Complex Analysis HW11

1. $\forall$ compact set $K \subset \mathbb{C}$, $\exists\, M>0$ s.t. $|z|<M$ $(\forall z \in K)$.

$\forall\, n \geq M+1$, $|\frac{z}{n}|<1 \Rightarrow \log(1+\frac{z}{n}) = \frac{z}{n} - \frac{1}{2}(\frac{z}{n})^2 + \cdots \Rightarrow \left|\log(1+\frac{z}{n}) \cdot n - z\right| = \left|\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} z (\frac{z}{n})^k\right|$

$\leq |z| \cdot \dfrac{\frac{|z|}{n}}{1-\frac{|z|}{n}} \leq M \cdot \dfrac{\frac{M}{n}}{1-\frac{M}{n}} = \dfrac{M^2}{n-M} \Rightarrow \log(1+\frac{z}{n})\cdot n \to z$ uniformly on K.

Note that $\exists\, C>0$ s.t. $\forall |z|<2M, |w|<2M$, $|e^z - e^w| \leq C|z-w|$, we have $\forall\, z\in K$, $n>2M$,

$|(1+\frac{z}{n})^n - e^z| \leq C\left|\log(1+\frac{z}{n})\cdot n - z\right| \leq \dfrac{CM^2}{n-M} \Rightarrow (1+\frac{z}{n})^n \to e^z$ uniformly on K.

2. (a) Note that $|n^{-z}| = n^{-\mathrm{Re}\, z}$ and $\mathrm{Re}\, z>1$, $\zeta(z)$ converges for $\mathrm{Re}\, z>1$.

(b) Take $C>0$ s.t. $|A_n|<C$ $(\forall n \in \mathbb{N})$. $\forall \varepsilon>0$, $\exists\, N>0$ s.t. $\forall M>N$, $m\in\mathbb{N}$, $\sum_{n=M}^{M+m} |b_n - b_{n+1}| < \varepsilon$ and $|b_M|<\varepsilon$.

So $\left|\sum_{n=M}^{M+m} a_n b_n\right| = \left|\sum_{n=M}^{M+m-1} A_n(b_n - b_{n+1}) + A_{M+m} b_{M+m} - A_{M-1} b_M\right| \leq C\varepsilon + C\varepsilon + C\varepsilon = 3C\varepsilon$.

$\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges.

Furthermore, if $K \subset \mathbb{C}$ is compact and $b_n(z)$ are entire functions s.t. $\lim_n b_n(z) = 0$ uniformly in K and $\sum_{n=1}^{\infty} |b_n(z) - b_{n+1}(z)| < B$ for some $B>0$ and $\forall z \in K$, then $\sum_{n=1}^{\infty} a_n b_n(z)$ converges uniformly in K:

Apply the above notations and note that $\left|\sum_{n=M}^{M+m} a_n b_n(z)\right| \leq 3C\varepsilon$.

(c) For $\mathrm{Re}\, z>1$, both sides converge absolutely. Hence $(1-\frac{2}{2^z})\zeta(z) = 1 + \frac{1}{2^z} + \cdots - 2(\frac{1}{2^z} + \frac{1}{4^z} + \cdots) = 1 - 2^{-z} + 3^{-z} - \cdots$

Let $a_n = (-1)^{n-1}$, $b_n(z) = n^{-z}$, then RHS $= \sum_{n=1}^{\infty} a_n b_n$ and $|A_n| \leq 1$, $b_n(z)$ are entire.

$\forall$ compact set $K \subset \{\mathrm{Re}\, z>0\}$, $\lim_n b_n(z) = 0$ uniformly ~~and $\sum_{n=1}^{\infty} |b_n(z) - b_{n+1}(z)|$~~

Take $0<a<A$, $0<B$ s.t. $\forall z \in K$, $a<\mathrm{Re}\, z<A$, $-B<\mathrm{Im}\, z<B$, then

$$|b_n(z) - b_{n+1}(z)| = \frac{|(n+1)^z - n^z|}{|n^z||(n+1)^z|} \leq \frac{|(n+1)^z - n^z|}{|n^{2z}|} \leq \frac{|(1+\frac{1}{n})^z - 1|}{n^a} \leq \frac{(1+\frac{1}{n})^{\mathrm{Re}\, z}|(1+\frac{1}{n})^{i\,\mathrm{Im}\, z} - 1|}{n^a} + \frac{|(1+\frac{1}{n})^{\mathrm{Re}\, z} - 1|}{n^a}$$

$$\leq \frac{2^A |(1+\frac{1}{n})^{iB} - 1|}{n^a} + \frac{(1+\frac{1}{n})^A - 1}{n^a} \leq \frac{2^A \frac{C}{n}}{n^a} + \frac{\frac{C}{n}}{n^a} = \frac{(2^A+1)C}{n^{a+1}}$$ for n large enough

and $C>0$ s.t. $|(1+\frac{1}{n})^{iB} - 1| \leq \frac{C}{n}$, $|(1+\frac{1}{n})^A - 1| \leq \frac{C}{n}$ if n large.

$\Rightarrow \sum_{n=1}^{\infty} |b_n(z) - b_{n+1}(z)| \leq (2^A+1) C \zeta(a+1) < \infty$

By (b), $\sum_{n=1}^{\infty} a_n b_n$ converges uniformly on $K \Rightarrow$ analytic.

3. If f is not identically 0, then the different zeros $z_1, \ldots, z_k$ of f are isolate.

Let $C_1 \ldots C_k$ be some small circles centered at $z_1 \ldots z_k$ s.t. $C_i \cap C_j = \emptyset$ $(i \neq j)$.

Since $|f|>0$ on $C_1 \cup \cdots \cup C_k$, which is compact, $\exists\, N \in \mathbb{N}$ s.t. $\forall n>N$, $|f_n|>0$ on $C_1 \cup \cdots \cup C_k$, and $f_n \rightrightarrows f$, $f_n' \rightrightarrows f'$ in these circle

So $\#\{\text{zeros of } f\} = \sum_{j=1}^{k} \int_{C_k} \frac{f'}{f} dz = \sum_{j=1}^{k} \int_{C_k} \lim_n \frac{f_n'}{f_n} dz = \lim_n \sum_{j=1}^{k} \int_{C_k} \frac{f_n'}{f_n} dz \leq m$.

4. (a) Note that $\frac{1}{e^z-1} - \frac{1}{z} + \frac{1}{2}$ is odd and has a removable singularity at $z=0$,

$\frac{1}{e^z-1}$ has Laurent expension $\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$ (B_k are Bernulli numbers)

(b) $\cot z = \frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} i = i + 2i \frac{1}{e^{2iz}-1} = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{2^{2k} B_k}{(2k)!} z^{2k+1}$

(c) $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \frac{1}{z} - \sum_{k=1}^{\infty} z^{2k-1} \sum_{n=1}^{\infty} \frac{2}{n^{2k}} = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{2^{2k} \pi^{2k} B_k}{(2k)!} z^{2k-1}$

$\Rightarrow \zeta(2k) = \frac{2^{2k-1} \pi^{2k} B_k}{(2k)!}$

5. (a) When $|z|<1$, $\sum_{n=0}^{\infty} |z^{2^n}| \leq \sum_{n=0}^{\infty} |z| = \frac{1}{1-|z|} \Rightarrow \prod_{n=0}^{\infty} (1+z^{2^n})$ converges absolutely.

Since $\prod_{n=0}^{N} (1+z^{2^n}) = \frac{1-z^{2^{N+1}}}{1-z}$, let $N \to \infty$ we get $\prod_{n=0}^{\infty} (1+z^{2^n}) = \frac{1}{1-z}$.

(b) $\forall$ compact set $K \subset \mathbb{C}$, $\exists M>0$ s.t. $|z|<M$ ($\forall z \in K$).

$\forall |z|<\frac{1}{2}$, $\log(1+z) = z + z^2 f(z)$ for some $f(z)$ analytic on $B(0, \frac{1}{2})$.

So $\exists C>0$ s.t. $|f(z)| \leq C$ $\forall z \in \overline{B(0, \frac{1}{3})}$.

Take $N > 3M$, then $\sum_{n=N}^{\infty} \left|\log(1+\frac{z}{n}) - \frac{z}{n}\right| \leq \sum_{n=N}^{\infty} \left|(\frac{z}{n})^2 f(\frac{z}{n})\right| \leq \sum_{n=N}^{\infty} \frac{M^2}{n^2} C$

$\Rightarrow \sum_{n=1}^{\infty} \log(1+\frac{z}{n}) - \frac{z}{n}$ converges absolutely and uniformly on K

$\Rightarrow \prod_{n=1}^{\infty} (1+\frac{z}{n}) e^{-\frac{z}{n}}$ converges absolutely and uniformly on K.