Differential Geometry

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Contents

1 Smooth Manifold

Definition 1.1 (Topological manifold)**.** A space M is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

Definition 1.2 (Smooth Manifold)**.** A smooth structure is given by an equivalence class of smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ *s.t.* $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth $\forall \alpha, \beta$. $M = \cup U_\alpha$.
A **smooth manifold** is a topological manifold with a smooth structure.
Define when a continuous map $f : M_1 \to M_2$ is smooth if $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$, we have $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is smooth.

Definition 1.3. Given $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$. A homeomorphism $f : M_1 \to M_2$ is called a diffeomorphism if f, f^{-1} is smooth.
In this case we say $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ are diffeomorphism.

Theorem 1.4 (Kervaire)**.** *$\exists$ 1 10-dimensional topological manifold without smooth manifold.*

Theorem 1.5 (Milnor)**.** *$\exists$ a smooth manifold M s.t. $M \cong S^7$ but not in diffeomorphism meaning.*

Theorem 1.6 (Kervaire-Milnor)**.** *$\exists$ 28 smooth structures (up to orientation preserving diffeomorphism) on S^7*

Theorem 1.7 (Morse-Birg)**.** *On S^7. If $n \leqslant 3$, then any n-dimensional topological manifold M has a unique smooth structure up to diffeomorphism.*

Theorem 1.8 (Stallings). *If $n \neq 4$, then $\exists$ a unique smooth structure on $\mathbb{R}^n$ up to diffeomorphism.*

Theorem 1.9 (Donaldson-Freedom-Gompf-Faubes). *$\exists$ uncountable smooth structures on $\mathbb{R}^4$ up to diffeomorphism.*

Definition 1.10 (topological manifold with boundary). A space M is called a topological manifold with boundary if

1. M is Hausdorff
2. M is second countable
3. $\forall p \in M$, $\exists$ a neighbourhood U of p and a homeomorphism $\varphi : U \to V$ where V is open in $\mathbb{H}^n$

We say a manifold M is closed if M is compact and ∂M is empty.

Our motivation for studying manifold is to study the space of solution for equations.

Question 1. Given $f : \mathbb{R}^n \to \mathbb{R}$ smooth, $q \in \mathbb{R}^n$, when is $f^{-1}(q)$ is a smooth manifold?

For $f : U \to \mathbb{R}^n$ smooth, U open in $\mathbb{R}^m$, the differential of f at $p \in U$ denoted as $\mathrm{d}f(p)$.

Definition 1.11. We say $p \in U$ is a **regular point** of f if $\mathrm{d}f(p)$ is surjective. Otherwise we say $p \in U$ is a **critical point**.
A point $q \in \mathbb{R}^n$ is called a **regular value** of f if $\forall p \in f^{-1}(q)$, p is a regular point of f.
A point $q \in \mathbb{R}^n$ is called a **critical value** of f if $\forall p \in f^{-1}(q)$, p is a critical point of f.

Theorem 1.12 (Implicit function theorem). *If $p \in U$ is a regular point of $f : U \to \mathbb{R}^n$. Then there exists*

- *An open neighbourhood V of p in U*
- *An open subset V' of $\mathbb{R}^m$*
- *A diffeomorphism $\varphi : V \to V'$ such that $P \circ \varphi = f$ where P is the projection from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

In other words, near a regular point, we can do local coordinate change to turn f into the projection.

Remark 1.13. In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \ldots, x_m) \in V' | (x_1, \cdots .x_n) = f(p)\}$$

i.e. if we set $M = f^{-1}(f(p))$, then $(M \cap V, \varphi_p)$ is a chart that contains p.

Corollary 1.14. *If q is a regular value of $f : U \to \mathbb{R}^n$ then $f^{-1}(q)$ is a smooth manifold.*

Remark 1.15. It suffices to show that the corresponding charts are compatible.

Theorem 1.16 (Sard). *If $f : U \to \mathbb{R}^n$ is a smooth map, then the set of critical values of f has measure 0.*

Remark 1.17. For a "generic" q, $f^{-1}(q)$ is a manifold of dimension $m - n$.

Corollary 1.18. *If $f : U \to \mathbb{R}^n$ is smooth and $m < n$ then $f(U)$ has measure 0.*

1.1 Lie groups and homogeneous spaces

Definition 1.19. We say G is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map $\cdot : G \times G \to G$ and the inverse map $G \rightsquigarrow G$ is smooth.

Example 1.20. $GL(n, \mathbb{R}) = \{n \times n$ matrices with non-zero determinant$\} \subset \mathbb{R}^{n\times n}$

$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$

$SO(n) = \{A \in O(n) | \det A = 1\}$

$U(n) = \{A \in GL(n, \mathbb{C}) | A\overline{A}^T = 1\}$

$SU(n) = \{A \in U(n) | \det A = 1\}$

Exercise 1.21.

$$O(1) \cong S^2 \qquad SO(1) \cong * \tag{1.1}$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \tag{1.2}$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \tag{1.3}$$

The last one is a diffeomorphism but do not preserve the multiplicatioin, *i.e.* not an isomorphism of Lie group.

Theorem 1.22 (Carton)**.** *Let H be a closed subgroup of Lie group G. Then H is a Lie group. More precisely, H is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also, G/H is a smooth manifold*

Definition 1.23. Let M be a smooth manifold. We say M is a **homogeneous space** if $\exists$ a Lie group G with a smooth transitive action $\rho : G \times M \to M$.

Definition 1.24. For M be a homogeneous space. The **isotropy** group of $x \in M$ is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of G

Given any $x, x' \in M$, $Iso(x) \cong Iso(x')$ because the group action is transitive.

Hence, we have a well-defined map

$$p : G/_{Iso(x)} \to M \tag{1.4}$$

$$g \mapsto gx \tag{1.5}$$

Theorem 1.25. *p is always a diffeomorphism.*

Therefore, we have this proposition

Proposition 1.26. *M is a homogeneous space $\Leftrightarrow$ $M = G/H$ for some closed subgroup H.*

Example 1.27. If $M = S^n$, let $G = SO(n+1)$.
Then $Iso(1, 0, \cdots, 0) \cong SO(n)$.
So $S^n \cong SO(n+1)/(SO(n))$.
Similarly, we can prove $\mathbb{RP}^n \cong SO(n+1)/(O(n))$, $\mathbb{CP}^n \cong SO(n+1)/(U(n))$
The isotropy k dimensional linear subspaces of $\mathbb{R}^n$ can be $O(k) \times O(n-k)$ if $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to $\mathbb{RP}^2, S^2, T^2$ and Klein bottle.

Theorem 1.28 (Whithead)**.** *Any smooth manifold has a triangulation.*

Theorem 1.29 (Poincare-Hopf)**.** *G is compact Lie group $\Rightarrow \chi(G) = 0$.*

Theorem 1.30 (Mostow2005)**.** *M is a compact homogeneous space $\Rightarrow \chi(M) \geqslant 0$.*

1.2 Bump Function and Partition of Unity

Theorem 1.31 (Urysohn smooth version)**.** *Given M, closed disjoint A, B, $\exists$ smooth $f : M \to [0,1]$ s.t. $f|_A = 0, f|_B = 1$.*

Theorem 1.32 (Tietze)**.** *Given M, closed A, smooth $f : A \to \mathbb{R}^n$, there exists smooth $\hat{f} : M \to \mathbb{R}^n$ s.t. $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

Lemma 1.33. *Let U be a neighbourhood of $p \in M$. Then $\exists$ smooth $\sigma : M \to [0, 1]$ s.t.*

1. *$\sigma \equiv 1$ near p*

2. *Supp $\sigma \subset U$*

*Such σ is called a **bump function** at p, supported in U.*

Definition 1.34. An open cover of a space X is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

Proposition 1.35. *Given compact $K \subset U$ and open neighbourhood U of K, $\exists$ a smooth $g : M \to [0, +\infty)$ s.t. $g|_K \equiv 1$ and Supp $g \subset U$.*

Definition 1.36. An **exhaust** of a space X is a sequence of open sets $\{U_i\}$ *s.t.*

1. $X = \bigcup\limits_{i=1}^{\infty} U_i$

2. $\overline{U_i}$ is compact and contained in U_{i+1}

Theorem 1.37. *Any topological manifold has an exhaust.*

Given two open covers $\mathcal{U}, \mathcal{V}$, we say $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if $\forall U_\alpha \in U$, $\exists V_\beta \in \mathcal{V}$ *s.t.* $V_\beta \subset U_\alpha$.

We say a space X is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact.(The proof is hard)

Proposition 1.38. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a topological manifold M. Then there exists countable open covers $\mathcal{W} = \{W_i\}$, $\mathcal{V} = \{V_i\}$ s.t.*

- *For any i, $\overline{V_i}$ is compact and $\overline{V_i} \subset W_i$*

- *$\mathcal{W}$ is locally finite.*

- *$\mathcal{W}$ is a refinement of $\mathcal{U}$.*

As a corollary, we have any topological manifold is paracompact.

Definition 1.39. Given open cover $\mathcal{U}$ of a smooth M, a partition of unity subordinate to $\mathcal{U}$ is a collection of smooth functions $\{\rho_\alpha : M \to [0,1]\}_{\alpha\in\mathcal{A}}$ *s.t.*

1. $\forall p \in M$, $\exists$ only finitely many $\alpha \in \mathcal{A}$ *s.t.* $p \in Supp\, \rho_\alpha$

2. $\sum_{\alpha\in\mathcal{A}} \rho_\alpha(p) = 1$

3. $Supp\, \rho_\alpha \subset U_\alpha$

Theorem 1.40 (Existence of P.O.U). *For any open cover $\mathcal{U}$ of smooth M, $\exists$ a P.O.U subordinate to $\mathcal{U}$*

Theorem 1.41 (Whitney approximation theorem). *Given any smooth M, any closed A and any continuous $f : M \to \mathbb{R}$, $\delta : M \to (0,+\infty)$. Suppose f is smooth on A. Then $\exists g : M \to \mathbb{R}$ smooth s.t.*

- $g|_A = f|_A$

- $\forall p \in M$, $|g(p) - f(p)| < \delta(p)$.

2 Tangent space and tangent vectors

2.1 Tangent Space

Given $p \in M$, consider the set $C_p^\infty(M) = \{\text{smooth function } V \to \mathbb{R}\}/_\sim$ where $f_1 \sim f_2$ if and only if $\exists$ neighbourhood U of p, $f_1|_U = f_2|_U$.

$C_p^\infty(M)$ is the space of **genus of smooth function** near p.

A **partial-derivative** of p is a $\mathbb{R}$-linear map $D : C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

Definition 2.1. A **tangent vector** of M at p is a partial-derivative at p.

Define the **tangent space** $T_pM = \{$all partial-derivative at p $\}$, which is a $\mathbb{R}$-vector space.

Proposition 2.2. *For $M = U \subset \mathbb{R}^n$ open. We have $\{\frac{\partial}{\partial x_i}\}$ is a basis for T_pU.*

Proposition 2.3.

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leqslant i \leqslant n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

M, N smooth manifolds, $C^\infty(N, M) = \{$smooth $F : N \to M\}$.

Given $F \in C^\infty(N, M)$, F induces $F^* : C^\infty_{F(p)}(M) \to C^\infty_p(N)$, $f \mapsto f \circ F$.

By taking dual, we get

$$F_* : T_pN \to T_{F(p)}M$$

we also write F_* as $F_{*,p}$, call it the **differential** of F at p.

where

$$F_*(\frac{\partial}{\partial x^i}|_p) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

Proposition 2.4. *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_pN \to T_{G \circ F(p)}W$$

Definition 2.5. A smooth **curve** is a smooth map $\gamma : (a, b) \to M$. We say γ starts at p if $\gamma(0) = p$. We define the **velocity** of γ at $\gamma(0)$ as $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)}M$

Take charts $(U, x^1, \cdots, x^n)$ about p, let $\gamma^i = x^i \circ \gamma$.

We say γ, δ are **tangent** to each other at p if $(\gamma^i)'(0) = (\delta^i)'(0)$.

Now we can define

$$(T_pM)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\ \}/_\sim$$

where $\gamma \sim \delta$ iff they are tangent to each other.

Then these definition is more geometric.

Lemma 2.6. *Given $F \in C^\infty(M, M)$, $p \in N$, the diagram commutes:*

$$\begin{array}{ccc} \gamma \in (T_pN)_{curve} & \xrightarrow{\cong} & T_pN \\ \downarrow & & \downarrow \\ F \circ \gamma \in (T_{F(p)}M)_{curve} & \xrightarrow{\cong} & T_{F(p)}M \end{array}$$

2.2 Tangent Bundle

Let $(M, \mathcal{A})$ be a smooth manifold, $TM = \bigcup_{p \in M} T_pM$, called the **tangent bundle**

Now we want to define a natural topology and smooth structure on TM. Take any chart $(U, \varphi) = (U, x^1, \cdots, x^n) \in \mathcal{A}$.

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \tag{2.1}$$

$$X \in T_pU \mapsto (\varphi(p), X^1, \cdots, X^n) \tag{2.2}$$

where $X = \sum X^i \frac{\partial}{\partial x^i}|_p$.

Then pull back standard topology on $\varphi(U) \times \mathbb{R}^n$ to a topology on TU.

$$\mathcal{B} = \{\hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- $\mathcal{B}$ is a basis
- $\mathcal{B}$ generates a Hausdorff, second countable topology on TM.

So TM is a topological manifold covered by charts $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$.

Given $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$, the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi}\circ\hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \tag{2.3}$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \tag{2.4}$$

So $\hat{\mathcal{A}}$ is a smooth atlas on TM, making TM into a smooth manifold.

Definition 2.7 (vector bundle)**.** Given a continuous map $f : E \to B$, we say f is a n-dimensional **vector bundle** if: $\exists$ an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of B and homeomorphisms $\{f^{-1}(U_\alpha) \underset{\cong}{\xrightarrow{\rho_\alpha}} U_\alpha \times \mathbb{R}\}$ *s.t.*

- $$\begin{tikzcd} f^{-1}(U_\alpha) \arrow[r, "\rho_\alpha"] \arrow[d, "f"] & U_\alpha \times \mathbb{R}^n \arrow[dl, "projection"] \\ U_\alpha & \end{tikzcd}$$ commutes for $\alpha \in I$

- $\forall\, p \in U_\alpha \cap U_\beta$, the map

  $$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

  is linear.

Call $f^{-1}(p)$ the **fiber** over p.

Proposition 2.8. *Given vector bundle $f : E \to B$, the fiber $f^{-1}(p)$ has a structure of a vector space.*

Example 2.9 (Product bundle). $E = \mathbb{R}^n \times B$

Example 2.10 (Tautological bundle).

$$B = \mathbb{CP}^n = \{\text{1-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map $(L, v) \mapsto L$

Given vector bundles $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$, a bundle map consists of $(\hat{f}, f)$ *s.t.*

- $\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$ commutes.
- $\forall b \in B$, $\hat{f} : \pi_1^{-1}(b) \to \pi_2^{-1}(f(b))$ is linear.

If $\hat{f}, f$ are diffeomorphisms, then we call $(\hat{f}, f)$ an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

Example 2.11. TS^1, TS^2 are both trivial.

$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$

Theorem 2.12. *If G is a Lie group, then TG is trivial.*

Proof. For $(x^1, x^2, \cdots, x^n)$ is a basis of T_eG The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG,\ (g, c^1, \cdots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \to G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism $(l_g)_*$

□

Proposition 2.13 (Adams, 1960s). *TS^n is trivial if and only if $n = 0, 1, 3, 7$.*

Proposition 2.14. *1. Given any $F \in C^\infty(M, N)$, $F_* : TM \to TN$ is a bundle map.*

2. TS^n is isomorphic to the following bundle:

$$B = s^n \qquad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

Definition 2.15 (smooth section). Given a smooth vector bundle $\pi : E \to B$, a **smooth section** is a smooth map $S : B \to E \quad s.t. \ \pi \circ S = id_b$.

$s_0 : B \to E, b \mapsto 0 \in$ 0-vector in $\pi^{-1}b$.

2.3 Vector Field, Curves and Flows

Definition 2.16. A (tangent) **vector field** is a smooth section of TM. *i.e.* a smooth map $M \xrightarrow{X} TM \quad s.t. \ X(p) \in T_pM, \forall p \in M$

Given any $f : \mathbb{R}^n \to \mathbb{R}$, define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leqslant i \leqslant n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

Example 2.17. $X = f^1 \partial x^1 + f^2 \partial x^2$ is a gradient field if and only if $\dfrac{\partial f^1}{\partial x^2} = \dfrac{\partial f^2}{\partial x^1}$

Theorem 2.18 (Poincare-Hopf). *For closed M, M has a nowhere vanishing vector field if and only if $\chi(M) = 0$.*

So S^n has a nowhere vanishing vector field if and only if n is odd.

Theorem 2.19 (MaoQiu). *S^2 has no no-where vanishing vector field.*

So We cannot smooth out all the hairs on a ball.

Given a vector field $X = \{X_p\}_{p \in M}$, a curve $\gamma : (a,b) \to M$ is called an **integral curve** of X if $\gamma'(t) = X_{\gamma(t)}$, $\forall t \in (a,b)$, where $\gamma'(t) = \gamma_*(\frac{\partial}{\partial t}) \in T_{\gamma(t)}M$.

We say γ is maximal if the domain cannot be extended to a larger interval. Denote the set of all smooth vector fields on M by $\mathfrak{X}M$

Recall that γ is maximal if it's domain can not be extended to a large open interval.

In a local chart $(U, x^1, \cdots, x^n)$, $X|_U = \sum_{i=1}^{n} a^i \partial x^i$. Then γ is an integral curve if and only if $(\gamma^i)'(t) = a^i(\gamma(t))$, $\forall 1 \leqslant i \leqslant n$, where $\gamma^i = x^i \circ \gamma : (a,b) \to \mathbb{R}$.

And in this case the initial value condition: $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$.

Locally, solving integral curve starting at p is equivalent to solving ODE with initial value $p^1, \cdots, p^n$. By existence and uniqueness of solutions of ODE, we have

Theorem 2.20 (Fundamental theorem of integral curve). *Let $X \in \mathfrak{X}M$, $p \in M$, then:*

(1) (Uniqueness) Given any two integral curves $\gamma_1, \gamma_2 : (a,b) \to M$, then we have:

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a,b) \Rightarrow \gamma_1 = \gamma_2$$

(2) there exists a unique max integral curve $\gamma : (a(p), b(p)) \to M$ starting at p.

(3) (integral curve smoothly depend on initial values) $\exists$ Nbh U of p, $\epsilon > 0$, and smooth $\varphi : (-\epsilon, \epsilon) \times U \to M$ s.t. $\forall q \in U$, $\varphi_\epsilon := \varphi(-, q) : (-\epsilon, \epsilon) \to M$ is an integral curve starting at p.

*we call such φ a local **flow** generated by X.*

Definition 2.21. Given $X \in \mathfrak{X}M$, a global **flow** generated by X is a smooth map $\varphi : \mathbb{R} \times M \to M \quad s.t.\ \forall q \in M, \varphi_q := \varphi(-, q)$ is the maximal integral curve of X starting at q.

$\Leftrightarrow \dfrac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s,p)}, \forall s \in \mathbb{R}, p \in M$ and $\varphi(0, p) = p, \forall p \in M$.

If such global flow exists, then we say X is **complete**.

Example 2.22.

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$ is complete, where global flow $\varphi : \mathbb{R} \times M \to M$, $\varphi(t, p) = p \cdot e^t$.
- $X = x^2 \partial x$ is not complete. Max integral curve starting at 1 is given by $\gamma(t) = \dfrac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$.

Given $X \in \mathfrak{X}M$, we define $\mathrm{Supp} X = \overline{\{p \in M : X_p \neq 0\}}$.

Theorem 2.23. *If a vector field X is compactly supported, then X is complete.*

Corollary 2.24. *Any vector field on closed manifold is complete.*

Lemma 2.25 (Escaping lemma)**.** *Suppose $\gamma : (a, b) \to M$ is a max integral curve, with $(a, b) \neq \mathbb{R}$. Then $\nexists$ compact $K \subset M \quad s.t.\ \gamma(a, b) \subset K$*

Proof. Otherwise, suppose $\gamma(a, b) \subset K$. WLOG, we may assume $b < +\infty$.

Take $(t_i) \to b$ from left. Then $\gamma(t_i) \in K$. After passing to subsequence, we may assume $(\gamma(t_i)) \to p \in K$.

Then $\exists\ U$ Nbh of p, local flow $\varphi : (-\epsilon, \epsilon) \times U \to M$. Take n large enough $\quad s.t.$ $b - t_n < \epsilon, \gamma(t_n) \in U$. Then $\gamma(- + t_n) : (a - t_n, b - t_n) \to M$, $\varphi(-, \gamma(t_n)) : (-\epsilon, \epsilon) \to M$ are both integral curves for X starting at $\gamma(t_n)$. By uniqueness, they coincide.

Let $\hat{\gamma} : (a, t_n + \epsilon) \to M$ be defined by $\hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \epsilon) \end{cases}$

Then $\hat{\gamma}$ is an integral curve with larger domain, then γ contradiction with the maxity of γ. $\square$

Proof of 2.23. Take any max integral curve $\gamma : (a,b) \to M$. Suppose $(a,b) \neq \mathbb{R}$. Then $X_{\gamma(s)} \neq 0$, $\forall s$. Otherwise, the constant map $\mathbb{R} \to M, t \mapsto \gamma(s)$ is an integral curve with lager domain.

So $\forall s$, $\gamma(s) \in \operatorname{Supp} X \Rightarrow \gamma(a,b) \subset \operatorname{Supp} X$ which is compact $\Rightarrow (a,b) = \mathbb{R}$ by the lemma. This causes contradiction! □

A smooth $\varphi : \mathbb{R} \times M \to M$ is called an **one-parameter transformation group** if

(1) $\varphi_0 := \varphi(0, -) = \mathrm{id}_M$

(2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}$. In particular, $\varphi_s^{-1} = \varphi_{-s}$.

Theorem 2.26. *$\varphi \in C^\infty(\mathbb{R} \times M, M)$, then φ is an one-parameter transformation group if and only if φ is the global flow generated by some $X \in \mathfrak{T}M$*

Lemma 2.27 (Translation lemma)**.** *If $\gamma : (a,b) \to M$ is an integral curve for some $X \in \mathfrak{T}M$, then $\forall s \in \mathbb{R}$, $\gamma(- + s) : (a - s, b - s) \to M$ is also an integral curve for X.*

Proof. Let $\iota = \gamma(- + s)$. Then $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$ □

Lemma 2.28. *Let $\varphi : (-\epsilon, \epsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{T}M$. Then $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$ provided that $s, t, s + t \in (-\epsilon, \epsilon), p, \varphi_r(p) \in U$.*

Proof. $\gamma_p = \varphi(-, p)$ is an integral curve for X.

$\Rightarrow \gamma_p(- + s)$ is an integral curve for X starting at $\gamma_p(s) = \varphi_s(p)$. But $\gamma_{\varphi_s(p)}$ is also an integral curve starting at $\varphi_s(p)$. Thus $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$ □

Lemma 2.29. *Let $\varphi : (-\epsilon, \epsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{T}M$. Then $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$ i.e. any vector field is invariant under its flow.*

Proof. Take $f \in C^\infty_{\varphi(p)}(M)$.

$$\varphi(s,*)(X_p)(f) = X_p(f \circ \varphi_s) \tag{2.5}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \tag{2.6}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \tag{2.7}$$
$$= X_{\varphi_s(p)}(f) \tag{2.8}$$

□

Proof of 2.26. "$\Leftarrow$" is because the lemma $\varphi_s \circ \varphi_r = \varphi_{s+r}$

"$\Rightarrow$" Let $X = \{X_p\}$ where $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$.

Leave it as an exercise. □

Time dependent vector field is a smooth map $X : \mathbb{R} \times M \to TM$ *s.t.* $X_{(t,p)} \in T_pM$.

A smooth curve $\gamma(a,b) \to M$ is the **integral curve** for X if $\gamma'(t) = X_{(t,\gamma(t))}$.

In local chart, solving γ is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness: γ_1, γ_2 are both integral curves for X, $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define $\mathrm{Supp}X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$.

Then X is compactly supported, then X is complete(*i.e.* a global flow $\varphi : \mathbb{R} \times M \to M$)

But something is not true for time dependent vector field:

- translation lemma is not true.

- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

2.4 Another definition of vector field

A derivation on M is a $\mathbb{R}$-linear map $C^\infty(M) \xrightarrow{D} C^\infty(M)$ that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

Theorem 2.30. *We have a bijection:*

$$\rho : \mathfrak{T}M \xrightarrow{1:1} \{\textit{derivation on } M\}$$

$$X \mapsto D_X : f \mapsto X(f)$$

Lemma 2.31. $D_p : \mathfrak{T}_pM \to$ *$\mathbb{R}$-linear map* $\mathbb{C}^\infty(M) \to \mathbb{R}$ *s.t.* $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$ *is an isomorphism of vector spaces.*

Proof. Leave it as an exercise. □

Lemma 2.32. *Given a vector field(not necessarily smooth)* $X = \{X_p\}_{p\in M}$ *, X is smooth* $\Leftrightarrow \forall f \in C^\infty(M), X(f)$ *is smooth.*

Proof. "$\Leftarrow$" $\forall p \in M$, take chart $(U, x^1, x^2, \cdots, x^n)$ around p. $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$ $f^i : U \to \mathbb{R}$, where $f^i = X|_U(x^i)$. Take $\varphi : M \to [0,1]$ *s.t.* $\varphi \equiv 1$ near p, Supp$\varphi \subset U$,$\varphi \cdot x^i \in C^\infty(M)$.

Then $X(\varphi \cdot x^i) = f^i$ near p. By assumption, f^i is smooth near p. So f^i is smooth, so X is smooth.

"$\Rightarrow$" Similar. □

Theorem 2.33. *The map* $\rho : \mathfrak{X}M \to \{\textit{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$ *is well-defined and bijective.*

Proof. ρ is well-defined: $X(f) \in C^\infty(M)$ by Lemma 2.32, and $D_x(fg) = D_x(f)g + fD_x(g)$ since X is a point-derivation.

ρ is injective: $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$ as maps $C^\infty(M)$ to $\mathbb{R}$. By Lemma 2.31, we have $X_p = Y_p$, $\forall p$. So $X = Y$.

ρ is surjective: Given $D : C^\infty(M) \to C^\infty(M)$. Define $D_p : C^\infty(M) \to \mathbb{R}$ by $D_p(f) := D(f)(p)$ satisfies the Leibniz rule. By Lemma 2.31, $D_p = D_{X_p}$ for some $X_p \in T_pM$. Define $X = \{X_p\}_{p\in M}$. Then $X(f) = D(f)$, $\forall f \in C^\infty(M)$. By Lemma**??**, X is a smooth vector field. ∎

2.5 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned}\{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \quad \textit{s.t.} \\ &\qquad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\}\end{aligned}$$

$$\begin{aligned}\{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\}\end{aligned}$$

Notation 2.34. We will identify $X \in \mathfrak{X}M$ with its derivation $D_x : C^\infty(M) \to C^\infty(M)$. So a vector field is just a $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M)$ *s.t.* $X(fg) = fX(g) + X(f)g$.

Definition 2.35 (Lie bracket)**.** Given two (smooth) vector field $X, Y : C^\infty(M) \to C^\infty(M)$, we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

Theorem 2.36. *For any $X, y \in \mathfrak{T}M$, $[X, Y] \in \mathfrak{T}M$*

Proof. Easy to check that $[X, Y]$ is linear.

By Leibuniz rule,

$$\begin{aligned}
[X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\
&= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\
&= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\
&= [X, Y](f) \cdot f \cdot [X, Y](g)
\end{aligned}$$

□

So What is the geometric meaning of $[X, Y]$? Non commutatiy of flows.

Fact 2.37. *Given $X, Y \in \mathfrak{T}M$, we say X, Y are commutative vector field if $[X, Y] = 0$*

X, Y are commuative iff for any local flows $\varphi^X : (-\epsilon, \epsilon) \times U \to M$, $\varphi^Y : (-\epsilon, \epsilon) \times U \to M$ we have $\varphi_s^X \circ \varphi_t^T = \varphi_t^Y \circ \varphi_s^X$

Proposition 2.38 (Calculation of $[V, W]$ using local charts)**.** *Chart $(U, x^1, \cdots, x^n)$, $V, W \in \mathfrak{T}M$, $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$, $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$. Then*

$$\begin{aligned}
[V, W]|_U &= \sum_{i=1}^n (V(W^i - W(V^i))) \frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^n (\sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial X^j}) \frac{\partial}{\partial x^j}
\end{aligned}$$

$$= \sum_{1\leqslant i,j\leqslant n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^j}$$

Example 2.39. $V = x\partial x + y\partial y$, $W = -y\partial x + x\partial y$ commutes.

Proposition 2.40 (Properties of Lie bracket)**.**

(a) Natuality under push-forword.

Given any $F \in \text{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, *we have* $[F_*V, F_*W] = F_*[V, W]$.

(b) $\mathbb{R}$*-linearity* $\forall a, b \in \mathbb{R}$

$$[aX + bV, W] = a[X, W] + b[V, W]$$
$$[W, aX + bV] = b[W, X] + a[W, V]$$

(c) anti-symmetric $[V, W] = -[W, V]$

(d) Jacobi identity

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

(f) Leibuniz rule

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

Definition 2.41. Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}N$. We say W is F**-related** to V if $\forall p \in M$, $F_{p,*}(V_p) = W_{F(p)}, F_{p,X} : T_pM \to T_{f(p)}N$

Example 2.42. $F : S^1 \to \mathbb{R}^2, \theta \mapsto (\cos\theta, \sin\theta)$, $V = \partial\theta, W = -y\partial x + x\partial y$.

note 1. In general, given $V \in \mathfrak{X}M$ and $F \in C^\infty(M, N)$. There may not exist $W \in \mathfrak{X}M$ *s.t.* V, W are F-related. Even such W exists, it may not be unique.

However, if F is a diffeomorphism, given any V, $\exists$ unique W *s.t.* V and W are F-related. Actually, $W_p = F_* V_{F^{-1}(p)}$.

Such W is called **push forward** of V along F, denoted by $F_* V$, only defined when F is a diffeomorphism.

Lemma 2.43. $\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N$, $F \in C^\infty(M, N)$. *Then W is F-related to V iff* $\forall f \in C^\infty(N)$, $V(f \circ F) = W(f) \circ F \in C^\infty(M)$

Proof. Check that $F_{p,*}(V_p)(f) = W_{F(p)}(f), \forall f \in C^\infty(N)$ $\square$

Proposition 2.44. *Given* $V_0, V_1 \in \mathfrak{X}M$, $W_0, W_1 \in \mathfrak{X}N$, $F \in C^\infty(M, N)$, W_i *is F-related to* V_i, $i = 0, 1 \Rightarrow [W_0, W_1]$ *is F-related to* $[V_0, V_1]$

Corollary 2.45 (Natuality of Lie bracket). *Given any* $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, *we have* $[F_* V, F_* W] = F_*[V, W]$

The rest of Proposition 2.40 is easy to check if it is viewed as a mapping $C^\infty(M) \to C^\infty(M)$.

2.6 Lie algebra of a Lie group

Definition 2.46. A **Lie algebra** g is $\mathbb{R}$-linear space g with map $[-,-] : g \times g \to g$ *s.t.* it is bilinear, anti-symmetric and satisfies the Jacobian identity.

Then $(\mathfrak{X}M, [-,-])$ is an infinite dimensional Lie algebra.

For G Lie group, $\forall g \in G$ we have diffeomorphism

$$l^g : G \to G, h \mapsto gh$$
$$r^g : G \to G, h \mapsto hg$$

We say $X \in \mathfrak{X}G$ is **left invariant** if $l^g_*(X) = X$, $\forall g \in G$. Similarly, X is **right invariant** if $r^g_*(X) = X$.

Proposition 2.47. *X, Y are left/right invariant $\Rightarrow [X, Y]$ is left/right invariant.*

Proof. $l^g_*[X, Y] = [l^g_* X, l^g_* Y] = [X, Y]$ □

So we can find a natural Lie algebra of G:

$$\mathrm{Lie}(G) := \{\text{left invariant vector fields on } G\}, \text{with } [-,-] \text{ restricted from } \mathfrak{X}G$$

Theorem 2.48. *Given any $V \in T_e G$, $\exists$ unique left invariant $\hat{V} \in \mathfrak{X}G$ s.t. $\hat{V}_e = V$.*

Corollary 2.49. $\mathrm{Lie}(G) \cong T_e G$ *as vector spaces.*

Proof of Theorem 2.48. **Uniqueness of $\hat{V}$:** $\hat{V}_g = l^g_{e,*}(\hat{V}_e) = l^g_{e,*}(V)$. So $\hat{V}$ is determined by V.

Existence of $\hat{V}$: Let $\hat{V} = \{\hat{V}_g\}_{g \in G}$ where $\hat{V}_g = l^g_{e,*}(\hat{V}_e)$.

$\hat{V}$ is left-invariant because

$$(l^h_*(\hat{V}))_g = l^h_{h^{-1}g,*}(\hat{h^{-1}g}) = l^h_{h^{-1}g,*}(l^{h^{-1}g}_{e,*}(V)) = l^g_{e,*}(V) = \hat{V}_g$$

$\hat{V}$ is smooth: Take any $f \in C^\infty(G)$ suffices to show $\hat{V}(f) \in C^\infty(G)$.

Take any smooth $\gamma : \mathbb{R} \to G$ *s.t.* $\gamma(0) = e, \gamma'(0) = V$. Then $l^g \circ \gamma : \mathbb{R} \to G$ satisfies $l^g \circ \gamma(0) = g, (l^g \circ \gamma)(0) = g, (l^g \circ \gamma)'(0) = l^g_{e,*}(V) = \hat{V}_g$

So

$$\hat{V}(f)(g) = \hat{V}_g(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(l^g \circ \gamma(t))|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(g \cdot \gamma(t))|_{t=0} \tag{2.9}$$

Consider the map

$$\hat{f} : G \times \mathbb{R} \xrightarrow{\mathrm{id} \times \gamma} G \times G \qquad\qquad \xrightarrow{\cdot} G \xrightarrow{f} \mathbb{R}$$

$$(g,t) \mapsto (g,\gamma(t)) \qquad\qquad \mapsto g\cdot\gamma(t) \mapsto f(g\cdot\gamma(t))$$

Then $\hat{f}$ s smooth, $\frac{\partial \hat{f}}{\partial t}|_{t=0} : G \to \mathbb{R}$ is smooth, but $\frac{\partial f}{\partial t}|_{t=0}(g) = \hat{V}(f)(g)$ by 2.9. So $\hat{V}(f) \in C^\infty(G)$. □

Example 2.50. $G = \mathrm{GL}(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) |\det A \neq 0\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^2$.

$\mathrm{gl}(n,\mathbb{R})$= $\mathrm{Lie}(\mathrm{GL}(n,\mathbb{R})) = T_I\mathrm{GL}(n,\mathbb{R}) = M_n(\mathbb{R})$

Theorem 2.51. $\forall A,B \in \mathrm{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$, $[A,B] = AB - BA$.

Remark 2.52. This theorem shows that the Lie bracket viewed as the Lie algebra and matrix are the same. In some sense, it means the Lie bracket defined in three sets $\mathrm{gl}(n,\mathbb{R}) = T_I\mathrm{GL}(n,\mathbb{R}) = M_n(\mathbb{R})$ can commute with those corresponding, or equivalently, are just the same.

Lemma 2.53. $\forall A \in \mathrm{gl}(n,\mathbb{R})$, *the left invariant vector field $\hat{A}$ is complete and generated the flow* $\varphi_t : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2t^2}{2!} + \cdots)$

Proof.

$$\hat{A}_g = g\cdot A \in T_gG = M_n(\mathbb{R})$$

$$\frac{\partial}{\partial t}\varphi_t(g) = \frac{\partial}{\partial t}(g(e^{At})) = ge^{At}A = \hat{A_{g\cdot e^{At}}} = \hat{A}_{\varphi_t(g)}$$

□

Proof of Theorem 2.51. Take $A,B \in \mathrm{gl}(n,\mathbb{R})$. Want to show $[\hat{A},\hat{B}]_I = AB - BA$.

Pick $f \in C_I^\infty(G)$, need to show $A(\hat{B}(f)) - B(\hat{A}(f)) = (AB - BA)(f)$

Further Simplification: Just need to focus on $f = x^{ij}$, where $x^{ij} : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}, E \mapsto (E - I)_{ij}$.

Such f satisfies $f(I + -)$ is $\mathbb{R}$-linear.

Recall that Given $W \in \mathfrak{X}M$, $W(f)(p) = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t^W(p))|_{t=0}$.

So $\hat{B}(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(ge^{tB})|_{t=0}$.

So

$$A(\hat{B}(f)) = \frac{\mathrm{d}}{\mathrm{d}t}(\hat{B}(f)(e^{As}))|_{s=0} = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t} f(I+sA+tB+\frac{s^2}{2}A^2+stAB+\frac{t^2}{2}B^2+\cdots)|_{s=t=0}$$

Similarly,

$$B(\hat{A}(f)) = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t} f(I+sA+tB+\frac{s^2}{2}A^2+stBA+\frac{t^2}{2}B^2+\cdots)|_{s=t=0}$$

So $A(\hat{B}(f)) - B(\hat{A}(f)) = f(I + (AB - BA)) = (AB - BA)(f)$ since f is $\mathbb{R}$-linear. □

Similarly, for $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{Lie}(G) = \mathrm{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$, we have $[A, B] = AB - BA$.

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group G, its universal cover $\hat{G}$ is simply-connected with $\pi^{-1}(G) \subset Z(\hat{G})$.

What is the meaning of Lie bracket. There is a fact about it:

Fact 2.54. *G is connected Lie group. G is abelian iff* $[-,-] = 0$ *on* $\mathrm{Lie}(G)$

2.7 Morphisms between Lie group and Lie algebras

A smooth map $F : G \to H$ between two Lie group is called a **morphism** if $F(gh) = F(g)F(h)$.

A linear map $L : g \to h$ between Lie algebra is called a **morphism** if $L[u, v] = [Lu, Lv]$.

Claim 2.55. *$\hat{W}_i$ is F-compatible with $\hat{V}_i$ for $i = 0, 1$.*

Proof. $\forall g \in G$, $F_*(\hat{V}_g) = F_*(l^g_*(V)) = (F \circ l^g)_*(V) = (l^{F(g)} \circ F)_*(V) = l^{F(g)}(W) = \hat{W}_{F(g)}$ □

In particular, $[W_0, W_1] = F_*([V_0, V_1])$.

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