In this homework, we define $|x-y| := d(x,y)$, if there is a metrc space with d.

E1

Proof. With K is compact, then for $n \in \mathbb{N}_+$, $K \subset \bigcup\limits_{x\in K} N_{\frac{1}{n}}(x)$ implies that exists a finite set $K_n \subset K$ such that $K \subset \bigcup\limits_{x\in K_n} N_{\frac{1}{n}}(x)$. Let $X = \bigcup\limits_{n\in\mathbb{N}}$, then each point in K is a limit point of X or a point of X (cause each point not in X is contained by $N_{\frac{1}{n}}(x)$ for some $x \in X$, $\forall n \in \mathbb{N}$). i.e. X is dense in K.

For K_n is finite, then X is countable, and dense in K. □

E2

Proof. For $\forall \epsilon > 0$, choose $\delta > 0$ such that $|f_n(x) - f_n(y)| < \frac{1}{3}\epsilon$, $\forall n \in \mathbb{N}, \forall x, y, \in K, |x-y| < \delta$.

Cause $\{f_n\}$ converges pointwise on K, i.e. $\forall x \in K$, $\{f_n(x)\}$ converges, and then is a Cauchy sequence. So choose $N_x \in \mathbb{N}$ such that $\forall n, m \geqslant N_x$, $|f_n(x) - f_m(x)| < \frac{1}{3}\epsilon$.

With $K = \bigcup\limits_{x\in K} N_\delta(x)$ and K is compact, then exists finite subset $P \subset K$, s.t. $K = \bigcup\limits_{x\in P} N_\delta(x)$.

Let $N := \max\limits_{x\in P} N_x$, then $\forall n, m \geqslant N, y \in K$, there exists $x \in P$ such that $y \in N_\delta(x)$. Now $|f_n(y) - f_m(y)| \leqslant |f_n(y) - f_n(x)| + |f_n(x) - f_m(x)| + |f_m(x) - f_m(y)| < 3 \times \frac{1}{3}\epsilon$

With Cauchy criterion, $\{f_n\}$ converges uniformly on K □

E3

Proof. S is compact implies that S is closed and bounded. Then there exists $f \in S and r \in \mathbb{R}$ such that $|f-g| \leqslant r, \forall g \in S$. Then $|g(x)| \leqslant |f(x)| + r$, $\forall g \in S, x \in K$. i.e. S is uniformly bounded.

For $\epsilon > 0$, choose $\delta_f, f \in S$ such that $|f(x) - f(y)| < \epsilon$, whenever $d(x,y) < \delta_f$.

Then if S is not equicontinuous, there exists $\epsilon > 0$ such that exists a series of $\{f_n\} \subset S$, $\lim\limits_{n\to\infty} \delta_{f_n} = 0$.

However, S is closed tells us that there exists a subsequence of $\{f_n\}$, $\{f_{n_k}\}$, such that $\{f_{n_k}\}$ converge at a point in S. With the metric, $\{f_{n_k}\}$ converges uniformly, and hense $\{f_{n_k}\}$ is equicontinuous (by the theorem 7.5.2), which contradict with the fact that $\lim\limits_{n\to\infty} \delta_{f_n} = 0$.

□

E4 (a)

Proof. The condition implies that $\int_0^1 f(x)P(x)\,\mathrm{d}x = 0$, $\forall P$ is a poolynomial

By the theorem 7.6.1, there exists a sequence of polynomials $\{P_n\}$, s.t. $\{P_n\} \to f$ uniformly on $[0,1]$. For f is continuous on a compact set, then f is bounded. Let $M \in \mathbb{R}$ be $\sup\limits_{x\in[0,1]} |f(x)|$. Thus for $\epsilon > 0$, there exists $N \in \mathbb{N}$, s.t.

$$\forall n \geqslant N,\, \forall x \in [0,1], |P_n(x) - f(x)| < \epsilon$$

Then

$$\forall n \geqslant N,\, \forall x \in [0,1], |f(x)P_n(x) - f^2(x)| < |f(x)|\epsilon < M\epsilon$$

So $\{fP_n\} \to f^2$ uniformly on $[0,1]$.

By the theorem 7.3.1, we have

$$0 = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty} f(x)P_n(x)\,\mathrm{d}x = \int_0^1 f^2(x)\,\mathrm{d}x$$

Cause $\int_0^1 f^2\,\mathrm{d}x \geqslant 0$, then $f \equiv 0$ □

(b)

Proof. For a sequence $\{\epsilon_n\} \to 0^+$, We try to choose a polynomial sequence $\{P_n\}$ such that $\int_a^b |f - P_n|^2\,\mathrm{d}x < \epsilon_n + \sqrt{\epsilon_n(b-a)}, \forall n \in \mathbb{N}$.

From E5 on HK7(b), we know there exists a continuous function g_n on $[a,b]$ such that $\int_a^b |f-g|^2\,\mathrm{d}x < \epsilon_n^2$. And By the theorem 7.3.1, we can find P_n such that $\max\limits_{x\in[a,b]} |f(x) - P_n(x)| < \sqrt{\epsilon_n}$. Then with E5 on HK7 (a),

$$\sqrt{\int_a^b |f-P_n|^2\,\mathrm{d}x} \leqslant \sqrt{\int_a^b |f-g|^2\,\mathrm{d}x} + \sqrt{\int_a^b |g-P_n|^2\,\mathrm{d}x} < \epsilon_n + \sqrt{\epsilon_n(b-a)}$$

Cause $\epsilon_n \to 0$, then $\lim\limits_{n\to\infty} \int_a^b |f-P_n|^2\,\mathrm{d}x = 0$ □

E5 (a)

Proof. Consider $\{f_n\}$ as a pointwise bounded sequence of complex functions on a countable set $E = [a,b] \cap \mathbb{Q}$. Then by the theorem 7.5.1, we know exists a subsequence converges at all rationals $r \in \mathbb{Q}$. □

(b)

Proof. With the definition of (a)(b), we know that $f(x)$ is increasing. Then $f(x) = \sup\limits_{r\in\mathbb{Q}, r<x} = \lim\limits_{r\to x^-} f(x)$, $x \notin \mathbb{Q} = f(x-)$

Now if $x \notin \mathbb{Q}$, x is a discontinuity of f. Then $f(x) \neq f(x+)$. i.e. $f(x+) > f(x)$. Choose a rational $p_x \in (f(x), f(x+))$. We only need to prove each p_x is unique. (Then $D_f/\mathbb{Q}$ is at most countable. Hence, D_f is at most countable)

If $p_x = p_y$, $x < y$, $x, y \in D_f/\mathbb{Q}$, then $f(x) < p_x < f(x+) < f(y) < p_y$, which causes contradiction. □

(c)

Proof. For $x \in \mathbb{Q}$, $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

If $x \notin \mathbb{Q}, x \notin D_f$, then $f(x-) = f(x) = f(x+)$.

For $\epsilon > 0$, choose $r_1 < x < r_2$ s.t. $r_1, r_2 \in \mathbb{Q}$, $|f(r_1) - f(x)| < \epsilon, |f(r_2) - f(x)| < \epsilon$. And $\exists N \in \mathbb{N}$, s.t. $\forall k \geqslant N$, $|f_{n_k}(r_i) - f(r_i)| < \epsilon$, $i = 1, 2$.

Then $\forall k \geqslant N$,

$$\begin{aligned}
|f_{n_k}(x) - f(x)| &\leqslant |f_{n_k}(x) - f_{n_k}(r_1)| + |f_{n_k}(r_1) - f(r_1)| + |f(r_1) - f(x)| \\
&\leqslant (f_{n_k}(r_2) - f_{n_k}(r_1)) + |f_{n_k}(r_1) - f(r_1)| + |f(r_1) - f(x)| \\
&< (f_{n_k}(r_2) - f_{n_k}(r_1)) + 2\epsilon \\
&\leqslant |f_{n_k}(r_2) - f(r_2)| + |f(r_2) - f(x)| + |f(x) - f(r_1)| + |f(r_1) - f_{n_k}(r_1)| + 2\epsilon \\
&< 6\epsilon
\end{aligned}$$

Let $\epsilon \downarrow 0$, then $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ ☐

(d)

Proof. Consider $\{f_{n_k}\}$ to be defined on D_f. Then by the theorem 7.5.1, there exists a subsequence $\{f_{m_j}\}$ converges for $\forall x \in D_f$. Hence, $\{f_{m_j}\}$ converges on $\mathbb{R}$. And we redefine $f(x) := \lim_{j\to\infty} f_{m_j}(x)$, $x \in \mathbb{R}$. ☐

E6 (a)

Proof. Consider the inequality below:

$$-x - \frac{1}{2}x^2 < \ln(1-x) < -x, \quad \forall x \in [0, 1)$$

Then $S_n - \ln n$ is decreasing, and

$$S_m - S_n + \ln n - \ln m = \sum_{i=n}^{m} (\frac{1}{i} + \ln(1 - \frac{1}{i})) > \sum_{i=n}^{m} -\frac{1}{2i^2} > -\frac{1}{2(n-1)}, \forall n, m \geqslant 2, m > n$$

With the Cauchy criterion, we know $\lim_{n\to\infty}(S_n - \ln n)$ exists. ☐

(b)

Proof.

$$\begin{aligned}
S_N &\leqslant \prod_{p:p\,prime,\ \leqslant N} (\sum_{i=0}^{+\infty} \frac{1}{p^i}) \\
&= \prod_{p:p\,prime, p\leqslant N} \frac{p}{p-1} \\
&\leqslant 2 \prod_{p:p\,prime, p\leqslant N} \frac{p+1}{p} \quad \text{for } \frac{p}{p-1} \leqslant \frac{q+1}{q} \text{ when p>q}
\end{aligned}$$

Then for $\{S_n\}$ diverges, we know that $\{\prod_{p:p\,prime,p\leqslant n}\frac{p+1}{p}\}_n \in \mathbb{N}$ diverges. By the E5 on HK4, we know that $\sum_{p:p\,prime}\frac{1}{p}$ diverges. $\square$