Tsinghua University

DISCRETE OPTIMISTIC

Fall 2024

Homework 5

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- **Collaborators:** I finish this homework by myself.

Problem 1. (a)

$$\lambda(G) = \lambda_{\min}(I + \frac{1}{d}A) = \min_{\vec{x}} \frac{\vec{x}^T(I + \frac{1}{d}A)\vec{x}}{\vec{x}^T\vec{x}} = \min_{\vec{x}} \frac{\frac{1}{d}\sum_{((i,j)\in E)}(x_i + x_j)^2}{\sum_{i=1}^n x_i^2} \le \min_{\vec{y}\in\{0,1,-1\}^n} \frac{\frac{1}{d}\sum_{((i,j)\in E)}(x_i + x_j)^2}{\sum_{i=1}^n x_i^2} = \beta(G)$$

For $\vec{x}$ such that

$$\lambda(G) = \frac{1}{d} \cdot \frac{\sum_{(i,j)\in E}(x_i + x_j)^2}{\sum_{i=1}^n x_i^2}$$

The algorithm $\mathcal{A}$ is defined as follows:

5.1. Randomly sample $t \in [0, 1]$.

5.2. Normalize $\|\vec{x}\| = 1$.

5.3. Let $y_i = \begin{cases} 1 & x_i > t \\ -1 & x_i < -t \\ 0 & \text{otherwise} \end{cases}$.

Then

$$\mathop{\mathbb{E}}_t \beta(G) = \mathop{\mathbb{E}}_t \frac{1}{d} \cdot \frac{\sum_{(i,j)\in E}(y_i + y_j)^2}{\sum_{i=1}^n y_i^2} = \frac{1}{d} \cdot \frac{\sum_{(i,j)\in E}(x_i + x_j)^2}{\sum_{i=1}^n x_i^2} = \lambda(G) \le \sqrt{2\lambda(G)}$$

So the rounding algorithm will return a feasible solution efficiently.

As a result,

$$\beta(G) \le \sqrt{2\lambda(G)}$$

(b)

Problem 2.

Problem 3.

Problem 4.

Problem 5.

Problem 6.

Problem 7. (a) They are actually all Fourier basis $\pm\chi_S$, including the constant function.

(b)

$$\{f:\{\pm 1\}^n \to \{\pm 1\} : |\mathcal{S}| = k\} = \{\sum_{i=1}^{k} \pm\chi_{S_i} : S_i \subset [n] \text{ different}\}$$

(c) For all S odd, χ_S satisfies $\chi_S(\vec{x}) = -\chi_S(\vec{x})$. As a result, those functions f are odd. Conversely, if a function is odd, then

$$f(-\vec{x}) = \sum_{S\subset[n]} \hat{f}(S)\chi_S(-\vec{x}) = -f(\vec{x}) = -\sum_{S\subset[n]} \hat{f}(S)\chi_S(\vec{x})$$

Therefore, its support $\mathcal{S}$ should contain only S with odd elements.

Problem 8. (a)

WLOG we assume g is the subfunction of f gotten by fixing $x_1 = x_2 = \cdots = x_c = 1$.

$$\mathop{\mathbb{E}}_{\vec{x},x_1=x_2=\cdots=x_c=1} f(\vec{x}) = \mathop{\mathbb{E}}_{\vec{x}} f(\vec{x}) \prod_{i=1}^{c}(x_i+1) = \sum_{S\subset[c]} \mathop{\mathbb{E}}_{\vec{x}} f(\vec{x})\chi_S(\vec{x}) = \sum_{S\subset[c]} \hat{f}(S)$$

is within an additive $\pm 2^c\sqrt{\epsilon}$ of the bias $\mathop{\mathbb{E}}_{\vec{x}} f(\vec{x}) = \hat{f}(\emptyset)$, since $\hat{f}(S) \le \sqrt{\epsilon}$ for all $S \subset [c]$.

(b) The hyperthesis implies that for all $\vec{y} \in \{\pm 1\}^c$,

$$\mathop{\mathbb{E}}_{\vec{x}} f(\vec{x}) \prod_{i=1}^{c}(x_i + y_i) = \sum_{S\subset[c]} \hat{f}(S)\chi_S(\vec{y})$$

is within an additive $\pm\sqrt{\epsilon}$ of the bias $\hat{f}(\emptyset)$. Then

$$h(\vec{y}) = \sum_{S\subset[c],S\neq\emptyset} \hat{f}(S)\chi_S(\vec{y})$$

satisfies that

$$|h(\vec{y})| \le \sqrt{\epsilon},\ \forall\vec{y}$$

So

$$\epsilon \ge \mathop{\mathbb{E}}_{\vec{y}} h(\vec{y})^2 = \sum_{S\subset[c]} \hat{h}(S)^2 = \sum_{S\subset[c],S\neq\emptyset} \hat{f}(S)^2$$

Therefore $\forall S \subset [c], |S| > 0$, $\hat{f}(S)^2 \le \epsilon$. Similarly, one can prove it for all $|S| \le c < \frac{1}{\delta}$.

So f is (ϵ, δ)-quasirandom.

(c) The bias of f is 0. By fixing c bits, the bias of g is at most

$$\frac{\sum_{-c\le t-(n-t)\le c} \binom{t}{n}}{2^n}$$

In particular, if $c = 1$, *i.e.* $1 > \delta > \frac{1}{2}$, we have the bias of g is at most

$$\frac{\binom{\frac{n-1}{2}}{n}}{2^n} \geq \sqrt{\frac{2}{\pi n}}$$

So it is $(\sqrt{\frac{2}{\pi n}}, \frac{1}{2})$-quasirandom

Problem 9.