Exercise 3.4 $\varphi(y+1-(x-1)^3)=0$. Let $l := y+1-(x-1)^3$

Then for $f \in \mathbb{C}[x,y]$, let $f = ml + n$, where $m \in \mathbb{C}[x,y], n \in \mathbb{C}[x]$. $(*)$

Then $\varphi(f) = \varphi(n)$. $\Rightarrow$

$$\varphi(f) = 0 \Leftrightarrow \varphi(n) = 0 \Leftrightarrow n(t+1) = 0 \Leftrightarrow n = 0$$

Hence $K = l\mathbb{C}[x,y] = (l)$.

By Correspondence theorem, the ideal I of $\mathbb{C}[x,y]$ that contains K gets an ideal I/K of $\mathbb{C}[x,y]/K$. And by $(*)$, $\mathbb{C}[x,y]/K = \mathbb{C}[x]$.

By Prop 11.3.22, $I/K = (\overline{x})$ for some $\overline{x} \in \mathbb{C}[x,y]/K$. Then $I = (x,l)$ since $\forall i \in I, \overline{i} = \overline{x} \cdot \overline{f}, \overline{f} \in \mathbb{C}[x,y]/K \Rightarrow i = (x+k_1)(f+k_2)+k_3 \in (x,l)$ for some $k_1, k_2, k_3 \in K = (l)$.

Exercise3.9 If x is nilpotent, assume $x^n = 0$. Then $(1-(-x))(1+(-x)+(-x)^2+\cdots+(-x)^{n-1}) = 1-(-x)^n = 1 \Rightarrow (1+x)^{-1} = (1-(-x))^{-1} = (1+(-x)+(-x)^2+\cdots+(-x)^{n-1})$ is a unit.

(b)Since a is nilpotent, then $a^n = 0$ for sufficient large n. In particular, $\exists m \in \mathbb{N}$ s.t. $a^{p^m} = 0$. Noticed that $p|\binom{p^m}{k}$ for all $1 \leq k \leq p^m - 1$. Then

$$(1+a)^{p^m} = \sum_{k=0}^{p^m} \binom{p^m}{k} a^k = 1 + a^{p^m} = 1$$

Exercise4.4 If there is an isomorphism $\varphi : \mathbb{Z}[x]/(2x^2+7) \to \mathbb{Z}[x]/(x^2+7)$, then $\varphi(\overline{1}) = \overline{1} \Rightarrow -\overline{7} = -7\varphi(\overline{1}) = \varphi(-\overline{7})(-7$ is the number -7).

$\Rightarrow -\overline{7} = \varphi(-\overline{7}) = \varphi(\overline{2x^2}) = 2\varphi(\overline{x^2})$

$\Rightarrow -7 = 2t + m(x^2+7)$ where $\varphi(\overline{x^2}) = \overline{t}, m \in \mathbb{Z}[x]$. But the sum of coefficients of RHS is even but LHS is odd. That's contradiction.

So there is no isomorphism.

Exercise5.6 (a) For every $\beta = \sum_{k=0}^n a_k\alpha^k$, $\beta = (a\alpha)^n \sum_{k=0}^n a_k\alpha^k = \alpha^n \cdot \sum_{k=0}^n a_k(a\alpha)^k \cdot a^{n-k} = \alpha^n \sum_{k=0}^n a_k \cdot a^{n-k} = \alpha^n b$ for some $b \in R$.

(b) For $b \in R$, $b = (\alpha a)^n b = \alpha^n a^n b$. Then if $a^n b = 0$, b equals 0 in R'. If $b = 0$ in R', i.e. $b = l(a\alpha - 1)$ for some $l \in R'$. $(a) \Rightarrow l = \alpha^k t$ where $t \in R$. Then $b = \alpha^k t(a\alpha - 1) \Rightarrow a^{k+1}b = t(a-a) = 0$.

(c)If a is a nilpotent, $(b) \Rightarrow b = 0$ in R' for $b \in R$. Thus $R' = R[x]/(ax-1) = \{0\}$.

If $R' = R[x]/(ax-1) = \{0\}$, then $1 = 0$ in $R' \Rightarrow \exists n, a^n \cdot 1 = 0$ by $(b) \Rightarrow a$ is a nilpotent.

Exercise8.3 $\mathbb{F}_2[x]/(x^3+x+1) = \{\overline{x^2}, \overline{x}, \overline{1}, \overline{x^2+x}, \overline{x+1}, \overline{x^2+1}, \overline{x^2+x+1}, 0\}$ and we have

$$\overline{x^2} \cdot \overline{x^2+x+1} = \overline{x^2+x} + \overline{x+1} + \overline{x^2} = 1, \overline{x} \cdot \overline{x^2+1} = -1 = 1$$

$$\overline{x+1} \cdot \overline{x^2+x} = \overline{x+1} + \overline{x^2} + \overline{x^2} + \overline{x} = 1$$

so every nonzero element in $\mathbb{F}_2[x]/(x^3+x+1)$ is a unit. Therefore $\mathbb{F}_2[x]/(x^3+x+1)$ is a field.

Noticed that $\overline{x^2+x+1} \cdot \overline{x^2+x-1} = \overline{x^2+x}^2 - 1 = \overline{x^4} + 2\overline{x^3} + \overline{x^2} - 1 = \overline{-x^2-x} + 2\overline{-x-1} + \overline{x^2} - 1 = 0$. Then $\overline{x^2+x+1}$ has no inverse $\Rightarrow \mathbb{F}_3[x]/(x^3+x+1)$ is not a field.

6.For $p = \sum_{n=0}^{\infty} a_n t^n, q = \sum_{n=0}^{b_n} t^n$,

$$p \cdot q = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} a_i \cdot b_{n-i}) t^n$$

Then p is a unit impiles that $a_0 b_0 = 1 \Rightarrow a_0$ is a unit in R.

Conversely, if a_0 is a unit in R, define a sequence $\{b_n\}$ s.t.

$$b_0 = a_0^{-1}, b_n = a_0^{-1} \cdot (-\sum_{i=1}^{n} a_i b_{n-i})$$

Then $\forall n \geq 1, \sum_{i=0}^{n} a_i \cdot b_{n-i} = 0$. Therefore $q = \sum_{n=0}^{\infty} b_n t^n = p^{-1} \Rightarrow p$ is a unit.

Conclusion: p is a unit if and only if $p = \sum_{n=0}^{\infty} a_n t^n$ where a_0 is a unit in R.