In this homework, we define $|x-y| := d(x,y)$, if there is a metrc space with d.

E1 Both of them are convergent.

Proof. (a)when $x \to \infty$, $e^x \to \infty$ .Then

$$\int_0^\infty \frac{\sin(e^x)}{e^x}\,\mathrm{d}e^x = \int_1^\infty \frac{\sin(e^x)}{e^x}\,\mathrm{d}x \quad \textbf{converges(ex in the Analysis16)}$$

so

$$\begin{aligned}\int_1^\infty \sin(e^x)\,\mathrm{d}x &= \int_1^\infty \frac{sin(e^x)}{e^x}\,e^x\,\mathrm{d}x\\ &= \int_0^\infty \frac{\sin(e^x)}{e^x}\,\mathrm{d}e^x \qquad\qquad converges\end{aligned}$$

(b)

$$\begin{aligned}\int_0^\infty \sin(x^2)\,\mathrm{d}x &= \int_0^\infty \frac{\sin(x^2)}{2x}\,2x\,\mathrm{d}x\\ &= \int_0^\infty \frac{\sin(x^2)}{2x}\,\mathrm{d}x^2\\ &= \int_0^\infty \frac{\sin(x)}{2\sqrt{x}}\,\mathrm{d}x \qquad \textbf{converges(use Abel-Dirichlet test)}\end{aligned}$$

□

E2

Proof. For ϕ is a continuous 1-1 mapping,then ϕ is monotonic ,and $\phi(c) = a \Rightarrow \phi$ is increasing. Then for each partition $P = a = x_0, x_1, x_2, \cdots, x_n = b$ of $[a,b]$,we can find uniquely parition $Q = c = y_0, y_1, \cdots, y_n = d$ s.t. $\phi(y_i) = x_i\,\forall\, i = 0, 1, \cdots, n$ then

$$\Lambda(\gamma_1) = \sup_p \sum_{i=1}^n |\gamma_1(x_i) - \gamma_1(x_{i-1})| = \sup_Q \sum_{i=1}^n |\gamma_2(y_i) - \gamma(y_{i-1})| = \Lambda(\gamma_2)$$

which implies that γ_1 is rectifiable iff γ_2 is rectifiable. And the lenth is the same. □

E3 (a)

Proof. $\forall \epsilon > 0, \exists n \in \mathbb{N} \quad s.t. \quad |f_n(x) - f(x)| < \epsilon, \quad \forall x \in E$ Then

$|f(x)-f(y)| \leqslant |f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)| < 2\epsilon+|f_n(x)-f_n(y)|$

f_n is continuous implies $\lim\limits_{y \to x} |f(x) - f(y)| \leqslant 2\epsilon + \lim\limits_{y \to x} |f_n(x) - f_n(y)| = 2\epsilon$. Let $\epsilon \to 0$,Then $\lim\limits_{y \to x} |f(x) - f(y)| = 0$

Which implies f is continuous;

Then $\forall \delta > 0$, exits $N \in \mathbb{N}$ s.t

$$\forall n \geqslant N,\, |f_n(x) - f(x)| < \frac{1}{2}\delta, \quad \forall n \geqslant N, x \in E$$

$$\forall n \geqslant N,\, |f(x_n) - f(x)| < \frac{1}{2}\delta$$

Then

$$\forall n \geqslant N,\, |f_n(x_n) - f(x)| \leqslant |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \delta$$

i.e. $\lim\limits_{n \to \infty} f_n(x_n) = f(x)$ $\square$

(b)The answer is no. Let we choose E be a set of isolated points. $E = x_1, x_2, \cdots$ Then (1) equals to $\lim\limits_{n \to \infty} f_n(x) = f(x), \quad \forall x \in E$. However we can't know $\{f_n\}$ converges uniformly on E from the previous condition.(Let $|f_n(x_m) - f(x_m)| < \epsilon \quad iff \quad n \geqslant m\epsilon$. then we can't fing a exactly N s.t $|f_n(x_m) - f(x_m)| < \epsilon \quad \forall n \geqslant N \quad and \quad m \in \mathbb{N}$)

(c)The answer is yes.

lemma. In the condition of (1),for each $x \in E$, we can find a neibourhood $N_r(x)$ in E such that $\{f_n\}$ uniformly converges in $N_r(x)$.

Proof. First, for $x \in E$, choose $\{x_n\}$ be the sequence whose terms are all x, then $\lim\limits_{n \to \infty} f_n(x) = f(x)$. If x is an isolated point of E, then we can find a neibourhood of x which contains only x. $\{f_n\}$ certainly converges uniformly in $N_r(x) = \{x\}$.

If x is a limit point of E , we assume that the lemma is not true for x.

i.e. we can't find $r > 0$ s.t. $\{f_n\}$ uniformly converges in $N_r(x)$. Then there exists ϵ such that $\forall N \in \mathbb{N}, r > 0$ there exists $n, m \geqslant N$, and $y \in N_r(x)$, $|f_n(y) - f_m(y)| < 2\epsilon$(use Cauchy criterion)

Then we try to construct a sequence $\{x_n\}$(converges at x)and increasing sequence$\{N_i\}$ such that $\forall n \in \mathbb{N}_+, |f_{N_n}(x_n) - f(x)| \geqslant \epsilon$. Then we can expand $\{x_n\}$ to $\{y_n\}$ such that

$$y_n = \begin{cases} x_i & n = N_i, i \in \mathbb{N} \\ x_{i-1} & N_{i-1} < n < N_i, i \geqslant 2; x \qquad n < N_1; \end{cases}$$

$\{y_n\}$ contradicts with (1), so the assumption is impossible;

First,if $\forall n \in \mathbb{N}, y \in E$, $|f_n(y) - f(x)| < \epsilon$, then $|f_n(y) - f_m(y)| \leqslant |f_n(y) - f(x)| + |f(x) - f_m(y)| < 2\epsilon$, $\forall n, m, \in \mathbb{N}, y \in E$, which imples the lemma is right. so with the assumption we can find $N_1 \in \mathbb{N}, x_1 \in E$, such that $|f_{N_1}(x_1) - f(x)| \geqslant \epsilon$. Let $r := |x_1 - x|$

If we have constructed $x_1, x_2, \cdots, x_n \in E$ and increasing sequence $\{N_i\}$($n \in \mathbb{N}_+$) such that $|f_{N_i}(x_i) - f(x)| \geqslant \epsilon$, $\forall 1 \leqslant i \leqslant n$ and $|x_i - x| \leqslant \frac{r}{2^{i-1}}$, $\forall 1 \leqslant i \leqslant n$. Then if $\forall n \in \mathbb{N}, n > N_n, y \in N_{\frac{r}{2^n}}(x)$, $|f_n(y) - f(x)| < \epsilon$, then $|f_n(y) - f_m(y)| \leqslant |f_n(y) - f(x)| + |f(x) - f_m(y)| < 2\epsilon$, $\forall n, m, \in \mathbb{N}, n, m > N_n, y \in N_{\frac{r}{2^n}}(x)$, which imples the lemma is right. so with the assumption we can find $N_{n+1} \in \mathbb{N}, N_{n+1} > N_n, x_{n+1} \in E$,such that $|f_{N_{n+1}}(x_{n+1}) - f(x)| \geqslant \epsilon$ and $|x_{n+1} - x| < \frac{r}{2^n}$.

So the construction exits, which cause the contradiction. □

With the lemma, define $\phi(x), x \in E$ be the neibourhood of x satisfying the lemma's condition. Then $E = \underset{x \in E}{\cup} \phi(x)$. With E is compact we know exists a finite set $K \subset E$ s.t. $E = \underset{x \in K}{\cup} \phi(x)$. Then for $\{f_n\}$ converges uniformly in each $\phi(x), x \in K$ and K is finite, then $\{f_n\}$ converges uniformly in $\underset{x \in K}{\cup} \phi(x) = E$.

E4

Proof. From Cauchy criterion we know (a) implies $\forall \delta > 0\, \exists N \in \mathbb{N}$ s.t. $\forall n, m \geqslant N, n < m \quad |\sum\limits_{i=n}^{m} f_i(x)| < \delta$ Then

$$
\begin{aligned}
|\sum_{i=n}^{m} f_i(x) g_i(x)| &= |\sum_{i=n}^{m-1} (g_i(x) - g_{i+1}(x)) \sum_{j=n}^{i} f_j(x) + g_m(x) \sum_{j=n}^{m} f_j(x)| \\
&\leqslant \sum_{i=n}^{m-1} |g_i(x) - g_{i+1}(x)| |\sum_{j=n}^{i} f_j(x)| + |g_m(x)| |\sum_{j=n}^{m} f_j(x)| \\
&< \delta (g_n(x) - g_m(x)) \qquad (*)
\end{aligned}
$$

Then (b) implies $\forall \epsilon > 0$, $\exists M \in \mathbb{M}$ s.t. $g_n(x) < \dfrac{\epsilon}{\delta} \quad \forall x \in E$,with (*) we can know $|\sum\limits_{i=n}^{m} f_i(x) g_i(x)| < \epsilon$ for $n, m \geqslant \max\{N, M\}$

use Cauchy criterion we know $\sum f_n g_n$ converges uniformly on E. □

E5 (a) we try to prove that the set of all discontinuities of f is $\mathbb{Q}$.

step1. f are discontinuous at All of rational numbers.

Proof. for $x = \frac{q}{p} \in \mathbb{Q}$, then $\forall \epsilon > 0$, choose $n \in \mathbb{N}$ s.t. $\epsilon > \frac{1}{n-1}$ then choose r be the minimun distance(except 0) between x and the number that have a form

$\frac{m}{n!}$, $m \in \mathbb{N}$. then for each $y \in (x-r,x)$,

$$
\begin{aligned}
|f(y)-f(x)| &= |\sum_{m=1}^{\infty} \frac{(my)-(mx)}{m^2}| \\
&= |-\sum_{p|m, m<n} \frac{1}{m^2} + \sum_{m=n}^{\infty} \frac{(my)-(mx)}{m^2}| \\
&\geqslant |\sum_{p|m, m<n} \frac{1}{m^2}| - \sum_{m=n}^{\infty} |\frac{(my)-(mx)}{m^2}| \\
&\geqslant |\sum_{p|m, m<n} \frac{1}{m^2}| - \sum_{m=n}^{\infty} \frac{1}{m^2} \\
&> |\sum_{p|m, m<n} \frac{1}{m^2}| - \frac{1}{n+1} \\
&> |\sum_{p|m, m<n} \frac{1}{m^2}| - \epsilon
\end{aligned}
$$

Let $\epsilon \to 0$, then $|f(y)-f(x)| > |\sum\limits_{p|m, m<n} \frac{1}{m^2}| > \frac{1}{p^2}, \forall y \in (x-r,x)$. i.e. $f(x-) \neq f(x)$. So f is discontinuous at x.

$\square$

step2. f are continuous at all of irrational numbers.

Proof. then $\forall \epsilon > 0$, choose $n \in \mathbb{N}$ s.t. $\epsilon > \frac{1}{n-1}$ then choose r be the minimun distance between x and the number that have a form $\frac{m}{n!}$, $m \in \mathbb{N}$, . then for each $y \in (x-r,x)$, $(my)-(mx) = 0 \, \forall m \leqslant n$. And

$$
\begin{aligned}
|f(y)-f(x)| &= |\sum_{m=1}^{\infty} \frac{(my)-(mx)}{m^2}| \\
&= |\sum_{m=n}^{\infty} \frac{(my)-(mx)}{m^2}| \\
&\leqslant \sum_{m=n}^{\infty} |\frac{(my)-(mx)}{m^2}| \\
&\leqslant \sum_{m=n}^{\infty} \frac{1}{m^2} \\
&\leqslant \frac{1}{n-1} \\
&< \epsilon
\end{aligned}
$$

Then let $\epsilon \to 0 \Rightarrow f(x-) = f(x)$.

Similarly, we can prove $f(x+) = f(x)$. Then f is continuous at x. $\square$

(b)

Proof. We need to prove that f is Rieman-integrable on $[a,b]$, $a,b\in\mathbb{R}$. For $\epsilon>0$, choose $N\in\mathbb{N}$ such that $\epsilon>2\frac{(Nb)-(Na)}{N!}+\frac{b-a}{N-1}$. Then let $x_i=a+\dfrac{i}{N!}, i=1,2,\cdots,M \quad M:=(b-a)N!$. partition $p=x_0,x_1,\cdots,x_M$.

So for $r_i,t_i\in[x_{i-1},x_i]\ i=1,2,\cdots,M$

$$\begin{aligned}\sum_{i=1}^{M}\sum_{n=1}^{\infty}\frac{(nr_i)-(nt_i)}{n^2}\Delta x_i &= \sum_{i=1}^{M}\sum_{n=1}^{\infty}\frac{(nr_i)-(nt_i)}{n^2}\frac{1}{N!}\\ &\leqslant \sum_{i=1}^{M}(\sum_{n=1}^{N}\frac{(nx_i)-(nx_{i-1})}{n^2}\frac{1}{N!}+\sum_{n=N}^{\infty}\frac{1}{n^2}\frac{1}{N!})\\ &< \sum_{n=1}^{N}(\frac{(nb)-(na)}{n^2})\frac{1}{N!}+M\frac{1}{N-1}\frac{1}{N!}\\ &< 2\frac{(Nb)-(Na)}{N!}+\frac{b-a}{N-1}\\ &<\epsilon\end{aligned}$$

i.e.

$$U(p,f)-L(p,f)=\sup_{r_i,t_i\in[x_{i-1},x_i]}\sum_{i=1}^{M}\sum_{n=1}^{\infty}\frac{(nr_i)-(nt_i)}{n^2}\Delta x_i<\epsilon$$

and from it we know f is Rieman-integrable. □

E6

Proof. For a closed interval $E=[a,b]$ $f_n\to f$ uniformly in E, so for $\forall\epsilon>0$,we can choose $N\in\mathbb{N}$ such that $\forall n\geqslant N$, $|f_n(x)-f(x)|<\dfrac{\epsilon}{b-a},\forall x\in E$. Then $\int_a^b f_n(x)-f(x)\,\mathrm{d}x<\int_a^b\dfrac{\epsilon}{b-a}\,\mathrm{d}x=\epsilon$. i.e. $\lim\limits_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x=\int_a^b f(x)\,\mathrm{d}x \quad (*)$
Then

$$\begin{aligned}|\lim_{n\to\infty}\int_0^\infty f_n(x)\,\mathrm{d}x-\int_0^\infty f(x)\,\mathrm{d}x| &\leqslant \lim_{n\to\infty}\Big[\lim_{a\to0^+,b\to+\infty}(|\int_a^b f_n(x)-f(x)\,\mathrm{d}x|+|\int_0^a f_n(x)-f(x)\,\mathrm{d}x|\\ &+|\int_b^\infty f_n(x)-f(x)|\,\mathrm{d}x)\Big]\\ &=\lim_{n\to\infty}\Big[\lim_{a\to0^+,b\to\infty}(|\int_0^a f_n(x)-f(x)\,\mathrm{d}x|\\ &+|\int_b^\infty f_n(x)-f(x)|\,\mathrm{d}x)\Big] \qquad \textbf{use(*)}\\ &\leqslant \lim_{n\to\infty}\Big[\lim_{a\to0^+,b\to\infty}(|\int_0^a 2g(x)\,\mathrm{d}x|+|\int_b^\infty 2g(x)\,\mathrm{d}x|)\Big] \qquad \textbf{use}(|f_n-f|\leqslant 2g)\end{aligned}$$

The last equal sign holds because $\int_0^{+\infty} g\,\mathrm{d}x < +\infty$ implies $\lim\limits_{a\to 0^+} \int_0^a g\,\mathrm{d}x = \lim\limits_{b\to\infty} \int_b^\infty g\,\mathrm{d}x = 0$ $\square$

E7 Each $(x_0, y_0) \in [0,1] \times [0,1]$ has the form :

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1},\ y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

with each $a_i \in 0, 1$

Let $t_0 = \sum\limits_{i=1}^{\infty} 3^{-1-i}(2a_i)$ Then we only need to prove $f(3^n t) = a_n$, $n \in \mathbb{N}_+$, which imples that $x(t_0) = x_0$ and $y(t_0) = y_0$.

Actually, $3^n t_0 = \sum\limits_{i=1}^{n-1} 3^{n-i-1}(2a_i) + \sum\limits_{i=n+1}^{\infty} 3^{n-i-1}(a_i) + 3^{-1}(2a_i)$. Notice that $\sum\limits_{i=1}^{n-1} 3^{n-i-1}(2a_i)$ is even, then $f(3^n t_0) = f(\sum\limits_{i=n+1}^{\infty} 3^{n-i-1}(a_i) + 3^{-1}(2a_i))$. However, $\sum\limits_{i=n+1}^{\infty} 3^{n-i-1}(a_i) \in (0, \frac{1}{3})$. With the definition of f we know that

$$f(3^n t_0) = \begin{cases} 1 & a_n = 1 \\ 0 & a_n = 0 \end{cases} = a_n$$