Complex Analysis HW3

1\. (a) $\Rightarrow$: Let $G=\bigcup_{x\in E} N_x$, then G is an open set of S. $\forall x\in E$, $x\in N_x \Rightarrow E\subseteq G \Rightarrow E\subseteq G\cap X$.

$X\cap G = X\cap(\bigcup_{x\in E}N_x)=\bigcup_{x\in E}(X\cap N_x)\subseteq E \Rightarrow E=X\cap G$.

$\Leftarrow$: Take $N_x=G$, clearly N_x is open in S and $N_x\cap X\subseteq E$.

(b) Let A, B be two regions. (Warning: In general topology, connected open sets may not be path-connected).

$\Rightarrow$: If $A\cap B=\emptyset$, then the region $A\cup B$ is a union of 2 disjoint non-trivial open subsets A, B, contradiction.

$\Leftarrow$: ~~$\exists x\in A\cap B$, and $A\cup B$~~ Let $x\in A\cap B$. If $A\cup B$ is not a region, then $\exists$ non-trivial open subsets U, V s.t. $U\cap V=\emptyset$ and $A\cup B=U\cup V$. Note that $A=(A\cap U)\cup(A\cap V)$ and $A\cap U$, $A\cap V$ are open, we have $A\cap U=\emptyset$ or $A\cap V=\emptyset$. We may assume $x\in U$, then $A\cap V=\emptyset$.

Similarly from $B=(B\cap U)\cup(B\cap V)$ and $x\in B\cap U$ we get $B\cap V=\emptyset \Rightarrow V\cap(A\cup B)=\emptyset$, contradiction.

2\. In general topological space it's not true: Consider $\mathbb{R}$ with trivial topology, i.e. $\emptyset$ and $\mathbb{R}$ are the only open sets, then $K_n=[n,+\infty)$ are compact sets with $K_n\supseteq K_{n+1}$, but $\bigcap K_n=\emptyset$.

We assume the space is Hausdorff. If $\bigcap K_n=\emptyset$, then $\bigcup K_n^c=X$ where X is the entire space.

In particular, $\bigcup_{n=2}^{\infty} K_n^c = K_1$. X Hausdorff $\Rightarrow K_n$ closed $\Rightarrow K_n^c$ open. K_1 compact $\Rightarrow \exists N\in\mathbb{N}$ s.t. $\bigcup_{n=2}^{N} K_n^c = K_1$

However, $K_1\supseteq K_2\supseteq\cdots \Rightarrow K_1^c\subseteq K_2^c\subseteq\cdots \Rightarrow K_1=K_N^c \Rightarrow K_N=K_N\cap K_1=\emptyset$, contradiction.

3\. (a) Positivity: Since only finitely $x_n\neq 0$, $d(x,y)=\max|x_n-y_n|\geq 0$ is well-defined.

If $d(x,y)=0$, then $\max|x_n-y_n|=0 \Rightarrow |x_n-y_n|=0\ (\forall n) \Rightarrow x=y$. Clearly $d(x,x)=0$.

Symmetry: $d(x,y)=\max|x_n-y_n|=\max|y_n-x_n|=d(y,x)$.

Triangle inequality: $d(x,y)=\max|x_n-y_n|\leq\max(|x_n-z_n|+|z_n-y_n|)\leq\max|x_n-z_n|+\max|z_n-y_n|=d(x,z)+d(z,y)$.

(b) Not complete. Consider a sequence $x^{(n)}=(1,\frac{1}{2},\dots,\frac{1}{n},0,0\dots)\in S$, then $\forall\varepsilon>0$, choose $N\in\mathbb{N}_+$ s.t. $\frac{1}{N}<\varepsilon$,

then $\forall m>n>N$, $d(x^{(m)},x^{(n)})=\frac{1}{n+1}<\frac{1}{N}<\varepsilon \Rightarrow \{x^{(n)}\}$ is a Cauchy sequence.

If $\lim_{n\to\infty}x^{(n)}=y$ in S, then $\exists M\in\mathbb{N}$ s.t. $\forall m>M$, $d(x^{(m)},y)<\varepsilon$.

However, since $y\in S$, $\exists K\in\mathbb{N}$ s.t. $y_n=0\ (\forall n\geq K)$. Take $\varepsilon=\frac{1}{K+2}$, $m=\max\{K,M\}$, then $d(x^{(m)},y)\geq\frac{1}{K}$, contradiction.

So $\lim x^{(n)}$ doesn't exist $\Rightarrow S$ is not complete.

(c) If $B(0,\delta)$ is totally bounded, then $\exists x^1,\dots,x^m\in S$ s.t. $B(0,\delta)\subseteq\bigcup_{i=1}^{m}B(x^i,\frac{\delta}{2})$.

Let $N\in\mathbb{N}$ s.t. $x_n^i=0\ (\forall i=1\dots m,\ n\geq N)$, $y=(\underset{1}{0},\dots,\underset{N}{0},\underset{N+1}{\tfrac{3}{4}\delta},0,0,\dots)$, then $y\in B(0,\delta)$.

However, $d(x^i,y)\geq\frac{3}{4}\delta \Rightarrow y\notin B(x^i,\frac{\delta}{2})$, contradiction.

Alternatively, consider $x^{(n)}=(0,\dots,0,\underset{n}{\tfrac{3}{4}\delta},0,0\dots)$, then $\forall n\neq m$, $d(x^{(m)},x^{(n)})=\frac{3}{4}\delta$.

If $x^{(m)},x^{(n)}\in B(y,\frac{\delta}{4})$ $(m\neq n)$, then $\frac{3}{4}\delta=d(x^{(m)},x^{(n)})\leq d(x^{(m)},y)+d(y,x^{(n)})<\frac{\delta}{2}$, contradiction.

So every ball with radius $\frac{\delta}{4}$ contains at most one $x^{(n)}$. Since $\{x^{(n)}\}$ is infinite, $B(0,\delta)$ is not totally bounded.

4\. Let $f:(x,y)\mapsto\left(\frac{x}{1-\sqrt{x^2+y^2}},\frac{y}{1-\sqrt{x^2+y^2}}\right)$, i.e. $z\mapsto\frac{z}{1-|z|}$, $(\rho,\theta)\mapsto\left(\frac{\rho}{1-\rho},\theta\right)$, then f is continuous.

Note that $f^{-1}: w\mapsto\frac{w}{1+|w|}$ is also continuous $\Rightarrow f:\mathbb{D}\to\mathbb{C}$ is homeomorphism.

5. In general topology space it's not true: let $X=[0,1]$ with ~~usual topology, $Y=[0,1]$ with~~ trivial topology,

$f: X \to X$ is ~~identity map~~ $x \mapsto \frac{1}{2}x$, then f is continuous one to one map of a compact space X.

However, $f([0,1]) = [0,\frac{1}{2}]$ is not open in $X \Rightarrow f$ is not homeomorphism.

Assume $f: X \to Y$ is continuous one to one map from compact space X to Hausdorff space Y, then f is homeomorphism.

$\forall$ closed set $A \subset X$, X compact $\Rightarrow A$ compact $\Rightarrow f(A)$ compact, Y Hausdorff $\Rightarrow f(A)$ closed $\Rightarrow f$ is a closed map.

So f^{-1} is continuous $\Rightarrow f$ is homeomorphism.

6. Define $f(z) = \sqrt{z} = e^{\frac{1}{2}\log z}$ on $\mathbb{C} \setminus \mathbb{R}_{\le 0}$, then f is analytic. Note that $\sqrt{1+z} = f(1+z)$ is defined on $\mathbb{C} \setminus \mathbb{R}_{\le -1}$, $\sqrt{1-z} = f(1-z)$ is defined on $\mathbb{C} \setminus \mathbb{R}_{\ge 1}$ and both analytic, $\sqrt{1+z} + \sqrt{1-z}$ is defined on $\mathbb{C} \setminus \{\mathbb{R}_{\ge 1} \cup \mathbb{R}_{\le -1}\}$ and analytic.

7. (a) since $f \in C^1(\Omega)$, z_0 is differentiable at t_0, by chain rule $w'(t_0) = \frac{\partial f}{\partial x}\Big|_{z_0} x'(t_0) + \frac{\partial f}{\partial y}\Big|_{z_0} y'(t_0)$ is well-defined.

(b) $\frac{w'(t_0)}{z'(t_0)} = \frac{1}{2}\left(\frac{\partial f}{\partial x}\Big|_{z_0} - i \frac{\partial f}{\partial y}\Big|_{z_0}\right) + \frac{1}{2}\left(\frac{\partial f}{\partial x}\Big|_{z_0} + i \frac{\partial f}{\partial y}\Big|_{z_0}\right) \frac{\overline{z'(t_0)}}{z'(t_0)} \Rightarrow \frac{\partial f}{\partial x}\Big|_{z_0} = -i \frac{\partial f}{\partial y}\Big|_{z_0}$

~~Alternatively, denote~~ $\cancel{\frac{\frac{\partial f}{\partial x}|_{z_0} x'(t_0) + \frac{\partial f}{\partial y}|_{z_0} y'(t_0)}{x'(t_0) + i y'(t_0)} = \frac{w'(t_0)}{z'(t_0)} \in \mathbb{C}}$,