2.2 Electric flux, Gauss' law, Stoke's theorem

A main reference of this section is David Tong's lecture notes on vector calculus http://www.damtp.cam.ac.uk/user/tong/vc.html.

2.2.1 Vector calculus

As we have seen in the previous section, electric fields are vector fields, which assign a vector for every point in space $\mathbb{R}^3$. More precisely, a vector field $\vec{F}$ in d dimensions is a map

$$\vec{F} : \mathbb{R}^d \to \mathbb{R}^d \,. \tag{2.2.1}$$

Fields are fundamental concepts in physics. Besides vector fields, later on, we will encounter another kind of field called a scalar field, which assigns a number for every point in space. More precisely, a scalar field ϕ in d dimensions is a map

$$\phi : \mathbb{R}^d \to \mathbb{R} \,. \tag{2.2.2}$$

Vector fields and scalar fields are related by three important operations: gradient, divergence, and curl.

Let us introduce the first operation, the gradient denoted by **Grad**. It is an operation that takes a scalar field to a vector field.

Definition 13 (Gradient)**.** *Given Cartesian coordinates x^i with $i = 1, \cdots, d$ on $\mathbb{R}^d$, the gradient is defined by*

$$\mathbf{Grad} : \phi(\vec{r}) \mapsto \vec{\nabla}\phi(\vec{r}) \equiv \left(\frac{\partial \phi(\vec{r})}{\partial x_1}, \cdots, \frac{\partial \phi(\vec{r})}{\partial x_d} \right) \tag{2.2.3}$$

where $\phi(\vec{r})$ is a scalar field.

The above definition relies on the choice of Cartesian coordinates x^i. A coordinate-free definition is given by considering the difference between the scalar field ϕ evaluated at two nearby points $\vec{r}$ and $\vec{r} + \vec{\epsilon}$ with $\epsilon = |\vec{\epsilon}| \ll 1$,

$$\phi(\vec{r} + \vec{\epsilon}) - \phi(\vec{r}) = \vec{\epsilon} \cdot \vec{\nabla}\phi(\vec{r}) + O(\epsilon^2) \,, \tag{2.2.4}$$

where $O(\epsilon^2)$ denotes the terms that are of order at least ϵ^2. (2.2.4) could be regarded as an alternative definition of the gradient. When picking a choice of Cartesian coordinates with $\vec{r} = (x_1, \cdots, x_d)$ and $\vec{\epsilon} = (\epsilon_1, \cdots, \epsilon_d)$, we recover the definition (2.2.3).

We have already seen many examples of the gradient when we discussed the conservative force in Section 1.4.1. In particular, the gradient of the gravitational potential energy of between two point particles is computed explicitly in equations (1.4.30) and (1.4.31).

We can view $\vec{\nabla}$ as an object in its own right, and call it the *gradient operator*.

Definition 14 (Gradient operator)**.** *Given Cartesian coordinates x^i with $i = 1, \cdots, d$ on $\mathbb{R}^d$, the gradient operator is defined by*

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d} \right) . \tag{2.2.5}$$

It is a vector whose entries are partial derivatives.

The gradient operator is an example of the differential operator. A differential operator broadly means a collection of derivatives, that can act on some functions. For example, the one-variable derivative $\frac{d}{dt}$ is a differential operator that can act on a function $f(t)$ and gives $\frac{df(t)}{dt}$.

Besides acting on scalar fields, the gradient operator $\vec{\nabla}$ can act on other fields in different ways. The divergence, denoted by **Div**, is a way for the gradient operator to act on a vector field and produces a scalar field.

Definition 15 (Divergence)**.** *Given Cartesian coordinates x^i with $i = 1, \cdots, d$ on $\mathbb{R}^d$, the divergence is defined by*

$$\mathbf{Div} : \vec{F}(\vec{r}) \mapsto \vec{\nabla} \cdot \vec{F}(\vec{r}) \equiv \sum_{i=1}^{d} \frac{\partial F_i(\vec{r})}{\partial x_i} , \tag{2.2.6}$$

where $\vec{F}(\vec{r})$ is a vector field.

As an example, let us compute the divergence of the electric field of a charged particle.

Example. *By Coulomb's law (2.1.1), the electric field at $\vec{r}$ due to a point charge q located at the origin $\vec{r} = 0$ is given by*

$$\vec{E}(\vec{r}) = \frac{kq}{r^3} \vec{r} . \tag{2.2.7}$$

Compute $\vec{\nabla} \cdot \vec{E}(\vec{r})$.

Solution: Let us first compute $\frac{\partial E_x}{\partial x}$,

$$\begin{aligned} \frac{\partial E_x}{\partial x} &= \frac{\partial}{\partial x} \frac{kqx}{r^3} \\ &= \frac{kq}{r^3} - \frac{3}{2} \times 2(x' - x) \times \frac{kqx}{r^5} \\ &= \frac{kq}{r^3} - \frac{3kqx^2}{r^5} . \end{aligned} \tag{2.2.8}$$

The $\frac{\partial E_y}{\partial y}$ and $\frac{\partial E_z}{\partial z}$ can be computed in a similar way. Now, we sum up these three terms and find

$$\begin{aligned} \vec{\nabla} \cdot \vec{E}(\vec{r}) &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\ &= \frac{3kq}{r^3} - \frac{3kq \left[x^2 + y^2 + z^2 \right]}{r^5} \\ &= 0 . \end{aligned} \tag{2.2.9}$$

Naively, we may conclude that $\vec{\nabla} \cdot \vec{E}(\vec{r})$ is a scalar field that is identically zero. However, we need to be careful about the point at the origin $\vec{r} = 0$, where the electric field diverges. We will see later that $\nabla \cdot \vec{E}(\vec{r})$ cannot be zero at the origin $\vec{r} = 0$, but instead we actually have

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = 4\pi k q \delta^3(\vec{r})\,, \tag{2.2.10}$$

where $\delta^3(\vec{r})$ is the three-dimensional Dirac delta function

$$\delta^3(\vec{r}) \equiv \delta(x)\delta(y)\delta(z)\,. \tag{2.2.11}$$

The Dirac delta function $\delta(x)$ can be loosely thought of as a function on the real line, which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) \approx \begin{cases} +\infty & \text{for } x = 0\,, \\ 0 & \text{for } x \neq 0\,, \end{cases} \tag{2.2.12}$$

and is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x) = 1\,. \tag{2.2.13}$$

A formula for the Dirac delta function as a limit is

$$\delta(x) = \lim_{b \to 0^+} \frac{1}{b\sqrt{\pi}} e^{-\left(\frac{x}{b}\right)^2} \tag{2.2.14}$$

We will give a derivation of the statement (2.2.10) when we discuss the divergence theorem in the next subsection.

Physically, the formula (2.2.10) tells us that the divergence is an operation that measures the source (the charged particle) of the electric field. $\vec{\nabla} \cdot \vec{E}$ is non-zero at the position of the charged particle, and is zero everywhere else. We will study the following two examples that will further confirm our physical intuition.

Let us consider the divergence of the electric field generated by n charged particles.

Example. *Consider n charged particles with charges q_1, q_2, $\cdots$, q_n at the positions $\vec{r}_1$, $\vec{r}_2$, $\cdots$, $\vec{r}_n$. The electric field generated by them is*

$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} k q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3}\,. \tag{2.2.15}$$

Compute $\vec{\nabla} \cdot \vec{E}(\vec{r})$.

Solution: We note that the divergence is a linear operation. That is, for a linear combination of two vector fields

$$a_1 \vec{F}_1(\vec{r}) + a_2 \vec{F}_2(\vec{r})\,, \tag{2.2.16}$$

with a_1 and a_2 two constants independent of the position vector $\vec{r}$, we have

$$\vec{\nabla}\cdot\left[a_1\vec{F}_1(\vec{r})+a_2\vec{F}_2(\vec{r})\right]=a_1\vec{\nabla}\cdot\vec{F}_1(\vec{r})+a_2\vec{\nabla}\cdot\vec{F}_2(\vec{r})\,. \tag{2.2.17}$$

Hence, we have

$$\begin{aligned}\vec{\nabla}\cdot\vec{E}(\vec{r})&=\sum_{i=1}^{n}kq_i\vec{\nabla}\cdot\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}\\&=4\pi k\sum_{i=1}^{n}q_i\delta^3(\vec{r}-\vec{r}_i)\,.\end{aligned} \tag{2.2.18}$$

We see again that the divergence of the electric field is only non-zero at the place where the charged particles reside.

We can go one step further to compute the divergence of the electric field generated by a charged object.

Example. *The electric field generated by a charged object with a charge density $\rho(\vec{r})$ is*

$$\vec{E}(\vec{r})=\int_D k\rho(\vec{r}')\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)dx'dy'dz'\,, \tag{2.2.19}$$

where D is the domain of the charged object. Compute $\vec{\nabla}\cdot\vec{E}(\vec{r})$.

Solution: By the linearity of the divergence (2.2.16), we have

$$\begin{aligned}\vec{\nabla}\cdot\vec{E}(\vec{r})&=\int_D k\rho(\vec{r}')\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)dx'dy'dz'\\&=\int k\rho(\vec{r}')\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)dx'dy'dz'\\&=\int 4\pi k\rho(\vec{r}')\delta^3(\vec{r}-\vec{r}')dx'dy'dz'\\&=4\pi k\rho(\vec{r})\,,\end{aligned} \tag{2.2.20}$$

where we have used the fact that $\rho(\vec{r}')$ is zero at $\vec{r}'\notin D$ at the second equality, and (2.2.13) at the forth equality. We see that $\vec{\nabla}\cdot\vec{E}(\vec{r})$ gives the charge density of the object. It again confirms our intuition that the divergence measures the source of the electric field.

In three dimensions, there is another way for the gradient operator $\vec{\nabla}$ to act on a vector field, that is, by taking the cross product.

Definition 16 (Curl)**.** *Given Cartesian coordinates (x,y,z), the curl, denoted as* **Curl***, is defined by*

$$\textbf{Curl}:\ \vec{F}(\vec{r})\mapsto\vec{\nabla}\times\vec{F}(\vec{r})\equiv\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\,, \tag{2.2.21}$$

where $\vec{F}(\vec{r})$ is a vector field.

We see that **Curl** is an operation that takes a vector field to another vector field. Alternatively, the curl can be defined by

$$(\vec{\nabla} \times \vec{F})_i = \sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} , \tag{2.2.22}$$

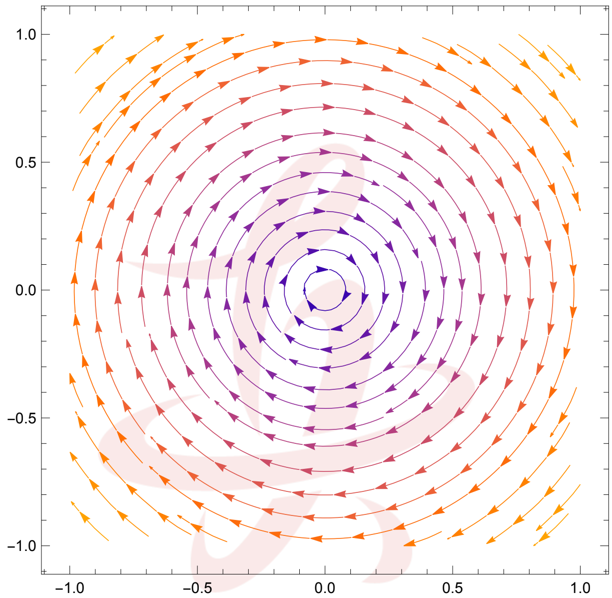
where ϵ_{ijk} is a totally antisymmetric tensor with $\epsilon_{123} = 1$.

The meaning of the curl is that it measures the rotation of a vector field. Let us try to understand this statement by looking at the following examples.

Example. *Compute the curl of the vector field*

$$\vec{F}(\vec{r}) = (y, -x, 0) , \tag{2.2.23}$$

whose field lines are plotted below



Solution: By a direct computation, we find

$$\vec{\nabla} \times \vec{F} = \left(0, 0, \frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\right) = (0, 0, -2) . \tag{2.2.24}$$

From the above figure, we see that $\vec{F}$ is a vector field that rotates clockwise on the x-y plane. Indeed, our computation shows that $\vec{\nabla} \times \vec{F}$ is a vector pointing in the negative z-direction, which is perpendicular to the x-y plane.

Next, we consider the curl on the electric fields. From the plots of the electric field lines in Figure 2.1, 2.2, and 2.3, we see that the static electric fields look in general not rotating. Let us verify our expectations.

Example. *Compute the curl of the electric field*

$$\vec{E}(\vec{r}) = kq\frac{\vec{r}}{r^3}\,. \tag{2.2.25}$$

Solution:Let us compute the z-component

$$\begin{aligned}(\vec{\nabla}\times\vec{E})_z &= kq\left(\frac{\partial}{\partial x}\frac{y}{r^3}-\frac{\partial}{\partial y}\frac{x}{r^3}\right)\\ &= kq\left(y\frac{-3x}{r^5}-x\frac{-3y}{r^5}\right)\\ &= 0\,.\end{aligned} \tag{2.2.26}$$

By similar computations, we find that the other two components of $\vec{\nabla}\times\vec{E}$ are also zero, and we conclude

$$\vec{\nabla}\times\vec{E}(\vec{r}) = 0\,. \tag{2.2.27}$$

We still need to worry about the point at $\vec{r}=0$ where the electric field diverges. We will see in Section 2.2.3 that $\vec{\nabla}\times\vec{E}$ is also zero at the origin.

Like divergence, curl is also a linear operation, i.e.

$$\vec{\nabla}\times(a_1\vec{F}_1+a_2\vec{F}_2) = a_1\vec{\nabla}\times\vec{F}_1+a_2\vec{\nabla}\times\vec{F}_2\,. \tag{2.2.28}$$

Then, by a similar argument as before for the divergence, we find that the electric fields of n charged particles or a charged object are nonrotational.

In Section 2.6, we will see that the electric fields can rotate when there are time-dependent magnetic fields.

2.2.2 Electric flux and Gauss's law

As we have learned from the previous subsection, the divergence provides a way to measure the sources of the electric fields. In this section, we will introduce Gauss's law, which is a very different way to measure the source of an electric field. The equivalence of these two ways leads to the divergence theorem.

The idea of Gauss's law is that given a closed surface S, we would like to know if there is any net charge inside S by analyzing the electric fields on S. To obtain more intuitions, let us consider the situation in which a charged particle is inside the closed surface S. When the charge of the particle is positive, the electric field lines of the particle are always pointing outward to the surface. Figure 2.4 shows the electric lines piercing a piece of the surface.

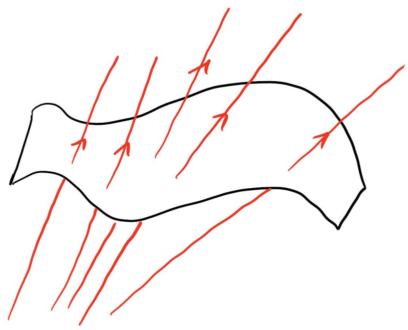


Figure 2.4: Electric field lines pierce a surface.

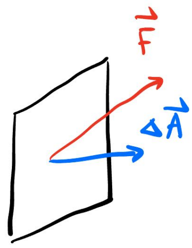
On the other hand, when the charge of the particle is negative, all the electric field lines are pointing inward to the surface. Hence, there should be a relation between the net number of electric field lines piercing a closed surface (the number of outward electric field lines minus the number of inward electric field lines) and the net electric charge inside the surface.

Let us try to make this relation more precise. First, we need to have a more precise definition of the "number of electric field lines piercing a surface". This leads to the following definition. We would like to make our discussion a bit more general, by working in d-dimensional space $\mathbb{R}^d$, and the "surfaces" in the following are specifically referred to the $(d-1)$-dimensional surfaces in $\mathbb{R}^d$. But, you could always fix $d = 3$ if you like.

Definition 17 (Flux)**.** *The flux Φ of a vector field $\vec{F}$ through an oriented surface S is defined as the integral*

$$\Phi = \int_S \vec{F} \cdot d\vec{A}\,. \tag{2.2.29}$$

Let us try to decode this definition. Consider a very small piece of the surface with area ΔA, which is small enough such that we could approximate it by a plane as shown in the following picture.



We define the area vector $\Delta\vec{A}$ by the following two conditions:

1. $|\Delta\vec{A}| = \Delta A\,,$
2. $\Delta\vec{A}$ is orthogonal to all the vectors along the plane, i.e. $\Delta\vec{A}\cdot\vec{v} = 0$ if $\vec{v}$ is along the plane.

The vector $d\vec{A}$ is defined as the limit when the area becomes infinitesimal. These two conditions fix the area vector $\Delta\vec{A}$ up to a sign; hence, infinitesimal area vector $d\vec{A}$ is also only defined up to a sign. This sign ambiguity is a $\mathbb{Z}_2$ valued function on the surface S. To make the flux well-behaved, we would like to require that the $d\vec{A}$ should vary continuously locally on the surface S. More precisely, we demand the condition:

3. In any open neighborhood on the surface S, the area vector $d\vec{A}$ is continuous.

Hence, the sign ambiguity is a constant $\mathbb{Z}_2$ valued function in any open neighborhood on S. If we could find a constant $\mathbb{Z}_2$ valued function on the entire surface S, then we have a way to consistently fix the sign for all the area vector $d\vec{A}$ on S. In fact, not every surface admits a constant $\mathbb{Z}_2$ valued function. The surfaces that admit constant $\mathbb{Z}_2$ valued functions are called *orientable surfaces*, otherwise are called *non-orientable surfaces*. We could only define the flux for orientable surfaces. For orientable surfaces, the constant $\mathbb{Z}_2$ valued functions are called the *orientations* of the surface. The orientable surfaces with a chosen orientation are called *oriented surfaces*. We would regard the orientation as part of the definition of an oriented surface. That is, two oriented surfaces that coincide in space are regarded as different oriented surfaces if they have different orientations. For a closed oriented surface, we choose our convention that its orientation is always pointing outward the surface.

In (2.2.29), we take the inner product between $d\vec{A}$ and the vector field $\vec{F}$ and integrate over the closed surface S. Let $d = 3$ and the vector field $\vec{F}$ be the electric field. The formula (2.2.29) defines the *electric flux*, which is our precise definition of the "number of electric field lines piercing a surface". We can see that the definition (2.2.29) of the electric flux agrees with our expectation. Namely, when we have a positively (negatively) charged object inside the closed surface S, we find a positive (negative) flux. We will see later a precise formula (Gauss' law) on the relation between the flux through a closed surface and the net charge inside that surface.

Now, from our discussions in the previous subsection and this subsection, we have seen two ways to find the sources (charged objects) of the electric fields, by the divergence and by the flux. These two ways are beautifully related by the divergence theorem, also known as Gauss' theorem.

Theorem 2.2.1 (Divergence theorem)**.** *For a vector field $\vec{F}$ over $\mathbb{R}^d$,*

$$\int_B \vec{\nabla}\cdot\vec{F}\, d^d x = \int_S \vec{F}\cdot d\vec{A}\,, \tag{2.2.30}$$

where B is a bounded region whose boundary $\partial B = S$ is å piecewise smooth closed $(d-1)$-dimensional surface.

Let us leave the proof of the divergence theorem to your analysis class. Instead, we will try to understand the physical meaning of the divergence theorem. We will again focus on $d = 3$ and $\vec{F} = \vec{E}$.

Example. *Consider a charged particle of charge q at the origin $\vec{r}=0$, which generates the electric field*

$$\vec{E}(\vec{r}) = kq\frac{\vec{r}}{|\vec{r}|^3}\,. \tag{2.2.31}$$

Compute the flux of the electric field through a round two-sphere S^2 of radius R centered at the origin.

Solution: By the spherical symmetry, the vector $d\vec{A}$ is along the radial direction, and we have

$$d\vec{A} = \hat{r}|d\vec{A}| = \hat{r}dA\,. \tag{2.2.32}$$

It is convenient to work in the spherical coordinates (r,θ,ϕ), which is related to the Cartesian coordinate (x,y,z) by

$$\begin{aligned} x &= r\sin\theta\cos\phi\,,\\ y &= r\sin\theta\sin\phi\,,\\ z &= r\cos\theta\,, \end{aligned} \tag{2.2.33}$$

where r, θ, ϕ are in the range $r\in[0,\infty)$, $\theta\in[0,\pi]$, $\phi\in[0,2\pi)$. We would like to know how to perform the volume integral and the surface integral in the spherical coordinates.

Let us consider a more general problem, the volume integral in a general coordinate system (u,v,w). Consider a small cube in the (u,v,w)-coordinate, whose six faces are on the constant u, v, or w planes. The sides of the cube have lengths Δu, Δv, and Δw. The area of the cube is not simply given by $\Delta u\Delta v\Delta w$, because the sides are not at necessarily right angles. When the cube is small enough, we have

$$\begin{aligned} \Delta x &= \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v + \frac{\partial x}{\partial w}\Delta w + \cdots\,,\\ \Delta y &= \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v + \frac{\partial y}{\partial w}\Delta w + \cdots\,,\\ \Delta z &= \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v + \frac{\partial z}{\partial w}\Delta w + \cdots\,. \end{aligned} \tag{2.2.34}$$

where the $\cdots$ are the second and higher order terms $O(\Delta u^2,\Delta v^2,\Delta w^2,\Delta u\Delta v,\Delta v\Delta w,\Delta u\Delta w)$. In the matrix form, we have

$$\begin{pmatrix}\Delta x\\ \Delta y\\ \Delta z\end{pmatrix} = \begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\\ \Delta w\end{pmatrix}\,. \tag{2.2.35}$$

Geometrically, this means that the cube in the (u,v,w)-coordinate is a parallelepiped in the (x,y,z)-coordinate with sides given by the vectors

$$\begin{aligned} \overrightarrow{\Delta u} &= \begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\end{pmatrix}\Delta u\,,\\ \overrightarrow{\Delta v} &= \begin{pmatrix}\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\end{pmatrix}\Delta v\,,\\ \overrightarrow{\Delta w} &= \begin{pmatrix}\frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w}\end{pmatrix}\Delta w\,. \end{aligned} \tag{2.2.36}$$

The volume of the parallelepiped is given by

$$(\overrightarrow{\Delta u} \times \overrightarrow{\Delta v}) \cdot \overrightarrow{\Delta w} = |J| \Delta u \Delta v \Delta w \,, \tag{2.2.37}$$

where J is called Jacobian and given by

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} . \tag{2.2.38}$$

Let us compute the Jacobian for the spherical coordinate

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \det \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2 \sin\theta \,. \tag{2.2.39}$$

The volume of a cube in the spherical coordinate is

$$r^2 \sin\theta \Delta r \Delta\theta \Delta\phi \,. \tag{2.2.40}$$

The integration measure in the spherical coordinate should be

$$dxdydz = r^2 \sin\theta dr d\theta d\phi \,. \tag{2.2.41}$$

Now, since a round two-sphere centered at the origin has a constant radial coordinate $r = R$, the area vector $d\vec{A}$ should be

$$d\vec{A} = \hat{r} dA = \hat{r}(r^2 \sin\theta d\theta d\phi) \,. \tag{2.2.42}$$

The flux is then computed by the integral

$$\Phi = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, kq \frac{\vec{r}}{r^3} \cdot \hat{r}(r^2 \sin\theta) = 4\pi kq \,. \tag{2.2.43}$$

Now, we could complete our argument that a three-dimensional Dirac delta function should sit at the right-hand side of the equation (2.2.10). From our previous computations (2.2.8) and (2.2.9), we know that the divergence of the electric field $\vec{\nabla} \cdot \vec{E}$ is zero except at the origin where the point charge sits. $\vec{\nabla} \cdot \vec{E}$ cannot be zero at the origin, because by the divergence theorem (2.2.30), we know that

$$\int_B \vec{\nabla} \cdot \vec{E} \, d^3x = 4\pi kq \,. \tag{2.2.44}$$

In fact, the volume integral on the left-hand side only receives a contribution from the point at the origin $\vec{r} = 0$. Hence, $\vec{\nabla} \cdot \vec{E}$ must be proportional to a Dirac delta function in order to give nonzero volume integral. The proportionality constant can be fixed by (2.2.44) to be $4\pi kq$.

The divergence theorem has a very profound consequence on the electric flux.

Corollary 2.2.2. *For a divergence-free vector field $\vec{F}$, i.e. $\vec{\nabla} \cdot \vec{F} = 0$, its flux through a surface S is invariant under local continuous deformation of S.*

To see this corollary, let us consider the following example.

Example. *A charged object of a charge density $\rho(\vec{r})$ is inside a closed surface S as shown in Figure 2.5.*

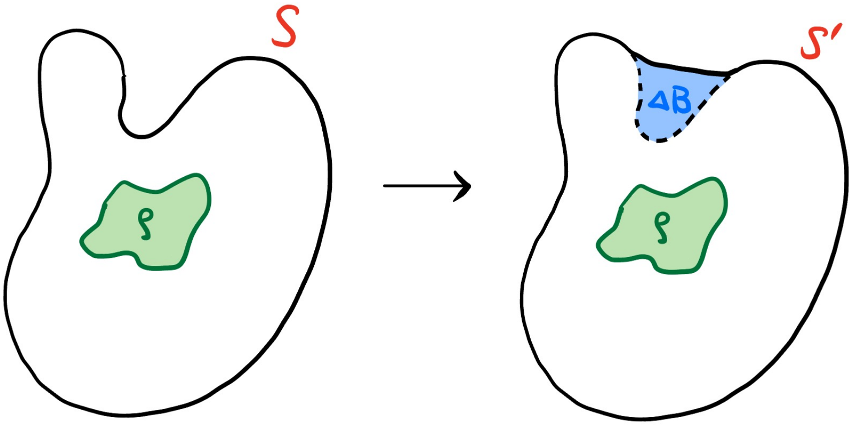


Figure 2.5: The electric flux is invariant under deformations on S as long as the deformation does not cross any charges.

Now let us consider continuously deforming the surface S to S', and let ΔB be the region bounded by the two surfaces S and S', i.e. $\partial(\Delta B) = S \cup \overline{S'}$, where $\overline{S'}$ denotes the orientation reversal of S'. Assume that there is no charged object inside ΔB. We can compute the difference of the electric flux through S and through S',

$$\Delta\Phi = \int_S \vec{E}\cdot d\vec{A} - \int_{S'} \vec{E}\cdot d\vec{A} = \int_{S\cup\overline{S'}} \vec{E}\cdot \vec{dA} = \int_{\Delta B} \vec{\nabla}\cdot\vec{E} d^3x = 0\,, \tag{2.2.45}$$

where at the last equality we used the fact that there is no charged object insider B so $\vec{\nabla}\cdot\vec{E} = 0$ inside B.

We can compute the electric flux directly using the divergence theorem and the formula (2.2.20). Let B be the region bounded by S. We have

$$\Phi = \int_S \vec{E}\cdot d\vec{A} = \int_B \vec{\nabla}\cdot\vec{E} d^3x = 4\pi k \int_B \rho(\vec{r}) d^3x = 4\pi k Q\,, \tag{2.2.46}$$

where Q is the total (net) charge inside the surface S. We have found that the electric flux through a surface S equals $4\pi k$ times the total net charge inside S. This statement is called *Gauss' law* of the electric field. The surface S is also called *Gaussian surface.*

Gauss' law has many important applications. For example, it implies that if an isolated conductor carries an excess charge, the charge would be entirely on the surface of the conductor. *Conductors* are materials in which charged particles (electrons) are free to move; examples include metals (such as copper in common lamp wire), the human body, and tap water. The charged particles in nonconductors (*insulators*) are not free to move; examples include rubber (such as the insulation on common lamp wire), plastic, glass, and chemically pure water. The net electric field inside a conductor must be zero, because, in a generic situation, the electric field would not always point in the normal direction to the surface of the conductor, and would exert forces on the charged particles to make them move and redistribute. Eventually, an equilibrium configuration would be achieved, such that there is no net force on any charged particles; hence, no net electric field inside the conductor.

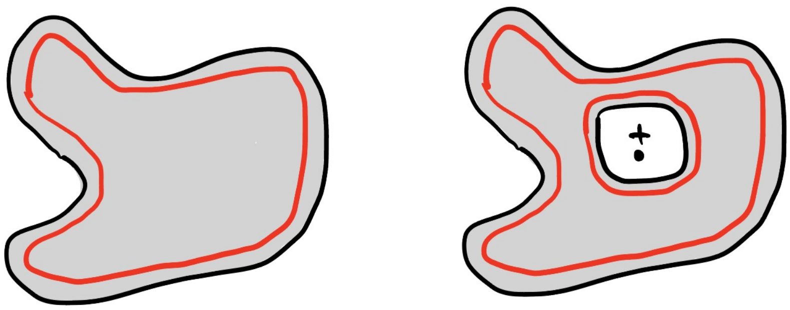


Figure 2.6: Cross-sections of conductors.

The left picture in Figure 2.6 shows the cross-section of a conductor. We can consider a Gaussian surface that is very close to the surface of the conductor, shown as the red curve in the picture. There is no electric flux through this Gaussian surface, because, as we just argued, there is no net electric field inside the conductor. By Gauss' law, the net charge inside the Gaussian surface must be zero. By shrinking the Gaussian surface to a smaller size inside the conductor, we can further argue that not just the net charge is zero, but the charge density at every interior point is zero. The left picture in Figure 2.6 shows a more complicated situation, a conductor with a cavity that contains a positive charge. In this case, there are non-zero electric fields in the cavity. We choose a Gaussian surface to be the union of two surfaces, shown as the red curves, that are very close to the inner and outer surfaces of the conductor. Again, there is no electric flux through either the Gaussian surface; hence, the excess charge in the conductor should be on the inner or outer surface of the conductor.

2.2.3 Stoke's theorem

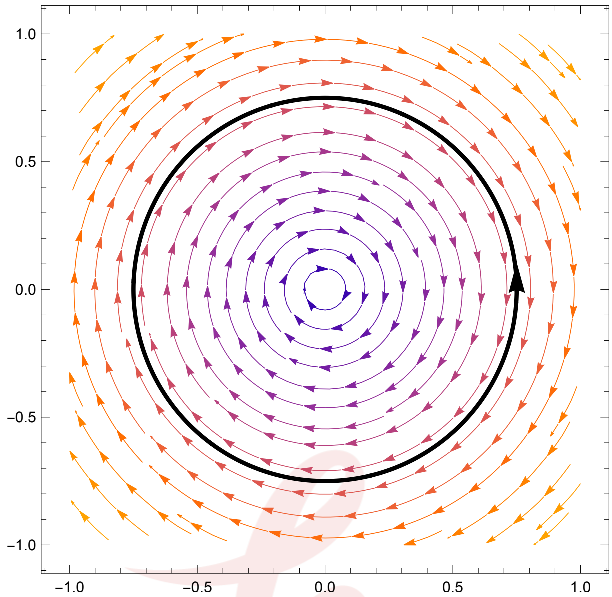
In Section 2.2.1, we have seen that curl measures the rotation of a vector field. Let us introduce a different way to measure the rotation of a vector field. The rotation of a vector field $\vec{F}$ along a closed curve $\mathcal{C}$ can be measured by the following loop integral:

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}. \tag{2.2.47}$$

Example. *Compute the loop integral (2.2.47) of the vector field*

$$\vec{F}(\vec{r}) = (y, -x, 0)\,, \tag{2.2.48}$$

along a curve $\mathcal{C}$ which is a counter-clockwise circle of radius r centered at the origin as shown in the following figure



Solution: We compute

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = -\int_{\mathcal{C}} |\vec{F}||d\vec{r}| = -\int_0^{2\pi} r^2 d\theta = -2\pi r^2\,, \tag{2.2.49}$$

where at the first equality we use $\vec{F}\cdot d\vec{r} = |\vec{F}||d\vec{r}|$ because the vector field $\vec{F}$ is always in the opposite direction as $d\vec{r}$. We see that the loop integral is indeed nonzero for a rotating vector field.

Let us also consider an example of the loop integral (2.2.47) for a nonrotating vector field

Example. *Compute the loop integral (2.2.47) of the electric field of a point charge q at the origin,*

$$\vec{E}(\vec{r}) = kq\frac{\vec{r}}{r^3}\,. \tag{2.2.50}$$

Solution: We compute

$$\int_{\mathcal{C}} \vec{E}\cdot d\vec{r} = \int_{\mathcal{C}} \vec{\nabla}\frac{kq}{r}\cdot d\vec{r} = 0\,, \tag{2.2.51}$$

where we have used the fact that the electric field (2.2.50) can be written as the gradient of a scalar field. We see that the loop integral indeed vanishes for the nonrotating field (2.2.50).

The equivalence of the two ways of measuring the rotation of the vector field, by taking curl and by the loop integral, leads to Stoke's theorem.

Theorem 2.2.3 (Stoke's theorem)**.** *Let S be a smooth surface in $\mathbb{R}^3$ with boundary $\mathcal{C} = \partial S$ a piecewise smooth curve. For any smooth vector field $\vec{F}(\vec{r})$, we have*

$$\int_S \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{A} = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} \,. \tag{2.2.52}$$

We will again leave the proof of Stoke's theorem to your analysis class.

Instead, let us verify Stoke's theorem for the examples (2.2.48) and (2.2.50). First, we have computed the curl of (2.2.48) previously in (2.2.24). We have

$$\int_S \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{A} = \int_S (0, 0, -2) \cdot d\vec{A} = \int_0^r dr' \int_0^{2\pi} d\theta(-2r) = -2\pi r^2 \,. \tag{2.2.53}$$

We got the same answer as the loop integral (2.2.49).

Next, we look at the example (2.2.50). We found that the curl of (2.2.50) was zero previously in (2.2.26) (except at the origin). Now, we can give an argument that $\vec{\nabla} \times \vec{E}$ is also zero at the origin. From Stoke's theorem (2.2.52) and the loop integral (2.2.51) we have

$$\int_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{A} = 0 \,. \tag{2.2.54}$$

We can take the surface $S = S_\epsilon$ to be a small disc of radius ϵ centering at the origin of the x-y plane. In the limit ϵ, we find

$$(\vec{\nabla} \times \vec{E})(0) = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \int_{S_\epsilon} \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{A} = \lim_{\epsilon \to 0} \frac{0}{\pi \epsilon^2} = 0 \,. \tag{2.2.55}$$

As discussed in Section (2.2.1), static electric fields are always nonrotational. There is a beautiful theorem that says that nonrotational is equivalent to conservative.

Theorem 2.2.4 (Poincaré lemma)**.** *For fields defined everywhere on $\mathbb{R}^3$, conservative is the same as nonrotational, i.e.*

$$\vec{F} = \vec{\nabla}\phi \Longleftrightarrow \vec{\nabla} \times \vec{F} = 0 \,. \tag{2.2.56}$$

Proof: First, let us prove the $\Rightarrow$ direction. We have

$$\vec{F} = \vec{\nabla}\phi \,, \tag{2.2.57}$$

or in component form

$$F_i = \frac{\partial \phi}{\partial x_i} \,. \tag{2.2.58}$$

Now, we compute the components of $\vec{\nabla} \times \vec{F}$ using (2.2.22),

$$(\vec{\nabla} \times \vec{F})_i = \sum_j \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} = \sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} = \sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} = 0 \,, \tag{2.2.59}$$

where we have used the fact that partial derivatives commute with each other at the last equality.

Next, let us prove the $\Leftarrow$ direction. We assume that $\vec{F}$ is a nonrotational field,

$$\vec{\nabla} \times \vec{F} = 0\,. \tag{2.2.60}$$

Let us define a scalar field $\phi(\vec{r})$ by the line integral

$$\phi(\vec{r}) = \int_{\mathcal{C}(\vec{r}_0,\vec{r})} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'\,, \tag{2.2.61}$$

where $\mathcal{C}(\vec{r}_0, \vec{r})$ is a curve from $\vec{r}_0$ to $\vec{r}$, and $\vec{r}_0$ is any fixed reference point. This line integral defines an unambiguous scalar field because it only depends on the boundary points of the curve but not the curve itself. To see this, let us consider the difference of the line integral along the curves $\mathcal{C}_1(\vec{r}_0, \vec{r})$ and $\mathcal{C}_2(\vec{r}_0, \vec{r})$,

$$\begin{aligned}\int_{\mathcal{C}_1(\vec{r}_0,\vec{r})} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,' - \int_{\mathcal{C}_2(\vec{r}_0,\vec{r})} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,' &= \int_{\mathcal{C}_1(\vec{r}_0,\vec{r})\cup\overline{\mathcal{C}_2(\vec{r}_0,\vec{r})}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,' \\ &= \int_S \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{A} \\ &= 0\,,\end{aligned} \tag{2.2.62}$$

where at the second equality we have used Stock's theorem for the surface S with boundary $\partial S = \mathcal{C}_1(\vec{r}_0, \vec{r}) \cup \overline{\mathcal{C}_2(\vec{r}_0, \vec{r})}$.

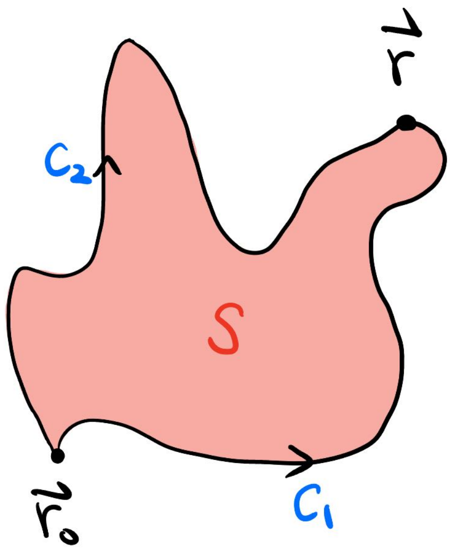


Figure 2.7: Changing the curve in the definition of the scalar field.

Now, it is easy to check that the scalar field $\phi(\vec{r})$ defined in (2.2.61) satisfies

$$\vec{F} = \vec{\nabla}\phi\,, \tag{2.2.63}$$

Hence, $\vec{F}$ is conservative. Q.E.D.

Since static electric fields are always nonrotational, they are always conservative. In other words, for a electric field $\vec{E}(\vec{r})$, there always exists a scalar field $V(\vec{r})$ such that

$$\vec{E} = -\vec{\nabla} V\,. \tag{2.2.64}$$

This scalar field $V(\vec{r})$ is called a *electric potential*.