

Complex Analysis

LIN150117

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

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1 Harmonic function

Definition 1.1 (Cauchy-Riemann equation).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 1.2 (Harmonic function). A function u is **harmonic** if it satisfied **Laplace equation** $\Delta u = 0$.

If two harmonic function u and v satisfies Cauchy-Riemann equations, then we say that v is **conjugate harmonic function of u** $\Rightarrow u$ is conjugate harmonic of $-v$.

1.1 Polynomial rational function

The polynomial $P(z) = \sum_{j=0}^n a_j z^j$ is analytic in \mathbb{C} .

We will prove the fundamental theorem of algebra

Theorem 1.3 (Fundamental Theorem of Algebra). *Every polynomial with degree $n > 0$ has at least one point.*

Theorem 1.4 (Gauss-Lucus theorem). *The smallest convex polygon that contain the zeros of P also contains the zeros of P' .*

Proof. Only need to check.

We can get this equation.

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{j=1}^n \frac{1}{\alpha - \alpha_j} = 0 \Rightarrow \sum_{j=1}^n \frac{\overline{\alpha - \alpha_j}}{|\alpha - \alpha_j|^2}$$

Hence α is linearly represented by α_j . □

Proposition 1.5. Let P and Q be two polynomials with no common zeros. Then the rational function $R(z) = \frac{P(z)}{Q(z)}$ is analytic away from the zeros of Q .

The zeros of Q are called **poles** of R , and the **order of a pole** is equal to the order of the corresponding zero of Q .

We often view R as a function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. $R_1(z) := R(\frac{1}{z})$.

If $R_1(0) = 0$, the order of the zero at ∞ (of R) is the order of the zero of $R_1(z)$ at $z = 0$.

If $R_1(0) = \infty$, the order of the pole at ∞ (of R) is the order of the pole of $R_1(z)$ at $z = 0$.

Suppose

$$R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, a_n \neq 0, b_m \neq 0$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$$

By discussing m and n , we can infer the situation of $R(z)$ at ∞ .

By adding the order of poles and zeros at ∞ , we can get the following theorem.

Theorem 1.6. The total number of zeros and poles of a rational function are the same.

Remark 1.7. This common number is called the **order of the rational function**.

Corollary 1.8. Suppose a rational function R has order p . Then every equation $R(z) = a$ has exactly p roots.

Proof. $\hat{R}(z) = R(z) - a$ has the same poles as R . □

A rational function of order 1 is a **linear fraction** $R(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$

Such fraction is often called **Möbius transformation**

Every rational function has a representation by **partial fractions**.

- If R has a pole at ∞ . Then we can write

$$R(z) = G(z) + H(z) \quad (*)$$

where G is a polynomial without constant term, and H is finite at ∞ .

The degree of G is the order of the pole of R at ∞ . G is called the **singular part** of R at ∞ .

- Let the distinct finite poles of R be β_1, \dots, β_k . Let $R_j(\psi) = R(\beta_j + \frac{1}{\psi})$. Then R_j is a rational function with a pole at ∞ . As in $(*)$, we can write

$$R_j = G_j + H_j$$

with H_j finite at ∞ . Then

$$R(z) = G_j\left(\frac{1}{z - \beta_j}\right) + H\left(\frac{1}{z - \beta_j}\right)$$

with G_j is a polynomial in $\frac{1}{z - \beta_j}$ without constant term called the **singular point** of R at β_j .

- Let $F(z) = R(z) - G(z) - \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right)$.

Then F is a rational function which can only have poles among β_j, ∞

Since by our construction, F is finite at every $\beta_j, 1 \leq j \leq k$ and ∞ .

So F is a constant.

In particular, $R(z) = G(z) + \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right) + c$.

2 Power Series

Theorem 2.1 (Abel's theorem). *If $\sum a_n$ converges, then $f(z) = \sum a_n z^n \rightarrow f(1)$ as $z \rightarrow 1$ in such a way that $\frac{|1-z|}{1-|z|}$ remains bounded.*

3 Exponential, Trigonometric and logarithmic functions

3.1 Exponential and Trigonometric function

The **exponential function** is defined as the solution of the differential equation

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1 \end{cases}$$

We denote $e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

The **trigonometric function** are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

3.2 Logarithmic Functions

The **logarithmic function** \ln is defined by $z = \ln w$ is a root of the equation $e^z = w$.

For $w \neq 0$, we write $z = x + iy$, then

$$e^{x+iy} = w \Leftrightarrow \begin{cases} e^x = |w| \\ e^{iy} = \frac{w}{|w|} \end{cases}$$

The first equation has a unique solution $x = \ln |w|$.

The second equation $e^{iy} = \frac{w}{|w|}$ has a unique solution $y_0 \in [0, 2\pi)$.

If we write $w = re^{i\theta}$, then $x = \ln w, y = \theta = \arg w$.

Thus, for $w \neq 0$, we have

$$\ln w = \ln |w| + i \arg w$$

The function \ln is actually not single-valued. But we can define a single-valued function Ln

We define

$$a^b = \exp(b \ln a)$$

We will prove Ln is analytic in $\mathbb{C} - (-\infty, 0]$ but not continuous in $(-\infty, 0]$.

Ln is the principle branch of the logarithm.

4 Conformal Mappings

4.1 Connectedness

Theorem 4.1. *A nonempty open set in \mathbb{C} is connected iff any two of its points can be joined by a polygon which lies in the set. (i.e. Connectedness is equivalent to Path Connectedness)*

An nonempty connected subset is called a **region**

4.2 Compactness

Definition 4.2. A set X is **totally bounded** if $\forall \epsilon > 0$, X can be covered by finitely many balls of radius ϵ

Theorem 4.3. *A set is compact iff it is complete and totally bounded.*

Theorem 4.4. *A subset $X \subset \mathbb{C}$ is compact iff every infinite sequence of X has a limit point in X .*

4.3 Continuous Functions

Theorem 4.5. *Continuous function maps connected space to connected space.*

Theorem 4.6. *Continuous function maps compact space to compact space.*

4.4 Arcs and closed curves

The equation of an **arc** r in \mathbb{C} can be represented by one of the terms

- $x = x(t), y = y(t), \alpha \leq t \leq \beta, x, y$ are continuous at t
- $z(t) = x(t) + iy(t), \alpha \leq t \leq \beta.$
- the continuous mapping $\gamma : [\alpha, \beta] \rightarrow \mathbb{C}.$

For a nondecreasing function $\varphi : [\alpha, \beta] \rightarrow [\alpha, \beta], z = z(\varphi(t)), \alpha' \leq \tau \leq \beta'$ is **change of parameter** of $z(t).$

The change is **reversible** iff φ is strictly increasing.

If γ is differentiable, then call γ a **curve**.

γ is **simple**, or a **Jordan curve**, if γ is injective.

γ is **closed curve** if $\gamma(0) = \gamma(1).$

4.5 Analytic Functions in Regions

A function f is analytic on an arbitrary set A if it is the restriction to A of a function which is analytic in some open set containing A .

Theorem 4.7. *An analytic function in a region(i.e. open and connected) Ω whose derivative is 0 must reduce to a constant. The same hold if the real part, the imaginary part, the modulus, or the argument is constant.*

4.6 Conformal Mappings

Suppose $f : \Omega \rightarrow \mathbb{C}$ is analytic in Ω . $r_1 = z_1(t), r_2 = z_2(t), \alpha \leq t \leq \beta$.

$$z_0 = z_1(t_0) = z_2(t'_0), z'_1(t_0) \neq 0, z'_2(\hat{t}_0) \neq 0, \alpha < t_0, \hat{t}_0 < \beta.$$

$$f'(z_0) \neq 0, w_1(t) = f(z_1(t)), w_2 = f(z_2(\hat{t}_0))$$

$$\Gamma_1 = \{w_1(t) | \alpha \leq t \leq \beta\}, \Gamma_2 = \{w_2(t) | \alpha \leq t \leq \beta\}$$

Then

$$w'_1(t) = f'(z_1(t))z'_1(t)$$

$$w'_2(t) = f'(z_2(\hat{t}))z'_2(\hat{t})$$

\Rightarrow

$$w'_1(t_0) \neq 0, w'_2(t_0) \neq 0$$

$$\arg w'_1(t_0) = \arg f'(z_1(t_0))z'_1(t_0)$$

$$\arg w'_2(t_0) = \arg f'(z_2(\hat{t}_0))z'_2(\hat{t}_0)$$

So the "angle" $\arg w'_1(t_0) - \arg w'_2(\hat{t}_0) = \arg z_1(t_0) - \arg z_2(\hat{t}_0)$ remains the same.

Now we give the definition.

Definition 4.8. $w = f(z)$ is said to be **conformal** in Ω if f is analytic in Ω and $f'(z) \neq 0$ for $\forall z \in \Omega$.

Easy to prove that linear change of scale at z_0 is independent of the direction.

$$i.e. |f'(z_0)| = \lim_{z \rightarrow z_0} \frac{\delta \sigma}{\delta s}$$

4.7 Length and Area

The **length** of a differentiable arc γ with the equation $z(t) = x(t) + iy(t)$, $a \leq t \leq b$

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b |z'(t)| dt$$

For $\Gamma = f(\gamma)$ where f conformal mapping.

Then

$$L(\Gamma) = \int_a^b |f'(z(t))| \cdot |z'(t)| dt$$

The **area** of $E \subset \mathbb{R}$ is $A(E) = \iint_E dx dy$

Then by the differentiable functional transformation, the area $\hat{E} = f(E)$ is

$$A(\hat{E}) = \int \int_E |u_x v_y - u_y v_x| dx dy$$

If f is the conformal mapping of an open set containing E , then by Cauchy-Riemann equation

$$A(\hat{E}) = \int \int_E |f'(z)|^2 dx dy$$

5 Möbius Transformation

Recall that a **Möbius transformation** is a function of the form

$$w = s(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Then it has an inverse $z = S^{-1}(w) = \frac{dw - b}{-cw + a}$.

We may define $S(\infty) = \lim_{z \rightarrow \infty} S(z) = \frac{a}{c}$, $S(\frac{-d}{c}) = \infty$

With these definition, $S : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a topological mapping. Here one may use the chordal metric to define the topology.

$$S'(z) = \frac{ad - bc}{(cz + d)^2}$$

Then S is conformal in $\hat{\mathbb{C}} - \{-\frac{d}{c}, \infty\}$.

$w = z + \alpha$ is called a **parallel translation**.

$w = kz$ with $|k| = 1$ is a **rotation**.

$w = kz$ with $k > 0$ is a **homothetic transformation**.

$x = \frac{1}{z}$ is called an **inversion**.

Proposition 5.1. *Every Möbius transformation is a composition of the above four operations.*

5.1 Cross ratio

For three distinct points $z_2, z_3, z_4 \in \hat{\mathbb{C}}$, we can find a Möbius transformation S such that $S(z_2) = 0, S(z_3) = 1, S(z_4) = \infty$.

Lemma 5.2. *The Möbius transformation satisfying the above conditions is unique.*

The **cross ratio** (z_1, z_2, z_3, z_4) is the image z_1 under the Möbius transformation which maps z_2 to 1, z_3 to 0 and z_4 to ∞ .

Theorem 5.3. *If $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ are distinct, and T is any Möbius transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Lemma 5.4. *Let T be a Möbius transformation, $T(\mathbb{R})$ is either a circle or a straight line.*

Theorem 5.5. *The cross ratio (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle or a straight line.*

Remark 5.6. One may prove the theorem by elementary geometry

Theorem 5.7. *A Möbius transformation maps circles into circles.*

5.2 Symmetry

Suppose T is a Möbius transformation which maps $\hat{\mathbb{R}}$ onto a circle C .

We say that $w = Tz$ and $w^* = T\bar{z}$ are **symmetric** w.r.t. C .

Remark 5.8. This definition is independent of T . Suppose S is another Möbius transformation which maps $\hat{\mathbb{R}}$ onto C , then $S^{-1}T$ maps $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$, and this $S^{-1}w = S^{-1}Tz$ and $S^{-1}w^* = S^{-1}T\bar{z}$ are conjugate.

The points z and z^* are **symmetric w.r.t C through** z_1, z_2, z_3 iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.

This can be another definition.

Note that only the points on C are symmetric to themselves.

The mapping $z \mapsto z^*$ is 1-1 and is called **reflection** w.r.t. C .

5.2.1 geometric meaning of symmetry

Case1: C is a straight line. We may assume $z_3 = \infty$.

z, z^* are symmetric w.r.t. C if and only if

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Then

$$|z^* - z_2| = |z - z_2|, \quad \forall z_2 \in C \text{ and } z_2 \neq \infty$$

$$\operatorname{Im} \frac{z^* - z_2}{z_1 - z_2} = \operatorname{Im} \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

So C is the bisecting normal of the segment between z and z^* .

Case2: C is the circle $|z - a| = R$.

$$\begin{aligned} \text{Then for } \forall \text{ distinct } z_1, z_2, z_3 \in \mathbb{C}, \quad & \overline{(z, z_1, z_2, z_3)} = \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)} \\ & = (\bar{z} - \bar{a}, \bar{z}_1 - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}) = (\bar{z} - \bar{a}, \frac{R^2}{\bar{z}_1 - a}, \frac{R^2}{\bar{z}_2 - a}, \frac{R^2}{\bar{z}_3 - a}) = (\frac{R^2}{\bar{z} - \bar{a}}, z_1 - a, z_2 - \end{aligned}$$

$$a, z_3 - a) \\ = \left(\frac{R^2}{\bar{z} - \bar{a}}, z_1, z_2, z_3 \right).$$

Then the symmetric point of z w.r.t. C is

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

or

$$(z^* - a)(\bar{z} - \bar{a}) = R^2$$

\Rightarrow

$$\begin{cases} |z^* - a| \cdot |z - a| = R^2 \\ \frac{z^* - a}{z - a} = \frac{(z^* - a)(\bar{z} - \bar{a})}{|z - a|^2} > 0 \end{cases}$$

Theorem 5.9 (The Symmetric principle). *If a Möbius transformation maps a circle C_1 onto a circle C_2 , then it transforms any pair of symmetric points w.r.t. C_1 into a pair of symmetric points w.r.t. C_2 .*

Proof. Case1: $C_1 = \hat{\mathbb{R}}$. Let T be the Möbius transformation which maps $\hat{\mathbb{R}}$ onto C_2 . $\forall z \in \mathbb{C}$, by definition, $w = Tz$ and $w^* = T\bar{z}$ are symmetric w.r.t. C_2 .

Case2: C_1 is a general circle. Let $T : C_1 \rightarrow C_2$ and $S : \mathbb{R} \rightarrow C_2$ be Möbius transformation.

Suppose w, w^* are symmetric w.r.t. C_1 . Then there exists z s.t. $w = Sz, w^* = S\bar{z}$.

Then we can find $Tw = TSz, Tw^* = TS\bar{z}$ are symmetric w.r.t. C_2 since $TS : \hat{\mathbb{R}} \rightarrow C_2$ □

Remark 5.10. 1. The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto$

$w, z_2 \mapsto w_2, z_3 \mapsto w_3$ where $z_1, z_2, z_3 \in C_1, w_1, w_2, w_3 \in C_2$ is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

2. The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w_1, z_2 \mapsto w_2$ where $z_1 \in C_1, z_2 \notin C_1, w_1 \in C_2, w_2 \notin C_2$ is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

5.3 Steiner Circles

For $S(z) = \frac{az+b}{cz+d}, S'(z) = \frac{ad-bc}{(cz+d)^2}$.

A point $z \notin$ a circle C is said to on the **right(left,resp.)** of C if $\text{Im}(z, z_1, z_2, z_3) > 0(\text{Im}(z, z_1, z_2, z_3) < 0)$

Remark 5.11.

1. This agrees with everyday use since $(i, 1, 0, \infty) = i$
2. This distinct between left and right is the same for all triples, while the meaning may be reversed.

(If $C = \hat{\mathbb{R}}$, then $(z, z_1, z_2, z_3) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R} \Rightarrow \text{Im}(z, z_1, z_2, z_3) = \frac{ad-bc}{|cz+d|^2} \text{Im}(z)$)

3. We can define an absolute positive orientation of all finite circles by requiring that ∞ should be lie to the right of the oriented circles.

Consider a Möbius transformation of the form

$$w = k \cdot \frac{z-a}{z-b}$$

Here, $z = a \mapsto w = 0, z = b \mapsto w = \infty$.

Then circles through a, b maps to straight line through $0, \infty$.

The concentric circle about the origin, $|w| = \rho$, correspond to circles with the equation

$$\left| \frac{z - a}{z - b} \right| = \frac{\rho}{|k|}$$

These are the circles of **Apollonius** with limit points a and b .

Denote by C_1 the circles through a, b and C_2 the circles of Apollonius with these limit points. The configuration formed by all the circles C_1 and C_2 is called the **Steiner circles**(or **circular net**)

Theorem 5.12.

- (a) *There is exactly one C_1 and one C_2 through each point in $\hat{\mathbb{C}} \setminus \{a, b\}$*
- (b) *Every C_1 meets every C_2 under right angle.*
- (c) *Reflection in a C_1 transforms every C_2 into itself and every C_1 into another C_1 .*
- (d) *The limit points a, b are symmetric w.r.t. each C_2 , but not w.r.t. other circles.*

Proof. If the limit points are $0, \infty$, those properties are trivial in the w -plane. The general case follows since all properties are invariant under Möbius transformations. □

6 Elementary Conformal mapping

Example 6.1. $w = z^\alpha$ where $\alpha > 0$.

Let $S(u_1, u_2)$ with $0 < \varphi_2 - \varphi_1 \leq 2\pi$ be $\{z \in \mathbb{C} : z \neq 0, \varphi_1 < \arg(z) < \varphi_2\}$ where $\arg(z)$ can be chosen as any value of it.

Then $S(\varphi_1, \varphi_2)$ is a region.

In this region, a unique value of $w = z^\alpha$ is defined by $\arg w = \alpha \arg z$.

This function is analytic with $\frac{dw}{dz} = \alpha \frac{w}{z}$.

This function is 1-1 only if $\alpha(\varphi_2 - \varphi_1) \leq 2\pi$.

Example 6.2. $w = e^z$ maps $\{z \in \mathbb{C} : -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2}\}$ onto $\{w \in \mathbb{C} : \text{Re}(w) > 0\}$

Example 6.3. $w = \frac{z-1}{z+1}$ maps $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$ onto $\{w \in \mathbb{C} : |w| < 1\}$

Example 6.4.

$$\mathbb{C} \setminus [-1, 1] \xrightarrow{z_1 = \frac{z+1}{z-1}} \mathbb{C} \setminus (-\infty, 0] \xrightarrow{z_2 = \sqrt{z_1}} \{\text{Re}(z_2) > 0\} \xrightarrow{w = \frac{z_2-1}{z_2+1}} \{w \in \mathbb{C} : |w| < 1\} \quad (6.1)$$

6.1 Elementary Riemann surfaces

Example 6.5. $w = z^n, n \in \mathbb{Z}_+$ and $n > 1$.

There is a 1-1 correspondence between each angle $\frac{(k-1)2\pi}{n} < \arg z < \frac{k \cdot 2\pi}{n}, k = 1, 2, \dots, n$ and while w -plane except for the positive real axis.

Example 6.6. $w = e^z$. This function maps each parallel strip $(k-1)2\pi < \text{Im } z < k \cdot 2\pi, k \in \mathbb{Z}$ onto a sheet with a cut along the positive axis.

7 Complex Integration

7.1 Fundamental Theorems

7.1.1 Line integral and rectifiable arcs

Let $f(t) = u(t) + iv(t)$ be a complex-valued defined on $t \in [a, b] \subset \mathbb{R}$ where u, v are real-valued functions. If f is continuous on $[a, b]$, we may define the **integral**

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt$$

Let γ be a piecewise differential arc in \mathbb{C} with the equation $z = z(t)$, $a \leq t \leq b$. If f is continuous on γ , then $f(z(t))$ is continuous on $[a, b]$, and we define

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt \quad (7.1)$$

The integral defined in 7.1 is independent of the parametrization of γ . Suppose that another parametrization of γ is $\gamma : (\alpha, \beta) \rightarrow \mathbb{C}, \tau \mapsto z(t(\tau))$, where $t : (\alpha, \beta) \rightarrow (a, b)$, $\tau \mapsto t(\tau)$ is piecewise differentiable. Then we have

$$\int_a^b f(z(t))z'(t)dt = \int_{\alpha}^{\beta} f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau = \int_{\alpha}^{\beta} f(z(t(\tau)))\frac{dz(t(\tau))}{d\tau}d\tau \quad (7.2)$$

For an arc γ with equation $z = z(t)$, $a \leq t \leq b$, we define $-\gamma$ by $z = z(-t)$, $-b \leq t \leq a$.

Then we have

$$\begin{aligned} \int_{-\gamma} f(z)dz &= \int_{-b}^{-a} f(z(-t))\frac{dz(-t)}{dt}dt \\ &= - \int_{-a}^{-b} f(z(-t))z'(-t)dt \\ &= - \int_a^b f(z(\tau))z'(\tau)d\tau \\ &= - \int_{\gamma} f(z)dz \end{aligned}$$

So we have those properties:

Proposition 7.1.

$$(a) \int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$$

(b) Let f and g be two continuous functions on the piecewise differentiable arc γ , then

$$\int_{\gamma} (\lambda_1 f + \lambda_2 g) dz = \lambda_1 \int_{\gamma} f dz + \lambda_2 \int_{\gamma} g dz, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

(c) If γ can be subdivided into two pieces differentiable arcs γ_1 and γ_2 , and f is continuous on γ_1 , then

$$\int_{\gamma} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz$$

(d) (c) implies that the integral of a closed curve doesn't depend on the starting point on the curve

Example 7.2. Evaluate $\int_{\gamma} \frac{1}{z-a} dz$ where γ is the circle centered at $a \in \mathbb{C}$ with radius R .

Let $z = z(t) = a + Re^{it}$. Then the integral is $2\pi i$

7.1.2 The fundamental theorem of Calculus for integrals in \mathbb{C}

The line integral w.r.t. \bar{z} is defined by

$$\int_{\gamma} f(z) d\bar{z} = \overline{\int_{\gamma} \overline{f(z)} dz}$$

With this notation, line integrals w.r.t. $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$ can be defined by

$$\int_{\gamma} f(z) dx = \frac{1}{2} \left[\int_{\gamma} f(z) dz + \int_{\gamma} f(z) d\bar{z} \right]$$

$$\int_{\gamma} f(z) dy = \frac{1}{2i} \left[\int_{\gamma} f(z) dz - \int_{\gamma} f(z) d\bar{z} \right]$$

if we write $f(z) = \mu + i\nu$, we have

$$\int_{\gamma} f(z) dz = \int_{\gamma} f(z) dx + i \int_{\gamma} f(z) dy = \int_{\gamma} (\mu dx - \nu dy) + i \int_{\gamma} (\nu dx + \mu dy)$$

Remark 7.3. It is followed by the intuition. We can view the integration as the multiplication between f and dz .

The integral w.r.t. **arc length** is defined by

$$\int_{\gamma} f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt$$

This integral is again independent of the parametrization. It is easy to check

$$\int_{-\gamma} f(z)|dz| = \int_{\gamma} f(z)|dz|$$

Now we define **length** of a curve γ : $L(\gamma) = \int_{\gamma} |dz|$

We have the inequality:

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| \cdot |dz| \leq L(\gamma) \cdot \sup_{z \in \gamma} |f(z)|$$

The length of an arc γ ($z = z(t)$) can also be defined as the least upper bound of all sums

$$\sum_{i=1}^n |z(t_i) - z(t_{i-1})|$$

where $a = t_0 < t_1 < \dots < t_n = b$ If this least upper bound is finite, we say that the arc is **rectifiable**

It is easy to show that piecewise differentiable arcs are rectifiable.

The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_{\gamma} f(z)dz = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(z(\psi_k)) [z(t_k) - z(t_{k-1})]$$

Theorem 7.4. Let $\Omega \subset \mathbb{C}$ be a region, and P, Q two (possibly complex-valued) functions that are continuous on Ω , γ closed curve. The integral $\int_{\gamma} p(x, y)dx + Q(x, y)dy$ depends only on the end point of γ iff there exists a function $U(x, y)$ on Ω with $\frac{\partial U}{\partial x} = P, \frac{\partial U}{\partial y} = Q$.

Proof. " \Leftarrow ": If such a U exists, then

$$\int_{\gamma} Pdx + Qdy = \int_{\gamma} \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = \int_{\gamma} \frac{dU}{dt}dt = U(\gamma(b)) - U(\gamma(a))$$

" \Rightarrow ": Fix a point $(x_0, y_0) \in \Omega$. We define $U(x, y) = \int_{\gamma} Pdx + Qdy$ where γ is any curve between (x_0, y_0) and (x, y) . Easy to check that it is true. \square

Theorem 7.5 (Fundamental theorem of Calculus for integrals on \mathbb{C}). Let f be continuous on a region Ω containing γ . $\int_{\gamma} f dz$ depends on the endpoints iff f is the derivative of an analytic function F in Ω .

Remark 7.6. We will prove $\int_{\gamma} f dz = F(\omega_2) - F(\omega_1)$ where γ begins at ω_1 and ends at ω_2 .

Proof. Transform the line integration into the composition of two real integration. \square

Corollary 7.7. If F is analytic on Ω with $F' = f$, and γ is a closed curve in Ω , then $\int_{\gamma} f dz = 0$. Conversely if f is continuous on Ω and $\int_{\gamma} f dz = 0$ for any closed curve in Ω , then f is the derivative of an analytic function F in Ω .

7.1.3 Cauchy's theorem for a rectangle

There is some notes in this section:

R is the rectangle in \mathbb{C} , $R = \{x + iy \in \mathbb{C} : a \leq x \leq b, c \leq y \leq d\}$. And ∂R is boundary curve oriented in the counterclockwise direction.

Theorem 7.8 (Cauchy's theorem for a rectangle). *If f is analytic on an open set which contains R , then $\int_{\partial R} f(z)dz = 0$*

Proof. For \forall rectangle \tilde{R} inside R , we define $Z(\tilde{R}) = \int_{\partial \tilde{R}} f(z)dz$. Then $Z(R) = Z(R_1) + Z(R_2)$ if R is divided into Z_1, Z_2 .

Since we can divide R into four equal rectangles, and find a rectangle with $|Z(R^{(1)})| \geq \frac{1}{4}|Z(R)|$. Then repeat the above steps and we obtain a sequence of nested rectangles $R \supset R^{(1)} \supset \dots$ with the property

$$|Z(R^{(n)})| \geq \frac{1}{4^n} |Z(R)| \geq \dots \geq \frac{1}{4^n} |Z(R)| \quad (7.3)$$

$\forall \delta > 0, \exists n \in \mathbb{N}$ s.t. $R^{(n)} \subset \{z \in \mathbb{C} : |z - z_0| < \delta\}$, $\forall n \geq N$, where z_0 is the limit of $R^{(n)}$ as $n \rightarrow \infty$.

f is analytic in $R \Rightarrow \forall \epsilon, \exists \delta > 0$ s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon, \forall z \text{ with } |z - z_0| < \delta \quad (7.4)$$

We assume that δ satisfies both conditions. We have

$$\begin{aligned} Z(R^{(n)}) &= \int_{\partial R^{(n)}} f(z)dz = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)]dz \\ &\Rightarrow |Z(R^{(n)})| \leq \epsilon \int_{\partial R^{(n)}} |z - z_0|dz \text{ by 7.4} \end{aligned}$$

Let d_n be the length of diagonal of $R^{(n)}$, L_n be the length of its perimeter. Then $|z - z_0| \leq d_n, \forall z \in \partial R^{(n)}$.

$\Rightarrow |Z(R^{(n)})| \leq \epsilon d_n L_n = \epsilon \frac{D}{2^n} \cdot \frac{L}{2^n}$ where D, L are the diameter and perimeter of R .

$$\Rightarrow |Z(R)| \stackrel{7.3}{\leq} 4^n |Z(R^{(n)})| \leq \epsilon DL \Rightarrow Z(R) = 0 \text{ since } \epsilon \text{ is arbitrary.} \quad \square$$

We will next prove the following stronger theorem:

Theorem 7.9 (stronger version of Cauchy's theorem for a rectangle). *Let f be analytic on $R' = R \setminus \{\psi_1, \dots, \psi_m\}$, $m \in \mathbb{N}$. If $\lim_{z \rightarrow \psi_j} (z - \psi_j)f(z) = 0$, $\forall 1 \leq j \leq m$, then*

$$\int_{\partial R} f(z)dz = 0.$$

Proof. WLOG, we may assume f is not analytic at only one point $\psi \in R$. If we put ψ into a small rectangle S_0 , then the previous theorem tells us $\int_{\partial R} f(z)dz = \int_{\partial S_0} f(z)dz$.

$\forall \epsilon > 0$, we may choose S_0 small enough such that $|f(z)| \leq \frac{\epsilon}{|z - \psi|}$, $\forall z \in \partial S_0$

$$\Rightarrow \left| \int_{\partial R} f(z)dz \right| \leq \epsilon \int_{\partial S_0} \frac{|dz|}{|z - \psi|} \leq \epsilon \frac{1}{\frac{l}{2}} \cdot 4l = 8\epsilon$$

$\Rightarrow \int_{\partial R} f(z)dz = 0$ since ϵ is arbitrary. □

7.1.4 Cauchy's Theorem for a disk

$\Delta := \{z \in \mathbb{C} : |z - z_0| < R\}$ where $R > 0$.

Theorem 7.10 (Cauchy's Theorem for a disk). *If f is analytic in an open disk Δ , then $\int_{\gamma} f(z)dz = 0$ for closed curve γ in Δ .*

Proof. Suppose the center of Δ is $z_0 = x_0 + iy_0$, $z = x + iy$. We define

$$F(z) = \int_{\gamma} f(z)dz$$

where γ is the horizontal line segment from z_0 to (x, y_0) added with vertical line segment from (x, y_0) to z . We have

$$\frac{\partial F}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{F(x, y + \delta y) - F(x, y)}{\delta y} = \lim_{\delta y \rightarrow 0} \frac{1}{\delta y} \int_{\delta \gamma} f(z)dz = if(z) \quad (7.5)$$

By Cauchy's thm on rectangles, one has $F(z) = -\int_{\tilde{\gamma}} f(z)dz$, where $\tilde{\gamma}$ is the vertical line segment from z_0 to (x_0, y) added with horizontal line segment from (x_0, y) to

z .

Similarly, $\frac{\partial F}{\partial x} = f(z)$.

$\Rightarrow \frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y} \Rightarrow F$ is analytic in Δ with derivative f .

By Fundamental Theorem 7.5 of Calculus $\Rightarrow \int_{\gamma} f(z)dz = 0$ for \forall closed curve in

Δ . □

Here is a stronger version.

Theorem 7.11 (stronger version of Cauchy's Theorem for a disk). *Let f be analytic in a region $\Delta' = \Delta \setminus \{\psi_1, \dots, \psi_m\}$ with $m \in \mathbb{N}$. If f satisfies $\lim_{z \rightarrow \psi_j} (z - \psi_j)f(z) = 0, \forall 1 \leq j \leq m$, then $\int_{\gamma} f(z)dz = 0, \forall \gamma$ closed in Δ'*

Proof. It is similar to the above proof.

For the case no ψ_j lies on $x = x_0$ and $y = y_0$, we can find a similar curve γ with last segment is a vertical one. Let $F(z) = \int_{\gamma} f(z)dz$. And continue the process of proof of the previous theorem.

For the case that $\exists \psi_j$ lies on the lines $x = x_0, y = y_0$, we actually can move the center to another point s.t. no ψ_j lies on the lines $x = x'_0, y = y'_0$. □

7.2 Cauchy's integral formula

7.2.1 Index of a point with respect to a closed curve

Lemma 7.12. *If the piecewise differentiable closed curve γ does not pass through $z \in \mathbb{C}$, then the value of the integral $\int_{\gamma} \frac{d\zeta}{\zeta - z}$ is a multiple of $2\pi i$.*

Proof. $\gamma : \zeta = \zeta(t), \alpha \leq t \leq \beta$. $h(t) = \int_{\alpha}^t \frac{\zeta'(s)}{\zeta(s) - z} ds$.

$z \in \gamma \Rightarrow h$ is defined and continuous on $[\alpha, \beta]$. For all t s.t. $\zeta'(t)$ is continuous, we have

$$h'(t) = \frac{\zeta'(t)}{\zeta(t) - z} \Rightarrow \frac{d}{dt} [e^{-h(t)}(\zeta(t) - z)] = 0$$



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