

Differential Geometry

Instructor: Jianfeng Lin

Notes Taker: Zejin Lin

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

lzmjmaths.github.io

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1 Smooth Manifold

Definition 1.1 (Topological manifold). A space M is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

Definition 1.2 (Smooth Manifold). A smooth structure is given by an equivalence class of smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ s.t. $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth $\forall \alpha, \beta$. $M = \cup U_\alpha$.

A **smooth manifold** is a topological manifold with a smooth structure.

Define when a continuous map $f : M_1 \rightarrow M_2$ is smooth if $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$, we have $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$ is smooth.

Definition 1.3. Given $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$. A homeomorphism $f : M_1 \rightarrow M_2$ is called a diffeomorphism if f, f^{-1} is smooth.

In this case we say $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ are diffeomorphism.

Theorem 1.4 (Kervaire). \exists 1 10-dimensional topological manifold without smooth manifold.

Theorem 1.5 (Milnor). \exists a smooth manifold M s.t. $M \cong S^7$ but not in diffeomorphism meaning.

Theorem 1.6 (Kervaire-Milnor). \exists 28 smooth structures (up to orientation preserving diffeomorphism) on S^7

Theorem 1.7 (Morse-Birg). On S^7 . If $n \leq 3$, then any n -dimensional topological manifold M has a unique smooth structure up to diffeomorphism.

Theorem 1.8 (Stallings). *If $n \neq 4$, then \exists a unique smooth structure on \mathbb{R}^n up to diffeomorphism.*

Theorem 1.9 (Donaldson-Freedman-Gompf-Faubes). *\exists uncountable smooth structures on \mathbb{R}^4 up to diffeomorphism.*

Definition 1.10 (topological manifold with boundary). A space M is called a topological manifold with boundary if

1. M is Hausdorff
2. M is second countable
3. $\forall p \in M, \exists$ a neighbourhood U of p and a homeomorphism $\varphi : U \rightarrow V$ where V is open in \mathbb{H}^n

We say a manifold M is closed if M is compact and ∂M is empty.

Our motivation for studying manifold is to study the space of solution for equations.

Question 1. Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth, $q \in \mathbb{R}^n$, when is $f^{-1}(q)$ is a smooth manifold?

For $f : U \rightarrow \mathbb{R}^n$ smooth, U open in \mathbb{R}^m , the differential of f at $p \in U$ denoted as $df(p)$.

Definition 1.11. We say $p \in U$ is a **regular point** of f if $df(p)$ is surjective. Otherwise we say $p \in U$ is a **critical point**.

A point $q \in \mathbb{R}^n$ is called a **regular value** of f if $\forall p \in f^{-1}(q)$, p is a regular point of f .

A point $q \in \mathbb{R}^n$ is called a **critical value** of f if $\forall p \in f^{-1}(q)$, p is a critical point of f .

Theorem 1.12 (Implicit function theorem). *If $p \in U$ is a regular point of $f : U \rightarrow \mathbb{R}^n$. Then there exists*

- *An open neighbourhood V of p in U*
- *An open subset V' of \mathbb{R}^m*
- *A diffeomorphism $\varphi : V \rightarrow V'$ such that $P \circ \varphi = f$ where P is the projection from \mathbb{R}^m to \mathbb{R}^n .*

In other words, near a regular point, we can do local coordinate change to turn f into the projection.

Remark 1.13. Inverse function theorem and Implicit function theorem gives a way to find the related from "a point" to "a beighbourhood"!

In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \dots, x_m) \in V' \mid (x_1, \dots, x_n) = f(p)\}$$

i.e. if we set $M = f^{-1}(f(p))$, then $(M \cap V, \varphi_p)$ is a chart that contains p .

Corollary 1.14. *If q is a regular value of $f : U \rightarrow \mathbb{R}^n$ then $f^{-1}(q)$ is a smooth manifold.*

Remark 1.15. It suffices to show that the corresponding charts are compatible.

Theorem 1.16 (Sard). *If $f : U \rightarrow \mathbb{R}^n$ is a smooth map, then the set of critical values of f has measure 0.*

Remark 1.17. For a "generic" q , $f^{-1}(q)$ is a manifold of dimension $m - n$.

Corollary 1.18. *If $f : U \rightarrow \mathbb{R}^n$ is smooth and $m < n$ then $f(U)$ has measure 0.*

1.1 Lie Groups and Homogeneous Spaces

Definition 1.19. We say G is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map $\cdot : G \times G \rightarrow G$ and the inverse map $G \rightsquigarrow G$ is smooth.

Example 1.20. $GL(n, \mathbb{R}) = \{n \times n \text{ matrices with non-zero determinant}\} \subset \mathbb{R}^{n \times n}$

$$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$$

$$SO(n) = \{A \in O(n) | \det A = 1\}$$

$$U(n) = \{A \in GL(n, \mathbb{C}) | A\bar{A}^T = I\}$$

$$SU(n) = \{A \in U(n) | \det A = 1\}$$

Exercise 1.21.

$$O(1) \cong S^2 \qquad SO(1) \cong * \qquad (1.1)$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \qquad (1.2)$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \qquad (1.3)$$

The last one is a diffeomorphism but do not preserve the multiplication, *i.e.* not an isomorphism of Lie group.

Theorem 1.22 (Cartan). *Let H be a closed subgroup of Lie group G . Then H is a Lie group. More precisely, H is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also, G/H is a smooth manifold*

Definition 1.23. Let M be a smooth manifold. We say M is a **homogeneous space** if \exists a Lie group G with a smooth transitive action $\rho : G \times M \rightarrow M$.

Definition 1.24. For M be a homogeneous space. The **isotropy** group of $x \in M$ is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of G

Given any $x, x' \in M$, $Iso(x) \cong Iso(x')$ because the group action is transitive.

Hence, we have a well-defined map

$$p : G/Iso(x) \rightarrow M \tag{1.4}$$

$$g \mapsto gx \tag{1.5}$$

Theorem 1.25. *p is always a diffeomorphism.*

Therefore, we have this proposition

Proposition 1.26. *M is a homogeneous space $\Leftrightarrow M = G/H$ for some closed subgroup H .*

Example 1.27. If $M = S^n$, let $G = SO(n+1)$.

Then $Iso(1, 0, \dots, 0) \cong SO(n)$.

So $S^n \cong SO(n+1)/(SO(n))$.

Similarly, we can prove $\mathbb{RP}^n \cong SO(n+1)/(O(n))$, $\mathbb{CP}^n \cong SO(n+1)/(U(n))$

The isotropy k dimensional linear subspaces of \mathbb{R}^n can be $O(k) \times O(n-k)$ if $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to \mathbb{RP}^2 , S^2 , T^2 and Klein bottle.

Theorem 1.28 (Whithead). *Any smooth manifold has a triangulation.*

Theorem 1.29 (Poincare-Hopf). *G is compact Lie group $\Rightarrow \chi(G) = 0$.*

Theorem 1.30 (Mostow2005). *M is a compact homogeneous space $\Rightarrow \chi(M) \geq 0$.*

1.2 Bump Function and Partition of Unity

Theorem 1.31 (Urysohn smooth version). *Given M , closed disjoint A, B , \exists smooth $f : M \rightarrow [0, 1]$ s.t. $f|_A = 0, f|_B = 1$.*

Theorem 1.32 (Tietze). *Given M , closed A , smooth $f : A \rightarrow \mathbb{R}^n$, there exists smooth $\hat{f} : M \rightarrow \mathbb{R}^n$ s.t. $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

Lemma 1.33. *Let U be a neighbourhood of $p \in M$. Then \exists smooth $\sigma : M \rightarrow [0, 1]$ s.t.*

1. $\sigma \equiv 1$ near p
2. $\text{Supp } \sigma \subset U$

Such σ is called a **bump function** at p , supported in U .

Definition 1.34. An open cover of a space X is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

Proposition 1.35. *Given compact $K \subset U$ and open neighbourhood U of K , \exists a smooth $g : M \rightarrow [0, +\infty)$ s.t. $g|_K \equiv 1$ and $\text{Supp } g \subset U$.*

Definition 1.36. An **exhaust** of a space X is a sequence of open sets $\{U_i\}$ s.t.

1. $X = \bigcup_{i=1}^{\infty} U_i$
2. $\overline{U_i}$ is compact and contained in U_{i+1}

Theorem 1.37. *Any topological manifold has an exhaust.*

Given two open covers \mathcal{U}, \mathcal{V} , we say \mathcal{V} is a **refinement** of \mathcal{U} if $\forall U_\alpha \in \mathcal{U}, \exists V_\beta \in \mathcal{V}$ s.t. $V_\beta \subset U_\alpha$.

We say a space X is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact. (The proof is hard)

Proposition 1.38. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a topological manifold M . Then there exists countable open covers $\mathcal{W} = \{W_i\}$, $\mathcal{V} = \{V_i\}$ s.t.*

- *For any i , $\overline{V_i}$ is compact and $\overline{V_i} \subset W_i$*
- *\mathcal{W} is locally finite.*
- *\mathcal{W} is a refinement of \mathcal{U} .*

As a corollary, we have any topological manifold is paracompact.

Definition 1.39. Given open cover \mathcal{U} of a smooth M , a partition of unity subordinate to \mathcal{U} is a collection of smooth functions $\{\rho_\alpha : M \rightarrow [0, 1]\}_{\alpha \in \mathcal{A}}$ s.t.

1. $\forall p \in M, \exists$ only finitely many $\alpha \in \mathcal{A}$ s.t. $p \in \text{Supp } \rho_\alpha$
2. $\sum_{\alpha \in \mathcal{A}} \rho_\alpha(p) = 1$
3. $\text{Supp } \rho_\alpha \subset U_\alpha$

Theorem 1.40 (Existence of P.O.U). *For any open cover \mathcal{U} of smooth M , \exists a P.O.U subordinate to \mathcal{U}*

Theorem 1.41 (Whitney approximation theorem). *Given any smooth M , any closed A and any continuous $f : M \rightarrow \mathbb{R}$, $\delta : M \rightarrow (0, +\infty)$. Suppose f is smooth on A . Then $\exists g : M \rightarrow \mathbb{R}$ smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M, |g(p) - f(p)| < \delta(p)$.

2 Tangent space and tangent vectors

2.1 Tangent Space

Given $p \in M$, consider the set $C_p^\infty(M) = \{\text{smooth function } V \rightarrow \mathbb{R}\} / \sim$ where $f_1 \sim f_2$ if and only if \exists neighbourhood U of p , $f_1|_U = f_2|_U$.

$C_p^\infty(M)$ is the space of **germs of smooth function** near p .

A **partial-derivative** of p is a \mathbb{R} -linear map $D : C_p^\infty(M) \rightarrow \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

Definition 2.1. A **tangent vector** of M at p is a partial-derivative at p .

Define the **tangent space** $T_p M = \{\text{all partial-derivative at } p\}$, which is a \mathbb{R} -vector space.

Proposition 2.2. For $M = U \subset \mathbb{R}^n$ open. We have $\{\frac{\partial}{\partial x_i}\}$ is a basis for $T_p U$.

Proposition 2.3.

$$\frac{\partial}{\partial x^i} |_p = \sum_{1 \leq i \leq n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i} |_p$$

Now we try to define differential of a smooth map.

M, N smooth manifolds, $C^\infty(N, M) = \{\text{smooth } F : N \rightarrow M\}$.

Given $F \in C^\infty(N, M)$, F induces $F^* : C_{F(p)}^\infty(M) \rightarrow C_p^\infty(N)$, $f \mapsto f \circ F$.

By taking dual, we get

$$F_* : T_p N \rightarrow T_{F(p)} M$$

we also write F_* as $F_{*,p}$, call it the **differential** of F at p .

where

$$F_*\left(\frac{\partial}{\partial x^i} |_p\right) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k} |_{F(p)}$$

Proposition 2.4. *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_p N \rightarrow T_{G \circ F(p)} W$$

Definition 2.5. A smooth **curve** is a smooth map $\gamma : (a, b) \rightarrow M$. We say γ starts at p if $\gamma(0) = p$. We define the **velocity** of γ at $\gamma(0)$ as $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)} M$

Take charts (U, x^1, \dots, x^n) about p , let $\gamma^i = x^i \circ \gamma$.

We say γ, δ are **tangent** to each other at p if $(\gamma^i)'(0) = (\delta^i)'(0)$.

Now we can define

$$(T_p M)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\} / \sim$$

where $\gamma \sim \delta$ iff they are tangent to each other.

Then these definition is more geometric.

Lemma 2.6. *Given $F \in C^\infty(M, M)$, $p \in N$, the diagram commutes:*

$$\begin{array}{ccc} \gamma \in (T_p N)_{curve} & \xrightarrow{\cong} & T_p N \\ \downarrow & & \downarrow \\ F \circ \gamma \in (T_{F(p)} M)_{curve} & \xrightarrow{\cong} & T_{F(p)} M \end{array}$$

2.2 Tangent Bundle

Let (M, \mathcal{A}) be a smooth manifold, $TM = \bigcup_{p \in M} T_p M$, called the **tangent bundle**

Now we want to define a natural topology and smooth structure on TM . Take any chart $(U, \varphi) = (U, x^1, \dots, x^n) \in \mathcal{A}$.

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \quad (2.1)$$

$$X \in T_p U \mapsto (\varphi(p), X^1, \dots, X^n) \quad (2.2)$$

where $X = \sum X^i \frac{\partial}{\partial x^i} |_p$.

Then pull back standard topology on $\varphi(U) \times \mathbb{R}^n$ to a topology on TU .

$$\mathcal{B} = \{ \hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- \mathcal{B} is a basis
- \mathcal{B} generates a Hausdorff, second countable topology on TM .

So TM is a topological manifold covered by charts $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$.

Given $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$, the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi} \circ \hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \quad (2.3)$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \quad (2.4)$$

So $\hat{\mathcal{A}}$ is a smooth atlas on TM , making TM into a smooth manifold.

Definition 2.7 (vector bundle). Given a continuous map $f : E \rightarrow B$, we say f is a n -dimensional **vector bundle** if: \exists an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of B and homeomorphisms $\{f^{-1}(U_\alpha) \xrightarrow[\cong]{\rho_\alpha} U_\alpha \times \mathbb{R}^n\}$ s.t.

$$\begin{array}{ccc} f^{-1}(U_\alpha) & \xrightarrow{\rho_\alpha} & U_\alpha \times \mathbb{R}^n \\ \downarrow f & \swarrow \text{projection} & \\ U_\alpha & & \end{array} \quad \text{commutes for } \alpha \in I$$

- $\forall p \in U_\alpha \cap U_\beta$, the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

is linear.

Call $f^{-1}(p)$ the **fiber** over p .

Proposition 2.8. *Given vector bundle $f : E \rightarrow B$, the fiber $f^{-1}(p)$ has a structure of a vector space.*

Example 2.9 (Product bundle). $E = \mathbb{R}^n \times B$

Example 2.10 (Tautological bundle).

$$B = \mathbb{CP}^n = \{1\text{-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map $(L, v) \mapsto L$

Given vector bundles $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$, a bundle map consists of (\hat{f}, f) s.t.

- $$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow \pi & & \downarrow \pi \\ B_1 & \xrightarrow{f} & B_2 \end{array} \text{ commutes.}$$
- $\forall b \in B, \hat{f} : \pi_1^{-1}(b) \rightarrow \pi_2^{-1}(f(b))$ is linear.

If \hat{f}, f are diffeomorphisms, then we call (\hat{f}, f) an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

Example 2.11. TS^1, TS^2 are both trivial.

$$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$$

Theorem 2.12. *If G is a Lie group, then TG is trivial.*

Proof. For (x^1, x^2, \dots, x^n) is a basis of $T_e G$ The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG, (g, c^1, \dots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \rightarrow G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism $(l_g)_*$

□

Proposition 2.13 (Adams, 1960s). TS^n is trivial if and only if $n = 0, 1, 3, 7$.

Proposition 2.14.

1. Given any $F \in C^\infty(M, N)$, $F_* : TM \rightarrow TN$ is a bundle map.
2. TS^n is isomorphic to the following bundle:

$$B = S^n \quad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

Definition 2.15 (smooth section). Given a smooth vector bundle $\pi : E \rightarrow B$, a **smooth section** is a smooth map $S : B \rightarrow E$ s.t. $\pi \circ S = id_B$.

$$s_0 : B \rightarrow E, b \mapsto 0 \in 0\text{-vector in } \pi^{-1}b.$$

2.3 Vector Field, Curves and Flows

Definition 2.16. A (tangent) **vector field** is a smooth section of TM . i.e. a smooth map $M \xrightarrow{X} TM$ s.t. $X(p) \in T_p M, \forall p \in M$

Given any $f : \mathbb{R}^n \rightarrow \mathbb{R}$, define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

Example 2.17. $X = f^1 \partial x^1 + f^2 \partial x^2$ is a gradient field if and only if $\frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}$

Theorem 2.18 (Poincare-Hopf). *For closed M , M has a nowhere vanishing vector field if and only if $\chi(M) = 0$.*

So S^n has a nowhere vanishing vector field if and only if n is odd.

Theorem 2.19 (MaoQiu). *S^2 has no no-where vanishing vector field.*

So We cannot smooth out all the hairs on a ball.

Given a vector field $X = \{X_p\}_{p \in M}$, a curve $\gamma : (a, b) \rightarrow M$ is called an **integral curve** of X if $\gamma'(t) = X_{\gamma(t)}, \forall t \in (a, b)$, where $\gamma'(t) = \gamma_*\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M$.

We say γ is maximal if the domain cannot be extended to a larger interval.

Denote the set of all smooth vector fields on M by $\mathfrak{T}M$

Recall that γ is maximal if it's domain can not be extended to a large open interval.

In a local chart (U, x^1, \dots, x^n) , $X|_U = \sum_{i=1}^n a^i \partial x^i$. Then γ is an integral curve if and only if $(\gamma^i)'(t) = a^i(\gamma(t)), \forall 1 \leq i \leq n$, where $\gamma^i = x^i \circ \gamma : (a, b) \rightarrow \mathbb{R}$.

And in this case the initial value condition: $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$.

Locally, solving integral curve starting at p is equivalent to solving ODE with initial value p^1, \dots, p^n . By existence and uniqueness of solutions of ODE, we have

Theorem 2.20 (Fundamental theorem of integral curve). *Let $X \in \mathfrak{T}M, p \in M$, then:*

(1) (Uniqueness) *Given any two integral curves $\gamma_1, \gamma_2 : (a, b) \rightarrow M$, then we have:*

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a, b) \Rightarrow \gamma_1 = \gamma_2$$

(2) *there exists a unique max integral curve $\gamma : (a(p), b(p)) \rightarrow M$ starting at p .*

(3) (integral curve smoothly depend on initial values) \exists Nbh U of $p, \varepsilon > 0$, and smooth

$\varphi : (-\varepsilon, \varepsilon) \times U \rightarrow M$ s.t. $\forall q \in U, \varphi_\varepsilon := \varphi(-, q) : (-\varepsilon, \varepsilon) \rightarrow M$ is an integral

curve starting at q .

we call such φ a local **flow** generated by X .

Definition 2.21. Given $X \in \mathfrak{X}M$, a global **flow** generated by X is a smooth map $\varphi : \mathbb{R} \times M \rightarrow M$ s.t. $\forall q \in M$, $\varphi_q := \varphi(-, q)$ is the maximal integral curve of X starting at q .

$$\Leftrightarrow \frac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s, p)}, \forall s \in \mathbb{R}, p \in M \text{ and } \varphi(0, p) = p, \forall p \in M.$$

If such global flow exists, then we say X is **complete**.

Example 2.22.

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$ is complete, where global flow $\varphi : \mathbb{R} \times M \rightarrow M$, $\varphi(t, p) = p \cdot e^t$.
- $X = x^2 \partial x$ is not complete. Max integral curve starting at 1 is given by $\gamma(t) = \frac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$.

Given $X \in \mathfrak{X}M$, we define $\text{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$.

Theorem 2.23. If a vector field X is compactly supported, then X is complete.

Corollary 2.24. Any vector field on closed manifold is complete.

Lemma 2.25 (Escaping lemma). Suppose $\gamma : (a, b) \rightarrow M$ is a max integral curve, with $(a, b) \neq \mathbb{R}$. Then \nexists compact $K \subset M$ s.t. $\gamma(a, b) \subset K$

Proof. Otherwise, suppose $\gamma(a, b) \subset K$. WLOG, we may assume $b < +\infty$.

Take $(t_i) \rightarrow b$ from left. Then $\gamma(t_i) \in K$. After passing to subsequence, we may assume $(\gamma(t_i)) \rightarrow p \in K$.

Then $\exists U$ Nbh of p , local flow $\varphi : (-\varepsilon, \varepsilon) \times U \rightarrow M$. Take n large enough s.t. $b - t_n < \varepsilon, \gamma(t_n) \in U$. Then $\gamma(- + t_n) : (a - t_n, b - t_n) \rightarrow M, \varphi(-, \gamma(t_n)) : (-\varepsilon, \varepsilon) \rightarrow M$ are both integral curves for X starting at $\gamma(t_n)$. By uniqueness, they coincide.

Let $\hat{\gamma} : (a, t_n + \varepsilon) \rightarrow M$ be defined by $\hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \varepsilon) \end{cases}$

Then $\hat{\gamma}$ is an integral curve with larger domain, then γ contradiction with the maximality of γ . \square

Proof of 2.23. Take any max integral curve $\gamma : (a, b) \rightarrow M$. Suppose $(a, b) \neq \mathbb{R}$. Then $X_{\gamma(s)} \neq 0, \forall s$. Otherwise, the constant map $\mathbb{R} \rightarrow M, t \mapsto \gamma(s)$ is an integral curve with larger domain.

So $\forall s, \gamma(s) \in \text{Supp} X \Rightarrow \gamma(a, b) \subset \text{Supp} X$ which is compact $\Rightarrow (a, b) = \mathbb{R}$ by the lemma. This causes contradiction! \square

A smooth $\varphi : \mathbb{R} \times M \rightarrow M$ is called an **one-parameter transformation group** if

$$(1) \quad \varphi_0 := \varphi(0, -) = \text{id}_M$$

$$(2) \quad \varphi_s \circ \varphi_t = \varphi_{s+t} \text{ for all } s, t \in \mathbb{R}. \text{ In particular, } \varphi_s^{-1} = \varphi_{-s}.$$

Theorem 2.26. $\varphi \in C^\infty(\mathbb{R} \times M, M)$, then φ is an one-parameter transformation group if and only if φ is the global flow generated by some $X \in \mathfrak{X}M$

Lemma 2.27 (Translation lemma). If $\gamma : (a, b) \rightarrow M$ is an integral curve for some $X \in \mathfrak{X}M$, then $\forall s \in \mathbb{R}, \gamma(- + s) : (a - s, b - s) \rightarrow M$ is also an integral curve for X .

Proof. Let $\iota = \gamma(- + s)$. Then $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$ \square

Lemma 2.28. Let $\varphi : (-\varepsilon, \varepsilon) \times U \rightarrow M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$ provided that $s, t, s + t \in (-\varepsilon, \varepsilon), p, \varphi_r(p) \in U$.

Proof. $\gamma_p = \varphi(-, p)$ is an integral curve for X .

$\Rightarrow \gamma_p(- + s)$ is an integral curve for X starting at $\gamma_p(s) = \varphi_s(p)$. But $\gamma_{\varphi_s(p)}$ is also an integral curve starting at $\varphi_s(p)$. Thus $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$ \square

Lemma 2.29. Let $\varphi : (-\varepsilon, \varepsilon) \times U \rightarrow M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$ i.e. any vector field is invariant under its flow.

Proof. Take $f \in C_{\varphi(p)}^\infty(M)$.

$$\begin{aligned}\varphi_{s,*}(X_p)(f) &= X_p(f \circ \varphi_s) \\ &= \frac{d}{dt}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \\ &= \frac{d}{dt}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \\ &= X_{\varphi_s(p)}(f)\end{aligned}$$

□

Proof of 2.26. " \Leftarrow " is because the lemma $\varphi_s \circ \varphi_r = \varphi_{s+r}$

" \Rightarrow " Let $X = \{X_p\}$ where $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$.

Leave it as an exercise.

□

Time dependent vector field is a smooth map $X : \mathbb{R} \times M \rightarrow TM$ s.t. $X_{(t,p)} \in T_pM$.

A smooth curve $\gamma(a, b) \rightarrow M$ is the **integral curve** for X if $\gamma'(t) = X_{(t,\gamma(t))}$.

In local chart, solving γ is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness: γ_1, γ_2 are both integral curves for X , $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define $\text{Supp}X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$.

Then X is compactly supported, then X is complete(i.e. a global flow $\varphi : \mathbb{R} \times M \rightarrow M$)

But something is not true for time dependent vector field:

- translation lemma is not true.
- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

2.4 Another Definition of Vector Field

A derivation on M is a \mathbb{R} -linear map $C^\infty(M) \xrightarrow{D} C^\infty(M)$ that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

Theorem 2.30. *We have a bijection:*

$$\begin{aligned} \rho : \mathfrak{X}M &\xrightarrow{1:1} \{\text{derivation on } M\} \\ X &\mapsto D_X : f \mapsto X(f) \end{aligned}$$

Lemma 2.31. $D_p : \mathfrak{X}_p M \rightarrow \mathbb{R}$ -linear map $C^\infty(M) \rightarrow \mathbb{R}$ s.t. $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$ is an isomorphism of vector spaces.

Proof. Leave it as an exercise. □

Lemma 2.32. *Given a vector field(not necessarily smooth) $X = \{X_p\}_{p \in M}$, X is smooth $\Leftrightarrow \forall f \in C^\infty(M)$, $X(f)$ is smooth.*

Proof. " \Leftarrow " $\forall p \in M$, take chart $(U, x^1, x^2, \dots, x^n)$ around p . $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$: $U \rightarrow \mathbb{R}$, where $f^i = X|_U(x^i)$. Take $\varphi : M \rightarrow [0, 1]$ s.t. $\varphi \equiv 1$ near p , $\text{Supp} \varphi \subset U$, $\varphi \cdot x^i \in C^\infty(M)$.

Then $X(\varphi \cdot x^i) = f^i$ near p . By assumption, f^i is smooth near p . So f^i is smooth, so X is smooth.

" \Rightarrow " Similar. □

Theorem 2.33. *The map $\rho : \mathfrak{T}M \rightarrow \{\text{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$ is well-defined and bijective.*

Proof. ρ is well-defined: $X(f) \in C^\infty(M)$ by Lemma 2.32, and $D_x(fg) = D_x(f)g + fD_x(g)$ since X is a point-derivation.

ρ is injective: $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$ as maps $C^\infty(M)$ to \mathbb{R} . By Lemma 2.31, we have $X_p = Y_p, \forall p$. So $X = Y$.

ρ is surjective: Given $D : C^\infty(M) \rightarrow C^\infty(M)$. Define $D_p : C^\infty(M) \rightarrow \mathbb{R}$ by $D_p(f) := D(f)(p)$ satisfies the Leibniz rule. By Lemma 2.31, $D_p = D_{X_p}$ for some $X_p \in T_p M$. Define $X = \{X_p\}_{p \in M}$. Then $X(f) = D(f), \forall f \in C^\infty(M)$. By Lemma 2.32, X is a smooth vector field. □

3 Lie group, Lie algebra and Lie bracket

3.1 Lie Bracket

In this section, we can actually find those identification:

$$\begin{aligned} \{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \text{ s.t.} \\ &\quad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\} \end{aligned}$$

$$\begin{aligned} \{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\} \end{aligned}$$

Notation 3.1. We will identify $X \in \mathfrak{X}M$ with its derivation $D_x : C^\infty(M) \rightarrow C^\infty(M)$. So a vector field is just a \mathbb{R} -linear map $X : C^\infty(M) \rightarrow C^\infty(M)$ s.t. $X(fg) = fX(g) + X(f)g$.

Definition 3.2 (Lie bracket). Given two (smooth) vector field $X, Y : C^\infty(M) \rightarrow C^\infty(M)$, we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \rightarrow C^\infty(M)$$

Theorem 3.3. For any $X, Y \in \mathfrak{X}M$, $[X, Y] \in \mathfrak{X}M$

Proof. Easy to check that $[X, Y]$ is linear.

By Leibuniz rule,

$$\begin{aligned} [X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\ &= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\ &= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\ &= [X, Y](f) \cdot g - f \cdot [X, Y](g) \end{aligned}$$

□

So What is the geometric meaning of $[X, Y]$? Non commutativity of flows.

Fact 3.4. Given $X, Y \in \mathfrak{X}M$, we say X, Y are commutative vector field if $[X, Y] = 0$

X, Y are commutative iff for any local flows $\varphi^X : (-\varepsilon, \varepsilon) \times U \rightarrow M$, $\varphi^Y : (-\varepsilon, \varepsilon) \times U \rightarrow M$ we have $\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X$

Proposition 3.5 (Calculation of $[V, W]$ using local charts). *Chart (U, x^1, \dots, x^n) , $V, W \in \mathfrak{X}M$, $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$, $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$. Then*

$$\begin{aligned} [V, W]|_U &= \sum_{i=1}^n (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} \\ &= \sum_{1 \leq i, j \leq n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i} \end{aligned}$$

Example 3.6. $V = x\partial x + y\partial y$, $W = -y\partial x + x\partial y$ commutes.

Proposition 3.7 (Properties of Lie bracket).

(a) *Natuality under push-forward.*

Given any $F \in \text{Diff}(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}M$, we have $[F_*V, F_*W] = F_*[V, W]$.

(b) *\mathbb{R} -linearity $\forall a, b \in \mathbb{R}$*

$$[aX + bV, W] = a[X, W] + b[V, W]$$

$$[W, aX + bV] = b[W, X] + a[W, V]$$

(c) *anti-symmetric $[V, W] = -[W, V]$*

(d) *Jacobi identity*

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

(f) Leibniz rule

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

Definition 3.8. Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}N$. We say W is **F -related** to V if $\forall p \in M$, $F_{p,*}(V_p) = W_{F(p)}$ where $F_{p,*} : T_pM \rightarrow T_{F(p)}N$

Example 3.9. $F : S^1 \rightarrow \mathbb{R}^2, \theta \mapsto (\cos \theta, \sin \theta)$, $V = \partial\theta$, $W = -y\partial x + x\partial y$.

Note 1. In general, given $V \in \mathfrak{X}M$ and $F \in C^\infty(M, N)$. There may not exist $W \in \mathfrak{X}N$ s.t. V, W are F -related. Even such W exists, it may not be unique.

However, if F is a diffeomorphism, given any V , \exists unique W s.t. V and W are F -related. Actually, $W_p = F_*V_{F^{-1}(p)}$.

Such W is called **push forward** of V along F , denoted by F_*V , only defined when F is a diffeomorphism.

Lemma 3.10. $\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N, F \in C^\infty(M, N)$. Then W is F -related to V iff $\forall f \in C^\infty(N)$, $V(f \circ F) = W(f) \circ F \in C^\infty(M)$

Proof. Check that $F_{p,*}(V_p)(f) = W_{F(p)}(f)$, $\forall f \in C^\infty(N)$ □

Proposition 3.11. Given $V_0, V_1 \in \mathfrak{X}M$, $W_0, W_1 \in \mathfrak{X}N$, $F \in C^\infty(M, N)$, W_i is F -related to V_i , $i = 0, 1 \Rightarrow [W_0, W_1]$ is F -related to $[V_0, V_1]$

Corollary 3.12 (Naturality of Lie bracket). Given any $F \in \text{Diff}(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}N$, we have $[F_*V, F_*W] = F_*[V, W]$

The rest of Proposition 3.7 is easy to check if it is viewed as a mapping $C^\infty(M) \rightarrow C^\infty(M)$.

3.2 Lie Algebra of a Lie Group

Definition 3.13. A Lie algebra \mathfrak{g} is \mathbb{R} -linear space \mathfrak{g} with map $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ s.t. it is bilinear, anti-symmetric and satisfies the Jacobian identity.

Then $(\mathfrak{TM}, [-, -])$ is an infinite dimensional Lie algebra.

For G Lie group, $\forall g \in G$ we have diffeomorphism

$$l^g : G \rightarrow G, h \mapsto gh$$

$$r^g : G \rightarrow G, h \mapsto hg$$

We say $X \in \mathfrak{X}G$ is **left invariant** if $l_*^g(X) = X, \forall g \in G$. Similarly, X is **right invariant** if $r_*^g(X) = X$.

Proposition 3.14. X, Y are left/right invariant $\Rightarrow [X, Y]$ is left/right invariant.

Proof. $l_*^g[X, Y] = [l_*^g X, l_*^g Y] = [X, Y]$ □

So we can find a natural Lie algebra of G :

$\text{Lie}(G) := \{\text{left invariant vector fields on } G\}$, with $[-, -]$ restricted from $\mathfrak{X}G$

Theorem 3.15. Given any $V \in T_e G, \exists$ unique left invariant $\hat{V} \in \mathfrak{X}G$ s.t. $\hat{V}_e = V$.

Corollary 3.16. $\text{Lie}(G) \cong T_e G$ as vector spaces.

Proof of Theorem 3.15.

Uniqueness of \hat{V} : $\hat{V}_g = l_{e,*}^g(\hat{V}_e) = l_{e,*}^g(V)$. So \hat{V} is determined by V .

Existence of \hat{V} : Let $\hat{V} = \{\hat{V}_g\}_{g \in G}$ where $\hat{V}_g = l_{e,*}^g(\hat{V}_e)$.

\widehat{V} is left-invariant because

$$(l_*^h(\widehat{V}))_g = l_{h^{-1}g,*}^h(\widehat{V}_{h^{-1}g}) = l_{h^{-1}g,*}^h(l_{e,*}^{h^{-1}g}(V)) = l_{e,*}^g(V) = \widehat{V}_g$$

\widehat{V} is smooth: Take any $f \in C^\infty(G)$ suffices to show $\widehat{V}(f) \in C^\infty(G)$.

Take any smooth $\gamma : \mathbb{R} \rightarrow G$ s.t. $\gamma(0) = e, \gamma'(0) = V$. Then $l^g \circ \gamma : \mathbb{R} \rightarrow G$ satisfies $l^g \circ \gamma(0) = g, (l^g \circ \gamma)(0) = g, (l^g \circ \gamma)'(0) = l_{e,*}^g(V) = \widehat{V}_g$

So

$$\widehat{V}(f)(g) = \widehat{V}_g(f) = \frac{d}{dt}f(l^g \circ \gamma(t))|_{t=0} = \frac{d}{dt}f(g \cdot \gamma(t))|_{t=0} \quad (3.1)$$

Consider the map

$$\begin{aligned} \widehat{f} : G \times \mathbb{R} &\xrightarrow{\text{id} \times \gamma} G \times G \xrightarrow{f} \mathbb{R} \\ (g, t) &\mapsto (g, \gamma(t)) \mapsto g \cdot \gamma(t) \mapsto f(g \cdot \gamma(t)) \end{aligned}$$

Then \widehat{f} is smooth, $\frac{\partial \widehat{f}}{\partial t}|_{t=0} : G \rightarrow \mathbb{R}$ is smooth, but $\frac{\partial \widehat{f}}{\partial t}|_{t=0}(g) = \widehat{V}(f)(g)$ by 3.1. So $\widehat{V}(f) \in C^\infty(G)$. \square

Example 3.17. $G = \text{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^2$.

$$\mathfrak{gl}(n, \mathbb{R}) = \text{Lie}(\text{GL}(n, \mathbb{R})) = T_I \text{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$$

Theorem 3.18. $\forall A, B \in \mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}), [A, B] = AB - BA$.

Remark 3.19. This theorem shows that the Lie bracket viewed as the Lie algebra and matrix are the same. In some sense, it means the Lie bracket defined in three sets $\mathfrak{gl}(n, \mathbb{R}) = T_I \text{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$ can commute with those corresponding, or equivalently, are just the same.

Lemma 3.20. $\forall A \in \mathfrak{gl}(n, \mathbb{R})$, the left invariant vector field \widehat{A} is complete and generate the flow $\varphi_t : \text{GL}(n, \mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2 t^2}{2!} + \dots)$

Proof.

$$\hat{A}_g = g \cdot A \in T_g G = M_n(\mathbb{R})$$

$$\frac{\partial}{\partial t} \varphi_t(g) = \frac{\partial}{\partial t} (g(e^{At})) = g e^{At} A = \hat{A}_{g \cdot e^{At}} = \hat{A}_{\varphi_t(g)}$$

□

Remark 3.21. This lemma tells how to compute $A(f) = \hat{A}(f)(I)$ as a tangent vector or a vector field, as we will see in the next proof.

Proof of Theorem 3.18. Take $A, B \in \mathfrak{gl}(n, \mathbb{R})$. Want to show $[\hat{A}, \hat{B}]_I = AB - BA$.

Pick $f \in C_I^\infty(G)$, need to show $A(\hat{B}(f)) - B(\hat{A}(f)) = (AB - BA)(f)$

Further Simplification: Just need to focus on $f = x^{ij}$, where $x^{ij} : \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}, E \mapsto (E - I)_{ij}$. Actually, ∂x^{ij} is what we choose as a basis of $T_I \text{GL}(n, \mathbb{R})$.

Such f satisfies $f(I + -)$ is \mathbb{R} -linear.

Recall that Given $W \in \mathfrak{Z}M$, $W(f)(p) = \frac{d}{dt} f(\varphi_t^W(p))|_{t=0}$.

So $\hat{B}(f)(g) = \frac{d}{dt} f(g e^{Bt})|_{t=0}$.

So since $A(\hat{B}(f)) = \hat{A}((\hat{B}(f)))(I) = \frac{d}{ds} (\hat{B}(f)(e^{As}))|_{s=0}$,

$$A(\hat{B}(f)) = \frac{d}{ds} (\hat{B}(f)(e^{As}))|_{s=0} = \frac{d^2}{ds dt} f(I + sA + tB + \frac{s^2}{2} A^2 + stAB + \frac{t^2}{2} B^2 + \dots)|_{s=t=0}$$

Similarly,

$$B(\hat{A}(f)) = \frac{d^2}{ds dt} f(I + sA + tB + \frac{s^2}{2} A^2 + stBA + \frac{t^2}{2} B^2 + \dots)|_{s=t=0}$$

So $A(\hat{B}(f)) - B(\hat{A}(f)) = f(I + (AB - BA)) = (AB - BA)(f)$ since f is \mathbb{R} -linear. □

Similarly, for $G = \text{GL}(n, \mathbb{C})$, $\text{Lie}(G) = \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$, we have $[A, B] = AB - BA$.

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group G , its universal cover \widehat{G} is simply-connected with $\pi^{-1}(G) \subset Z(\widehat{G})$.

What is the meaning of Lie bracket. There is a fact about it:

Fact 3.22. G is connected Lie group. G is abelian iff $[-, -] = 0$ on $\text{Lie}(G)$

3.3 Morphisms between Lie group and Lie algebras

A smooth map $F : G \rightarrow H$ between two Lie group is called a **morphism** if $F(gh) = F(g)F(h)$.

A linear map $L : g \rightarrow h$ between Lie algebra is called a **morphism** if $L[u, v] = [Lu, Lv]$.

Proposition 3.23. Let $F : G \rightarrow H$ be a morphism of Lie groups. Then $F_{e,*} : \text{Lie}(G) \rightarrow \text{Lie}(H)$ is a morphism of Lie algebra.

Proof. $V_0, V_1 \in \text{Lie}(G) = T_e G$, $W_i = F_{e,*}(V_i) \in \text{Lie}(H) = T_e H$. Let \widehat{V}, \widehat{W} be left-invariant vector fields.

Claim. \widehat{W}_i is F -compatible with \widehat{V}_i for $i = 0, 1$.

Proof of Claim. $\forall g \in G$, $F_*(\widehat{V}_g) = F_*(l_g^*(V)) = (F \circ l_g)_*(V) = (l^{F(g)} \circ F)_*(V) = l^{F(g)}(W) = \widehat{W}_{F(g)}$ □

So $[\widehat{W}_0, \widehat{W}_1]$ is F -compatible with $[\widehat{V}_0, \widehat{V}_1]$. In particular, $[W_0, W_1] = F_*([V_0, V_1])$. □

4 Vector Field

4.1 Canonical form of a Field

Recall that $V \in \mathfrak{X}M$, $p \in M$ is called a **regular point** if $V_p \neq 0$, and is called a **singular point** if $V_p = 0$.

Theorem 4.1 (Canonical Form Theorem). *Let p be a regular point of V . Then \exists local chart (U, x^1, \dots, x^n) around p s.t. $V|_U = \partial x^1$*

Proof. This is a local problem. We may assume $M \subset \mathbb{R}^n$ open. We may also assume $p = 0$, $V_0 = \partial r^1|_0$ where r^i coordinate function.

Let $\varphi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)^n \rightarrow M$ be the local flow of V .

Define $\psi : (-\varepsilon, \varepsilon)^n \rightarrow M$ by $\psi(t, r^2, \dots, r^n) = \varphi(t, (0, r^2, \dots, r^n))$. Then $\psi(-, r^2, \dots, r^n)$ is an integral curve for V . Therefore, $\psi_*(\partial t) = V$.

At $\vec{0}$, we have $\psi_{\vec{0},*}(\partial t) = V_{\vec{0}} = \partial r^1$, $\psi_{\vec{0},*}(\partial r^i) = \partial r^i$.

So $\psi_{*,\vec{0}} : T_{\vec{0}}(-\varepsilon, \varepsilon)^n \rightarrow T_{\vec{0}}M$ is an isomorphism.

By the inverse function theorem, $\exists U' \subset (-\varepsilon, \varepsilon)^n$, $U \subset M$ s.t. $\psi|_{U'} : U' \rightarrow U$ is a diffeomorphism.

Then $(U, (\psi|_{U'})^{-1})$ is the local chart what we need. □

Remark 4.2. Regular point in a vector field is simple, as we can view it in the standard chart locally. However, behavior of V at a singular point can be complicated. For example, for $f(x, y) = x^2 - y^2$, $\nabla f = 2x\partial x - 2y\partial y$, $g : \mathbb{C} \rightarrow \mathbb{C}$, $z \mapsto z^n$, they behave differently at $\vec{0}$.

4.2 Lie Derivative of Vector Field

$V, W \in \mathfrak{X}M$, $\mathcal{L}_V W$ is the directional derivative of W in the direction of V .

Definition 4.3. The **Lie derivative** $\mathcal{L}_V W \in \mathfrak{T}M$ is defined as follows: $\forall p \in M$, let $\{\theta_t : U \rightarrow M\}_{t \in (-\varepsilon, \varepsilon)}$ be the local flow for V . Then

$$(\mathcal{L}_V W)_p = \lim_{t \rightarrow 0} \frac{(\theta_{-t})_* W_{\theta_t(p)} - W_p}{t}$$

Remark 4.4. This defintion is actually a difference between $T_{\theta_t(p)}$ and T_p , which need pullback.

Lemma 4.5. $\mathcal{L}_V W$ is well-defined and smooth.

Proof. For $p \in M$, take local chart (U, x^1, \dots, x^n) . Let $\theta : (-\varepsilon, \varepsilon) \times U \rightarrow M$ be the flow of V . Take $J_0 \subset (-\varepsilon, \varepsilon)$, $U_0 \subset U$. Let $\theta^i = x^i \circ \theta : J_0 \times U_0 \rightarrow \mathbb{R}$, $W|_U = \sum_{i=1}^n W^i \partial x^i$.

Under the basis $\{\partial x^i\}$, $(\theta_{-t})_* : T_{\theta_t(p)} M \rightarrow T_p M$ is represented by

$$\left(\frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j} \right)_{i,j}$$

So $(\theta_{-t})_* W_{\theta_t(x)} = \sum_{i,j} \frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j} W^j(\theta(t, x)) \cdot \partial x^i$ is smooth in t, x . So

$$(\mathcal{L}_V W)_x = \frac{\partial((\theta_{-t})_*(W_{\theta_t(x)}))}{\partial t} \Big|_{t=0}$$

is well-defined and smooth. □

Theorem 4.6. For all $V, W \in \mathfrak{T}M$, $\mathcal{L}_V W = [V, W]$.

Proof. For p is a regular point of V . By canonical form theorem 4.1, \exists local chart (U, x^1, \dots, x^n) around p s.t. $V|_U = \partial x^1$. Let $W|_U = \sum_{i=1}^n W^i \partial x^i$.

Then $\theta_t(x_1, \dots, x_n) = (x_1 + t, x_2, \dots, x_n)$. So

$$\mathcal{L}_V W|_U = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

$$[V, W]|_U = \sum_i V(W^i) \partial x^i - \sum_i W(V^i) \partial x^i = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

Then $[V, W]|_U = \mathcal{L}_V W$.

For p is a singular point but $p \in \text{Supp}(V)$. Then $\exists p_i \rightarrow p$ s.t. $V_p \neq 0$. By the previous case $(\mathcal{L}_V W)_{p_i} = [V, W]|_{p_i}$. By continuity, We have $(\mathcal{L}_V W)_p = [V, W]_p$.

For $p \notin \text{Supp}(V)$, \exists Nbd U of p s.t. $V|_U = 0$. Then $\theta_t(q) = q$. So

$$(\mathcal{L}_V W)|_U = 0 = [V, W]|_U$$

□

Corollary 4.7.

- $\mathcal{L}_V W$ is \mathbb{R} -linear with respect to V, W .
- $\mathcal{L}_V W = -\mathcal{L}_W V$.
- (Jacobian identity) $\mathcal{L}_V [W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.
- (Jacobian identity) $\mathcal{L}_{[V, W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.
- $\mathcal{L}_V (fW) = (Vf) \cdot W + f \mathcal{L}_V W$
- Let $F : M \rightarrow N$ be a diffeomorphism. Then $F_*(\mathcal{L}_V W) = \mathcal{L}_{F_*(V)} F_*(W)$.

4.3 Commuting Vector Fields

Definition 4.8. We say $V, W \in \mathfrak{X}M$ **commutes** if $[V, W] = 0$.

Theorem 4.9. TFAE:

- 1 V, W commutes.

2 W is invariant under the flow generated by V , i.e. $\theta_{t,*}(W_p) = W_{\theta_t(p)}$

3 The flow for V, W commutes, i.e. $\theta_t \circ \eta_s = \eta_s \circ \theta_t$ whenever either side is defined or equivalently, whose the domain is compatible.

Lemma 4.10. Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}N$. Then W is F -related to V if and only if $\forall t \in \mathbb{R}, \eta_t \circ F = F \circ \theta_t$ on the domain of θ_t , which means

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow \theta_t & & \downarrow \eta_t \\ M & \xrightarrow{F} & N \end{array} \text{ commutes.}$$

Proof. " \Rightarrow " Let $\gamma = F \circ \theta^p : J \rightarrow N$ satisfies

$$\gamma'(t) = (F \circ \theta^p)'(t) = F_*((\theta^p)'(t)) = F_*(V_{\theta^p(t)}) = W_{F(\theta^p(t))} = W_{\gamma(t)}$$

So γ is an inetgral curve of W starting at $\gamma(0) = F(p)$ i.e. $F \circ \theta^p = \gamma(t) = \eta^{F(p)}(t)$ i.e. $F \circ \theta_t = \eta \circ F$.

" \Leftarrow " Suppose $F \circ \theta_t = \eta \circ F$. Then $(F \circ \theta^p)(t) = \eta^{F(p)}(t)$.

Then $F_*V_p = F_*((\theta^p)'(0)) = (F \circ \theta^p)'(0) = (\eta^{F(p)})'(0) = W_{F(p)}$. So W is F -related to V . □

Proof of Theorem 4.9. $2 \Rightarrow 1$: $(\theta_{-t})_*(W_{\theta_t(p)}) = W_p$. So

$$\mathcal{L}_V W = \lim_{t \rightarrow 0} \frac{(\theta_{-t})_*(W_{\theta_t(p)}) - W_p}{t} = 0$$

$1 \Rightarrow 2$: Let $X(t) = (\theta_{-t})_*(W_{\theta_t(p)})$, $p \in M$.

Want to show that $X(t) = X_p$ for all t . Suffices to show $\frac{d}{dt}|_{t=t_0} X(t) = 0$.

For $t_0 = 0$, $\frac{d}{dt}|_{t=0} X(t) = (\mathcal{L}_V W)_p = 0$.

In general, set $s = t - t_0$, $X(t) = (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))})$. Then

$$\frac{d}{dt}|_{t=t_0} X(t) = \frac{d}{ds}|_s X(s + t_0)$$

$$\begin{aligned}
&= \frac{d}{ds} \Big|_s (\theta_{-t_0})_* \circ (\theta_{-s})_* (W_{\theta_s(\theta_{t_0}(p))}) \\
&= (\theta_{t_0})_* \frac{d}{ds} \Big|_{s=0} (\theta_{-s})_* (W_{\theta_s(\theta_{t_0}(p))}) \\
&= (\theta_{t_0})_* (\mathcal{L}_V W)_{\theta_{t_0}(p)} \\
&= 0
\end{aligned}$$

2 \Rightarrow 3. For simplicity, assume V, W are complete. $F = \theta_s : M \rightarrow M$. By 2, W is F -related to W . So by the lemma,

$$\begin{array}{ccc}
M & \xrightarrow{F} & M \\
\downarrow \theta_t & & \downarrow \eta_t \\
M & \xrightarrow{F} & M
\end{array} \text{ commutes.}$$

η_t is flow for W . i.e. $\theta_s \circ \eta_t = \eta \circ \theta_s$

3 \Rightarrow 2 is similar. The diagram commutes, so W is F -related to W . \square

4.3.1 Canonical Form of Commuting Vector Field

Theorem 4.11. *Given $V_1, \dots, V_k \in \mathfrak{X}M$, s.t.*

- 1) $[V_i, V_j] = 0, \forall i, j$.
- 2) $V_{1,p}, V_{2,p}, \dots, V_{k,p}$ linearly independent at some $p \in M$

Then \exists local chart (U, x^1, \dots, x^n) around p s.t. $V_i|_U = \frac{\partial}{\partial x^i}, \forall 1 \leq i \leq k$

We prove it using the inverse function theorem.

Proof. This is a local problem. So we may assume $M \subset \mathbb{R}^m$ be open with coordinate function $r^i : M \rightarrow \mathbb{R}, 1 \leq i \leq m$.

After translation and linear transformation, we may assume $p = \vec{0}, V_{i,\vec{0}} = \frac{\partial}{\partial x^i} \Big|_{\vec{0}}, 1 \leq i \leq k$.

Take local flow $\{\theta_t^i : (-\varepsilon, \varepsilon)^m \rightarrow M\}_{t \in (-\varepsilon, \varepsilon)}$ for V_i .

Define $\psi : (-\varepsilon, \varepsilon)^k \times (-\varepsilon, \varepsilon)^{m-k} \rightarrow M$, $\psi(t^1, \dots, t^k, r^{k+1}, \dots, r^m) = \theta_{t_1}^1 \circ \theta_{t_2}^2 \cdots \circ \theta_{t_k}^k(0, 0, \dots, 0, r^{k+1}, \dots, r^m)$, where θ^i commutes with each other.

So if we fix $t^j, j \neq i$ except t^i , $\psi(t^1, \dots, t^{i-1}, -, t^{i+1}, \dots, t^k, r^{k+1}, \dots, r^m)$ is an integral curve for V^i . Then V^i is ψ -related to ∂t^i .

On the other hand. $\psi(0, 0, \dots, 0, r^{k+1}, \dots, r^m) = (0, 0, \dots, 0, r^{k+1}, \dots, r^m)$. So $\psi_{0,*} : T_{0,*} : T_0(-\varepsilon', \varepsilon')^m \rightarrow T_0 M$, $\partial t^i \mapsto V_{i,0} = \partial x^i|_0$ and $\partial r^i \mapsto \partial r^i, k+1 \leq i \leq m$.

So $\psi_{0,*}$ is an isomorphism.

By the inverse function theorem, there exists Nbh $U' \subset (-\varepsilon', \varepsilon')^m$ s.t. $\psi : U' \rightarrow U$ is a diffeomorphism and $U \subset M$ open.

Then $(U, (\psi|_U)^{-1})$ is the local chart we need. □

4.4 The Constant Rank Theorem

$F \in C^\infty(M, N)$, $p \in M$. The **rank** of F at p is

$$\begin{aligned} \text{rank}_p F &:= \text{rank}(F_{p,*} : T_p M \rightarrow T_{F(p)} N) \\ &= \text{rank} \left(\frac{\partial F^i(p)}{\partial x^j} \right)_{i,j} \end{aligned}$$

We say F has **constant rank** k near p if \exists Nbh U of p s.t. $\text{rank}_q F = k, \forall q \in U$

Proposition 4.12.

$$\text{rank}_q(F) \leq \min(\dim(M), \dim(N))$$

Theorem 4.13 (The constant rank theorem). *Suppose $F : M \rightarrow N$ has constant rank k near $p \in M$, then \exists local charts $U \xrightarrow[\cong]{\varphi} \mathbb{R}^m$ around p , $V \xrightarrow[\cong]{\psi} \mathbb{R}^n$ around $F(p)$ s.t.*

$$\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \rightarrow \mathbb{R}^n \text{ is given by } (x^1, \dots, x^m) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$$

Proof. This is a local problem. So we may assume $M = \mathbb{R}^m, N = \mathbb{R}^n$ by restricting

to local charts. And $p = 0, F(p) = 0$. After changing orders of coordinates, may assume $\left(\frac{\partial F^i}{\partial x^j}(0)\right)_{1 \leq i, j \leq k}$ is invertible. Write $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}, \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Then $F(x, y) = (Q(x, y), R(x, y))$. Consider $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m, (x, y) \mapsto (Q(x, y), y)$.

Then

$$\varphi_{(0,0),*} = \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}(0) & 0 \\ \frac{\partial Q^i}{\partial y^j}(0) & I_{m-k} \end{bmatrix} \quad (4.1)$$

is invertible.

By inverse function theorem, \exists Nbh $U_0 \subset \mathbb{R}^m, \tilde{U}_0 \subset \mathbb{R}^m$ of 0 s.t. $\varphi : U_0 \rightarrow \tilde{U}_0$ is a diffeomorphism.

$$\begin{aligned} \tilde{U}_0 &\xrightleftharpoons[\varphi]{\varphi^{-1}} U_0 \xrightarrow{F} \mathbb{R}^n \\ (Q(x, y), y) &\leftarrow (x, y) \mapsto (Q(x, y), R(x, y)) \end{aligned}$$

So $F \circ \varphi^{-1} : \tilde{U}_0 \rightarrow \mathbb{R}^n, (x, y) \mapsto (x, A(x, y))$. And

$$(F \circ \varphi^{-1})_{p,*} = \begin{bmatrix} I_k & 0 \\ \frac{\partial A}{\partial x}(p) & \frac{\partial A}{\partial y}(p) \end{bmatrix} \quad (4.2)$$

Since $\text{rank}(F \circ \varphi^{-1})$ is k , $\frac{\partial A}{\partial y}(p) = 0$. i.e. $A(x, y) = A(x)$.

We can find a map $\psi : (x, y) \mapsto (x, y - A(x))$ in a smaller neighborhood of 0 by the inverse theorem similarly.

And $\psi \circ F \circ \varphi$ maps (x, y) to $(x, 0)$. So we end the proof. \square

Definition 4.14. $F \in C^\infty(M, N)$.

We say F is **submersion** if $F_{p,*}$ is surjective $\forall p \in M$.

We say F is **immersion** if $F_{p,*}$ is injective $\forall p \in M$.

We say F is **embedding** if F is immersion and F is a topological embedding.(i.e. $F : M \rightarrow F(M)$ is a homeomorphism)

If F is embedding(immersion resp.), we say M or $F(M)$ is an **embedded submanifold**(**immersed submanifold**, resp.) of N .

Denote $M \looparrowright N$ be the immersion. $M \hookrightarrow N$ be the embedding.

Example 4.15.

- There is an example $F : S^1 \rightarrow \mathbb{R}^2$ where F is an immersion but not an embedding.
- Projection $M \times N \rightarrow M$ is a submersion.
- $E \xrightarrow{p} B$ is a smooth vector bundle, then p is a submersion.
- $\gamma : \mathbb{R} \rightarrow M$ is an immersion $\Leftrightarrow \gamma'(t) \neq 0, \forall t$.
- There is an example $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ is injective immersion but not an embedding
- $\gamma : \mathbb{R} \rightarrow \mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}, x \mapsto (x, cx), c \notin \mathbb{Q}$ is injective immersion but not embedding.

Definition 4.16. For $F : X \rightarrow Y$, we say F is **proper** if for any compact set $K \subset N$, $F^{-1}(K)$ is compact.

Lemma 4.17. X is compact, Y Hausdorff, then $F : X \rightarrow Y$ is proper.

Proposition 4.18. $F \in C^\infty(M, N)$ is an injective immersion, and F is proper. Then F is an embedding.

Proof. $F : M \rightarrow F(M)$ is a closed map. □

Definition 4.19. For $F \in C^\infty(M, N)$.

$p \in M$ is called **regular point** if $F_{p,*} : T_p M \rightarrow T_{F(p)} N$ is surjective.

$p \in M$ is called **critical point** if $F_{p,*} : T_p M \rightarrow T_{F(p)} N$ is not surjective.

$q \in N$ is called **regular value** if $\forall p \in F^{-1}(q), p$ is a regular point.

$q \in N$ is called **critical value**(or **singular value**) if $\exists p \in F^{-1}(q), p$ is a critical point.

Theorem 4.20 (Sard). *Singular value has measure 0.*

Proof. We will not prove it in this lecture. □

Theorem 4.21. *M is an embedded submanifold of N if and only if $\forall p \in M \subset N, \exists$ local chart (U, x^1, \dots, x^n) around p of N s.t. $M \cap U = \{(x^1, \dots, x^m, 0, \dots, 0)\}$*

Proof. " \Rightarrow ": $F : M \rightarrow N$ is embedding $\Rightarrow F$ has constant rank m . Apply constant rank theorem near p , and we finish the proof of " \Rightarrow "

The converse is trivial. □

Theorem 4.22. *$F \in C^\infty(M, N), q$ is a regular value of F . Then $F^{-1}(q)$ is an embedded submanifold of M . And*

$$\forall p \in F^{-1}(q), T_p F^{-1}(q) = \ker(F_{p,*} : T_p M \rightarrow T_{F(p)} N)$$

Proof. q is regular value $\Rightarrow \text{rank}_p F = n, \forall p \in F^{-1}(q)$.

$\Rightarrow \text{rank}_{p'} F = n, \forall p' \text{ near } p$, since we know the rank of p' near p should not be less than that of p

So by the constant rank theorem, $F^{-1}(q)$ is a submanifold near p . □

Denote

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) | A + A^T = 0\}$$

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) | A + A^* = 0\}$$

$$\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) | \text{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \text{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) | \text{tr} A = 0\}$$

Theorem 4.23. *Those above sets are the Lie algebra of the corresponding Lie group. For instance, $\mathfrak{su}(n) = \text{Lie}(\text{SU}(n))$.*

5 Differential forms

5.1 Introduction

Our goal is to define the integration $\int_M \alpha$ s.t.

- Works for any smooth manifold M , without embedding M into \mathbb{R}^n
- Generalize two types of surface integral, i.e. $\int_{\Sigma} f dS$ and $\int_{\Sigma} f dx \wedge dy$

For Cartan's idea, α is a "differential k -form" on M s.t.

- $\forall F \in C^\infty(N, M)$, $F^*(\alpha)$ is a k -form on N
- If $k = \dim M$, then $\int_M \alpha \in \mathbb{R}$

5.2 Alternating Vector Linear Algebra

For V_1, \dots, V_n, W be \mathbb{R} -vector spaces, $f : V_1 \times \dots \times V_n \rightarrow W$ is called **multi \mathbb{R} -linear** if

$$\begin{aligned} f(v_1, \dots, v_{i-1}, av_i + bv'_i, v_{i+1}, \dots, v_n) &= af(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n) \\ &+ bf(v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_n) \end{aligned} \quad (5.1)$$

Example 5.1.

- Inner product $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\cdot} \mathbb{R}$.
- Matrix multiplication $M_{n \times m}(\mathbb{R}) \times M_{m \times k}(\mathbb{R}) \rightarrow M_{n \times k}(\mathbb{R})$.
- Cross product $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\times} \mathbb{R}^3$.
- Bilinear form.

We hope that we can construct a vector space $V_1 \otimes \dots \otimes V_n$ s.t. we have canonical isomorphism:

$$\{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \dots \times V_n \rightarrow W\} \cong \{\text{linear map } V_1 \otimes \dots \otimes V_n \rightarrow W\} \quad (5.2)$$

Then we can transform the study of multilinear algebra into the study of the normal linear algebra.

For any set S , let

$$\mathbb{R} \langle S \rangle = \left\{ \text{formal linear combination } \sum_{i=1}^n a_i s_i \mid a_i \in \mathbb{R}, s_i \in S, n < \infty \right\} \quad (5.3)$$

Consider $\mathbb{R} \langle V_1 \times \cdots \times V_n \rangle = \left\{ \sum_{i=1}^k a^i(V_{i,1}, \dots, V_{i,n}) | a^i \in \mathbb{R}, v_{i,j} \in V_j \right\}$. Denote

$$W = \text{Span}\{(\cdots, av_i + bv'_i, \cdots) - a(\cdots, v_i, \cdots) - b(\cdots, v'_i, \cdots) | a, b \in \mathbb{R}, v_i, v'_i \in V_i\} \quad (5.4)$$

Define $V_1 \otimes \cdots \otimes V_n = \mathbb{R} \langle V_1 \times \cdots \times V_n \rangle / W$, write $[(v_1, \dots, v_n)]$ as $v_1 \otimes \cdots \otimes v_n$, called a **n -tensor**.

Proposition 5.2 (Universal Property). *We have a multi \mathbb{R} -linear map $q : V_1 \times \cdots \times V_n \rightarrow V_1 \otimes \cdots \otimes V_n$, $(v_1, v_2, \dots, v_n) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$. It satisfies the universal property:*

\forall multi \mathbb{R} -linear map $f : V_1 \times \cdots \times V_n \rightarrow W$, \exists unique linear map $\tilde{f} : V_1 \otimes \cdots \otimes V_n \rightarrow W$ s.t. $\tilde{f} \circ q = f$. i.e. The diagram commutes:

$$\begin{array}{ccc} V_1 \otimes \cdots \otimes V_n & & \\ \uparrow \rho & \searrow \exists! \tilde{f} & \\ V_1 \times \cdots \times V_n & \xrightarrow{f} & W \end{array}$$

Corollary 5.3.

$$\{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V_n \rightarrow W\} \cong \{\text{linear map } V_1 \otimes \cdots \otimes V_n \rightarrow W\} \quad (5.5)$$

$$f \leftrightarrow \tilde{f}$$

Proposition 5.4.

- Any element in $V_1 \otimes \cdots \otimes V_n$ can be written as $\sum a_i v_i^1 \otimes \cdots \otimes v_i^n$ for some $a_i \in \mathbb{R}$.
- If $(e_i^j)_{j \in \mathcal{A}_i}$ is a basis for V_i , then $\{e_1^{j_1} \otimes e_2^{j_2} \otimes \cdots \otimes e_n^{j_n} | j_i \in \mathcal{A}_i\}$ is a basis of $V_1 \otimes \cdots \otimes V_n$.
- $\dim(V_1 \otimes \cdots \otimes V_n) = \prod_{i=1}^n \dim(V_i)$

Proposition 5.5. Denote $W^* = \text{Hom}(W, \mathbb{R})$, then we have an injection

$$\begin{aligned} V \otimes W^* &\xrightarrow{e} \text{Hom}(W, V) \\ v \otimes f &\mapsto (w \mapsto f(w) \cdot v) \end{aligned} \quad (5.6)$$

If $\dim V$ or $\dim W$ is finite, then e is an isomorphism.

Indeed, if $\dim V = \infty$, then $\text{id}_V \notin e(V \otimes V^*)$

Given any $l_i \in \text{Hom}(V_i, W_i)$, $1 \leq i \leq n$, we define

$$\begin{aligned} l_1 \otimes \cdots \otimes l_n &\in \text{Hom}(V_1 \otimes \cdots \otimes V_n, W_1 \otimes \cdots \otimes W_n) \\ (l_1 \otimes \cdots \otimes l_n)(v_1 \otimes \cdots \otimes v_n) &= l_1(v_1) \otimes \cdots \otimes l_n(v_n) \end{aligned} \quad (5.7)$$

Proposition 5.6. If $\dim V_i < \infty$, $\forall 1 \leq i \leq n$, then we have isomorphism

$$\begin{aligned} V_1^* \otimes \cdots \otimes V_n^* &\xrightarrow{\cong} (V_1 \otimes \cdots \otimes V_n)^* \\ f_1 \otimes \cdots \otimes f_n &\mapsto \left((v_1 \otimes \cdots \otimes v_n \mapsto \prod_{i=1}^n f_i(v_i)) \right) \end{aligned} \quad (5.8)$$

For $\bigotimes_n V = \underbrace{V \otimes \cdots \otimes V}_n$, $S_n = \{\text{bijection on } \{1, 2, \dots, n\}\}$ acts on $\bigotimes_n V$, where

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \quad (5.9)$$

A tensor $T \in \bigotimes_n V$ is called **symmetric** if $\sigma(T) = T$, $\forall \sigma \in S_n$.

T is called **anti-symmetric** if $\sigma(T) = \text{sgn}(\sigma) \cdot T$, $\forall \sigma \in S_n$.

Define

$$\begin{aligned} \text{Sym}^n(V) &= \{\text{symmetric tensors in } \bigotimes_n V\} \\ \bigwedge^n(V) &= \{\text{anti-symmetric tensors in } \bigotimes_n V\} \end{aligned} \quad (5.10)$$

which are both in $\bigotimes_n V$. And

$$\dim(\text{Sym}^n(V)) = \binom{\dim(V) + n - 1}{n} \quad \dim(\bigwedge^n V) = \binom{\dim(V)}{n} \quad (5.11)$$

From now on, **we may assume** $\dim V < \infty$. Define

$$L^n(V) = \left(\bigotimes_n V \right)^* \cong \bigotimes_n V^* \cong \{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V \rightarrow \mathbb{R}\} \quad (5.12)$$

And by the assumption we can obtain

$$\begin{aligned} \text{Sym}^n(V^*) &\cong \{\text{symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \rightarrow \mathbb{R}\} \\ \bigwedge^n(V^*) &\cong \{\text{anti-symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \rightarrow \mathbb{R}\} \end{aligned} \quad (5.13)$$

We will mainly focus on $\bigwedge^n(V^*)$, also denoted as $\text{Alt}^k(V) = \bigwedge^n(V^*)$. An element in $\text{Alt}^k(V)$ is called a (linear) **k -form** on V . Now for $V = \mathbb{R}\langle e_1, \dots, e_n \rangle$, $V^* = \mathbb{R}\langle e_1^*, \dots, e_n^* \rangle$. Then

$$\begin{aligned} L^2(V) &= \{\text{all bilinear forms on } V\} \\ L^2(V) &\cong \text{Sym}^2(V^*) \oplus \bigwedge^2(V^*) \end{aligned}$$

And $\text{Sym}^2(V^*) = \mathbb{R}\langle e_i^* \otimes e_j^* + e_j^* \otimes e_i^* | 1 \leq i \leq i \leq n \rangle$ is symmetric bilinear form

$\text{Alt}^2(V) = \bigwedge^2(V^*) = \mathbb{R}\langle e_i^* \otimes e_j^* - e_j^* \otimes e_i^* | 1 \leq i \leq i \leq n \rangle$ is anti-symmetric bilinear form.

The determinant $\det \in \text{Alt}^n(\mathbb{R}^n)$.

Definition 5.7 (Exterior product).

$$\bigwedge : \text{Alt}^k(V) \times \text{Alt}^l(V) \rightarrow \text{Alt}^{k+l}(V)$$

$$\begin{aligned}
\omega_1 \wedge \omega_2(v_1, \dots, v_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \omega_2(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\
&= \sum_{\sigma \in S_{k,l}} \text{sgn}(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \omega_2(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})
\end{aligned}$$

where $S_{k,l} = \{\sigma \in S_{k+l} \mid \sigma(1) < \dots < \sigma(k), \sigma(k+1) < \dots < \sigma(k+l)\} \subset S_{k+l}$.

Then we have those properties:

Proposition 5.8.

- (1) $\omega_1 \wedge \omega_2 = (-1)^{|\omega_1| \cdot |\omega_2|} \omega_2 \wedge \omega_1$, $|\omega| = k$ is $\omega \in \text{Alt}^k(V)$. In particular, $\omega \wedge \omega = 0$ if $|\omega|$ is odd.
- (2) $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$
- (3) Given any $\omega_1, \dots, \omega_k \in \text{Alt}^1(V) = V^*$, $v_1, \dots, v_k \in V$. Then

$$(\omega_1 \wedge \dots \wedge \omega_k)(v_1, \dots, v_k) = \det [w_i(v_j)]_{i,j} \quad (5.14)$$

Moreover, $\omega_1 \wedge \dots \wedge \omega_n \neq 0$ iff ω_i are linearly independent.

- (4) $V = \mathbb{R} \langle e_1, \dots, e_n \rangle$. Then

$$\text{Alt}^k(V) = \mathbb{R} \langle e_{i_1}^* \wedge \dots \wedge e_{i_k}^* \mid i_1 < \dots < i_k \rangle \quad (5.15)$$

In particular, $\text{Alt}^n(V) = \mathbb{R} \langle e_1^* \wedge \dots \wedge e_n^* \rangle$. And we denote $\text{Alt}^0(V) = \mathbb{R}$, $\text{Alt}^k(V) = 0$, $k > n$.

- (5) Any $f \in \text{Hom}(V, W)$ induces $\text{Alt}^k(f) \in \text{Hom}(\text{Alt}^k(V), \text{Alt}^k(W))$, where

$$\text{Alt}^k(f)(\omega)(w_1, \dots, w_k) = \omega(f(w_1), \dots, f(w_k)) \quad (5.16)$$

We have $\text{Alt}^k(f \circ g) = \text{Alt}^k(g) \circ \text{Alt}^k(f)$, $\text{Alt}^k(\text{id}_V) = \text{id}_{\text{Alt}^k(V)}$. Such $\text{Alt}^k(-)$ is called a **contravariant functor**.

Proof.

(1) By definition,

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{k+l}) = \omega_2 \wedge \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(k+l)})$$

$$\text{where } \sigma(i) = \begin{cases} i + k & 1 \leq i \leq l \\ i - l & l + 1 \leq i \leq k + l \end{cases} \cdot \text{sgn}(\sigma) = (-1)^{k+l}.$$

(2) By definition.

(3) By linearity, we assume $\omega_i = e_{a(i)}^*$, $v_j = e_{b(j)}$ for some $a(i), b(j)$. Further more, can assume $\{a(i)\} = \{b(i)\}$. (Otherwise, $LHS = RHS = 0$.)

Then $e_{a(i)}^*(e_{b(j)}) = \delta_{a(i), b(j)}$. After permutation, may assume $a(i) = b(i), \forall i$. It is direct to check $LHS = 1 = RHS$.

(4) If $\omega_1, \dots, \omega_k$ are linear independent. Then \exists basis e_1^*, \dots, e_n^* of V^* , basis e_1, \dots, e_n of V s.t. $\omega_i = e_i^*, \forall 1 \leq i \leq n$.

$$(\omega_1 \wedge \dots \wedge \omega_n)(e_1, \dots, e_n) = \det(I) = 1 \neq 0 \Rightarrow \omega_1 \wedge \dots \wedge \omega_n \neq 0$$

If $\omega_1, \dots, \omega_k$ are linearly dependent. WLOG, we assume $\omega_k = \sum_{i=1}^{k-1} a_i \omega_i$.

$$(\omega_1 \wedge \dots \wedge \omega_k)(e_1, \dots, e_n) = \sum_{i=1}^{k-1} a_i (\omega_1 \wedge \dots \wedge \omega_{k-1} \wedge \omega_i)(e_1, \dots, e_n) = 0$$

(5) For $i_1 < \cdots < i_k, j_1 < \cdots < j_n$ we have

$$(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)(e_{j_1}, \cdots, e_{j_k}) = \begin{cases} 1 & j_t = i_t, \forall 1 \leq t \leq k \\ 0 & \text{otherwise} \end{cases} \quad (5.17)$$

Since $\dim \text{Alt}(V) = \dim \bigwedge^k(V^*) = \binom{n}{k} = |\{e_{i_1} \wedge \cdots \wedge e_{i_k} | i_1 < \cdots < i_k\}|$.

(6) For $\omega \in \text{Alt}^k(W), f \in \text{Hom}(V, W)$, define $\text{Alt}^k(f)(\omega) \in \text{Alt}^k(V)$ by

$$\text{Alt}^k(f)(\omega(V_1, \cdots, V_k)) = \omega(f_* V_1, \cdots, f_* V_k) \in \mathbb{R}$$

□

Definition 5.9.

An **\mathbb{R} -algebra** consists of an \mathbb{R} -vector space A with a bilinear map $\mu : A \times A \rightarrow A$ that is associate, i.e. $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$.

Say A is **unitary** if $\exists 1 \in A$ s.t. $\mu(a, 1) = \mu(1, a) = a, \forall a \in A$

Say A is **graded** if $A = \bigoplus_{k \in \mathbb{Z}} A_k$ as vector space, and $\mu(A_k \times A_l) \subset A_{k+l}$. Elements in A_k are called **homogeneous elements** of degree k .

If A is graded \mathbb{R} -algebra, we say A is **anticommutative** if $\mu(a, b) = (-1)^{k+l} \mu(b, a), \forall a \in A_k, b \in A_l$. And say A is **commutative** if $\mu(a, b) = \mu(b, a), \forall a, b$.

If A is graded \mathbb{R} -algebra, say A is **connected** if \exists unit $1 \in A_0$ s.t. the map $\varepsilon : \mathbb{R} \rightarrow A_0, r \mapsto r \cdot 1$ is an isomorphism.

Given vector space V , let

$$\begin{array}{ccc} \text{Alt}^k(V) & \xlongequal{\quad} & \bigoplus_{k \geq 0} \text{Alt}^k(V) \\ \parallel & & \parallel \\ \text{Alt}^*(V^*) & \xlongequal{\quad} & \bigoplus_{k \geq 0} \wedge^k(V^*) \end{array}$$

By Proposition 5.8, we have the theorem

Theorem 5.10. $(\text{Alt}^*(V), \wedge)$ is a graded connected anticommutative \mathbb{R} -algebra, called the exterior algebra of V or exterior algebra of V

5.3 Operation on Vector Bundles

Given $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$, meaning a vector bundle $E \xrightarrow{\pi} M$ of dimension n , local trivialization $\left\{ U_\alpha, \varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n \right\}_{\alpha \in \mathcal{A}}$. By shrinking U_α , we may assume we have an smooth atlas $\left\{ \varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^m \right\}_{\alpha \in \mathcal{A}}$.

For $x \in M$, use E_x to denote $\pi^{-1}(x)$, fiber over x , which is a vector space of dimension n .

Then **Dual bundle of a vector bundle** $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$ is

$$E^* := \{(x, l) | x \in M, l \in (E_x)^*\}, \pi' : E^* \rightarrow M, (x, l) \mapsto x, (\pi')^{-1}(x) = (E_x)^* \quad (5.18)$$

Define topology or smooth structure on E^* s.t. $\pi' : E^* \rightarrow M$ is a smooth vector bundle.

For $\alpha \in \mathcal{A}$, let $E_\alpha^* = \pi'^{-1}(U_\alpha)$, we have a bijection

$$\tilde{\varphi}_\alpha : E_\alpha^* \xrightarrow{\text{bijection}} \mathbb{R}^m \times (\mathbb{R}^n)^* \xrightarrow{\cong} \mathbb{R}^{m+n}$$

$$(x, l) \longmapsto (\psi_\alpha(x), (\varphi_{\alpha,x})^{-1}(l))$$

We can check that

(1) $\{\tilde{\varphi}_\alpha^{-1} | \alpha \in \mathcal{A}, V \subset \mathbb{R}^{m+n} \text{ open}\}$ is a basis, we use it to generate a topology on E^* .

(2) Use $\tilde{\varphi}_\alpha : E_\alpha^* \xrightarrow{\cong} \mathbb{R}^{m+n}, \alpha \in \mathcal{A}$ as an atlas to give E^* a smooth structure.

(3) $E^* \xrightarrow{\pi'} M$ is a smooth vector bundle, called the **dual vector bundle** of $E \xrightarrow{\pi} M$, where

$$(E^*)_x = E_x^*$$

We can define other operations on vector bundles in similar way:

Given $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$, $\mathbb{R}^m \hookrightarrow F \xrightarrow{\pi} M$, we can define

$$\mathbb{R}^{m+n} \hookrightarrow E \oplus F \xrightarrow{\pi} M \text{ with } (E \oplus F)_x = E_x \oplus F_x$$

$$\mathbb{R}^{mn} \hookrightarrow E \otimes F \xrightarrow{\pi} M \text{ with } (E \otimes F)_x = E_x \otimes F_x$$

$$\mathbb{R}^{mn} \hookrightarrow \text{Hom}(E, F) \xrightarrow{\pi} M \text{ with } \text{Hom}(E, F)_x = \text{Hom}(E_x, F_x)$$

$$\mathbb{R}^{\binom{n}{k}} \hookrightarrow \text{Alt}^k(E) \rightarrow M \text{ with}$$

$$\text{Alt}^k(E)_x = \text{Alt}^k(E_x) = \{\text{alternating } k\text{-linear } l : E_x \times \cdots \times E_x \rightarrow \mathbb{R}\}$$

$$\text{Then } \text{Alt}^k(TM) = \bigwedge^k(T^*M).$$

$$\text{Alt}^k(M)_x = \{\text{alternating } k\text{-linear } l : T_x M \times \cdots \times T_x M \rightarrow \mathbb{R}\} = \{\text{linear } k\text{-form on } T_x M\}$$

Define

$$\Gamma(E) := \{\text{smooth sections of } E\} = \{s \in C^\infty(M, E) : \pi \circ s = \text{id}_M\}$$

Definition 5.11. Given smooth M , define a differential k -form on M to be an element in $\Gamma(\text{Alt}^k(TM))$ is a differential k -form α assigns each $x \in M$ a linear k -form $\alpha(x) \in \text{Alt}^k(T_x M)$.

Denote $\Omega^k(M)$ be the set of all the differential k -forms.

Then $\Omega^0(M) = C^\infty(M, \mathbb{R})$. $\text{Alt}^1(TM) = T^*M \Rightarrow$ a 1-form on M is just a "cotangent vector field" on M .

$$\Omega^k(M) = 0 \text{ if } k \geq \dim(M).$$

5.4 Differential forms using local chart

Given local chart (U, x^1, \dots, x^n) of M .

For any $p \in U$, $\{\frac{\partial}{\partial x^i}|_p\}_{1 \leq i \leq n}$ is a basis of $T_x M$.

We denote the dual basis of $T_x^* M$ by $\{dx^i|_p\}_{1 \leq i \leq n}$.

For any $\alpha \in \Omega^1(M)$, $\alpha|_U$ can be written as $\sum_{i=1}^n f_i dx^i$, where $f^i \in C^\infty(U, \mathbb{R})$.

Similarly, $\{dx^{i_1}|_p \wedge \dots \wedge dx^{i_k}|_p | i_1 < \dots < i_k\}$ is a basis for $\bigwedge^k(T_x^* M)$, so $\forall \alpha \in \Omega^k(M)$,

$$\alpha|_U = \sum_{i_1 < \dots < i_k} f_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad f_{i_1, \dots, i_k} \in C^\infty(U, \mathbb{R})$$

We give the notation that $I = (i_1, \dots, i_k)$, write $f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$ as $f^I dx^I$.

Change of coordinate If (U, x^1, \dots, x^n) and (V, y^1, \dots, y^n) two charts of M and $p \in U \cap V$, then

$$dy^i = \sum_{1 \leq j \leq n} \frac{\partial y_i}{\partial x^j} dx^j. \quad (5.19)$$

5.5 Exterior Differential

For $k = 0$, define $d : \Omega^0(M) \rightarrow \Omega^1(M)$ as follows:

$$\forall p \in M, X_p \in T_p M, df|_p(X_p) = X_p(f) \in \mathbb{R}. \text{ In local chart, } df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

Theorem 5.12. \exists linear operator $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ s.t. For $\alpha \in \Omega^k(M)$,

$$\alpha|_U = \sum_I f^I dx^I \Rightarrow d\alpha|_U = \sum_I df^I \wedge dx^I \quad (5.20)$$

Called the *exterior differential*

Proof. It suffices to prove that (5.20) is compatible for two charts $(U, x^1, \dots, x^n), (V, y^1, \dots, y^n)$, i.e. the diagram is commutative.

$$\begin{array}{ccc}
 f dy^1 \wedge \dots \wedge dy^k & \xleftarrow{\quad} & \sum_{1 \leq i_1, i_2, \dots, i_k \leq n} f \frac{\partial y^{i_1}}{\partial x^1} \dots \frac{\partial y^{i_k}}{\partial x^k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \\
 \downarrow d & & \downarrow ?d \\
 \sum_{1 \leq i \leq n} \frac{\partial f}{\partial y^i} dy^i \wedge dy^{i_1} \wedge \dots \wedge dy^{i_k} & \xleftarrow{?} & \sum_{1 \leq i_1, i_2, \dots, i_k \leq n} \frac{\partial f}{\partial x^j} \cdot \frac{\partial y^{i_1}}{\partial x^1} \dots \frac{\partial y^{i_k}}{\partial x^k} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}
 \end{array}$$

□

Theorem 5.13.

(1) $d^2 = 0$.

(2) $\forall \alpha \in \Omega^k(M), \beta \in \Omega^l(M), d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$.

Proof.

(1) If $\alpha|_U = \sum_U f^I dx^I$. By linearity suffices to check

$$\begin{aligned}
 d \circ d(f dx^I) &= d \left(\sum_{1 \leq i \leq n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) \\
 &= \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i \wedge dx^I \\
 &= 0
 \end{aligned}$$

(2) By linearity, suffices to assume $\alpha = f dx^I, \beta = g dx^J$.

$$\begin{aligned}
 d(\alpha \wedge \beta) &= d(f g dx^I \wedge dx^J) \\
 &= \sum_{1 \leq i \leq n} \frac{\partial(fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\
 &= \sum_{1 \leq i \leq n} \left(\frac{f}{\partial x^i} g + f \frac{\partial g}{\partial x^i} \right) dx^i \wedge dx^I \wedge dx^J
 \end{aligned}$$

And

$$d\alpha \wedge \beta = \sum_i \frac{\partial f}{\partial x^i} g dx^i \wedge dx^I \wedge dx^J$$

$$\alpha \wedge d\beta = \sum_i \frac{\partial g}{\partial x^i} f dx^I \wedge dx^i \wedge dx^J = \sum_i (-1)^k \frac{\partial g}{\partial x^i} f dx^i \wedge dx^I \wedge dx^J$$

□

Example 5.14. For $M = \mathbb{R}^3$,

$$\begin{array}{ccc} \Omega^0(\mathbb{R}^3) & \xlongequal{\quad} & C^\infty(\mathbb{R}^3) \\ \downarrow d & & \downarrow \text{gradient} \\ \Omega^1(\mathbb{R}^3) & \longleftrightarrow & \mathfrak{T}\mathbb{R}^3, & f dx + g dy + h dz \longleftrightarrow f \partial x + g \partial y + h \partial z \\ \downarrow d & & \downarrow \text{curl} \\ \Omega^2(\mathbb{R}^3) & \longleftrightarrow & \mathfrak{T}\mathbb{R}^3, & f dx \wedge dy + g dx \wedge dz + h dy \wedge dz \longleftrightarrow f \partial z + g \partial x + h \partial y \\ \downarrow d & & \downarrow \text{divergent} \\ \Omega^3(\mathbb{R}^3) & \longleftrightarrow & C^\infty(\mathbb{R}^3), & f dx \wedge dy \wedge dz \longleftrightarrow f \end{array}$$

5.6 Pull Back of Differential Forms

For $F \in C^\infty(M, N)$, $\alpha \in \Omega^k(N)$, define the **pullback** $F^*(\alpha) \in \Omega^k(M)$ as follows:

$$\forall p \in M, V_1 \cdots, V_k \in T_p M, F^*(\alpha)|_p(V_1, \cdots, V_k) = \alpha|_{F(p)}(F_{p,*}(V_1), \cdots, F_{p,*}(V_k)) \in \mathbb{R}$$

Actually, $F^*|_p = \text{Alt}^k(F_{p,*}) : \text{Alt}^k(T_{F(p)}N) \rightarrow \text{Alt}^k(T_p M)$.

Proposition 5.15. For $F : M \rightarrow N$, $G : N \rightarrow L$.

$$(1) \quad f \in \Omega^0(N), F^*(f) = f \circ F \in \Omega^0(M).$$

$$(2) \quad F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta).$$

$$(3) \quad F^*(d\alpha) = dF^*(\alpha).$$

$$(4) \quad (G \circ F)^* = F^* \circ G^*$$

Proof.

$$(1) \quad \begin{array}{ccc} \text{Alt}^0(T_{F(p)}N) & \xrightarrow{\text{Alt}^0(F_p,*)} & \text{Alt}^0(T_pM) \\ \parallel & & \parallel \\ \mathbb{R} & \xrightarrow{id} & \mathbb{R} \end{array} \quad \text{commutes.}$$

$$(2) \quad \begin{array}{ccc} \text{Alt}^k(T_{F(p)}N) \times \text{Alt}^l(T_{F(p)}N) & \xrightarrow{\wedge} & \text{Alt}^{k+l}(T_pM) \\ \downarrow \text{Alt}^k(F_p,*) \times \text{Alt}^l(F_p,*) & & \downarrow \text{Alt}^{k+l}(F_p,*) \\ \text{Alt}^k(T_pM) \times \text{Alt}^l(T_pM) & \xrightarrow{\wedge} & \text{Alt}^{k+l}(T_pM) \end{array} \quad \text{commutes.}$$

(3) By linearity it suffices to check

$$dF^*(f dx^I) = F^*d(f dx^I)$$

By Leibniz rule for d and (2), it suffices to show

$$(a) \quad dF^*(df) = F^*(df)$$

$$(b) \quad dF^*(dx^i) = F^*(d(dx^i))$$

Which leaves to the readers.

(4) By definition.

□

Definition 5.16. A k -form ω is **closed** if $\omega \in \ker \left(\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \right)$.

A k -form ω is **exact** if there exists a $(k-1)$ -form η such that $d\eta = \omega$, or equivalently, $\omega \in \text{Im} \left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \right)$.

By Proposition 5.15 (1), exact k -form are all closed.

So we may define the k -th **de Rham cohomology** of M

$$H_{\text{DR}}^k(M) := \frac{\ker \left(d : \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \right)}{\text{Im} \left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \right)} \quad (5.21)$$

By Proposition 5.15 (2), we have $\forall F \in C^\infty(M, N), \omega \in \Omega^k(N)$.

Then ω closed $\Rightarrow F^*\omega$ is closed. ω exact $\Rightarrow F^*\omega$ exact.

So F induces a linear map

$$F^* : H_{\text{DR}}^k(N) \rightarrow H_{\text{DR}}^k(M)$$

$$[\omega] \mapsto [F^*\omega]$$

Proposition 5.17 (Key properties of $H_{\text{DR}}^k(M)$).

$$(1) (F \circ G)^* = G^* \circ F^*$$

$$(2) (\text{id})^* = \text{id}.$$

$$(3) F, G \in C^\infty(M, N), F \text{ homotopic to } G \Rightarrow F^* = G^*$$

$$(4) \text{ If } F \text{ is a homotopy equivalence} \Rightarrow F^* : H_{\text{DR}}^k(M) \rightarrow H_{\text{DR}}^k(N) \text{ is an isomorphism.}$$

Remark 5.18. Properties (3),(4) are nontrivial, which is the essential part of the theory of de Rham cohomology

Proposition 5.19. $H_{\text{DR}}^k(M) \cong \mathbb{R} \langle \pi_0(M) \rangle$, where $\pi_0(M) = \{\text{path component of } M\}$.

It suffices to prove the lemma that

Lemma 5.20. $\alpha \in \Omega^0(M) = C^\infty(M, \mathbb{R}^n)$. Then α is closed iff α is constant on each component of M .

Proof. The inverse part is trivial.

Assume α is closed. Pick $p, q \in M$ in some path component. \exists smooth path

$$\gamma : \mathbb{R} \rightarrow M, \gamma(0) = p, \gamma(1) = q.$$

$$d\alpha = 0 \Rightarrow d(\gamma^*\alpha) = 0 \Rightarrow d(\alpha \circ \gamma) = 0 \Rightarrow \frac{d(\alpha \circ \gamma)}{dt} = 0 \Rightarrow \alpha \circ \gamma(1) = \alpha \circ \gamma(0).$$

$$\text{So } \alpha(p) = \alpha(q)$$

□

We have $H_{\text{DR}}^k(M) \cong \text{Ab}(\pi_1(M)) \otimes_{\mathbb{Z}} \mathbb{R}$. $\text{Ab}(\pi_1(M))$ is the Abelian group of $\pi_1(M)$. In particular, $H_{\text{DR}}^1(\mathbb{R}^2) = 0$, $H_{\text{DR}}^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$.

Let us stop the discussion of de Rham cohomology for a moment, and move on to the next topic.

6 Orientation and Integration of Differential Form

6.1 Orientation on Manifold

An orientation on a finite dimensional vector space V is an equivalent class of ordered basis

$$\alpha = (\alpha_1, \dots, \alpha_n)^T \sim \beta = (\beta_1, \dots, \beta_n)^T \Leftrightarrow \det(\alpha\beta^T) > 0$$

Each vector space has exactly two orientations. And we actually have the 1-1 correspondence

$$\{\text{orientation on } V\} \leftrightarrow (\text{Alt}^n(V) \setminus \{0\}) / \mathbb{R}_+$$

$$[(e_1, \dots, e_n)] \leftrightarrow [e_1^* \wedge \dots \wedge e_n^*]$$

An **orientation form** on M of dimension n is a nowhere vanishing $\omega \in \Omega^n(M)$ *i.e.* an orientation form is a nowhere vanishing section of $\text{Alt}^n(TM)$.

Two orientation forms ω_1, ω_2 are equivalent if $\exists f \in C^\infty(M, \mathbb{R}^+) \text{ s.t. } \omega_1 = f\omega_2$.

An **orientation** on M is an equivalent class of orientation form.

An **orientation manifold** is a manifold that has an orientation.

An **oriented manifold** is a manifold equipped with an orientation.

Example 6.1. $|\pi_0(M)| = k \Rightarrow M$ has 2^k orientations or no orientations.

Example 6.2.

1. $U \subset \mathbb{R}^n$ open. U has a standard orientation, represented by the form $dx^1 \wedge \cdots \wedge dx^n$. Denote this standard orientation as \mathcal{O}_{std}
2. $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ oriented manifolds $\Rightarrow (M \times N, \mathcal{O}_M \times \mathcal{O}_N)$. If $\mathcal{O}_M = [\omega_M], \mathcal{O}_N = [\omega_N]$, then $\mathcal{O}_M \times \mathcal{O}_N$ is defined by $[\pi_M^*(\omega_M) \wedge \pi_N^*(\omega_N)]$. (π_M, π_N is the pullback of the projection map)
3. T^n, S^n are orientable.
4. \mathbb{RP}^n orientable iff n is odd.

Proposition 6.3. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M . Suppose we have an orientation \mathcal{O}_α on each U_α s.t. $\mathcal{O}_\alpha|_{U_\alpha \cap U_\beta} = \mathcal{O}_\beta|_{U_\alpha \cap U_\beta}, \forall \alpha, \beta$. Then \exists unique orientation \mathcal{O}_M on M s.t. $\mathcal{O}_M|_{U_\alpha} = \mathcal{O}_\alpha$.

Proof. For each α , we have $\omega_\alpha \in \Omega^n(U_\alpha)$ nowhere-vanishing. And

$$\omega_\alpha|_{U_\alpha \cap U_\beta} = f_{\alpha\beta} \cdot \omega_\beta|_{U_\alpha \cap U_\beta}, f_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{R}^+ \quad (6.1)$$

Take partition of unity subordinate to $\mathcal{U}, \{\varphi_\alpha\}$.

Set $\omega = \sum_{\alpha} \varphi_\alpha \cdot \omega_\alpha$. Then ω is nowhere-vanishing by (6.1).

The uniqueness follows from the fact that n -form is equivalent if and only if it is equivalent on each chart. □

Definition 6.4. Given $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$. $f \in \text{Diff}(M, N)$. Say f is **orientation preserving** if $f^*(\mathcal{O}_N) = \mathcal{O}_M$. f is **orientation reversing** if $f^*(\mathcal{O}_N) = -\mathcal{O}_M$.

Lemma 6.5. $U_1, U_2 \subset \mathbb{R}^n$ open. Then $f : (U_1, \mathcal{O}_{\text{std}}) \rightarrow (U_2, \mathcal{O}_{\text{std}})$ is orientation preserving iff

$$\forall p \in U_1, \det(Df|_p) > 0 \quad Df = \left(\frac{\partial f^i}{\partial x^j} \right)_{1 \leq i, j \leq n}$$

Proof. For $\mathcal{O}_{\text{std}} = [dx^1 \wedge \cdots \wedge dx^n]$,

$$\begin{aligned} f^*(dx^1 \wedge \cdots \wedge dx^n) &= df^1 \wedge \cdots \wedge df^n, \quad df^i = \sum_{j=1}^n \frac{\partial f^i}{\partial x^j} dx^j \\ &= \det(Df) dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

Then

$$\begin{aligned} \det(Df) > 0 &\Leftrightarrow f^*(dx^1 \wedge \cdots \wedge dx^n) \sim dx^1 \wedge \cdots \wedge dx^n \\ &\Leftrightarrow f^*\mathcal{O}_{\text{std}} = \mathcal{O}_{\text{std}} \\ &\Leftrightarrow f \text{ is orientation preserving} \end{aligned}$$

□

Given (M, \mathcal{O}) , $p \in M$, a basis e_1, \dots, e_n of $T_p M$ is called **oriented** if $\mathcal{O}_p = [(e_1, \dots, e_n)]$.

A chart $U \xrightarrow{\varphi} V \subset^{\text{open}} \mathbb{R}^n$ is **oriented** if $\varphi^*(\mathcal{O}_{\text{std}}) = \mathcal{O}|_U$.

A smooth atlas $\left\{ U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \right\}_{\alpha \in \mathcal{A}}$ is **oriented** if each chart $U_\alpha \xrightarrow{\varphi_\alpha}$ is oriented.

A smooth atlas $\left\{ U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \right\}_{\alpha \in \mathcal{A}}$ is called **positive** if $\forall \alpha, \beta \in \mathcal{A}$,

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi'_\alpha : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta) \text{ is orientation preserving}$$

By Lemma 6.5, this is equivalent to $\det(D\varphi_{\alpha\beta}|_p) > 0$ for any $p \in \varphi_\alpha(U_\alpha \cap U_\beta)$.

6.2 Integration on Oriented Manifold

Goal: Given M, \mathcal{O} , $\omega \in \Omega_c^n(M) = \{\text{compactly supported } n\text{-form on } M\}$. Then $\text{Supp}(\omega) = \overline{\{p \in M | \omega_p \neq 0 \in \text{Alt}^n(T_p M)\}}$ is compact.

We hope to define $\int_M \omega \in \mathbb{R}$.

For $M \overset{\text{open}}{\subset} \mathbb{R}^n$, $\mathcal{O} = \mathcal{O}_{\text{std}}$. Then $\forall \omega \in \Omega_c^n(M)$,

$$\omega = f dx^1 \wedge \cdots \wedge dx^n, f \in C_c^\infty(M)$$

Define $\int_M \omega = \int_M f d\mu$ where μ is the standard Lebesgue measure on \mathbb{R}^n .

Lemma 6.6. $U, V \overset{\text{open}}{\subset} \mathbb{R}^n$, $\varphi : U \xrightarrow{\cong} V$ is orientation preserving. Then $\forall \omega \in \Omega_c^n(V)$, we have $\int_U \varphi^*(\omega) = \int_V \omega$.

Proof. If $\omega = f dx^1 \wedge \cdots \wedge dx^n$, then

$$\begin{aligned} \varphi^*(\omega) &= \varphi^*(f) \wedge d\varphi^1 \wedge \cdots \wedge d\varphi^n \\ &= (f \circ \varphi) \det \left(\frac{\partial \varphi^i}{\partial x^j} \right)_{1 \leq i, j \leq n} dx^{1^1} \wedge \cdots \wedge dx^{1^n} \end{aligned} \tag{6.2}$$

$$\text{So } \int_U \varphi^*(\omega) = \int_U (f \circ \varphi) \det \left(\frac{\partial \varphi^i}{\partial x^j} \right)_{1 \leq i, j \leq n} d\mu = \int_V f d\mu = \int_V \omega$$

□

So we can define the integral over special ω and general M .

Definition 6.7. If $\omega \in \Omega_s^n(M) = \{n\text{-forms with "small" support}\}$
 $= \{\omega \in \Omega_c^n(M) | \exists \text{ oriented chart } \varphi : U \xrightarrow{\cong} \text{ s.t. } \text{Supp}(\omega) \subset U\}$.

We define $\int_M \omega := \int_V \varphi^{-1,*}(\omega)$

Claim. If $\text{Supp}(\omega) \subset U_\alpha \cap U_\beta$, then $\int_{V_\alpha} \varphi_\alpha^{-1,*}(\omega) = \int_{V_\beta} \varphi_\beta^{-1,*}(\omega)$

Proof.

$$\begin{aligned} \varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) &\xrightarrow{\cong} \varphi_\beta(U_\alpha \cap U_\beta) \\ \varphi_\alpha^{-1,*}(\omega) &\mapsto \varphi_\beta^*(\omega) \end{aligned}$$

By Lemma 6.6, we have $\int_{V_\alpha} \varphi_\alpha^{-1,*}(\omega) = \int_{V_\beta} \varphi_\beta^{-1,*}(\omega)$

□

Theorem 6.8. For any oriented (M, \mathcal{O}) , \exists unique linear map $\int_M : \Omega_c^n(M) \rightarrow \mathbb{R}$ that extends $\int_M : \Omega_s^n(M) \rightarrow \mathbb{R}$.

Proof.

Step1: There exists an oriented atlas $\mathcal{U} = \{\varphi_\alpha : U_\alpha \rightarrow V_\alpha \stackrel{\text{open}}{\subset} \mathbb{R}^n\}_{\alpha \in \mathcal{A}}$. Indeed, pick any smooth atlas $\mathcal{U} = \{U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha\}_{\alpha \in \mathcal{A}}$. By replacing φ_α with $r \circ \varphi_\alpha$ where $r(x_1, \dots, x_n) = (-x_1, \dots, x_n)$. We can get the oriented atlas \mathcal{U}' .

Step2. Pick a partition of unity subordinate to \mathcal{U} , $\{\varphi_\alpha : M \rightarrow [0, 1]\}$

Now we begin the main proof:

Let $\omega_\alpha = \rho_\alpha \cdot \omega$. $\text{Supp}(\omega_\alpha) \subset \text{Supp}(\rho_\alpha) \cap \text{Supp}(\omega) \subset U_\alpha$. And $\omega_\alpha \in \Omega_s^n(M)$

Claim. $\omega_\alpha \neq 0$ for only finite many $\alpha \in \mathcal{A}$

Proof. $\forall p \in \text{Supp}(\omega)$, \exists neighbourhood W_p only intersects $\text{Supp}(\rho_\alpha)$ for finitely many α .

Since $\{W_p\}_{p \in \text{Supp}(\omega)}$ is an open cover of $\text{Supp}(\omega)$, by compactness, $\text{Supp}(\omega)$ only intersects $\text{Supp}(\rho_\alpha)$ for finitely many α .

Therefore, $\omega_\alpha \neq 0$ for only finitely many α □

By this claim, since $\sum_{\alpha \in \mathcal{A}} \rho_\alpha = 1 \Rightarrow \omega = \omega_{\alpha_1} + \dots + \omega_{\alpha_n}$ for some $\alpha_1, \dots, \alpha_n \in \mathcal{A}$.

We may define

$$\int_M \omega = \sum_{\alpha \in \mathcal{A}} \int_M \omega_\alpha = \int_M \omega_{\alpha_1} + \dots + \int_M \omega_{\alpha_n} \in \mathbb{R} \quad (6.3)$$

This proves existence $\int_{(M, \mathcal{O}, \mathcal{U}, \{\rho_\alpha\})} : \Omega_c^n(M) \rightarrow \mathbb{R}$.

Uniqueness: $\forall \omega \in \Omega_c^n(M)$, $\exists \omega_1, \dots, \omega_k \in \Omega_s^n(M)$, $\omega = \omega_1 + \dots + \omega_k$ as the claim proved.

So $\int_M \omega = \sum_{i=1}^n \int_M \omega_i$ is uniquely defined. □

Remark 6.9. We actually obtain that each $\omega \in \Omega_c^n(M)$ can be expressed as $\sum_{k=1}^n \omega_k$ where $\omega_k \in \Omega_s^n(M)$.

Proposition 6.10.

1. $f : (M, \mathcal{O}_M) \xrightarrow{\cong} (N, \mathcal{O}_N)$. If f is orientation preserving, then $\int_M f^*(\omega) = \int_N \omega$. If f is orientation reversing, then $\int_M f^*(\omega) = -\int_N \omega$.
2. $\int_{(M, \mathcal{I})} \omega = -\int_{(M, \overline{\mathcal{O}})} \omega$.
3. If $\text{Supp}(\omega) \subset U \stackrel{\text{open}}{\subset} M$, then $\int_M \omega = \int_U \omega$.

Proof. Leave as exercise. □

6.3 Smooth Manifold with Boundary

Now let M be the smooth manifold with boundary, $\partial M = N$. i.e. M has a smooth atlas $\{\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha \stackrel{\text{open}}{\subset} \mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}\}$.

We can define $T_p M, TM, \text{Alt}^k(M), \Omega^k(M), \Omega_c^k(M), \Omega_s^k(M)$ and orientation similar as before.

For $p \in \partial M$, $X \in T_p M$ is called **outward** if \exists local chart $\varphi : U \xrightarrow{\cong} V$ around p s.t.

$$\varphi_{p,*}(X) = a_1 \partial x^1 + \cdots + a_n \partial x^n \text{ with } a_1 > 0$$

Recall that if M is n dimensional manifold with boundary, then $N = \partial M$ is a $n - 1$ dimensional manifold without boundary.

Proposition 6.11. For any orientation \mathcal{O}_M on M , \exists a unique induced orientation \mathcal{O}_N on $N = \partial M$ s.t. $\forall p \in N$, $X = T_p M$ outward, $e_2, \dots, e_n \in T_p N$ is an oriented basis. Moreover, (X, e_2, \dots, e_n) is oriented basis of $T_p M$.

Proof. Take oriented atlas $\mathcal{U} = \{\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha\}_{\alpha \in \mathcal{A}}$.

We have $\varphi_\alpha(U_\alpha \cap U_\beta) \subset \{0\} \times \mathbb{R}^{n-1}$.

Define $\psi_\alpha : N \cap U_\alpha \xrightarrow{\cong} (\{0\} \times \mathbb{R}^{n-1}) \cap U_\alpha$. Then $\mathcal{U}'\{\psi_\alpha\}$ is a smooth atlas for N .

\mathcal{U} is oriented implies \mathcal{U} is positive, so is \mathcal{U} . So there exists the unique \mathcal{O}_N s.t. \mathcal{U}' is oriented. \square

Theorem 6.12 (Stokes' Theorem). *M is n dimensional manifold with boundary, oriented by \mathcal{O}_M . $N = \partial M$, with induced orientation \mathcal{O}_N . $\iota : N \hookrightarrow M$ is the inclusion map. Then $\forall \omega \in \Omega_c^{n-1}(M)$, we have*

$$\int_M d\omega = \int_N \iota^*(\omega) \quad (6.4)$$

Proof. By Remark 6.9 $\forall \omega \in \Omega_c^{n-1}(M)$, $\omega = \omega_1 + \dots + \omega_k$, $\omega_j \in \Omega_s^n(M)$.

By linearity, we may assume $\omega \in \Omega_s^n(M)$, $\text{Supp}(\omega) \subset U_\alpha$ for chart $\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha$.

$\varphi_\alpha^{-1,*}(\omega) \in \Omega_c^n(V_\alpha)$ induces $\omega' \in \Omega_c^n(\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1})$ if we extend it by 0.

By considering ω' instead of ω , we may assume $M = \mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$ and we just need to prove

$$\int_{\mathbb{R}^{n-1}} \iota^*(\omega') = \int_{\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}} d\omega'$$

Let $\omega' = \sum_{k=1}^n f^k dx^1 \wedge \dots \wedge \widehat{dx^k} \wedge \dots \wedge dx^n$.

By linearity, we may assume $\omega' = f^k dx^1 \wedge \dots \wedge \widehat{dx^k} \wedge \dots \wedge dx^n$.

$$\begin{aligned} d\omega' &= \left(\sum_{i=1}^n \frac{\partial f^k}{\partial x^i} dx^i \right) \\ &= \frac{\partial f^k}{\partial x^k} (-1)^{k-1} dx^1 \wedge \dots \wedge dx^n \end{aligned} \quad (6.5)$$

For $k = 1$,

$$\begin{aligned}
\int_{\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}} d\omega' &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}_{\leq 0}} \frac{\partial f^1}{\partial x^1} dx^1 \right) dx^2 \wedge \cdots \wedge x^n \\
&= \int_{\mathbb{R}^{n-1}} \left(f^1(0, x_2, \dots, x_n) - \lim_{x_1 \rightarrow \infty} f^1(x_1, x_2, \dots, x_n) \right) dx^2 \cdots \wedge dx^n \\
&= \int_{\mathbb{R}^{n-1}} f^1(0, x^2, \dots, x^n) dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n \\
&= \int_{\mathbb{R}^{n-1}} \iota^*(\omega')
\end{aligned}$$

For $k \neq 1$,

$$\begin{aligned}
\int_{\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}} d\omega' &= \int_{\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}} \left(\int_{\mathbb{R}_{\leq 0}} \frac{\partial f^k}{\partial x^k} dx^k \right) \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{x_k \rightarrow +\infty} f^k(x_1, x_2, \dots, 0, \dots, x_n) - \lim_{x_k \rightarrow -\infty} f^k(x_1, x_2, \dots, x_n) \right) \\
&\quad dx^1 \wedge dx^2 \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx^n \\
&= 0 \\
&= \int_{\partial M} \iota^*(\omega')
\end{aligned}$$

□

Theorem 6.13. *For any smooth orientable compact manifold M , ∂M is not a **retract** of M , i.e. there is no continuous map $r : M \rightarrow \partial M$ s.t. $r|_{\partial M} = \text{id}_{\partial M}$*

Proof. Assume such retraction r exists. After homotopy, we may assume r is smooth followed from Whitney approximation Theorem [1.41](#)

Pick an orientation on M and such an orientation induces orientation on ∂M and hence, there exists an orientation form $\omega \in \Omega^{n-1}(\partial M)$ s.t. ω is nowhere vanishing.

Then $\int_{\partial M} \omega \neq 0$.

Noticed that $r \circ i = \text{id}$, we have $\int_{\partial M} \omega = \int_{\partial M} \iota^*(r^*(\omega)) = \int_M d(r^*(\omega)) = \int_M r^*(d\omega) = 0$ \square

Corollary 6.14. $f : D^n \rightarrow D^n$ continuous. Then $\exists x \in D^n$ s.t. $f(x) = x$.

Proof. Suppose there is no x s.t. $f(x) = x$. Let l_x be the unique ray from $f(x)$ to x .

Define $r : D^n \rightarrow S^{n-1}$ by $r(x) = l_x \cap S^{n-1}$. Then r is continuous, $r(x) = x$ for $x \in S^{n-1}$, which is contradiction to the previous theorem. \square

6.4 Riemannian Metric

A **Riemannian metric** g on M is a smooth section of $T^*M \otimes T^*M$ s.t. $\forall p \in M$, $g_p : T_pM \otimes T_pM \rightarrow \mathbb{R}$ is symmetric and positive definite.

Or equivalently, g_p is a bilinear map from $T_pM \times T_pM \rightarrow \mathbb{R}$ s.t.

$$g_p(v, w) = g_p(w, v), \quad g_p(v, v) \geq 0 \text{ with equality holds iff } v = 0$$

We also write $g_p(v, w)$ as $\langle v, w \rangle_p$, $v, w \in T_pM$.

In a local chart (U, x^1, \dots, x^n) ,

$$g|_U = \sum_{1 \leq i, j \leq n} g_{i,j} dx^i \otimes dx^j$$

where $g_{i,j} \in C^\infty(U, \mathbb{R})$, $g_{i,j}(p) = \left\langle \frac{\partial}{\partial x^i} \Big|_p, \frac{\partial}{\partial x^j} \Big|_p \right\rangle_p$.

We will abbreviate Riemannian metric to just metric in this lecture, just for convenience.

Example 6.15.

(1) Euclidean metric on \mathbb{R}^n , $g_{std} = \sum_{i=1}^n dx^i \otimes dx^i$.

- (2) Given any $N \xrightarrow{f} M$, any metric g_M on M , we can pullback g_M to metric on g_N by $\langle V, W \rangle_p := \langle f_{p,*}(V), f_{p,*}(W) \rangle_{f(p)} \in \mathbb{R}, \forall p \in N, V, W \in T_p N$.

When f is an embedding, also write $f^*(g_M)$ as $g_M|_N$

- (3) For $S^{n-1} \hookrightarrow \mathbb{R}^n$, $g_{std}|_{S^{n-1}} = g_{S^{n-1}}, T_p S^{n-1} \cong p^\perp$. So $\langle v, w \rangle_p = \langle v, w \rangle_{\mathbb{R}^n}$. Hence S^{n-1} has **constant positive sectional curvature**

- (4) For $M = \overset{\circ}{D}^n$, $g = \frac{\sum_{i=1}^n dx^i \otimes dx^i}{(1 - \sum_{i=1}^n x^i)^2}$ called **Poincaré disk model**, has **constant sectional curvature -1**.

Proposition 6.16. *Any smooth manifold has a Riemannian metric,*

Proof. Take a smooth atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$, take partition of unity $\rho_\alpha : M \rightarrow [0, 1]$ subordinate to \mathcal{U} .

On each $(U_\alpha, x^1, \dots, x^n)$, take the standard metric

$$g_\alpha = \sum_{i=1}^n dx^i \otimes dx^i$$

Then $g = \sum_{\alpha} \rho_\alpha g_\alpha \in \Gamma(T^*M \otimes T^*M)$.

For $\forall p \in \alpha$, there exists finitely many α_i s.t. $g_p = \sum_{i=1}^m \rho_{\alpha_i}(p) g_{\alpha_i}$. Positive linear combination of positive definite symmetric form is still positive definite and symmetric. Thus, g is what we need. \square

Consider $\mathbb{R}^n \hookrightarrow E \rightarrow M$ a smooth vector bundle.

A **Riemannian metric** on E is a smooth section $g \in \Gamma(E^* \otimes E^*)$ s.t. $\forall p \in M, g_p$ is a symmetric, positive definite bilinear form.

Proposition 6.17. *Any smooth vector bundle has a metric.*

Proof. It suffices to replace $T_p M$ to vector bundle in the previous proof. \square

Corollary 6.18. *For any smooth vector bundle E , E is isomorphic to E^* . i.e. $\exists \rho$ diffeomorphism s.t. ρ restricts to linear isomorphism $E_p \rightarrow E_p^* M$ for any $p \in M$.*

Proof. Pick a metric g on E . Define $E \xrightarrow{\rho} E^*, v \mapsto g_p(v, -)$. It is easy to check ρ is an isomorphism. \square

In particular, $TM \cong T^*M$.

Theorem 6.19. *Let M be oriented manifold. Then any Riemannian metric g determines an oriented n -form $\text{Vol} \in \Omega^n(M)$, called **volume form**.*

Proof. Take oriented chart $(U_\alpha, x^1, \dots, x^n)$, we get vector fields ∂x^i on U_α .

Apply Gram-Schmidt process,

$$e_1 = \frac{\partial x^1}{\|\partial x^1\|_g}, e_2 = \frac{\partial x^2 - \langle \partial x^2, e_1 \rangle_g}{\|\partial x^2 - \langle \partial x^2, e_1 \rangle_g\|}, e_3 = \dots$$

We get smooth orthonormal vector fields e_1, \dots, e_n on U_α . Let $e_1^*, \dots, e_n^* \in \Omega^1(U_\alpha)$ be the dual of $\{e_i\}$.

Set $\omega_\alpha := e_1^* \wedge \dots \wedge e_n^* \in \Omega^n(U_\alpha)$.

Now suppose we have another chart $(U_\beta, y^1, \dots, y^n) \rightsquigarrow e_\beta = e_1'^* \wedge \dots \wedge e_n'^*$.

Then $\forall p \in U_\alpha \cap U_\beta$, $\{e_{i,p}\}, \{e'_{i,p}\}$ are both oriented orthogonal basis of $(T_p M, g_p)$.

Then there exists $A \in SO(n)$ s.t.

$$\begin{pmatrix} e_{1,p} \\ \vdots \\ e_{n,p} \end{pmatrix} = A \begin{pmatrix} e'_{1,p} \\ \vdots \\ e'_{n,p} \end{pmatrix}$$

Then

$$\begin{pmatrix} e_{1,p}^* \\ \vdots \\ e_{n,p}^* \end{pmatrix} = A^T \begin{pmatrix} e'_{1,p} \\ \vdots \\ e'_{n,p} \end{pmatrix}$$

So $e'_{1,p} \wedge \cdots \wedge e'_{n,p} = \det(A^T) e_{1,p}^* \wedge \cdots \wedge e_{n,p}^* \Rightarrow \omega_{\alpha,p} = \omega_{\beta,p}$.

Therefore, $\omega_\alpha|_{U_\alpha \cap U_\beta} = \omega_\beta|_{U_\alpha \cap U_\beta}$. So $\{\omega_\alpha\}$ induces a unique $\text{Vol} \in \Omega^n(M)$ s.t.

$$\text{Vol}|_{U_\alpha} = \omega_\alpha \quad \square$$

Remark 6.20. In this proof, we know that the wedge product of orthonormal basis at any point are all the same except signs.

Proposition 6.21 (Calculation of volume form in local chart). *In a local chart*

$(U, x^1, \dots, x^n), g|_U = \sum_{1 \leq i, j \leq n} g_{ij} dx^i \otimes dx^j$, then

$$\text{Vol} = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n \quad (6.6)$$

Proof. If $\begin{pmatrix} \partial x^1 \\ \vdots \\ \partial x^n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$, where $A : U \rightarrow \text{GL}_n(\mathbb{R})$.

Then $\begin{pmatrix} e_1^* \\ \vdots \\ e_n^* \end{pmatrix} = A^T \begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix} \Rightarrow$

$$\text{Vol} = e_1^* \wedge \cdots \wedge e_n^* = \det(A^T) dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$$

The final equality is because

$$(g_{ij}) = \begin{pmatrix} \partial x^1 \\ \vdots \\ \partial x^n \end{pmatrix}_g \begin{pmatrix} \partial x^1 & \cdots & \partial x^n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}_g \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} A^T = AA^T$$

The multiplication of matrix is defined by the Riemannian metric $\langle -, - \rangle_{g_p}$ □

Example 6.22. For $\omega_0 \in \Omega^{n-1}(\mathbb{R}^n)$ defined by

$$\omega_{0,x}(v_1, \dots, v_{n-1}) = \det(x, v_1, \dots, v_{n-1})$$

Then $\omega_{0,x}(\partial x^1, \dots, \widehat{\partial x^k}, \dots, \partial x^n) = (-1)^{i-1} x^i$. So

$$\omega_0 = \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

And we have $\omega_0|_{S^{n-1}} = \text{Vol}_{S^{n-1}}$.

Proof. For $x \in S^{n-1}$, pick an orthonormal basis (e_2, \dots, e_n) for $T_x S^{n-1} \cong x^\perp \subset \mathbb{R}^n$. Then (x, e_2, \dots, e_n) is an oriented orthonormal basis for \mathbb{R}^n

Then $\omega_{0,x}(e_2, \dots, e_n) = \det(x, e_2, \dots, e_n) = 1$. So $(\omega_0|_{S^{n-1}})_x = e_2^* \wedge \cdots \wedge e_n^* = (\text{Vol}_{S^{n-1}})_x$. So $\omega_0|_{S^{n-1}} = \text{Vol}_{S^{n-1}}$ □

Remark 6.23. In this example, we see that it is usual to view the differential n -form as the anti-symmetric map $T^n M \rightarrow \mathbb{R}$ and it is uniquely determined by the image of the orthonormal basis of tangent space.

ω_0 is not closed since $d\omega_0 = n dx^1 \wedge \cdots \wedge dx^n$.

Actually, there is no closed $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ s.t. $\omega|_{S^{n-1}} = \text{Vol}_{S^{n-1}}$, since $\text{Vol}(S^{n-1}) > 0$ but $\int_{S^{n-1}} \omega|_{S^{n-1}} = \int_{D^n} d\omega = 0$.

However, there exists $\omega_1 \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ s.t. $d\omega_1 = 0$. $\omega_1|_{S^{n-1}} = \text{Vol}_{S^{n-1}}$.

Indeed, for $\gamma : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$, $x \mapsto \frac{x}{|x|}$, let $\omega_1 = \gamma^*(\text{Vol}_{S^{n-1}})$. Then $d\omega_1 = 0$, $\omega_1|_{S^{n-1}} = \text{Vol}_{S^{n-1}}$.

Exercise 6.24.

$$\omega_1 = \sum_{i=1}^n (-1)^{i-1} \frac{1}{|x|^n} x_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

6.5 Orientability of \mathbb{RP}^{n-1}

Define the $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \mapsto -x$, $A|_{S^{n-1}} : S^{n-1} \rightarrow S^{n-1}$. Then

$$\begin{aligned} (A^*(\omega_0))_x(V_2, \dots, V_n) &= \omega_{0,-x}(-V_2, \dots, -V_n) \\ &= \det(-x, -V_2, \dots, V_n) \\ &= (-1)^n \det(x, V_2, \dots, V_n) \\ &= (-1)^n \omega_{0,x}(V_2, \dots, V_n) \end{aligned} \tag{6.7}$$

Therefore, $A^*\omega_0 = (-1)^n \omega_0 \Rightarrow (A|_{S^{n-1}})^*(\text{Vol}_{S^{n-1}}) = (-1)^n \text{Vol}_{S^{n-1}}$.

① If n is even, $(A|_{S^{n-1}})^*(\text{Vol}_{S^{n-1}}) = \text{Vol}_{S^{n-1}}$.

Define $\mathbb{RP}^{n-1} = S^{n-1}/_{x \sim Ax}$. Then $\text{Vol}_{S^{n-1}}$ induces a nowhere vanishing form $\omega \in \Omega^{n-1}(\mathbb{RP}^{n-1})$ if n is even. So \mathbb{RP}^{n-1} is orientable, ω is the volume form $\text{Vol}_{\mathbb{RP}^{n-1}}$.

Proposition 6.25. *We have the identification:*

$$\mathbb{RP}^1 = S^1, \mathbb{RP}^3 \cong \text{SO}(3) \cong \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | u \perp v, |u| = |v| = 1\}$$

② If n is odd, $(A|_{S^{n-1}})^*(\text{Vol}|_{S^{n-1}}) = -\text{Vol}|_{S^{n-1}}$. We claim that

Proposition 6.26. \mathbb{RP}^{n-1} is unorientable for n odd.

Proof. Suppose \mathbb{RP}^{n-1} is orientable. Then \exists nowhere vanishing orientation form

$\omega' \in \Omega^{n-1}(\mathbb{RP}^{n-1})$. Define

$$q : S^{n-1} \rightarrow \mathbb{RP}^{n-1}$$

$$x \mapsto [x]$$

Then $q^*(\omega') = f \text{Vol}_{S^{n-1}} \in \Omega^{n-1}(S^{n-1})$ for $f \in C^\infty(S^{n-1})$ nowhere vanishing.

By $q \circ A = q$, $A^*(q^*(\omega')) = q^*(\omega') \Rightarrow A^*(f \text{Vol}_{S^{n-1}}) = f \text{Vol}_{S^{n-1}}$. Then

$$(f \circ A) \cdot A^*(\text{Vol}_{S^{n-1}}) = f \text{Vol}_{S^{n-1}} \Rightarrow f \circ A = -f \quad (6.8)$$

So $f(-x) = -f(x)$, $\forall x \in S^{n-1}$. $\Rightarrow f$ can't be nowhere vanishing, which causes contradiction! □

Theorem 6.27. \mathbb{RP}^n is orientable iff n is odd.

6.6 Tensor Field and Lie Derivative

For M smooth manifold, $a, n \in \mathbb{N}$.

A (a, b) -**tensor field** is a smooth section τ of $\left(\bigotimes_a TM \right) \otimes \left(\bigotimes_b T^*M \right)$.

A (a, b) -tensor field is called **covariant tensor field** if $a = 0$.

A (a, b) -tensor field is called **contravariant tensor field** if $b = 0$.

A (a, b) -tensor field is called **mixed tensor field** if $a \neq 0, b \neq 0$.

Under local chart (U, x^1, \dots, x^n) , a (a, b) -tensor field τ can be written as

$$\tau|_U = \sum_{\substack{1 \leq j_1 < \dots < j_b \leq n \\ 1 \leq i_1 < \dots < i_a \leq n}} \tau_{j_1, \dots, j_b}^{i_1, \dots, i_a} \partial x^{i_1} \otimes \dots \otimes \partial x^{i_a} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_b} \quad (6.9)$$

Example 6.28.

(1) $f \in C^\infty(M)$ is a $(0, 0)$ -tensor field.

- (2) Vector field is a $(1, 0)$ -tensor field.
- (3) k -form is an anti-symmetric $(0, k)$ -tensor field.
- (4) A Riemannian metric is a symmetric $(0, 2)$ -tensor field.

Given vector field X , (a, b) -tensor field τ , one can define the **Lie derivative** $\mathcal{L}_X \tau$ be a (a, b) -tensor field.

We focus on the case $a = 0$, *i.e.* covariant vector field. (Indeed, we have defined the Lie derivative for the contravariant field when $b = 0$)

For $p \in M$, $\{\varphi_t : U \rightarrow M\}_{t \in (-\varepsilon, \varepsilon)}$ local flow for X . Define $\mathcal{L}_X \tau$ as follows:

For $X \in \Gamma(TM)$, $\varphi_{t,*} : T_p M \rightarrow T_{\varphi_t(p)} M$ induces $\varphi_t^* : T_{\varphi_t(p)}^* M \rightarrow T_p^* M$ and

$$\varphi_t^* : \bigotimes_b T_{\varphi_t(p)}^* M \rightarrow \bigotimes_b T_p^* M$$

Since $\tau_{\varphi_t(p)} \in \bigotimes_b T_{\varphi_t(p)}^* M$, $\tau_p \in \bigotimes_b T_p^* M$, define

$$(\mathcal{L}_X \tau)_p = \lim_{t \rightarrow 0} \frac{\varphi_t^*(\tau_{\varphi_t(p)}) - \tau_p}{t} \quad (6.10)$$

Equivalently, $\mathcal{L}_X \tau|_U = \lim_{t \rightarrow 0} \frac{\varphi_t^*(\tau)|_U - \tau|_U}{t}$.

Lemma 6.29. X, τ smooth $\Rightarrow \mathcal{L}_X \tau$ smooth.

Proof. See Lee's Book □

$\mathcal{L}_X \tau$ describes the change of τ under the (local) flow generated by X .

For simplicity, let's assume X is complete. Let $\varphi_t : M \rightarrow M$ be the global flow for X . We say a covariant tensor field $\tau \in \Gamma\left(\bigotimes_b T^* M\right)$ is invariant under φ_t or invariant under X if $\varphi_t^*(\tau) = \tau$.

Proposition 6.30.

- τ is invariant under X if and only if $\mathcal{L}_X \tau = 0$
- $\frac{d}{dt} \varphi_t^*(\tau)|_{t=0} = \mathcal{L}_X \tau$

Proof. The second part and the necessary of the first part is direct from the definition.

We only prove the sufficiency for the first part. Assume $\mathcal{L}_X \tau = 0$.

Consider the map $\mathbb{R} \xrightarrow{f} \bigotimes_b T_p^* M, t \mapsto \varphi_t^*(\tau_{\varphi_t(p)})$.

Then $\forall t_0 \in \mathbb{R}, f(t) = \varphi_{t_0}^*(\varphi_{t-t_0}^*(\tau_{\varphi_t(p)}))$. So

$$\begin{aligned}
 \frac{df(t)}{dt} \Big|_{t=t_0} &= \lim_{t \rightarrow t_0} \frac{\varphi_{t_0}^*(\varphi_{t-t_0}^*(\tau_{\varphi_t(p)})) - \varphi_{t_0}^*(\tau_{\varphi_{t_0}(p)})}{t - t_0} \\
 &\stackrel{s=t-t_0}{=} \varphi_{t_0}^* \left(\lim_{s \rightarrow 0} \frac{\varphi_s^*(\tau_{\varphi_s(\varphi_{t_0}(p))}) - \tau_{\varphi_{t_0}(p)}}{s} \right) \\
 &= \varphi_{t_0}^* (\mathcal{L}_X \tau)_{\varphi_{t_0}(p)} \\
 &= 0
 \end{aligned} \tag{6.11}$$

□

Definition 6.31. Given Riemannian manifold (M, g) , a vector field X is called a **Killing vector field** if $\mathcal{L}_X g = 0$. Or equivalently, the flow generated by X is an isometry.

Example 6.32.

$$\begin{aligned}
 \{\text{Killing vector field on } (S^2, g_{S^2})\} &\cong \mathfrak{so}(3) \\
 &= \text{Span}(x\partial y - y\partial x, y\partial z - z\partial y, z\partial x - x\partial z)
 \end{aligned}$$

Lemma 6.33.

- (1) $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta$
- (2) $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$

Proof. (1)

$$\begin{aligned}
(\mathcal{L}_X(\omega \wedge \eta))_p &= \lim_{t \rightarrow 0} \frac{\varphi_t^*(\omega \wedge \eta)_{\varphi_t(p)} - (\omega \wedge \eta)_p}{t} \\
&= \lim_{t \rightarrow 0} \frac{\varphi_t^*(\omega_{\varphi_t(p)}) \wedge \varphi_t^*(\eta_{\varphi_t(p)}) - \varphi_t^*(\omega_{\varphi_t(p)}) \wedge \eta_p + \varphi_t^*(\omega_{\varphi_t(p)}) \wedge \eta_p - \omega_p \wedge \eta_p}{t} \\
&= \lim_{t \rightarrow 0} \frac{\varphi_t^*(\omega_{\varphi_t(p)}) - \omega_p}{t} \wedge \eta_p + \lim_{t \rightarrow 0} \varphi_t^*(\omega_{\varphi_t(p)}) \wedge \lim_{t \rightarrow 0} \frac{\varphi_t^*(\eta_{\varphi_t(p)}) - \eta_p}{t} \\
&= (\mathcal{L}_X \omega)_p \wedge \eta_p + \omega_p \wedge (\mathcal{L}_X \eta)_p
\end{aligned} \tag{6.12}$$

(2)

$$\begin{aligned}
\mathcal{L}_X d\omega &= \lim_{t \rightarrow 0} \frac{\varphi_t^*(d\omega) - d(\omega)}{t} \\
&= \lim_{t \rightarrow 0} \frac{d(\varphi_t^*(\omega) - \omega)}{t} \\
&= d \left(\lim_{t \rightarrow 0} \frac{\varphi_t^*(\omega) - \omega}{t} \right) \\
&= d\mathcal{L}_X(\omega)
\end{aligned}$$

□

Given $X \in \Gamma(TM)$, define **contraction** $X \lrcorner : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ by

$$(X \lrcorner \alpha)(Y_2, \dots, Y_k) := \alpha(X, Y_2, \dots, Y_k) \text{ for } Y_2, \dots, Y_k \in \Gamma(TM)$$

Often abbreviated to $\iota_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$.

Example 6.34. $X = f \partial x^i$, $\alpha = g dx^{i_1} \wedge \dots \wedge dx^{i_n}$. Then

$$x \lrcorner \alpha = \begin{cases} 0, & i \notin \{i_1, \dots, i_n\} \\ (-1)^{k-1} f g dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_k}} \wedge \dots \wedge dx^{i_n}, & i = i_k \end{cases} \tag{6.13}$$

Theorem 6.35 (Cartan's Magical Formula). $\mathcal{L}_X(\omega) = X \lrcorner d\omega + d(X \lrcorner \omega)$. Also written as

$$\mathcal{L}_X(\omega) = \iota_X(d\omega) + d(\iota_X\omega)$$

Proof. Need to show $(\mathcal{L}_X\omega)_p = (X \lrcorner (d\omega))_p + (d(X \lrcorner \omega))_p$.

For $X_p \neq 0$. By canonical form theorem 4.1, there exists local chart around p s.t. $X|_U = \partial x^1$.

Let $\omega = \sum_{1 \leq i_1 < \dots < i_n \leq n} f^{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$. By linearity we may assume $\omega = f dx^{i_1} \wedge \dots \wedge dx^{i_k}, i_1 < \dots < i_k$.

For $i_1 = 1$,

$$\varphi_t(x^1, \dots, x^n) = (x^1 + t, \dots, x^n)$$

$$\begin{aligned} X \lrcorner d\omega &= \frac{\partial}{\partial x^1} \lrcorner \left(\sum_{1 \leq j \leq n} \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \right) \\ &\stackrel{(6.13)}{=} - \sum_{j \notin \{i_1, \dots, i_k\}} \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} \\ (X \lrcorner \omega) &\stackrel{(6.13)}{=} f dx^{i_2} \wedge \dots \wedge dx^{i_k} \end{aligned}$$

Then

$$d(X \lrcorner \omega) = \sum_{j \notin \{i_2, \dots, i_k\}} \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$$

So

$$(d(X \lrcorner \omega) + X \lrcorner \omega)_p = \frac{\partial f}{\partial x^1} dx^1 \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} = (\mathcal{L}_X\omega)_p$$

For the case $i_1 \neq 1$,

$$\begin{aligned} X \lrcorner d\omega &= \frac{\partial}{\partial x^1} \lrcorner \left(\sum_{1 \leq j \leq n} \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) \\ &\stackrel{(6.13)}{=} \frac{\partial f}{\partial x^1} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ (X \lrcorner \omega) &\stackrel{(6.13)}{=} 0 \end{aligned}$$

Thus,

$$(d(X \lrcorner \omega) + X \lrcorner \omega)_p = \frac{\partial f}{\partial x^1} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = (\mathcal{L}_X \omega)_p$$

Now we consider the case that $X_p = 0$. For $p \in \text{Supp}(X)$, it is true using continuity. If $p \notin \text{Supp}(X)$, then \exists neighborhood U of p s.t. $X|_U \equiv 0$. So the two sides of the equation equals 0. \square

6.6.1 Divergence of vector fields

Definition 6.36. For Riemannian metric space (M, g) , $X \in \Gamma(TM)$. Define $\text{div} X \in C^\infty(M)$ by

$$d(\iota_X \text{Vol}) = \text{div}(X) \text{Vol} \tag{6.14}$$

Then by Cartan's formula,

$$\mathcal{L}_X \text{Vol} = d(\iota_X \text{Vol}) = \text{div}(X) \text{Vol} \tag{6.15}$$

Let $\varphi_t : U \rightarrow M$ be the flow for X .

$$\frac{d}{dt} \varphi_t^*(\text{Vol})|_{t=t_0} = \varphi_{t_0}^*(\mathcal{L}_X \text{Vol}) = \varphi_{t_0}^*(\text{div} X \cdot \text{Vol}) \tag{6.16}$$

If D is an integration domain of M , a small ball for example. Then define

$$\text{Vol}(D) = \int_D \text{Vol} \quad (6.17)$$

Now $\text{Vol}(\varphi_t(D)) = \int_{\varphi_t(D)} \text{Vol} = \int_D \varphi_t^*(\text{Vol})$.

$$\begin{aligned} \frac{d}{dt} \text{Vol}(\varphi_t(D)) &= \int_D \varphi_t^*(\text{div} X \text{Vol}) \\ &= \int_{\varphi_t(D)} \text{div} X \text{Vol} \end{aligned} \quad (6.18)$$

If $\text{div} X|_p > 0$ for all p , then φ_t is volume increasing. If $\text{div} X|_p = 0$ for all p , then φ_t is volume preserving. If $\text{div} X|_p < 0$ for all p , then φ_t is volume decreasing.

6.6.2 Hamiltonian vector fields on symplectic manifolds

A **symplectic structure** on M is a 2-form $\omega \in \Omega^2(M)$, called **symplectic form** s.t. (1) $d\omega = 0$ (2) ω is non-degenerate everywhere. i.e. For $\forall v \in T_p M \neq 0$, there exists $w \in T_p M$ such that $\omega(v, w) \neq 0$, ω nowhere vanishing equivalently.

Example 6.37. For $M = T^*N$, we have a canonical form $\alpha \in \Omega^1(M)$ defined as follows:

$\forall p \in T_q^* N \subset M, v \in T_p M$, it induces a canonical map $\pi : M \rightarrow N, p \mapsto q$. Then its pullback $\pi_* : T_p M \rightarrow T_q N, v \mapsto \pi_* v$ induces $\alpha_p(v) = p(\pi_*(v)) \in \mathbb{R}$.

Then $\omega = d\alpha$ is a canonical symplectic structure on T^*N .

Proof. Take local chart (U, x^1, \dots, x^n) of N ,

$$U \xrightarrow{x^i} \tilde{U} \overset{\text{open}}{\subset} \mathbb{R}^n$$

$$M \supset T^*U \xrightarrow{\cong} T^*\tilde{U} \cong \tilde{U} \times \mathbb{R}^n \overset{\text{open}}{\subset} \mathbb{R}^{2n}$$

$$\sum_{i=1}^n p_i dx^i \longmapsto (x^1(q), \dots, x^n(q), p_1, \dots, p_n)$$

$$\text{Then } \alpha = \sum_{i=1}^n p_i dx^i, \omega = \sum_{i=1}^n dp_i \wedge dx^i.$$

$$\omega^n = n! dp_1 \wedge dx_1 \wedge dp_2 \wedge dx_2 \wedge \dots \wedge dp_n \wedge dx_n$$

So ω is non-degenerate everywhere. □

Definition 6.38. For (M, ω) symplectic, $f \in C^\infty(M)$, $df \in \Omega^1(M)$. ω is non-degenerate everywhere. Define

$$\iota_{-}\omega : \Gamma(TM) \rightarrow \Gamma(T^*M), X \mapsto \mathcal{L}_X\omega \quad (6.19)$$

Define the **Hamiltonian vector field** X_f by $\iota_{X_f}\omega = df$. The flow generated by X_f is called the **Hamiltonian flow** $\phi : U \rightarrow M$.

By Cartan's formula

$$\mathcal{L}_{X_f}\omega = \iota_{X_f}d\omega + d(\iota_{X_f}\omega) = d(df) = 0 \quad (6.20)$$

So the symplectic form is preserved under the Hamiltonian flow.

The motivation to study Hamiltonian flow is that in classical mechanics, $N = \mathbb{R}^n$, $M = T^*N = \mathbb{R}^n \times \mathbb{R}^n$ configuration space. The movement of a system of m

partial is given by $\gamma : \mathbb{R} \rightarrow M$, γ satisfies the Hamiltonian equation

$$\gamma'(t) = X_H|_{\gamma(t)}$$

where $H : M \rightarrow \mathbb{R}$ is the Hamiltonian. *i.e.* γ is an integral curve of the Hamiltonian flow.

6.7 Frobenius Theorem

The motivation is to solve the PDE equation for $f : U \rightarrow \mathbb{R}$ such that

$$\begin{cases} \frac{\partial f}{\partial x} = \alpha(x, y, f(x, y)) \\ \frac{\partial f}{\partial y} = \beta(x, y, f(x, y)) \end{cases} \quad (6.21)$$

where initial value is $f(x_0, y_0) = z_0$.

Consider vector fields on \mathbb{R}^3 , $X_1 = \frac{\partial}{\partial x} + \alpha(x, y, z) \frac{\partial}{\partial z}$, $X_2 = \frac{\partial}{\partial y} + \beta(x, y, z) \frac{\partial}{\partial z}$. Then f is a solution iff $N : \text{graph}(f) = \{(x, y, f(x, y))\} \hookrightarrow \mathbb{R}^3$ satisfies $(x_0, y_0, z_0) \in N$, $T_p N = \text{Span}(X_{1,p}, X_{2,p})$.

Definition 6.39. M^n is a smooth manifold. A k -dimensional **tangent distribution** D is a k -dimensional linear subspace $D_p \subset T_p M$ at each $p \in M$ s.t. $D = \bigsqcup_{p \in M} D_p$ is a smooth subbundle of TM .

Equivalently, this means for any $p \in M$, there exists a neighborhood U of p , and smooth vector fields Y_1, \dots, Y_k on U s.t. $D_q = \text{Span} \langle Y_{1,q}, \dots, Y_{k,q} \rangle$, $\forall q \in U$.

An immersed submanifold $\varphi : N \hookrightarrow M$ is called an **integral manifold** for D if $\forall q \in N$, $\varphi_*(T_q N) = D_{\varphi(q)}$. Identify N with $\varphi(N)$ to simplify notation, It can be abbreviated as $T_q N = D_q$,

Example 6.40. Every nowhere vanishing vector field is a 1-dimensional distribution, and the integral curve of it is the integral manifold.

Example 6.41. For $M = \mathbb{T}^2 = \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{Z}$, $D = \text{Span}(\frac{\partial}{\partial x})$. Then the integral submanifold is $S^1 \times \{y\}$. For $D = \text{Span}(\frac{\partial}{\partial x} + \sqrt{2} + \frac{\partial}{\partial y})$, integral submanifold $\{[y, x] | y - y_0 = \sqrt{2}(x - x_0)\}$ is an immersed but not embedding submanifold dense in \mathbb{T}^2 .

Example 6.42. $M = \mathbb{R}^n$, $D = \text{Span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}\right\}$ has integral manifold $N = \mathbb{R}^k \times \{\mathbf{c}\}$, $\mathbf{c} \in \mathbb{R}^{n-k}$.

Example 6.43. $M = \mathbb{R}^n \setminus \{0\}$, $D_p = p^\perp$ is an $n - 1$ dimensional distribution with integral submanifold $N = S_r^{n-1} = \{\mathbf{x} : \|\mathbf{x}\| = r\}$.

Example 6.44. $M = \mathbb{R}^3$, $D = \text{Span}\left\{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right\}$. We claim that D has no integral submanifold.

Proof of the Claim. Let N be an integral submanifold, $(0, 0, 0) \in N$. $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n, t \mapsto (t, 0, 0)$ is an integral curve for $X_1 = \frac{\partial}{\partial x}$. So $(x, 0, 0) \in N$, for $x \in (-\varepsilon, \varepsilon)$.

Let $\eta : (-\varepsilon', \varepsilon') \rightarrow \mathbb{R}^3, t \mapsto (x, t, 0)$ be an integral curve for $X_2 = \frac{\partial}{\partial y}$ starting at $(x, 0, 0)$. So for $(x, y, 0) \in N$ for $x \in (-\varepsilon, \varepsilon), y \in (-\varepsilon', \varepsilon')$.

That implies that N contains $X - Y$ plane near $(0, 0, 0)$ but now $T_p N = \text{Span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\} \neq D_p$. □

Remark 6.45. This example shows that a distribution doesn't have integral manifold always. This property that have integral manifold implies something more than linearity, or in some sense, it should be closed under the action by local flows in that case.

Here we introduce Frobenius theorem to explain it.

Definition 6.46. Let D be a k -dimensional distribution on M .

We say a chart (U, φ) on M is **flat** for D if $\varphi(U)$ is a product of connected open sets $U' \times U'' \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, and at points of U , D is spanned by the first k coordinate vector fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}$

- (1) We say D is **involutive** if for any open $U \subset M$, any smooth sections $X, Y \in \Gamma(D|_U)$, we have $[X, Y] \in \Gamma(D|_U)$.
- (2) We say D is **integrable** for any $p \in M$, there exists an integral submanifold N for D s.t. $p \in N$
- (3) We say D is **completely integrable** if $\forall p \in M$, there exists local chart (U, x^1, \dots, x^n) around p s.t. $D|_U = \text{Span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k} \right\}$. i.e. There exists a flat chart for D in a neighborhood of every point of M .

Theorem 6.47 (Local Frobenius). *Those definitions above is equivalent.*

Proof. Completely integrable \Rightarrow Integrable: In U , all submanifolds of the form $\mathbb{R}^k \times \{c\}$ are integral submanifold for D .

Integrable \Rightarrow Involutive: For $X, Y \in \Gamma(D)$, $p \in M$. Let $\iota : N \looparrowright M$ be integral submanifold s.t. $p \in N$.

Then $[X, Y]|_p = [X|_N, Y|_N]_p \in T_p N = D_p$, since X, Y is ι -related to $X|_N, Y|_N$ respectively. So $[X, Y] \in \Gamma(D)$.

Involutive \Rightarrow Completely integrable. $\forall p \in M$, take local chart (V, y^1, \dots, y^n) . WLOG, $y(p) = 0$. Then $\pi = (y^1, \dots, y^k) : V \rightarrow \mathbb{R}^k, p \mapsto 0$. It induces $\pi_{q,*} : T_q V \rightarrow T_{\pi(q)} \mathbb{R}^k = \mathbb{R}^k$.

Since D_p has dimension k , by shrinking V , and reordering the coordinate such that D_q is disjoint with $\frac{\partial}{\partial x^{k+1}}, \dots, \frac{\partial}{\partial x^n}$, we may assume $\pi_{q,*}|_{D_q} : D_q \xrightarrow{\cong} \mathbb{R}^k, \forall q \in V$.

Consider $\partial y^i \in \Gamma(T\mathbb{R}^k)$, $1 \leq i \leq k$. Let X_i be the unique section of $D|_V$ s.t. $\pi_*(X_i) = \partial y^i$. $X_i \in \Gamma(D|_V) \subset \Gamma(TM)$. Then ∂y^i is π -related to X_i .

Then $0 = [\partial y^i, \partial y^j]$ is π -related to $[X_i, X_j]$. So $\pi_*[X_i, X_j]_q = 0, \forall q \in V \Rightarrow [X_i, X_j]_q = 0, \forall q \in V$.

Thus, we obtain linear independent vector fields with $[X_i, X_j] = 0, i \neq j$.

By canonical form of commuting vector fields 4.11, there exists local charts (U, x^1, \dots, x^n) s.t. $X_i = \partial x^i, 1 \leq i \leq k$. So $D|_U = \text{Span}\{\partial x^1, \dots, \partial x^k\}$ \square

Remark 6.48. Indeed, completely integrable means more. We can find an embedding integrable manifold locally for completely integrable.

Lemma 6.49. Let D be a k -dimensional distribution. Then D is involutive iff there exists an open cover $\mathcal{U}, \forall U \in \mathcal{U}, \exists X_1, \dots, X_k \in \Gamma(TU)$ s.t. $D|_U = \text{Span}\{X_1, \dots, X_k\}$ and $[X_i, X_j] \in \Gamma(D|_U)$

Proof. " \Rightarrow " is followed by the definition.

" \Leftarrow " Just need to show $D|_U$ is involutive for each U .

$\forall X, Y \in \Gamma(D|_U), X = \sum_{i=1}^n f^i X_i, Y = \sum_{i=1}^n g^i X_i, f^i, g^i : U \rightarrow \mathbb{R}$. Then

$$[X, Y] = \sum_{i,j} f^i g^j [X_i, X_j] + \sum_{i,j} f^i X_i(g^j) X_j - \sum_{i,j} g^j X_j(f^i) X_i \in \Gamma(D) \quad (6.22)$$

\square

Corollary 6.50. $\begin{cases} \frac{\partial f}{\partial x} = \alpha(x, y, f(x, y)) \\ \frac{\partial f}{\partial y} = \beta(x, y, f(x, y)) \end{cases}, f(x_0, y_0) = z_0$ has local solution for

$\forall (x_0, y_0, z_0)$ iff

$$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z}$$

Proof. \Rightarrow is because of $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

" \Leftarrow " Consider $D = \text{Span}\{\frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \beta \frac{\partial}{\partial z}\}$ 2 dimensional distribution on \mathbb{R}^3 .

$$[\frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \beta \frac{\partial}{\partial z}] = \left(\frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z} - \frac{\partial \alpha}{\partial y} - \beta \frac{\partial \alpha}{\partial z} \right) \frac{\partial}{\partial z} = 0$$

So D is involutive $\Rightarrow \forall (x_0, y_0, z_0), \exists N \looparrowright \mathbb{R}^3$ s.t. $(x_0, y_0, z_0) \in N$, N is an integrable manifold of D .

$N \looparrowright \mathbb{R}^3 \xrightarrow{(x,y,z) \mapsto (x,y)} \mathbb{R}^2$ is a submersion. So N can be locally written as $(x, y, f(x, y))$ for $f : U \rightarrow \mathbb{R}$ □

A k -dimensional **foliation** is a decomposition $M = \bigcup_s N_s$ s.t. (1) Each N_s is an injective immersed k -dimensional submanifold. (2) $\forall p \in M, \exists$ local chart (U, x^1, \dots, x^n) s.t. $\forall s \in S, N_s \cap U = \mathbb{R}^k \times A_s$. A_s is a countable subset of \mathbb{R}^{n-k} .

Example 6.51. $\mathbb{T}^2 = \bigcup_s N_s, N_s = \mathbb{R} \hookrightarrow \mathbb{T}^2, t \mapsto (x_0 + t, \sqrt{2}t)$

Theorem 6.52 (Global Frobenius). D is an involutive k -dimensional distribution $\Rightarrow D$ induces a k -dimensional foliation $M = \bigcup_s N_s$ such that each N_s is a maximal integral submanifold of D .

7 de-Rham Cohomology

7.1 Basic Definition

A **cochain complex** over \mathbb{Z} is a graded abelian group $C = \bigotimes_{b \in \mathbb{Z}} C^b$ with a deg-1 map $d = \bigotimes_n (d^n : C^n \rightarrow C^{n+1})$ s.t. $d^2 = 0$.

d is called **differential** or **boundary map**

The k -th **homology**

$$H^k(C, d) := \frac{\ker(d^k : C^k \rightarrow C^{k+1})}{\text{Im}(d^{k-1} : C^{k-1} \rightarrow C^k)} \quad (7.1)$$

The denominator is the set of k -**boundary**. The numerator is the set of k -**cycles**

Example 7.1. $C_{DR}^*(M) = \bigotimes_{n \in \mathbb{Z}} \Omega^n(M)$ is homology equipped with the exterior differential d . And

$$H^k(C_{DR}^*(M), d) = H_{DR}^*(M) \cong H^k(M; \mathbb{R}) \quad (7.2)$$

where $H^k(M; \mathbb{R})$ is the singular cohomology.

Given cochain complexed $(A, d_A), (B, d_B)$, a **chain map** $f : (A, d_A) \rightarrow (B, d_B)$ is a degree 0 (i.e. $f(A^n) \subset B^n$) group homomorphism s.t. $d_B \circ f = f \circ d_A$.

If f is a chain map, then f maps k -cycles to k -cycles and k -boundaries to k -boundaries. So f induces $f^* : H^k(A, d_A) \rightarrow H^k(B, d_B)$.

Example 7.2. $f : C^\infty(M, N) \Rightarrow f^* : \Omega^k(N) \rightarrow \Omega^k(M)$ induces a chain map $f^* : C_{DR}^*(N) \rightarrow C_{DR}^*(M) \Rightarrow f^* : H_{DR}^*(N) \rightarrow H_{DR}^*(M)$ satisfies $(f \circ g)^* = g^* \circ f^*, (\text{id})^* = \text{id}$

Given chain maps $f, g : (A, d_A) \rightarrow (B, d_B)$, a chain homotopy from f to g is a deg-(-1) map $s : A \rightarrow B$ s.t. $d_B s + d d_A = f - g$. In this case we write $f \simeq g$.

$$\begin{array}{ccccccc} A^{k-1} & \longrightarrow & A^k & \longrightarrow & A^{k+1} \\ g \downarrow f & \swarrow s & g \downarrow f & \swarrow s & g \downarrow f \\ B^{k-1} & \longrightarrow & B^k & \longrightarrow & B^{k+1} \end{array}$$

Lemma 7.3. If $f \simeq g$, then $f^* = g^* H^*(A, d_A) \mapsto H^*(B, d_B)$

Proof. Take $[a] \in H^*(A, d_A)$, $d_* a = 0$. Then $f(a) - g(a) = d_B(sa) + s(d_A a) = d_B(sa) \in \text{Im} d_B$. So $f^*[a] = g^*[a]$. \square

Definition 7.4. Given the map $f : (A, d_A) \rightarrow (B, d_B)$, we say f is a chain homotopy equivalence if $\exists g : (B, d_B) \rightarrow (A, d_A)$ s.t. $g \circ f \simeq \text{id}_A, f \circ g \simeq \text{id}_B$. We call such g the **chain homotopy inverse** of f . And we say $(A, d_A), (B, d_B)$ are **chain homotopic**, denoted as $(A, d_A) \stackrel{f}{\simeq} (B, d_B)$.

Corollary 7.5. If $(A, d_A) \stackrel{f}{\simeq} (B, d_B)$, then $f^* : H^*(A, d_A) \rightarrow H^*(B, d_B)$ is an isomorphism.

Proof. Let $g : (B, d_B) \rightarrow (A, d_A)$ be the chain homotopy inverse of f . Then $g^* \circ f^* = \text{id}_{H^*(A, d_A)}, f^* \circ g^* = \text{id}_{H^*(B, d_B)}$. So f^* is an isomorphism. \square

Recall that given smooth manifold M (with boundary), define $C_{dR}^*(M) = \bigotimes_{i \in \mathbb{Z}} \Omega^i(M)$, $d : \Omega^i(M) \rightarrow \Omega^{i+1}(M)$ the exterior derivative, $H_{dR}^*(M) = H^*(C_{dR}, d)$.
 $f \in C^\infty(M, N) \rightarrow$ chain map $f^* : C_{dR}(N) \rightarrow C_{dR}(M)$, $a \mapsto f^*(a) \rightarrow$ induced map $f^* : H_{dR}^*(N) \rightarrow H_{dR}^*(M)$.

Theorem 7.6. *Given $f, g : C^\infty(M, N)$. Then $f \simeq g \Rightarrow f^* \simeq g^* : C_{dR}^*(N) \rightarrow C_{dR}^*(M)$. Hence, $f^* = g^* : H_{dR}^*(N) \rightarrow H_{dR}^*(M)$ by Corollary 7.5*

We will use the Whitney approximation theorem over manifolds.

Theorem 7.7 (Whitney Approximation Theorem). *Given M, N smooth manifolds. Given embedded submanifold $L \hookrightarrow M$. Given a continuous map $f : M \rightarrow N$ s.t. $f|_L : L \rightarrow N$ is smooth. Then there is a homotopy H from f to g related to L s.t. $g : M \rightarrow N$ is smooth, $g|_L = f|_L$.*

7.2 Integration along Fibers

For F compact oriented smooth manifold possibly with boundary, $\dim F = l$.
 M is an n -dimensional smooth manifold possibly with boundaries.

Our goal is to define $\int_F - : \Omega^*(F \times M) \rightarrow \Omega^{*-l}(M)$ s.t. when $l = *$, this map $\int_F : \Omega^l(F) \rightarrow \mathbb{R}$ is the usual integration.

Now take $\alpha \in \Omega^k(F \times M)$, we will define $\int_F \alpha \in \Omega^{k-l}(M)$ step by step.

Case i: If \exists chart (U, x^1, \dots, x^n) for M s.t. $\text{Supp}(\alpha) \subset V \times U$ and \exists oriented chart (V, t^1, \dots, t^l) for F . Then

$$\alpha = \sum_{\substack{I \subset \{1, \dots, l\} \\ J \subset \{1, \dots, n\} \\ |I| + |J| = k}} \alpha_{IJ} dt^I \wedge dx^J, \alpha_{IJ} \in C^\infty(V \times U, \mathbb{R}) \quad (7.3)$$

If $I_0 = \{1, \dots, l\}$, define

$$\int_F \alpha_{I_0, J} dt^{I_0} : U \rightarrow \mathbb{R}$$

$$p \mapsto \int_{F \times \{p\}} : \alpha_{I_0, J}|_{F \times \{p\}} dt^1 \wedge \dots \wedge dt^l$$

Now define

$$\int_F \alpha = \sum_{\substack{J \subset \{1, \dots, m\} \\ |J| = k-l}} \left(\int_F \alpha_{I_0, J} dt^{I_0} \right) \cdot dx^I \in \Omega^{k-l}(M) \quad (7.4)$$

It can be generalized to all α and easy to check it is independent with the choice of chart.

Case ii: \exists local chart (U, x^1, \dots, x^n) for M s.t. $\text{Supp}(\alpha) \subset F \times U$. Take an oriented atlas $\mathcal{V} = \{(V_i, \varphi_i)\}_{1 \leq i \leq n}$ for F , and partition of unity $\rho_i : F \rightarrow [0, 1]$ subordinate to \mathcal{V} . Let $\tilde{\rho}_i$ be composition $F \times M \xrightarrow{p} F \xrightarrow{\rho_i} [0, 1]$. Then $\text{Supp}(\tilde{\rho}_i \alpha) \subset V_i \times U$. We define $\int_F \alpha = \sum_{1 \leq i \leq n} \int_F (\tilde{\rho}_i \alpha)$. Easy to check it is independent with the choice of $U, \mathcal{V}, \{\rho_i\}$.

Case iii: For a general $\alpha \in \Omega^k(F \times M)$, take an atlas $\mathcal{U} = \{U_i, \psi_i\}_{i \in I}$ of M and partition of unity $\tau_i : M \rightarrow [0, 1]$ subordinate to \mathcal{U} . Let $\tilde{\tau}_i$ be composition $F \times M \xrightarrow{p} M \xrightarrow{\tau_i} [0, 1]$.

Then $\text{Supp}(\tilde{\tau}_i \alpha) \subset F \times U_i$. Define $\int_F \alpha = \sum_{i \in I} \int_F (\tilde{\tau}_i \alpha)$. Easy to check it is independent with the choice of $\mathcal{U}, \{\tau_i\}$.

Remark 7.8. $\pi : E \rightarrow M$ proper submersion. One theorem tells us that π is exactly a smooth fiber with compact fiber $F = \pi^{-1}(*)$. For instance, the **Hopf map**

$$\pi : S^3 \rightarrow S^2, (z_1, z_2) \in \mathbb{C}^2 \mapsto \frac{z_1}{z_2} \in \mathbb{C} \cup \{\infty\}$$

is a smooth fiber with compact fiber.

Then we can generalize above constnution and define $\pi_1 = \int_F - : \Omega^*(E) \rightarrow \Omega^{*-\dim(F)}(M)$

Remark 7.9. If also works if F is not compact,

$$\int_F - : \Omega_{c,F}^*(F \times M) \rightarrow \Omega^*(M) \quad (7.5)$$

where $\Omega_{c,F}^*(F \times M) = \{\alpha \in \Omega^*(F \times M) | \text{Supp}(\alpha) \xrightarrow{\text{Projection}} M \text{ is proper}\}$, viewed as the set of forms with compact fiber.

Theorem 7.10. F is compact, then for any $\alpha \in \Omega^k(F \times M)$, we have

$$d \left(\int_F \alpha \right) + \int_F (d\alpha) = \int_{\partial F} (\alpha|_{\partial F \times M}) \quad (7.6)$$

For the special case $M = *$, $\int_F d\alpha = \int_{\partial F} \alpha|_{\partial F}$ is exactly the stokes theorem.

For the special case $F = [0, 1]$, $d \left(\int_F \alpha \right) + \int_F (d\alpha) = \int_M \alpha|_{\{1\} \times M} - \int_M \alpha|_{\{0\} \times M}$.

We prove only this.

By partition of unity and linearity, we may assume $\text{Supp}(\alpha) \subset [0, 1] \times U$ for some chart (U, x^1, \dots, x^m) of M .

$$\alpha = \sum_{\substack{J \subset \{1, \dots, m\} \\ \|J\|=k}} \alpha_J dx^J + \sum_{\substack{J \subset \{1, \dots, m\} \\ \|J\|=k-1}} \alpha_J dt^1 \wedge dx^J \quad (7.7)$$

By linearity agian, we may assume $\alpha = \alpha_J dx^J$ or $\alpha = \alpha_J dt \wedge dJ$.

For $\alpha = \alpha_J dt \wedge dJ$, we have $\int_F \alpha = 0$, $d \left(\int_F \alpha \right) = 0$. $d\alpha = \frac{\partial \alpha_J}{\partial t} dt \wedge d\alpha^J + \sum_{1 \leq i_0 \leq m} \frac{\partial \alpha_J}{\partial x^{i_0}} dx^{i_0} \wedge dx^J$. Then

$$\int_F d\alpha = \left(\int_0^1 \frac{\partial \alpha_J}{\partial t} dt \right) \cdot dx^J = \alpha_J|_{\{1\} \times M} \cdot dx^J - \alpha_J|_{\{0\} \times M} dx^J = \alpha|_{\{1\} \times M} - \alpha|_{\{0\} \times M} \quad (7.8)$$

So $d(\int_F \alpha) + \int_F(d\alpha) = \int_M \alpha|_{\{1\} \times M} - \int_M \alpha|_{\{0\} \times M}$.

For $\alpha = \alpha_J dt \wedge dx^J$, $\int_F \alpha = \left(\int_0^1 \alpha_J dt \right) dx^J$,

$$d \int_F \alpha = \sum_{1 \leq i_0 \leq m} \frac{\partial}{\partial x^{i_0}} \left(\int_0^1 \alpha_J dt \right) dx^{i_0} \wedge dx^J = \sum_{1 \leq i_0 \leq m} \left(\int_0^1 \frac{\partial \alpha_J}{\partial x^{i_0}} dt \right) dx^{i_0} \wedge dx^J \quad (7.9)$$

$$d\alpha = - \sum_{1 \leq i_0 \leq m} \frac{\partial \alpha_J}{\partial x^{i_0}} dt \wedge dx^{i_0} \wedge dx^J \quad (7.10)$$

So $\int_F d\alpha = - \sum_{1 \leq i_0 \leq m} \left(\int_0^1 \frac{\partial \alpha_J}{\partial x^{i_0}} dt \right) dx^{i_0} \wedge dx^J$. Hence $d \int_F \alpha + \int_F d\alpha = 0 = 0 - 0$.

Proof of Theorem 7.6. Let $H : [0, 1] \times M \rightarrow N$ be a homotopy from f to g . By Whitney approximation theorem 7.7, we may assume H is a smooth map. Define $s : \Omega^*(N) \rightarrow \Omega^{*-1}(M)$, $s(\alpha) = \int_I H^*(\alpha)$. $H^*(\alpha) \in \Omega^*(I \times M)$. Then

$$\begin{aligned} (ds + sd)\alpha &= d \int_I H^* \alpha + \int_I H^*(d\alpha) \\ &= d \int_I H^* \alpha + \int_I dH^*(\alpha) \\ &= H^* \alpha|_{\{1\} \times M} - H^* \alpha|_{\{0\} \times M} \\ &= g^*(\alpha) - f^*(\alpha) \end{aligned} \quad (7.11)$$

So s is a chain homotopy between $f^*, g^* : C_{dR}^*(N) \rightarrow C_{dR}^*(M)$. □

Corollary 7.11. $f \in C^\infty(M, N)$, f is a homotopy equivalence $\Rightarrow f^* : H_{dR}^*(N) \rightarrow H_{dR}^*(M)$ is an isomorphism.

Corollary 7.12 (Poincaré Lemma). $H_{dR}^k(\mathbb{R}^n) = \begin{cases} 0 & k > 0 \\ \mathbb{R} & k = 0 \end{cases}$. i.e. Any closed k -form

on \mathbb{R}^n with $k > 0$ is exact.

Proof. Take constant $c : \mathbb{R}^n \xrightarrow{\cong} *$ being a homotopy equivalence

Then $c^* : H_{dR}^*(\text{point}) \rightarrow H_{dR}^k(\mathbb{R}^n)$ is an isomorphism. But $\Omega^k(\text{point}) =$

$$\begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}. \text{ So}$$

$$H^k(\mathbb{R}^n) = H_{dR}^k(\text{point}) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

□

7.3 Mayer-Vietoris sequence

7.3.1 Some algebraic constructions

A sequence of maps between groups $\cdots \rightarrow G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \rightarrow \cdots$ is **exact** at G_i if $\ker(f_i) = \text{Im}(f_{i-1})$. We say the sequence is **exact** if it is exact at every G_i .

A **short exact** is an exact sequence of the form $0 \rightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \rightarrow 0$. *i.e.* f is injective, g is surjective and $\ker(f) = \text{Im}(g)$.

A **short exact sequence** of cochain complexed is a sequence $0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0$ where A^*, B^*, C^* are cochain complexes, f, g are cochain maps, and $\forall k$, the sequence $0 \rightarrow A^k \xrightarrow{f} B^k \xrightarrow{g} C^k \rightarrow 0$ is exact.

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & A^{k-1} & \xrightarrow{f} & B^{k-1} & \xrightarrow{g} & C^{k-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A^k & \xrightarrow{f} & B^k & \xrightarrow{g} & C^k \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A^{k+1} & \xrightarrow{f} & B^{k+1} & \xrightarrow{g} & C^{k+1} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

Given short exact sequence of cochain complexed $0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0$, we define a **boundary map** $\partial : H^k(C^*) \rightarrow H^{k+1}(A^*)$ as follows:

Take any $[c_0] \in H^k(C^*)$, $c_0 \in C^k$, $dc_0 = 0$. g is surjective so $\exists b_0 \in B^k$ s.t. $g(b_0) = c_0$, and $g(db_0) = dc_0 = 0$. So $\exists a_0 \in A^{k+1}$ s.t. $f(a_0) = db_0$, $f(da_0) = df(a_0) = d^2b_0 = 0$. Since f is injective, $da_0 = 0$. Set $\partial[c_0] = [a_0]$.

$$\begin{array}{ccccc}
& & b_0 & \longrightarrow & c_0 \\
& & \downarrow & & \downarrow \\
a_0 & \xrightarrow{f} & db_0 & \xrightarrow{g} & 0 \\
\downarrow & & \downarrow & & \\
da_0 & \xrightarrow{f} & 0 & &
\end{array}$$

Lemma 7.13. $\partial : H^k(C^*) \rightarrow H^{k+1}(A^*)$ is well-defined

Proof. If we choose another b_1 s.t. $g(b_1) = g(b_0) = c_0$. Then

$$\begin{aligned}
g(b_0 - b_1) = 0 &\Rightarrow b_0 - b_1 = f(a_2) \text{ for some } a_2 \in A^k \\
&\Rightarrow db_0 - db_1 = df(a_2) = f(da_2) \\
f(a_0) = db_0, f(a_1) = db_1 &\Rightarrow f(a_0 - a_1) = db_0 - db_1 = f(da_2)
\end{aligned}$$

$$\Rightarrow a_0 - a_1 = da_2$$

$$\Rightarrow [a_0] = [a_1] \in H^{k+1}(A^*)$$

If we choose c_1 with $[c_1] = [c_0] \in H^k(C^*)$. Then $\exists c_2 \in C^{k-1}$ s.t. $c_1 = c_0 + dc_2$. Take any $b_2 \in B^{k-1}$ s.t. $g(b_2) = c_2$. Then $g(db_2) = dc_2 = c_1 - c_0$. Pick b_0 s.t. $g(b_0) = c_0 \Rightarrow g(b_0 + db_2) = c_1$. So we may pick $b_1 = b_0 + db_2$. Then $db_1 = db_0 \Rightarrow a_1 = a_0$. \square

Lemma 7.14 (Snake Lemma). *Given short exact sequence of cochain complexes $0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0$, the sequence*

$$\begin{array}{ccccccc} H^{k+1}(C^*) & \xleftarrow{g^*} & H^{k+1}(B^*) & \xleftarrow{f^*} & H^{k+1}(A^*) & \xleftarrow{\quad} & \\ & & \searrow \partial & & \searrow \partial & & \\ H^k(C^*) & \xleftarrow{g^*} & H^k(B^*) & \xleftarrow{f^*} & H^k(A^*) & \xleftarrow{\quad} & \\ & & \searrow \partial & & \searrow \partial & & \\ H^{k-1}(C^*) & \xleftarrow{g^*} & H^{k-1}(B^*) & \xleftarrow{f^*} & H^{k-1}(A^*) & \xleftarrow{\quad} & \\ & & \searrow \partial & & \searrow \partial & & \\ \dots & \xleftarrow{\quad} & \dots & \xleftarrow{\quad} & \dots & \xleftarrow{\quad} & 0 \end{array}$$

is exact. For simplicity, short exact sequence of cochain complexes induces long exact sequence on homotopy.

Proof. **Exactness at $H^k(B^*)$:** $g \circ f = 0 \Rightarrow g^* \circ f^* = 0 \Rightarrow \text{Im } f \subset \ker g^*$.

Take any $[b] \in \ker g^*$, $b \in B^*$, $db = 0$, then $[g(b)] = g^*([b]) = 0$. So $g(b) \in \text{Im } d$ i.e. $\exists c \in C^{k-1}$ s.t. $dc = g(b)$.

Pick $b' \in B^{k-1}$ s.t. $g(b') = c$. Then $[b] = [b - db']$. $g(b - db') = 0 \Rightarrow \exists a$ s.t. $f(a) = b - db'$. $f(da) = 0 \Rightarrow da = 0$. Then $f^*(a) = [b - db'] = [b]$. So $\ker g^* \subset \text{Im } f^*$.

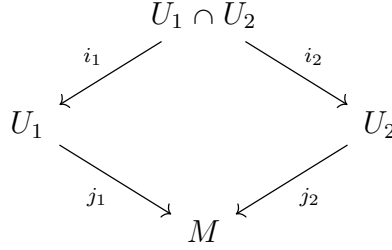
Exactness at $H^k(C^*)$: If $[c_0] \in \ker(\partial)$. Then $[a_0] = 0 \in H^{k+1}(A^*)$ i.e. $\exists a_2 \in A^k$ s.t. $a_0 = da_2$. Set $b_2 = b_0 - f(a_2)$. Then $db_2 = db_0 - f(a_0) = 0$. So $[b_2] \in H^k(B^*)$, $g(b_2) = g(b_0) = c_0$. So $[c_0] = g^*([b_2]) \in \text{Im } g^*$.

If $[c_0] \in \text{Im}(g^*)$, then $\exists [b] \in H^k(B^*)$ s.t. $g^*[b] = [c_0]$. Set $c_1 = g(b)$, $b_1 = b$. Then $[c_1] = [c_0]$. Then $db_1 = 0$. So $a_1 = 0 \Rightarrow \partial[c_1] = \partial[c_0] = [a_1] = 0$. Then $\text{Im}(g^*) \subset \ker(\partial)$.

Exactness at

□

Now we consider $M = U_1 \cup U_2$ for U_1, U_2 open subsets. The inclusion maps are i_1, i_2, j_1, j_2 .



Lemma 7.15. The sequence $0 \rightarrow \Omega^*(M) \xrightarrow{(j_1^*, j_2^*)} \Omega^*(U_1) \oplus \Omega^*(U_2) \xrightarrow{i_1^* - i_2^*} \Omega^*(U_1 \cap U_2) \rightarrow 0$ is exact.

Proof. (j_1^*, j_2^*) is injective. $\alpha \in \Omega^k(M)$. If $(j_1^*, j_2^*)(\alpha) = 0$, then $\alpha|_{U_1} = \alpha|_{U_2} = 0$. So $\alpha = 0$.

Exactness at $\Omega^*(U_1) \oplus \Omega^*(U_2)$: If $(\alpha_1, \alpha_2) \in \ker(i_1^* - i_2^*)$, then $\alpha_1|_{U_1 \cap U_2} = \alpha_2|_{U_1 \cap U_2}$. Set $\alpha \in \Omega^*(M)$ s.t. $\alpha|_{U_1} = \alpha_1, \alpha|_{U_2} = \alpha_2$. Then $(\alpha_1, \alpha_2) = (j_1^* \alpha, j_2^* \alpha)$.

If $(\alpha_1, \alpha_2) \in \text{Im}(j_1^*, j_2^*)$. Let $\alpha_1 = \alpha|_{U_1}, \alpha_2 = \alpha|_{U_2}$, then $(i_1^* - i_2^*)(\alpha_1, \alpha_2) = \alpha|_{U_1 \cap U_2} - \alpha|_{U_1 \cap U_2} = 0$. So $(\alpha_1, \alpha_2) \in \ker(i_1^* - i_2^*)$.

$i_1^* - i_2^*$ is surjective. Take any $\alpha_{12} \in \Omega^*(U_1 \cap U_2)$. Take a partition of unity $\{\rho_i : M \rightarrow [0, 1]\}_{i=1,2}, \rho_1 + \rho_2 \equiv 1, \text{Supp}(\rho_0) \subset U_i$. Then $\text{Supp}(\rho_i) \cap U_2 \subset U_1 \cap U_2$. So we may define $\alpha_i \in \Omega^*(U_i)$ by

$$\alpha_{i,p} = \begin{cases} (-1)^{i-1} \rho_i(p) \cdot \alpha_{12,p} & p \in U_1 \cap U_2 \\ 0 & p \in U_i \setminus (U_1 \cap U_2) \end{cases} \quad (7.12)$$

Then $\alpha_1|_{U_1 \cap U_2} - \alpha_2|_{U_1 \cap U_2} = \rho_1(p)\alpha_{12} + \rho_2(p)\alpha_{12} = \alpha_{12}$.

□

Apply the snake lemma 7.14, we obtain

Theorem 7.16 (Mayer-Vietoris sequence). *The sequence*

$$\begin{array}{ccccccc}
 \cdots & \longleftarrow & H_{dR}^{k+1}(U) \oplus H_{dR}^{k+1}(V) & \longleftarrow & H_{dR}^{k+1}(M) & \longleftarrow & \\
 & & \searrow d & & \searrow & & \\
 & & H_{dR}^k(U \cap V) & \longleftarrow & H_{dR}^k(U) \oplus H_{dR}^k(V) & \longleftarrow & H_{dR}^k(M) \\
 & & \searrow d & & \searrow & & \\
 & & H_{dR}^{k-1}(U \cap V) & \longleftarrow & H_{dR}^{k-1}(U) \oplus H_{dR}^{k-1}(V) & \longleftarrow & \cdots \longleftarrow 0
 \end{array}$$

is exact

Example 7.17. For $S^n = U \cap V$ where

$$U = S^n \setminus \{(1, 0, \dots, 0)\}, V = S^n \setminus \{(-1, 0, \dots, 0)\}, U \cap V = S^n \setminus \{(\pm 1, 0, \dots, 0)\}$$

$U, V \simeq *, U \cap V \simeq S^{n-1}$. So the M.V. sequence

$$0 \longrightarrow H_{dR}^0(S^1) \longrightarrow H^0(*) \oplus H^0(*) \xrightarrow{\delta} H_{dR}^0(S^0) \longrightarrow H_{dR}^1(S^1) \longrightarrow 0$$

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0$$

Then $H_{dR}^0(S^1) = \ker \delta = \mathbb{R}$, $H_{dR}^1(S^1) = \operatorname{coker} \delta = \mathbb{R}$.

And for $k \geq 1$, the Mayer-sequence

$$0 \longrightarrow H_{dR}^k(S^{n-1}) \longrightarrow H_{dR}^{k+1}(S^n) \longrightarrow 0$$

implies $H_{dR}^k(S^{n-1}) \cong H_{dR}^{k+1}(S^n)$. Therefore, by induction, we have

$$H_{dR}^k(S^n) = \begin{cases} \mathbb{R} & n = k \\ \mathbb{R} & k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (7.13)$$

Example 7.18. For $M = \mathbb{CP}^n$, $U_1 = M \setminus \{[1, 0, \dots, 0]\}$, $U_2 = M \setminus \{[0, *, *, \dots, *]\}$.

Then

$$\begin{aligned} U_1 &= \{[x_0, \dots, x_n] | x_1, \dots, x_n \text{ not all zero}\} \\ &\simeq \{[0, x_1, \dots, x_n] | x_1, \dots, x_n \text{ not all zero}\} \\ &\simeq \mathbb{CP}^{n-1} \end{aligned}$$

$$U_2 = M \setminus \{[0, *, \dots, *]\} = \{[1, *, \dots, *]\} \cong \mathbb{C}^n$$

$$U_1 \cap U_2 \cong U_2 \setminus \{[1, 0, \dots, 0]\} \cong \mathbb{C}^n \setminus \{0\} \simeq S^{2n-1}$$

$$\text{Claim. } H_{dR}^k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{R} & k \text{ even } 0 \leq k \leq 2n \\ 0 & \text{otherwise} \end{cases}.$$

Prove the claim by induction. For $n = 1$, $\mathbb{CP}^1 \cong S^2$ is true.

Suppose we have proved it for \mathbb{CP}^{n-1} . Apply Mayer-Vietoris sequence to $M = U_1 \cup U_2$,

$$H_{dR}^{i-1}(S^{2n-1}) \rightarrow H_{dR}^i(\mathbb{CP}^n) \rightarrow H_{dR}^i(\mathbb{CP}^{n-1}) \oplus H_{dR}^i(\mathbb{C}^n) \rightarrow H_{dR}^i(S^{2n-1}) \quad (7.14)$$

Then for $i \neq 2n-1, 2n$,

$$0 \rightarrow H_{dR}^i(\mathbb{CP}^n) \xrightarrow{\cong} H_{dR}^i(\mathbb{CP}^{n-1}) \rightarrow 0 \quad (7.15)$$

For $2n-1, 2n$,

$$0 = H_{dR}^{2n-1}(\mathbb{CP}^{n-1}) \rightarrow H_{dR}^{2n-1}(S^{2n-1}) \xrightarrow{\cong} H_{dR}^{2n}(\mathbb{CP}^n) \rightarrow H_{dR}^{2n}(\mathbb{CP}^{n-1}) = 0 \quad (7.16)$$

$$0 = H_{dR}^{2n-2}(S^{2n-1}) \rightarrow H_{dR}^{2n-1}(\mathbb{CP}^n) \rightarrow H_{dR}^{2n-1}(\mathbb{CP}^{n-1}) = 0$$

Remark 7.19. $H^*(M)$ is a ring where

$$[\alpha] \cdot [\beta] = [\alpha \wedge \beta] \quad (7.17)$$

It's graded commutative since

$$[\alpha] \cdot [\beta] = (-1)^{kl} [\beta] \cdot [\alpha] \text{ for } [\alpha] \in H^k(M), \beta \in H^l(M) \quad (7.18)$$

Any $f : M \rightarrow N$ induces ring homomorphism $f^* : H_{dR}^*(N) \rightarrow H_{dR}^*(M)$.

Indeed, $H_{dR}^*(\mathbb{CP}^n) \cong \mathbb{R}[x]/(x^{n+1})$, where x generates $H_{dR}^2(\mathbb{CP}^n)$.

Proposition 7.20. $A \subsetneq \mathbb{R}^n$ not closed. Then

$$H_{dR}^{i+1}(\mathbb{R}^{n+1} \setminus \{0\} \times A) \cong H_{dR}^i(\mathbb{R}^n \setminus A), \quad i > 0 \quad (7.19)$$

$$H_{dR}^1(\mathbb{R}^{n+1} \setminus \{0\} \times A) \cong H_{dR}^0(\mathbb{R}^n \setminus A) / \mathbb{R} \cdot 1 \quad (7.20)$$

Proof. $\mathbb{R}^{n+1} \setminus \{0\} \times A = U_1 \cup U_2$ where

$$U_1 = (\mathbb{R}_{<0} \times \mathbb{R}^n) \cup ([0, 1] \times (\mathbb{R}^n \setminus A))$$

$$U_2 = (\mathbb{R}_{>0} \times \mathbb{R}^n) \cup ((-1, 0] \times (\mathbb{R}^n \setminus A))$$

$$U_1 \cap U_2 \cong (-1, 1) \times (\mathbb{R}^n \setminus A)$$

Then there exists a deformation retraction $[0, 1] \times U_1 \rightarrow U_1$, $(t, (x_0, x_1, \dots, x_n)) \mapsto (tx_0 + (1-t)(-1), x_1, \dots, x_n)$. Then $U_1 \cong \{-1\} \times \mathbb{R}^n \simeq *$. $U_2 \simeq *$, $U_1 \cap U_2 \simeq \mathbb{R}^n \setminus A$.

The Mayer-Vietoris sequence gives

$$H_{dR}^i(U_1) \oplus H_{dR}^i(U_2) \rightarrow H_{dR}^i(U_1 \cap U_2) \xrightarrow{\partial} H_{dR}^{i+1}(\mathbb{R}^{n+1} \setminus \{0\} \times A) \rightarrow H_{dR}^{i+1}(U_1) \oplus H_{dR}^{i+1}(U_2)$$

for $i > 0$.

The case $i = 0$ is similar. □

Lemma 7.21. $A \subset \mathbb{R}^n, B \subset \mathbb{R}^n, f : A \xrightarrow{\cong} B$. Then $\mathbb{R}^{n+m} \setminus \{0\} \times B \cong \mathbb{R}^{n+m} \setminus \{(x, f(x)) | x \in A\} \cong \mathbb{R}^{n+m} \setminus A \times \{0\}$.

Proof. $A \xrightarrow{\cong} fB \hookrightarrow \mathbb{R}^n$. By Tietz extension theorem, there exists $\tilde{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ s.t. $\tilde{f}|_A = f$.

Consider $\hat{f} : \mathbb{R}^m \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m \times \mathbb{R}^n, (x, y) \mapsto (x, y + \hat{f}(x))$. Then $\hat{f}(A \times \{0\}) = \{(x, f(x)) | x \in A\} \Rightarrow \mathbb{R}^{n+m} \setminus \{0\} \times B \cong \mathbb{R}^{n+m} \setminus \{(x, f(x)) | x \in A\} \cong \mathbb{R}^{n+m} \setminus A \times \{0\}$. □

Special case, if $A, B \subset \mathbb{R}^n$ closed, $A \cong B$, then $\mathbb{R}^{2n} \setminus A \times \{0\} \cong \mathbb{R}^{2n} \setminus B \times \{0\}$.

It directly follows that

Theorem 7.22. $A, B \subset \mathbb{R}^n$ closed, $A \cong B$ under homeomorphism. $\Rightarrow H_{dR}^*(\mathbb{R}^n \setminus A) \cong H_{dR}^*(\mathbb{R}^n \setminus B)$.

Proof. By proposition 7.20

$$H_{dR}^*(\mathbb{R}^n \setminus A) \cong H_{dR}^{*+n}(\mathbb{R}^{2n} \setminus A \times \{0\}) \cong H_{dR}^{*+n}(\mathbb{R}^{2n} \setminus B \times \{0\}) \cong H^*(\mathbb{R}^n \setminus B)$$

□

Example 7.23. A knot is an embedded $S^1 \hookrightarrow \mathbb{R}^3$. So \forall knot K ,

$$H_{dR}^i(\mathbb{R}^3 \setminus K) \cong H_{dR}^i(\mathbb{R}^3 \text{ circle})_{\text{cong}} \begin{cases} \mathbb{R} & i = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases} \quad (7.21)$$

Corollary 7.24. $A \subset \mathbb{R}^n$ closed, $A \cong S^{n-1} \Rightarrow \mathbb{R}^n \setminus A$ has two components U_1, U_2 where U_1 is bounded and U_2 is unbounded. Moreover, $\partial U_1 = \partial U_2 = A$.

Proof. $H_{dR}^0(\mathbb{R}^n \setminus A) \cong H_{dR}^0(\mathbb{R}^n \setminus S^{n-1}) \cong \mathbb{R} \langle \pi_0(\mathbb{R}^n \setminus S^{n-1}) \rangle = \mathbb{R} \oplus \mathbb{R}$.

So $\mathbb{R}^n \setminus A$ has two components U_1, U_2 .

Take $L = \max\{\|x\| \mid x \in A\} + 1$, $V = \{x \in \mathbb{R}^n \mid \|x\| > L\}$. $V \subset \mathbb{R}^n \setminus A$ connected and unbounded. $V \subset U_1$ so U_1 is unbounded $\Rightarrow U_2 \subset \mathbb{R}^n \setminus V$ is unbounded.

The proof of $\partial U_1 = \partial U_2 = A$ is omitted. See Madsen-Trnehave. \square

Corollary 7.25. $A \subset \mathbb{R}^n$, $A \subset D^k \Rightarrow \mathbb{R}^n \setminus A$ is connected.

Theorem 7.26 (Invariance of domain). *Let U be an open subset of \mathbb{R}^n . Let $f : U \rightarrow \mathbb{R}^n$ be continuous and injective map. Then $f(U)$ is also open in \mathbb{R}^n . And f sends U homeomorphically to $f(U)$*

Proof. It suffices to show $f(U)$ is open. Since for any $W \subset U$ open, $f(W)$ is open in $f(U)$. So $f|_U : U \rightarrow f(U)$ is open.

Take any $x_0 \in U$, want to show $f(x_0)$. Take $D = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq \delta\} \subset U$. Then $\Sigma = f(\partial D) \cong S^{n-1}$. So $\mathbb{R}^n \setminus \Sigma$ has 2 components U_1, U_2 , where U_1 is bounded and U_2 is unbounded.

$\mathbb{R}^n \setminus f(D)$ is connected so $\mathbb{R}^n \setminus f(D) \subset U_2$. So $U_1 \cup \Sigma = \mathbb{R}^n \setminus U_2 \subset f(D) = f(\text{int}(D)) \sqcup \Sigma \Rightarrow U_1 \subset f(\text{int}(D))$.

Since $f(\text{int}(D))$ is connected, $f(\text{int}(D)) \subset U \Rightarrow f(\text{int}(D)) = U_1$. So $f(x_0) \in U_1 \subset \text{int} f(U)$. \square

Corollary 7.27. *If $m > n$, $U \subset \mathbb{R}^m$ open, then there is no injective continuous map $U \rightarrow \mathbb{R}^n$*

Proof. If $f : U \rightarrow \mathbb{R}^n \hookrightarrow \mathbb{R}^m$. Then $f(U)$ not open in \mathbb{R}^m , which causes contradiction. \square

Corollary 7.28. $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^m$ open. $U \cong V \Rightarrow m = n$.

7.4 Compact supported de Rham cohomology

Define $H_{dR,c}^*(M) = H^*(\Omega^0(M) + \Omega^1(M) + \cdots)$, abbreviated to $H_c^*(M)$.

If M is compact, then $H_{dR,c}^*(M) \cong H_{dR}^*(M)$. But it is not true for M not compact. The following is a counterexample.

Theorem 7.29. *Let M be connected n -dimensional manifold without boundary. Then the map $H_c^n(M) \xrightarrow{\cong} \mathbb{R}, [\alpha] \mapsto \int_M \alpha$ is a well-defined isomorphism. Moreover, if M is closed and connected, then $H_{dR}^n(M) \cong \mathbb{R}$*

Proof. Well-defineness: If $[\alpha] = [\alpha']$, then $\alpha - \alpha' = d\beta$. By Stokes theorem 6.12, $\int_M \alpha = \int_M \alpha'$.

Proposition 7.30. *If $\alpha \in \Omega_c^n(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \alpha = 0 \Rightarrow \exists \beta \in \Omega_c^{n-1}(\mathbb{R}^n)$ s.t. $d\beta = \alpha$.*

Note that if $\alpha = f dx_1 \wedge \cdots \wedge dx_n$, $\beta = \sum_{i=1}^n (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

Then $\alpha = d\beta$ iff $f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$.

So the proposition is equivalent to $\forall f \in C_c^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} f dx_1 \wedge \cdots \wedge dx_n = 0 \Rightarrow \exists f_1, \dots, f_n \in C_c^\infty(\mathbb{R}^n)$ s.t. $f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$.

Prove it by induction. For $n = 1$, $f_1(x) = \int_{-\infty}^x f(t) dt$. $f_1 \in C_c^\infty(\mathbb{R})$, $f_1' = f$.

Suppose n is proved, $f \in C_c^\infty(\mathbb{R}^{n+1})$, $\int_{\mathbb{R}^{n+1}} f dx_1 \wedge \cdots \wedge dx_{n+1} = 0$.

Define $g \in C_c^\infty(\mathbb{R}^n)$ by $g(x_1, \dots, x_n) = \int_{-\infty}^{\infty} f(x_1, \dots, x_{n+1}) dx_{n+1}$. Then $\int_{\mathbb{R}^n} g dx_1 \wedge \cdots \wedge dx_n = 0$.

By induction, $\exists g_1, \dots, g_n \in C_c^\infty(\mathbb{R}^n)$ s.t. $g = \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}$.

Take $\rho \in C_c^\infty(\mathbb{R})$ s.t. $\int_{-\infty}^{\infty} \rho dx = 0$. Define $f_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by $f_i(x_1, \dots, x_{n+1}) = g_i(x_1, \dots, x_n) \cdot \rho(x_{n+1})$.

Set $h = f - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$. Then for $\forall (x_1, \dots, x_n)$,

$$\int_{-\infty}^{\infty} h(x_1, \dots, x_{n+1}) dx_{n+1} = \int_{-\infty}^{\infty} f(x_1, \dots, x_{n+1}) - \int_{-\infty}^{\infty} \rho(x_{n+1}) dx_{n+1} \cdot \left(\sum_{i=1}^n \frac{\partial g_i}{\partial x_i} \right) = 0$$

Set $f_{n+1}(x_1, \dots, x_{n+1}) = \int_{-\infty}^{x_{n+1}} h(x_1, \dots, x_n, t) dt$. Then $\frac{\partial f_{n+1}}{\partial x_{n+1}} = h$ and $f = \sum_{i=1}^{n+1} \frac{\partial f_i}{\partial x_i}$ \square

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