# Differential Geometry

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# Contents

# 1 Smooth Manifold

**Definition 1.1** (Topological manifold)**.** A space $M$ is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

**Definition 1.2** (Smooth Manifold)**.** A smooth structure is given by an equivalence class of smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ *s.t.* $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth $\forall \alpha, \beta$. $M = \cup U_\alpha$.
A **smooth manifold** is a topological manifold with a smooth structure.
Define when a continuous map $f : M_1 \to M_2$ is smooth if $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$, we have $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is smooth.

**Definition 1.3.** Given $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$. A homeomorphism $f : M_1 \to M_2$ is called a diffeomorphism if $f, f^{-1}$ is smooth.
In this case we say $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ are diffeomorphism.

**Theorem 1.4** (Kervaire)**.** *$\exists$ 1 10-dimensional topological manifold without smooth manifold.*

**Theorem 1.5** (Milnor)**.** *$\exists$ a smooth manifold $M$ s.t. $M \cong S^7$ but not in diffeomorphism meaning.*

**Theorem 1.6** (Kervaire-Milnor)**.** *$\exists$ 28 smooth structures (up to orientation preserving diffeomorphism) on $S^7$*

**Theorem 1.7** (Morse-Birg)**.** *On $S^7$. If $n \leqslant 3$, then any $n$-dimensional topological manifold $M$ has a unique smooth structure up to diffeomorphism.*

**Theorem 1.8** (Stallings). *If $n \neq 4$, then $\exists$ a unique smooth structure on $\mathbb{R}^n$ up to diffeomorphism.*

**Theorem 1.9** (Donaldson-Freedom-Gompf-Faubes). *$\exists$ uncountable smooth structures on $\mathbb{R}^4$ up to diffeomorphism.*

**Definition 1.10** (topological manifold with boundary)**.** A space $M$ is called a topological manifold with boundary if

1. $M$ is Hausdorff
2. $M$ is second countable
3. $\forall p \in M$, $\exists$ a neighbourhood $U$ of $p$ and a homeomorphism $\varphi : U \to V$ where $V$ is open in $\mathbb{H}^n$

We say a manifold $M$ is closed if $M$ is compact and $\partial M$ is empty.

Our motivation for studying manifold is to study the space of solution for equations.

**Question 1.** Given $f : \mathbb{R}^n \to \mathbb{R}$ smooth, $q \in \mathbb{R}^n$, when is $f^{-1}(q)$ is a smooth manifold?

For $f : U \to \mathbb{R}^n$ smooth, $U$ open in $\mathbb{R}^m$, the differential of $f$ at $p \in U$ denoted as $\mathrm{d}f(p)$.

**Definition 1.11.** We say $p \in U$ is a **regular point** of $f$ if $\mathrm{d}f(p)$ is surjective. Otherwise we say $p \in U$ is a **critical point**.
A point $q \in \mathbb{R}^n$ is called a **regular value** of $f$ if $\forall p \in f^{-1}(q)$ , $p$ is a regular point of $f$.
A point $q \in \mathbb{R}^n$ is called a **critical value** of $f$ if $\forall p \in f^{-1}(q)$ , $p$ is a critical point of $f$.

**Theorem 1.12** (Implicit function theorem). *If $p \in U$ is a regular point of $f : U \to \mathbb{R}^n$. Then there exists*

- *An open neighbourhood $V$ of $p$ in $U$*
- *An open subset $V'$ of $\mathbb{R}^m$*
- *A diffeomorphism $\varphi : V \to V'$ such that $P \circ \varphi = f$ where $P$ is the projection from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

*In other words, near a regular point, we can do local coordinate change to turn $f$ into the projection.*

**Remark 1.13.** Inverse function theorem and Implicit function theorem gives a way to find the related from "a point" to "a beibourhood"!

In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \ldots, x_m) \in V' | (x_1, \cdots .x_n) = f(p)\}$$

*i.e.* if we set $M = f^{-1}(f(p))$, then $(M \cap V, \varphi_p)$ is a chart that contains $p$.

**Corollary 1.14.** *If $q$ is a regular value of $f : U \to \mathbb{R}^n$ then $f^{-1}(q)$ is a smooth manifold.*

**Remark 1.15.** It suffices to show that the corresponding charts are compatible.

**Theorem 1.16** (Sard). *If $f : U \to \mathbb{R}^n$ is a smooth map, then the set of critical values of $f$ has measure $0$.*

**Remark 1.17.** For a "generic" $q$, $f^{-1}(q)$ is a manifold of dimension $m - n$.

**Corollary 1.18.** *If $f : U \to \mathbb{R}^n$ is smooth and $m < n$ then $f(U)$ has measure $0$.*

## 1.1 Lie groups and homogeneous spaces

**Definition 1.19.** We say $G$ is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map $\cdot : G \times G \to G$ and the inverse map $G \rightsquigarrow G$ is smooth.

**Example 1.20.** $GL(n, \mathbb{R}) = \{n \times n$ matrices with non-zero determinant$\} \subset \mathbb{R}^{n\times n}$
$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$
$SO(n) = \{A \in O(n) | \det A = 1\}$
$U(n) = \{A \in GL(n, \mathbb{C}) | A\overline{A}^T = I\}$
$SU(n) = \{A \in U(n) | \det A = 1\}$

**Exercise 1.21.**

$$O(1) \cong S^2 \qquad SO(1) \cong * \tag{1.1}$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \tag{1.2}$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \tag{1.3}$$

The last one is a diffeomorphism but do not preserve the multiplicatioin, *i.e.* not an isomorphism of Lie group.

**Theorem 1.22** (Carton)**.** *Let $H$ be a closed subgroup of Lie group $G$. Then $H$ is a Lie group. More precisely, $H$ is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also, $G/H$ is a smooth manifold*

**Definition 1.23.** Let $M$ be a smooth manifold. We say $M$ is a **homogeneous space** if $\exists$ a Lie group $G$ with a smooth transitive action $\rho : G \times M \to M$.

**Definition 1.24.** For $M$ be a homogeneous space. The **isotropy** group of $x \in M$ is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of $G$

Given any $x, x' \in M$, $Iso(x) \cong Iso(x')$ because the group action is transitive.

Hence, we have a well-defined map

$$p : G/_{Iso(x)} \to M \tag{1.4}$$

$$g \mapsto gx \tag{1.5}$$

**Theorem 1.25.** *$p$ is always a diffeomorphism.*

Therefore, we have this proposition

**Proposition 1.26.** *$M$ is a homogeneous space $\Leftrightarrow$ $M = G/H$ for some closed subgroup $H$.*

**Example 1.27.** If $M = S^n$, let $G = SO(n+1)$.

Then $Iso(1, 0, \cdots, 0) \cong SO(n)$.

So $S^n \cong SO(n+1)/(SO(n))$.

Similarly, we can prove $\mathbb{RP}^n \cong SO(n+1)/(O(n))$, $\mathbb{CP}^n \cong SO(n+1)/(U(n))$

The isotropy k dimensional linear subspaces of $\mathbb{R}^n$ can be $O(k) \times O(n-k)$ if $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to $\mathbb{RP}^2, S^2, T^2$ and Klein bottle.

**Theorem 1.28** (Whithead)**.** *Any smooth manifold has a triangulation.*

**Theorem 1.29** (Poincare-Hopf)**.** *$G$ is compact Lie group $\Rightarrow$ $\chi(G) = 0$.*

**Theorem 1.30** (Mostow2005)**.** *$M$ is a compact homogeneous space $\Rightarrow$ $\chi(M) \geqslant 0$.*

## 1.2 Bump Function and Partition of Unity

**Theorem 1.31** (Urysohn smooth version)**.** *Given $M$, closed disjoint $A, B$, $\exists$ smooth $f : M \to [0,1]$ s.t. $f|_A = 0, f|_B = 1$.*

**Theorem 1.32** (Tietze)**.** *Given $M$, closed $A$, smooth $f : A \to \mathbb{R}^n$, there exists smooth $\hat{f} : M \to \mathbb{R}^n$ s.t. $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

**Lemma 1.33.** *Let $U$ be a neighbourhood of $p \in M$. Then $\exists$ smooth $\sigma : M \to [0,1]$ s.t.*

1. *$\sigma \equiv 1$ near $p$*

2. *Supp $\sigma \subset U$*

*Such $\sigma$ is called a **bump function** at $p$, supported in $U$.*

**Definition 1.34.** An open cover of a space $X$ is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

**Proposition 1.35.** *Given compact $K \subset U$ and open neighbourhood $U$ of $K$, $\exists$ a smooth $g : M \to [0, +\infty)$ s.t. $g|_K \equiv 1$ and Supp $g \subset U$.*

**Definition 1.36.** An **exhaust** of a space $X$ is a sequence of open sets $\{U_i\}$ *s.t.*

1. $X = \bigcup_{i=1}^{\infty} U_i$

2. $\overline{U_i}$ is compact and contained in $U_{i+1}$

**Theorem 1.37.** *Any topological manifold has an exhaust.*

Given two open covers $\mathcal{U}, \mathcal{V}$, we say $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if $\forall U_\alpha \in U$, $\exists V_\beta \in \mathcal{V}$ *s.t.* $V_\beta \subset U_\alpha$.

We say a space $X$ is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact.(The proof is hard)

**Proposition 1.38.** *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a topological manifold $M$. Then there exists countable open covers $\mathcal{W} = \{W_i\}$, $\mathcal{V} = \{V_i\}$ s.t.*

- *For any $i$, $\overline{V_i}$ is compact and $\overline{V_i} \subset W_i$*
- *$\mathcal{W}$ is locally finite.*
- *$\mathcal{W}$ is a refinement of $\mathcal{U}$.*

As a corollary, we have any topological manifold is paracompact.

**Definition 1.39.** Given open cover $\mathcal{U}$ of a smooth $M$, a partition of unity subordinate to $\mathcal{U}$ is a collection of smooth functions $\{\rho_\alpha : M \to [0,1]\}_{\alpha \in \mathcal{A}}$ *s.t.*

1. $\forall p \in M$, $\exists$ only finitely many $\alpha \in \mathcal{A}$ *s.t.* $p \in Supp\,\rho_\alpha$
2. $\sum\limits_{\alpha \in \mathcal{A}} \rho_\alpha(p) = 1$
3. $Supp\,\rho_\alpha \subset U_\alpha$

**Theorem 1.40** (Existence of P.O.U)**.** *For any open cover $\mathcal{U}$ of smooth $M$, $\exists$ a P.O.U subordinate to $\mathcal{U}$*

**Theorem 1.41** (Whitney approximation theorem)**.** *Given any smooth $M$, any closed $A$ and any continuous $f : M \to \mathbb{R}$, $\delta : M \to (0, +\infty)$. Suppose $f$ is smooth on $A$. Then $\exists g : M \to \mathbb{R}$ smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M$, $|g(p) - f(p)| < \delta(p)$.

# 2 Tangent space and tangent vectors

## 2.1 Tangent Space

Given $p \in M$, consider the set $C_p^\infty(M) = \{\text{smooth function } V \to \mathbb{R}\}/_\sim$ where $f_1 \sim f_2$ if and only if $\exists$ neighbourhood $U$ of $p$, $f_1|_U = f_2|_U$.

$C_p^\infty(M)$ is the space of **genus of smooth function** near $p$.

A **partial-derivative** of $p$ is a $\mathbb{R}$-linear map $D : C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

**Definition 2.1.** A **tangent vector** of $M$ at $p$ is a partial-derivative at $p$.

Define the **tangent space** $T_pM = \{\text{all partial-derivative at } p\}$, which is a $\mathbb{R}$-vector space.

**Proposition 2.2.** *For $M = U \subset \mathbb{R}^n$ open. We have $\{\frac{\partial}{\partial x_i}\}$ is a basis for $T_pU$.*

**Proposition 2.3.**

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leqslant i \leqslant n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

$M, N$ smooth manifolds, $C^\infty(N, M) = \{\text{smooth } F : N \to M\}$.

Given $F \in C^\infty(N, M)$, $F$ induces $F^* : C_{F(p)}^\infty(M) \to C_p^\infty(N)$, $f \mapsto f \circ F$.

By taking dual, we get

$$F_* : T_pN \to T_{F(p)}M$$

we also write $F_*$ as $F_{*,p}$, call it the **differential** of $F$ at $p$.

where

$$F_*(\frac{\partial}{\partial x^i}|_p) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

**Proposition 2.4.** *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_pN \to T_{G\circ F(p)}W$$

**Definition 2.5.** A smooth **curve** is a smooth map $\gamma : (a, b) \to M$. We say $\gamma$ starts at $p$ if $\gamma(0) = p$. We define the **velocity** of $\gamma$ at $\gamma(0)$ as $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)}M$

Take charts $(U, x^1, \cdots, x^n)$ about $p$, let $\gamma^i = x^i \circ \gamma$.

We say $\gamma, \delta$ are **tangent** to each other at $p$ if $(\gamma^i)'(0) = (\delta^i)'(0)$.

Now we can define

$$(T_pM)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\}/_\sim$$

where $\gamma \sim \delta$ iff they are tangent to each other.

Then these definition is more geometric.

**Lemma 2.6.** *Given $F \in C^\infty(M, M)$, $p \in N$, the diagram commutes:*

$$\begin{tikzcd}
\gamma \in (T_pN)_{curve} \arrow[r, "\cong"] \arrow[d] & T_pN \arrow[d] \\
F \circ \gamma \in (T_{F(p)}M)_{curve} \arrow[r, "\cong"] & T_{F(p)}M
\end{tikzcd}$$

## 2.2 Tangent Bundle

Let $(M, \mathcal{A})$ be a smooth manifold, $TM = \bigcup\limits_{p\in M} T_pM$, called the **tangent bundle**

Now we want to define a natural topology and smooth structure on $TM$. Take any chart $(U, \varphi) = (U, x^1, \cdots, x^n) \in \mathcal{A}$.

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \tag{2.1}$$

$$X \in T_pU \mapsto (\varphi(p), X^1, \cdots, X^n) \tag{2.2}$$

where $X = \sum X^i \frac{\partial}{\partial x^i}|_p$.

Then pull back standard topology on $\varphi(U) \times \mathbb{R}^n$ to a topology on $TU$.

$$\mathcal{B} = \{\hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- $\mathcal{B}$ is a basis
- $\mathcal{B}$ generates a Hausdorff, second countable topology on $TM$.

So $TM$ is a topological manifold covered by charts $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$.

Given $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$, the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi} \circ \hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \tag{2.3}$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \tag{2.4}$$

So $\hat{\mathcal{A}}$ is a smooth atlas on $TM$, making $TM$ into a smooth manifold.

**Definition 2.7** (vector bundle)**.** Given a continuous map $f : E \to B$, we say $f$ is a $n$-dimensional **vector bundle** if: $\exists$ an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of $B$ and homeomorphisms $\{f^{-1}(U_\alpha) \underset{\cong}{\xrightarrow{\rho_\alpha}} U_\alpha \times \mathbb{R}\}$ *s.t.*

- $\begin{tikzcd} f^{-1}(U_\alpha) \arrow[r, "\rho_\alpha"] \arrow[d, "f"] & U_\alpha \times \mathbb{R}^n \arrow[dl, "projection"] \\ U_\alpha & \end{tikzcd}$ commutes for $\alpha \in I$
- $\forall\, p \in U_\alpha \cap U_\beta$, the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

is linear.

Call $f^{-1}(p)$ the **fiber** over $p$.

**Proposition 2.8.** *Given vector bundle $f : E \to B$, the fiber $f^{-1}(p)$ has a structure of a vector space.*

**Example 2.9** (Product bundle)**.** $E = \mathbb{R}^n \times B$

**Example 2.10** (Tautological bundle)**.**

$$B = \mathbb{CP}^n = \{\text{1-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map $(L, v) \mapsto L$

Given vector bundles $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$, a bundle map consists of $(\hat{f}, f)$ *s.t.*

- $\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$ commutes.
- $\forall b \in B$, $\hat{f} : \pi_1^{-1}(b) \to \pi_2^{-1}(f(b))$ is linear.

If $\hat{f}, f$ are diffeomorphisms, then we call $(\hat{f}, f)$ an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

**Example 2.11.** $TS^1, TS^2$ are both trivial.

$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$

**Theorem 2.12.** *If $G$ is a Lie group, then $TG$ is trivial.*

*Proof.* For $(x^1, x^2, \cdots, x^n)$ is a basis of $T_eG$ The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG,\ (g, c^1, \cdots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \to G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism $(l_g)_*$

□

**Proposition 2.13** (Adams, 1960s). *$TS^n$ is trivial if and only if $n = 0, 1, 3, 7$.*

**Proposition 2.14.**

1. *Given any $F \in C^\infty(M, N)$, $F_* : TM \to TN$ is a bundle map.*

2. *$TS^n$ is isomorphic to the following bundle:*

$$B = s^n \qquad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

**Definition 2.15** (smooth section)**.** Given a smooth vector bundle $\pi : E \to B$, a **smooth section** is a smooth map $S : B \to E \quad s.t. \ \pi \circ S = id_b$.

$s_0 : B \to E, b \mapsto 0 \in$ 0-vector in $\pi^{-1}b$ .

## 2.3 Vector Field, Curves and Flows

**Definition 2.16.** A (tangent) **vector field** is a smooth section of $TM$. *i.e.* a smooth map $M \xrightarrow{X} TM \quad s.t. \ X(p) \in T_pM, \forall p \in M$

Given any $f : \mathbb{R}^n \to \mathbb{R}$, define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leqslant i \leqslant n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

**Example 2.17.** $X = f^1 \partial x^1 + f^2 \partial x^2$ is a gradient field if and only if $\dfrac{\partial f^1}{\partial x^2} = \dfrac{\partial f^2}{\partial x^1}$

**Theorem 2.18** (Poincare-Hopf). *For closed $M$, $M$ has a nowhere vanishing vector field if and only if $\chi(M) = 0$.*

So $S^n$ has a nowhere vanishing vector field if and only if $n$ is odd.

**Theorem 2.19** (MaoQiu). *$S^2$ has no no-where vanishing vector field.*

So We cannot smooth out all the hairs on a ball.

Given a vector field $X = \{X_p\}_{p \in M}$, a curve $\gamma : (a, b) \to M$ is called an **integral curve** of $X$ if $\gamma'(t) = X_{\gamma(t)}$, $\forall t \in (a, b)$, where $\gamma'(t) = \gamma_*(\dfrac{\partial}{\partial t}) \in T_{\gamma(t)}M$.

We say $\gamma$ is maximal if the domain cannot be extended to a larger interval. Denote the set of all smooth vector fields on $M$ by $\mathfrak{X}M$

Recall that $\gamma$ is maximal if it's domain can not be extended to a large open interval.

In a local chart $(U, x^1, \cdots, x^n)$, $X|_U = \sum_{i=1}^{n} a^i \partial x^i$. Then $\gamma$ is an integral curve if and only if $(\gamma^i)'(t) = a^i(\gamma(t))$, $\forall 1 \leqslant i \leqslant n$, where $\gamma^i = x^i \circ \gamma : (a, b) \to \mathbb{R}$.

And in this case the initial value condition: $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$.

Locally, solving integral curve starting at $p$ is equivalent to solving ODE with initial value $p^1, \cdots, p^n$. By existence and uniqueness of solutions of ODE, we have

**Theorem 2.20** (Fundamental theorem of integral curve). *Let $X \in \mathfrak{X}M$, $p \in M$, then:*

*(1) (Uniqueness) Given any two integral curves $\gamma_1, \gamma_2 : (a, b) \to M$, then we have:*

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a, b) \ \Rightarrow \gamma_1 = \gamma_2$$

*(2) there exists a unique max integral curve $\gamma : (a(p), b(p)) \to M$ starting at $p$.*

*(3) (integral curve smoothly depend on initial values) $\exists$ Nbh $U$ of $p$, $\varepsilon > 0$, and smooth $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ s.t. $\forall q \in U$, $\varphi_\varepsilon := \varphi(-, q) : (-\varepsilon, \varepsilon) \to M$ is an integral*

*curve starting at $q$.*

*we call such $\varphi$ a local **flow** generated by $X$.*

**Definition 2.21.** Given $X \in \mathfrak{X}M$, a global **flow** generated by $X$ is a smooth map $\varphi : \mathbb{R} \times M \to M \quad s.t.\ \forall q \in M$, $\varphi_q := \varphi(-, q)$ is the maximal integral curve of $X$ starting at $q$.
$\Leftrightarrow \dfrac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s,p)}, \forall s \in \mathbb{R}, p \in M$ and $\varphi(0, p) = p, \forall p \in M$.

If such global flow exists, then we say $X$ is **complete**.

**Example 2.22.**

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$ is complete, where global flow $\varphi : \mathbb{R} \times M \to M$, $\varphi(t, p) = p \cdot e^t$.
- $X = x^2 \partial x$ is not complete. Max integral curve starting at $1$ is given by $\gamma(t) = \dfrac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$.

Given $X \in \mathfrak{X}M$, we define $\mathrm{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$.

**Theorem 2.23.** *If a vector field $X$ is compactly supported, then $X$ is complete.*

**Corollary 2.24.** *Any vector field on closed manifold is complete.*

**Lemma 2.25** (Escaping lemma). *Suppose $\gamma : (a, b) \to M$ is a max integral curve, with $(a, b) \neq \mathbb{R}$. Then $\nexists$ compact $K \subset M \quad s.t.\ \gamma(a, b) \subset K$*

*Proof.* Otherwise, suppose $\gamma(a, b) \subset K$. WLOG, we may assume $b < +\infty$.

Take $(t_i) \to b$ from left. Then $\gamma(t_i) \in K$. After passing to subsequence, we may assume $(\gamma(t_i)) \to p \in K$.

Then $\exists\ U$ Nbh of $p$, local flow $\varphi : (-\varepsilon, \varepsilon) \times U \to M$. Take $n$ large enough $\quad s.t.$ $b - t_n < \varepsilon, \gamma(t_n) \in U$. Then $\gamma(- + t_n) : (a - t_n, b - t_n) \to M$, $\varphi(-, \gamma(t_n)) : (-\varepsilon, \varepsilon) \to M$ are both integral curves for $X$ starting at $\gamma(t_n)$. By uniqueness, they coincide.

Let $\hat{\gamma} : (a, t_n + \varepsilon) \to M$ be defined by $\hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \varepsilon) \end{cases}$

Then $\hat{\gamma}$ is an integral curve with larger domain, then $\gamma$ contradiction with the maxity of $\gamma$. □

*Proof of 2.23.* Take any max integral curve $\gamma : (a, b) \to M$. Suppose $(a, b) \neq \mathbb{R}$. Then $X_{\gamma(s)} \neq 0$, $\forall s$. Otherwise, the constant map $\mathbb{R} \to M, t \mapsto \gamma(s)$ is an integral curve with lager domain.

So $\forall s$, $\gamma(s) \in \mathrm{Supp}X \Rightarrow \gamma(a, b) \subset \mathrm{Supp}X$ which is compact $\Rightarrow (a, b) = \mathbb{R}$ by the lemma. This causes contradiction! □

A smooth $\varphi : \mathbb{R} \times M \to M$ is called an **one-parameter transformation group** if

(1) $\varphi_0 := \varphi(0, -) = \mathrm{id}_M$

(2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}$. In particular, $\varphi_s^{-1} = \varphi_{-s}$.

**Theorem 2.26.** *$\varphi \in C^\infty(\mathbb{R} \times M, M)$, then $\varphi$ is an one-parameter transformation group if and only if $\varphi$ is the global flow generated by some $X \in \mathfrak{X}M$*

**Lemma 2.27** (Translation lemma). *If $\gamma : (a, b) \to M$ is an integral curve for some $X \in \mathfrak{X}M$, then $\forall s \in \mathbb{R}$, $\gamma(- + s) : (a - s, b - s) \to M$ is also an integral curve for $X$.*

*Proof.* Let $\iota = \gamma(- + s)$. Then $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$ □

**Lemma 2.28.** *Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$ provided that $s, t, s + t \in (-\varepsilon, \varepsilon), p, \varphi_r(p) \in U$.*

*Proof.* $\gamma_p = \varphi(-, p)$ is an integral curve for $X$.

$\Rightarrow \gamma_p(- + s)$ is an integral curve for $X$ starting at $\gamma_p(s) = \varphi_s(p)$. But $\gamma_{\varphi_s(p)}$ is also an integral curve starting at $\varphi_s(p)$. Thus $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$ □

**Lemma 2.29.** *Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$ i.e. any vector field is invariant under its flow.*

*Proof.* Take $f \in C^\infty_{\varphi(p)}(M)$.

$$\begin{aligned}\varphi_{s,*}(X_p)(f) &= X_p(f \circ \varphi_s) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \\ &= X_{\varphi_s(p)}(f)\end{aligned}$$

□

*Proof of 2.26.* "$\Leftarrow$" is because the lemma $\varphi_s \circ \varphi_r = \varphi_{s+r}$

"$\Rightarrow$" Let $X = \{X_p\}$ where $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$.

Leave it as an exercise. □

**Time dependent** vector field is a smooth map $X : \mathbb{R} \times M \to TM$ *s.t.* $X_{(t,p)} \in T_pM$.

A smooth curve $\gamma(a, b) \to M$ is the **integral curve** for $X$ if $\gamma'(t) = X_{(t,\gamma(t))}$.

In local chart, solving $\gamma$ is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness: $\gamma_1, \gamma_2$ are both integral curves for $X$, $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define $\mathrm{Supp}X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$.

Then $X$ is compactly supported, then $X$ is complete( *i.e.* a global flow $\varphi : \mathbb{R} \times M \to M$)

But something is not true for time dependent vector field:

- translation lemma is not true.
- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

## 2.4 Another definition of vector field

A derivation on $M$ is a $\mathbb{R}$-linear map $C^\infty(M) \xrightarrow{D} C^\infty(M)$ that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

**Theorem 2.30.** *We have a bijection:*

$$\rho : \mathfrak{X}M \xrightarrow{1:1} \{\textit{derivation on } M\}$$

$$X \mapsto D_X : f \mapsto X(f)$$

**Lemma 2.31.** $D_p : \mathfrak{X}_pM \to$ *$\mathbb{R}$-linear map* $C^\infty(M) \to \mathbb{R}$ *s.t.* $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$ *is an isomorphism of vector spaces.*

*Proof.* Leave it as an exercise. $\square$

**Lemma 2.32.** *Given a vector field(not necessarily smooth)* $X = \{X_p\}_{p\in M}$ *, X is smooth* $\Leftrightarrow \forall f \in C^\infty(M), X(f)$ *is smooth.*

*Proof.* "$\Leftarrow$" $\forall p \in M$, take chart $(U, x^1, x^2, \cdots, x^n)$ around $p$. $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$ $f^i : U \to \mathbb{R}$, where $f^i = X|_U(x^i)$. Take $\varphi : M \to [0,1]$ *s.t.* $\varphi \equiv 1$ near $p$, Supp$\varphi \subset U$,$\varphi \cdot x^i \in C^\infty(M)$.

Then $X(\varphi \cdot x^i) = f^i$ near $p$. By assumption, $f^i$ is smooth near $p$. So $f^i$ is smooth, so $X$ is smooth.

"$\Rightarrow$" Similar. $\square$

**Theorem 2.33.** *The map $\rho : \mathfrak{X}M \to \{\text{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$ is well-defined and bijective.*

*Proof.* $\rho$ is well-defined: $X(f) \in C^\infty(M)$ by Lemma 2.32, and $D_x(fg) = D_x(f)g + fD_x(g)$ since $X$ is a point-derivation.

$\rho$ is injective: $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$ as maps $C^\infty(M)$ to $\mathbb{R}$. By Lemma 2.31, we have $X_p = Y_p$, $\forall p$. So $X = Y$.

$\rho$ is surjective: Given $D : C^\infty(M) \to C^\infty(M)$. Define $D_p : C^\infty(M) \to \mathbb{R}$ by $D_p(f) := D(f)(p)$ satisfies the Leibniz rule. By Lemma 2.31, $D_p = D_{X_p}$ for some $X_p \in T_pM$. Define $X = \{X_p\}_{p\in M}$. Then $X(f) = D(f), \forall f \in C^\infty(M)$. By Lemma2.32, $X$ is a smooth vector field. $\square$

# 3 Lie group, Lie algebra and Lie bracket

## 3.1 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned}
\{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\
&= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \quad s.t. \\
&\qquad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\}
\end{aligned}$$

$$\begin{aligned}
\{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\
&= \{\text{derivation on } M\}
\end{aligned}$$

**Notation 3.1.** We will identify $X \in \mathfrak{T}M$ with its derivation $D_x : C^\infty(M) \to C^\infty(M)$. So a vector field is just a $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M) \quad s.t. \ X(fg) = fX(g) + X(f)g$.

**Definition 3.2** (Lie bracket)**.** Given two (smooth) vector field $X, Y : C^\infty(M) \to C^\infty(M)$, we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

**Theorem 3.3.** *For any $X, Y \in \mathfrak{T}M$, $[X, Y] \in \mathfrak{T}M$*

*Proof.* Easy to check that $[X, Y]$ is linear.

By Leibuniz rule,

$$\begin{aligned}
[X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\
&= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\
&= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\
&= [X, Y](f) \cdot g - f \cdot [X, Y](g)
\end{aligned}$$

□

So What is the geometric meaning of $[X, Y]$? Non commutatiy of flows.

**Fact 3.4.** *Given $X, Y \in \mathfrak{T}M$, we say $X, Y$ are commutative vector field if $[X, Y] = 0$*

*$X, Y$ are commuative iff for any local flows $\varphi^X : (-\varepsilon, \varepsilon) \times U \to M$, $\varphi^Y : (-\varepsilon, \varepsilon) \times U \to M$ we have $\varphi^X_s \circ \varphi^T_t = \varphi^Y_t \circ \varphi^X_s$*

**Proposition 3.5** (Calculation of $[V, W]$ using local charts)**.** *Chart* $(U, x^1, \cdots, x^n)$, $V, W \in \mathfrak{X}M$, $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$, $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$. *Then*

$$
\begin{aligned}
[V, W]|_U &= \sum_{i=1}^n (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^n (\sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i} \\
&= \sum_{1 \leqslant i,j \leqslant n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i}
\end{aligned}
$$

**Example 3.6.** $V = x\partial x + y\partial y$, $W = -y\partial x + x\partial y$ commutes.

**Proposition 3.7** (Properties of Lie bracket)**.**

*(a) Natuality under push-forword.*

*Given any* $F \in \mathrm{Diff}(M, N)$*,* $V \in \mathfrak{X}M, W \in \mathfrak{X}M$*, we have* $[F_*V, F_*W] = F_*[V, W]$*.*

*(b)* $\mathbb{R}$*-linearity* $\forall a, b \in \mathbb{R}$

$$[aX + bV, W] = a[X, W] + b[V, W]$$

$$[W, aX + bV] = b[W, X] + a[W, V]$$

*(c) anti-symmetric* $[V, W] = -[W, V]$

*(d) Jacobi identity*

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

(f) *Leibuniz rule*

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

**Definition 3.8.** Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}N$. We say $W$ is $F$**-related** to $V$ if $\forall p \in M$, $F_{p,*}(V_p) = W_{F(p)}$ where $F_{p,*} : T_pM \to T_{f(p)}N$

**Example 3.9.** $F : S^1 \to \mathbb{R}^2, \theta \mapsto (\cos\theta, \sin\theta)$, $V = \partial\theta, W = -y\partial x + x\partial y$.

*Note* 1. In general, given $V \in \mathfrak{X}M$ and $F \in C^\infty(M, N)$. There may not exist $W \in \mathfrak{X}M \quad s.t.\ V, W$ are $F$-related. Even such $W$ exists, it may not be unique.

However, if $F$ is a diffeomorphism, given any $V$, $\exists$ unique $W \quad s.t.\ V$ and $W$ are $F$-related. Actually, $W_p = F_* V_{F^{-1}(p)}$.

Such $W$ is called **push forward** of $V$ along $F$, denoted by $F_*V$, only defined when $F$ is a diffeomorphism.

**Lemma 3.10.** $\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N$, $F \in C^\infty(M, N)$. *Then $W$ is $F$-related to $V$ iff* $\forall f \in C^\infty(N)$, $V(f \circ F) = W(f) \circ F \in C^\infty(M)$

*Proof.* Check that $F_{p,*}(V_p)(f) = W_{F(p)}(f), \forall f \in C^\infty(N)$ □

**Proposition 3.11.** *Given $V_0, V_1 \in \mathfrak{X}M$, $W_0, W_1 \in \mathfrak{X}N$, $F \in C^\infty(M, N)$, $W_i$ is $F$-related to $V_i$, $i = 0, 1 \Rightarrow [W_0, W_1]$ is $F$-related to $[V_0, V_1]$*

**Corollary 3.12** (Naturality of Lie bracket)**.** *Given any $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, we have $[F_*V, F_*W] = F_*[V, W]$*

The rest of Proposition 3.7 is easy to check if it is viewed as a mapping $C^\infty(M) \to C^\infty(M)$.

## 3.2 Lie algebra of a Lie group

**Definition 3.13.** A **Lie algebra** $g$ is $\mathbb{R}$-linear space $g$ with map $[-,-] : g \times g \to g$ *s.t.* it is bilinear, anti-symmetric and satisfies the Jacobian identity.

Then $(\mathfrak{X}M, [-,-])$ is an infinite dimensional Lie algebra.

For $G$ Lie group, $\forall g \in G$ we have diffeomorphism

$$l^g : G \to G, h \mapsto gh$$
$$r^g : G \to G, h \mapsto hg$$

We say $X \in \mathfrak{X}G$ is **left invariant** if $l_*^g(X) = X$, $\forall g \in G$. Similarly, $X$ is **right invariant** if $r_*^g(X) = X$.

**Proposition 3.14.** *$X, Y$ are left/right invariant $\Rightarrow [X, Y]$ is left/right invariant.*

*Proof.* $l_*^g[X, Y] = [l_*^g X, l_*^g Y] = [X, Y]$ □

So we can find a natural Lie algebra of $G$:

$$\mathrm{Lie}(G) := \{\text{left invariant vector fields on } G\}, \text{with } [-,-] \text{ restricted from } \mathfrak{X}G$$

**Theorem 3.15.** *Given any $V \in T_eG$, $\exists$ unique left invariant $\hat{V} \in \mathfrak{X}G$ s.t. $\hat{V}_e = V$.*

**Corollary 3.16.** $\mathrm{Lie}(G) \cong T_eG$ *as vector spaces.*

*Proof of Theorem 3.15.*

**Uniqueness of $\hat{V}$:** $\hat{V}_g = l_{e,*}^g(\hat{V}_e) = l_{e,*}^g(V)$. So $\hat{V}$ is determined by $V$.

**Existence of $\hat{V}$:** Let $\hat{V} = \{\hat{V}_g\}_{g \in G}$ where $\hat{V}_g = l_{e,*}^g(\hat{V}_e)$.

$\hat{V}$ is left-invariant because

$$(l^h_*(\hat{V}))_g = l^h_{h^{-1}g,*}(\hat{V}_{h^{-1}g}) = l^h_{h^{-1}g,*}(l^{h^{-1}g}_{e,*}(V)) = l^g_{e,*}(V) = \hat{V}_g$$

$\hat{V}$ is smooth: Take any $f \in C^\infty(G)$ suffices to show $\hat{V}(f) \in C^\infty(G)$.

Take any smooth $\gamma : \mathbb{R} \to G \quad s.t.\ \gamma(0) = e, \gamma'(0) = V$. Then $l^g \circ \gamma : \mathbb{R} \to G$ satisfies $l^g \circ \gamma(0) = g, (l^g \circ \gamma)(0) = g, (l^g \circ \gamma)'(0) = l^g_{e,*}(V) = \hat{V}_g$

So

$$\hat{V}(f)(g) = \hat{V}_g(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(l^g \circ \gamma(t))|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(g \cdot \gamma(t))|_{t=0} \tag{3.1}$$

Consider the map

$$\hat{f} : G \times \mathbb{R} \xrightarrow{\mathrm{id}\times\gamma} G \times G \xrightarrow{\cdot} G \xrightarrow{f} \mathbb{R}$$

$$(g, t) \mapsto (g, \gamma(t)) \mapsto g \cdot \gamma(t) \mapsto f(g \cdot \gamma(t))$$

Then $\hat{f}$ is smooth, $\frac{\partial \hat{f}}{\partial t}|_{t=0} : G \to \mathbb{R}$ is smooth, but $\frac{\partial f}{\partial t}|_{t=0}(g) = \hat{V}(f)(g)$ by 3.1. So $\hat{V}(f) \in C^\infty(G)$. $\square$

**Example 3.17.** $G = \mathrm{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det A \neq 0\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^2$.

$\mathrm{gl}(n, \mathbb{R})$= $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) = T_I \mathrm{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$

**Theorem 3.18.** $\forall A, B \in \mathrm{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$, $[A, B] = AB - BA$.

**Remark 3.19.** This theorem shows that the Lie bracket viewed as the Lie algebra and matrix are the same. In some sense, it means the Lie bracket defined in three sets $\mathrm{gl}(n, \mathbb{R}) = T_I \mathrm{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$ can commute with those corresponding, or equivalently, are just the same.

**Lemma 3.20.** $\forall A \in \mathrm{gl}(n, \mathbb{R})$, *the left invariant vector field $\hat{A}$ is complete and generate the flow* $\varphi_t : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2t^2}{2!} + \cdots)$

*Proof.*

$$\hat{A}_g = g \cdot A \in T_g G = M_n(\mathbb{R})$$

$$\frac{\partial}{\partial t}\varphi_t(g) = \frac{\partial}{\partial t}(g(e^{At})) = ge^{At}A = \hat{A}_{g\cdot e^{At}} = \hat{A}_{\varphi_t(g)}$$

$\square$

**Remark 3.21.** This lemma tells how to compute $A(f) = \hat{A}(f)(I)$ as a tangent vector or a vector field, as we will see in the next proof.

*Proof of Theorem 3.18.* Take $A, B \in \mathrm{gl}(n, \mathbb{R})$. Want to show $[\hat{A}, \hat{B}]_I = AB - BA$.

Pick $f \in C_I^\infty(G)$, need to show $A(\hat{B}(f)) - B(\hat{A}(f)) = (AB - BA)(f)$

Further Simplification: Just need to focus on $f = x^{ij}$, where $x^{ij} : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}, E \mapsto (E - I)_{ij}$. Actually, $\partial x^{ij}$ is what we choose as a basis of $T_I \mathrm{GL}(n, \mathbb{R})$.

Such $f$ satisfies $f(I + -)$ is $\mathbb{R}$-linear.

Recall that Given $W \in \mathfrak{X}M$, $W(f)(p) = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t^W(p))|_{t=0}$.

So $\hat{B}(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(ge^{Bt})|_{t=0}$.

So since $A(\hat{B}(f)) = \hat{A}((\hat{B})(f))(I) = \frac{\mathrm{d}}{\mathrm{d}s}(\hat{B}(f)(e^{As}))|_{s=0}$,

$$A(\hat{B}(f)) = \frac{\mathrm{d}}{\mathrm{d}s}(\hat{B}(f)(e^{As}))|_{s=0} = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t} f(I + sA + tB + \frac{s^2}{2}A^2 + stAB + \frac{t^2}{2}B^2 + \cdots)|_{s=t=0}$$

Similarly,

$$B(\hat{A}(f)) = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t} f(I + sA + tB + \frac{s^2}{2}A^2 + stBA + \frac{t^2}{2}B^2 + \cdots)|_{s=t=0}$$

So $A(\hat{B}(f)) - B(\hat{A}(f)) = f(I + (AB - BA)) = (AB - BA)(f)$ since $f$ is $\mathbb{R}$-linear. $\square$

Similarly, for $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{Lie}(G) = \mathrm{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$, we have $[A, B] = AB - BA$.

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group $G$, its universal cover $\hat{G}$ is simply-connected with $\pi^{-1}(G) \subset Z(\hat{G})$.

What is the meaning of Lie bracket. There is a fact about it:

**Fact 3.22.** *$G$ is connected Lie group. $G$ is abelian iff* $[-,-] = 0$ *on* $\mathrm{Lie}(G)$

## 3.3 Morphisms between Lie group and Lie algebras

A smooth map $F : G \to H$ between two Lie group is called a **morphism** if $F(gh) = F(g)F(h)$.

A linear map $L : g \to h$ between Lie algebra is called a **morphism** if $L[u, v] = [Lu, Lv]$.

**Proposition 3.23.** *Let $F : G \to H$ be a morphism of Lie groups. Then $F_{e,*} : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a morphism of Lie algebra.*

*Proof.* $V_0, V_1 \in \mathrm{Lie}(G) = T_eG$, $W_i = F_{e,*}(V_i) \in \mathrm{Lie}(H) = T_eH$. Let $\hat{V}, \hat{W}$ be left-invariant vector fields.

*Claim.* $\hat{W}_i$ is $F$-compatible with $\hat{V}_i$ for $i = 0, 1$.

*Proof of Claim.* $\forall g \in G$, $F_*(\hat{V}_g) = F_*(l^g_*(V)) = (F \circ l^g)_*(V) = (l^{F(g)} \circ F)_*(V) = l^{F(g)}(W) = \hat{W}_{F(g)}$ □

So $[\hat{W}_0, \hat{W}_1]$ is $F$-compatible with $[\hat{V}_0, \hat{V}_1]$. In particular, $[W_0, W_1] = F_*([V_0, V_1])$. □

# 4 Vector Field

## 4.1 Canonical form of a field

Recall that $V \in \mathfrak{X}M$, $p \in M$ is called a **regular point** if $V_p \neq 0$, and is called a **singular point** if $V_p = 0$.

**Theorem 4.1** (Canonical Form Theorem). *Let $p$ be a regular point of $V$. Then $\exists$ local chart $(U, x^1, \cdots, x^n)$ around $p$ s.t. $V|_U = \partial x^1$*

*Proof.* This is a local problem. We may assume $M \subset \mathbb{R}^n$ open. We may also assume $p = 0, V_0 = \partial r^1|_0$ where $r^i$ coordinate function.

Let $\varphi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)^n \to M$ be the local flow of $V$.

Define $\psi : (-\varepsilon, \varepsilon)^n \to M$ by $\psi(t, r^2, \cdots, r^n) = \varphi(t, (0, r^2, \cdots, r^n))$. Then $\psi(-, r^2, \cdots, r^n)$ is an integral curve for $V$. Therefore, $\psi_*(\partial t) = V$.

At $\vec{0}$, we have $\psi_{\vec{0},*}(\partial t) = V_{\vec{0}} = \partial r^1$, $\psi_{\vec{0},*}(\partial r^i) = \partial r^i$.

So $\psi_{*,\vec{0}} : T_{\vec{0}}(-\varepsilon, \varepsilon)^n \to T_{\vec{0}}M$ is an isomorphism.

By the inverse function theorem, $\exists U' \subset (-\varepsilon, \varepsilon)^n$, $U \subset M$ s.t. $\psi|_{U'} : U' \to U$ is a diffeomorphism.

Then $(U, (\psi|_{U'})^{-1})$ is the local chart what we need. $\square$

**Remark 4.2.** Regular point in a vector field is simple, as we can view it in the standard chart locally. However, behavior of $V$ art a singular point can be complicated. For example, for $f(x, y) = x^2 - y^2$, $\nabla f = 2x\partial x - 2y\partial y$, $g : \mathbb{C} \to C, z \mapsto z^n$, they behave differently at $\vec{0}$.

## 4.2 Lie derivative of vector field

$V, W \in \mathfrak{X}M$, $\mathcal{L}_V W$ is the directional derivative of $W$ in the direction of $V$.

**Definition 4.3.** The **Lie derivative** $\mathcal{L}_V W \in \mathfrak{X}M$ is defined as follows: $\forall p \in M$, let $\{\theta_t : U \to M\}_{t\in(-\varepsilon,\varepsilon)}$ be the local flow for $V$. Then

$$(\mathcal{L}_V W)_p = \lim_{t\to 0} \frac{(\theta_{-t})_* W_{\theta_t(p)} - W_p}{t}$$

**Remark 4.4.** This defintion is actually a difference between $T_{\theta_t(p)}$ and $T_p$, which need pullback.

**Lemma 4.5.** *$\mathcal{L}_V W$ is well-defined and smooth.*

*Proof.* For $p \in M$, take local chart $(U, x^1, \cdots, x^n)$. Let $\theta : (-\varepsilon, \varepsilon) \times U \to M$ be the flow of $V$. Take $J_0 \subset (-\varepsilon, \varepsilon)$, $U_0 \subset U$. Let $\theta^i = x^i \circ \theta : J_0 \times U_0 \to \mathbb{R}$, $W|_U = \sum_{i=1}^n W^i \partial x^i$.

Under the basis $\{\partial x^i\}$, $(\theta_{-t})_* : T_{\theta_t(p)}M \to T_pM$ is represented by

$$\left(\frac{\partial \theta^i(-t, \theta(t,x))}{\partial x^j}\right)_{i,j}$$

So $(\theta_{-t})_* W_{\theta_t(x)} = \sum_{i,j} \frac{\partial \theta^i(-t,\theta(t,x))}{\partial x^j} W^j(\theta(t,x)) \cdot \partial x^i$ is smooth in $t, x$. So

$$(\mathcal{L}_V W)_x = \frac{\partial((\theta_{-t})_*(W_{\theta_t(x)}))}{\partial t}|_{t=0}$$

is well-defined and smooth. □

**Theorem 4.6.** *For all $V, W \in \mathfrak{X}M$, $\mathcal{L}_V W = [V, W]$.*

*Proof.* For $p$ is a regular point of $V$. By canonical form theorem 4.1, $\exists$ local chart $(U, x^1, \cdots, x^n)$ around $p$ *s.t.* $V|_U = \partial x^1$. Let $W|_U = \sum_{i=1}^n W^i \partial x^i$.

Then $\theta_t(x_1, \cdots, x_n) = (x_1 + t, x_2, \cdots, x_n)$. So

$$\mathcal{L}_V W|_U = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

.

$$[V,W]|_U = \sum_i V(W^i)\partial x^i - \sum_i W(V^i)\partial x^i = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

Then $[V,W]|_U = \mathcal{L}_V W$.

For $p$ is a singular point but $p \in \mathrm{Supp}(V)$. Then $\exists\, p_i \to p \quad s.t.\ V_p \neq 0$. By the previous case $(\mathcal{L}_V W)_{p_i} = [V,W]|_{p_i}$. By continuity, We have $(\mathcal{L}_V W)_p = [V,W]_p$.

For $p \notin \mathrm{Supp}(V)$, $\exists$ Nbd $U$ of $p \quad s.t.\ V|_U = 0$. Then $\theta_t(q) = q$. So

$$(\mathcal{L}_V W)|_U = 0 = [V,W]|_U$$

$\square$

**Corollary 4.7.**

- *$\mathcal{L}_V W$ is $\mathbb{R}$-linear with respect to $V, W$.*
- *$\mathcal{V}W = -\mathcal{L}_W V$.*
- *$\mathcal{L}_V[W,X]$.*
- *(Jacobian identity) $\mathcal{L}_V[W,X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.*
- *(Jacobian identity) $\mathcal{L}_{[V,W]}X = \mathcal{L}_V\mathcal{L}_W X - \mathcal{L}_W\mathcal{L}_V X$.*
- *$\mathcal{L}_V(fW) = (Vf)\cdot W + f\mathcal{L}_V W$*
- *Let $F : M \to N$ be a diffeomorphism. Then $F_*(\mathcal{L}_V W) = \mathcal{L}_{F_*(V)}F_*(W)$.*

## 4.3 Commuting vector fields

**Definition 4.8.** We say $V, W \in \mathfrak{T}M$ **commutes** if $[V,W] = 0$.

**Theorem 4.9.** *TFAE:*

*1 $V, W$ commutes.*

*2 $W$ is invariant under the flow generated by $V$, i.e. $\theta_{t,*}(W_p) = W_{\theta_t(p)}$*

*3 The flow for $V, W$ commutes, i.e. $\theta_t \circ \eta_s = \eta_s \circ \theta_t$ whenever either side is defined or equivalently, whose the domain is compatible.*

**Lemma 4.10.** *Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}N$. Then $W$ is $F$-related to $V$ if and only if $\forall t \in \mathbb{R}$, $\eta_t \circ F = F \circ \theta_t$ on the domain of $\theta_t$, which means*

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow{\scriptstyle \theta_t} & & \downarrow{\scriptstyle \eta_t} \\ M & \xrightarrow{F} & N \end{array} \text{ commutes.}$$

*Proof.* "$\Rightarrow$" Let $\gamma = F \circ \theta^p : J \to N$ satisfies

$$\gamma'(t) = (F \circ \theta^p)'(t) = F_*((\theta^p)'(t)) = F_*(V_{\theta^p(t)}) = W_{F(\theta^p(t))} = W_{\gamma(t)}$$

So $\gamma$ is an inetgral curve of $W$ starting at $\gamma(0) = F(p)$ *i.e.* $F \circ \theta^p = \gamma(t) = \eta^{F(p)}(t)$ *i.e.* $F \circ \theta_t = \eta \circ F$.

"$\Leftarrow$" Suppose $F \circ \theta_t = \eta \circ F$. Then $(F \circ \theta^p)(t) = \eta^{F(p)}(t)$.

Then $F_* V_p = F_*((\theta^p)'(0)) = (F \circ \theta^p)'(0) = (\eta^{F(p)})'(0) = W_{F(p)}$. So $W$ is $F$-related to $V$. $\square$

*Proof of Theorem 4.9.* $2 \Rightarrow 1$: $(\theta_{-t})_*(W_{\theta_t(p)}) = W_p$. So

$$\mathcal{L}_V W = \lim_{t \to 0} \frac{(\theta_{-t})_*(W_{\theta_t(p)}) - W_p}{t} = 0$$

$1 \Rightarrow 2$: Let $X(t) = (\theta_{-t})_*(W_{\theta_t(p)})$, $p \in M$.

Want to show that $X(t) = X_p$ for all $t$. Suffices to show $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} X(t) = 0$.

For $t_0 = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} X(t) = (\mathcal{L}_V W)_p = 0$.

In general, set $s = t - t_0$, $X(t) = (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))})$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} X(t) &= \frac{\mathrm{d}}{\mathrm{d}s}|_s X(s+t_0)\\ &= \frac{\mathrm{d}}{\mathrm{d}s}|_s (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))})\\ &= (\theta_{t_0})_* \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))})\\ &= (\theta_{t_0})_* (\mathcal{L}_V W)_{\theta_{t_0}(p)}\\ &= 0\end{aligned}$$

$2 \Rightarrow 3$. For simplicity, assume $V, W$ are complete. $F = \theta_s : M \to M$. By 2, $W$ is $F$-related to $W$. So by the lemma,

$$\begin{array}{ccc} M & \xrightarrow{F} & M \\ \Big\downarrow{\scriptstyle \theta_t} & & \Big\downarrow{\scriptstyle \eta_t} \\ M & \xrightarrow{F} & M \end{array} \quad \text{commutes.}$$

$\eta_t$ is flow for $W$. *i.e.* $\theta_s \circ \eta_t = \eta \circ \theta_s$

$3 \Rightarrow 2$ is similar. The diagram commutes, so $W$ is $F$-related to $W$. $\square$

""""""

### 4.3.1 Canonical form of commuting vector field

**Theorem 4.11.** *Given $V_1, \cdots, V_k \in \mathfrak{X}M$, s.t.*

*1) $[V_i, V_j] = 0, \forall i, j$.*

*2) $V_{1,p}, V_{2,p}, \cdots, V_{k,p}$ linearly independent at some $p \in M$*

*Then $\exists$ local chart $(U, x^1, \cdots, x^n)$ around $p$ s.t. $V_i|_U = \dfrac{\partial}{\partial x^i}, \forall 1 \leqslant i \leqslant k$*

We prove it using the inverse function theorem.

*Proof.* This is a local problem. So we may assume $M \subset \mathbb{R}^m$ be open with coordinate function $r^i : M \to \mathbb{R}, 1 \leqslant i \leqslant m$.

After translation and linear transformation, we may assume $p = \vec{0}$, $V_{i,\vec{0}} = \left.\dfrac{\partial}{\partial x^i}\right|_{\vec{0}}, 1 \leqslant i \leqslant k$.

Take local flow $\{\theta_t^i : (-\varepsilon, \varepsilon)^m \to M\}_{t \in (-\varepsilon,\varepsilon)}$ for $V_i$.

Define $\psi : (-\varepsilon, \varepsilon)^k \times (-\varepsilon, \varepsilon)^{m-k} \to M$, $\psi(t^1, \cdots, t^k, r^{k+1}, \cdots, r^m) = \theta_{t_1}^1 \circ \theta_{t_2}^2 \cdots \circ \theta_{t_k}^k(0, 0, \cdots, 0, r^{k+1}, \cdots, r^m)$, where $\theta^i$ commutes with each other.

So if we fix $t^j, j \neq i$ except $t^i$, $\psi(t^1, \cdots, t^{i-1}, -, t^{i+1}, \cdots, t^k, r^{k+1}, \cdots, r^m)$ is an integral curve for $V^i$. Then $V^i$ is $\psi$-related to $\partial t^i$.

On the other hand. $\psi(0, 0, \cdots, 0, r^{k+1}, \cdots, r^m) = (0, 0, \cdots, 0, r^{k+1}, \cdots, r^m)$. So $\psi_{\vec{0},*} : T_{\vec{0},*} : T_{\vec{0}}(-\varepsilon', \varepsilon')^m \to T_{\vec{0}}M, \partial t^i \mapsto V_{i,0} = \partial x^i|_0$ and $\partial r^i \mapsto \partial r^i, k+1 \leqslant i \leqslant m$.

So $\psi_{\vec{0},*}$ is an isomorphism.

By the inverse function theorem, there exists Nbh $U' \subset (-\varepsilon', \varepsilon')^m$ *s.t.* $\psi : U' \to U$ is a diffeomorphism and $U \subset M$ open.

Then $(U, (\psi|_U)^{-1})$ is the local chart we need. □

## 4.4 The constant rank theorem

$F \in C^\infty(M, N)$, $p \in M$. The **rank** of $F$ at $p$ is

$$\begin{aligned}\operatorname{rank}_p F &:= \operatorname{rank}(F_{p,*} : T_pM \to T_{F(p)}N) \\ &= \operatorname{rank}\left(\frac{\partial F^i(p)}{\partial x^j}\right)_{i,j}\end{aligned}$$

We say $F$ has **constant rank** $k$ near $p$ if $\exists$ Nbh $U$ of $p$ *s.t.* $\operatorname{rank}_q F = k$, $\forall q \in U$

**Proposition 4.12.**

$$\operatorname{rank}_q(F) \leqslant \min(\dim(M), \dim(N))$$

**Theorem 4.13** (The constant rank theorem)**.** *Suppose $F : M \to N$ has constant rank $k$ near $p \in M$, then $\exists$ local charts $U \xrightarrow[\cong]{\varphi} \mathbb{R}^m$ around $p$, $V \xrightarrow[\cong]{\psi} \mathbb{R}^n$ around $F(p)$ s.t.*

$$\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \text{ is given by } (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^k, 0, \cdots, 0)$$

*Proof.* This is a local problem. So we may assume $M = \mathbb{R}^m, N = \mathbb{R}^n$ by restricting to local charts. And $p = 0, F(p) = 0$. After changing orders of coordinates, may assume $\left(\frac{\partial F^i}{\partial x^j}(0)\right)_{1 \leqslant i,j \leqslant k}$ is invertible. Write $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}, \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Then $F(x, y) = (Q(x, y), R(x, y))$. Consider $\varphi : \mathbb{R}^m \to \mathbb{R}^m, (x, y) \mapsto (Q(x, y), y)$. Then

$$\varphi_{(0,0),*} = \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}(0) & 0 \\ \\ \frac{\partial Q^i}{\partial y^j}(0) & I_{m-k} \end{bmatrix} \tag{4.1}$$

is invertible.

By inverse function theorem, $\exists$ Nbh $U_0 \subset \mathbb{R}^m, \tilde{U}_0 \subset \mathbb{R}^m$ of $0$ *s.t.* $\varphi : U_0 \to \tilde{U}_0$ is a diffeomorphism.

$$\tilde{U}_0 \underset{\varphi}{\overset{\varphi^{-1}}{\rightleftarrows}} U_0 \xrightarrow{F} \mathbb{R}^n$$

$$(Q(x, y), y) \leftarrow\!\shortmid (x, y) \mapsto (Q(x, y), R(x, y))$$

So $F \circ \varphi^{-1} : \tilde{U}_i \to \mathbb{R}^n, (x, y) \mapsto (x, A(x, y))$. And

$$(F \circ \varphi^{-1})_{p,*} = \begin{bmatrix} I_k & 0 \\ \\ \frac{\partial A}{\partial x}(p) & \frac{\partial A}{\partial y}(p) \end{bmatrix} \tag{4.2}$$

Since $\operatorname{rank}(F \circ \varphi^{-1})$ is k, $\dfrac{\partial A}{\partial y}(p) = 0$. *i.e.* $A(x, y) = A(x)$.

We can find a map $\psi : (x, y) \mapsto (x, y - A(x))$ in a smaller neighborhood of $0$ by the inverse theorem similarly.

And $\psi \circ F \circ \varphi$ maps $(x, y)$ to $(x, 0)$. So we end the proof. $\square$

**Definition 4.14.** $F \in C^\infty(M, N)$.

We say $F$ is **submersion** if $F_{p,*}$ is surjective $\forall p \in M$.

We say $F$ is **immersion** if $F_{p,*}$ is injective $\forall p \in M$.

We say $F$ is **embedding** if $F$ is immersion and $F$ is a topological embedding.(i.e. $F : M \to F(M)$ is a homeomorphism)

If $F$ is embedding(immersion resp.), we say $M$ or $F(M)$ is an **embedded submanifold(immersed submanifold**, resp.) of $N$.

Denote $M \looparrowright N$ be the immersion. $M \hookrightarrow N$ be the embedding.

**Example 4.15.**

- There is an example $F : S^1 \to \mathbb{R}^2$ where $F$ is an immersion but not an embedding.
- Projection $M \times N \to M$ is a submersion.
- $E \xrightarrow{p} B$ is a smooth vector bundle, then $p$ is a submersion.
- $\gamma : \mathbb{R} \to M$ is an immersion $\Leftrightarrow \gamma'(t) \neq 0$, $\forall t$.
- There is an example $\gamma : \mathbb{R} \to \mathbb{R}^2$ is injective immersion but not an embedding
- $\gamma : \mathbb{R} \to \mathrm{T}^2 = \mathbb{R}/_{\mathbb{Z}} \times \mathbb{R}/_{\mathbb{Z}}$, $x \mapsto (x, cx)$, $c \notin \mathbb{Q}$ is injective immersion but not embedding.

**Definition 4.16.** For $F : X \to Y$, we say $F$ is **proper** if for any compact set $K \subset N$, $F^{-1}(K)$ is compact.

**Lemma 4.17.** *$X$ is compact, $Y$ Hausdorff, then $F : X \to Y$ is proper.*

**Proposition 4.18.** *$F \in C^\infty(M, N)$ is an injective immersion, and $F$ is proper. Then $F$ is an embedding.*

*Proof.* $F : M \to F(M)$ is a closed map. □

**Definition 4.19.** For $F \in C^\infty(M, N)$.

$p \in M$ is called **regular point** if $F_{p,*} : T_pM \to T_{F(p)}N$ is surjective.

$p \in M$ is called **critical point** if $F_{p,*} : T_pM \to T_{F(p)}N$ is not surjective.

$q \in N$ is called **regular value** if $\forall p \in F^{-1}(q)$, $p$ is a regular point.

$q \in N$ is called **critical value**(or **singular value**) if $\exists p \in F^{-1}(q)$, $p$ is a critical point.

**Theorem 4.20** (Sard)**.** *Singular value has measure $0$.*

*Proof.* We will not prove it in this lecture. □

**Theorem 4.21.** *$M$ is an embedded submanifold of $N$ if and only if $\forall p \in M \subset N$, $\exists$ local chart $(U, x^1, \cdots, x^n)$ around $p$ of $N$ s.t. $M \cap U = \{(x^1, \cdots, x^m, 0, \cdots, 0)\}$*

*Proof.* "$\Rightarrow$": $F : M \to N$ is embedding $\Rightarrow$ $F$ has constant rank $m$. Apply constant rank theorem near $p$, and we finish the proof of "$\Rightarrow$"

The converse is trivial.

□

**Theorem 4.22.** *$F \in C^\infty(M, N)$, $q$ is a regular value of $F$. Then $F^{-1}(q)$ is an embedded submanifold of $M$. And*

$$\forall p \in F^{-1}(q), T_pF^{-1}(q) = \ker(F_{p,*} : T_pM \to T_{F(p)}N)$$

*Proof.* $q$ is regular value $\Rightarrow \mathrm{rank}_pF = n$, $\forall p \in F^{-1}(q)$.

$\Rightarrow \operatorname{rank}_{p'} F = n$, $\forall p'$ near $p$, since we know the rank of $p'$ near $p$ should not be less than that of $p$

So by the constant rank theorem, $F^{-1}(q)$ is a submanifold near $p$.

$\square$

Denote

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) | A + A^T = 0\}$$

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) | A + A^* = 0\}$$

$$\mathfrak{su}(n) = \{A \in u(n) | \mathrm{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \mathrm{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) | \mathrm{tr} A = 0\}$$

**Theorem 4.23.** *Those above sets are the Lie algebra of the corresponding Lie group. For instance,* $\mathfrak{su}(n) = \mathrm{Lie}(\mathrm{SU}(n))$.

# 5 Differential forms

## 5.1 Introduction

Our goal is to define the integration $\displaystyle\int_M \alpha$ *s.t.*

- Works for any smooth manifold $M$, without embedding $M$ into $\mathbb{R}^n$
- Generalize two types of surface integral, *i.e.* $\int_\Sigma f \mathrm{d}S$ and $\int_\Sigma f \mathrm{d}x \wedge \mathrm{d}y$

For Canton's idea, $\alpha$ is a "differential $k$-form" on $M$ *s.t.*

- $\forall F \in C^\infty(N, M)$, $F^*(\alpha)$ is a $k$-form on $N$

- If $k = \dim M$, then $\int_M \alpha \in \mathbb{R}$

## 5.2 Alternating vector linear algebra

For $V_1, \cdots, V_n, W$ be $\mathbb{R}$-vector spaces, $f : V_1 \times \cdots \times V_n \to W$ is called **multi $\mathbb{R}$-linear** if

$$\begin{aligned} f(v_1, \cdots, v_{i-1}, av_i + bv_i', v_{i+1}, \cdots, v_n) = & af(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_n) \\ & + bf(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_n) \end{aligned} \tag{5.1}$$

**Example 5.1.**

- Inner product $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\cdot} \mathbb{R}$.
- Matrix multiplication $M_{n\times m}(\mathbb{R}) \times M_{m\times k}(\mathbb{R}) \to M_{n\times k}(\mathbb{R})$.
- Cross product $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\times} \mathbb{R}^3$.
- Bilinear form.

We hope that we can construct a vector space $V_1 \otimes \cdots \otimes V_n \quad s.t.$ we have canonical isomorphism:

$$\{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V_n \to W\} \cong \{\text{linear map } V_1 \otimes \cdots \otimes V_n \to W\} \tag{5.2}$$

Then we can transform the study of multilinear algebra into the study of the normal linear algebra.

For any set $S$, let

$$\mathbb{R}\langle S\rangle = \left\{ \text{formal linear combination } \sum_{i=1}^{n} a_i s_i | a_i \in \mathbb{R}, s_i \in S, n < \infty \right\} \tag{5.3}$$

Consider $\mathbb{R}\langle V_1 \times \cdots \times V_n\rangle = \left\{\sum_{i=1}^{k} a^i(V_{i,1}, \cdots, V_{i,n}) | a^i \in \mathbb{R}, v_{i,j} \in V_j\right\}$. Denote

$$W = \mathrm{Span}\{(\cdots, av_i + bv_i', \cdots) - a(\cdots, v_i, \cdots) - b(\cdots, v_i', \cdots) | a, b \in \mathbb{R}, v_i, v_i' \in V_i\} \tag{5.4}$$

Define $V_1 \otimes \cdots \otimes V_n = \mathbb{R}\langle V_1 \times \cdots \times V_n\rangle /_W$, write $[(v_1, \cdots, v_n)]$ as $v_1 \otimes \cdots \otimes v_n$, called a $n$-**tensor**.

**Proposition 5.2** (Universal Property). *We have a multi $\mathbb{R}$-linear map $q : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$, $(v_1, v_2, \cdots, v_n) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$. It satisfies the universal property:*

*$\forall$ multi $\mathbb{R}$-linear map $f : V_1 \times \cdots \times V_n \to W$, $\exists$ unique linear map $\tilde{f} : V_1 \otimes \cdots \otimes V_n \to W$ s.t. $\tilde{f} \circ q = f$. i.e. The diagram commutes:*

$$\begin{tikzcd}
V_1 \otimes \cdots \otimes V_n \arrow[dr, "\exists ! \tilde{f}"] & \\
V_1 \times \cdots \times V_n \arrow[u, "\rho"] \arrow[r, "f"'] & W
\end{tikzcd}$$

**Corollary 5.3.**

$$\begin{gathered}\{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V_n \to W\} \cong \{\text{linear map } V_1 \otimes \cdots \otimes V_n \to W\} \\ f \leftrightarrow \tilde{f}\end{gathered} \tag{5.5}$$

**Proposition 5.4.**

- *Any element in $V_1 \otimes \cdots \otimes V_n$ can be written as $\sum_{i=1}^{n} a_i \otimes v_i$ for some $a_i \in \mathbb{R}$.*
- *If $(e_i^j)_{j \in \mathcal{A}_i}$ is a basis for $V_i$, then $\{e_1^{j_1} \otimes e_2^{j_2} \otimes \cdots \otimes e_n^{j_n} | j_i \in \mathcal{A}_i\}$ is a basis of $V_1 \otimes \cdots \otimes V_n$.*
- $\dim(V_1 \otimes \cdots \otimes V_n) = \prod_{i=1}^{n} \dim(V_i)$

**Proposition 5.5.** *Denote $W^* = \mathrm{Hom}(W, \mathbb{R})$, then we have an injection*

$$\begin{aligned} V \otimes W^* &\xrightarrow{e} \mathrm{Hom}(W, V) \\ v \otimes f &\mapsto (w \mapsto f(w) \cdot v) \end{aligned} \tag{5.6}$$

*If $\dim V$ or $\dim W$ is finite, then $e$ is an isomorphism.*

*Indeed, if $\dim V = \infty$, then $\mathrm{id}_V \notin e(V \otimes V^*)$*

Given any $l_i \in \mathrm{Hom}(V_i, W_i)$, $1 \leqslant i \leqslant n$, we define

$$\begin{aligned} l_1 \otimes \cdots \otimes l_n &\in \mathrm{Hom}(V_1 \otimes \cdots \otimes V_n, W_1 \otimes \cdots W_n) \\ (l_1 \otimes \cdots \otimes l_n)(v_1 \otimes \cdots \otimes v_n) &= l_1(v_1) \otimes \cdots \otimes l_n(v_n) \end{aligned} \tag{5.7}$$

**Proposition 5.6.** *If $\dim V_i < \infty$, $\forall 1 \leqslant i \leqslant n$, then we have isomorphism*

$$\begin{aligned} V_1^* \otimes \cdots \otimes V_n^* &\xrightarrow{\cong} (V_1 \otimes \cdots \otimes V_n)^* \\ f_1 \otimes \cdots \otimes f_n &\mapsto \left( (v_1 \otimes \cdots \otimes v_n \mapsto \prod_{i=1}^{n} f_i(v_i)) \right) \end{aligned} \tag{5.8}$$

For $\bigotimes_n V = \underbrace{V \otimes \cdots \otimes V}_{n}$, $S_n = \{\text{bijection on } \{1, 2, \cdots, n\}\}$ acts on $\bigotimes_n V$, where

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \tag{5.9}$$

A tensor $T \in \bigotimes_n V$ is called **symmetric** if $\sigma(T) = T$, $\forall \sigma \in S_n$.

$T$ is called **anti-symmetric** if $\sigma(T) = \mathrm{sgn}(\sigma) \cdot T$, $\forall \sigma \in S_n$.

Define

$$\begin{aligned} \mathrm{Sym}^n(V) &= \{\text{symmetric tensors in } \bigotimes_n V\} \\ \bigwedge\nolimits^n(V) &= \{\text{anti-symmetric tensors in } \bigotimes_n V\} \end{aligned} \tag{5.10}$$

which are both in $\bigotimes_n V$. And

$$\dim(\mathrm{Sym}^n(V)) = \binom{\dim(V)+n-1}{n} \quad \dim(\bigwedge{}^n V) = \binom{\dim(V)}{n} \tag{5.11}$$

From now on, **we may assume** $\dim V < \infty$. Define

$$L^n(V) = \left(\bigotimes_n V\right)^* \cong \bigotimes_n V^* \cong \{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V \to \mathbb{R}\} \tag{5.12}$$

And by the assumption we can obtain

$$\begin{aligned} \mathrm{Sym}^n(V^*) &\cong \{\text{symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \to \mathbb{R}\} \\ \bigwedge{}^n(V^*) &\cong \{\text{anti-symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \to \mathbb{R}\} \end{aligned} \tag{5.13}$$

We will mainly focus on $\bigwedge^n(V^*)$, also denoted as $\mathrm{Alt}^k(V)$= $\bigwedge^n(V^*)$. An element in $\mathrm{Alt}^k(V)$ is called a (linear) $k$-**form** on $V$ Now for $V = \mathbb{R}\langle e_1, \cdots, e_n\rangle$, $V^* = \mathbb{R}\langle e_1^*, \cdots, e_n^*\rangle$. Then

$$\begin{aligned} L^2(V) &= \{\text{all bilinear forms on } V\} \\ L^2(V) &\cong \mathrm{Sym}^2(V^*) \oplus \bigwedge{}^2(V^*) \end{aligned}$$

And $\mathrm{Sym}^2(V^*) = \mathbb{R}\left\langle e_i^* \otimes e_j^* + e_j^* \otimes e_i^* | 1 \leqslant i \leqslant i \leqslant n\right\rangle$ is symmetric bilinear form

$\mathrm{Alt}^2(V) = \bigwedge^2(V^*) = \mathbb{R}\left\langle e_i^* \otimes e_j^* - e_j^* \otimes e_i^* | 1 \leqslant i \leqslant i \leqslant n\right\rangle$ is anti-symmetric bilinear form.

The determinant $\det \in \mathrm{Alt}^n(\mathbb{R}^n)$.

**Definition 5.7** (Exterior product).

$$\bigwedge : \mathrm{Alt}^k(V) \times \mathrm{Alt}^l(V) \to \mathrm{Alt}^{k+l}(V)$$

$$\omega_1 \wedge \omega_2(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\omega_2(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$
$$= \sum_{\sigma \in S_{k,l}} \operatorname{sgn}(\sigma)\omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\omega_2(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

where $S_{k,l} = \{\sigma \in S_{k+l} | \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\} \subset S_{k+l}$.

Then we have those properties:

**Proposition 5.8.**

*(1)* $\omega_1 \wedge \omega_2 = (-1)^{|\omega_1|\cdot|\omega_2|}\omega_2 \wedge \omega_1$, $|\omega| = k$ *is* $\omega \in \operatorname{Alt}^k(V)$. *In particular,* $\omega \wedge \omega = 0$ *if* $|\omega|$ *is odd.*

*(2)* $(\omega_1 \wedge \omega_2) \wedge w_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$

*(3) Given any* $\omega_1, \cdots, \omega_k \in \operatorname{Alt}^1(V) = V^*$, $v_1, \cdots, v_k \in V$. *Then*

$$(\omega_1 \wedge \cdots \wedge \omega_k)(v_1, \cdots, v_k) = \det\left[w_i(v_j)\right]_{i,j} \tag{5.14}$$

*Moreover,* $\omega_1 \wedge \cdots \wedge_n \neq 0$ *iff* $\omega_i$ *are linearly independent.*

*(4)* $V = \mathbb{R}\langle e_1, \cdots, e_n\rangle$. *Then*

$$\operatorname{Alt}^k(V) = \mathbb{R}\left\langle e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* | i_1 < \cdots < i_k\right\rangle \tag{5.15}$$

*In particular,* $\operatorname{Alt}^n(V) = \mathbb{R}\langle e_1^* \wedge \cdots \wedge e_n^*\rangle$. *And we denote* $\operatorname{Alt}^0(V) = \mathbb{R}$, $\operatorname{Alt}^k(V) = 0$, $k > n$.

*(5) Any* $f \in \operatorname{Hom}(V, W)$ *induces* $\operatorname{Alt}^k(f) \in \operatorname{Hom}(\operatorname{Alt}^k(V), \operatorname{Alt}^k(W))$, *where*

$$\operatorname{Alt}^k(f)(\omega)(w_1, \cdots, w_k) = \omega(f(w_1), \cdots, f(w_k)) \tag{5.16}$$

*We have* $\mathrm{Alt}^k(f \circ g) = \mathrm{Alt}^k(g) \circ \mathrm{Alt}^k(f)$, $\mathrm{Alt}^k(\mathrm{id}_V) = \mathrm{id}_{\mathrm{Alt}^k(V)}$. *Such* $\mathrm{Alt}^k(-)$ *is called a* ***contravariant functor****.*

*Proof.*

(1) By definition,

$$\omega_1 \wedge \omega_2(v_1, \cdots, v_{k+l}) = \omega_2 \wedge \omega_1(v_{\sigma(1)}), \cdots, v_{\sigma(k+l)}$$

where $\sigma(i) = \begin{cases} i+k & 1 \leqslant i \leqslant l \\ i-l & l+1 \leqslant i \leqslant k+l \end{cases}$. $\mathrm{sgn}(\sigma) = (-1)^{k+l}$.

(2) By definition.

(3) By linearity, we assume $\omega_i = e^*_{a(i)}, v_j = e_{b(j)}$ for some $a(i), b(j)$. Further more, can assume $\{a(i)\} = \{b(i)\}$. (Otherwise, $LHS = RHS = 0$.)

Then $e^*_{a(i)}(e_{b(j)}) = \delta_{a(i),b(j)}$. After permutation, may assume $a(i) = b(i), \forall i$. It is direct to check $LHS = 1 = RHS$.

(4) If $\omega_1, \cdots, \omega_k$ are linear independent. Then $\exists$ basis $e^*_1, \cdots, e^*_n$ of $V^*$, basis $e_1, \cdots, e_n$ of $V$ *s.t.* $\omega_i = e^*_i$, $\forall 1 \leqslant i \leqslant n$.

$$(\omega_1 \wedge \cdots \wedge \omega_n)(e_1, \cdots, e_n) = \det(I) = 1 \neq 0 \Rightarrow \omega_1 \wedge \cdots \wedge \omega_n \neq 0$$

If $\omega_1, \cdots, \omega_k$ are linearly dependent. WLOG, we assume $\omega_k = \sum\limits_{i=1}^{k-1} a_i \omega_i$.

$$(\omega_1 \wedge \cdots \wedge \omega_k)(e_1, \cdots, e_n) = \sum_{i=1}^{k-1} a_i (\omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega_i)(e_1, \cdots, e_n) = 0$$

(5) For $i_1 < \cdots < i_k, j_1 < \cdots < j_n$ we have

$$(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)(e_{j_1}, \cdots, e_{j_k}) = \begin{cases} 1 & j_t = i_t, \forall 1 \leqslant t \leqslant k \\ 0 & \text{otherwise} \end{cases} \tag{5.17}$$

Since $\dim \operatorname{Alt}(V) = \dim \bigwedge^k(V^*) = \binom{n}{k} = |\{e_{i_1} \wedge \cdots e_{i_k} | i_1 < \cdots < i_k\}|$.

(6) For $\omega \in \operatorname{Alt}^k(W), f \in \operatorname{Hom}(V, W)$, define $\operatorname{Alt}^k(f)(\omega) \in \operatorname{Alt}^k(V)$ by

$$\operatorname{Alt}^k(f)(\omega(V_1, \cdots, V_k)) = \omega(fV_1, \cdots, fV_k) \in \mathbb{R}$$

$\square$

**Definition 5.9.**

An $\mathbb{R}$**-algebra** consists of an $\mathbb{R}$-vector space $A$ with a bilinear map $\mu : A \times A \to A$ that is associate, *i.e.* $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$.

Say $A$ is **unitary** if $\exists 1 \in A$ *s.t.* $\mu(a, 1) = \mu(1, a) = a, \forall a \in A$

Say $A$ is **graded** if $A = \bigoplus_{k \in \mathbb{Z}} A_k$ as vector space, and $\mu(A_k \times A_l) \subset A_{k+l}$. Elements in $A_k$ are called **homogeneous elements** of degree $k$.

If $A$ is graded $\mathbb{R}$-algebra, we say $A$ is **anticommutative** if $\mu(a, b) = (-1)^{k+l}\mu(b, a), \forall a \in A_k, b \in A_l$. And say $A$ is **commutative** if $\mu(a, b) = \mu(b, a), \forall a, b$.

If $A$ is graded $\mathbb{R}$-algebra, say $A$ is **connected** if $\exists$ unit $1 \in A_0$ *s.t.* the map $\varepsilon : \mathbb{R} \to A_0, r \mapsto r \cdot 1$ is an isomorphism.

Given vector space $V$, let

$$\begin{tikzcd} \operatorname{Alt}^k(V) \arrow[r, equal] \arrow[d, equal] & \bigoplus_{k \geqslant 0} \operatorname{Alt}^k(V) \arrow[d, equal] \\ \operatorname{Alt}^*(V^*) \arrow[r, equal] & \bigoplus_{k \geqslant 0} \wedge^k(V^*) \end{tikzcd}$$

By Proposition 5.8, we have the theorem

**Theorem 5.10.** $(\mathrm{Alt}^*(V), \wedge)$ *is a graded connected anticommutative* $\mathbb{R}$*-algebra, called the* ***exterior algebra*** *of* $V$ *or* ***exterior algebra*** *of* $V$

## 5.3 Operation on vector bundles

Given $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$, meaning a vector bundle of dimension $n$, local trivialization $\left\{U_\alpha, \varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n\right\}_{\alpha\in\mathcal{A}}$. By shrinking $U_\alpha$, we may assume we have an smooth atlas $\{\varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^m\}_{\alpha\in\mathcal{A}}$.

For $x \in M$, use $E_x$ to denote $\pi^{-1}(x)$, fiber over $x$, which is a vector space of dimension $n$.

Then **Dual bundle of a vector bundle** $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$ is

$$E^* := \{(x,l)|x \in M, l \in (E_x)^*\}, \pi' : E^* \to M, (x,l) \mapsto x, (\pi')^{-1}(x) = (E_x)^* \quad (5.18)$$

Define topology or smooth structure on $E^*$ *s.t.* $\pi' : E^* \to M$ is a smooth vector bundle.

For $\alpha \in \mathcal{A}$, let $E^*_\alpha = \pi'^{-1}(U_\alpha)$, we have a bijection

$$\tilde{\varphi_\alpha} : E^*_\alpha \xrightarrow{bijection} \mathbb{R}^m \times (\mathbb{R}^n)^* \xrightarrow{\quad\cong\quad} \mathbb{R}^{m+n}$$

$$(x,l) \longmapsto (\psi_\alpha(x), (\varphi_{\alpha,x})^{-1}(l))$$

We can check that

(1) $\{\tilde{\varphi_\alpha}^{-1}|\alpha \in \mathcal{A}, V \subset \mathbb{R}^{m+n} \text{ open}\}$ is a basis, we use it to generate a topology on $E^*$.

(2) Use $\tilde{\varphi_\alpha} : E^*_\alpha \xrightarrow{\cong} \mathbb{R}^{m+n}, \alpha \in \mathcal{A}$ as an atlas to give $E^*$ a smooth structure.

(3) $E^* \xrightarrow{\pi'} M$ is a smooth vector bundle, called the **dual vector bundle** of $E \xrightarrow{pi} M$, where

$$(E^*)_x = E^*_x$$

We can define other operations on vector bundles in similar way:

Given $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$, $\mathbb{R}^m \hookrightarrow F \xrightarrow{\pi} M$, we can define

$$\mathbb{R}^{m+n} \hookrightarrow E \oplus F \xrightarrow{\pi} M \text{ with } (E \oplus F)_x = E_x \oplus F_x$$

$$\mathbb{R}^{mn} \hookrightarrow E \otimes F \xrightarrow{\pi} M \text{ with } (E \otimes F)_x = E_x \otimes F_x$$

$$\mathbb{R}^{mn} \hookrightarrow \operatorname{Hom}(E, F) \xrightarrow{\pi} M \text{ with} \operatorname{Hom}(E, F)_x = \operatorname{Hom}(E_x, F_x)$$

$$\mathbb{R}^{\binom{n}{k}} \hookrightarrow \operatorname{Alt}^k(E) \to M \text{ with}$$

$$\operatorname{Alt}^k(E)_x = \operatorname{Alt}^k(E_l) = \{\text{alternating } k\text{-linear } l : E_x \times \cdots \times E_x \to \mathbb{R}\}$$

Then $\operatorname{Alt}^k(TM) = \bigwedge^k(T^*M)$.

$\operatorname{Alt}^k(M)_x = \{\text{alternating } k\text{-linear } l : T_xM \times \cdots \times T_xM \to \mathbb{R}\} = \{\text{linear } k\text{-form on } T_xM\}$

Define

$$\Gamma(E) := \{\text{smooth sections of } E\} = \{s \in C^\infty(M, E) : \pi \circ s = \mathrm{id}_M\}$$

**Definition 5.11.** Given smooth $M$, define a differential $k$-form on $M$ to be an element in $\Gamma(\operatorname{Alt}^k(TM))$ is a differential $k$-form $\alpha$ assigns each $x \in M$ a linear $k$-form $\alpha(x) \in \operatorname{Alt}^k(T_xM)$.

Denote $\Omega^k(M)$ be the set of all the differential $k$-forms.

Then $\Omega^0(M) = C^\infty(M, \mathbb{R})$. $\operatorname{Alt}^1(TM) = T^*M \Rightarrow$ a 1-form on $M$ is just a "cotangent vector field" on $M$.

$\Omega^k(M) = 0$ if $k \geqslant \dim(M)$.

## 5.4 Differential forms using local chart

Given local chart $(U, x^1, \cdots, x^n)$ of $M$.

For any $p \in U$, $\{\frac{\partial}{\partial x^i}|_p\}_{1 \leqslant i \leqslant n}$ is a basis of $T_x M$.

We denote the dual basis of $T_x^* M$ by $\{\mathrm{d}x^i|_p\}_{1 \leqslant i \leqslant n}$.

For any $\alpha \in \Omega^1(M)$, $\alpha|_U$ can be written as $\sum_{i=1}^{n} f_1 \mathrm{d}x^i$, where $f^i \in C^\infty(U, \mathbb{R})$.

Similarly, $\{\mathrm{d}x^{i_1}|_1 \wedge \cdots \wedge \mathrm{d}x^{i_k}|_p | i_1 < \cdots < i_k\}$ is a basis for $\bigwedge^k(T_x^* M)$, so $\forall \alpha \in \Omega^k(M)$,

$$\alpha|_U = \sum_{i_1 < \cdots < i_k} f_{i_1, \cdots, i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},\ f_{i_1, \cdots, i_k} \in C^\infty(U, \mathbb{R})$$

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# List of Theorems