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# Physics-0 Lecture Notes

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## 1.6 The Theory of Gravitation

One of the earliest goals of physics is to understand the gravitational force that holds us to Earth, holds the Moon in orbit around Earth, and holds Earth in orbit around the Sun. It also reaches out through the whole Milky Way galaxy, holding together the billions and billions of stars in the Galaxy and the countless molecules and dust particles between stars. The gravitational force also reaches across intergalactic space, holding together the Local Group of galaxies, which includes, in addition to the Milky Way, the Andromeda Galaxy at a distance of $2.3 \times 10^6$ light-years away from Earth, plus several closer dwarf galaxies. The Local Group is part of the Local Supercluster of galaxies that is being drawn by the gravitational force toward an exceptionally massive region of space called the Great Attractor. This region appears to be about $3.0 \times 10^8$ light-years from Earth, on the opposite side of the Milky Way. And the gravitational force is even more far-reaching because it attempts to hold together the entire universe, which is still expanding.

Gravitation is the first fundamental force whose law is known to people due to Newton's great contribution, i.e., Newton's law of universal gravitation, which is the focus of the current section. However, the gravitational force remains to be one of the most mysterious forces in nature and is still not fully understood by physicists so far.

### 1.6.1 Newton's law of Gravitation

If the myth were true, in 1665, a falling apple inspired the 23-year-old Isaac Newton to his law of universal gravitation. Newton recognized that this force pulling the apple to the ground is also responsible for holding the Moon in its orbit. He further made a far-reaching generalization that every body in the universe attracts every other body. This tendency of bodies to move toward one another is called *gravitation*, which originates from the mass of each body.

**Physics law 5** (Newton's Law of Gravitation)**.** *Any point particle in the universe attracts any point particle with a gravitational force whose magnitude is*

$$F = G\frac{m_1 m_2}{r^2},$$

*where $m_1$ and $m_2$ are the masses of the two particles, $r$ is the distance between them, and*

$$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$$

*is the gravitational constant.*

Here, particle 1 at $\vec{r}_1$ attracts particle 2 at $\vec{r}_2$ means the gravitational force on particle 2 is along the direction from $\vec{r}_2$ to $\vec{r}_1$, i.e., the gravitation force on particle 2 in vector form is

$$\vec{F}_{12} = G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2}\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}. \tag{1.6.1}$$

In reality, two objects can be treated as point particles only when the distance between them is sufficiently large. When the two objects are close to each other compared to their own scales, the formula (1.6.1) cannot be applied directly. In particular, $\vec{r}_1$ and $\vec{r}_2$ are not well-defined for the two objects anymore. In general, let $D \subset \mathbb{R}^3$ denote a domain occupied by an object. We decompose it into a union of many small parts (called differential elements), each of which can be treated as a point particle. A small differential element at $\vec{r} = (x, y, z)$ can be thought of as a box with side lengths $\Delta x$, $\Delta y$, and $\Delta z$. Suppose the density of the object at $\vec{r} = (x, y, z)$ is given by $\rho(\vec{r})$. Then, the gravitation of this differential element on a point particle of mass $m$ at position $\vec{r}'$ is given by

$$Gm\rho(\vec{r})\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\Delta x \Delta y \Delta z.$$

Then, by the principle of superposition for forces, the total gravitational force is given by

$$\sum_{\text{differential elements in } D} Gm\rho(\vec{r})\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\Delta x \Delta y \Delta z.$$

Taking $\Delta x$, $\Delta y$ and $\Delta z$ to zero, the summation above becomes a triple integral

$$\iiint_D Gm\rho(\vec{r})\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \vec{r} = (x, y, z). \tag{1.6.2}$$

You will learn how to evaluate such multi-variable integrals in your calculus courses; it is not the focus of this course.

In applications, the integral is (1.6.2) can be very hard to calculate analytically. We hope to find a simpler way to calculate the gravitation between two objects. Intuitively, it may be natural to conjecture that we can still apply (1.6.1) directly with $\vec{x}_1$ and $\vec{x}_2$ being the centers of mass of these two objects. Unfortunately, this is not true in general. However, there are some special cases where this is true, which simplifies the calculations greatly.

**Proposition 1.6.1.** *Consider a ball with uniform density (mass per unit volume). Suppose the center of the ball is at $\vec{r}_1$ and the ball has mass $m_1$. The magnitude of the gravitational force from this ball on a particle of mass $m_2$, located outside the ball at $\vec{r}_2$, is then given by* (1.6.1).

In other words, when calculating the gravitation, a uniform ball can be treated as a point particle of the same mass located at its center. From this proposition, we can easily derive that when calculating the gravitational force between two balls, they can be treated as two point particles located at their respective centers (think about why). This means when we consider gravitation between planets, it is safe to treat them as point particles no matter how far they are apart from each other.

Let us assume that Earth is a uniform sphere of mass $M$. The magnitude of the gravitational force from Earth on a particle of mass $m$, located outside Earth a distance $r$ from Earth's center, is then given by $GMm/r^2$. If the particle is released, it will fall toward the center of Earth with the gravitational acceleration $g = GM/r^2$. In particular, at the surface of Earth, the gravitational acceleration is 9.8 m/s$^2$. Moreover, the radius of Earth is about $r = 6371$km. Then, we can calculate the mass of Earth as

$$M = \frac{gr^2}{G} \approx 5.96 \times 10^{24} \text{ kg}.$$

This is a simple example of a way to measure the masses of planets: we first observe the acceleration caused by the gravitation of a planet and measure various distances with astronomical observations, and then calculate the mass using Newton's law of gravitation.

**Example.** *A geostationary satellite is a geosynchronous satellite, which has a geostationary orbit—a circular orbit directly above the Earth's equator and with an orbital period the same as the Earth's rotation period. Determine the height of a geostationary satellite.*

**Solution**: Let $R$ be the distance from the geostationary satellite to the center of Earth, $r = 6371$km be the radius of Earth, and $g = 9.8$ m/s$^2$ be the gravitational acceleration at the ground. We have seen that $GM = gr^2$. The angular speed of the satellite is

$$\omega = \frac{2\pi}{24 \times 3600} \text{ rad/s}.$$

Since the size of the satellite is very small compared with $r$, it can be treated as a point particle. Then, by Newton's second law, we have

$$m\omega^2 R = \frac{GMm}{R^2} = \frac{mgr^2}{R^2} \Rightarrow R = \left(\frac{gr^2}{\omega^2}\right)^{1/3}.$$

The height is then given by $R - r$.

We have an immediate generalization of Proposition 1.6.1 to the radially symmetrical case.

**Proposition 1.6.2.** *Proposition 1.6.1 also holds for a ball with radially symmetrical density, i.e., the density depends only on the distance to the center.*

In particular, this proposition holds for the gravitational forces outside a uniform shell of matter—the region between two spheres with the same center. On the other hand, we also have the following simple fact inside the shell:

**Proposition 1.6.3.** *A shell of matter with radially symmetric density exerts no net gravitational force on a particle located inside it.*

By Proposition 1.6.2 and Proposition 1.6.3, to calculate the gravitational force at a point $\vec{x}$ inside a ball with radially symmetric density, we can ignore the shell outside the $\vec{x}$ and treat the matter inside the sphere containing $\vec{x}$ as a point particle located at the center of the ball.

**Example.** *Assume that Earth is a uniform ball. Suppose there is a tunnel connecting the north pole and the south pole. Drop a ball in the tunnel from the north pole. Determine the time the ball needs to reach the south pole.*

**Solution**: Let $R$ be the radius of Earth, $M$ the mass of earth, and $\rho$ the density of Earth. We set up an axis along the tunnel pointing from the north pole to the south pole. Let the origin be the center of Earth, with the coordinate of the north and south poles being $-R$ and $R$ respectively. Denote by $r(t)$ be the coordinate of the ball, with $r(0) = -R$. By Proposition 1.6.2 and Proposition 1.6.3, the gravitation acceleration of the ball at $r(t)$ is equal to

$$a(t) = -\mathrm{sgn}(r(t))\frac{G\rho\frac{4}{3}\pi|r(t)|^3}{|r(t)|^2} = -\frac{4}{3}\pi G\rho r(t) = -\frac{GM}{R^3}r(t) = -\frac{g}{R}r(t), \tag{1.6.3}$$

where the sign function

$$\mathrm{sgn}(r(t)) = \begin{cases} 1, & r(t) > 0 \\ 0, & r(t) = 0 \\ -1, & r(t) < 0 \end{cases}$$

is due to the fact that the gravitation points to the center of Earth, and we used $GM = gR^2$ in the last step. Thus, we get the differential equation

$$\ddot{r}(t) = -\frac{g}{R}r(t),$$

with initial conditions $r(0) = -R$ and $\dot{r}(0) = 0$. This again gives a harmonic oscillator, which has the solution

$$r(t) = -R\cos\left(\sqrt{\frac{g}{R}}t\right).$$

To reach the south pole with $r(t) = R$, we need half of the period of the oscillation, which is

$$t = \pi\sqrt{\frac{R}{g}}.$$

It is possible to prove Propositions 1.6.1–1.6.3 by evaluating the integral (1.6.2) directly. But there is a much simpler way to prove them based on a deep theorem in multi-variable calculus, called *Gauss' law*; see Section 1.6.5 below.

### 1.6.2 Gravitational potential energy

In Section 1.4, we have mentioned that the gravitational force is a conservative force and is associated with some potential energy. There we were careful to keep the particle near Earth's surface so that we could regard the gravitational force as constant. By Newton's Law of Gravitation, we know that the gravitational force is generally not constant. Is it still conservative, and, if yes, what is the associated potential energy?

Suppose we already know that the gravitational force is conservative. We now calculate the potential energy. Consider a point particle of $M$ located at the origin, let $m$ be a particle located at $\vec{r} = (x, y, z)$. Suppose the system has energy 0 when $r \to \infty$. Now, suppose we move the particle from $\infty$ to $\vec{r}$ along the radial direction, i.e., from $\infty\vec{e}_r$ to $\vec{r}$ with $\vec{e}_r = \vec{r}/|\vec{r}|$. Then, it is easy to calculate the work done by the gravitational force during this process:

$$W = \int_r^\infty \frac{GMm}{x^2}\mathrm{d}x = \frac{GMm}{r}.$$

Thus, the potential energy, if exists, must be equal to $-W = -GMm/r$.

**Proposition 1.6.4.** *Gravitational force is conservative. Furthermore, the potential energy of a system of two particles with masses $m_1$ and $m_2$ is*

$$V = -\frac{Gm_1m_2}{r} + C, \tag{1.6.4}$$

*where $r$ is the distance between the two particles. In particular, we can choose $C = 0$ if the reference potential energy at $\infty$ is taken to be 0.*

*Proof.* To show that $V$ is a potential energy, it suffices to show that its gradient is equal to the gravitational force. Suppose particle 1 is at origin, and particle 2 is at $\vec{r} = (x, y, z)$. Then,

$$V(\vec{r}) = -\frac{Gm_1m_2}{(x^2+y^2+z^2)^{1/2}} + C.$$

We can calculate its gradient as

$$-\nabla V(\vec{r}) = -(\partial_x V, \partial_y V, \partial_z V) = -\frac{Gm_1m_2}{(x^2+y^2+z^2)^{3/2}}(x, y, z) = -\frac{Gm_1m_2}{r^2}\frac{\vec{r}}{r},$$

which is exactly the gravitational force on particle 2. Hence, $V$ is indeed the potential energy of the gravitation between the two particles, which implies that the gravitation force is conservative. $\square$

By this proposition, the gravitational force satisfies the path independence property: if we move the pair of particles from one configuration to another, the net work done by the gravitational force during this process is equal to the negative of the change of the potential energy and does not depend on the path taken by the particles.

We remark that the potential energy given by Equation (1.6.4) is a property of the system of two particles rather than of either particle alone. There is no way to divide this energy and say that so much belongs to one particle and so much to the other. However, if $m_1 \gg m_2$, as is true for the problems concerning Earth (mass $m_1$) and an object (mass $m_2$) near its surface, we often speak of "the potential energy of particle 2", because, when the object moves in the vicinity of Earth, changes in the potential energy of the system appear almost entirely as changes in the kinetic energy of the baseball, while changes in the kinetic energy of Earth are too small to be measured. When we speak of the potential energy of bodies of comparable mass, however, we have to be careful to treat them as a system.

In general, if our system contains more than two particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair as if the other particles were not there, and then algebraically sum the results. For example, for a system of three particles with masses $m_1$, $m_2$ and $m_3$ and distances $r_{12}$, $r_{23}$ and $r_{13}$ between each pair of them, its potential energy is

$$V = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_1m_3}{r_{13}}.$$

More generally, as discussed above (1.6.2), the potential energy between an object occupying $D \subset \mathbb{R}^3$ and a point particle of mass $m$ at position $\vec{r}'$ is calculated as an integral:

$$V = -\iiint_D \frac{Gm\rho(\vec{r})}{|\vec{r}-\vec{r}'|}\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \vec{r} = (x, y, z). \tag{1.6.5}$$

Again, the potential energy between a radially symmetric ball and a paticle can be evaluated in a much simpler way by using Propositions 1.6.2 and 1.6.3.

**Example.** *Assume that Earth is a uniform ball. Determine the gravitational potential energy of a point particle of mass $m$ outside and inside Earth.*

**Solution**: Let $R$ be the radius of Earth, $M$ the mass of earth, and $\rho$ the density of Earth. First, suppose the location $\vec{r}$ of the point particle is outside Earth. Then, with Proposition 1.6.2, we can obtain the work done by the gravitational force from $\infty$ to $\vec{r}$ along the radial direction as $W = GMm/r$, thus giving a potential

$$V(r) = -\frac{GMm}{r}, \quad \text{for} \quad r \geq R.$$

Now, suppose $\vec{r}$ is inside Earth, i.e., $r < R$. In Equation (1.6.3), we have seen that the gravitational force on the particle at $x\vec{e}_r$ for some $r \leq x \leq R$ has a magnitude

$$\frac{GMm}{R^3}x.$$

Thus, the work done by the gravitational force from $R\vec{e}_r$ to $\vec{r}$ along the radial direction is equal to

$$\int_r^R \frac{GMm}{R^3}x\mathrm{d}x = \frac{1}{2}\frac{GMm}{R^3}(R^2 - r^2),$$

which gives a change in the potential energy as

$$V(r) - V(R) = -\frac{1}{2}\frac{GMm}{R^3}(R^2 - r^2), \quad \text{for} \quad r < R.$$

To summarize, we have

$$V(r) = \begin{cases} -\frac{GMm}{r} = -\frac{mgR^2}{r}, & r \geq R \\ -\frac{GMm}{R} - \frac{1}{2}\frac{GMm}{R^3}(R^2 - r^2) = -mgR - \frac{mg}{2R}(R^2 - r^2), & 0 \leq r < R \end{cases},$$

where we also rewrote the results using $GM = gR^2$.

With a similar argument, we obtain that the gravitational potential energy between two radially symmetric balls with masses $m_1$, $m_2$ and distance $r$ between them is given by (1.6.4).

**Example.** *A satellite with mass $m$ moves in a circular orbit around Earth with radius $r$. Determine its mechanical energy $E$.*

**Solution**: The potential energy of the satellite is

$$V = -\frac{GMm}{r}.$$

To find the kinetic energy, we write Newton's second law as

$$\frac{GMm}{r^2} = m\frac{v^2}{r}.$$

where $v^2/r$ is the centripetal acceleration of the satellite. From this equation, we can get the kinetic energy

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r}.$$

Therefore, the total mechanical energy is

$$E = K + V = -\frac{GMm}{2r}. \tag{1.6.6}$$

Following Section 1.4, we have the celebrated energy conservation for an isolated system interacting only through gravitational forces: *the mechanical energy of the system, i.e., the kinetic energy plus the gravitational potential energy, does not change.* We now use the mechanical energy conservation to study an important concept called *escape speed*.

If you fire a projectile upward, it will slow, stop momentarily, and return to Earth if the initial speed is too slow. There is a certain minimum initial speed that will cause it to move upward forever, theoretically coming to rest only at infinity. This minimum initial speed is called the *(Earth) escape speed*. We now determine the escape speed of Earth using the concept of energy conservation. Suppose the escape speed is $v_2$. Then, a projectile of mass $m$ has mechanical energy

$$E = \frac{1}{2}mv_2^2 - \frac{GMm}{R} = \frac{1}{2}mv_2^2 - mgR,$$

in which $M$ is the mass of Earth and $R$ is its radius. If the projectile can reach infinity, it has zero potential energy and at least zero kinetic energy there. Thus, its mechanical energy at infinity is zero, so

$$\frac{1}{2}mv_2^2 - mgR = 0 \;\Rightarrow\; v_2 = \sqrt{2gR}.$$

Plugging into $g = 9.8\ \mathrm{m/s^2}$ and $R = 6371$ km, we get the escape speed for Earth as $v_2 = 11.2$ km/s, which is also called the second cosmic velocity for Earth. The "third cosmic velocity" is the speed that a spacecraft needs to attain in order to be able to leave our solar system, i.e., to escape the gravitation of Earth and Sun. With a similar method, one can obtain the third cosmic velocity as 16.7 km/s. Finally, let's determine the "first cosmic velocity".

**Example.** *Calculate the first cosmic velocity $v_1$ of Earth, defined as the minimum initial speed for a projectile to move in a circular orbit around Earth.*

**Solution**: By (1.6.6), the mechanical energy of a projectile moving in a circular orbit around Earth is

$$E = -\frac{GMm}{2r},$$

where $m$ is the mass of the projectile and $r$ is the radius of the orbit. Note that this energy is minimum when $r$ is equal to the radius $R$ of Earth. Moreover, the potential energy at the surface of Earth is $-GMm/R$. Thus, by energy conservation,

$$\frac{1}{2}mv_1^2 - \frac{GMm}{R} = -\frac{GMm}{2R} \;\Rightarrow\; v_1 = \sqrt{\frac{GM}{R}} = \sqrt{gR} \approx 7.9 \text{ km/s}.$$

We have seen that the escape speed of a planet of mass $M$ and radius $R$ is

$$\sqrt{\frac{2GM}{R}},$$

which is larger as $M$ becomes larger and $R$ becomes smaller. In particular, when $M$ is large and $R$ is small enough that the escape speed is larger than the speed of light, neither particles nor light can escape from its surface, thus giving some of the most mysterious structures in the universe: "black holes". A black hole may form when a star considerably larger than our Sun burns out, the gravitational force between all its particles can cause the star to collapse in on itself. Any star coming too near a black hole can be ripped apart by the strong gravitational force (i.e, the tidal force) and pulled into the hole. Enough captures like this yields a supermassive black hole. Such mysterious monsters appear to be common in the universe, although observing them is very difficult.

### 1.6.3 Kepler's laws

The motions of the planets, as they seemingly wander against the background of the stars, have been a puzzle since the dawn of history. Johannes Kepler (1571–1630), after a lifetime of study of the extensive data of the planetary motions in the solar system, worked out the empirical laws that govern these motions that now bear Kepler's name.

**Physics law 6** (Kepler's first law: the law of orbits)**.** *All planets move in elliptical orbits, with the Sun at one focus.*

Note that circular orbit is a special case of this law in which the two foci merge to a single central point. Our Earth is indeed on an elliptical orbit around the Sun, although the eccentricity $e$ of the orbit is not large: $e \approx 0.0167$. Recall that eccentricity is defined as

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}},$$

where $a$ is the semi-major axis, $b$ is the semi-minor axis, and $c$ is the distance from a focus to the center.

**Physics law 7** (Kepler's second law: the law of areas)**.** *A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals; that is, the rate* $\mathrm{d}A/\mathrm{d}t$ *at which it sweeps out area* $A$ *is constant.*

Qualitatively, this second law tells us that the planet will move most slowly when it is farthest from the Sun and most rapidly when it is nearest to the Sun. Kepler's second law is actually equivalent to the law of *conservation of angular momentum*, and we will prove it below using Newton's second law and law of gravitation.

**Physics law 8** (Kepler's third law: the law of periods)**.** *The square of the period of any planet is proportional to the cube of the semi-major axis of its orbit.*

We now illustrate the third law with a circular orbit. Applying Newton's second law to the orbiting planet with mass $m$ yields

$$\frac{GMm}{r^2} = m\omega^2 r,$$

where $M$ is the mass of Sun and $r$ is the radius of the orbit. On the other hand, given the period $T$, the angular speed is equal to $2\pi/T$. Thus, the above equation gives

$$\frac{GM}{r^2} = \frac{4\pi^2}{T^2} r \ \Rightarrow\ \frac{T^2}{r^3} = \frac{4\pi^2}{GM}.$$

The quantity on the right-hand side is a constant that depends only on the mass $M$ of Sun. The above equation holds also for elliptical orbits, provided we replace $r$ with $a$, the semi-major axis of the ellipse.

Although Kepler's laws are about planets orbiting the Sun, they hold equally well for satellites, either natural or artificial, orbiting Earth or any other massive central body.

Kepler's laws are phenomenological laws and indeed can be derived from Newton's second law and law of gravitation (although historically, Newton's law of gravitation is inspired by Kepler's three laws). We now prove Kepler's second law, while the proof of Kepler's first and third laws is more advanced and is not required in this course.

Choose the coordinate such that the origin is at the center of the Sun. Suppose at time $t$, the planet is at $\vec{r} = (x, y)$ and its velocity is $\vec{v} = (v_x, v_y)$. We can choose the direction of $\vec{v}$ such that $xv_y - yv_x \geq 0$ (otherwise, we can reverse the directions of the coordinate axes). During an infinitesimal time $\Delta t$, the planet travels $\vec{v}\Delta t$. Then, using cross product formula for the area of a triangle, the area swept by the planet during $\Delta t$ is

$$\Delta A = \frac{1}{2}|\vec{x} \times (\vec{v}\Delta t)| = \frac{1}{2}(xv_y - yv_x)\Delta t.$$

Hence, the instantaneous rate at which the area is being swept out is equal to

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}(xv_y - yv_x).$$

Then, Kepler's second law is equivalent to that this rate is constant, i.e.,

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} = 0. \tag{1.6.7}$$

In fact, for this system, we can define the *angular momentum* of the planet as

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = m(xv_y - yv_x)\vec{k},$$

where $\vec{k}$ is the basis unit vector along the positive $z$ direction. Then, equation (1.6.7) is equivalent to the conservation of $\vec{L}$:

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = 0. \tag{1.6.8}$$

To check (1.6.8), we take the derivative of $\vec{L}$ and use that $\dot{\vec{r}} = \vec{v}$ and $\dot{\vec{p}} = \vec{F}_g$ by Newton's second law, where $\vec{F}_g$ is the gravitational force on the planet. In this way, we get

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F}_g = 0,$$

where, in the last step, we use that $\vec{v}$ is parallel to $\vec{p} = m\vec{v}$ and $\vec{r}$ is parallel to $\vec{F}_g$ by Newton's law of gravitation. This shows (1.6.8), hence concluding Kepler's second law.

### 1.6.4 Gravitation on Earth

So far, we have assumed that Earth is a uniform ball and an inertial frame by neglecting its rotation. This simplification has allowed us to assume that the free-fall acceleration $g$ of a particle is the same as the particle's gravitational acceleration. Furthermore, we assumed that $g$ has the constant value $g = GM/r^2$ any place on Earth's surface. However, any $g$ value measured at a given location will differ from the value $a_g = GM/r^2$ for that location for the following three reasons:

- Earth's mass is not uniformly distributed. The density of Earth varies radially, and the density of the crust (outer section) varies from region to region over Earth's surface. Thus, $g$ varies from region to region over the surface.
- Earth is not a sphere. Earth is approximately an ellipsoid, flattened at the poles and bulging at the equator. Its equatorial radius (from its center point out to the equator) is greater than its polar radius (from its center point out to either north or south pole) by 21 km. Thus, the free-fall acceleration $g$ increases if you were to measure it while moving at sea level from the equator toward the north or south pole. As you move, you are actually getting closer to the center of Earth, and thus, by Newton's law of gravitation, $g$ increases.
- Earth is rotating, so its ground is *not* an inertial frame. The rotation axis runs through the north and south poles of Earth. An object located on Earth's surface anywhere except at those poles must rotate in a circle about the rotation axis and thus must have a centripetal acceleration directed toward the center of the circle. This centripetal acceleration requires a centripetal net force that is also directed toward that center.

To see how Earth's rotation causes a difference in gravitational acceleration, we consider a particle of mass $m$ at a point with latitude $\pi/2 - \theta \in [0, \pi/2]$. Denote the normal force on the particle by $\vec{F}_N$, and the tangent force by $\vec{T}$. The gravitation from Earth is $GMm/r^2 = ma_g$ pointing to the center of Earth. Since the particle is doing a circular motion around the rotation axis with radius $r\sin\theta$ and angular speed $\omega$, it has a centripetal acceleration $\omega^2 r\sin\theta$ directed toward the rotation axis between the north and south poles.

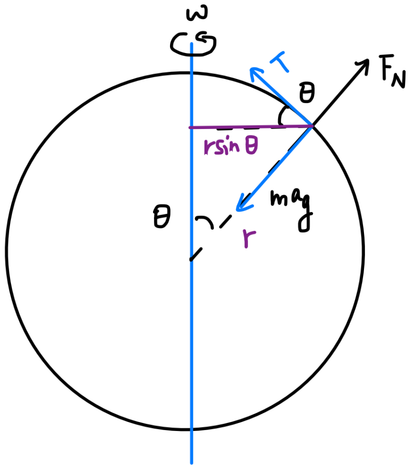


Figure 1.32: The rotation frame on Earth.

By Newton's second law, we have

$$(ma_g - F_N)\sin\theta + T\cos\theta = m\omega^2 r\sin\theta, \quad (ma_g - F_N)\cos\theta = T\sin\theta,$$

solving which gives

$$F_N = m(a_g - \omega^2 r\sin^2\theta).$$

The magnitude of the normal force is equal to the weight $mg$ read on the scale, meaning that the free-fall acceleration $g$ measured in the frame of the rotating Earth is

$$g = \frac{F_N}{m} = a_g - \omega^2 r\sin^2\theta.$$

The difference $\omega^2 r\sin^2\theta$ is largest at the equator (i.e., $\theta = \pi/2$), where one can calculate that $\omega^2 r$ is approximately 0.034 m/s$^2$, much smaller than 9.8 m/s$^2$. Therefore, neglecting this difference is often justified.

### 1.6.5 Gauss's law for gravity

Gauss's law for gravity, also known as Gauss's flux theorem for gravity, states that: *the flux (surface integral) of the gravitational field (i.e., the field of gravitational acceleration) over any closed surface is equal to the mass enclosed times* $-4\pi G$.

To explain the statement, we consider an object of mass $M$ and enclosed by a surface $S$. This object generates a gravitational acceleration $\vec{a}_g$ at any point of space, which is called the gravitational field. Its flux over the surface is defined as follows. We divide the surface into a union of small area elements $\mathrm{d}\vec{A}$, each of which can be regarded as a flat area element with direction. More precisely, $\mathrm{d}\vec{A}$ is a vector that is perpendicular to the element and points *outside* the surface and has a magnitude equal to the area $\mathrm{d}A$ of the element. Then, the flux of the gravitational field $\vec{a}_g$ through this area element is given by $\vec{a}_g \cdot \mathrm{d}\vec{A}$, with $\vec{a}_g$ taking the value at the location of the element. Adding together the flux over all area elements yields the total flux through the whole surface, which can be expressed as a surface integral

$$\oiint_S \vec{a}_g \cdot \mathrm{d}\vec{A}.$$

Gauss's law tells us that this integral is equal to $-4\pi GM$.

Gauss's law for gravity is equivalent to Newton's law of gravitation, and it is often more convenient to work from than Newton's law. Going from Gauss's law to Newton's law of gravitation is simple (you can think about it by yourself). The other direction needs to use Gauss's theorem in vector calculus. We will return to Gauss's law when we discuss electric fields of electrically charged bodies. We now use it to prove Propositions 1.6.2 and 1.6.3.

We consider a radially symmetric ball, whose center is at the origin. We want to calculate the gravitational field it generates at a location $\vec{r}$ outside the ball. By *radial symmetry*, the field points toward the origin, i.e., it has direction $-\vec{e}_r = -\vec{r}/r$. We only need to determine the magnitude of the field, denoted as $a_g(r)$. For this purpose, we consider a sphere of radius $r$ that encloses the ball. Again, by radial symmetry, the magnitude of the field is the same at any point of the sphere, perpendicular to the sphere, and points inside the sphere. Hence, the flux (surface integral) of the gravitational field over the sphere is

$$-a_g(r) \cdot 4\pi r^2 = -4\pi GM,$$

where $M$ is the mass of the ball. Solving this equation gives

$$a_g(r) = \frac{GM}{r^2}.$$

This concludes Proposition 1.6.2. Proposition 1.6.3 follows from the same argument with $M = 0$, because there is no mass enclosed by the sphere containing $\vec{r}$ inside the shell.