Complex Analysis HW12

1. By Weierstrass Factorization Theorem, $\exists$ entire function g s.t. $\{a_n\}$ are exactly the simple zeros of g.

By Mittag-Leffler Theorem, $\exists$ meromorphic function h on $\mathbb{C}$ s.t. $\{a_n\}$ are exactly the simple poles of h with $\operatorname{Res}_{z=a_n} h(z) = A_n$.

let $f=gh$, then f is analytic outside $\{a_n\}$. Note that $\lim_{z\to a_n} f(z) = A_n$, f has removable singularity at $\{a_n\}$.

$\Rightarrow f(a_n) = A_n$. Remark: We may consider $f=\sum_{n=1}^{\infty} g(z)\frac{e^{r_n(z-a_n)}}{z-a_n}\frac{A_n}{g'(a_n)}$ for suitable $\{r_n\}$.

2. The zeros of $\cos\sqrt{z}$ are $(n+\frac{1}{2})^2\pi^2$ $(n\in\mathbb{N})$. Since $\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^2\pi^2}<\infty$, the genus of canonical product is 0.

From $\cos\pi z = \prod_{n=0}^{\infty}\left(1-\frac{4z^2}{(2n+1)^2}\right)$ or $\cos\sqrt{z}$ has growth order $\frac{1}{2}$ (by Hadamard Thm) we know $\deg g = 0 \Rightarrow$ genus of f is 0.

Remark: $\cos z$ is even, i.e. $\cos z = 1-\frac{z^2}{2}+\dots \Rightarrow \cos\sqrt{z}$ is entire.

3. ① genus 0: $f = cz^m\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) \Rightarrow \log f = m\log z + \sum\log\left(1-\frac{z}{a_n}\right) \Rightarrow \frac{f'}{f} = \frac{m}{z}+\sum\frac{1}{z-a_n} = \frac{m\bar{z}}{|z|^2}+\sum\frac{\bar{z}-a_n}{|z-a_n|^2}$

$\Rightarrow \operatorname{Im}\left(\frac{f'}{f}\right) = -\operatorname{Im}(z)\left(\frac{m}{|z|^2}+\sum\frac{1}{|z-a_n|^2}\right)$, i.e. zeros of f' has $\operatorname{Im} z = 0 \Rightarrow z\in\mathbb{R}$.

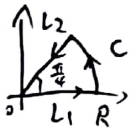
② genus 1, $f = cz^m e^{\alpha z}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)$: $\operatorname{Im}\left(\frac{f'}{f}\right) = -\operatorname{Im}(z)\left(\frac{m}{|z|^2}+\sum\frac{1}{|z-a_n|^2}\right)+\operatorname{Im}\alpha$

Since $f(\mathbb{R})\subseteq\mathbb{R}$, $c,\alpha\in\mathbb{R} \Rightarrow$ zeros of f' are real.

③ genus 1. $f = cz^m e^{\alpha z}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}$: $\operatorname{Im}\left(\frac{f'}{f}\right) = -\operatorname{Im}(z)\left(\frac{m}{|z|^2}+\sum\frac{1}{|z-a_n|^2}\right) \Rightarrow$ zeros of f' are real.

4. By Legendre's duplication formula: $\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$, take $z=\frac{1}{6}$ we get

$$\Gamma\left(\tfrac{1}{6}\right) = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{3})}{2^{-\frac{2}{3}}\Gamma(\frac{2}{3})} = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{3})^2}{2^{-\frac{2}{3}}\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{3})^2}{2^{-\frac{2}{3}}\frac{\pi}{\sin(\pi\cdot\frac{1}{3})}} = 2^{-\frac{1}{3}}\left(\frac{3}{\pi}\right)^{\frac{1}{2}}\Gamma\left(\tfrac{1}{3}\right)^2.$$

5. 

By Cauchy's theorem, $\int_{L_1}+\int_C+\int_{L_2} e^{iz^2}dz = 0$, where $R>$

Since $\int_{L_2} e^{iz^2}dz = \int_R^0 e^{i(\sqrt{i}x)^2}d\sqrt{i}x = -\frac{1+i}{\sqrt{2}}\int_0^R e^{-x^2}dx$ and

$$\left|\int_C e^{iz^2}dz\right| \le \int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}R\,d\theta \le \int_0^{\frac{\pi}{4}} e^{-R^2\cdot\theta}R\,d\theta = \frac{1-e^{-R^2\frac{\pi}{4}}}{R} \to 0$$

we have $\int_0^{\infty} e^{iz^2}dz = \frac{1+i}{\sqrt{2}}\int_0^{\infty}e^{-x^2}dx = \frac{1+i}{\sqrt{2}}\cdot\frac{\sqrt{\pi}}{2}$, i.e. $\int_0^{\infty}\sin t^2\,dt = \int_0^{\infty}\cos t^2\,dt = \frac{\sqrt{2\pi}}{4}$.

6. let h_a, h_b be the genera defined in (a), (b). Clearly $h_a=\infty \Leftrightarrow h_b=\infty$.

If $f(z) = z^m e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left[\sum_{j=1}^{h}\frac{1}{j}\left(\frac{z}{a_n}\right)^j\right]$, then $h_a = \max\{h, \deg g\}$.

$\forall\ h+1\le j\le h_a$, $\sum\frac{1}{a_n^j}$ converges $\Rightarrow h(z) = g(z) - \sum_{j=h+1}^{h_a}\sum_{n=1}^{\infty}\frac{1}{j}\left(\frac{z}{a_n}\right)^j$ is well-defined polynomial, and

$f(z) = z^m e^{h(z)}\prod\left(1-\frac{z}{a_n}\right)\exp\left[\sum_{j=1}^{h_a}\frac{1}{j}\left(\frac{z}{a_n}\right)^j\right]$ and $\deg h\le h_a \Rightarrow h_a \ge h_b$.

If $f(z) = z^m e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left[\sum_{j=1}^{h_b}\frac{1}{j}\left(\frac{z}{a_n}\right)^j\right]$, then $h_b\ge h$. Let $h(z) = g(z)+\sum_{j=h+1}^{h_b}\frac{1}{j}\left(\frac{z}{a_n}\right)^j$, then

$f(z) = z^m e^{h(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left[\sum_{j=1}^{h}\frac{1}{j}\left(\frac{z}{a_n}\right)^j\right]$ and $\deg h\le h_b \Rightarrow h_a\le h_b$

7. $\max_{|z|=r}|f(z)| = r^m \max_{|z|=r} \prod \left|1-\frac{z}{a_n}\right| \le r^m \max_{|z|=r} \prod \left(1+\frac{|z|}{a_n}\right) = |g(-r)| \le \max_{|z|=r} |g(z)|$

$\min_{|z|=r}|f(z)| = r^m \min_{|z|=r} \prod \left|1-\frac{z}{a_n}\right| \ge r^m \min_{|z|=r} \prod \left|1-\left|\frac{z}{a_n}\right|\right| = |g(r)| \ge \min_{|z|=r} |g(z)|.$