Complex Analysis HW5

1.(a) Let $z = re^{i\theta}$, then $\int_{|z|=r} x\,dz = \int_0^{2\pi} r\cos\theta \cdot rie^{i\theta}\,d\theta = ir^2\int_0^{2\pi} \cos^2\theta + i\sin\theta\cos\theta\,d\theta = ir^2\int_0^{2\pi} \cos^2\theta\,d\theta = ir^2\pi$

(b) $\int_{|z|=r} x\,dz = \int_{|z|=r} \left(z + \frac{r^2}{z}\right)\cdot\frac{1}{2}\,dz = \frac{r^2}{2}\int_{|z|=r}\frac{1}{z}\,dz = \frac{r^2}{2}\cdot 2\pi i = r^2\pi i$

2. The roots of $z^2=1$ lie in $\{|z|\le 2\}$. $\int_{|z|=2}\frac{1}{z^2-1}\,dz = \int_{|z|=2}\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz = \frac{1}{2}\int_{|z|=2}\frac{dz}{z-1} - \frac{1}{2}\int_{|z|=2}\frac{dz}{z+1} = \pi i - \pi i = 0$

3. $\int_{|z|=1}|z-1|\cdot|dz| = \int_0^{2\pi}|e^{i\theta}-1|\cdot|ie^{i\theta}d\theta| = \int_0^{2\pi}\left((1-\cos\theta)^2+\sin^2\theta\right)^{\frac{1}{2}}d\theta = \int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta = 2\int_0^{2\pi}\left|\sin\frac{\theta}{2}\right|d\theta = 8$

4. ① Let $\gamma:[0,1]\to\mathbb{C}$, then $\int_\gamma \overline{f(z)}\,f'(z)\,dz = \int_0^1 \overline{f(z(t))}\,f'(z(t))\,z'(t)\,dt = \int_0^1 \overline{f(z(t))}\frac{d}{dt}(f(z(t)))\,dt$

$= |f(z(t))|^2\Big|_0^1 - \int_0^1 f(z(t))\,d\overline{f(z(t))} = -\overline{\int_0^1 \overline{f(z(t))}\frac{d}{dt}f(z(t))\,dt} = -\overline{\int_\gamma \overline{f(z)}\,f'(z)\,dz} \Rightarrow \int_\gamma \overline{f(z)}\,f'(z)\,dz$ is purely imaginary.

② $\int_\gamma \overline{f(z)}\,f'(z)\,dz = \int_\gamma (u-iv)(u_x+iv_x)(dx+idy) \Rightarrow \mathrm{Re}\int_\gamma \overline{f(z)}\,f'(z)\,dz = \int_\gamma (uu_x+vv_x)\,dx + (vu_x-uv_x)\,dy$

$= \int_\gamma (uu_x+vv_x)\,dx + (uu_y+vv_y)\,dy = \frac{1}{2}\int_\gamma d(u^2+v^2) = 0.$

5. Taylor expension of P at a is $P(z) = a_n(z-a)^n + \cdots + a_1(z-a) + a_0$

Note that on $\{|z-a| = R\}$ we have $\overline{(z-a)}^k = \frac{R^{2k}}{(z-a)^k}$

Hence $\int_C \overline{P(z)}\,dz = \int_C \overline{a_n}\,\overline{(z-a)}^n + \cdots + \overline{a_0}\,dz = \int_C \overline{a_n}\frac{R^{2n}}{(z-a)^n} + \cdots + \overline{a_1}\frac{R^2}{(z-a)} + a_0\,dz$

$= \int_C \frac{R^2}{(z-a)}\overline{a_1}\,dz = 2\pi i R^2\overline{a_1} \Rightarrow \int_C P(z)\,\overline{dz} = -2\pi i R^2 a_1 = -2\pi i R^2 P'(a).$