Tsinghua University

Numerical Analysis

Fall 2024

Homework 1

Lin Zejin March 11, 2025

- **Collaborators:** I finish this homework by myself.

Problem 1. Assume there exists $x_1, x_2, \cdots, x_{2n+1} \in [a, a+2\pi) \quad s.t.$

$$\begin{pmatrix} 1 \\ \cos x_i \\ \sin x_i \\ \vdots \\ \cos nx_i \\ \sin nx_i \end{pmatrix}$$

linearly dependent.

i.e. $\exists a_1, \cdots, a_{2n+1} \in \mathbb{R}$, such that

$$\sum_{i=1}^{2n+1} a_i \begin{pmatrix} 1 \\ \cos x_i \\ \sin x_i \\ \vdots \\ \cos nx_i \\ \sin nx_i \end{pmatrix} = 0$$

Since $e^{ix} = \cos x + i \sin x$, we have

$$\sum_{j=1}^{n+1} (a_{2j-1} + a_{2j}) \begin{pmatrix} 1 \\ e^{ix_1} \\ \vdots \\ e^{ix_n} \end{pmatrix} = 0$$

which implies that $a_{2j-1} + a_{2j} = 0$ for all j since we know that the Vandermonde determinant is invertible. (In this equation, $a_{2n+2} = 0$)

Noticed that we can rearrange a_j, it actually proves that $a_m + a_n = 0$ for all $m \neq n$. So $a_j = 0, \forall j$.

Problem 2. Assume $\exists a = x_1 < x_2 < \cdots < x_N \leq b$ such that $|\epsilon(x_i)| = \Delta(P), \epsilon(x_j) = (-1)^{j-1}\epsilon(x_1),\ j = 0, 1, \cdots,$.n

Then $\forall Q \in \text{Span}\{g_1, \cdots, g_N\}$, if $\Delta(Q) < \Delta(P)$, let

$$\eta(x) = P(x) - Q(x) = (P(x) - f(x)) - (Q(x) - f(x))$$

Then

$$\text{sgn}(\eta(x_j)) = \eta(P(x_j) - f(x_j)) = \eta(\epsilon(x_j)) = (-1)^{j-1}, j = 0, 1, \cdots, n$$

So Q has at least n roots on $[a, b]$. Since $\{g_1, \cdots, g_n\}$ satisfies the Haar condition, $Q \equiv 0$.

So P is the best approximation of f.

Conversely, if P is the best approximation. If the result is not true, then we can divide $[a, b]$ into

$$[a, \zeta_1], [\zeta_1, \zeta_2], \cdots, [\zeta_N, b]$$

such that on each interval $\Delta(P)$ satisfies $N \leq n - 1$ and

$$-\Delta(P) \leq \epsilon(x) < \Delta(P) - \alpha$$

or

$$-\Delta(P) + \alpha \leq \epsilon(x) < \Delta(P)$$

Denote $\Phi(x)$ as an element with roots $\zeta_1, \cdots, \zeta_N$. (The existence because of Haar condition)

Then $Q(x) := P(x) + \omega\Phi(x)$ with difference

$$Q(x) - f(x) = P(x) - f(x) + \omega\Phi(x)$$

On $[a, b]$, $\Phi(x)$ is bounded. Take $|\omega|$ sufficiently small, and choose the signature of ω properly, we have

$$\Delta(Q) < \Delta(P)$$

which causes contradiction.

Here we end the proof.

Problem 3. Replace f with $f - p_n$. WLOG we assume the best approximation polynomial is 0.

If $\exists q_n$ such that

$$\|f - q_n\| < \|f\| + \lambda\|q_n\|$$

where $\lambda < \frac{1}{2}$.

Now if $\forall \lambda_m = \frac{1}{m}, m \geq 2$, $\exists\ q_m$ such that

$$\|f - q_m\| < \|f\| + \lambda_m\|q_m\|$$

Since $\|f - q_m\| \geq \|q_m\| - \|f\|$, we have $\|q_m\| < \frac{2}{1-\lambda_m}\|f\| < 4\|f\|$.

So $\|q_m\|$ are uniformly bounded. Hence, $\{q_m\}$ is precompact in the polynomial space, or equivalently, there exists $q \in \mathbb{P}_n$ such that some subsequence $\{q_{m_i}\}$ converges to q.

As $m \to 0$, $\lambda_m \to 0$, then

$$\|f - q\| \leq \|f\|$$

So $q \equiv 0$.

So $\exists N > 0$ such that $\forall i \geq N$, $\|q_{m_i}\| < \|f\|$. Choose q be some q_{m_i}, $i \geq N$.

Now for $x^i = \arg\max |f(x) - q_{m_i}(x)|$, since $|f(x^i) - q_{m_i}(x^i)| \geq \|f\|$, $q_{m_i}(x^i)$ and $f(x^i)$ have different signature. So $|f(x^i)| \geq \|f\| - |q_{m_i}(x^i)|$

Since there exists $a \leq x_0 < x_1 < \cdots < x_{n+1} \leq b$ such that $f(x_i) = \delta(-1)^i \|f\|$ where $\delta = \pm 1$, in the finite dimentional polynomial space, the norm is all equivalent. Therefore $\exists \lambda > 0 \quad s.t.$

$$\max_{0 \leq i \leq n+1} |q(x_i)| > \lambda \|q\|$$

But if $q(x_i)f(x_i) > 0$, then q has root on each interval (x_i, x_{i+1}), $i = 0, 1, \cdots, n$, which contradicts with the fact that q has degree of n.

So $\exists i \quad s.t.\ q(x_i)f(x_i) \leq 0$. Then $|f(x_i) - q(x_i)| = |f(x_i)| + |q(x_i)| \geq \|f\| + \lambda \|q\|$.

Problem 4. For $x \in [a, b]$, WLOG assume $x \neq x_i$. ($x = x_i$ is trivial) Define

$$G(t) = R_{2n+1}(t) - \frac{\omega_{n+1}^2(t)}{\omega_{n+1}^2(x)} R_{2n+1}(x)$$

Then

$$G(x_i) = 0, G(x) = 0$$

So there are $n + 2$ roots on $[a, b]$.

By Rolle's theorem, there are $n + 1$ roots on $[a, b] \setminus \{x_0, \cdots, x_n, x\}$.

Since $G'(t) = R'_{2n+1}(t) - \frac{\omega_{n+1}(t)\omega'_{n+1}(t)}{\omega_{n+1}^2} R_{2n+1}(x)$, $G'(x_i) = 0$.

So there are at least $2n + 2$ roots on $[a, b]$ of G'.

Apply $2n + 1$ times of Rolle's theorem to G', we obtain there is at least one root on $[a, b]$ of $G^{(2n+2)}$.

So $\exists \zeta \in [a, b]$, $0 = G^{(2n+2)}(\zeta) = f^{(2n+2)}(\zeta) - \frac{(2n+2)!}{\omega_{n+1}^2(x)} R_{2n+1}(x)$.

So $\exists \zeta \in [a, b]$, $R_{2n+1}(x) = \dfrac{f^{(2n+2)}(zeta)}{(2n+2)!} \omega_{n+1}(x)$

Problem 5. By partition of unity, we could find $\|f\| = 1$ such that $f(t) \cdot D_n(t) = |D_n(t)|$ except for a small enough set E with $m(E) < \epsilon$. Then

$$\|s_n\| \geq |\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t)\, \mathrm{d}t| \geq \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)|\, \mathrm{d}t - \epsilon \cdot \|D_n\| = \lambda_n - \epsilon \cdot \|D_n\|$$

Hence if let $\epsilon \to 0$, $\|s_n\| \geq \lambda_n$. Therefore, $\|s_n\| = \lambda_n$.

Problem 6. Define the inner product in $C^2[a, b]$ as

$$\langle f, g \rangle = \int_a^b \langle f'', g'' \rangle\, \mathrm{d}x$$

Easy to check that it is a inner product.

Let $g = f - s$.
Then $\int_a^b g''(x)\,\mathrm{d}x = \left.(f'(x) - s'(x))\right|_a^b = 0$.
$\int_a^b xg''(x)\,\mathrm{d}x = \left.(xg'(x))\right|_a^b - \int_a^b g'(x)\,\mathrm{d}x = 0$.
Therefore $\int_a^b p(x)g''(x)\,\mathrm{d}x = 0$ for all p polynomial of degree 1.
Since $s'' \in \mathbb{P}_1$, $\langle s, g\rangle = 0$. Therefore,

$$\|f\| = \|s\| + \|f - s\| \geq \|s\|$$

It is what we need.

Problem 7. Noticed that $f(x) = -\frac{3}{4}(x-1)(x-2)(x+\frac{2}{3}) + 1$ satisfies

$$f(1) = f(2) = 1,\ f(0) = 0$$

and

$$f'(0) = 0$$

So $f(x)$ is what we need.