

# Complex Analysis

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# 1 Harmonic function

**Definition 1.1** (Cauchy-Riemann equation).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**Definition 1.2** (Harmonic function). A function  $u$  is **harmonic** if it satisfied **Laplace equation**  $\Delta u = 0$ .

If two harmonic function  $u$  and  $v$  satisfies Cauchy-Riemann equations, then we say that  $v$  is **conjugate harmonic function of  $u$**   $\Rightarrow u$  is conjugate harmonic of  $-v$ .

## 1.1 Polynomial rational function

The polynomial  $P(z) = \sum_{j=0}^n a_j z^j$  is analytic in  $\mathbb{C}$ .

We will prove the fundamental theorem of algebra

**Theorem 1.3** (Fundamental Theorem of Algebra). *Every polynomial with degree  $n > 0$  has at least one point.*

**Theorem 1.4** (Gauss-Lucus theorem). *The smallest convex polygon that contain the zeros of  $P$  also contains the zeros of  $P'$ .*

*Proof.* Only need to check.

We can get this equation.

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{j=1}^n \frac{1}{\alpha - \alpha_j} = 0 \Rightarrow \sum_{j=1}^n \frac{\overline{\alpha - \alpha_j}}{|\alpha - \alpha_j|^2}$$

Hence  $\alpha$  is linearly represented by  $\alpha_j$ . □

**Proposition 1.5.** Let  $P$  and  $Q$  be two polynomials with no common zeros. Then the rational function  $R(z) = \frac{P(z)}{Q(z)}$  is analytic away from the zeros of  $Q$ .

The zeros of  $Q$  are called **poles** of  $R$ , and the **order of a pole** is equal to the order of the corresponding zero of  $Q$ .

We often view  $R$  as a function from  $\hat{\mathbb{C}}$  to  $\hat{\mathbb{C}}$ .  $R_1(z) := R(\frac{1}{z})$ .

If  $R_1(0) = 0$ , the order of the zero at  $\infty$  (of  $R$ ) is the order of the zero of  $R_1(z)$  at  $z = 0$ .

If  $R_1(0) = \infty$ , the order of the pole at  $\infty$  (of  $R$ ) is the order of the pole of  $R_1(z)$  at  $z = 0$ .

Suppose

$$R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, a_n \neq 0, b_m \neq 0$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$$

By discussing  $m$  and  $n$ , we can infer the situation of  $R(z)$  at  $\infty$ .

By adding the order of poles and zeros at  $\infty$ , we can get the following theorem.

**Theorem 1.6.** The total number of zeros and poles of a rational function are the same.

**Remark 1.7.** This common number is called the **order of the rational function**.

**Corollary 1.8.** Suppose a rational function  $R$  has order  $p$ . Then every equation  $R(z) = a$  has exactly  $p$  roots.

*Proof.*  $\hat{R}(z) = R(z) - a$  has the same poles as  $R$ . □

A rational function of order 1 is a **linear fraction**  $R(z) = \frac{az+b}{cz+d}$ ,  $ad - bc \neq 0$

Such fraction is often called **Möbius transformation**

Every rational function has a representation by **partial fractions**.

- If  $R$  has a pole at  $\infty$ . Then we can write

$$R(z) = G(z) + H(z) \quad (*)$$

where  $G$  is a polynomial without constant term, and  $H$  is finite at  $\infty$ .

The degree of  $G$  is the order of the pole of  $R$  at  $\infty$ .  $G$  is called the **singular part** of  $R$  at  $\infty$ .

- Let the distinct finite poles of  $R$  be  $\beta_1, \dots, \beta_k$ . Let  $R_j(\psi) = R(\beta_j + \frac{1}{\psi})$ . Then  $R_j$  is a rational function with a pole at  $\infty$ . As in (\*), we can write

$$R_j = G_j + H_j$$

with  $H_j$  finite at  $\infty$ . Then

$$R(z) = G_j\left(\frac{1}{z - \beta_j}\right) + H\left(\frac{1}{z - \beta_j}\right)$$

with  $G_j$  is a polynomial in  $\frac{1}{z - \beta_j}$  without constant term called the **singular point** of  $R$  at  $\beta_j$ .

- Let  $F(z) = R(z) - G(z) - \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right)$ .

Then  $F$  is a rational function which can only have poles among  $\beta_j, \infty$

Since by our construction,  $F$  is finite at every  $\beta_j, 1 \leq j \leq k$  and  $\infty$ .

So  $F$  is a constant.

In particular,  $R(z) = G(z) + \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right) + c$ .

## 2 Power Series

**Theorem 2.1** (Abel's theorem). *If  $\sum a_n$  converges, then  $f(z) = \sum a_n z^n \rightarrow f(1)$  as  $z \rightarrow 1$  in such a way that  $\frac{|1-z|}{1-|z|}$  remains bounded.*

## 3 Exponential, Trigonometric and logarithmic functions

### 3.1 Exponential and Trigonometric function

The **exponential function** is defined as the solution of the differential equation

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1 \end{cases}$$

We denote  $e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ .

The **trigonometric function** are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

### 3.2 Logarithmic Functions

The **logarithmic function**  $\ln$  is defined by  $z = \ln w$  is a root of the equation  $e^z = w$ .

For  $w \neq 0$ , we write  $z = x + iy$ , then

$$e^{x+iy} = w \Leftrightarrow \begin{cases} e^x = |w| \\ e^{iy} = \frac{w}{|w|} \end{cases}$$

The first equation has a unique solution  $x = \ln |w|$ .

The second equation  $e^{iy} = \frac{w}{|w|}$  has a unique solution  $y_0 \in [0, 2\pi)$ .

If we write  $w = re^{i\theta}$ , then  $x = \ln w, y = \theta = \arg w$ .

Thus, for  $w \neq 0$ , we have

$$\ln w = \ln |w| + i \arg w$$

The function  $\ln$  is actually not single-valued. But we can define a single-valued function  $Ln$

We define

$$a^b = \exp(b \ln a)$$

We will prove  $Ln$  is analytic in  $\mathbb{C} - (-\infty, 0]$  but not continuous in  $(-\infty, 0]$ .

$Ln$  is the principal branch of the logarithm.

## 4 Conformal Mappings

### 4.1 Connectedness

**Theorem 4.1.** *A nonempty open set in  $\mathbb{C}$  is connected iff any two of its points can be joined by a polygon which lies in the set. ( i.e. Connectedness is equivalent to Path Connectedness)*

An nonempty connected subset is called a **region**

### 4.2 Compactness

**Definition 4.2.** A set  $X$  is **totally bounded** if  $\forall \varepsilon > 0$ ,  $X$  can be covered by finitely many balls of radius  $\varepsilon$

**Theorem 4.3.** *A set is compact iff it is complete and totally bounded.*

**Theorem 4.4.** *A subset  $X \subset \mathbb{C}$  is compact iff every infinite sequence of  $X$  has a limit point in  $X$ .*

### 4.3 Continuous Functions

**Theorem 4.5.** *Continuous function maps connected space to connected space.*

**Theorem 4.6.** *Continuous function maps compact space to compact space.*

### 4.4 Arcs and closed curves

The equation of an **arc**  $r$  in  $\mathbb{C}$  can be represented by one of the terms

- $x = x(t), y = y(t), \alpha \leq t \leq \beta, x, y$  are continuous at  $t$
- $z(t) = x(t) + iy(t), \alpha \leq t \leq \beta.$
- The continuous mapping  $\gamma : [\alpha, \beta] \rightarrow \mathbb{C}.$

For a non-decreasing function  $\varphi : [\alpha, \beta] \rightarrow [\alpha, \beta], z = z(\varphi(t)), \alpha' \leq \tau \leq \beta'$  is **change of parameter** of  $z(t).$

The change is **reversible** iff  $\varphi$  is strictly increasing.

If  $\gamma$  is differentiable, then call  $\gamma$  a **curve**.

$\gamma$  is **simple**, or a **Jordan curve**, if  $\gamma$  is injective.

$\gamma$  is **closed curve** if  $\gamma(0) = \gamma(1).$

### 4.5 Analytic Functions in Regions

A function  $f$  is analytic on an arbitrary set  $A$  if it is the restriction to  $A$  of a function which is analytic in some open set containing  $A$ .



**Theorem 4.7.** *An analytic function in a region( i.e. open and connected)  $\Omega$  whose derivative is 0 must reduce to a constant. The same hold if the real part, the imaginary part, the modulus, or the argument is constant.*

## 4.6 Conformal Mappings

Suppose  $f : \Omega \rightarrow \mathbb{C}$  is analytic in  $\Omega$ .  $r_1 = z_1(t), r_2 = z_2(t), \alpha \leq t \leq \beta$ .

$$z_0 = z_1(t_0) = z_2(t'_0), z'_1(t_0) \neq 0, z'_2(\hat{t}_0) \neq 0, \alpha < t_0, \hat{t}_0 < \beta.$$

$$f'(z_0) \neq 0, w_1(t) = f(z_1(t)), w_2 = f(z_2(\hat{t}_0))$$

$$\Gamma_1 = \{w_1(t) | \alpha \leq t \leq \beta\}, \Gamma_2 = \{w_2(t) | \alpha \leq t \leq \beta\}$$

Then

$$w'_1(t) = f'(z_1(t))z'_1(t)$$

$$w'_2(t) = f'(z_2(\hat{t}))z'_2(\hat{t})$$

$\Rightarrow$

$$w'_1(t_0) \neq 0, w'_2(t_0) \neq 0$$

$$\arg w'_1(t_0) = \arg f'(z_1(t_0))z'_1(t_0)$$

$$\arg w'_2(t_0) = \arg f'(z_2(\hat{t}_0))z'_2(\hat{t}_0)$$

So the "angle"  $\arg w'_1(t_0) - \arg w'_2(\hat{t}_0) = \arg z_1(t_0) - \arg z_2(\hat{t}_0)$  remains the same.

Now we give the definition.

**Definition 4.8.**  $w = f(z)$  is said to be **conformal** in  $\Omega$  if  $f$  is analytic in  $\Omega$  and  $f'(z) \neq 0$  for  $\forall z \in \Omega$ .

Easy to prove that linear change of scale at  $z_0$  is independent of the direction.

$$i.e. |f'(z_0)| = \lim_{z \rightarrow z_0} \frac{\delta \sigma}{\delta s}$$

## 4.7 Length and Area

The **length** of a differentiable arc  $\gamma$  with the equation  $z(t) = x(t) + iy(t)$ ,  $a \leq t \leq b$

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b |z'(t)| dt$$

For  $\Gamma = f(\gamma)$  where  $f$  conformal mapping.

Then

$$L(\Gamma) = \int_a^b |f'(z(t))| \cdot |z'(t)| dt$$

The **area** of  $E \subset \mathbb{R}$  is  $A(E) = \iint_E dx dy$

Then by the differentiable functional transformation, the area  $\hat{E} = f(E)$  is

$$A(\hat{E}) = \iint_E |u_x v_y - u_y v_x| dx dy$$

If  $f$  is the conformal mapping of an open set containing  $E$ , then by Cauchy-Riemann equation

$$A(\hat{E}) = \iint_E |f'(z)|^2 dx dy$$

## 5 Möbius Transformation

Recall that a **Möbius transformation** is a function of the form

$$w = s(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Then it has an inverse  $z = S^{-1}(w) = \frac{dw - b}{-cw + a}$ .

We may define  $S(\infty) = \lim_{z \rightarrow \infty} S(z) = \frac{a}{c}$ ,  $S(-\frac{d}{c}) = \infty$

With these definition,  $S : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is a topological mapping. Here one may use the chordal metric to define the topology.

$$S'(z) = \frac{ad - bc}{(cz + d)^2}$$

Then  $S$  is conformal in  $\hat{\mathbb{C}} - \{-\frac{d}{c}, \infty\}$ .

$w = z + \alpha$  is called a **parallel translation**.

$w = kz$  with  $|k| = 1$  is a **rotation**.

$w = kz$  with  $k > 0$  is a **homothetic transformation**.

$x = \frac{1}{z}$  is called an **inversion**.

**Proposition 5.1.** *Every Möbius transformation is a composition of the above four operations.*

## 5.1 Cross ratio

For three distinct points  $z_2, z_3, z_4 \in \hat{\mathbb{C}}$ , we can find a Möbius transformation  $S$  such that  $S(z_2) = 0, S(z_3) = 1, S(z_4) = \infty$ .

**Lemma 5.2.** *The Möbius transformation satisfying the above conditions is unique.*

The **cross ratio**  $(z_1, z_2, z_3, z_4)$  is the image  $z_1$  under the Möbius transformation which maps  $z_2$  to 1,  $z_3$  to 0 and  $z_4$  to  $\infty$ .

**Theorem 5.3.** *If  $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$  are distinct, and  $T$  is any Möbius transformation, then  $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$ .*

**Lemma 5.4.** *Let  $T$  be a Möbius transformation,  $T(\mathbb{R})$  is either a circle or a straight line.*

**Theorem 5.5.** *The cross ratio  $(z_1, z_2, z_3, z_4)$  is real iff the four points lie on a circle or a straight line.*

**Remark 5.6.** One may prove the theorem by elementary geometry

**Theorem 5.7.** *A Möbius transformation maps circles into circles.*

## 5.2 Symmetry

Suppose  $T$  is a Möbius transformation which maps  $\hat{\mathbb{R}}$  onto a circle  $C$ .

We say that  $w = Tz$  and  $w^* = T\bar{z}$  are **symmetric** w.r.t.  $C$ .

**Remark 5.8.** This definition is independent of  $T$ . Suppose  $S$  is another Möbius transformation which maps  $\hat{\mathbb{R}}$  onto  $C$ , then  $S^{-1}T$  maps  $\hat{\mathbb{R}}$  to  $\hat{\mathbb{R}}$ , and this  $S^{-1}w = S^{-1}Tz$  and  $S^{-1}w^* = S^{-1}T\bar{z}$  are conjugate.

The points  $z$  and  $z^*$  are **symmetric w.r.t  $C$  through**  $z_1, z_2, z_3$  iff  $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$ .

This can be another definition.

Note that only the points on  $C$  are symmetric to themselves.

The mapping  $z \mapsto z^*$  is 1-1 and is called **reflection** w.r.t.  $C$ .

### 5.2.1 Geometric Meaning of Symmetry

Case1:  $C$  is a straight line. We may assume  $z_3 = \infty$ .

$z, z^*$  are symmetric w.r.t.  $C$  if and only if

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Then

$$|z^* - z_2| = |z - z_2|, \quad \forall z_2 \in C \text{ and } z_2 \neq \infty$$

$$\operatorname{Im} \frac{z^* - z_2}{z_1 - z_2} = \operatorname{Im} \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

So  $C$  is the bisecting normal of the segment between  $z$  and  $z^*$ .

Case2:  $C$  is the circle  $|z - a| = R$ .

$$\begin{aligned} \text{Then for } \forall \text{ distinct } z_1, z_2, z_3 \in \mathbb{C}, \quad & \overline{(z, z_1, z_2, z_3)} = \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)} \\ & = (\bar{z} - \bar{a}, \bar{z}_1 - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}) = (\bar{z} - \bar{a}, \frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}) = (\frac{R^2}{\bar{z} - \bar{a}}, z_1 - a, z_2 - \end{aligned}$$

$$a, z_3 - a) \\ = \left( \frac{R^2}{\bar{z} - \bar{a}}, z_1, z_2, z_3 \right).$$

Then the symmetric point of  $z$  w.r.t.  $C$  is

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

or

$$(z^* - a)(\bar{z} - \bar{a}) = R^2$$

$\Rightarrow$

$$\begin{cases} |z^* - a| \cdot |z - a| = R^2 \\ \frac{z^* - a}{z - a} = \frac{(z^* - a)(\bar{z} - \bar{a})}{|z - a|^2} > 0 \end{cases}$$

**Theorem 5.9** (The Symmetric principle). *If a Möbius transformation maps a circle  $C_1$  onto a circle  $C_2$ , then it transforms any pair of symmetric points w.r.t.  $C_1$  into a pair of symmetric points w.r.t.  $C_2$ .*

*Proof.* Case1:  $C_1 = \hat{\mathbb{R}}$ . Let  $T$  be the Möbius transformation which maps  $\hat{\mathbb{R}}$  onto  $C_2$ .  $\forall z \in \mathbb{C}$ , by definition,  $w = Tz$  and  $w^* = T\bar{z}$  are symmetric w.r.t.  $C_2$ .

Case2:  $C_1$  is a general circle. Let  $T : C_1 \rightarrow C_2$  and  $S : \mathbb{R} \rightarrow C_2$  be Möbius transformation.

Suppose  $w, w^*$  are symmetric w.r.t.  $C_1$ . Then there exists  $z$  s.t.  $w = Sz, w^* = S\bar{z}$ .

Then we can find  $Tw = TSz, Tw^* = TS\bar{z}$  are symmetric w.r.t.  $C_2$  since  $TS : \hat{\mathbb{R}} \rightarrow C_2$  □

**Remark 5.10.** (1). The Möbius transformation from  $C_1$  to  $C_2$  satisfies  $z_1 \mapsto$

$w, z_2 \mapsto w_2, z_3 \mapsto w_3$  where  $z_1, z_2, z_3 \in C_1, w_1, w_2, w_3 \in C_2$  is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

- (2). The Möbius transformation from  $C_1$  to  $C_2$  satisfies  $z_1 \mapsto w_1, z_2 \mapsto w_2$  where  $z_1 \in C_1, z_2 \notin C_1, w_1 \in C_2, w_2 \notin C_2$  is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

### 5.3 Steiner Circles

For  $S(z) = \frac{az+b}{cz+d}, S'(z) = \frac{ad-bc}{(cz+d)^2}$ .

A point  $z \notin$  a circle  $C$  is said to on the **right(left, resp.)** of  $C$  if  $\text{Im}(z, z_1, z_2, z_3) > 0(\text{Im}(z, z_1, z_2, z_3) < 0)$

**Remark 5.11.**

- (1). This agrees with everyday use since  $(i, 1, 0, \infty) = i$
- (2). This distinct between left and right is the same for all triples, while the meaning may be reversed.

(If  $C = \hat{\mathbb{R}}$ , then  $(z, z_1, z_2, z_3) = \frac{az+b}{cz+d}$  with  $a, b, c, d \in \mathbb{R} \Rightarrow \text{Im}(z, z_1, z_2, z_3) = \frac{ad-bc}{|cz+d|^2} \text{Im}(z)$ )

- (3). We can define an absolute positive orientation of all finite circles by requiring that  $\infty$  should be lie to the right of the oriented circles.

Consider a Möbius transformation of the form

$$w = k \cdot \frac{z-a}{z-b}$$

Here,  $z = a \mapsto w = 0, z = b \mapsto w = \infty$ .

Then circles through  $a, b$  maps to straight line through  $0, \infty$ .

The concentric circle about the origin,  $|w| = \rho$ , correspond to circles with the equation

$$\left| \frac{z - a}{z - b} \right| = \frac{\rho}{|k|}$$

These are the circles of **Apollonius** with limit points  $a$  and  $b$ .

Denote by  $C_1$  the circles through  $a, b$  and  $C_2$  the circles of Apollonius with these limit points. The configuration formed by all the circles  $C_1$  and  $C_2$  is called the **Steiner circles**(or **circular net**)

**Theorem 5.12.**

- (a) *There is exactly one  $C_1$  and one  $C_2$  through each point in  $\hat{\mathbb{C}} \setminus \{a, b\}$*
- (b) *Every  $C_1$  meets every  $C_2$  under right angle.*
- (c) *Reflection in a  $C_1$  transforms every  $C_2$  into itself and every  $C_1$  into another  $C_1$ .*
- (d) *The limit points  $a, b$  are symmetric w.r.t. each  $C_2$ , but not w.r.t. other circles.*

*Proof.* If the limit points are  $0, \infty$ , those properties are trivial in the  $w$ -plane. The general case follows since all properties are invariant under Möbius transformations. □

## 6 Elementary Conformal mapping

**Example 6.1.**  $w = z^\alpha$  where  $\alpha > 0$ .

Let  $S(u_1, u_2)$  with  $0 < \varphi_2 - \varphi_1 \leq 2\pi$  be  $\{z \in \mathbb{C} : z \neq 0, \varphi_1 < \arg(z) < \varphi_2\}$  where  $\arg(z)$  can be chosen as any value of it.

Then  $S(\varphi_1, \varphi_2)$  is a region.

In this region, a unique value of  $w = z^\alpha$  is defined by  $\arg w = \alpha \arg z$ .

This function is analytic with  $\frac{dw}{dz} = \alpha \frac{w}{z}$ .

This function is 1-1 only if  $\alpha(\varphi_2 - \varphi_1) \leq 2\pi$ .

**Example 6.2.**  $w = e^z$  maps  $\{z \in \mathbb{C} : -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2}\}$  onto  $\{w \in \mathbb{C} : \text{Re}(w) > 0\}$

**Example 6.3.**  $w = \frac{z-1}{z+1}$  maps  $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$  onto  $\{w \in \mathbb{C} : |w| < 1\}$

**Example 6.4.**

$$\mathbb{C} \setminus [-1, 1] \xrightarrow{z_1 = \frac{z+1}{z-1}} \mathbb{C} \setminus (-\infty, 0] \xrightarrow{z_2 = \sqrt{z_1}} \{\text{Re}(z_2) > 0\} \xrightarrow{w = \frac{z_2-1}{z_2+1}} \{w \in \mathbb{C} : |w| < 1\} \quad (6.1)$$

## 6.1 Elementary Riemann surfaces

**Example 6.5.**  $w = z^n, n \in \mathbb{Z}_+$  and  $n > 1$ .

There is a 1-1 correspondence between each angle  $\frac{(k-1)2\pi}{n} < \arg z < \frac{k \cdot 2\pi}{n}, k = 1, 2, \dots, n$  and while  $w$ -plane except for the positive real axis.

**Example 6.6.**  $w = e^z$ . This function maps each parallel strip  $(k-1)2\pi < \text{Im } z < k \cdot 2\pi, k \in \mathbb{Z}$  onto a sheet with a cut along the positive axis.

## 7 Complex Integration

### 7.1 Fundamental Theorems

#### 7.1.1 Line integral and rectifiable arcs

Let  $f(t) = u(t) + iv(t)$  be a complex-valued defined on  $t \in [a, b] \subset \mathbb{R}$  where  $u, v$  are real-valued functions. If  $f$  is continuous on  $[a, b]$ , we may define the **integral**

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt$$



Let  $\gamma$  be a piecewise differential arc in  $\mathbb{C}$  with the equation  $z = z(t)$ ,  $a \leq t \leq b$ . If  $f$  is continuous on  $\gamma$ , then  $f(z(t))$  is continuous on  $[a, b]$ , and we define

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt \quad (7.1)$$

The integral defined in 7.1 is independent of the parametrization of  $\gamma$ . Suppose that another parametrization of  $\gamma$  is  $\gamma : (\alpha, \beta) \rightarrow \mathbb{C}, \tau \mapsto z(t(\tau))$ , where  $t : (\alpha, \beta) \rightarrow (a, b), \tau \mapsto t(\tau)$  is piecewise differentiable. Then we have

$$\int_a^b f(z(t))z'(t)dt = \int_{\alpha}^{\beta} f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau = \int_{\alpha}^{\beta} f(z(t(\tau)))\frac{dz(t(\tau))}{d\tau}d\tau \quad (7.2)$$

For an arc  $\gamma$  with equation  $z = z(t), a \leq t \leq b$ , we define  $-\gamma$  by  $z = z(-t), -b \leq t \leq a$ .

Then we have

$$\begin{aligned} \int_{-\gamma} f(z)dz &= \int_{-b}^{-a} f(z(-t))\frac{dz(-t)}{dt}dt \\ &= - \int_{-a}^{-b} f(z(-t))z'(-t)dt \\ &= - \int_a^b f(z(\tau))z'(\tau)d\tau \\ &= - \int_{\gamma} f(z)dz \end{aligned}$$

So we have those properties:

**Proposition 7.1.**

$$(a) \int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$$

(b) Let  $f$  and  $g$  be two continuous functions on the piecewise differentiable arc  $\gamma$ , then

$$\int_{\gamma} (\lambda_1 f + \lambda_2 g) dz = \lambda_1 \int_{\gamma} f dz + \lambda_2 \int_{\gamma} g dz, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

(c) If  $\gamma$  can be subdivided into two pieces differentiable arcs  $\gamma_1$  and  $\gamma_2$ , and  $f$  is continuous on  $\gamma_1$ , then

$$\int_{\gamma} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz$$

(d) (c) implies that the integral of a closed curve doesn't depend on the starting point on the curve

**Example 7.2.** Evaluate  $\int_{\gamma} \frac{1}{z-a} dz$  where  $\gamma$  is the circle centered at  $a \in \mathbb{C}$  with radius  $R$ .

Let  $z = z(t) = a + Re^{it}$ . Then the integral is  $2\pi i$

### 7.1.2 The fundamental theorem of Calculus for integrals in $\mathbb{C}$

The line integral w.r.t.  $\bar{z}$  is defined by

$$\int_{\gamma} f(z) d\bar{z} = \overline{\int_{\gamma} \overline{f(z)} dz}$$

With this notation, line integrals w.r.t.  $x = \operatorname{Re}(z)$  and  $y = \operatorname{Im}(z)$  can be defined by

$$\int_{\gamma} f(z) dx = \frac{1}{2} \left[ \int_{\gamma} f(z) dz + \int_{\gamma} f(z) d\bar{z} \right]$$

$$\int_{\gamma} f(z) dy = \frac{1}{2i} \left[ \int_{\gamma} f(z) dz - \int_{\gamma} f(z) d\bar{z} \right]$$

if we write  $f(z) = \mu + i\nu$ , we have

$$\int_{\gamma} f(z) dz = \int_{\gamma} f(z) dx + i \int_{\gamma} f(z) dy = \int_{\gamma} (\mu dx - \nu dy) + i \int_{\gamma} (\nu dx + \mu dy)$$

**Remark 7.3.** It is followed by the intuition. We can view the integration as the multiplication between  $f$  and  $dz$ .

The integral w.r.t. **arc length** is defined by

$$\int_{\gamma} f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt$$

This integral is again independent of the parametrization. It is easy to check

$$\int_{-\gamma} f(z)|dz| = \int_{\gamma} f(z)|dz|$$

Now we define **length** of a curve  $\gamma$ :  $L(\gamma) = \int_{\gamma} |dz|$

We have the inequality:

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| \cdot |dz| \leq L(\gamma) \cdot \sup_{z \in \gamma} |f(z)|$$

The length of an arc  $\gamma$  ( $z = z(t)$ ) can also be defined as the least upper bound of all sums

$$\sum_{i=1}^n |z(t_i) - z(t_{i-1})|$$

where  $a = t_0 < t_1 < \dots < t_n = b$  If this least upper bound is finite, we say that the arc is **rectifiable**

It is easy to show that piecewise differentiable arcs are rectifiable.

The integral of a continuous function  $f$  on a rectifiable arc may be defined as

$$\int_{\gamma} f(z)dz = \lim \sum_{k=1}^n f(z(\psi_k))[z(t_k) - z(t_{k-1})]$$

**Theorem 7.4.** Let  $\Omega \subset \mathbb{C}$  be a region, and  $P, Q$  two (possibly complex-valued) functions that are continuous on  $\Omega$ ,  $\gamma$  closed curve. The integral  $\int_{\gamma} p(x, y)dx + Q(x, y)dy$  depends only on the end point of  $\gamma$  iff there exists a function  $U(x, y)$  on  $\Omega$  with  $\frac{\partial U}{\partial x} = P, \frac{\partial U}{\partial y} = Q$ .

*Proof.* " $\Leftarrow$ ": If such a  $U$  exists, then

$$\int_{\gamma} Pdx + Qdy = \int_{\gamma} \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = \int_{\gamma} \frac{dU}{dt}dt = U(\gamma(b)) - U(\gamma(a))$$

" $\Rightarrow$ ": Fix a point  $(x_0, y_0) \in \Omega$ . We define  $U(x, y) = \int_{\gamma} Pdx + Qdy$  where  $\gamma$  is any curve between  $(x_0, y_0)$  and  $(x, y)$ . Easy to check that it is true.  $\square$

**Theorem 7.5** (Fundamental theorem of Calculus for integrals on  $\mathbb{C}$ ). Let  $f$  be continuous on a region  $\Omega$  containing  $\gamma$ .  $\int_{\gamma} f dz$  depends on the endpoints iff  $f$  is the derivative of an analytic function  $F$  in  $\Omega$ .

**Remark 7.6.** We will prove  $\int_{\gamma} f dz = F(\omega_2) - F(\omega_1)$  where  $\gamma$  begins at  $\omega_1$  and ends at  $\omega_2$ .

*Proof.* Transform the line integration into the composition of two real integration.  $\square$

**Corollary 7.7.** If  $F$  is analytic on  $\Omega$  with  $F' = f$ , and  $\gamma$  is a closed curve in  $\Omega$ , then  $\int_{\gamma} f dz = 0$ . Conversely if  $f$  is continuous on  $\Omega$  and  $\int_{\gamma} f dz = 0$  for any closed curve in  $\Omega$ , then  $f$  is the derivative of an analytic function  $F$  in  $\Omega$ .

### 7.1.3 Cauchy's theorem for a rectangle

There is some notes in this section:

$R$  is the rectangle in  $\mathbb{C}$ ,  $R = \{x + iy \in \mathbb{C} : a \leq x \leq b, c \leq y \leq d\}$ . And  $\partial R$  is boundary curve oriented in the counterclockwise direction.

**Theorem 7.8** (Cauchy's theorem for a rectangle). *If  $f$  is analytic on an open set which contains  $R$ , then  $\int_{\partial R} f(z)dz = 0$*

*Proof.* For  $\forall$  rectangle  $\tilde{R}$  inside  $R$ , we define  $Z(\tilde{R}) = \int_{\partial \tilde{R}} f(z)dz$ . Then  $Z(R) = Z(R_1) + Z(R_2)$  if  $R$  is divided into  $Z_1, Z_2$ .

Since we can divide  $R$  into four equal rectangles, and find a rectangle with  $|Z(R^{(1)})| \geq \frac{1}{4}|Z(R)|$ . Then repeat the above steps and we obtain a sequence of nested rectangles  $R \supset R^{(1)} \supset \dots$  with the property

$$|Z(R^{(n)})| \geq \frac{1}{4^n} |Z(R)| \geq \dots \geq \frac{1}{4^n} |Z(R)| \quad (7.3)$$

$\forall \delta > 0, \exists n \in \mathbb{N}$  s.t.  $R^{(n)} \subset \{z \in \mathbb{C} : |z - z_0| < \delta\}$ ,  $\forall n \geq N$ , where  $z_0$  is the limit of  $R^{(n)}$  as  $n \rightarrow \infty$ .

$f$  is analytic in  $R \Rightarrow \forall \varepsilon, \exists \delta > 0$  s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon, \forall z \text{ with } |z - z_0| < \delta \quad (7.4)$$

We assume that  $\delta$  satisfies both conditions. We have

$$\begin{aligned} Z(R^{(n)}) &= \int_{\partial R^{(n)}} f(z)dz = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)]dz \\ &\Rightarrow |Z(R^{(n)})| \leq \varepsilon \int_{\partial R^{(n)}} |z - z_0|dz \text{ by 7.4} \end{aligned}$$

Let  $d_n$  be the length of diagonal of  $R^{(n)}$ ,  $L_n$  be the length of its perimeter. Then  $|z - z_0| \leq d_n, \forall z \in \partial R^{(n)}$ .

$\Rightarrow |Z(R^{(n)})| \leq \varepsilon d_n L_n = \varepsilon \frac{D}{2^n} \cdot \frac{L}{2^n}$  where  $D, L$  are the diameter and perimeter of  $R$ .

$\Rightarrow |Z(R)| \stackrel{7.3}{\leq} 4^n |Z(R^{(n)})| \leq \varepsilon DL \Rightarrow Z(R) = 0$  since  $\varepsilon$  is arbitrary.  $\square$

We will next prove the following stronger theorem:

**Theorem 7.9** (stronger version of Cauchy's theorem for a rectangle). *Let  $f$  be analytic on  $R' = R \setminus \{\psi_1, \dots, \psi_m\}$ ,  $m \in \mathbb{N}$ . If  $\lim_{z \rightarrow \psi_j} (z - \psi_j)f(z) = 0$ ,  $\forall 1 \leq j \leq m$ , then*

$$\int_{\partial R} f(z)dz = 0.$$

*Proof.* WLOG, we may assume  $f$  is not analytic at only one point  $\psi \in R$ . If we put  $\psi$  into a small rectangle  $S_0$ , then the previous theorem tells us  $\int_{\partial R} f(z)dz = \int_{\partial S_0} f(z)dz$ .

$\forall \varepsilon > 0$ , we may choose  $S_0$  small enough such that  $|f(z)| \leq \frac{\varepsilon}{|z - \psi|}$ ,  $\forall z \in \partial S_0$

$$\Rightarrow \left| \int_{\partial R} f(z)dz \right| \leq \varepsilon \int_{\partial S_0} \frac{|dz|}{|z - \psi|} \leq \varepsilon \frac{1}{\frac{l}{2}} \cdot 4l = 8\varepsilon$$

$\Rightarrow \int_{\partial R} f(z)dz = 0$  since  $\varepsilon$  is arbitrary. □

### 7.1.4 Cauchy's Theorem for a disk

$\Delta := \{z \in \mathbb{C} : |z - z_0| < R\}$  where  $R > 0$ .

**Theorem 7.10** (Cauchy's Theorem for a disk). *If  $f$  is analytic in an open disk  $\Delta$ , then  $\int_{\gamma} f(z)dz = 0$  for closed curve  $\gamma$  in  $\Delta$ .*

*Proof.* Suppose the center of  $\Delta$  is  $z_0 = x_0 + iy_0$ ,  $z = x + iy$ . We define

$$F(z) = \int_{\gamma} f(z)dz$$

where  $\gamma$  is the horizontal line segment from  $z_0$  to  $(x, y_0)$  added with vertical line segment from  $(x, y_0)$  to  $z$ . We have

$$\frac{\partial F}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{F(x, y + \delta y) - F(x, y)}{\delta y} = \lim_{\delta y \rightarrow 0} \frac{1}{\delta y} \int_{\delta \gamma} f(z)dz = if(z) \quad (7.5)$$

By Cauchy's theorem on rectangles, one has  $F(z) = -\int_{\tilde{\gamma}} f(z)dz$ , where  $\tilde{\gamma}$  is the vertical line segment from  $z_0$  to  $(x_0, y)$  added with horizontal line segment from

$(x_0, y)$  to  $z$ .

Similarly,  $\frac{\partial F}{\partial x} = f(z)$ .

$\Rightarrow \frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y} \Rightarrow F$  is analytic in  $\Delta$  with derivative  $f$ .

By Fundamental Theorem 7.5 of Calculus  $\Rightarrow \int_{\gamma} f(z)dz = 0$  for  $\forall$  closed curve in  $\Delta$ . □

Here is a stronger version.

**Theorem 7.11** (stronger version of Cauchy's Theorem for a disk). *Let  $f$  be analytic in a region  $\Delta' = \Delta \setminus \{\psi_1, \dots, \psi_m\}$  with  $m \in \mathbb{N}$ . If  $f$  satisfies  $\lim_{z \rightarrow \psi_j} (z - \psi_j)f(z) = 0, \forall 1 \leq j \leq m$ , then  $\int_{\gamma} f(z)dz = 0, \forall \gamma$  closed in  $\Delta'$*

*Proof.* It is similar to the above proof.

For the case no  $\psi_j$  lies on  $x = x_0$  and  $y = y_0$ , we can find a similar curve  $\gamma$  with last segment is a vertical one. Let  $F(z) = \int_{\gamma} f(z)dz$ . And continue the process of proof of the previous theorem.

For the case that  $\exists \psi_j$  lies on the lines  $x = x_0, y = y_0$ , we actually can move the center to another point s.t. no  $\psi_j$  lies on the lines  $x = x'_0, y = y'_0$ . □

## 7.2 Cauchy's integral formula

### 7.2.1 Index of a point with respect to a closed curve

**Lemma 7.12.** *If the piecewise differentiable closed curve  $\gamma$  does not pass through  $z \in \mathbb{C}$ , then the value of the integral  $\int_{\gamma} \frac{d\zeta}{\zeta - z}$  is a multiple of  $2\pi i$ .*

*Proof.*  $\gamma : \zeta = \zeta(t), \alpha \leq t \leq \beta$ .  $h(t) = \int_{\alpha}^t \frac{\zeta'(s)}{\zeta(s) - z} ds$ .

$z \in \gamma \Rightarrow h$  is defined and continuous on  $[\alpha, \beta]$ . For all  $t$  s.t.  $\zeta'(t)$  is continuous, we have

$$h'(t) = \frac{\zeta'(t)}{\zeta(t) - z} \Rightarrow \frac{d}{dt} [e^{-h(t)}(\zeta(t) - z)] = 0$$

So  $e^{-h(t)}(\zeta(t) - z)$  is constant on  $[\alpha, \beta]$ .

Then  $e^{h(t)} = \frac{\zeta(t) - z}{\zeta(\alpha) - z} \Rightarrow e^{h(\beta)} = 1 \Rightarrow h(\beta) \in \{2k\pi i : k \in \mathbb{Z}\}$ . □

The **index of the point**  $z$  w.r.t. the closed curve  $\gamma$  is the number

$$n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z}$$

$n$  is also called the **winding number**.

**Theorem 7.13.** *Let  $\gamma$  be a piecewise differentiable closed curve. The function  $z \mapsto n(\gamma, z)$  is constant on each connected set of  $\mathbb{C} \setminus \gamma$ , and zero if this set is unbounded.*

*Proof.* Define  $f : \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}, z \mapsto n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z}$ .

Then

$$|f(z) - f(z_0)| = \frac{1}{2\pi} \left| \int_{\gamma} \frac{z - z_0}{(\zeta - z)(\zeta - z_0)} d\zeta \right| \leq \frac{|z - z_0|}{2\pi} \int_{\gamma} \frac{1}{|\zeta - z| \cdot |\zeta - z_0|} |d\zeta|$$

$\Rightarrow f$  is continuous on each open connected set of  $\mathbb{C} \setminus \gamma$ . Let  $\Omega$  be any open connected set of  $\mathbb{C} \setminus \gamma$ . We have  $f(\Omega)$  is connected  $\xRightarrow{f(\Omega) \subset \mathbb{Z}} f(\Omega)$  contains at most one point  $\Rightarrow f$  is constant on  $\Omega$ .

If  $|z|$  is sufficient large,  $\exists$  a disk of radius  $R$ ,  $B(0, R)$ , s.t.  $\gamma \subset B(0, R)$  but  $z \notin B(0, R)$ . Cauchy's theorem for a disk 7.10 tells us that  $f(z) = n(\gamma, z) = 0$ . So it is zero if this set is unbounded. □

**Lemma 7.14.** *Let  $z_1, z_2$  be two points on a closed curve  $\gamma$  and  $0 \notin \gamma$ .*

*Suppose  $z_1$  in the lower half space and  $z_2$  in upper half space. If  $\gamma_1 \cap \{(x, 0) : x \leq 0\} = \emptyset$ , and  $\gamma_2 \cap \{(x, 0) : x \geq 0\} = \emptyset$ , then  $n(\gamma, 0) = 1$ .*

**Remark 7.15.** One method to prove this lemma is to create two segment from  $z_i$  to the point in the unit circle. By divide the curve into two parts, we can easily



remove the part of previous curve by using the theorem 7.13, since 0 is in the unbounded set.

In this proof, we can find that Theorem 7.13 is such powerful that we can change any curve to a more simple curve easily!

### 7.2.2 Cauchy's integral formula

**Theorem 7.16** (Cauchy's integral formula). *Suppose that  $f$  is analytic in an open disk  $\Delta$ , and let  $\gamma$  be a closed curve in  $\Delta$ . For  $\forall z \notin \gamma$ ,*

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where  $n(\gamma, z)$  is the index of  $z$  w.r.t.  $\gamma$ .

*Proof.* If  $z \notin \Delta$ , The both sides of the equation is 0.

So we may assume  $z \in \Delta$  and  $z \notin \gamma$ . Define  $F : \Delta \setminus \{z\} \rightarrow \mathbb{C}, \zeta \mapsto \frac{f(\zeta) - f(z)}{\zeta - z}$ .

Then  $F$  is analytic in  $\Delta \setminus \{z\}$ , and  $\lim_{\zeta \rightarrow z} (\zeta - z)F(\zeta) = 0$ .

By Cauchy's Theorem 7.9  $\Rightarrow \int_{\gamma} F(\zeta) d\zeta = 0 \Rightarrow \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \int_{\gamma} \frac{1}{\zeta - z} d\zeta = f(z) \cdot 2\pi i \cdot n(\gamma, z)$  □

**Remark 7.17.** This proof let us find that for a good-enough function, its integral over a closed curve is a constant.

The theorem still holds if  $f$  is analytic except at a finite number of  $\zeta_j$  s.t.

$$\lim_{\zeta \rightarrow \zeta_j} (\zeta - \zeta_j)f(\zeta) = 0$$

and  $z \neq \zeta_j$  for each  $j$ , since Cauchy's theorem is still applicable.

**Theorem 7.18** (The mean value property for analytic functions).  *$f$  is analytic in a*

region  $\Omega$  which contain  $\overline{B(z, R)}$ . Then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\theta}) dt$$

*Proof.* The previous theorem 7.16  $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta-z|=R} \frac{f(\zeta)}{\zeta - z} d\zeta \stackrel{\zeta=z+Re^{it}}{=} \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{it}) dt$$

□

If  $f$  is analytic in an open disk  $\Delta$ , and  $\gamma$  is a closed curve in  $\Delta$ . And  $n(\gamma, z) = 1$ . Then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

This is usually referred to as **Cauchy's integral formula**

### 7.2.3 Higher derivatives

**Lemma 7.19.** Let  $\Omega \subset \mathbb{C}$  be a region and  $\gamma$  be an arc in  $\Omega$ . If  $\varphi$  is continuous on  $\gamma$ , then the function

$$F_n(z) := \int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z)^n} d\zeta$$

is analytic in each of the regions  $\Omega \setminus \gamma$ , and its derivative is  $F'_n(z) = nF_{n+1}(z)$

*Proof.* We prove it by induction.

The lemma is true if  $n = 0$ :  $F_0(z) = \int_{\gamma} \varphi(\zeta) d\zeta$  and  $F'_0(z) = 0 = 0 \cdot F_1(z)$ .

We suppose that the lemma holds for  $n - 1$  with  $n \in \mathbb{N}$ :  $\forall$  continuous  $\varphi$  on  $\gamma$ ,  $F_{n-1}$  is analytic in  $\Omega \setminus \gamma$  and  $F'_{n-1}(z) = (n - 1)F_n(z)$ ,  $\forall z \in \Omega \setminus \gamma$ .

Fix  $z_0 \in \Omega \setminus \gamma$ . For  $\forall z \in B(z_0, \frac{\delta}{2})$ , with  $B(z_0, \delta) \subset \Omega \setminus \gamma$ , we have  $|\zeta - z| > \frac{\delta}{2}$ ,  $\forall \zeta \in \gamma$ .

For  $\forall$  continuous  $\varphi$  on  $\gamma$ ,

$$\begin{aligned} F_n(z) - F_n(z_0) &= \int_{\gamma} \frac{\varphi(\zeta)(\zeta - z + z - z_0)}{(\zeta - z)^n(\zeta - z_0)} d\zeta - \int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta \\ &= \left[ \int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z_0)^{n-1}(\zeta - z)} d\zeta \right] \\ &\quad + (z - z_0) \int_{\gamma} \frac{\varphi(\zeta) d\zeta}{(\zeta - z)^n(\zeta - z_0)} \end{aligned}$$

Let  $\psi(\zeta) = \frac{\varphi(\zeta)}{\zeta - z_0}$ , which is continuous except  $\gamma$ .

Using the induction condition to  $\psi$ , we can finish the proof.  $\square$

**Theorem 7.20.** *An analytic function on a region  $\Omega$  has derivatives of all orders which are analytic in  $\Omega$ . More precisely,  $\forall z_0 \in \Omega$ , choose  $B(z, \delta) \subset \Omega$  and a circle  $C \subset B(z_0, \delta)$  with center  $z_0$ . For  $\forall z$  in the interior of  $C$ , Cauchy's integral formula gives*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Then the previous lemma implies  $f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$  is analytic in the interior of  $C$ . More generally, for  $\forall n \in \mathbb{N}$ ,

$$f^{(n)} = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad (7.6)$$

#### 7.2.4 Consequences of Cauchy

**Theorem 7.21 (Morera's Theorem).** *If  $f$  is continuous in a region  $\Omega$ , and if  $\int_{\gamma} f(z) dz$  for  $\forall$  closed curve  $\gamma$  in  $\Omega$ . Then  $f$  is analytic in  $\Omega$ .*

*Proof.* We proved in Corollary 7.7 that under the hypothesis of theorem,  $f = F'$  where  $F$  is analytic in  $\Omega$ . The last theorem  $\Rightarrow f$  is analytic.  $\square$

Suppose  $f$  is analytic in a disk,  $\overline{B(z_0, R)}$ , and bounded on the circle  $\gamma$  given by

$|z - z_0| = R$ . Then  $\forall z \in \gamma, |f(z)| \leq M$  for some  $M \geq 0$ . By (7.6),

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_C \frac{f(\zeta)}{|\zeta - z_0|^{n+1}} |d\zeta| \leq \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = MR^{-n}n! \quad (7.7)$$

This inequality is known as **Cauchy's estimate**.

**Theorem 7.22** (Liouville's Theorem). *A bounded entire function ( i.e. analytic in  $\mathbb{C}$ ) is constant.*

*Proof.* Suppose  $|f(z)| \leq M, \forall z \in \mathbb{C}$ . Cauchy's estimate  $\Rightarrow$

$$|f'(z)| \leq \frac{M}{R}, \forall z \in \mathbb{C}, \forall R > 0 \quad (7.8)$$

□

$$\xrightarrow{R \rightarrow \infty} f'(z) = 0 \text{ for } z \in \mathbb{C} \Rightarrow f = 0.$$

**Theorem 7.23** (Fundamental Theorem for Algebra). *Every polynomial of degree  $n \geq 1$  has  $n$  roots.*

*Proof.* It suffices to prove it has at least one root.

Suppose  $P(z) = a_n z^n + \cdots a_1 z + a_0$  with  $a_0 \neq 0$  does not have a root.

Then  $f(z) := \frac{1}{P(z)}$  is an entire function. As  $z \rightarrow \infty, \lim_{|z| \rightarrow \infty} \frac{|P(z)|}{|z|^n} = |a_n| \Rightarrow$   
 $\lim_{|z| \rightarrow \infty} \frac{1}{|P(z)|} = 0.$

So  $f$  is bounded. By Liouville's Theorem,  $f$  is a constant. Where  $f = f(\infty) = 0$ .

That causes contradiction. □

**Theorem 7.24** (Power series). *If  $f$  is analytic in a region  $\Omega$  which contains a closed disk  $\overline{B(z_0, R)}$ , then  $f$  has a power series expansion at  $z_0$ ,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \quad \forall z \in B(z_0, R) \quad (7.9)$$

*Proof.*  $\forall z \in B(z_0, R), \forall \zeta$  with  $|\zeta - z_0| = R$ .

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\
&= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n \\
&= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}
\end{aligned} \tag{7.10}$$

This series converges uniformly in  $\zeta$  with  $|\zeta - z_0| = R$ .

For  $\forall z \in B(z, R)$ ,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi i} \int_{|\zeta - z| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta \\
&\stackrel{\text{uniformly}}{=} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \cdot (z - z_0)^n \\
&\stackrel{(7.6)}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\end{aligned} \tag{7.11}$$

□

## 7.3 Local properties of analytic functions

### 7.3.1 Removable Singularities and Taylor's Theorem

We remarked that Cauchy's integral formula holds if  $f$  is analytic except at a finite number of point  $\zeta_j$  s.t.  $\lim_{\zeta \rightarrow \zeta_j} (\zeta - \zeta_j)(f\zeta) = 0$ . We will prove  $f$  can be extended to an analytic function in  $\Delta$ . In other word,  $\zeta_j$  are **removable singularities**.

**Theorem 7.25** (Riemann's Removable Singularities Theorem). *Suppose that  $f$  is*

analytic in the region  $\Omega' = \Omega \setminus \{\zeta_0\}$  where  $\Omega$  is also a region. Then there exists an analytic function in  $\Omega$  which coincides with  $f$  in  $\Omega'$  if and only if  $\lim_{z \rightarrow \zeta_0} (z - \zeta_0)f(z) = 0$ .

*Proof.* The uniqueness and " $\Rightarrow$ " part is trivial since the extended function is continuous at  $\psi_0$ .

" $\Leftarrow$ ": Cauchy's integral formula  $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta, \quad \forall z \in \Delta \text{ and } z \neq \zeta_0 \quad (7.12)$$

Lemma 7.19  $\Rightarrow$  the RHS of the last equation 7.12 is analytic in  $z \in \Delta$ . Then

$$\hat{f}(z) = \begin{cases} f(z), & z \neq \zeta_0 \\ \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - \zeta_0} d\zeta, & z = \zeta_0 \end{cases} \quad (7.13)$$

is analytic in  $\Omega$ . □

We apply Theorem 7.25 to the function  $F(z) = \frac{f(z) - f(\zeta)}{z - \zeta}$ , where  $f$  is analytic in a region  $\Omega$ . Note that

$$\lim_{z \rightarrow \zeta_0} (z - \zeta)F(z) = 0, \quad \lim_{z \rightarrow \zeta} F(z) = f'(\zeta) \quad (7.14)$$

Theorem 7.25  $\Rightarrow \exists$  analytic function  $f_1$  on  $\Omega$  s.t.

$$f_1(z) = \begin{cases} F(z), & z \neq \zeta_0 \\ f'(\zeta), & z = \zeta_0 \end{cases} \quad (7.15)$$

we may thus write  $f(z) = f(\zeta) + (z - \zeta)f_1(z)$ .

Repeating this process for  $f_1$ , we get an analytic function  $f_2$  on  $\Omega$  s.t.

$$f_1(z) = f_1(\zeta) + (z - \zeta)f_2(z) \quad (7.16)$$

where

$$f_2(z) = \begin{cases} \frac{f_1(z) - f_1(\zeta)}{z - \zeta}, & z \neq \zeta \\ f_2'(\zeta), & z = \zeta \end{cases} \quad (7.17)$$

Continuing the recursion, we have the general form

$$f_{n-1}(z) = f_{n-1}(\zeta) + (z - \zeta)f_n(z) \quad (7.18)$$

$\Rightarrow$

$$f(z) = f(\zeta) + (z - \zeta)f_1(\zeta) + \cdots + (z - \zeta)^{n-1}f_n(\zeta) + (z - \zeta)^n f_n(z) \quad (7.19)$$

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