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Physics-0 Lecture Notes

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1.3 Forces in Nature, Statics

1.3.1 The four fundamental forces in nature

Our universe is governed by the **four fundamental forces** that play a crucial role in how everything interacts with one another. These four forces are

- gravitational force
- electromagnetic force
- weak nuclear force
- strong nuclear force.

Each of these forces plays a unique role in the workings of the universe, from the behavior of objects on a planetary scale to the interactions of subatomic particles.

In quantum theory, the concept of forces as we know it in classical physics becomes somewhat ambiguous. Instead, what we call forces are usually interpreted as the exchange of particles between objects, which is more accurately described as an interaction. Therefore, it is more appropriate to refer to the "**four fundamental interactions**" rather than the "four fundamental forces".

Here is a brief summary of these four fundamental forces:

1. **Gravitational Force.** Gravity is the force that governs the behavior of objects on a large or macroscopic scale. It is responsible for keeping planets in orbit around stars, and causing apples and other objects to fall towards the ground. This force is described by Newton's law of universal gravitation, which states that every particle in the universe is attracted to every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them:

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}. \tag{1.3.1}$$

Comparing to other forces, the Newtonian constant of gravitation $G \approx 6.67 \times 10^{-11}\mathrm{m}^3/\mathrm{kg}{\cdot}\mathrm{s}^2$ is relatively small. We will discuss Newton's theory of gravity in detail in section 1.6 of this lecture.

The best theory for gravity to date is Einstein's theory of General Relativity. It describes gravity as the curvature of spacetime manifold caused by the presence of massive objects. And masses move according to the curvature of the manifold. The basic equation of general relativity is Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.3.2}$$

where the left-hand side describes the geometry of spacetime and the right-hand side describes the distribution of matter and energy. J. A. Wheeler's famous statement summarizes the core idea of general relativity as *"Spacetime tells matter how to move; matter tells spacetime how to curve."*

2. **Electromagnetic Force.** Electromagnetic force unifies the electric force, which describes how electric charges attract or repel each other and interact with electric fields, and the magnetic force, which describes the interaction of moving charged objects and magnetic fields. For example, the Coulomb's law states that the force between two stationary, electrically charged particles is

$$\vec{F} = \frac{kq_1q_2}{r^2}\hat{r}, \tag{1.3.3}$$

which is very similar to Newton's law of universal gravitation Eq. (1.3.1). Because almost all everyday objects have electric charges, the electromagnetic force plays a significant role in many everyday phenomena, including the operation of electronic devices, the behavior of a compass, the colors we see, and the heat and light we feel from the sun. Our lecture will delve deeper into the topic of electromagnetism in the second half, offering a more detailed and comprehensive discussion on the subject.

3. **Weak Nuclear Force.** The weak interaction plays a critical role in governing the decay of unstable subatomic particles, initiating nuclear fusion reactions in stars like the Sun, and underlying some forms of radioactivity. It involves the exchange of force-carrier particles called W and Z particles, which are relatively heavy with masses around 100 times that of a proton. Interestingly, the weak interaction is the only fundamental interaction known to break parity symmetry. The idea of parity violation was proposed in the mid-1950s by Chen-Ning Yang and Tsung-Dao Lee and later confirmed experimentally by Chien Shiung Wu. The weak force violates parity symmetry, which means that it treats left-handed and right-handed particles differently. As a result, the concept of left-handedness and right-handedness has physical significance in the weak interaction, making them fundamentally different: The nature is left/right handed.

4. **Strong Nuclear Force.** The strong interaction is the force that binds protons and neutrons together in the nucleus of an atom. The strong interaction is responsible for the stability of atomic nuclei, and without it, the nucleus would quickly fall apart. The strong force is mediated by particles known as gluons, which are exchanged between quarks to hold them together. The strong force is very strong at very short distances, but it quickly decreases in strength at longer distances.

These four fundamental forces behave very differently. For example, the interaction range of them is different. The strong nuclear force has a very short range, only acting over distances of the order of a few femtometers (10^{-15} meters). The weak nuclear force has a range of about 10^{-18} meters, while the electromagnetic force and gravity have infinite range. These differences originate from the masses of the mediating particles. As a result, in our everyday life, we can only observe and experience the electromagnetic force and the gravitational force easily.

Besides the interaction ranges, the strengths of these four fundamental forces are also different:

$$\text{strong nuclear force} > \text{electromagnetic force} > \text{weak nuclear force} > \text{gravitational force}.$$

A notable example that illustrates the relative strength of electromagnetic force compared to gravitational force is that we are able to stand on the ground, rather than being pulled directly to the center of the Earth by gravity. This is because the electromagnetic force between the atoms in our feet and the atoms in the ground (only atoms near the feet!) is much stronger than the gravitational attraction between us and the Earth (the whole Earth!).

The history of physics is a history of unification, where theories are developed to explain multiple phenomena with a single framework. For example, Newton's theory of gravity unified the falling of an apple and the motion of the moon around the Earth. Maxwell's theory unified electricity and magnetism. While three of the four fundamental forces have been successfully unified by gauge theory, gravity remains incompatible with quantum theory. Despite decades of effort, it is still unknown how to reconcile Einstein's theory of gravity with quantum mechanics.

1.3.2 Some particular forces

There are several types of forces that we encounter in our daily life quite frequently.

Gravitational force and weight

The gravitational force on Earth is a fundamental force that is exerted by the Earth on all objects with mass, including human beings, animals, and objects. The weight of an object near the surface of the Earth is defined by Newton's law of universal gravitation (1.3.1) as:

$$W = mg = \frac{GMm}{R^2}, \tag{1.3.4}$$

where M is the mass of the Earth, m is the mass of the object, R is the radius of the Earth, and g is the acceleration due to gravity. The direction of the gravitational force is always towards the center of the Earth. From the expression of the free-fall acceleration

$$g = \frac{GM}{R^2}, \tag{1.3.5}$$

we find notably that this acceleration is independent of the mass of the object, meaning that all objects will experience the same acceleration under the influence of gravity near the Earth's surface. Numerically, the free-fall acceleration is approximately $g \approx 9.8\ \mathrm{m/s^2}$.

Normal force

When an object is placed on a table, it experiences a normal force with direction perpendicular to the table. Essentially, this normal force comes from the electromagnetic repulsion between the closely spaced molecules of the object and the molecules of the table. The magnitude of the force is not fixed.

Friction

Friction is a force that opposes motion between two surfaces in contact. When two objects are in contact, the roughness of their surfaces causes them to "stick" together slightly. This makes it harder to move one object relative to the other. Friction can be thought of as a microscopic "drag" force that acts opposite to the direction of motion or the direction of an applied force.

There are two types of friction: *static friction* and *kinetic (or sliding) friction.* Static friction occurs when an object is stationary and is about to be moved. The force of static friction prevents the object from being moved until a sufficient external force is applied. Once the object starts to move, kinetic friction takes over, which is generally less than static friction.

The magnitude of static friction has a maximum value, which is proportional to the normal force. So we have

$$f_s \leq \mu_s F_N, \tag{1.3.6}$$

where μ_s is a dimensionless coefficient of static friction and F_N is the magnitude of the normal force. On the other hand, if the body begins to slide along the surface, the magnitude of the kinetic frictional force is

$$f_k = \mu_k F_N, \tag{1.3.7}$$

where μ_k is the coefficient of kinetic friction. Usually, the maximum static friction is bigger than the kinetic friction: $\mu_s > \mu_k$.

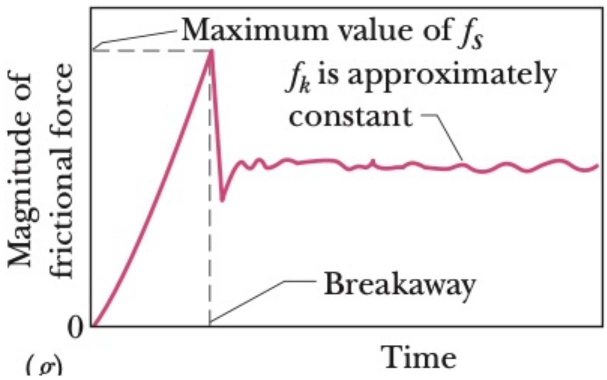

Figure 1.6: Friction.

Tension

Tension is a pulling force that is transmitted through a flexible material, such as a rope, cable, or string, when it is pulled tight by opposing forces. It is a type of force that acts along the length of the material, and its magnitude is proportional to the amount of force being applied. Tension is used in a wide range of applications, such as in bridges, cables, pulleys, and other structures that rely on the strength of flexible materials.

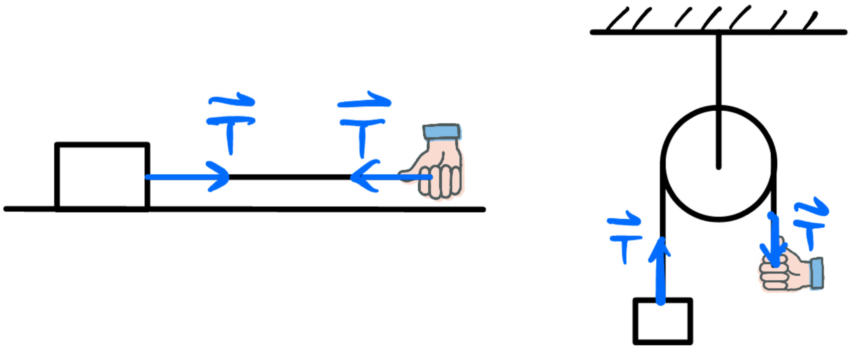

Figure 1.7: Tensions of ropes. (a) Pulling an object on a table. (b) The rope runs around a pulley.

Spring force

Spring force is a force exerted by a compressed or stretched spring, which tends to restore the spring to its equilibrium length. The magnitude of the spring force is proportional to the displacement of the spring from its equilibrium position. It can be expressed mathematically as

$$\vec{F} = -k\vec{x}, \tag{1.3.8}$$

where F is the spring force, k is the spring constant, and x is the displacement from the equilibrium position.

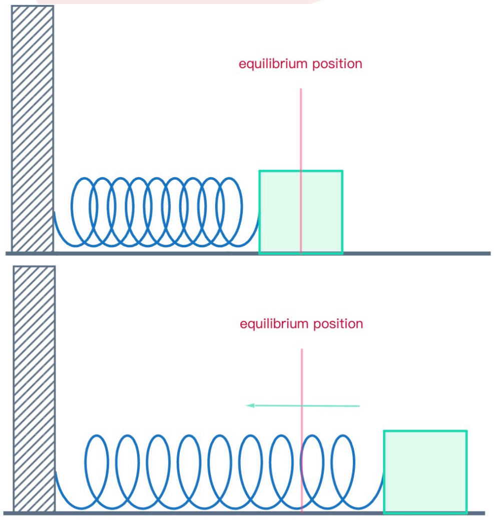

Figure 1.8: Spring force.

1.3.3 Statics I: forces

When an object or system remains unchanged over time in a particular reference frame, it is said to be in a state of equilibrium or static equilibrium. The branch of classical mechanics that studies systems in this state is known as statics.

Definition 8 (Body)**.** *A body refers to a physical object, which can consist of one or multiple parts, that can be treated as a single entity for the purpose of analysis.*

The concept of body is useful for simplifying the description and calculation of the motion, forces, and other properties of objects, especially when the individual components are too numerous or complicated to handle separately. By treating a complex system as a single body, we can apply the principles of mechanics, such as Newton's laws, to describe its behavior.

In the field of statics, we analyze an object by considering the forces acting on it as a whole and by examining each of its individual parts, which can be thought of as distinct bodies. The selection of a particular body to analyze is an engineering decision that is made based on the specific goals of the analysis. For instance, when designing the foundation of a high-rise building, we may consider the entire building as a single body. However, when evaluating the strength of the building's individual components, such as columns and beams, we would examine them separately to ensure that they can effectively perform their intended functions.

Newton's second law states that the net force acting on a body is equal to its mass times its acceleration. Therefore, if a body is not accelerating, the net force acting on it must be zero. In other words, if a body is in static equilibrium, the total force acting on it should be zero:

$$\vec{F}_{\text{tot}} = m\vec{a} = \vec{0}. \tag{1.3.9}$$

So we have the following result:

The first condition of equilibrium. *In order for a body to be in static equilibrium, the net force acting on it must be zero.*

According to the principle of superposition of forces (or addition of vectors in a vector space), the total force acting on an object is the sum of all the individual forces acting on it from the environment. Mathematically, we can express this as:

$$\vec{F}_{\text{tot}} = \sum_a \vec{F}_a, \tag{1.3.10}$$

where $\vec{F}_a$ represents an individual force acting on the body from the environment. If we connect the tails of the arrows representing the force vectors to their heads, then the condition that the total force is zero is equivalent to saying that the resulting path formed by all the arrows is closed (see Figure 1.9).

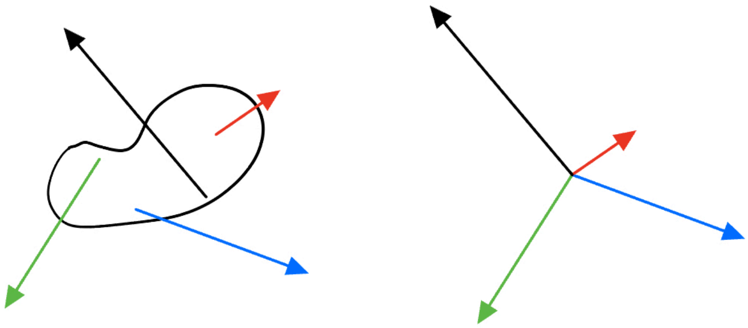

Figure 1.9: The first condition of equilibrium: Total force of a static body should be zero.

When considering a body as consisting of several parts, each part will exert internal interacting forces $\vec{F}_{ij}$ on each other. The total force acting on the body should include all external forces $\vec{F}_{aj}$ and internal forces $\vec{F}_{ij}$:

$$\vec{F}_{\rm tot} = \sum_j \left(\sum_a \vec{F}_{aj} + \sum_i \vec{F}_{ij} \right). \tag{1.3.11}$$

Because of the Newton's third law/action-reaction law $\vec{F}_{ij} = -\vec{F}_{ji}$, the internal forces $\sum_j \sum_i \vec{F}_{ij}$ sum up to zero. Therefore, we obtain again Eq. (1.3.10) by identifying the external force as the sum of $\vec{F}_a = \sum_j \vec{F}_{aj}$.

Example. *A box is at rest on an inclined plane with an angle of inclination θ. The weight of the box is mg. The static friction coefficient for the plane and the box is μ. Determine the normal force and friction exerted by the inclined plane on the box. What is the maximum angle $\theta_{\max}$ at which the box remains in equilibrium on the plane?*

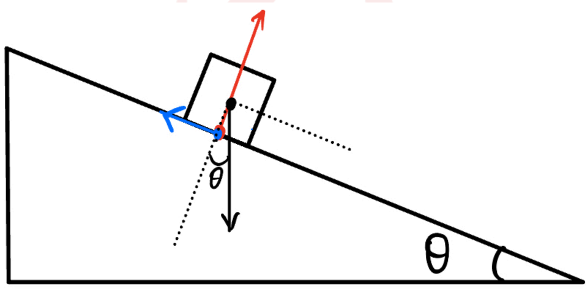

Figure 1.10: Box on an inclined plane.

Solution: We can set up a coordinate system such that the x-axis is parallel to the surface of the inclined plane and pointing down the slope, the y-axis is perpendicular to the surface and pointing upwards, and the origin is at a convenient location such as the position of the box.

The box on a slope experiences three forces: the gravitational force $m\vec{g}$, the normal force $\vec{F}_N$, and the friction $\vec{f}$. To satisfy the condition of total force, $\vec{F}_{\text{tot}} = m\vec{g} + \vec{F}_N + \vec{f}$, being zero, we need to consider two equations along the two axes:

$$f = mg\sin\theta, \tag{1.3.12}$$

$$F_N = mg\cos\theta. \tag{1.3.13}$$

These two equations completely determine the normal force and the frictional force.

Since the static friction should satisfy $f \leq \mu F_N$, the maximum angle satisfy $\tan\theta_{\max} = \mu$, so

$$\theta_{\max} = \arctan\mu. \tag{1.3.14}$$

Example. *Calculating forces in the pulley system shown in Figure 1.11.*

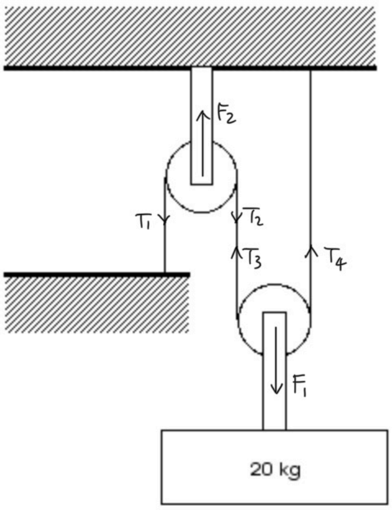

Figure 1.11: A box hanging by ropes.

Example. *If the crate has mass m in Figure 1.12, determine the forces in the boom and in the topping lift.*

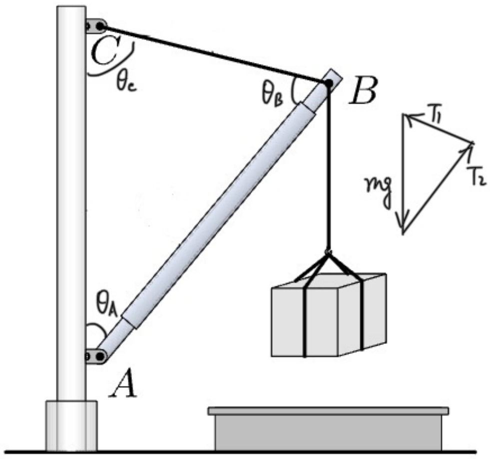

Figure 1.12: Cargo boom.

Solution: The triangle formed by the three forces acting on the static point B is similar to the triangle ABC. Therefore, we can establish the following equation:

$$\frac{T_1}{\sin\theta_A} = \frac{T_2}{\sin\theta_C} = \frac{mg}{\sin\theta_B}, \tag{1.3.15}$$

which allows us to determine all three forces.

Example. *A small ball with mass m is hanging still by ropes (see Figure 1.13). The angles between the first two ropes and the vertical direction are θ_1 and θ_2, respectively. Find the tension in each of the three ropes.*

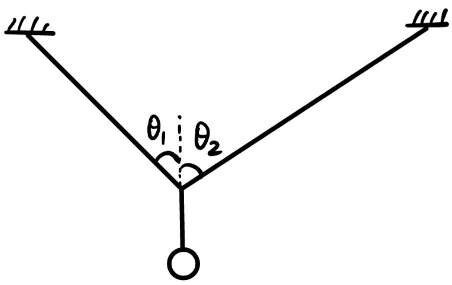

Figure 1.13: A small ball hanging by ropes.

Solution: The total force of the ball should be zero, so we have

$$T_3 = mg. \tag{1.3.16}$$

The total force acting on the connecting point of the three ropes should also be zero. The two equations in horizontal and vertical directions are

$$T_1 \sin\theta_1 = T_2 \sin\theta_2 \tag{1.3.17}$$
$$T_1 \cos\theta_1 + T_2 \cos\theta_2 = T_3. \tag{1.3.18}$$

The solutions for the three equations are

$$T_1 = mg\frac{\sin\theta_2}{\sin(\theta_1+\theta_2)}, \tag{1.3.19}$$
$$T_2 = mg\frac{\sin\theta_1}{\sin(\theta_1+\theta_2)}, \tag{1.3.20}$$
$$T_3 = mg. \tag{1.3.21}$$

Example. *Consider the cable and pulley arrangement shown in Figure 1.14. The lower block has mass M, and the upper block has mass m. The coefficient of friction between the two blocks is μ, and the coefficient of friction between the lower block and the floor is also μ. What is the maximum horizontal force F that can be exerted on the lower block before it moves? And what is the tension T in the cable?*

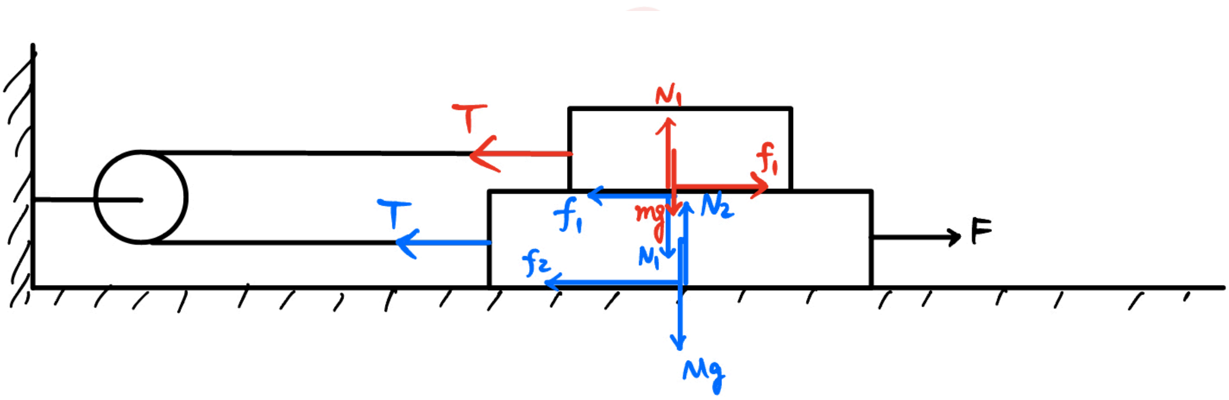

Figure 1.14

Solution: The maximum external force F that can be exerted on the lower block before it moves occurs when the friction forces between the two blocks and between the lower block and the floor are at their maximum values,

$$f_1 = \mu N_1, \tag{1.3.22}$$
$$f_2 = \mu N_2. \tag{1.3.23}$$

Applying the first condition of equilibrium on the upper block, we can set the total forces along the x and y directions to zero, giving us:

$$f_1 - T = 0, \tag{1.3.24}$$
$$N_1 - mg = 0. \tag{1.3.25}$$

For the lower block, we have

$$F - T - f_1 - f_2 = 0, \tag{1.3.26}$$
$$N_2 - N_1 - Mg = 0. \tag{1.3.27}$$

By solving these equations, we can find that the maximum external force F and the tension in the cable T are given by:

$$F = \mu(3m + M)g, \tag{1.3.28}$$
$$T = \mu mg. \tag{1.3.29}$$

1.3.4 Statics II: torques

When analyzing the motion of a body made up of several parts, it can be useful to separate their motions into the motion of the center of mass and the relative motions of the different parts. However, in order to simplify our analysis and ignore the relative motion, we introduce the following concept.

Definition 9 (Rigid body)**.** *A rigid body is a physical object that maintains its shape and size under external forces, meaning that it does not undergo deformation, bending, stretching, or twisting.*

In reality, no object is truly rigid, and all objects can deform under the application of forces. However, for many practical purposes, it is sufficient to treat an object as a rigid body. For a rigid body, we can assume that all points of it move together, so we only need to consider the motion of the object as a whole rather than the motion of each individual point on the object. This simplifies the equations of motion and makes it easier to analyze the object's behavior.

We can ask the question: what is the condition for a rigid body to be at rest or in static equilibrium? The condition that the total force acting on a rigid body is zero is not sufficient to ensure that the body is static. There is another condition related to the moment acting on the body.

Definition 10 (Cross product)**.** *Given two vectors $\vec{A}$ and $\vec{B}$ in three-dimensional Euclidean space, the cross product of them is another vector $\vec{C}$ that is perpendicular to both $\vec{A}$ and $\vec{B}$ and whose magnitude is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ multiplied by the sine of the angle between them. The direction of the resulting vector $\vec{C}$ is given by the right-hand rule.*

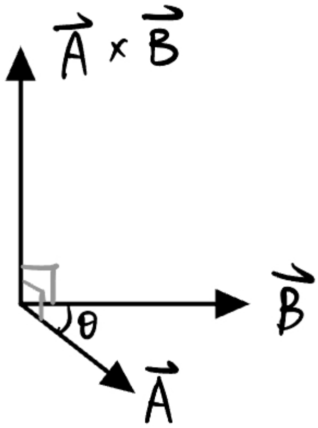

Figure 1.15: Cross product.

The cross product of $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$ in a coordinate system is given by:

$$\begin{aligned} \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} &= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z} \\ &= (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x), \end{aligned} \tag{1.3.30}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors in the x, y, and z directions, respectively. One can show directly that

$$|\vec{A} \times \vec{B}| = |\vec{A}| \cdot |\vec{B}| \cdot \sin\theta, \tag{1.3.31}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. If both $\vec{A}$ and $\vec{B}$ are coplanar and lie in the xy plane, their cross product is perpendicular to the plane:

$$\vec{A} \times \vec{B} = (A_x, A_y, 0) \times (B_x, B_y, 0) = (0, 0, A_xB_y - A_yB_x). \tag{1.3.32}$$

Therefore, the cross product of $\vec{A}$ and $\vec{B}$ in the xy plane is a vector that only has a z-component, given by $A_xB_y - A_yB_x$.

The cross product is anticommutative (that is, $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$) and is distributive over addition (that is, $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$). It is not associative, but satisfies the Jacobi identity $\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B}) = 0$. It also satisfies $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$ and $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$.

The cross product allows us to define the physical concept of the *moment of a force*, also known as *torque*. The moment of a force measures the rotational effect of a force about a specific point or axis. Unlike a linear force, which causes translational motion of a body, a force that creates a moment must be applied in a way that causes the body to begin to rotate or twist. This happens when the force does not act through the centroid or center of mass of the body.

Definition 11 (Moment of a force/torque)**.** *The moment of force, or torque, is a vector defined as*

$$\vec{\tau} = \vec{r} \times \vec{F}, \tag{1.3.33}$$

where $\vec{r}$ is the position vector from the rotation center to the point where the force $\vec{F}$ is applied.

By definition, the magnitude of $\vec{\tau}$ is

$$|\vec{\tau}| = |\vec{r}| \cdot |\vec{F}| \cdot \sin\theta, \tag{1.3.34}$$

where θ is the angle between the two vectors $\vec{r}$ and $\vec{F}$. The direction of τ is given by the right-hand rule.

One can define the moment of a force with respect to a rotational axis using either $\vec{r} \times \vec{F}_{\perp}$ (see Figure 1.16b) or $(\vec{r} \times \vec{F})_{\parallel}$ (see Figure 1.16c). These two definitions coincide with each other and can be used interchangeably (show that).

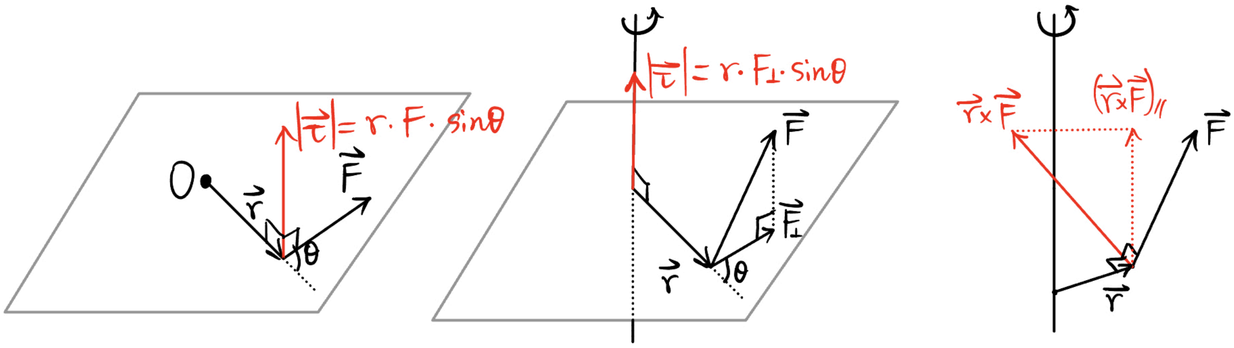

Figure 1.16: Moment of a force/t/orque with respect to (a) a point or (b) (c) a rotational axis.

The second condition of equilibrium. *In order for a rigid body to be in static equilibrium, the net torque acting on it with respect to any chosen axis or point must be zero.*

If we have several forces acting on different positions of a static rigid body, the total torque can be calculated as:

$$\vec{\tau}_{\text{tot}} = \sum_i \vec{r}_i \times \vec{F}_i. \tag{1.3.35}$$

The second condition of equilibrium states that the total torque $\vec{\tau}_{\text{tot}}$ acting on the rigid body as a vector is zero, which ensures that the body does not rotate.

Example. *Design a steelyard balance to find mass with torque.*

Figure 1.17: A steelyard balance.

Example. *A uniform pole of length L and weight mg is pivoted at one end to a wall. It is held at an angle of θ above the horizontal by a horizontal guy wire attached l units from the end attached to the wall. A load of Mg hangs from the upper end of the pole. Calculate the tension in the guy wire and determine the force exerted on the pole by the wall.*

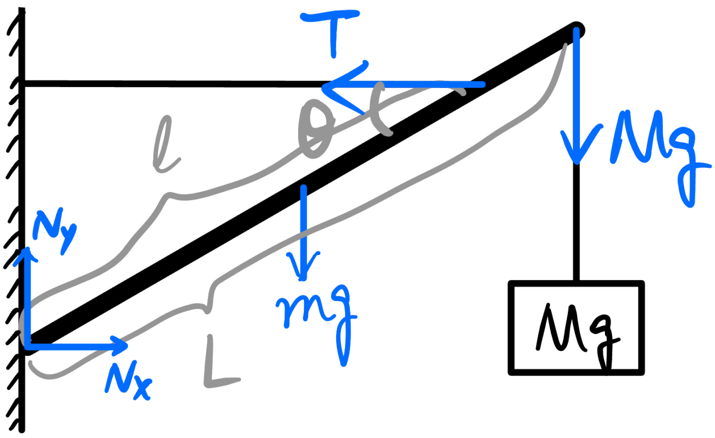

Figure 1.18: A pole at static equilibrium.

Solution: The first condition of equilibrium gives us two equations relating the forces on the pole:

$$N_x - T = 0, \tag{1.3.36}$$
$$N_y - mg - Mg = 0. \tag{1.3.37}$$

However, we have three unknown forces: N_x, N_y, and T. Therefore, we need an additional equation to solve for all three forces. This equation can be obtained from the second condition of equilibrium.

Before calculating torques, we need to choose a reference point around which we will calculate the torques. Choosing the lower left end of the pole is a good choice as it will result in two forces, N_x and N_y, having zero torque. This simplifies the equations. Using this reference point, the total torque can be written as:

$$\tau_{\text{tot}} = mg\frac{L}{2}\cos\theta + MgL\cos\theta - Tl\sin\theta = 0. \tag{1.3.38}$$

Using all the three equations from the first and second conditions of equilibrium, we can solve for the unknowns as:

$$N_y = mg + Mg, \tag{1.3.39}$$

$$N_x = T = mg\frac{L}{2l}\cos\theta + Mg\frac{L}{l}\cos\theta. \tag{1.3.40}$$

1.4 Energy and Momentum Conservation

Nowadays, we know that Newton's laws are only approximations of the laws of nature. New theories are necessary when objects move at very high speeds (special relativity), are very massive (general relativity), or are very small (quantum mechanics). However, some consequences followed by Newton's law actually hold more universally. In this section, we introduce two of such important consequences: energy and momentum conservation.

1.4.1 Energy conservation

Let us consider a point particle moving in one dimension under a force which can be written as

$$F(x) = -\frac{dU(x)}{dx}\,, \tag{1.4.1}$$

where the function $U(x)$ is called the *potential energy*. Newton's second law tells us that

$$\frac{dU(x)}{dx} = -m\frac{dv}{dt}\,. \tag{1.4.2}$$

Multiplying both sides of the equation by the velocity v, we obtain

$$-\frac{dU(x)}{dt} = -\frac{dx}{dt}\frac{dU(x)}{dx} = mv\frac{dv}{dt} = \frac{1}{2}m\frac{dv^2}{dt}\,, \tag{1.4.3}$$

which can be simplified to

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + U(x)\right) = 0\,. \tag{1.4.4}$$

We see that Newton's second law implies that a particular combination of the square of the velocity and the potential energy is conserved (invariant under time evolution). Let us define the *kinetic energy* K of the point particle as

$$K = \frac{1}{2}mv^2\,, \tag{1.4.5}$$

and the *total energy* E of the point particle as

$$E = \frac{1}{2}mv^2 + U(x)\,. \tag{1.4.6}$$

Newton's second law implies that the total energy E is conserved.

Example. *The force of a spring*

$$F = -kx\,, \tag{1.4.7}$$

is conservative. The potential energy of the spring is

$$U(x) = kx^2\,. \tag{1.4.8}$$

Show that the energy of a simple harmonic oscillator (1.2.4) is conserved.

Solution: The position and velocity of the box are

$$\begin{aligned} x(t) &= x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)\,, \\ \dot{x}(t) &= -x_0 \omega \sin(\omega t) + v_0 \cos(\omega t)\,, \end{aligned} \tag{1.4.9}$$

where x_0 and v_0 are the initial position and velocity. The kinetic energy K is

$$K = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \left(-x_0 \omega \sin(\omega t) + v_0 \cos(\omega t)\right)^2 . \tag{1.4.10}$$

The potential energy U is

$$U = \frac{1}{2} k x^2 = \frac{1}{2} k \left(x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)\right)^2 . \tag{1.4.11}$$

They separately are not conserved. The total energy is

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2} \Big[m\omega^2 x_0^2 \sin^2(\omega t) + m v_0^2 \cos^2(\omega t) - 2m\omega x_0 v_0 \sin(\omega t)\cos(\omega t) \\ &\quad + k x_0^2 \cos^2(\omega t) + k \frac{v_0^2}{\omega^2} \sin^2(\omega t) + 2k \frac{x_0 v_0}{\omega} \sin(\omega t)\cos(\omega t) \Big] \\ &= \frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2 . \end{aligned} \tag{1.4.12}$$

We see that the total energy is indeed conserved.

In the example of the simple harmonic oscillator, if initially the spring is relaxed and the box has no velocity, i.e. $x_0 = 0$ and $v_0 = 0$, then the system would be at rest with zero energy forever. We could apply an external force to push the spring to increase the potential and start the motion. We would give a more general discussion of this process in the following. Consider another force F' that balances the force F such that the net force is zero, i.e.

$$F' = -F = \frac{dU}{dx} . \tag{1.4.13}$$

Suppose the point particle travels from $x = x_i$ to $x = x_f$ while remaining zero net force (1.4.13). We define the *work* that is done by the force F' on the point particle as

$$W = \int_{x_i}^{x_f} F' dx = \int_{x_i}^{x_f} \frac{dU}{dx} dx = U(x_f) - U(x_i) . \tag{1.4.14}$$

We see that the work equals to the change of the potential energy.

Let us generalize the above discussion to higher dimensions.

Definition 12 (Conservative force). *A force $\vec{F}$ is conservative if it only depends on the position $\vec{r}$ (but not the velocity $\vec{v}$) and can be written as the gradient of the potential energy $U(\vec{r})$, more precisely,*

$$\vec{F} = -\vec{\nabla} U = \left(\frac{\partial U}{\partial x_1}, \cdots, \frac{\partial U}{\partial x_d}\right) . \tag{1.4.15}$$

The notation $\frac{\partial}{\partial x_i}$ in (1.4.15) is the partial derivative, which means that we only take the derivative on x_i while fixing the other x_j for $j \neq i$. The kinetic energy and the total energy of a point particle moving in higher dimensions are

$$K = \frac{1}{2}m|\vec{v}|^2\,, \quad E = \frac{1}{2}m|\vec{v}|^2 + U(\vec{r})\,. \tag{1.4.16}$$

Let us check that the total energy is conserved,

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m|\vec{v}|^2 + U(\vec{r})\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt}\cdot\vec{\nabla}U(\vec{r}) = \vec{v}\cdot\left(m\vec{a} - \vec{F}\right) = 0\,. \tag{1.4.17}$$

Example. *Consider a free-falling ball near Earth's surface. It receives a gravitational force as*

$$\vec{F} = (0, -mg)\,, \tag{1.4.18}$$

which is conservative, and can be written as

$$\vec{F} = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)\,, \quad U = mgy\,, \tag{1.4.19}$$

where U is called the gravitational potential energy. The motion of the ball is given by (1.1.42)*. The velocity of the ball is given by* (1.1.43)*. Check that the total energy is conserved.*

Solution: Let us compute the total energy

$$\begin{aligned} E &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy \\ &= \frac{1}{2}mv_x^2 + \frac{1}{2}m(v_y - gt)^2 + mg\left(y_0 + v_y t - \frac{1}{2}gt^2\right) \\ &= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + mgy_0\,, \end{aligned} \tag{1.4.20}$$

which is independent of the time t.

Consider an external force $\vec{F}'$ that balances the gravitational force and pulls the ball from $\vec{r}_i$ to $\vec{r}_f$ following the curve $\mathcal{C}_{\vec{r}_i,\vec{r}_f}$ as shown in figure 1.19. What is the work done by the force $\vec{F}'$?

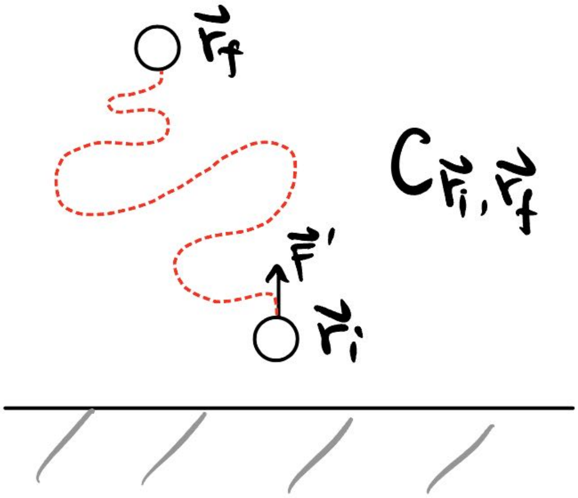

Figure 1.19: An external force $\vec{F}'$ pulls the ball upward from $\vec{r}_i$ to $\vec{r}_f$.

Let us consider the case with a general potential energy $U(\vec{r})$. The force $\vec{F}'$ balances the conservative force $\vec{F}$ as

$$\vec{F}' = -\vec{F} = \vec{\nabla} U\,. \tag{1.4.21}$$

The work done by the force $\vec{F}'$ along a curve $\mathcal{C}_{\vec{r}_i,\vec{r}_f}$ starting at $\vec{r}_i$ and ending at $\vec{r}_f$ is

$$W = \oint_{\mathcal{C}_{\vec{r}_i,\vec{r}_f}} \vec{F}' \cdot d\vec{r} = \oint_{\mathcal{C}_{\vec{r}_i,\vec{r}_f}} \vec{\nabla} U \cdot d\vec{r} = U(\vec{r}_f) - U(\vec{r}_i) \equiv \Delta U\,. \tag{1.4.22}$$

We see interestingly that the work only depends on the starting and ending positions $\vec{r}_i$, $\vec{r}_f$ and the curve $\mathcal{C}_{\vec{r}_i,\vec{r}_f}$. In particular, if the curve is a closed loop $\mathcal{C}$, then the work done by the force $\vec{F}'$ is zero,

$$W = \oint_{\mathcal{C}} \vec{F}' \cdot d\vec{r} = \oint_{\mathcal{C}} \vec{\nabla} U \cdot d\vec{r} = 0\,. \tag{1.4.23}$$

Note that this property is only true if $\vec{F}'$ is against a conservative force.

Now, let us consider the situation that the external force $\vec{F}'$ does not balance the conservative force $\vec{F} = -\vec{\nabla} U$. The net force of the system is

$$\vec{F}' + \vec{F} = \vec{F}' - \vec{\nabla} U\,. \tag{1.4.24}$$

Newton's second law implies

$$\vec{F}' - \vec{\nabla} U = m\vec{a}\,. \tag{1.4.25}$$

Let us integrate this equation along the contour $\mathcal{C}_{\vec{r}_i,\vec{r}_f}$. We find

$$\begin{aligned}
\oint_{\mathcal{C}_{\vec{r}_i,\vec{r}_f}} \vec{F}' \cdot d\vec{r} &= \oint_{\mathcal{C}_{\vec{r}_i,\vec{r}_f}} \vec{\nabla} U \cdot d\vec{r} + \oint_{\mathcal{C}_{\vec{r}_i,\vec{r}_f}} m\frac{d\vec{v}}{dt} \cdot d\vec{r} \\
&= \oint_{\mathcal{C}_{\vec{r}_i,\vec{r}_f}} \vec{\nabla} U \cdot d\vec{r} + \int_{t_i}^{t_f} m\frac{d\vec{v}}{dt} \cdot \vec{v} dt \\
&= \oint_{\mathcal{C}_{\vec{r}_i,\vec{r}_f}} \vec{\nabla} U \cdot d\vec{r} + \int_{t_i}^{t_f} \frac{d\frac{1}{2}m|\vec{v}|^2}{dt} dt \\
&= [U(\vec{r}_f) + K(t_f)] - [U(\vec{r}_i) + K(t_i)] \\
&\equiv \Delta E\,.
\end{aligned} \tag{1.4.26}$$

We have learned that when the external force $\vec{F}'$ does not balance the conservative force, the integral of the external force over a contour $\mathcal{C}_{\vec{r}_i,\vec{r}_f}$ equals to the energy difference between the initial and final configuration. It is important to note that now the integral of the external force depends on the entire contour $\mathcal{C}_{\vec{r}_i,\vec{r}_f}$ not just on the initial and final positions.

Energy conservation can be further generalized to multi-particle systems. The kinetic energy of a system of n particles is the sum of the kinetic energies of the individual particles,

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i |\vec{v}_i|^2\,, \tag{1.4.27}$$

The potential energy is a function of the position vectors $\vec{r}_i$ of the particles, $U(\vec{r}_1, \cdots, \vec{r}_n)$. The force acting on the i-th particle is given by the i-th divergence of the potential energy,

$$\vec{F}_i = -\vec{\nabla}_i U(\vec{r}_1, \cdots, \vec{r}_n)\,, \tag{1.4.28}$$

where $\vec{\nabla}_i$ is the differential operator acting on $\vec{r}_i$. It is straightforward to check that the total energy is conserved.

Example. *The gravitational potential of two stars* 1 *and* 2 *is*

$$U(\vec{r}_1, \vec{r}_2) = -\frac{G m_1 m_2}{|\vec{r}_{12}|}\,. \tag{1.4.29}$$

The force on star 1 is[4]

$$\vec{F}_1 = -\vec{\nabla}_1 \left(-\frac{G m_1 m_2}{|\vec{r}_{12}|} \right) = -\frac{G m_1 m_2}{|\vec{r}_{12}|^2} \hat{r}_{12}\,, \tag{1.4.31}$$

[4]More explicitly, we have used

$$\begin{aligned}
\vec{\nabla} \frac{1}{|\vec{r}|} &= \left(\frac{\partial}{\partial x} \frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial}{\partial y} \frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial}{\partial z} \frac{1}{\sqrt{x^2+y^2+z^2}} \right) \\
&= -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} (x, y, z) \\
&= -\frac{1}{|\vec{r}|^2} \hat{r}\,.
\end{aligned} \tag{1.4.30}$$

and the force on star 2 is

$$\vec{F}_1 = -\vec{\nabla}_2 \left(-\frac{Gm_1m_2}{|\vec{r}_{12}|} \right) = \frac{Gm_1m_2}{|\vec{r}_{12}|^2}\hat{r}_{12} \,. \tag{1.4.32}$$

Consider the circular motion of the two-star system studied in section 1.2.5. The total energy is

$$E = \frac{1}{2}m_1(r_1\omega)^2 + \frac{1}{2}m_2(r_2\omega)^2 + \frac{Gm_1m_2}{r} \,, \tag{1.4.33}$$

where r_1, r_2 and r are given in (1.2.2) and (1.2.3). However, this example is not very interesting since the kinetic energy and the potential energy are separately conserved. In the next section, we will see the elliptic motion of the two-star system, where the kinetic energy and the potential energy are not conserved separately while the total energy is conserved.

1.4.2 Momentum conservation

Let us consider a system of two point particles, whose position vectors are $\vec{r}_1$ and $\vec{r}_2$. We assume that no external force (or no net external force) is acting on the system, but the two particles in the system can interact with each other. More explicitly, particle 1 gives a force $\vec{F}_{12}$ to particle 2, and particle 2 gives a force $\vec{F}_{21}$ to particle 1. Newton's second law gives

$$\vec{F}_{21} = m_1\frac{d^2\vec{r}_1}{dt^2} \,, \quad \vec{F}_{12} = m_2\frac{d^2\vec{r}_2}{dt^2} \,, \tag{1.4.34}$$

where m_1 and m_2 are the masses of the two particles. Newton's third law implies

$$\vec{F}_{12} = -\vec{F}_{21} \,. \tag{1.4.35}$$

Now, let us consider

$$0 = \vec{F}_{12} + \vec{F}_{21} = m_1\frac{d^2\vec{r}_1}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2} = \frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2) \,. \tag{1.4.36}$$

We see that the quantity $m_1\vec{v}_1 + m_2\vec{v}_2$ is conserved.

The above discussion can be generalized to more complicated systems. Before we do that, let us introduce some new concepts. First, the *momentum* of a point particle is defined by its mass times velocity,

$$\vec{p} \equiv m\vec{v}, \tag{1.4.37}$$

The *total momentum* $\vec{P}$ of a system is defined by the sum over all the momenta. For example, in our previous two-particle system, the total momentum is

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \,, \tag{1.4.38}$$

which is conserved if there is no net external force by (1.4.36). Now, let us generalize the above discussion to a system of n particles without net external force. The i-th particle acts on the j-th particle by a force $\vec{F}_{ij}$. Newton's third law implies

$$0 = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \vec{F}_{ij} = \sum_{i=1}^{n} m_i \frac{d^2\vec{r}_i}{dt^2} = \frac{d}{dt}\left(\sum_{i=1} m_i \vec{v}_i\right) \equiv \frac{d}{dt}\left(\sum_{i=1} p_i\right) = \frac{d\vec{P}}{dt} . \tag{1.4.39}$$

Therefore, the total momentum of the system is conserved if there is no net external force.

When there is a net external force $\vec{F}_{\text{ext}}$, Newton's second law tells us that the total momentum $\vec{P}$ is not conserved, and its time-derivative is

$$\vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt} . \tag{1.4.40}$$

The above equation looks like the Newton's second law for a point particle. Hence, it is convenient to interpret the total momentum as the momentum of an effective point particle whose mass equals the total mass $M = m_1 + \cdots + m_n$, i.e.

$$\vec{P} = M\vec{v}_{\text{com}} , \tag{1.4.41}$$

where $\vec{v}_{\text{com}}$ is the velocity of the effective point particle which is a weighted sum of the velocities of individual particles

$$\vec{v}_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{v}_i . \tag{1.4.42}$$

From this formula, we can further infer that the effective point particle is located at the position $\vec{r}_{\text{com}}$ given by

$$\vec{r}_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i . \tag{1.4.43}$$

The position specified by the position vector $\vec{r}_{\text{com}}$ is called the *center of mass*.

The center of mass provides a way to describe the "overall motion" of a system of particles. Namely, a system of particles with a net external force $\vec{F}_{\text{ext}}$ can be viewed as a point particle with mass M sitting at the center of mass $\vec{r}_{\text{com}}$. In other words, the center of mass motion of a system of particles only depends on the net external force $\vec{F}_{\text{ext}}$ via Newton's second law and is independent of the interactions among the particles in the system. Using translation and Galilean transformation, we could go to an inertial frame that the center of mass is at the origin and has zero velocity, i.e. $\vec{r}_{\text{com}} = 0 = \vec{v}_{\text{com}}$. Such an inertial frame is called the *center of mass frame*.

1.4.3 Elastic scattering

A $m \to n$ scattering is a physical process where m objects come in from infinity, interact via contact or non-contact forces at finite distance, and n objects go out to infinity. The initial momenta $\vec{p}_1, \cdots, \vec{p}_m$ of the objects change to the final momenta $\vec{p}_1{}', \cdots, \vec{p}_n{}'$ during the scattering process. The interactions (forces) between the objects can be very complicated and we usually do not know their form. We will only assume that the interactions are "local", i.e. they decay sufficiently fast with the distance between the objects so that the objects are free when they come in or go out to infinity. During the scattering process, the total momentum and energy should be conserved. While they are usually not sufficient to fully determine the scattering process, they give important constraints to the system.

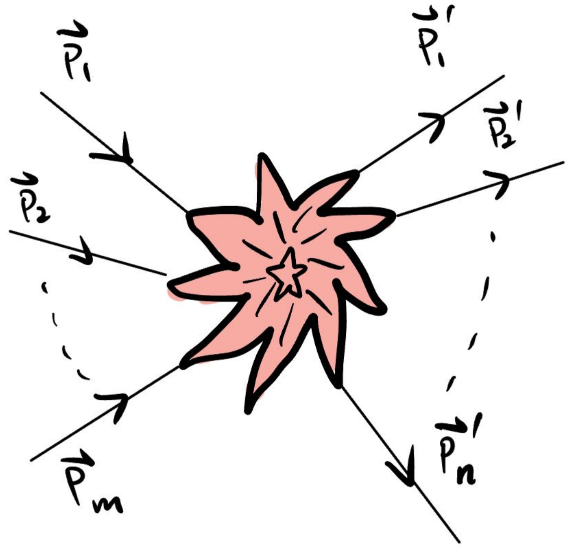

Figure 1.20: A $m \to n$ scattering process.

In a scattering process, while the total energy is always conserved, the form of the energy could change from one to the other. For example, the objects coming in from infinity carry kinetic energy, which could transfer to potential energy, heat (thermal energy), or energy of sound and light. In this subsection, we will first consider an idealistic situation, where the kinetic energy during the scattering process is conserved. Such a scattering is called an *elastic scatterings*. The scattering processes that do not conserve kinetic energy are called *inelastic scatterings*. On the other hand, momentum should always be conserved if the system is closed.

We will focus on the $2 \to 2$ scattering process, where we have two incoming particles or objects and two outgoing ones.

Example. *Consider a one-dimensional $2 \to 2$ elastic scattering process. Two objects A and B of masses m_A, m_B, and velocities v_A, v_B come in from the left infinity. They collide and go out to the right infinity. Compute the velocities of the objects A and B after the scattering.*

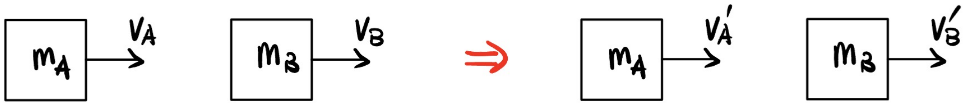

Figure 1.21: One-dimensional elastic scattering process.

Solution: The momentum conservation of the scattering process gives

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \,. \tag{1.4.44}$$

The conservation of the kinetic energy gives

$$\frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A v_A'^2 + \frac{1}{2} m_B v_B'^2 \,. \tag{1.4.45}$$

Solving the above two equations, we find two solutions

$$v_A' = v_A \,, \quad v_B' = v_B \,, \tag{1.4.46}$$

and

$$v_A' = \frac{m_A - m_B}{m_A + m_B} v_A + \frac{2m_B}{m_A + m_B} v_B \,, \quad v_B' = \frac{2m_A}{m_A + m_B} v_A + \frac{m_B - m_A}{m_A + m_B} v_B \,. \tag{1.4.47}$$

The initial and final velocities in the first solution are the same. This means that the scattering actually does not happen, which could be realized when $v_A \leq v_B$. The velocities in the second solution change under the scattering.

It is interesting to go to the center of mass frame. Let us first compute the velocity of the center of mass

$$v_{\text{com}} = \frac{m_A v_A + m_B v_B}{m_A + m_B} \,. \tag{1.4.48}$$

In the center of mass frame, the objects A and B have initial velocities

$$v_A - v_{\text{com}} = \frac{m_B (v_A - v_B)}{m_A + m_B} \,, \quad v_B - v_{\text{com}} = \frac{m_A (v_B - v_A)}{m_A + m_B} \,, \tag{1.4.49}$$

and final velocities

$$v_A' - v_{\text{com}} = -\frac{m_B (v_A - v_B)}{m_A + m_B} \,, \quad v_B' - v_{\text{com}} = -\frac{m_A (v_B - v_A)}{m_A + m_B} \,. \tag{1.4.50}$$

We see that in the center of mass frame, the final velocities are just given by flipping the signs of the initial velocities. This can be explained by the time-reversal symmetry of the system.

Now, let us generalize our study to the scatterings in higher dimensions.

Example. *Consider a* $2 \to 2$ *elastic scattering in* d *dimensions. Let the velocities of the incoming particles be* $\vec{v}_A$ *and* $\vec{v}_B$ *and of the outgoing particles be* $\vec{v}_A{}'$ *and* $\vec{v}_B{}'$.

1. *Show that the momentum conservation and the energy conservation are invariant under the Galilean transformation.*

2. *Using Galilean transformation, translation, and rotation, show that we can always go to an inertial frame such that the total momentum is zero, and the scattering is two-dimensional, i.e. all the incoming and outgoing objects or particles are moving in the x-y plane.*

3. *Show that any Galilean invariant quantities (quantities that are invariant under transformations in the Galilean group) can be written as functions of two Galilean invariant variables. Explicitly construct such two variables.*

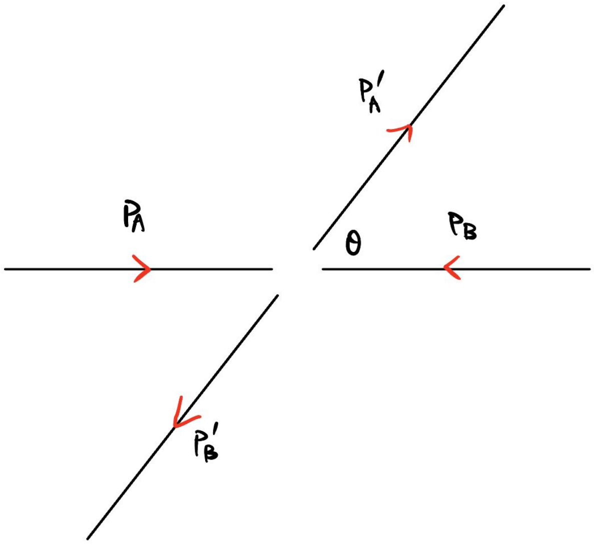

Figure 1.22: A d-dimensional $2 \to 2$ scattering can always be put on a 2-plane.

Solution:

1. Let us start with the momentum conservation

$$m_A\vec{v}_A + m_B\vec{v}_B = \vec{p}_A + \vec{p}_B = \vec{p}_A{}' + \vec{p}_B{}' = m_A\vec{v}_A{}' + m_B\vec{v}_B{}' \,. \tag{1.4.51}$$

Under the Galilean transformation

$$\vec{v}_A \to \vec{v}_A + \vec{v}\,, \quad \vec{v}_B \to \vec{v}_B + \vec{v}\,, \quad \vec{v}_A{}' \to \vec{v}_A{}' + \vec{v}\,, \quad \vec{v}_B{}' \to \vec{v}_B{}' + \vec{v}\,, \tag{1.4.52}$$

the momentum conservation becomes

$$m_A(\vec{v}_A + \vec{v}) + m_B(\vec{v}_B + \vec{v}) = m_A(\vec{v}_A{}' + \vec{v}) + m_B(\vec{v}_B{}' + \vec{v})\,, \tag{1.4.53}$$

which is equivalent to the momentum conservation before the Galilean transformation.

Next, let us consider the energy conservation,

$$\frac{1}{2}m_A|\vec{v}_A|^2 + \frac{1}{2}m_B|\vec{v}_B|^2 = \frac{1}{2}m_A|\vec{v}_A{}'|^2 + \frac{1}{2}m_B|\vec{v}_B{}'|^2 . \tag{1.4.54}$$

Under the Galilean transformation, the energy conservation becomes

$$\frac{1}{2}m_A|\vec{v}_A + \vec{v}|^2 + \frac{1}{2}m_B|\vec{v}_B + \vec{v}|^2 = \frac{1}{2}m_A|\vec{v}_A{}' + \vec{v}|^2 + \frac{1}{2}m_B|\vec{v}_B{}' + \vec{v}|^2 . \tag{1.4.55}$$

Expanding both sides of the equation, we find

$$\begin{aligned} &\frac{1}{2}m_A|\vec{v}_A|^2 + \frac{1}{2}m_B|\vec{v}_B|^2 + \vec{v}\cdot(m_A\vec{v}_A + m_B\vec{v}_B) + \frac{1}{2}(m_A+m_B)|\vec{v}|^2 \\ &= \frac{1}{2}m_A|\vec{v}_A{}'|^2 + \frac{1}{2}m_B|\vec{v}_B{}'|^2 + \vec{v}\cdot(m_A\vec{v}_A{}' + m_B\vec{v}_B{}') + \frac{1}{2}(m_A+m_B)|\vec{v}|^2 . \end{aligned} \tag{1.4.56}$$

Using the momentum conservation, we see that it is equivalent to the energy conservation before the Galilean transformation.

2. Let us first use translation and Galilean transformation to set the center of mass to be at the origin with zero velocity. Since the total momentum is zero, the momenta of the two objects, labeled by A and B, have the same magnitude and opposite directions, i.e. the incoming momenta satisfy

$$\vec{p}_A = -\vec{p}_B , \tag{1.4.57}$$

and the outgoing momenta satisfy

$$\vec{p}_A{}' = -\vec{p}_B{}' . \tag{1.4.58}$$

Next, we can rotate our frame to make $\vec{p}_A$ and $\vec{p}_A{}'$ to lie in the x-y plane.

In this frame, the scattering process is characterized by only two quantities: the center of mass energy E_{com} and the scattering angle θ. They are given by

$$E_{\text{com}} = \frac{|\vec{p}_A|^2}{2m_A} + \frac{|\vec{p}_B|^2}{2m_B} = \frac{m_A+m_B}{2m_Am_B}|\vec{p}_A|^2 , \tag{1.4.59}$$

and

$$\vec{p}_A\cdot\vec{p}_A{}' = |\vec{p}_A||\vec{p}_A{}'|\cos\theta . \tag{1.4.60}$$

3. Consider a generic inertial frame. The momenta $\vec{p}_A$, $\vec{p}_B$, $\vec{p}_A{}'$, and , $\vec{p}_B{}'$, subject to the momentum conservation

$$\vec{p}_A + \vec{p}_B = \vec{p}_A{}' + \vec{p}_B{}' , \tag{1.4.61}$$

span a three-dimensional vector space. There is a two-dimensional subspace that is invariant under the Galilean transformation and spanned by the vectors

$$\vec{v}_{AB} \equiv \vec{v}_A - \vec{v}_B = \frac{\vec{p}_A}{m_A} - \frac{\vec{p}_B}{m_B} , \quad \vec{v}_{B'B} \equiv \vec{v}_B{}' - \vec{v}_B = \frac{1}{m_B}(\vec{p}_B{}' - \vec{p}_B) . \tag{1.4.62}$$

Using these two vectors, we can construct three rotational invariant quadratic combinations

$$\alpha = |\vec{v}_{AB}|^2\,, \quad \beta = |\vec{v}_{B'B}|^2\,, \quad \gamma = \vec{v}_{AB} \cdot \vec{v}_{B'B}\,. \tag{1.4.63}$$

By the energy and momentum conservation, one can find one linear relation between α, β, and γ. To find such a relation, let us use Galilean transformation to go to the frame with $\vec{v}_B = 0$. In this frame, $\vec{v}_{AB} = -\vec{v}_A$, $\vec{v}_{B'B} = -\vec{v}_B{}'$, and the energy and momentum conservation become

$$\begin{aligned} m_A\vec{v}_A &= m_A\vec{v}_A{}' + m_B\vec{v}_B{}'\,, \\ \frac{1}{2}m_A|\vec{v}_A|^2 &= \frac{1}{2}m_A|\vec{v}_A{}'|^2 + \frac{1}{2}m_B|\vec{v}_B{}'|^2\,. \end{aligned} \tag{1.4.64}$$

We can solve for $\vec{v}_A{}'$ using the momentum conservation as

$$\vec{v}_A{}' = \vec{v}_A - \frac{m_B}{m_A}\vec{v}_B{}'\,. \tag{1.4.65}$$

The energy conservation now can be written as

$$\begin{aligned} \frac{1}{2}m_A\alpha &= \frac{1}{2}m_A|\vec{v}_A{}'|^2 + \frac{1}{2}m_B\beta \\ &= \frac{1}{2}m_A\left|\vec{v}_A - \frac{m_B}{m_A}\vec{v}_B{}'\right|^2 + \frac{1}{2}m_B\beta \\ &= \frac{1}{2}m_A\alpha - m_B\gamma + \frac{m_B^2}{2m_A}\beta + \frac{1}{2}m_B\beta\,, \end{aligned} \tag{1.4.66}$$

which simplifies as

$$(m_A + m_B)\beta - 2m_A\gamma = 0\,. \tag{1.4.67}$$

Therefore, any Galilean invariant quantities can be written as functions of the Galilean invariant variables α and γ.[5]

Now, let us go to the center of mass frame. We find that α and γ are related to the center of mass energy E_{com} and the scattering angle θ by

$$\begin{aligned} \alpha &= \left|\frac{\vec{p}_A}{m_A} - \frac{\vec{p}_B}{m_B}\right|^2 = \left(\frac{1}{m_A} + \frac{1}{m_B}\right)^2 |\vec{p}_A|^2 = \frac{m_A + m_B}{2m_A m_B}E_{\text{com}}\,, \\ \gamma &= \left(\frac{\vec{p}_A}{m_A} - \frac{\vec{p}_B}{m_B}\right) \cdot \frac{(\vec{p}_B{}' - \vec{p}_B)}{m_B} = \left(\frac{1}{m_A} + \frac{1}{m_B}\right)\frac{1}{m_B}\vec{p}_A \cdot (-\vec{p}_A{}' + \vec{p}_A) \\ &= \left(\frac{1}{m_A} + \frac{1}{m_B}\right)\frac{1}{m_B}(-|\vec{p}_A||\vec{p}_A{}'|\cos\theta + |\vec{p}_A|^2) \\ &= \left(\frac{1}{m_A} + \frac{1}{m_B}\right)\frac{1}{m_B}|\vec{p}_A|^2(1 - \cos\theta) \\ &= \left(\frac{1}{m_A} + \frac{1}{m_B}\right)\frac{1}{m_B}\frac{2m_A m_B}{m_A + m_B}E_{\text{com}}(1 - \cos\theta) \\ &= \frac{2}{m_B}E_{\text{com}}(1 - \cos\theta)\,. \end{aligned} \tag{1.4.68}$$

[5]The variables α, β and γ are the analogs of the Mandelstam variables of $2 \to 2$ scatterings in quantum field theory.

1.4.4 Inelastic scattering

As we discussed in the previous section, an inelastic scattering process does not conserve the kinetic energy, while the momentum is conserved. Let us consider the following examples of the inelastic scatterings if the system is closed.

Example. *Show that* $2 \to 1$ *scatterings cannot be elastic.*

Solution: Let the momenta of the incoming particles be $\vec{p}_A$ and $\vec{p}_B$. By the momentum conservation, the momentum of the outgoing particle is $\vec{p}_A + \vec{p}_B$. The total kinetic energies of the incoming particles and the outgoing particle are

$$\frac{|\vec{p}_A|^2}{2m_A} + \frac{|\vec{p}_B|^2}{2m_B}\,, \quad \frac{|\vec{p}_A + \vec{p}_B|^2}{2(m_A + m_B)}\,. \tag{1.4.69}$$

Now, Let us go to the center of mass frame. The outgoing particle has zero momentum and zero kinetic energy. However, the total kinetic energy of the incoming particles is positive.

Example. *Consider a one-dimensional scattering process between a box* A *and a system of two boxes* B *and* C *connected by a spring. Initially, the box* A *has velocity* $\vec{v}_A$*, the boxes* B *and* C *are at rest, and the spring is in the relaxed state with the spring constant* k*. The scattering between the boxes* A *and* B *is elastic, but the scattering between the box* A *and the system of the boxes* B *and* C *is inelastic as some of the kinetic energy changes to the "internal energy", the potential energy of the spring plus the kinetic energy of the relative velocity between boxes* B *and* C*. Find the total internal energy.*

Figure 1.23: An inelastic scattering between a box and two boxes connected by a spring.

Solution: Since the scattering between the boxes A and B is elastic, we can use the formulae in (1.4.47) and find

$$v'_A = \frac{m_A - m_B}{m_A + m_B} v_A\,, \quad v'_B = \frac{2m_A}{m_A + m_B} v_A\,. \tag{1.4.70}$$

The box B would then push the spring to compress, and the spring would push the box C to move. The system of the boxes B and C has a center of mass velocity $\vec{v}_{BC,\text{com}}$, which can be computed by the momentum conservation

$$m_B v'_B = (m_B + m_C) v_{BC,\text{com}}\,. \tag{1.4.71}$$

The internal energy is given by the initial kinetic energy of the box B minus the kinetic energy of the center of mass velocity, i.e.

$$\begin{aligned}
&\frac{1}{2}m_B v_B'^2 - \frac{1}{2}(m_B+m_C)v_{BC,\mathrm{com}}^2 \\
&= \frac{1}{2}m_B\left(\frac{2m_A}{m_A+m_B}v_A\right)^2 - \frac{1}{2}(m_B+m_C)\left(\frac{m_B}{m_B+m_C}\frac{2m_A}{m_A+m_B}v_A\right)^2 \\
&= \frac{m_A^2 m_B m_C}{(m_A+m_B)^2(m_B+m_C)}v_A^2 \,.
\end{aligned} \tag{1.4.72}$$