2.7 Maxwell's equations, Electromagnetic Waves

2.7.1 Overview of Maxwell's equations

Maxwell's equations are a set of four differential equations that describe how electric and magnetic fields interact. Named after the Scottish physicist James Clerk Maxwell, these equations form the foundation of classical electrodynamics, optics, and electric circuits. They can also be used to derive the wave equation for light, illustrating that light is a form of electromagnetic radiation.

The equations are typically written in the following form:

Gauss's Law for Electricity:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{2.7.1}$$

Gauss's Law for Magnetism:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.7.2}$$

Faraday's Law of Induction:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \,. \tag{2.7.3}$$

Ampere's Circuital Law with Maxwell's Addition:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \,. \tag{2.7.4}$$

Here, $\vec{E}$ and $\vec{B}$ are the electric and magnetic field vectors, ρ is the electric charge density, $\vec{J}$ is the current density, ε_0 is the permittivity of free space, and μ_0 is the permeability of free space.

Gauss's Law for Electricity Gauss's law for electricity states that the electric flux through any closed surface is proportional to the total charge enclosed within the surface. Mathematically, this is represented by the equation:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{2.7.5}$$

The left-hand side of this equation, $\vec{\nabla} \cdot \vec{E}$, represents the divergence of the electric field $\vec{E}$. The divergence is a measure of the 'outgoingness' of a vector field at a given point. In the context of an electric field, it measures how much electric field is emanating from a particular point in space.

The right-hand side, $\frac{\rho}{\varepsilon_0}$, represents the charge density ρ divided by the permittivity of free space ε_0. The permittivity of free space is a constant that characterizes the amount of electric field that can exist in a vacuum for a given electric charge.

Gauss's Law for Magnetism Gauss's law for magnetism states that the net magnetic flux out of any closed surface is zero. This implies that there are no magnetic monopoles and that every magnetic field line that begins at some point must end at another point. Mathematically, this is represented as:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.7.6}$$

Here, $\vec{B}$ is the magnetic field, and $\vec{\nabla} \cdot \vec{B} = 0$ represents the divergence of $\vec{B}$. Since the divergence of the magnetic field is zero, this implies that the magnetic field lines are continuous loops.

Faraday's Law of Induction Faraday's law of induction states that a changing magnetic field induces an electromotive force (EMF) in a closed loop of wire. This induced EMF creates an electric field, represented by the equation:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,. \tag{2.7.7}$$

The left-hand side, $\vec{\nabla} \times \vec{E}$, represents the curl of the electric field, which measures the 'circulation' or 'rotationality' of the field. The right-hand side, $-\frac{\partial \vec{B}}{\partial t}$, represents the rate of change of the magnetic field over time. The negative sign indicates that the induced EMF and, therefore, the induced electric field, oppose the change in the magnetic field, as stated by Lenz's law.

Ampere's Circuital Law with Maxwell's Addition Ampere's Circuital Law with Maxwell's addition, also known as Maxwell's law of electromagnetic induction, states that a magnetic field is induced by both the current density and the rate of change of the electric field. The law is represented as:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\,. \tag{2.7.8}$$

The left-hand side, $\vec{\nabla} \times \vec{B}$, represents the curl of the magnetic field. This measures the 'circulation' or 'rotationality' of the field. On the right-hand side, the term $\mu_0 \vec{J}$ represents the current density $\vec{J}$ multiplied by the permeability of free space μ_0. The term $\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ represents the rate of change of the electric field over time, scaled by the constants μ_0 and ε_0. The second term on the right-hand side is Maxwell's addition, which accounts for the creation of a magnetic field due to a changing electric field. This term is what allows for electromagnetic waves to propagate through space, as each changing field induces a change in the other, creating a self-sustaining wave.

2.7.2 Maxwell's Addition

The original form of Ampère's law, which relates the circulation of the magnetic field around a closed loop to the electric current passing through the loop, worked well for static fields. However, it failed to account for situations where the electric field changes with time. It was James Clerk Maxwell who recognized this inconsistency and added a crucial term to Ampère's law, now referred to as "Maxwell's Addition".

Maxwell realized that a changing electric field produces a magnetic field, just as a changing magnetic field produces an electric field (as described by Faraday's law of induction). This was a significant insight because it established symmetry between electric and magnetic fields.

Maxwell's addition is stated mathematically as follows:

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) . \tag{2.7.9}$$

The term $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ is Maxwell's addition, where ε_0 is the permittivity of free space and $\frac{\partial \vec{E}}{\partial t}$ is the rate of change of the electric field. This term is known as the displacement current. It isn't a current in the traditional sense of charges moving in a conductor; rather, it's a "current" of changing electric field. It's essential for the consistency of Ampère's law with charge conservation.

The Necessity of Maxwell's Addition: The Displacement Current

Consider a charging capacitor. There is a current i inside the wires connected to the plates, but in the region outside the wire and between the plates, there is no conduction current. According to the original Ampère's law, the integral of the magnetic field $\vec{B}$ around the red loop $\mathcal{C}$ in Figure 2.30 equals the integral of the current density $\vec{J}$ over the surface S, whose boundary is $\mathcal{C}$, i.e. $S = \mathcal{C}$,

$$\int_{\mathcal{C}} \vec{B} \cdot d\vec{r} = \mu_0 \int_S \vec{J} \cdot d\vec{A} . \tag{2.7.10}$$

Now, there is a paradox that we could choose the surface to be the orange surface S_{orange} or the blue surface S_{blue}. We obtained different values of the surface integral

$$\mu_0 \int_{S_{\text{orange}}} \vec{J} \cdot d\vec{A} = \mu_0 i \neq 0 = \mu_0 \int_{S_{\text{blue}}} \vec{J} \cdot d\vec{A} . \tag{2.7.11}$$

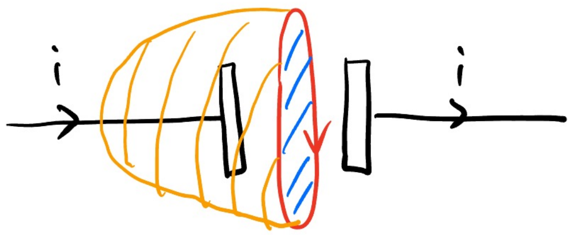

Figure 2.30: Charging a capacitor.

Maxwell explained this paradox by introducing the concept of displacement current. He argued that a changing electric field between the plates of the capacitor induces a "current" - not of moving charges, but of the changing electric field itself. This displacement current, $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$, is present in the region between the capacitor plates.

Recall, the electric field between two conducting plates of surface charge density σ is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{\varepsilon_0 A}\,. \tag{2.7.12}$$

Taking a time derivative gives

$$\mu_0 i = \mu_0\varepsilon_0 A\frac{dE}{dt} = \mu_0\varepsilon_0\frac{d}{dt}\int \vec{E}\cdot d\vec{A}\,, \tag{2.7.13}$$

where we have used the fact that the electric field is approximately uniform when the plate is sufficiently large. Now, we find

$$\int_{S_{\text{orange}}}\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{A} = \int_{S_{\text{blue}}}\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{A}\,. \tag{2.7.14}$$

The integral form of Ampère's Law with Maxwell's Addition is

$$\int_{\mathcal{C}}\vec{B}\cdot d\vec{r} = \mu_0\int_S\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{A}\,, \tag{2.7.15}$$

which is independent of the choice of the surface S as long as the boundary of S is the curve $\mathcal{C}$, i.e. $\partial S = \mathcal{C}$. Using Stoke's theorem, we have

$$\int_{\mathcal{C}}\vec{B}\cdot d\vec{r} = \int_S(\vec{\nabla}\times\vec{B})\cdot d\vec{A}\,. \tag{2.7.16}$$

Taking S to be infinitesimal, we arrive at the differential form of Ampère's Law with Maxwell's Addition,

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\,. \tag{2.7.17}$$

Consistency with charge conservation

Maxwell's addition to Ampère's law is also crucial for the conservation of electric charge. Consider an electric charge distribution of density $\rho(\vec{r})$ inside a region B, and the charges are flowing out of B by a current density $\vec{J}(\vec{r})$ over the boundary $S = \partial B$. The conservation of electric charge gives the equation

$$\int_S \vec{J}(\vec{r})\cdot d\vec{A} = -\frac{d}{dt}\int_B \rho(\vec{r})d^3x\,. \tag{2.7.18}$$

By the divergence theorem, we have

$$\int_S \vec{J}(\vec{r})\cdot d\vec{A} = \int_B \vec{\nabla}\cdot\vec{J}(\vec{r})d^3x\,. \tag{2.7.19}$$

Taking B to be infinitesimal, we arrive at the charge conservation equation

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}\,. \tag{2.7.20}$$

On the other hand, in the homework problem, you have found the identity

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0\,. \tag{2.7.21}$$

Using Ampère's Law with Maxwell's Addition and the Gauss law, we recover the charge conservation equation

$$0 = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \varepsilon_0 \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \frac{\partial \rho}{\partial t}\,. \tag{2.7.22}$$

We note that Maxwell's Addition is crucial for the above derivation. Without Maxwell's Addition, Ampère's Law would be inconsistent with the charge conservation.

2.7.3 Relativistic formulations of Maxwell equations

Maxwell's equations can be written in the index form as

$$\begin{aligned}
\partial_i E_i &= \frac{1}{\varepsilon_0}\rho\,, \\
\partial_i B_i &= 0\,, \\
\epsilon_{ijk}\partial_j E_k &= -\frac{\partial}{\partial t} B_i\,, \\
\epsilon_{ijk}\partial_j B_k &= \mu_0 J_i + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} E_i\,,
\end{aligned} \tag{2.7.23}$$

where $i,\, j = 1,\, 2,\, 3$ and $\partial_i \equiv \frac{\partial}{\partial x^i}$. We have used the Einstein summation convention, where the repeated indices are summed over. We do not distinguish the upper and the lower i, j indices, i.e. $V^i = V_i$. ϵ_{ijk} is the rank-3 Levi-Civita symbol, a totally anti-symmetric tensor with $\epsilon_{123} = 1$. The other components of ϵ_{ijk} are determined by the total anti-symmetricity.

Now, we define

$$F^{ij} \equiv \frac{1}{\mu_0}\epsilon_{ijk} B_k\,, \quad F^{0i} \equiv c\varepsilon_0 E_i\,, \quad F^{i0} \equiv -c\varepsilon_0 E_i\,, \quad F^{00} \equiv 0\,, \quad J^0 \equiv c\rho\,, \tag{2.7.24}$$

where c is a constant that will be determined later.

Exercise. *Show that* $B_i = \frac{\mu_0}{2}\epsilon_{ijk} F^{jk}$.

The electric and magnetic fields can be assembled into a rank-2 anti-symmetric tensor

$$F^{\mu\nu} = -F^{\nu\mu}\,, \tag{2.7.25}$$

where we have combined 0 and i forming a new index $\mu = 0, 1, 2, 3$. Writing in the matrix form, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & c\varepsilon_0 E_1 & c\varepsilon_0 E_2 & c\varepsilon_0 E_3 \\ -c\varepsilon_0 E_1 & 0 & \frac{1}{\mu_0} B_3 & -\frac{1}{\mu_0} B_2 \\ -c\varepsilon_0 E_2 & -\frac{1}{\mu_0} B_3 & 0 & \frac{1}{\mu_0} B_1 \\ -c\varepsilon_0 E_3 & \frac{1}{\mu_0} B_2 & -\frac{1}{\mu_0} B_1 & 0 \end{pmatrix}\,. \tag{2.7.26}$$

$F^{\mu\nu}$ is called a field strength.

Now, let us rewrite Maxwell's equation in terms of $F^{\mu\nu}$. First, let us define

$$x^0 \equiv ct\,, \quad \partial_0 \equiv \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}\,. \tag{2.7.27}$$

We need to stress here that the upper and lower 0-indices are different, i.e. $V^0 \neq V_0$, and we will determine how they are related later. Maxwell's equation now becomes

$$\begin{aligned} \partial_j F^{0j} &= J^0\,, \\ \partial_j F^{ij} &= J^i + \partial_0 F^{0i}\,, \\ \epsilon_{ijk}\partial^i F^{jk} &= 0\,, \\ \epsilon_{ijk}\partial^j F^{0k} &= -\frac{c^2\mu_0\varepsilon_0}{2}\epsilon_{ijk}\partial_0 F^{jk}\,. \end{aligned} \tag{2.7.28}$$

The first two equations can be nicely written as

$$\partial_\nu F^{\mu\nu} = J^\mu\,. \tag{2.7.29}$$

To simplify the last two equations, let us examine the equation

$$\epsilon_{\sigma\mu\nu\rho}\partial^\mu F^{\nu\rho} = 0\,, \tag{2.7.30}$$

where $\epsilon_{\sigma\mu\nu\rho}$ is the rank-4 Levi Civita symbol, which is a totally anti-symmetric tensor with $\epsilon_{0123} = 1$, and in particular, $\epsilon_{0ijk} = \epsilon_{ijk}$. When $\sigma = 0$, the equation (2.7.30) becomes

$$\epsilon_{ijk}\partial^i F^{jk} = 0\,, \tag{2.7.31}$$

which is the same as the third equation in (2.7.28). When $\sigma = i$, the equation (2.7.30) becomes

$$\epsilon_{ijk}\partial^j F^{0k} = \frac{1}{2}\epsilon_{ijk}\partial^0 F^{jk}\,. \tag{2.7.32}$$

To match with the fourth equation in (2.7.28), we first set

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\,. \tag{2.7.33}$$

Next, we define

$$x_0 \equiv -x^0\,, \quad \partial^0 \equiv \frac{\partial}{\partial x_0} = -\frac{\partial}{\partial x^0} = -\partial_0\,. \tag{2.7.34}$$

With these definitions, we see that (2.7.32) and the fourth equation of (2.7.28) become the same. In summary, Maxwell's equations become (2.7.29) and (2.7.30).

The equation (2.7.30) can be solved by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,, \tag{2.7.35}$$

for arbitrary vector field A^μ. We check that

$$\epsilon_{\sigma\mu\nu\rho}\partial^\mu(\partial^\nu A^\rho - \partial^\rho A^\nu) = \epsilon_{\sigma\mu\nu\rho}\partial^\mu\partial^\nu A^\rho - \epsilon_{\sigma\mu\nu\rho}\partial^\mu\partial^\rho A^\nu = 0\,, \tag{2.7.36}$$

where we have used the fact that partial derivatives commutes, i.e. $\partial^\mu\partial^\nu A^\rho = \partial^\nu\partial^\mu A^\rho$. The vector field A^μ is called a gauge potential.

We note that the gauge potential is not entirely physical. Consider two gauge potentials that differ by a total derivative as

$$A'^\nu = A^\mu + \partial^\mu\lambda\,. \tag{2.7.37}$$

We have

$$F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = \partial^\mu A^\nu + \partial^\mu\partial^\nu\lambda - \partial^\nu A^\mu - \partial^\nu\partial^\mu\lambda = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}\,. \tag{2.7.38}$$

(2.7.37) is called a gauge transformation or a gauge ambiguity. The gauge ambiguity can be eliminated by imposing gauge conditions. A commonly used gauge condition is the Lorentz gauge

$$\partial_\mu A^\mu = 0\,, \tag{2.7.39}$$

which actually eliminates only part of the gauge ambiguity (2.7.37).

2.7.4 Electromagnetic wave

We have seen that Maxwell's equations (2.7.23) simplify to (2.7.29) and (2.7.29) after treating time t as the fourth coordinate $x^0 = ct$. An important question remains: What is the constant c?

Exercise. *Show that the unit of c is m/s from the units of μ_0 and ε_0. Hence, x^0 has the unit m, so it makes sense to combine x^0 with x^i into x^μ.*

Therefore, c should be the velocity of something. We show that the "something" is the electromagnetic wave. Let us plug (2.7.35) into (2.7.30),

$$\partial_\nu\partial^\mu A^\nu - \partial_\nu\partial^\nu A^\mu = J^\mu\,. \tag{2.7.40}$$

Using the Lorentz gauge (2.7.39), we find

$$-\partial_\nu\partial^\nu A^\mu = J^\mu\,. \tag{2.7.41}$$

For simplicity, we consider the case without any charge and current. We have

$$\left[\left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2\right] A^\mu = 0\,. \tag{2.7.42}$$

This equation is a wave equation that describes electromagnetic waves. The equation can be solved by the ansatz

$$A^\mu = \varepsilon^\mu \cos(k_\mu x^\mu + \phi)\,, \tag{2.7.43}$$

where k_μ is called the momentum and ε^μ is called the polarization of the electromagnetic wave. The wave equation (2.7.42) and the Lorentz gauge (2.7.39) implies

$$k_\mu k^\mu = k_\mu \varepsilon^\mu = 0\,. \tag{2.7.44}$$

For simplicity, we choose

$$k^\mu = (k, 0, 0, k)\,, \quad \varepsilon^\mu = (0, 1, 0, 0)\,. \tag{2.7.45}$$

The nonzero component of the gauge potential is

$$A^1 = \cos(k(x^0 - x^3) + \phi)\,. \tag{2.7.46}$$

The nonzero components of the field strength are

$$\begin{aligned} F^{01} &= -\partial_0 A^1 - \partial_1 A^0 = k\sin(k(x^0 - x^3) + \phi)\,, \\ F^{13} &= \partial_1 A^3 - \partial_3 A^1 = -k\sin(k(x^0 - x^3) + \phi)\,. \end{aligned} \tag{2.7.47}$$

We find the nonzero components of the electro and magnetic fields

$$E_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} k\sin(k(x^0 - x^3) + \phi)\,, \quad B_2 = \mu_0 k \sin(k(x^0 - x^3) + \phi)\,. \tag{2.7.48}$$

Now, using $x^0 = ct$, we find

$$E_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} k\sin(k(ct - x^3) + \phi)\,, \quad B_2 = \mu_0 k \sin(k(ct - x^3) + \phi)\,. \tag{2.7.49}$$

We see that the wavefronts of both the electric and magnetic fields have speed c.

2.7.5 Symmetry of Maxwell's equations

We have seen before in many examples that our laws of physics are invariant under Galilean transformations. Are Maxwell's equations also invariant under Galilean transformations?

First, Maxwell's equations are invariant under rotation $R^i{}_j \in O(3)$,

$$x'^i = R^i{}_j x^j\,, \quad \partial'_j = \frac{\partial}{\partial x'^j} = \frac{\partial x^i}{\partial x'^j}\frac{\partial}{\partial x^i} = (R^{-1})^i{}_j \partial_i = (R^T)^i{}_j \partial_i\,. \tag{2.7.50}$$

Exercise. *Show that* $\epsilon_{ijk} R^i{}_l R^j{}_m R^k{}_n = \epsilon_{lmn} \det(R)$.

The field strengths and the current transform as

$$F'^{0j} = R^i{}_j F^{0j}\,, \quad F'^{ij} = R^i{}_k R^j{}_l F^{kl}\,, \quad J'^i = R^i{}_j J^j\,, \quad J'^0 = J^0\,. \tag{2.7.51}$$

It is now easy to see that Maxwell's equations (2.7.28) are invariant under the $O(3)$ transformations.

However, Maxwell's equations are not invariant under the Galilean transformations,

$$x'^i = x^i + \frac{v^i}{c} x^0\,, \quad x'^0 = x^0\,. \tag{2.7.52}$$

The derivatives transform as

$$
\begin{aligned}
\frac{\partial}{\partial x'^i} &= \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} + \frac{\partial x^0}{\partial x'^i}\frac{\partial}{\partial x^0} = \frac{\partial}{\partial x^i}\,, \\
\frac{\partial}{\partial x'^0} &= \frac{\partial x^j}{\partial x'^0}\frac{\partial}{\partial x^j} + \frac{\partial x^0}{\partial x'^0}\frac{\partial}{\partial x^0} = -\frac{v^i}{c}\frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^0}\,.
\end{aligned}
\tag{2.7.53}
$$

There is no good way for the field strength F^{ij} and F^{0i} to transform and make Maxwell's equations (2.7.28) invariant.

Instead of Galilean transformations, Maxwell's equations are invariant under Lorentz transformations. Let us see how Lorentz transformations come about. In the previous sections, we use the speed c of the electromagnetic wave to turn time t into $x^0 = ct$, and combine it with x^i to form x^μ. It is natural to treat x^μ as a *4-vector*. We have upper μ and lower μ indices. They are related by

$$
x^\mu = \eta^{\mu\nu} x_\nu\,, \quad x_\mu = \eta_{\mu\nu} x^\nu\,, \tag{2.7.54}
$$

where $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ are 4×4 matrices,

$$
\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta_{\mu\nu}\,. \tag{2.7.55}
$$

The norm of the 4-vector x^μ is defined by

$$
|x|^2 \equiv x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu\,. \tag{2.7.56}
$$

Recall that the $O(3)$ transformations of the 3-vector x^i leave the norm of x^i invariant. Analogously, we consider transformations that leave the norm (2.7.56) of the 4-vector invariant as

$$
\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma = \eta_{\mu\nu} x^\mu x^\nu\,. \tag{2.7.57}
$$

The transformation matrix $\Lambda^\mu{}_\rho$ must satisfy

$$
\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}\,. \tag{2.7.58}
$$

In matrix notation, we have

$$
\Lambda^T \eta \Lambda = \eta\,. \tag{2.7.59}
$$

Such matrices form a group called $O(1,3)$. The group $O(3)$ is a subgroup of $O(1,3)$. These transformations are called *Lorentz transformations*.

For simplicity, let us consider the transformations that leave x^2 and x^3 invariant. They take the form

$$
\Lambda = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7.60}
$$

We find that x^0 and x^1 transform as

$$
\begin{aligned}
x'^0 &= x^0 \cosh\zeta + x^1 \sinh\zeta\,,\\
x'^1 &= x^0 \sinh\zeta + x^1 \cosh\zeta\,.
\end{aligned}
\tag{2.7.61}
$$

It can be written using $x = x^1$ and $t = x^0/c$ as

$$
\begin{aligned}
t' &= \gamma\left(t + \frac{v}{c^2}x\right)\,,\\
x' &= \gamma(x + vt)\,,
\end{aligned}
\tag{2.7.62}
$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, and we have changed the variable ζ to v by

$$
\cosh\zeta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,.
\tag{2.7.63}
$$

Now, we can consider the limit $c \gg v$, and the Lorentz transformation reduces to the Galilean transformation

$$
\begin{aligned}
t' &= t\,,\\
x' &= x + vt\,.
\end{aligned}
\tag{2.7.64}
$$

We see that the parameter v in the Lorentz transformation becomes the velocity in the Galilean transformation. The Galilean transformation can be regarded as the low-velocity limit of the Lorentz transformation.

The Lorentz transformation has very profound physical implications. Although Lorentz first found the Lorentz transformation, it was Einstein who first explain their meaning to the rest of the world. Now, this subject is called Einstein's special relativity.

In the following two sections, we will introduce two of the most important implications of the Lorentz transformations: 1. Time Dilation, 2. Length Contraction.

2.7.6 Time Dilation

Time dilation, as described by special relativity, posits that an observer in one inertial frame will perceive time to be passing more slowly in another frame that is moving relative to the first. This is a counterintuitive phenomenon that runs counter to our everyday experiences but is a natural consequence of the Lorentz transformations and the constant speed of light.

Let's consider two observers, Alice who is stationary and Bob who is moving at a velocity v relative to Alice. Suppose Bob carries a light clock with him, which measures time by bouncing a beam of light between two mirrors.

Now, from Bob's perspective, the light in the clock travels a distance of $2d$ (where d is the distance between the mirrors) in a time interval $\Delta t'$ (one tick of the clock). Using the invariant speed of light, c, we have

$$
2d = c\Delta t'\,.
\tag{2.7.65}
$$

However, from Alice's perspective, who sees Bob moving, the light beam in Bob's clock follows a diagonal path, forming a right triangle with the vertical distance of $2d$ and horizontal distance of $v\Delta t$. According to Pythagoras' theorem, the hypotenuse of this triangle (the path of the light) should be:

$$c^2\Delta t^2 = (2d)^2 + (v\Delta t)^2. \tag{2.7.66}$$

Here, Δt is the time interval as measured by Alice for one tick of Bob's clock. Substituting $2d = c\Delta t'$ into the above equation, we get:

$$c^2\Delta t^2 = c^2\Delta t'^2 + v^2\Delta t^2. \tag{2.7.67}$$

Solving for Δt, we find:

$$\Delta t = \gamma\Delta t', \tag{2.7.68}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor. This is the time dilation formula, indicating that Alice observes Bob's clock to tick more slowly by a factor of γ.

It's important to note that this time dilation effect is not due to any mechanical or optical flaws in the clock, but is a genuine effect of relative motion. We can also directly derive the time dilation (2.7.68) from the Lorentz transformation. Let's begin with the Lorentz transformation:

$$\begin{aligned} t' &= \gamma\left(t + \frac{vx}{c^2}\right), \\ x' &= \gamma\left(x + vt\right), \end{aligned} \tag{2.7.69}$$

The time intervals $\Delta t'$, Δt and space intervals $\Delta x'$, Δx would satisfy

$$\begin{aligned} \Delta t' &= \gamma\left(\Delta t + \frac{v}{c^2}\Delta x\right), \\ \Delta x' &= \gamma\left(\Delta x + v\Delta t\right), \end{aligned} \tag{2.7.70}$$

Here:

- $\Delta t'$ and $\Delta x'$ are the time and space intervals measured in the moving frame (Bob's frame), respectively
- Δt and Δx are the time and space intervals measured in the stationary frame (Alice's frame), resepctively
- v is the relative velocity between the two frames.

Now, let's imagine a scenario where a single event (such as the tick of a clock) happens at the origin of the moving frame (Bob's frame). In this case, $\Delta x' = 0$ for that event. Substituting $\Delta x' = 0$ into the Lorentz transformation for time, we have:

$$\Delta t = \gamma\Delta t'. \tag{2.7.71}$$

This equation states that the time interval for an event at the origin, as measured in the moving frame (Bob's frame), is γ times the time interval as measured in the stationary frame (Alice's frame).

This means Bob measures the time interval to be longer than Alice by a factor of γ, which is the essence of time dilation.

Time dilation has been experimentally confirmed in numerous tests, such as time-dilated decay of muons in cosmic rays and precision measurements using atomic clocks on board GPS satellites.

Time dilation, as such, highlights the flexible nature of time under special relativity - a stark contrast to our everyday perception of time as a constant, unchanging entity.

2.7.7 Length Contraction

Length contraction, much like time dilation, is a crucial and counterintuitive prediction of special relativity. According to this phenomenon, the length of an object in its direction of motion is observed to be shorter when viewed from a frame that is in motion relative to the object. This can be derived from the Lorentz transformations.

Consider an object at rest in the frame of observer Bob. The length L' of the object as measured in Bob's frame is given by the difference between the coordinates of its endpoints, x'_1 and x'_2: $L' = x'_2 - x'_1 = \Delta x'$.

From the perspective of another observer, Alice, who is moving at a speed v relative to Bob, the coordinates of the endpoints of the object are transformed by the Lorentz transformations as follows:

$$\begin{aligned} \Delta t' &= \gamma \left(\Delta t + \frac{v}{c^2} \Delta x \right) , \\ \Delta x' &= \gamma \left(\Delta x + v \Delta t \right) , \end{aligned} \tag{2.7.72}$$

where x denotes coordinates in Alice's frame, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor, and t' is the time as observed by Bob.

However, Alice measures the length of the object at a particular time, so for her, the times at the two ends of the object are the same, i.e., $\Delta t = 0$. Substituting this into the equations, we have:

$$\begin{aligned} \Delta t' &= \gamma \frac{v}{c^2} \Delta x , \\ \Delta x' &= \gamma \Delta x, \end{aligned} \tag{2.7.73}$$

The length $L = \Delta x$ of the object as observed by Alice is then given by $L = \Delta x = \gamma^{-1} \Delta x' = \gamma^{-1} L'$.

This result is counterintuitive because it suggests that an object's length can change depending on the observer's state of motion, which contradicts our everyday experiences. However, it's important to note that length contraction, like time dilation, is a real effect that has been confirmed in numerous experiments, and it is essential for maintaining the consistency of the laws of physics in all inertial frames as required by the principle of relativity.