

# Complex Analysis

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# 1 Harmonic function

**Definition 1.1** (Cauchy-Riemann equation).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**Definition 1.2** (Harmonic function). A function  $u$  is **harmonic** if it satisfied **Laplace equation**  $\Delta u = 0$ .

If two harmonic function  $u$  and  $v$  satisfies Cauchy-Riemann equations, then we say that  $v$  is **conjugate harmonic function of  $u$**   $\Rightarrow u$  is conjugate harmonic of  $-v$ .

## 1.1 Polynomial rational function

The polynomial  $P(z) = \sum_{j=0}^n a_j z^j$  is analytic in  $\mathbb{C}$ .

We will prove the fundamental theorem of algebra

**Theorem 1.3** (Fundamental Theorem of Algebra). *Every polynomial with degree  $n > 0$  has at least one point.*

**Theorem 1.4** (Gauss-Lucus theorem). *The smallest convex polygon that contain the zeros of  $P$  also contains the zeros of  $P'$ .*

*Proof.* Only need to check.

We can get this equation.

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{j=1}^n \frac{1}{\alpha - \alpha_j} = 0 \Rightarrow \sum_{j=1}^n \frac{\overline{\alpha - \alpha_j}}{|\alpha - \alpha_j|^2}$$

Hence  $\alpha$  is linearly represented by  $\alpha_j$ .

□

**Proposition 1.5.** Let  $P$  and  $Q$  be two polynomials with no common zeros. Then the rational function  $R(z) = \frac{P(z)}{Q(z)}$  is analytic away from the zeros of  $Q$ .

The zeros of  $Q$  are called **poles** of  $R$ , and the **order of a pole** is equal to the order of the corresponding zero of  $Q$ .

We often view  $R$  as a function from  $\hat{\mathbb{C}}$  to  $\hat{\mathbb{C}}$ .  $R_1(z) := R(\frac{1}{z})$ .

If  $R_1(0) = 0$ , the order of the zero at  $\infty$  (of  $R$ ) is the order of the zero of  $R_1(z)$  at  $z = 0$ .

If  $R_1(0) = \infty$ , the order of the pole at  $\infty$  (of  $R$ ) is the order of the pole of  $R_1(z)$  at  $z = 0$ .

Suppose

$$R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, a_n \neq 0, b_m \neq 0$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$$

By discussing  $m$  and  $n$ , we can infer the situation of  $R(z)$  at  $\infty$ .

By adding the order of poles and zeros at  $\infty$ , we can get the following theorem.

**Theorem 1.6.** The total number of zeros and poles of a rational function are the same.

**Remark 1.7.** This common number is called the **order of the rational function**.

**Corollary 1.8.** Suppose a rational function  $R$  has order  $p$ . Then every equation  $R(z) = a$  has exactly  $p$  roots.

*Proof.*  $\hat{R}(z) = R(z) - a$  has the same poles as  $R$ . □

A rational function of order 1 is a **linear fraction**  $R(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$

Such fraction is often called **Möbius transformation**

Every rational function has a representation by **partial fractions**.

- If  $R$  has a pole at  $\infty$ . Then we can write

$$R(z) = G(z) + H(z) \quad (*)$$

where  $G$  is a polynomial without constant term, and  $H$  is finite at  $\infty$ .

The degree of  $G$  is the order of the pole of  $R$  at  $\infty$ .  $G$  is called the **singular part** of  $R$  at  $\infty$ .

- Let the distinct finite poles of  $R$  be  $\beta_1, \dots, \beta_k$ . Let  $R_j(\psi) = R(\beta_j + \frac{1}{\psi})$ . Then  $R_j$  is a rational function with a pole at  $\infty$ . As in  $(*)$ , we can write

$$R_j = G_j + H_j$$

with  $H_j$  finite at  $\infty$ . Then

$$R(z) = G_j\left(\frac{1}{z - \beta_j}\right) + H\left(\frac{1}{z - \beta_j}\right)$$

with  $G_j$  is a polynomial in  $\frac{1}{z - \beta_j}$  without constant term called the **singular point** of  $R$  at  $\beta_j$ .

- Let  $F(z) = R(z) - G(z) - \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right)$ .

Then  $F$  is a rational function which can only have poles among  $\beta_j, \infty$

Since by our construction,  $F$  is finite at every  $\beta_j, 1 \leq j \leq k$  and  $\infty$ .

So  $F$  is a constant.

In particular,  $R(z) = G(z) + \sum_{j=1}^k G_j\left(\frac{1}{z - \beta_j}\right) + c$ .

## 2 Power Series

**Theorem 2.1** (Abel's theorem). *If  $\sum a_n$  converges, then  $f(z) = \sum a_n z^n \rightarrow f(1)$  as  $z \rightarrow 1$  in such a way that  $\frac{|1-z|}{1-|z|}$  remains bounded.*

## 3 Exponential, Trigonometric and logarithmic functions

### 3.1 Exponential and Trigonometric function

The **exponential function** is defined as the solution of the differential equation

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1 \end{cases}$$

We denote  $e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ .

The **trigonometric functions** are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

### 3.2 Logarithmic Functions

The **logarithmic function**  $\ln$  is defined by  $z = \ln w$  is a root of the equation  $e^z = w$ .

For  $w \neq 0$ , we write  $z = x + iy$ , then

$$e^{x+iy} = w \Leftrightarrow \begin{cases} e^x = |w| \\ e^{iy} = \frac{w}{|w|} \end{cases}$$

The first equation has a unique solution  $x = \ln |w|$ .

The second equation  $e^{iy} = \frac{w}{|w|}$  has a unique solution  $y_0 \in [0, 2\pi)$ .

If we write  $w = re^{i\theta}$ , then  $x = \ln w, y = \theta = \arg w$ .

Thus, for  $w \neq 0$ , we have

$$\ln w = \ln |w| + i \arg w$$

The function  $\ln$  is actually not single-valued. But we can define a single-valued function  $Ln$

We define

$$a^b = \exp(b \ln a)$$

We will prove  $Ln$  is analytic in  $\mathbb{C} - (-\infty, 0]$  but not continuous in  $(-\infty, 0]$ .

$Ln$  is the principle branch of the logarithm.

## 4 Conformal Mappings

### 4.1 Connectedness

**Theorem 4.1.** *A nonempty open set in  $\mathbb{C}$  is connected iff any two of its points can be joined by a polygon which lies in the set. ( i.e. Connectedness is equivalent to Path Connectedness)*

An nonempty connected subset is called a **region**

### 4.2 Compactness

**Definition 4.2.** A set  $X$  is **totally bounded** if  $\forall \epsilon > 0$ ,  $X$  can be covered by finitely many balls of radius  $\epsilon$

**Theorem 4.3.** *A set is compact iff it is complete and totally bounded.*

**Theorem 4.4.** *A subset  $X \subset \mathbb{C}$  is compact iff every infinite sequence of  $X$  has a limit point in  $X$ .*

### 4.3 Continuous Functions

**Theorem 4.5.** *Continuous function maps connected space to connected space.*

**Theorem 4.6.** *Continuous function maps compact space to compact space.*

### 4.4 Arcs and closed curves

The equation of an **arc**  $r$  in  $\mathbb{C}$  can be represented by one of the terms

- $x = x(t), y = y(t), \alpha \leq t \leq \beta, x, y$  are continuous at  $t$
- $z(t) = x(t) + iy(t), \alpha \leq t \leq \beta.$
- the continuous mapping  $\gamma : [\alpha, \beta] \rightarrow \mathbb{C}.$

For a nondecreasing function  $\varphi : [\alpha, \beta] \rightarrow [\alpha, \beta], z = z(\varphi(t)), \alpha' \leq \tau \leq \beta'$  is **change of parameter** of  $z(t).$

The change is **reversible** iff  $\varphi$  is strictly increasing.

If  $\gamma$  is differentiable, then call  $\gamma$  a **curve**.

$\gamma$  is **simple**, or a **Jordan curve**, if  $\gamma$  is injective.

$\gamma$  is **closed curve** if  $\gamma(0) = \gamma(1).$

### 4.5 Analytic Functions in Regions

A function  $f$  is analytic on an arbitrary set  $A$  if it is the restriction to  $A$  of a function which is analytic in some open set containing  $A$ .



**Theorem 4.7.** *An analytic function in a region( i.e. open and connected)  $\Omega$  whose derivative is 0 must reduce to a constant. The same hold if the real part, the imaginary part, the modulus, or the argument is constant.*

## 4.6 Conformal Mappings

Suppose  $f : \Omega \rightarrow \mathbb{C}$  is analytic in  $\Omega$ .  $r_1 = z_1(t), r_2 = z_2(t), \alpha \leq t \leq \beta$ .

$$z_0 = z_1(t_0) = z_2(t'_0), z'_1(t_0) \neq 0, z'_2(\hat{t}_0) \neq 0, \alpha < t_0, \hat{t}_0 < \beta.$$

$$f'(z_0) \neq 0, w_1(t) = f(z_1(t)), w_2 = f(z_2(\hat{t}_0))$$

$$\Gamma_1 = \{w_1(t) | \alpha \leq t \leq \beta\}, \Gamma_2 = \{w_2(t) | \alpha \leq t \leq \beta\}$$

Then

$$w'_1(t) = f'(z_1(t))z'_1(t)$$

$$w'_2(t) = f'(z_2(\hat{t}))z'_2(\hat{t})$$

$\Rightarrow$

$$w'_1(t_0) \neq 0, w'_2(t_0) \neq 0$$

$$\arg w'_1(t_0) = \arg f'(z_1(t_0))z'_1(t_0)$$

$$\arg w'_2(t_0) = \arg f'(z_2(\hat{t}_0))z'_2(\hat{t}_0)$$

So the "angle"  $\arg w'_1(t_0) - \arg w'_2(\hat{t}_0) = \arg z'_1(t_0) - \arg z'_2(\hat{t}_0)$  remains the same.

Now we give the definition.

**Definition 4.8.**  $w = f(z)$  is said to be **conformal** in  $\Omega$  if  $f$  is analytic in  $\Omega$  and  $f'(z) \neq 0$  for  $\forall z \in \Omega$ .

Easy to prove that linear change of scale at  $z_0$  is independent of the direction.

$$i.e. |f'(z_0)| = \lim_{z \rightarrow z_0} \frac{\delta \sigma}{\delta s}$$

## 4.7 Length and Area

The **length** of a differentiable arc  $\gamma$  with the equation  $z(t) = x(t) + iy(t)$ ,  $a \leq t \leq b$

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b |z'(t)| dt$$

For  $\Gamma = f(\gamma)$  where  $f$  conformal mapping.

Then

$$L(\Gamma) = \int_a^b |f'(z(t))| \cdot |z'(t)| dt$$

The **area** of  $E \subset \mathbb{R}$  is  $A(E) = \iint_E dx dy$

Then by the differentiable functional transformation, the area  $\hat{E} = f(E)$  is

$$A(\hat{E}) = \int \int_E |u_x v_y - u_y v_x| dx dy$$

If  $f$  is the conformal mapping of an open set containing  $E$ , then by Cauchy-Riemann equation

$$A(\hat{E}) = \int \int_E |f'(z)|^2 dx dy$$

## 5 Möbius Transformation

Recall that a **Möbius transformation** is a function of the form

$$w = s(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Then it has an inverse  $z = S^{-1}(w) = \frac{dw - b}{-cw + a}$ .

We may define  $S(\infty) = \lim_{z \rightarrow \infty} S(z) = \frac{a}{c}$ ,  $S(-\frac{d}{c}) = \infty$

With these definition,  $S : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is a topological mapping. Here one may use the chordal metric to define the topology.

$$S'(z) = \frac{ad - bc}{(cz + d)^2}$$

Then  $S$  is conformal in  $\hat{\mathbb{C}} - \{-\frac{d}{c}, \infty\}$ .

$w = z + \alpha$  is called a **parallel translation**.

$w = kz$  with  $|k| = 1$  is a **rotation**.

$w = kz$  with  $k > 0$  is a **homothetic transformation**.

$x = \frac{1}{z}$  is called an **inversion**.

**Proposition 5.1.** *Every Möbius transformation is a composition of the above four operations.*

## 5.1 Cross ratio

For three distinct points  $z_2, z_3, z_4 \in \hat{\mathbb{C}}$ , we can find a Möbius transformation  $S$  such that  $S(z_2) = 0, S(z_3) = 1, S(z_4) = \infty$ .

**Lemma 5.2.** *The Möbius transformation satisfying the above conditions is unique.*

The **cross ratio**  $(z_1, z_2, z_3, z_4)$  is the image  $z_1$  under the Möbius transformation which maps  $z_2$  to 1,  $z_3$  to 0 and  $z_4$  to  $\infty$ .

**Theorem 5.3.** *If  $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$  are distinct, and  $T$  is any Möbius transformation, then  $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$ .*

**Lemma 5.4.** *Let  $T$  be a Möbius transformation,  $T(\mathbb{R})$  is either a circle or a straight line.*

**Theorem 5.5.** *The cross ratio  $(z_1, z_2, z_3, z_4)$  is real iff the four points lie on a circle or a straight line.*

**Remark 5.6.** One may prove the theorem by elementary geometry

**Theorem 5.7.** *A Möbius transformation maps circles into circles.*

## 5.2 Symmetry

Suppose  $T$  is a Möbius transformation which maps  $\hat{\mathbb{R}}$  onto a circle  $C$ .

We say that  $w = Tz$  and  $w^* = T\bar{z}$  are **symmetric** w.r.t.  $C$ .

**Remark 5.8.** This definition is independent of  $T$ . Suppose  $S$  is another Möbius transformation which maps  $\hat{\mathbb{R}}$  onto  $C$ , then  $S^{-1}T$  maps  $\hat{\mathbb{R}}$  to  $\hat{\mathbb{R}}$ , and this  $S^{-1}w = S^{-1}Tz$  and  $S^{-1}w^* = S^{-1}T\bar{z}$  are conjugate.

The points  $z$  and  $z^*$  are **symmetric w.r.t  $C$  through**  $z_1, z_2, z_3$  iff  $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$ .

This can be another definition.

Note that only the points on  $C$  are symmetric to themselves.

The mapping  $z \mapsto z^*$  is 1-1 and is called **reflection** w.r.t.  $C$ .

### 5.2.1 geometric meaning of symmetry

Case1:  $C$  is a straight line. We may assume  $z_3 = \infty$ .

$z, z^*$  are symmetric w.r.t.  $C$  if and only if

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Then

$$|z^* - z_2| = |z - z_2|, \quad \forall z_2 \in C \text{ and } z_2 \neq \infty$$

$$\operatorname{Im} \frac{z^* - z_2}{z_1 - z_2} = \operatorname{Im} \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

So  $C$  is the bisecting normal of the segment between  $z$  and  $z^*$ .

Case2:  $C$  is the circle  $|z - a| = R$ .

$$\begin{aligned} &\text{Then for } \forall \text{ distinct } z_1, z_2, z_3 \in \mathbb{C}, \overline{(z, z_1, z_2, z_3)} = \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)} \\ &= (\bar{z} - \bar{a}, \bar{z}_1 - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}) = (\bar{z} - \bar{a}, \frac{R^2}{\bar{z}_1 - \bar{a}}, \frac{R^2}{\bar{z}_2 - \bar{a}}, \frac{R^2}{\bar{z}_3 - \bar{a}}) = (\frac{R^2}{\bar{z} - \bar{a}}, z_1 - a, z_2 - a, z_3 - a) \end{aligned}$$

$$a, z_3 - a) \\ = \left( \frac{R^2}{\bar{z} - \bar{a}}, z_1, z_2, z_3 \right).$$

Then the symmetric point of  $z$  w.r.t.  $C$  is

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

or

$$(z^* - a)(\bar{z} - \bar{a}) = R^2$$

$\Rightarrow$

$$\begin{cases} |z^* - a| \cdot |z - a| = R^2 \\ \frac{z^* - a}{z - a} = \frac{(z^* - a)(\bar{z} - \bar{a})}{|z - a|^2} > 0 \end{cases}$$

**Theorem 5.9** (The Symmetric principle). *If a Möbius transformation maps a circle  $C_1$  onto a circle  $C_2$ , then it transforms any pair of symmetric points w.r.t.  $C_1$  into a pair of symmetric points w.r.t.  $C_2$ .*

*Proof.* Case1:  $C_1 = \hat{\mathbb{R}}$ . Let  $T$  be the Möbius transformation which maps  $\hat{\mathbb{R}}$  onto  $C_2$ .  $\forall z \in \mathbb{C}$ , by definition,  $w = Tz$  and  $w^* = T\bar{z}$  are symmetric w.r.t.  $C_2$ .

Case2:  $C_1$  is a general circle. Let  $T : C_1 \rightarrow C_2$  and  $S : \mathbb{R} \rightarrow C_2$  be Möbius transformation.

Suppose  $w, w^*$  are symmetric w.r.t.  $C_1$ . Then there exists  $z$  s.t.  $w = Sz, w^* = S\bar{z}$ .

Then we can find  $Tw = TSz, Tw^* = TS\bar{z}$  are symmetric w.r.t.  $C_2$  since  $TS : \hat{\mathbb{R}} \rightarrow C_2$  □

**Remark 5.10.** 1. The Möbius transformation from  $C_1$  to  $C_2$  satisfies  $z_1 \mapsto$

$w, z_2 \mapsto w_2, z_3 \mapsto w_3$  where  $z_1, z_2, z_3 \in C_1, w_1, w_2, w_3 \in C_2$  is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

2. The Möbius transformation from  $C_1$  to  $C_2$  satisfies  $z_1 \mapsto w_1, z_2 \mapsto w_2$  where  $z_1 \in C_1, z_2 \notin C_1, w_1 \in C_2, w_2 \notin C_2$  is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

### 5.3 Steiner Circles

For  $S(z) = \frac{az+b}{cz+d}, S'(z) = \frac{ad-bc}{(cz+d)^2}$ .

A point  $z \notin$  a circle  $C$  is said to on the **right(left,resp.)** of  $C$  if  $\text{Im}(z, z_1, z_2, z_3) > 0(\text{Im}(z, z_1, z_2, z_3) < 0)$

**Remark 5.11.**

1. This agrees with everyday use since  $(i, 1, 0, \infty) = i$
2. This distinct between left and right is the same for all triples, while the meaning may be reversed.

(If  $C = \hat{\mathbb{R}}$ , then  $(z, z_1, z_2, z_3) = \frac{az+b}{cz+d}$  with  $a, b, c, d \in \mathbb{R} \Rightarrow \text{Im}(z, z_1, z_2, z_3) = \frac{ad-bc}{|cz+d|^2} \text{Im}(z)$ )

3. We can define an absolute positive orientation of all finite circles by requiring that  $\infty$  should be lie to the right of the oriented circles.

Consider a Möbius transformation of the form

$$w = k \cdot \frac{z-a}{z-b}$$

Here,  $z = a \mapsto w = 0, z = b \mapsto w = \infty$ .

Then circles through  $a, b$  maps to straight line through  $0, \infty$ .

The concentric circle about the origin,  $|w| = \rho$ , correspond to circles with the equation

$$\left| \frac{z - a}{z - b} \right| = \frac{\rho}{|k|}$$

These are the circles of **Apollonius** with limit points  $a$  and  $b$ .

Denote by  $C_1$  the circles through  $a, b$  and  $C_2$  the circles of Apollonius with these limit points. The configuration formed by all the circles  $C_1$  and  $C_2$  is called the **Steiner circles**(or **circular net**)

**Theorem 5.12.**

- (a) *There is exactly one  $C_1$  and one  $C_2$  through each point in  $\hat{\mathbb{C}} \setminus \{a, b\}$*
- (b) *Every  $C_1$  meets every  $C_2$  under right angle.*
- (c) *Reflection in a  $C_1$  transforms every  $C_2$  into itself and every  $C_1$  into another  $C_1$ .*
- (d) *The limit points  $a, b$  are symmetric w.r.t. each  $C_2$ , but not w.r.t. other circles.*

*Proof.* If the limit points are  $0, \infty$ , those properties are trivial in the  $w$ -plane. The general case follows since all properties are invariant under Möbius transformations. □

## 6 Elementary Conformal mapping

**Example 6.1.**  $w = z^\alpha$  where  $\alpha > 0$ .

Let  $S(u_1, u_2)$  with  $0 < \varphi_2 - \varphi_1 \leq 2\pi$  be  $\{z \in \mathbb{C} : z \neq 0, \varphi_1 < \arg(z) < \varphi_2\}$  where  $\arg(z)$  can be chosen as any value of it.

Then  $S(\varphi_1, \varphi_2)$  is a region.

In this region, a unique value of  $w = z^\alpha$  is defined by  $\arg w = \alpha \arg z$ .

This function is analytic with  $\frac{dw}{dz} = \alpha \frac{w}{z}$ .

This function is 1-1 only if  $\alpha(\varphi_2 - \varphi_1) \leq 2\pi$ .

**Example 6.2.**  $w = e^z$  maps  $\{z \in \mathbb{C} : -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2}\}$  onto  $\{w \in \mathbb{C} : \text{Re}(w) > 0\}$

**Example 6.3.**  $w = \frac{z-1}{z+1}$  maps  $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$  onto  $\{w \in \mathbb{C} : |w| < 1\}$

**Example 6.4.**

$$\mathbb{C} \setminus [-1, 1] \xrightarrow{z_1 = \frac{z+1}{z-1}} \mathbb{C} \setminus (-\infty, 0] \xrightarrow{z_2 = \sqrt{z_1}} \{\text{Re}(z_2) > 0\} \xrightarrow{w = \frac{z_2-1}{z_2+1}} \{w \in \mathbb{C} : |w| < 1\} \quad (6.1)$$

## 6.1 Elementary Riemann surfaces

**Example 6.5.**  $w = z^n, n \in \mathbb{Z}_+$  and  $n > 1$ .

There is a 1-1 correspondence between each angle  $\frac{(k-1)2\pi}{n} < \arg z < \frac{k \cdot 2\pi}{n}, k = 1, 2, \dots, n$  and while  $w$ -plane except for the positive real axis.

**Example 6.6.**  $w = e^z$ . This function maps each parallel strip  $(k-1)2\pi < \text{Im } z < k \cdot 2\pi, k \in \mathbb{Z}$  onto a sheet with a cut along the positive axis.

## 7 Complex Integration

### 7.1 Fundamental Theorems

#### 7.1.1 Line integral and rectifiable arcs

Let  $f(t) = u(t) + iv(t)$  be a complex-valued defined on  $t \in [a, b] \subset \mathbb{R}$  where  $u, v$  are real-valued functions. If  $f$  is continuous on  $[a, b]$ , we may define the **integral**

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt$$



Let  $\gamma$  be a piecewise differential arc in  $\mathbb{C}$  with the equation  $z = z(t)$ ,  $a \leq t \leq b$ . If  $f$  is continuous on  $\gamma$ , then  $f(z(t))$  is continuous on  $[a, b]$ , and we define

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt \quad (7.1)$$

The integral defined in 7.1 is independent of the parametrization of  $\gamma$ . Suppose that another parametrization of  $\gamma$  is  $\gamma : (\alpha, \beta) \rightarrow \mathbb{C}, \tau \mapsto z(t(\tau))$ , where  $t : (\alpha, \beta) \rightarrow (a, b)$ ,  $\tau \mapsto t(\tau)$  is piecewise differentiable. Then we have

$$\int_a^b f(z(t))z'(t)dt = \int_{\alpha}^{\beta} f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau = \int_{\alpha}^{\beta} f(z(t(\tau)))\frac{dz(t(\tau))}{d\tau}d\tau \quad (7.2)$$

For an arc  $\gamma$  with equation  $z = z(t)$ ,  $a \leq t \leq b$ , we define  $-\gamma$  by  $z = z(-t)$ ,  $-b \leq t \leq a$ .

Then we have

$$\begin{aligned} \int_{-\gamma} f(z)dz &= \int_{-b}^{-a} f(z(-t))\frac{dz(-t)}{dt}dt \\ &= - \int_{-a}^{-b} f(z(-t))z'(-t)dt \\ &= - \int_a^b f(z(\tau))z'(\tau)d\tau \\ &= - \int_{\gamma} f(z)dz \end{aligned}$$

So we have those properties:

**Proposition 7.1.**

$$(a) \int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz$$

(b) Let  $f$  and  $g$  be two continuous functions on the piecewise differentiable arc  $\gamma$ , then

$$\int_{\gamma} (\lambda_1 f + \lambda_2 g) dz = \lambda_1 \int_{\gamma} f dz + \lambda_2 \int_{\gamma} g dz, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

(c) If  $\gamma$  can be subdivided into two pieces differentiable arcs  $\gamma_1$  and  $\gamma_2$ , and  $f$  is continuous on  $\gamma_1$ , then

$$\int_{\gamma} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz$$

(d) (c) implies that the integral of a closed curve doesn't depend on the starting point on the curve

**Example 7.2.** Evaluate  $\int_{\gamma} \frac{1}{z-a} dz$  where  $\gamma$  is the circle centered at  $a \in \mathbb{C}$  with radius  $R$ .

Let  $z = z(t) = a + Re^{it}$ . Then the integral is  $2\pi i$

### 7.1.2 The fundamental theorem of Calculus for integrals in $\mathbb{C}$

The line integral w.r.t.  $\bar{z}$  is defined by

$$\int_{\gamma} f(z) d\bar{z} = \overline{\int_{\gamma} \overline{f(z)} dz}$$

With this notation, line integrals w.r.t.  $x = \operatorname{Re}(z)$  and  $y = \operatorname{Im}(z)$  can be defined by

$$\int_{\gamma} f(z) dx = \frac{1}{2} \left[ \int_{\gamma} f(z) dz + \int_{\gamma} f(z) d\bar{z} \right]$$

$$\int_{\gamma} f(z) dy = \frac{1}{2i} \left[ \int_{\gamma} f(z) dz - \int_{\gamma} f(z) d\bar{z} \right]$$

if we write  $f(z) = \mu + i\nu$ , we have

$$\int_{\gamma} f(z) dz = \int_{\gamma} f(z) dx + i \int_{\gamma} f(z) dy = \int_{\gamma} (\mu dx - \nu dy) + i \int_{\gamma} (\nu dx + \mu dy)$$

**Remark 7.3.** It is followed by the intuition. We can view the integration as the multiplication between  $f$  and  $dz$ .

The integral w.r.t. **arc length** is defined by

$$\int_{\gamma} f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt$$

This integral is again independent of the parametrization. It is easy to check

$$\int_{-\gamma} f(z)|dz| = \int_{\gamma} f(z)|dz|$$

Now we define **length** of a curve  $\gamma$ :  $L(\gamma) = \int_{\gamma} |dz|$

We have the inequality:

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| \cdot |dz| \leq L(\gamma) \cdot \sup_{z \in \gamma} |f(z)|$$

The length of an arc  $\gamma$  ( $z = z(t)$ ) can also be defined as the least upper bound of all sums

$$\sum_{i=1}^n |z(t_i) - z(t_{i-1})|$$

where  $a = t_0 < t_1 < \dots < t_n = b$  If this least upper bound is finite, we say that the arc is **rectifiable**

It is easy to show that piecewise differentiable arcs are rectifiable.

The integral of a continuous function  $f$  on a rectifiable arc may be defined as

$$\int_{\gamma} f(z)dz = \lim \sum_{k=1}^n f(z(\psi_k))[z(t_k) - z(t_{k-1})]$$

**Theorem 7.4.** Let  $\gamma \subset \mathbb{C}$  be a region, and  $P, Q$  two (possibly complex-valued) functions that are continuous on  $\Omega$ ,  $\gamma$  closed curve. The integral  $\int_{\gamma} p(x, y)dx + Q(x, y)dy$  depends only on the end point of  $\gamma$  iff there exists a function  $U(x, y)$  on  $\Omega$  with  $\frac{\partial U}{\partial x} = P, \frac{\partial U}{\partial y} = Q$ .

*Proof.* " $\Leftarrow$ ": If such a  $U$  exists, then

$$\int_{\gamma} Pdx + Qdy = \int_{\gamma} \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = \int_{\gamma} \frac{dU}{dt}dt = U(\gamma(b)) - U(\gamma(a))$$

" $\Rightarrow$ ": Fix a point  $(x_0, y_0) \in \Omega$ . We define  $U(x, y) = \int_{\gamma} Pdx + Qdy$  where  $\gamma$  is any curve between  $(x_0, y_0)$  and  $(x, y)$ . Easy to check that it is true.  $\square$

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