Tsinghua University

Numerical Analysis

Fall 2024

Homework 1

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- **Collaborators:** I finish this homework by myself.

Problem 1. Assume there exists $x_1, x_2, \cdots, x_{2n+1} \in [a, a+2\pi) \quad s.t.$

$$\begin{pmatrix} 1 \\ \cos x_i \\ \sin x_i \\ \vdots \\ \cos nx_i \\ \sin nx_i \end{pmatrix}$$

linearly dependent.

$i.e.\quad \exists a_1, \cdots, a_{2n+1} \in \mathbb{R}$, such that

$$\sum_{i=1}^{2n+1} a_i \begin{pmatrix} 1 \\ \cos x_i \\ \sin x_i \\ \vdots \\ \cos nx_i \\ \sin nx_i \end{pmatrix} = 0$$

Since $e^{ix} = \cos x + i \sin x$, we have

$$\sum_{j=1}^{n+1} (a_{2j-1} + a_{2j}) \begin{pmatrix} 1 \\ e^{ix_1} \\ \vdots \\ e^{ix_n} \end{pmatrix}$$

which is impossible since we know that the Vandermonde determinant is invertible. (In this equation, $a_{2n+2} = 0$)

Problem 2. Assume $\exists a = x_1 < x_2 < \cdots < x_N \le b$ such that $|\epsilon(x_i)| = \Delta(P), \epsilon(x_j) = (-1)^{j-1}\epsilon(x_1),\ j = 0, 1, \cdots,$.n Then $\forall Q \in \mathrm{Span}\{g_1, \cdots, g_N\}$, if $\Delta(Q) < \Delta(P)$, let

$$\eta(x) = P(x) - Q(x) = (P(x) - f(x)) - (Q(x) - f(x))$$

Then

$$\operatorname{sgn}(\eta(x_j)) = \eta(P(x_j) - f(x_j)) = \eta(\epsilon(x_j)) = (-1)^{j-1}, j = 0, 1, \cdots, n$$

So Q has at least n roots on $[a, b]$. Since $\{g_1, \cdots, g_n\}$ satisfies the Haar condition, $Q \equiv 0$.

So P is the best approximation of f.

Conversely, if P is the best approximation. If the result is not true, then we can divide $[a, b]$ into

$$[a, \zeta_1], [\zeta_1, \zeta_2], \cdots, [\zeta_N, b]$$

such that on each interval $\Delta(P)$ satisfies $N \leq n - 1$ and

$$-\Delta(P) \leq \epsilon(x) < \Delta(P) - \alpha$$

or

$$-\Delta(P) + \alpha \leq \epsilon(x) < \Delta(P)$$

Denote $\Phi(x)$ as an element with roots $\zeta_1, \cdots, \zeta_N$. (The existence because of Haar condition)

Then $Q(x) := P(x) + \omega\Phi(x)$ with difference

$$Q(x) - f(x) = P(x) - f(x) + \omega\Phi(x)$$

On $[a, b]$, $\Phi(x)$ is bounded. Take $|\omega|$ sufficiently small, and choose the signature of ω properly, we have

$$\Delta(Q) < \Delta(P)$$

which causes contradiction.

Here we end the proof.

Problem 3. Replace f with $f - p_n$. WLOG we assume the best approximation polynomial is 0.

If $\exists q_n$ such that

$$\|f - q_n\| < \|f\| + \lambda\|q_n\|$$

where $\lambda < \frac{1}{2}$.

For $\omega > 1$, if $|f(x)| < |q_n(x)|$, then $q_n(x)$, $f(x)$ have different signature or $q_n(x) - f(x), f(x), q_n(x)$ have the same signature. Therefore,

$$
\begin{aligned}
|f(x)-\omega q_n(x)| &= \begin{cases} \omega|q_n(x)|-|f(x)|, & \mathrm{sgn}(f(x))=\mathrm{sgn}(q_n(x)) \\ \omega|q_n(x)|+|f(x)|, & \mathrm{sgn}(f(x))\neq\mathrm{sgn}(q_n(x)) \end{cases} \\
&= \begin{cases} \omega|f(x)-q_n(x)|+(\omega-1)|f(x)|, & \mathrm{sgn}(f(x))=\mathrm{sgn}(q_n(x)) \\ \omega|f(x)-q_n(x)|-(\omega-1)|f(x)|, & \mathrm{sgn}(f(x))\neq\mathrm{sgn}(q_n(x)) \end{cases}
\end{aligned} \tag{3.1}
$$

Now if $\forall\lambda_m=\frac{1}{m}, m\geq 2$, $\exists\ q_m$ such that

$$\|f-q_m\|<\|f\|+\lambda_m\|q_m\|$$

Since $\|f-q_m\|\geq\|q_m\|-\|f\|$, we have $\|q_m\|<\frac{2}{1-\lambda_m}\|f\|<4\|f\|$.

So $\|q_m\|$ are uniformly bounded. Hence, $\{q_m\}$ is precompact in the polynomial space, or equivalently, there exists $q\in\mathbb{P}_n$ such that some subsequence $\{q_{m_i}\}$ converges to q.

As $m\to 0$, $\lambda_m\to 0$, then

$$\|f-q\|\leq\|f\|$$

So $q\equiv 0$.

So $\exists N>0$ such that $\forall i\geq N$, $\|q_{m_i}\|<\|f\|$.

Now for $x^i=\arg\max|f(x)-q_{m_i}(x)|$, since $|f(x^i)-q_{m_i}(x^i)|\geq\|f\|$, $q_{m_i}(x^i)$ and $f(x^i)$ have different signature. So $|f(x^i)|\geq\|f\|-|q_{m_i}(x^i)|$

By (3.1), we have for $\omega>1$,

$$
\begin{aligned}
|f(x^i)-\omega q_n(x^i)| &= \omega|f(x^i)-q_n(x^i)|-(\omega-1)|f(x^i)| \\
&< \omega(\|f\|+\lambda_{m_i}\|q_n\|)-(\omega-1)(\|f\|-|q_{m_i}(x^i)|) \\
&= \|f\|+\lambda_{m_i}\|\omega q_n\|+(\omega-1)|q_{m_i}(x^i)|
\end{aligned}
$$

Problem 4. For $x\in[a,b]$, WLOG assume $x\neq x_i$. ($x=x_i$ is trivial) Define

$$G(t)=R_{2n+1}(t)-\frac{\omega_{n+1}^2(t)}{\omega_{n+1}^2(x)}R_{2n+1}(x)$$

Then

$$G(x_i)=0, G(x)=0$$

So there are $n+2$ roots on $[a,b]$.

By Rolle's theorem, there are $n+1$ roots on $[a,b]\setminus\{x_0,\cdots,x_n,x\}$.

Since $G'(t)=R'_{2n+1}(t)-\frac{\omega_{n+1}(t)\omega'_{n+1}(t)}{\omega_{n+1}^2}R_{2n+1}(x)$, $G'(x_i)=0$.

So there are at least $2n+2$ roots on $[a,b]$ of G'.

Apply $2n+1$ times of Rolle's theorem to G', we obtain there is at least one root on $[a,b]$ of $G^{(2n+2)}$.

So $\exists\zeta \in [a,b]$, $0 = G^{(2n+2)}(\zeta) = f^{(2n+2)}(\zeta) - \frac{(2n+2)!}{\omega_{n+1}^2(x)}R_{2n+1}(x)$.

So $\exists\zeta \in [a,b]$, $R_{2n+1}(x) = \dfrac{f^{(2n+2)}(zeta)}{(2n+2)!}\omega_{n+1}(x)$

Problem 5.

Problem 6. 1

Problem 7. Noticed that $f(x) = -\frac{3}{4}(x-1)(x-2)(x+\frac{2}{3})+1$ satisfies

$$f(1) = f(2) = 1,\ f(0) = 0$$

and

$$f'(0) = 0$$

So $f(x)$ is what we need.