Exercise 3.8

Proof. Noticed that $p|\binom{p}{k}$ for any $1 \leq k \leq p-1$. So $(x+y)^p = \sum_{k=0}^{p} \binom{p}{k} x^k y^{p-k} = x^p + y^p$ for any $x, y \in R$. Combining with the fact that $(xy)^p = x^p y^p$ for any $x, y \in R$, and $1^p = 1$, we have this map is a ring homomorphism. $\square$

Exercise 6.8

(a)

Proof. By definition of ideal, $ij \in I \cap J, \forall i \in I, j \in J$. $\Rightarrow i_1 j_1 + \cdots + i_n j_n \in I \cap J$ since $I \cap J$ is an ideal by Ex3.13. $\Rightarrow IJ \subset I \cap J$

$\forall t \in I \cap J$, since $1 \in R = I + J \Rightarrow 1 = i + j$ for some $i \in I, j \in J \Rightarrow t = t \cdot 1 = t \cdot (i+j) = ti + tj$ where $ti \in I, tj \in J \Rightarrow t \in IJ$ $\square$

(b)

Proof. Let $a = i_a + j_a, b = i_b + j_b$ where $i_a, i_b \in I, j_a, j_b \in J$

Then $j_a + i_b - a = i_b - i_a \in I$, $j_a + i_b - b = j_a - j_b \in J \Rightarrow x = j_a + i_b$ satisfies that $x \equiv a$ modulo I and $x \equiv b$ modulo J. $\square$

(c)

Proof. If $IJ = 0$, then $I \cap J = 0$.

Then $\forall r \in R$, if $r = i + j = i' + j'$, $i, i' \in I. j, j' \in J$, then $i - i' = j' - j \in I \cap J \Rightarrow i = i' = j' - j = 0$. So r can be uniquely written as $r = i + j, i \in I, j \in J$. $(\star)$

Define $\varphi : R \to (R/I) \times (R/J), r \mapsto (r + I, r + J)$. Easy to check that φ is a ring homomorphism.(In fact, $\varphi(rs) = (rs + I, rs + J) = (r + I, r + J)(s + I, s + J), \varphi(r + s) = (r + s + I, r + s + J) = (r + I, r + J) + (s + I, s + J)$)

If $(r + I, r + J) = (r' + I, r' + J)$, then $r - r' \in I, r - r' \in J \Rightarrow r - r' \in I \cap J$, $\Rightarrow r - r' = 0$. So φ is injective.

$\forall m + I \in R/I, n + J \in R/J$, by Chinese Remainder Theorem we have $x \in R$ s.t. $x + I = m + I, x + J = n + J \Rightarrow \varphi(x) = (m + I, n + J) \Rightarrow \varphi$ is surjective. Thus φ is isomorphic. $\square$

(d)Idempotents in $(R/I) \times (R/J)$ is $(\overline{e_1}, \overline{e_2})$ where $\overline{e_1}^2 = \overline{e_1}$ in R/I, $\overline{e_2}^2 = \overline{e_2}$ in R/J.

Exercise 7.4

The answer is no. By Prop7.7.7(a), a ring R is a cyclic group under the law of composition $+$. So $(id+id+id)\cdot(id+id+id+id+id) = 15\cdot id = 0$ where id is the identity element of R (under multiplication). However $(3id), (5id)$ is not zero since R is a cyclic group, hence R is not a domain. i.e. every ring of order 15 is not a domain.

M.3 Define $M_n = \{a \in R : \textbf{n-th component of a equals 0}\}$. Easy to check that M_n is an ideal. And for $a \notin M_n$, we have $(0, 0, \cdots, a_n^{-1}, 0, \cdots)\cdot a = (0, 0, \cdots, 1, \cdots)$. Then $(a) + M_n$ contains a basis of R(under addition), hence $(a) + M_n = R \Rightarrow M$ is maximal ideal.

If M maximal ideal but not one of M_n. We prove that $a = (a_1, \cdots) \in M$, $0 = a_n = a_{n+1} = \cdots$. Otherwise, R/M is a field $\Rightarrow$ There exsits b s.t. $ab - (1, 1, 1, \cdots) \in M$. i.e. $(a_1', \cdots, a_{n-1}', -1, -1, \cdots) \in M$. Since $M \neq M_n$, and $M \not\subset M_n$, there exists $B^m \in M$ s.t. m-th component of B^m, $B^m_m \neq 0$. By Gauss elimation, any element $t \in R$ can be expressed as $m + t'$, $m' inM$, $t_i' = 0, \forall 1 \leq i \leq n-1$. Then $t' = (0, 0, \cdots, -t_n', -t_{n+1}', \cdots)\cdot(a_1', \cdots, a_{n-1}', -1, -1, \cdots) \in M$ $\Rightarrow M = R \Rightarrow$ contradiction.

Therefore $M \supset (a : a_i = 1, \forall 1 \leq i \leq n-1; a_i = 0, \forall i \geq n; n \in \mathbb{Z}_+)$. Actually, $N = (a : a_i = 1, \forall 1 \leq i \leq n-1; a_i = 0, \forall i \geq n; n \in \mathbb{Z}_+)$ is a maximal ideal. That's because: for $t \notin M$, $t = (t_1, cdots), 0 \neq t_n = t_{n+1} = \cdots$. Then $t\cdot(1, 1, \cdots, t_n^{-1}, t_{n+1}^{-1}, \cdots) - (1, 1, 1, \cdots) \in N \Rightarrow t$ is a unit in $R/N \Rightarrow R/N$ is a field $\Rightarrow N$ is a maximal ideal.

Therefore, M is a maximal ideal if and only if $M = M_n$ or $M = N = (a : a_i = 1, \forall 1 \leq i \leq n-1; a_i = 0, \forall i \geq n; n \in \mathbb{Z}_+)$

5. (a)Every ideal contains 0 so $V(0) = Spec\, R$. R is not prime ideal$\Rightarrow V(R) = \emptyset$

(b) $V(\cup_{\lambda\in\Lambda}E_\lambda) = \{p \in Spec\, R | p \supset \cup_{\lambda\in\Lambda}E_\lambda\} = \{p \in Spec\, R | p \supset E_\lambda \forall\lambda \in \Lambda\} = \cap_{\lambda\in\Lambda}\{p \in Spec\, R | p \supset E_\lambda\} = \cap_{\lambda\in\Lambda}V(E_\lambda)$

(c) $\forall p \supset IJ$, $p \in Spec\, R$, then $\forall t \in I \cap J$, $t^2 \in IJ \subset p \Rightarrow t \in p$ since p is prime ideal. So $I \cap J \subset p \Rightarrow p \in V(I \cap J)$. $\forall p \supset I \cap J$, $p \in Spec\, R$, then $IJ \subset I \cap J \subset p \Rightarrow p \in V(IJ)$. Therefore $V(I \cap J) = V(IJ)$.

$\forall p \in V(IJ)$, if $\exists i \in I, i \notin p$, then $\forall j \in J, ij \in IJ \subset p \Rightarrow j \in p$ since p prime ideal. $\Rightarrow J \subset p \Rightarrow p \in V(J)$. If not, then $I \subset P \Rightarrow p \in V(I)$. Thus, $V(IJ) \subset V(I) \cup V(J)$.

$\forall p \in V(I)$, $p \supset I \supset I \cap J \Rightarrow p \in V(I \cap J) = V(IJ)$. Similarly for $p \in V(J), p \in V(IJ)$. $\Rightarrow V(I) \cup V(J) \subset V(IJ)$.

Therefore $V(IJ) = V(I) \cup V(J)$.

(d)For any decreasing net of nonempty closed subsets $(V(E_\lambda))_{\lambda\in\Lambda}$, we have $p_\lambda \in V(E_\lambda)$. Then $\cup_{\lambda\in\Lambda}E_\lambda \subset \cup_{\lambda\in\Lambda}p_\lambda$. Since $\forall xy \in \cup_{\lambda\in\Lambda}p_\lambda$, $xy \in p_\lambda$ for some $\lambda \in \Lambda$, $\Rightarrow x \in \cup_{\lambda\in\Lambda}p_\lambda$ or $\cup_{\lambda\in\Lambda}p_\lambda$. And since $1 \notin p_\lambda \Rightarrow 1 \notin \cup_{\lambda\in\Lambda}p_\lambda$. Thus $\cup_{\lambda\in\Lambda}p_\lambda$ is a prime ideal containing $\cup_{\lambda\in\Lambda}E_\lambda$. i.e. $\cup_{\lambda\in\Lambda}p_\lambda \in V(\cup_{\lambda\in\Lambda}E_\lambda) = \cap_{\lambda\in\Lambda}V(E_\lambda)$ $\Rightarrow \cap_{\lambda\in\Lambda}V(E_\lambda) \neq \emptyset$.

By decreasing chain property, we have $X = Spec\, R$ is compact.

Note: you can check dereasing chain property in `https://binguimath.github.io/Files/2023_Analysis.pdf`, page136, Propsition 8.15.