$

- Compute the integral $\int_{S^n} \omega$ and conclude that ω is not exact.
- Regarding r as a function on $\mathbb{R}^{n+1}-0$, show that $(dr)\cdot\omega = dx_1 \cdots dx_{n+1}$. Thus ω is the Euclidean volume form on the sphere $S^n(r)$.

From (a) we obtain an explicit formula for the generator of the top cohomology of S^n. For example, the generator of $H^2(S^2)$ is represented by

$$\sigma = \frac{1}{4\pi}(x_1 dx_2 dx_3 - x_2 dx_1 dx_3 + x_3 dx_1 dx_2)$$

4.2 The Poincaré Lemma for Compactly Supported Cohomology

The computation of the compactly supported cohomology $H^*(\mathbb{R}^n)$ is again by induction; we will show that there is an isomorphism

$$H_c^{*+1}(\mathbb{R}^n \times \mathbb{R}) \simeq H_c^*(\mathbb{R}^n)$$

But the dimension is shifted by one.

More generally consider the projection $\pi : M \times \mathbb{R} \to M$. Pull back of a form necessarily has no compact support, However, there is a push-forward map $\pi_* : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^*(M)$, called integration along fiber, defined as folows. First note that a compactly supported form on $M \times \mathbb{R}$ is a linear conbination of two types of forms:

$$(1)\ (\pi^*\phi)f(x,t),$$

$$(2)\ (\pi^*\phi)f(x,t)\mathrm{d}t$$

where ϕ is a form on the base (not necessarily with compact support), and f(x,t) is a function with compact support. We define π_* by

$$\begin{aligned}&(1)\ (\pi^*\phi)f(x,t) \mapsto 0,\\ &(2)\ (\pi^*\phi)f(x,t)\mathrm{d}t \mapsto \phi\int_{-\infty}^{\infty} f(x,t)\,\mathrm{d}t.\end{aligned} \tag{4.4}$$

Exercise 4.5 $d\pi_* = \pi_* d$; in other words, π_* is a chain map.

By the exercise π_* induce a map in cohomopogy $\pi_* : H_c^* \to H_c^{*-1}$. To produce a map in the reverse direction, let $e = e(t)dt$ be a compactly supported 1-form on $\mathbb{R}$ with total integral 1 and define

$$e_* : \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R})$$

by

$$\phi \mapsto (\pi^*\phi) \wedge e$$

The map e_* clearly commutes with d, so it also induces a map in cohomology. And $\pi_* \circ e_* = 1$ on $\Omega_c^*(\mathbb{R}^n)$ is trivial. Now we shall produce a homotopy operator K between 1 and $e_* \circ \pi_*$; then it will follow that $e_* \circ \pi_* = 1$ in cohomology.

To streamline the notation, write ϕ for $\pi^*\phi$ (in right place) and $\int f$ for $\int f(x,t)dt$. Then $K : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M \times \mathbb{R})$ is defined by

$$\begin{aligned}&(1)\phi \cdot f \mapsto 0,\\ &(2)\phi \cdot f\, dt \mapsto \phi\int_{-\infty}^{t} f - \phi A(t)\int_{-\infty}^{\infty} f \qquad \text{where } A(t) = \int_{-\infty}^{t} e\end{aligned}$$

Proposition 4.6. *$1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$ on $\Omega_c^q(M \times \mathbb{R})$*

证明. On forms of type (1), assuming $\deg\phi = q$, we have:

$$\begin{aligned}(1 - e_*\pi_*)\phi f &= \phi f\\ (dK - Kd)\phi f &= -K(d\phi f + (-1)^q\phi df + (-1)^q\phi\frac{\partial f}{\partial t}dt)\\ &= (-1)^{q-1}\phi f \quad \text{Here} \int_{-\infty}^{\infty}\frac{\partial f}{\partial t} = f(x,+\infty) - f(x,-\infty) = 0\end{aligned}$$

So $1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$

On forms of type(2), now assuming $\deg\phi = q-1$, we have

$$(1 - e_*\pi_*)\phi f\,dt = \phi f\,dt - \phi(\int_{-\infty}^{\infty} f)e$$

$$\begin{aligned}
(dK)(\phi f\,dt) &= (d\phi)\int_{-\infty}^{t} f + (-1)^{q-1}\phi d(\int_{-\infty}^{t} f) + (-1)^{q-1}\phi f dt \\
&- (d\phi)A(t)\int_{-\infty}^{\infty} f - (-1)^{q-1}\phi\left[e\int_{-\infty}^{\infty} f + A(t)(\int_{-\infty}^{\infty} df)\right] \\
(Kd)(\phi f\,dt) &= K((d\phi)f\,dt + (-1)^{q-1}\phi df\,dt) \\
&= (d\phi)\int_{-\infty}^{t} f - (d\phi)A(t)\int_{-\infty}^{\infty} f + (-1)^{q-1}\left[\phi(\int_{-\infty}^{t} df) - \phi A(t)(\int_{-\infty}^{\infty} df)\right]
\end{aligned}$$

So

$$(dK - Kd)\phi\, f\,dt = (-1)^{q-1}[\phi f\,dt - \phi(\int_{-\infty}^{\infty} f)e]$$

and the formula again holds. □

This conclude the proof of the folowing

Proposition 4.7. *The maps*

$$H_c^*(M \times \mathbb{R}) \rightleftarrows H_c^{*-1}(M)$$

are isomorphisms.

Corollary 4.7.1 (Poincaré Lemma for Compact Supports)**.**

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n \\ 0 & \text{otherwise} \end{cases}$$

Here the isomorphism $H_c^*(\mathbb{R}^n) \simeq \mathbb{R}$ is given by iterated π_*, i.e. by integration over $\mathbb{R}^n$. To determine a generator for $H_c^n(\mathbb{R})$, we start with the constant function 1 on a point and iterated with e_*. So a generator for $H_C^*(\mathbb{R}^n)$ is a bump n-form $\alpha(x)\,dx_1 \cdots dx_n = 1$ with

$$\int_{\mathbb{R}^n} \alpha(x)\,dx_1 \cdots dx_n = 1$$

The support of α can be made as small as we like.

Remark 1. *It shows that the compactly supported cohomology is not invariant under homotopy equivalence, although it is of course invariant under diffeomorphisms.*

Exercise 4.8. *Compute the cohomology groups $H^*(M)$ and $H_c^*(M)$ of the open Möbius strip M.*

4.3 The degree of a Proper Map

对光滑流形 X, 记 $C^\infty(X)$ 为所有光滑函数 $f\colon X \to \mathbb{R}$ 构成的实数向量空间.

Definition 4.3.1 (切空间)**.** 设 X 是光滑流形, 设 $x \in X$. 则

- X 在 x 处的**切向量**是指一个线性函数
$$v\colon C^\infty(X) \to \mathbb{R},$$
满足对任意 $f, g \in C^\infty(X)$, 有 *Leibniz* 法则
$$v(fg) = f(x) \cdot v(g) + g(x) \cdot v(f).$$
这里, 我们把 $v(f)$ 理解成函数 f 在点 x 处沿着向量 v 的方向导数.
- X 在 x 处的**切空间** $\mathrm{T}_x X$ 是 X 在 x 处的所有切向量构成的向量空间.

对 Euclid 空间的切映射

Definition 4.3.2. 设 m, n 是自然数, $U \subset \mathbb{R}^m$ 和 $V \subset \mathbb{R}^n$ 是开集, $f\colon U \to V$ 是连续可微映射. 设 $x \in U$. 则 f 在 x 处的**切映射**是指切空间之间的线性映射
$$df_x\colon \mathrm{T}_x U \longrightarrow \mathrm{T}_{f(x)} V,$$
由 *Jacobi* 矩阵
$$\begin{pmatrix} \dfrac{\partial f^1}{\partial x^1}(x) & \cdots & \dfrac{\partial f^1}{\partial x^m}(x) \\ \vdots & & \vdots \\ \dfrac{\partial f^n}{\partial x^1}(x) & \cdots & \dfrac{\partial f^n}{\partial x^m}(x) \end{pmatrix}$$
给出. 这里 $f^1, \ldots, f^n$ 是 f 的各分量, 而 $x^1, \ldots, x^m$ 是 U 上的坐标. 我们自然地将 $\mathrm{T}_x U$ 与 $\mathbb{R}^m$ 等同起来, 将 $\mathrm{T}_{f(x)} V$ 与 $\mathbb{R}^n$ 等同起来.

对流形的切映射

Definition 4.3.3. 设 X, Y 是光滑流形, $f\colon X \to Y$ 是光滑映射. 设 $x \in X$. 则 f 在 x 处的**切映射**是指切空间之间的线性映射

$$\begin{aligned} df_x\colon \mathrm{T}_x X &\longrightarrow \mathrm{T}_{f(x)} Y, \\ v &\longmapsto df_x(v), \end{aligned}$$

这里 $df_x(v)$ 定义如下: 对任何 $g \in C^\infty(Y)$, 有

$$df_x(v)\,(g) = v(g \circ f).$$

更一般地, 若 X, Y 是微分流形 C^1 流形, 而 f 是连续可微映射 C^1 映射, 则上述定义仍然有效, 只需将 $g \in C^\infty(Y)$ 换成 $g \in C^1(Y)$.

Theorem 4.9 (the Inverse Function theorem)**.** *There is a neighborhood around a regular point such that f is local diffeomorphism on it .*

A map is proper if the inverse image of every compact set is compact.
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a proper map. Then the pullback $f^* : H_c^*(\mathbb{R}^n) \to H_c^*(\mathbb{R}^n)$ is defined. It carries a generator of $H_c^*(\mathbb{R}^n)$ to some multiple of the generator. Then the multiple is defined to be the degree. If α is a generator of $H_c^*(\mathbb{R}^n)$,

$$\deg f = \int_{\mathbb{R}^n} f^* \alpha$$

We now prove it to be an integer.
Recall the <u>critical point</u> of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$ is a point p where the differential $(f_*)_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^n$ is not surjective, and a <u>critical value</u> is the image of a critical point. A point of $\mathbb{R}^n$ which is not a critical value is called a regular value.

Theorem 4.10 (Sard's theorem for $\mathbb{R}^n$)**.** *The set of critical values of a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ has measure zero in $\mathbb{R}^n$ for any integers m and n.*

Proposition 4.11. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a proper map. If f is not surjective, then it has degree 0.*

证明. Since the image of a proper map is closed (use each infinite subset of the compact set has a limit point), if f misses a point q, then it must miss some neighborhood U of q. Choose a bump n-form α whose supported lies in U. Then $f^*\alpha \equiv 0$ so that $\deg f = 0$ $\square$

Now we just need to look at surjective proper maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. By Sard's theorem, we can pick one regular value q. By the inverse function theorem, around any point in the pre-image of q, f is a local diffeomorphism. It implies the $f^{-1}(q)$ is a discrete set of points and hence a finite set. Choose a generator α whose support is localized near q. Then $f^*\alpha$ is a n-form whose support is localized near the points of $f^{-1}(q)$. As f^* is a local diffeomorphism, then the integral of f^* is $\pm \int \alpha = \pm 1$. Thus

$$\int_{\mathbb{R}^n} = \sum_{f-1(q)} \pm 1$$

Moreover, it shows that the number of the points, counted with multiplicity ± 1, in the inverse image of any regular value is the same.