FMM: 1-D Example

$x_j,\ y_i \in [0,1]$, $\quad i, j = 1 \cdots N$

x_j, y_i both uniformly distributed in $[0,1]$

we want to compute

$$v_i = \sum_{j=1}^{N} G(x_j - y_i)\, g_j . \qquad i = 1 \cdots N$$

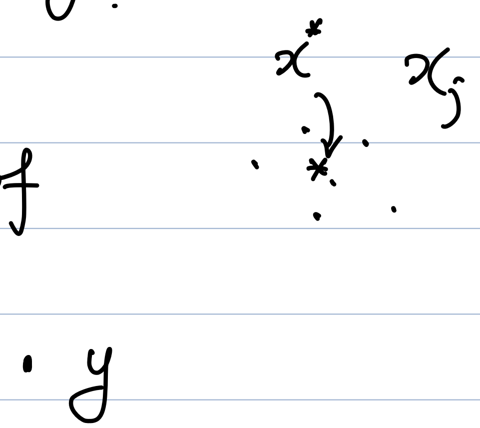
$$G(x_j - y_i) = |x_j - y_i|^{-2}$$

Here we assume $x_j \neq y_i$, $\forall\, i, j$

- Multipole expansion and far field effect.

If the source point $\{x_j,\ j \in T\}$ is far away from target point y.

Let x^* be a center point of $\{x_j,\ j \in T\}$



y is far away from $\{x_j, j \in T\}$ in the sense that

$$\max_{j \in T} |x_j - x^*| \leq \delta |y - x^*|, \quad 0 \leq \delta \leq \frac{1}{2}$$

Then $\sum_{j \in T} G(x_j - y) q_j$ can be approximated by a function of $x^* - y$.

$$G(x_j - y) = G(x_j - x^* + x^* - y)$$

$$= G(x^* - y)\, G\left(1 + \frac{x_j - x^*}{x^* - y}\right)$$

$$= G(x^* - y) \sum_{m=0}^{p} \frac{G^{(m)}(1)}{m!} \frac{(x_j - x^*)^m}{(x^* - y)^m} + O(\delta^{p+1})$$

$$= \sum_{m=0}^{p} a_m(x_j - x^*)\, S_m(x^* - y) + O(\delta^{p+1})$$

Where

$$a_m(x_j - x^*) = \frac{G^{(m)}(1)}{m!} (x_j - x^*)^m$$

$$S_m(x^* - y) = \frac{G(x^* - y)}{(x^* - y)^m}$$

Then

$$v(y) = \sum_{j \in T} G(x_j - y)\, g_j \qquad (1)$$

$$\approx \sum_{m=0}^{p} \left[\sum_{j \in T} g_j\, a_m (x_j - x^*) \right] S_m(x^* - y)$$

The integer p is chosen such that $\delta^{p+1} \le \varepsilon$ for a given accuracy ε.

- **Tree Algorithm**

We construct a binary tree of interval $[0, 1]$, denoted as

$T_{l,k}$, $\quad l = 1 \cdots J, \quad k = 1 \cdots 2^l$

$T_{l,k} = [(k-1) \cdot 2^{-l}, \; k \cdot 2^{-l})$

For each cell $T_{l,k}$, define

$$w_{l,k,m} = \sum_{x_j \in T_{l,k}} q_j\, a_m(x_j - x^*_{l,k}) \quad (2)$$

$x^*_{l,k}$ is the center of $T_{l,k}$.

For any $y_i \in T_{l,k}$ $x_j \in T_{l,s}$

$s \neq k-1, k, k+1$ which means $T_{l,s}$ is not adjacient to $T_{l,k}$. We call $T_{l,s}$ is far field cell of $T_{l,k}$

Then $|y_i - x^*_{l,s}| \geq 3 \cdot 2^{-(l+1)}$

$$|x_j - x^*_{l,s}| \leq 2^{-(l+1)}$$

$$\Rightarrow \quad |x_j - x^*_{l,s}| \leq \frac{1}{3} |y_i - x^*_{l,s}|$$

From (1) and (2)

$$v_i \approx \sum_{m=0}^{p} \sum_{l=1}^{J} \sum_{k \in I(l,i)} w_{l,k,m}\, S_m(x^*_{l,k} - y_i)$$

+ near field interaction

Here $I(l,i)$ is the index of interaction cells of y_i at level l:

$$I(l, i) = \left\{ k = 1 \cdots 2^l : T_{l,k} \text{ is far field cell of } y_i \text{ and parent cell of } T_{l,k} \text{ at level } l-1 \text{ is not far field cell of } y_i \right\}$$

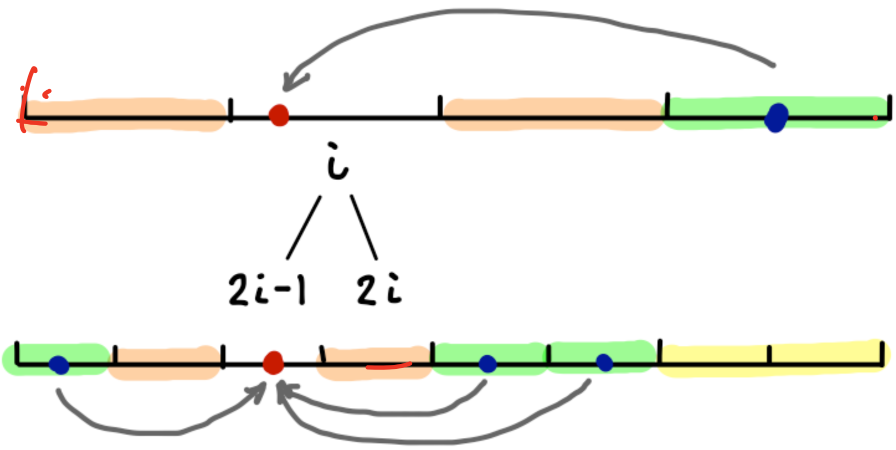


FIGURE 5. Interaction list in a coarse and fine grid. The orange one is the near field cells and the green cells are in interaction list. The yellow cells are far field cells whose parent is also the far-field of the target in the coarse grid.

"Near field interaction" is computed directly by

$$\sum_{k \in M(J,i)} \sum_{x_j \in T_{J,k}} G(x_j - y_i)\, q_j$$

$M(J,i)$ is the near field cells of y_i in level J.

- optimal complexity

Take $T_{\ell-1,1}$ and its two children cells $T_{\ell,1}$, $T_{\ell,2}$. Let $h_\ell = 2^{-\ell}$ be the length of cells in level ℓ.

For $x_j \in T_{\ell,1}$

$$(x_j - x^*_{\ell-1,1})^m = (x_j - x^*_{\ell,1} + x^*_{\ell,1} - x^*_{\ell-1,1})^m$$

$$= \sum_{s=0}^{m} C_m^s (x^*_{\ell,1} - x^*_{\ell-1,1})^{m-s} (x_j - x^*_{\ell,1})^s$$

$$= \sum_{s=0}^{m} C_m^s \left(-\frac{h_\ell}{2}\right)^{m-s} (x_j - x^*_{\ell,1})^s$$

Using this expansion

$$w_{\ell-1,1,m} = \sum_{x_j \in T_{\ell-1,1}} q_j (x_j - x^*_{\ell-1,1})^m$$

$$= \sum_{j \in T_{\ell,1}} q_j (x_j - x^*_{\ell-1,1})^m + \sum_{j \in T_{\ell,2}} q_j (x_j - x^*_{\ell-1,1})^m$$

$$= \sum_{s=0}^{m} C_m^s \left(\frac{h_\ell}{2}\right)^{m-s} \left[(-1)^{m-s} w_{\ell,1,s} + w_{\ell,2,s}\right]$$

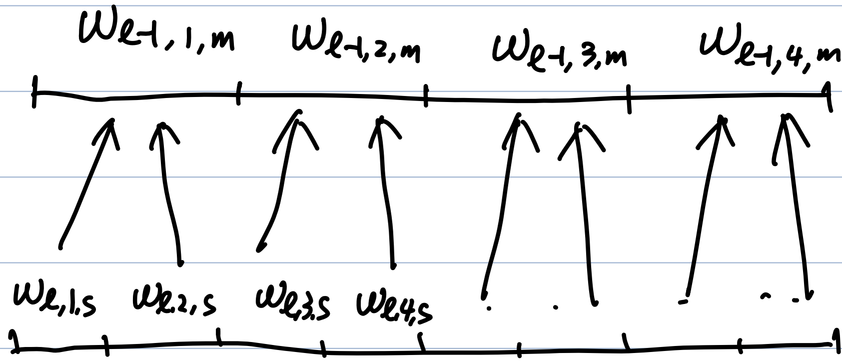


Fig. $O(N)$ bottom to top algorithm to compute $w_{\ell,k,m}$

In this algorithm, N point source x_j only visited at the finest level and in each level the computational cost is $O(2^\ell)$. Then computational cost of all $w_{\ell,k,m}$ is

$$O\left(\sum_{\ell=1}^{I} 2^\ell\right) = O(N)$$

Recall that

$$v_i \simeq \sum_{m=0}^{P} \sum_{l=1}^{J} \sum_{k \in I(l,i)} w_{l,k,m} S_m(x^*_{l,k} - y_i)$$

Define $s(l,i)$ be the index of cell such that $y_i \in T_{l,s(l,i)}$

Then for any $k \in I(l,i)$

$$|y_i - y^*_{l,s(l,i)}| \leq 2^{-(l+1)}$$

$$|y^*_{l,s(l,i)} - x^*_{l,k}| \geq 2^{-(l-1)}$$

which imply

$$\frac{|y_i - y^*_{l,s(l,i)}|}{|y^*_{l,s(l,i)} - x^*_{l,k}|} \leq \delta \leq \frac{1}{4}$$

Define $\psi_J(y_i)$

$$= \sum_{m=0}^{p} \sum_{l=1}^{J} \sum_{k \in I(l,i)} w_{l,k,m} S_m(x_{l,k}^* - y_i)$$

Then $\psi_J(y_i;\ y_{J,1}^*, \cdots y_{J,2^J}^*)$

$$= \sum_{m=0}^{p} \sum_{l=1}^{J} \sum_{k \in I(l,i)} w_{l,k,m} S_m(x_{l,k}^* - y_i)$$

$$= \sum_{m=0}^{p} \sum_{l=1}^{J} \sum_{k \in I(l,i)} w_{l,k,m} S_m(x_{l,k}^* - y_{l,s(l,i)}^* + y_{l,s(l,i)}^* - y_i)$$

Notice that

$$S_m(x_{l,k}^* - y_{l,s(l,i)}^* + y_{l,s(l,i)}^* - y_i)$$

$$= S_m(x_{l,k}^* - y_{l,s(l,i)}^*)\, S_m\left(1 - \frac{y - y_{l,s(l,i)}^*}{x_{l,k}^* - y_{l,s(l,i)}^*}\right)$$

$$= \sum_{n=0}^{p} \frac{S_m^{(n)}(1)}{n!} \frac{S_m(x_{l,k}^* - y_{l,s(l,i)}^*)}{(x_{l,k}^* - y_{l,s(l,i)}^*)^n} (y_i - y_{l,s(l,i)}^*)^n + O(\delta^{p+1})$$

Then

$$\psi_J(y_i)$$

$$= \sum_{n=0}^{p} \sum_{\ell=1}^{J} b_{\ell, s(\ell,i), n} \left(y_i - y^*_{\ell, s(\ell,i)}\right)^n$$

$$+ O(\delta^{p+1})$$

where

$$b_{\ell,s,n}$$

$$= \sum_{m=0}^{p} \sum_{k \in I(\ell,s)} w_{\ell,k,m} \frac{S_m^{(n)}(1)}{n!} \frac{S_m(x^*_{\ell,k} - y^*_{\ell,s})}{(x^*_{\ell,k} - y^*_{\ell,s})^n}$$

Further, notice that

$$\left(y_i - y^*_{\ell, s(\ell,i)}\right)^n \quad (3)$$

$$= \left(y_i - y^*_{\ell+1, s(\ell+1,i)} + y^*_{\ell+1, s(\ell+1,i)} - y^*_{\ell, s(\ell,i)}\right)^n$$

$$= \sum_{t=0}^{n} C_n^t \left(\pm \frac{h_e}{2}\right)^{m-t} \left(y_i - y^*_{\ell+1, S(\ell+1, i)}\right)^t$$

The sigh '$\pm$' is the sign of

$$\left(y^*_{\ell+1, s(\ell+1, i)} - y^*_{\ell, s(\ell, i)}\right)$$

Using this expansion, by deduction we can show for any J, there exist $\xi_{J,s,n}$ such that

$$\psi_J(y_i) = \sum_{n=0}^{p} \xi_{J, s(J,i), n} \left(y_i - y^*_{J, s(J,i)}\right)^n \quad (4)$$

when $J = 1$,

$$\psi_1(y_i) = \sum_{n=0}^{p} b_{1, s(1,i), n} \left(y_i - y^*_{1, s(1,i)}\right)^n$$

Choose $\xi_{1,s,n} = b_{1,s,n}$

Assume it is true for $J \le J'$

for $J = J'+1$, by defination of ψ_J

$$\psi_{J'+1}(y_i) = \psi_{J'}(y_i) + \sum_{n=0}^{p} b_{J'+1,\, s(J'+1,i),\, n}\left(y_i - y^*_{J'+1,\, s(J'+1,i)}\right)^n$$

$$= \sum_{n=0}^{p} \xi_{J',\, s(J',i),\, n}\left(y_i - y^*_{J',\, s(J',i)}\right)^n + \sum_{n=0}^{p} b_{J'+1,\, s(J'+1,i),\, n}\left(y_i - y^*_{J'+1,\, s(J'+1,i)}\right)^n$$

$$= \sum_{n=0}^{p} \xi_{J',\, s(J',i),\, n} \sum_{t=0}^{n} C_n^t \left(\pm\frac{h_{J'}}{2}\right)^{n-t} \left(y_i - y^*_{J'+1,\, s(J'+1,i)}\right)^t + \sum_{n=0}^{p} b_{J'+1,\, s(J'+1,i),\, n}\left(y_i - y^*_{J'+1,\, s(J'+1,i)}\right)^n$$

$$= \sum_{n=0}^{p} \left[\left(\sum_{t=n}^{p} C_t^n \left(\pm\frac{h_{J'}}{2}\right)^{t-n} \xi_{J',\, s(J',i),\, t}\right) + b_{J'+1,\, s(J'+1,i),\, n}\right] \left(y_i - y_{J'+1,\, s(J'+1,i)}\right)^n$$

Then (4) is proved and

$$\xi_{J+1,s,n} = \left(\sum_{t=n}^{p} C_t^n \left(\pm \frac{h_J}{2}\right)^{t-n} \xi_{J,\bar{s}(J,s),t} \right) + b_{J+1,s,n}$$

where $\bar{s}(J,s)$ is the parent cell of $T_{J+1,s}$ in level J and the sign '$\pm$' is $-$ $(+)$ if $T_{J+1,s}$ is left (right) child cell of $T_{J,\bar{s}(J,s)}$

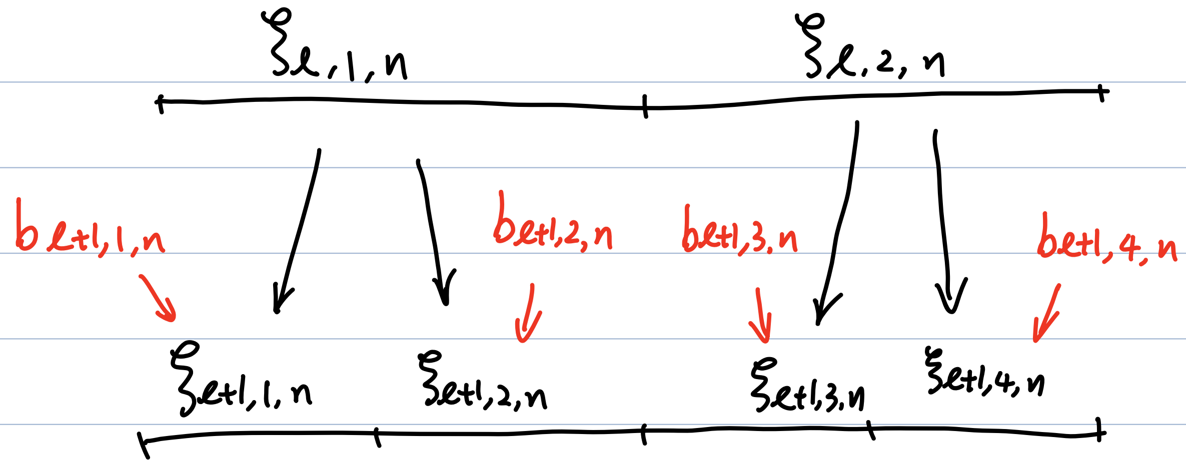


Fig. $O(N)$ top to bottom algorithm to compute $\xi_{\ell,s,n}$

Finally.

$$V(y_i) \approx \sum_{n=0}^{p} \xi_{J,\,s(J,i),\,n} \left(y_i - y^*_{J,\,s(J,i)} \right)^n + \sum_{k \in M(J,i)} \sum_{x_j \in T_{J,k}} G(x_j - y_i)\, \xi_j$$

with $J = \log(N)$ and $M(J,i)$ are near field cells of y_i in level J