Complex Analysis HW7.

1. It's equivalent to say that $e^{\frac{1}{z}}$ has an essential singularity at $z=0$.

$\forall \alpha \in \mathbb{R}$, $\lim\limits_{n\to\infty} |\frac{1}{n}|^{\alpha} |e^{n}| = +\infty$, $\lim\limits_{n\to\infty} |\frac{-1}{n}|^{\alpha} |e^{-n}| = 0 \Rightarrow \lim\limits_{z\to 0} |z|^{\alpha} |e^{\frac{1}{z}}|$ doesn't exist $\Rightarrow z=0$ is essential singularity.

2. ① f has a non-essential singularity at ∞ implies that $\exists\, n>0$ s.t. $\lim\limits_{z\to 0} |z|^{n} |f(\frac{1}{z})| = 0$.

Since f is holomorphic on $\mathbb{C}$, we have Taylor expension $f(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + g(z) z^{n}$ for some holomorphic function g on $\mathbb{C}$, $\lim\limits_{z\to 0} |z|^{n} |f(\frac{1}{z})| = \lim\limits_{z\to 0} |a_0 z^{n} + \dots + a_{n-1} z + g(\frac{1}{z})| = \lim\limits_{z\to 0} |g(\frac{1}{z})| = 0$.

In particular, g is bounded on $\mathbb{C} \Rightarrow g$ is constant $\Rightarrow g \equiv 0 \Rightarrow f$ is a polynomial.

② $\exists\, \varepsilon > 0$ s.t. $|z|^{n} |f(\frac{1}{z})| < 1$ $(\forall\, |z| < \varepsilon) \Rightarrow |f(z)| \leq |z|^{n}$ $(\forall\, |z| > \frac{1}{\varepsilon})$.

By HW6. Ex 3, f is a polynomial.

3. Since $\hat{\mathbb{C}}$ is compact and the poles are isolated, f has only finitely many poles, say $\{a_1, \dots, a_n\}$ with multiplicity $\{N_1, \dots, N_n\}$. Define $F(z) = f(z) \prod\limits_{i=1}^{n} (z - a_i)^{N_i}$, then F is analytic on $\mathbb{C}$ and has a non-essential singularity at ∞. By Ex 2 we know F is a polynomial $\Rightarrow f$ is rational.

4. let $T: z \mapsto \dfrac{z-(M-1)}{z-(M+1)}$, then T maps $\{Re(z) \leq M\}$ to $\{|z| \leq 1\}$.

So $\forall z \in \Omega \setminus \{a\}$, $|T\circ f(z)| \leq 1 \Rightarrow \lim\limits_{z\to a} |z-a| \cdot |T\circ f(z)| = 0 \Rightarrow a$ is a removable singularity of $T\circ f$.

Note that $f = T^{-1} \circ (T\circ f)$. ~~and T^{-1} is holomorphic~~ If $T\circ f(a) \neq 1$, then T^{-1} is holomorphic near $T\circ f(a)$

$\Rightarrow f$ has a removable singularity at a.

If $T\circ f(a) = 1$, then $\lim\limits_{z\to a} f(z) = \infty$, i.e. a is a pole of f. Assume $f(z) = (z-a)^{-n} g(z)$ for some $n \geq 1$ and holomorphic function $g(z)$ near $z=a$ with $g(a) \neq 0$.

Denote by $g(a) = b + ic$. We may assume $b > 0$. Then $\exists\, \varepsilon > 0$ s.t. $\forall\, |z-a| < \varepsilon$, $|Re(g(z)) - b| < \frac{b}{2}$.

Take $z \in \{|z-a| < \varepsilon\}$ s.t. $z - a = r \in \mathbb{R}_+$, then $Re(f(z)) = Re(r^{-n} g(z)) = r^{-n}(Re(g(z))) \geq r^{-n} \frac{b}{2}$.

let $r \to \infty$ we get $Re(f(z)) \to \infty$, a contradiction. So $T\circ f(a) \neq 1$.

5. If a is a pole, write $f(z)=g(z)(z-a)^{-n}$ for some $n\geq 1$ and $g(a)\neq 0$. We may assume $g(a)=b+ic$ with $b>0$. $\forall \alpha\in\mathbb{R}$, $\lim\limits_{m\to\infty}|\frac{1}{m}|^{\alpha}\cdot|e^{f(a+\frac{1}{m})}|=\lim\limits_{m\to\infty}|\frac{1}{m}|^{\alpha}\cdot|e^{g(a+\frac{1}{m})\frac{1}{m^{n}}}|=+\infty$.

Take $w\in\mathbb{C}$ s.t. $w^n=-1$, then $\lim\limits_{m\to\infty}|\frac{w}{m}|^{\alpha}\cdot|e^{f(a+\frac{w}{m})}|=\lim\limits_{m\to\infty}|\frac{1}{m}|^{\alpha}\cdot|e^{g(a+\frac{w}{m})\frac{-1}{m^n}}|=0$

So e^f has an essential singularity at $z=a$.

If a is an essential singularity, then $\exists\ \{z_i\},\{z_i'\}$ near a s.t. $z_i\to a$, $z_i'\to a$, $f(z_i)\to 0$, $f(z_i')\to 1$.

Since $\lim\limits_{z_i\to a}e^{f(z_i)}=1$, $\lim\limits_{z_i'\to a}e^{f(z_i')}=e$, $\lim\limits_{z\to a}e^{f(z)}$ doesn't exists and $\lim\limits_{z\to a}e^{f(z)}\neq\infty$.

Note that e^f is holomorphic outside $z=a$, a is an essentially singularity of e^f.

6. (a) Since f is analytic in Ω except for poles, if the poles $\{a_n\}$ have an accumulation point $a\in\Omega$, then a is a pole or f is analytic at a.

If a is a pole, by definition $\exists\ r>0$ s.t. f is analytic on $\{0<|z-a|<r\}$, which contradicts to $a_n\to a$.

If f is analytic at a, then from $\lim\limits_{z\to a_n}f(z)=\infty$ we may find $z_n\in\{|z-a_n|<|a-a_n|\}$ s.t. $|f(z_n)|>n$ $(\forall n\geq 1)$. Note that $z_n\to a$ and f is continuous near a, $f(a)=\lim\limits_{n\to\infty}f(z_n)=\infty$, contradiction.

We may also use the fact that f has no poles near a to get a contradiction.

Thus the poles of f can not have an accumulation point in Ω.

(b) If $\nexists\ z_n\to a$ in Ω s.t. $f(z_n)\to w$, then $\exists\ \begin{matrix}\varepsilon>0\\ \delta>0\end{matrix}$ s.t. $|f(z)-w|>\varepsilon$ $(\forall\, 0<|z-a|<\delta)$

Let $g(z)=\frac{1}{f(z)-w}$, then g is analytic on $\{0<|z-a|<\delta\}$ for $g(a_n)=0$ implies a_n are removable singularity.

Note that $|g(z)|\leq\frac{1}{\varepsilon}$ on $\{0<|z-a|<\delta\}$, a is a removable singularity of g.

So g is analytic on $\{|z-a|<\delta\}$ with zeros $a_n\to a\Rightarrow g(a)=0$, which is impossible by (a) since a is a pole of $f(z)$ in this case.

Thus $\exists\ z_n\to a$ in Ω s.t. $f(z_n)=w$ $(\forall w\in\mathbb{C})$.