2.5 Magnetic Fields, Magnetic Fields Due to Currents

Magnetism is a fundamental force of nature that is observed in certain materials, notably iron, cobalt, and nickel, as well as some alloys and compounds. Magnets produce a magnetic field, which is a region of space in which magnetic forces are exerted on other magnets, magnetic materials, or charged particles.

Magnetism plays a critical role in both our natural world and our daily lives. It is responsible for the Earth's magnetic field, which protects our planet from the harmful effects of solar wind and cosmic rays. In addition, magnetism has a profound impact on our technology, from the small magnets used in hard drives and credit card strips to the giant magnets used in medical imaging machines and particle accelerators. Understanding magnetism is therefore essential for both advancing our understanding of the natural world and developing new technologies that improve our lives.

2.5.1 Magnetic fields

An important concept in magnetism is the source of magnetic field. While electric charges are the source of electric fields, the existence of magnetic charges, or magnetic monopoles, has not been experimentally observed. Instead, a magnet with both north and south poles can produce a magnetic field. When two magnets are brought close to each other, the north pole of one magnet will attract the south pole of the other magnet, while the north pole of both magnets will repel each other. However, it is not possible to isolate a single north pole or south pole of a magnet, as attempting to do so will split the magnet into two separate magnets with both poles.

It is crucial to understand that a magnetic field can also be generated by the motion of electric charges, as observed in electric currents. This principle gave rise to the development of electromagnets, which will be covered in detail in subsequent sections. When an electric current flows through a wire coil, a magnetic field is produced in the surrounding space. As moving electric charges have a time-varying electric field, we can conclude that the magnetic field arises due to the time-varying electric field.

Definition of magnetic field

Electric fields are defined as the Coulomb force acting on a test electric charge divided by the value of its electric charge $\vec{E} = \vec{F}_E/q$. The concept of electric fields can be extended to magnetic fields by defining them using the magnetic force. Magnetic fields do not affect static electric charges, but they do exert a force on moving charges known as the Lorentz force. This force is perpendicular to the direction of the velocity of the charged particle and the magnetic field it is moving through. The magnetic force can be described using the cross product of the charge's velocity and the magnetic field, and can be written as

$$\vec{F}_B = q\vec{v} \times \vec{B}, \tag{2.5.1}$$

where q is the electric charge of the charged particle and $\vec{v}$ is its velocity. This allows us to define the magnetic field as a vector that satisfies this equation. Therefore, the magnetic field can be thought of as a quantity that measures the force exerted on a moving charge in a magnetic field.

By definition of the cross product, the magnitude of the Lorentz force is given by

$$|\vec{F}_B| = q|\vec{v}||\vec{B}|\sin\theta, \tag{2.5.2}$$

where θ is the angle between the velocity and the magnetic field. As the magnetic force is always perpendicular to the velocity of the particle, it does not do any work on the particle. It only changes the direction of the velocity and does not alter the magnitude of the velocity. This means that a magnetic field can only change the trajectory of a charged particle and not its speed.

The unit of the magnetic field is the Tesla (T), which is defined as the magnetic field that exerts a force of one Newton on a charge of one Coulomb moving perpendicularly to the magnetic field with a velocity of one meter per second: $1\text{T} = 1\frac{\text{N}\cdot\text{s}}{\text{C}\cdot\text{m}}$. One Tesla is equal to 10^4 Gauss (G), which is commonly used as another unit of magnetic field strength. For example, the Earth's magnetic field has a strength of about 0.5 Gauss. The choice of unit depends on the context and magnitude of the magnetic field being measured. In everyday applications, milliTesla (mT) is often used, while in scientific research and high-tech industries, the Tesla is the standard unit.

Magnetic field line

Magnetic field lines are a useful tool for visualizing the direction and strength of a magnetic field. The field lines are drawn in such a way that the tangent to the line at any point gives the direction of the magnetic field at that point. The strength of the field is proportional to the density of the field lines. Magnetic field lines always form closed loops, which means that they originate from the north pole of a magnet and terminate at the south pole. In addition, magnetic field lines never intersect, which means that they cannot cross each other. The closer the field lines are, the stronger the magnetic field is at that point.

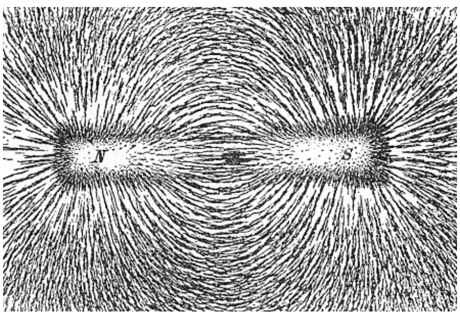

Figure 2.16

Michael Faraday, a renowned English physicist and chemist, revolutionized our understanding of magnetic fields in the mid-19th century by introducing the concept of magnetic field lines. Through a series of groundbreaking experiments on electromagnetism, he discovered that magnetic fields possess direction and shape, leading him to propose the idea of visualizing magnetic fields with lines. Faraday's seminal work formed the basis of modern electromagnetic theory and transformed the study of magnetic fields from a vague notion to a concrete and measurable phenomenon. Today, fields are a fundamental concept in quantum field theory and are considered as real physical degrees of freedom rather than merely conceptual lines.

The absence of magnetic monopoles is a fundamental property of magnetic fields, which is visually represented through closed magnetic field line loops that do not have any endpoints or sources/sinks. Unlike electric fields, where positive and negative charges act as sources and sinks of electric field lines, the magnetic field is divergenceless, meaning that its divergence is always zero. Using vector calculus, this is expressed as $\nabla \cdot \vec{B} = 0$, which is different from the electric field with $\nabla \cdot \vec{E} = \rho/\epsilon_0$.

2.5.2 Magnetic forces

When a charged particle is in motion and exposed to both electric and magnetic fields, the net force acting on it is given by the equation

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}. \tag{2.5.3}$$

This is the fundamental formula for forces of electric and magnetic fields acting on a charged particle.

The Hall effect

We can consider the field configuration in which the electric and magnetic fields are perpendicular to each other, and the charged particle is initially moving with a velocity perpendicular to both fields. In this scenario, there is a possibility that the total force acting on the particle is zero, and the particle will continue moving with a constant velocity (see Figure 2.17). The condition for this equilibrium is given by

$$q\vec{E} + q\vec{v} \times \vec{B} = 0, \tag{2.5.4}$$

from which we find the speed of the particle should be

$$v = \frac{E}{B}, \tag{2.5.5}$$

where E and B are the magnitudes of the electric and magnetic fields, respectively.

The Hall effect is a phenomenon that occurs when two conductor plates are exposed to a magnetic field, as shown in Figure 2.17. Electrons move between the plates with an initial velocity,

v_d, in the absence of an electric field. As time goes on, one plate accumulates more electrons than the other, as illustrated in Figure 2.17(a), resulting in a potential difference between the two plates and an induced electric field. By using the equilibrium condition derived earlier, we can determine the electric potential difference between the two plates as

$$V = Ed = v_d Bd, \tag{2.5.6}$$

where E is the electric field induced by the accumulated electrons, v_d is the drift speed of the electrons, and d is the distance between the two plates. Meanwhile, the current is given by $i = JA = nev_d A$, where n is the electron density and A is the cross-sectional area of the electron current. Therefore, the magnetic field can be expressed as

$$B = \frac{V}{v_d d} = \frac{neVA}{id}. \tag{2.5.7}$$

If all the physical quantities on the right-hand side are known, we can use this formula of the Hall effect to measure the magnetic field.

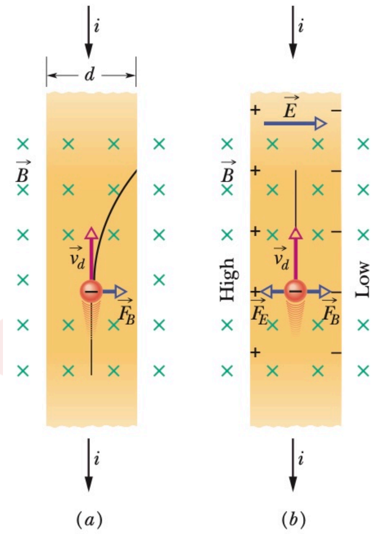

Figure 2.17: (a) A moving electron in magnetic field. (b) A moving electron in electric and magnetic fields which are perpendicular to each other.

Circulating charged particle in uniform magnetic field

When a charged particle moves in a uniform magnetic field, it experiences a force perpendicular to both the direction of motion and the direction of the magnetic field. This force causes the particle to move in a circular or helical path.

First, let's consider a charged particle moving in a uniform magnetic field $\vec{B}$ with an initial velocity $\vec{v}$ perpendicular to the magnetic field. In this case, the magnetic force $\vec{F}_B = q\vec{v} \times \vec{B}$ acting on the particle is perpendicular to both $\vec{v}$ and $\vec{B}$, so the particle moves in a circular path with a radius r governed by the centripetal force:

$$F_B = |q|vB = \frac{mv^2}{r}, \tag{2.5.8}$$

where m is the mass of the particle. The radius of the circle is given by:

$$r = \frac{mv}{|q|B}, \tag{2.5.9}$$

This equation tells us that the radius of the circle depends on the speed of the particle and the strength of the magnetic field. The period of the motion, which is the time it takes for the particle to complete one full circle, is given by:

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{|q|B}. \tag{2.5.10}$$

This equation tells us that the period of the motion is independent of the velocity of the particle and depends only on its mass, charge, and the strength of the magnetic field.

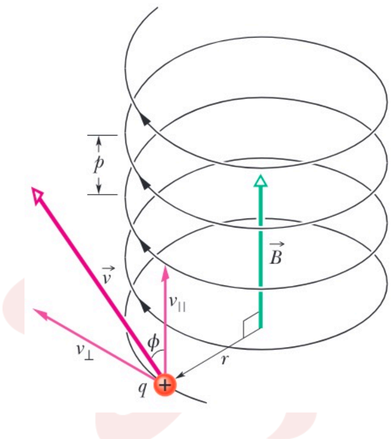

Figure 2.18

In the more general case, when the initial velocity of a charged particle is not perpendicular to a uniform magnetic field, the particle moves in a helical path around the direction of the field vector. To understand this motion, we can resolve the velocity vector $\vec{v}$ into two components, one parallel and one perpendicular to the magnetic field $\vec{B}$. The parallel component $v_{\|} = v\cos\phi$ determines the pitch p of the helix, which is the distance between adjacent turns, while the perpendicular component $v_{\perp} = v\sin\phi$ determines the radius of the helix. The radius of the circular path can be calculated using the perpendicular velocity component and Eq. (2.5.9):

$$r = \frac{mv_{\perp}}{|q|B}. \tag{2.5.11}$$

The pitch of the helix can be calculated using the parallel velocity component and the period T of the motion:

$$p = v_{\parallel} T = \frac{2\pi m v_{\parallel}}{|q|B}. \tag{2.5.12}$$

It is worth noting that the pitch of the helix is independent of the perpendicular velocity $v_{\perp}$, since the period T only depends on the mass, charge, and strength of the magnetic field.

In general, a charged particle moving in a magnetic field experiences a force that causes it to move in a spiral path along the direction of the magnetic field lines. This phenomenon is crucial for understanding the protective nature of the Earth's magnetic field. The Earth's magnetic field acts as an invisible shield that deflects charged particles, especially those present in the solar wind. The solar wind primarily consists of electrons and protons that approach the Earth. When these charged particles encounter the Earth's magnetic field lines, they undergo a magnetic force that alters their trajectory. Instead of moving straight towards the Earth's surface, they spiral along the field lines. This spiraling motion redirects the charged particles towards the Earth's north or south poles.

Magnetic force acting on a wire

Let us now consider a straight wire carrying a current i placed in a uniform magnetic field $\vec{B}$. We want to understand the magnetic force acting on this wire with length L.

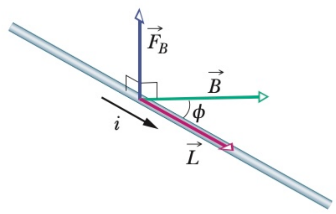

Figure 2.19

Suppose the wire has electric current i, which is induced by charged particles with charge q and drift velocity $\vec{v}_d$. Then the total moving charge in the wire of length L should be $q = it = iL/v_d$. The magnetic force acting on this wire is given by

$$\vec{F} = q\vec{v}_d \times \vec{B} = \frac{iL}{v_d}\vec{v}_d \times \vec{B} = i\vec{L} \times \vec{B}. \tag{2.5.13}$$

Here, $\vec{L}$ is the vector of length L and pointing along the direction of the current or drift velocity. From this expression, we see that the magnetic force acting on the current-carrying wire is perpendicular to both the direction of the current and the direction of the magnetic field.

Torque on a current loop

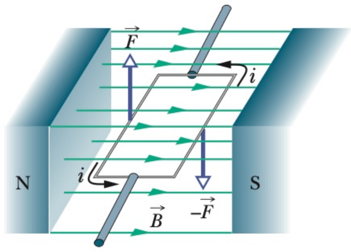

Figure 2.20

The figure above depicts a simple motor that consists of a single current-carrying loop placed in a uniform magnetic field $\vec{B}$. The loop experiences two magnetic forces, denoted by $\vec{F}$ and $-\vec{F}$, which act in opposite directions along the loop. As a result, the loop experiences a torque that tends to rotate it about its central axis.

Since two of the edges are always perpendicular to the direction of the magnetic field, the force acting on them has a magnitude

$$F = ibB, \tag{2.5.14}$$

where b is the length of the edge. The torque of these forces with respect to the central axis is then

$$\tau = Fa\sin\theta = iBab\sin\theta \tag{2.5.15}$$

where a is the length of the other edge of the current loop, and θ is the angle between the normal direction $\hat{n}$ of the loop plane and the direction of the magnetic field.

Suppose we replace the single loop of current with a coil of N tightly wound loops. The torque on the coil due to the magnetic field is given by

$$\tau = N\tau_{single\text{-}loop} = NiBA\sin\theta, \tag{2.5.16}$$

where $A = ab$ is the area enclosed by each loop. We see that the torque on the coil is proportional to the number of turns N, the current i, and the enclosed area A. This demonstrates that increasing any of these parameters will result in an increase in the torque of the simple motor.

Magnetic dipole moment

The magnetic dipole moment is a fundamental concept in magnetism and plays a crucial role in understanding the behavior of current-carrying coils in magnetic fields. Similar to an electric dipole consisting of positive and negative charges separated by a distance, a magnetic dipole arises from a current loop or coil. The magnitude of the magnetic dipole moment of a coil is given by the expression

$$|\vec{\mu}| = NiA, \tag{2.5.17}$$

Here, N represents the number of turns in the coil, i is the current flowing through the coil, and A is the area enclosed by each turn. The magnetic dipole moment is a measure of the strength of the dipole and is typically expressed in units of ampere-square meters ($\mathrm{A} \cdot \mathrm{m}^2$). The direction of the magnetic dipole moment vector, denoted as $\vec{\mu}$, is determined by the normal vector $\hat{n}$ to the plane of the coil. The right-hand rule can be used to determine the direction of $\vec{\mu}$ by aligning the fingers of your right hand with the current flow in the coil, and the thumb points in the direction of $\vec{\mu}$.

With the magnetic dipole moment defined, we can now apply it to understand the torque experienced by a current-carrying coil in a magnetic field. The torque on the coil is given by the cross product of the magnetic dipole moment vector and the magnetic field vector, as shown in Eq. (2.5.16):

$$\vec{\tau} = \vec{\mu} \times \vec{B}. \tag{2.5.18}$$

This expression reveals that the coil experiences a torque perpendicular to both the magnetic dipole moment vector and the magnetic field vector. This is similar to the torque experienced by an electric dipole in an electric field, given by:

$$\vec{\tau} = \vec{p} \times \vec{E}, \tag{2.5.19}$$

where $\vec{p} = q(\vec{r}_+ - \vec{r}_-)$ is the electric dipole moment.

A magnetic dipole placed in an external magnetic field possesses energy that is influenced by the orientation of the dipole with respect to the field. Drawing an analogy to electric dipoles, where the energy is given by

$$U = -\vec{p} \cdot \vec{E}, \tag{2.5.20}$$

we can write a similar expression for magnetic dipoles as:

$$U = -\vec{\mu} \cdot \vec{B}. \tag{2.5.21}$$

This relationship emphasizes the similarity in the energy calculations between electric and magnetic dipoles, where the energy depends on the alignment of the dipole moment with the corresponding field.

Not only coils possess magnetic dipole moments, but fundamental particles such as the electron also exhibit their intrinsic magnetic dipole moments. This phenomenon arises from the intrinsic spin of the electron, which can be envisioned as a minute charged particle spinning on its axis. As a result of this spin, the electron generates a magnetic moment akin to that of a miniature bar magnet, complete with north and south poles. The direction of the electron's magnetic dipole moment aligns with its spin angular momentum vector. When subjected to an external magnetic field, the electron's magnetic dipole moment interacts with the field, leading to a torque and energy associated with the interaction. This interaction can give rise to various magnetic effects, including precession and alignment of the electron's spin with the applied field. Accurate determination of the electron's magnetic dipole moment has been the focus of extensive experimental and theoretical investigations, playing a significant role in advancing our comprehension of quantum field theories and the properties of matter at the microscopic level.

2.5.3 Magnetic fields from currents

A magnetic field is not only capable of exerting a force on moving charged particles but also of being produced by the motion of charged particles or currents. This reciprocal relationship between magnetic fields and moving charges is a fundamental aspect of electromagnetism.

Biot-Savart law

The Biot-Savart law summarizes the experimental findings regarding the magnetic field produced at a point P, located at a distance $\vec{r}$ from a current-carrying length element $\mathrm{d}\vec{s}$. This law states that the magnetic field $\mathrm{d}\vec{B}$ is equal to $\frac{\mu_0}{4\pi}$ multiplied by the product of the current i, $\mathrm{d}\vec{s}$, $\sin\theta$, and inversely proportional to the square of the distance $|\vec{r}|$. Here, θ represents the angle between the directions of $\mathrm{d}\vec{s}$ and the unit vector $\hat{r}$, which points from $\mathrm{d}\vec{s}$ towards P. The constant μ_0, known as the permeability constant, has a value of $\mu_0 \approx 4\pi \times 10^{-7}\mathrm{T}\cdot\mathrm{m/A} \approx 1.26 \times 10^{-6}\mathrm{T}\cdot\mathrm{m/A}$.

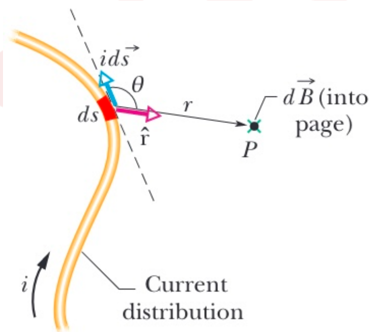

Figure 2.21

The direction of the magnetic field $\mathrm{d}\vec{B}$, as illustrated in the above figure, is perpendicular to both $\mathrm{d}\vec{s}$ and $\hat{r}$ and can be determined using the cross product $\mathrm{d}\vec{s} \times \hat{r}$. Consequently, the vector form of the Biot-Savart law is expressed as

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{i\mathrm{d}\vec{s} \times \hat{r}}{|\vec{r}|^2} \tag{2.5.22}$$

The magnitude of a magnetic field is proportional to the current and inversely proportional to the square of the distance from the source. This is similar to the electric field generated by a charged particle, which is proportional to the electric charge and inversely proportional to the square of the distance.

We can now use the law of Biot and Savart to derive that the magnetic field at a perpendicular distance R from a long (infinite) straight wire carrying a current i. To find the total magnetic field due to the entire infinite wire, we integrate the contributions from all infinitesimal current elements along the wire

$$\begin{aligned}\vec{B} = \int \mathrm{d}\vec{B} &= \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi} \frac{i \sin\theta \mathrm{d}s}{r^2}(\hat{\imath} \times \hat{r}_0) = \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi} \frac{i(R/\sqrt{s^2+R^2})\mathrm{d}s}{s^2+R^2}(\hat{\imath} \times \hat{r}_0) \\ &= \frac{\mu_0 i R}{4\pi} \frac{s}{R^2\sqrt{s^2+R^2}}\bigg|_{s=-\infty}^{\infty}(\hat{\imath} \times \hat{r}) = \frac{\mu_0 i}{2\pi R}(\hat{\imath} \times \hat{r}_0),\end{aligned} \tag{2.5.23}$$

Here, $\hat{\imath}$ is the unit vector along the direction of the current, $\hat{r}_0$ is the unit vector pointing from the wire to the point, θ is the angle between $\hat{\imath}$ (or $\mathrm{d}\vec{s}$) and $\vec{r}$. The magnetic field lines produced by the central wire form circular patterns around the wire, and their magnitude is inversely proportional to the distance R. This can be observed in the figure below, which shows iron filings aligned with the magnetic field lines.

Figure 2.22

As an additional example, let's examine the magnetic field generated by a current in a circular arc of wire. Consider an arc-shaped wire with a central angle ϕ, radius R, and center C, carrying a current i. The magnetic field at the center C due to any differential segment is directed along the same direction. Therefore, we can simplify the problem by considering the magnitude of the magnetic field using the scalar integral form of the Biot-Savart law:

$$B = \int \mathrm{d}B = \int_0^{\phi} \frac{\mu_0}{4\pi} \frac{iR\mathrm{d}\phi'}{R^2} = \frac{\mu_0 i \phi}{4\pi R}. \tag{2.5.24}$$

If the arc becomes a complete circle, the magnetic field at its center can be simplified further. In this case, the central angle ϕ is replaced by 2π. The expression for the magnetic field at the center of a circular wire becomes

$$B = \frac{\mu_0 i}{R}. \tag{2.5.25}$$

And its direction is perpendicular to the circle plane with right-hand rule.

Force between two parallel currents

Since moving charged particles can generate a magnetic field, it follows that magnetic fields can exert forces on moving charged particles. This leads us to the intriguing question of what happens when two currents are present. Consider the simplest scenario where two currents flow through infinitely long wires that are parallel to each other. We want to determine the force between these parallel currents.

To calculate the force between two parallel currents, we can apply Biot-Savart law to determine the magnetic field created by one current and then use the Lorentz force law to find the force experienced by the other current. For two parallel long wires carrying currents i_1 and i_2, separated by a distance d, the magnetic field created by the first wire at the location of the second wire is given by

$$B = \frac{\mu_0 i_1}{2\pi d}. \tag{2.5.26}$$

The force experienced by the second wire can be obtained by multiplying the magnetic field B with the length of the wire L_2 and the current i_2:

$$F = i_2 L_2 B = \frac{\mu_0 i_1 i_2 L_2}{2\pi d}. \tag{2.5.27}$$

A simple analysis of the direction of the force reveals that parallel currents attract each other, while antiparallel currents repel each other.

The force between two parallel currents is not only a fundamental concept in electromagnetism, but it also played a crucial role in defining the unit of electric current, the ampere. Prior to 2019, the ampere was defined as the constant current that, when maintained in two parallel conductors of infinite length, placed one meter apart in a vacuum, would exert a force of 2×10^{-7} newtons per meter of length between them. This basically fix the value of μ_0 to be $4\pi \times 10^{-7} \mathrm{T} \cdot \mathrm{m/A}$.

However, with the redefinition of the SI base units in 2019, the ampere is now defined in terms of the elementary charge e. The elementary charge has a fixed numerical value of $1.602176634 \times 10^{-19}$ coulombs (C), where 1 ampere (A) is equal to 1 coulomb per second (C/s). The second, in turn, is defined in terms of the unperturbed ground state hyperfine transition frequency of the caesium-133 atom.

2.5.4 Ampere's law

When determining the net electric field resulting from a charge distribution, we consider the sum of the differential electric fields contributed by each element. In cases where the distribution exhibits planar, cylindrical, or spherical symmetry, Gauss' law provides a more efficient method to calculate the total electric field.

Similarly, when calculating the net magnetic field caused by a distribution of currents, we sum the contributions of the differential magnetic fields using the Biot-Savart law. If the distribution possesses symmetry, Ampere's law can be utilized for a simpler analysis. Ampere's law, derived from the Biot-Savart law, is commonly associated with Ampere, but it was actually derived and generalized by James Clerk Maxwell. Ampere's law can be expressed as:

$$\oint \vec{B} \cdot \mathrm{d}\vec{s} = \mu_0 i_{\mathrm{enc}}. \tag{2.5.28}$$

In this equation, the integral is taken around a closed loop, known as an Amperian loop, and the dot product represents the magnetic field along the loop integrated over the differential path length. The quantity i_{enc} denotes the net current enclosed by the loop.

Symmetry transformation of magnetic field

To explore the symmetry transformation of the magnetic field, let's first examine the cross product under O(3) rotation. It can be easily shown that

$$(R\vec{A}) \times (R\vec{B}) = \det(R) \cdot R(\vec{A} \times \vec{B}) \tag{2.5.29}$$

for $R \in \mathrm{O}(3)$. In fact, the cross product is unique in that it is orthogonal to both factors, has a length equal to the area of the parallelogram they form, and forms a right-handed triple with them. These properties remain invariant under rotations in SO(3). Therefore, the cross product is invariant under SO(3) transformation. However, if $R \in \mathrm{O}(3)$ has the property $\det R = -1$, the right-hand rule will transform into a left-hand rule, resulting in a minus sign in the previous equation.

Now, let's delve into understanding the transformation rule for the magnetic field due to a current density that is invariant under an $R \in \mathrm{O}(3)$ transformation. Starting from the Biot-Savart law, we have:

$$\vec{B}(\vec{r}) = \int \mathrm{d}\vec{B} = \int \frac{\mu_0}{4\pi} \frac{i(\vec{s})\mathrm{d}\vec{s} \times (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^3}. \tag{2.5.30}$$

The magnetic field at position $\vec{r'} = R\vec{r}$ is then given by:

$$\begin{aligned}
\vec{B}(\vec{r'}) = \vec{B}(R\vec{r}) &= \int \frac{\mu_0}{4\pi} \frac{i(\vec{s})\mathrm{d}\vec{s} \times (R\vec{r} - \vec{s})}{|R\vec{r} - \vec{s}|^3} = \int \frac{\mu_0}{4\pi} \frac{i(R^{-1}\vec{s})\mathrm{d}(R^{-1}\vec{s}) \times R(\vec{r} - R^{-1}\vec{s})}{|\vec{r} - R^{-1}\vec{s}|^3} \\
&= \int \frac{\mu_0}{4\pi} \frac{i(\vec{s})\mathrm{d}\vec{s} \times R(\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^3} = \int \frac{\mu_0}{4\pi} \frac{R[i(\vec{x})\mathrm{d}\vec{s}] \times R(\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^3} \\
&= \det(R) \cdot R \int \frac{\mu_0}{4\pi} \frac{i(\vec{s})\mathrm{d}\vec{s} \times (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^3} = \det(R) \cdot R\vec{B}(\vec{r}).
\end{aligned} \tag{2.5.31}$$

Here, we used the symmetry property of the current, $R(i\mathrm{d}\vec{s}) = i\mathrm{d}\vec{s}$. From this transformation rule, we observe that the magnetic field is transformed in the same way under rotation if $R \in \mathrm{SO}(3)$ and has an additional minus sign for $R \in \mathrm{O}(3)$ with $\det(R) = -1$.

A physical quantity is classified as a vector if it follows the transformation rule

$$\vec{v}' = R\vec{v}, \tag{2.5.32}$$

under a rotation $R \in \mathrm{O}(3)$. On the other hand, a quantity is considered a pseudovector (or axial vector) if its transformation rule is

$$\vec{v}' = \det(R) \cdot R\vec{v}. \tag{2.5.33}$$

Performing usual operations on vectors or pseudovectors does not change their nature. For example, the position vector $\vec{r}$ is a vector, and its derivative $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ (velocity) is also a vector. Similarly, the acceleration is a vector. As demonstrated earlier, the electric field is a vector. However, the cross product of two vectors results in a pseudovector. This explains why the magnetic field is a pseudovector, as it is fundamentally related to $\mathrm{d}\vec{s} \times \vec{r}$, where $\mathrm{d}\vec{s}$ can be viewed as the direction of the drift velocity of charge carriers in a wire, which is a vector. Similarly, angular momentum $\vec{L} = \vec{r} \times \vec{p}$ and torque $\tau = \vec{r} \times \vec{F}$ are pseudovectors.

Combining Ampere's law with symmetry

When there are symmetries in the current density or distribution, Ampere's law and symmetry can be utilized to simplify the derivation of the magnetic field generated by the current.

For example, let's consider a long, straight wire carrying a steady current i. Applying Ampere's law to a circular loop of radius r centered on the wire, we utilize the rotational and translational symmetries. The magnetic field $\vec{B}$ has a constant magnitude along the loop, with direction determined by the right-hand rule. Ampere's law yields

$$\oint \vec{B} \cdot \mathrm{d}s = B \cdot 2\pi r = \mu_0 i, \tag{2.5.34}$$

allowing us to solve for the magnetic field of an infinite wire

$$B = \frac{\mu_0 i}{2\pi r}. \tag{2.5.35}$$

This demonstrates the inverse relationship between the magnetic field and distance r, while emphasizing the utility of Ampere's law in analyzing infinitely long wires.

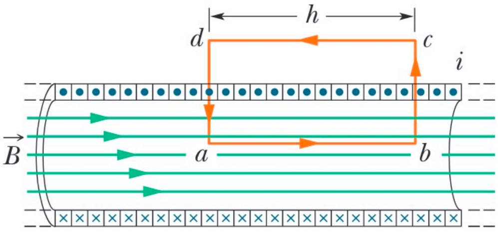

Figure 2.23

Similarly, for a long ideal solenoid carrying a current i, we can choose the Amperian loop to be the loop in Fig. 2.23. Then the Ampere's law gives us

$$\oint \vec{B} \cdot \mathrm{d}s = Bh = \mu_0 inh, \tag{2.5.36}$$

where n is the number of turns per unit length of the solenoid. The magnetic field inside the solenoid is uniform and has magnitude

$$B = \mu_0 ih. \tag{2.5.37}$$

2.6 Faraday's Law, Induction and Inductance

2.6.1 Faraday's law of induction

In the last section, we showed that a current produces a magnetic field. A more surprising physical discovery of Faraday is the reverse effect: *a changing magnetic field can induce an electric field that drives a current.* This link between a magnetic field and the electric field it induces is now called *Faraday's law of induction.*

Faraday originally stated his law of induction as "an emf is induced in a loop when the number of magnetic field lines that pass through the loop is changing". The number of magnetic field lines passing through the loop is quantitatively described by the magnetic flux. Suppose a loop enclosing an area A is placed in a magnetic field $\vec{B}$. Then, the magnetic flux through the loop is defined as

$$\Phi_B := \iint_A \vec{B} \cdot \mathrm{d}\vec{A}.$$

Since the magnetic field $\vec{B}$ is divergence-free, i.e., $\vec{\nabla} \cdot \vec{B} = 0$, by Corollary 2.2.2, its flux through a surface S is invariant under local continuous deformation of S. As a consequence, the flux Φ_B is well-defined and does not depend on the particular surface enclosed by the loop that we have chosen. The SI unit for magnetic flux is called the weber (Wb): $1\ \mathrm{Wb} = 1\ \mathrm{T} \cdot \mathrm{m}^2$.

Physics law 12 (Faraday's law of induction)**.** *The magnitude of the emf $\mathscr{E}$ induced in a conducting loop is equal to the rate at which the magnetic flux Φ_B through that loop changes with time. Moreover, the induced emf $\mathscr{E}$ tends to oppose the flux change. In formula, it writes that*

$$\mathscr{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}. \tag{2.6.1}$$

If we change the magnetic flux through a coil of n turns, an induced emf appears in every turn and the total emf induced in the coil is the sum of these individual induced emfs:

$$\mathscr{E} = -n\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}.$$

Here are some common means by which we can change the magnetic flux through a coil:

- Change the magnitude of the magnetic field within the coil.
- Change the area of the coil or the portion of that area that lies within the magnetic field.
- Change the angle between the direction of the magnetic field and the plane of the coil.

The rule of thumb to determine the directions of the induced currents is as follows:

An induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic flux that induces the current.

This is called Lenz's law. We illustrate it with the following figure.

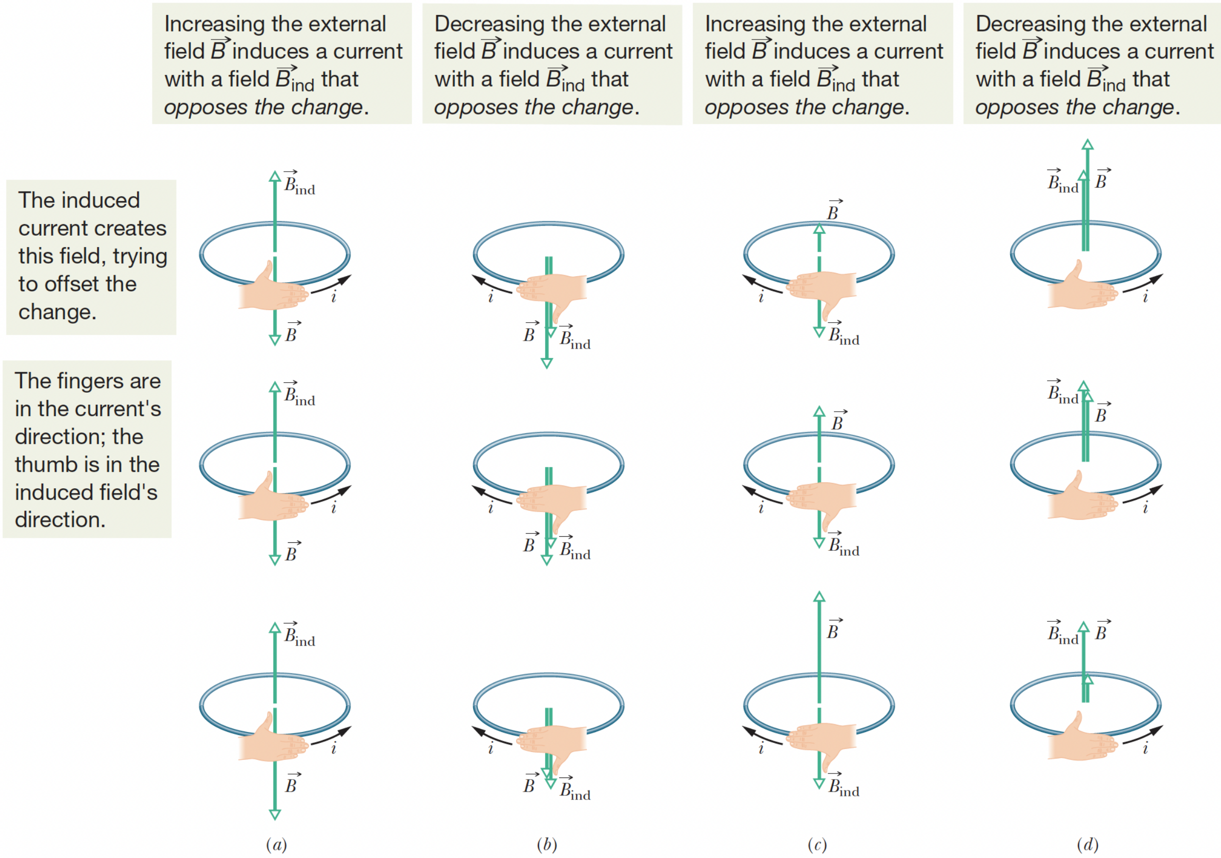

Figure 2.24: The direction of the current i induced in a loop is such that the current's magnetic field $\vec{B}_{\text{ind}}$ opposes the change in the magnetic field $\vec{B}$ inducing i. The field $\vec{B}_{\text{ind}}$ is always directed opposite an increasing field $\vec{B}$ (figures a, c) and in the same direction as a decreasing field $\vec{B}$ (figures b, d). The curled–straight right-hand rule gives the direction of the induced current based on the direction of the induced field.

Example. *Suppose we pull a closed conducting loop out of a uniform magnetic field $\vec{B}$ (pointing inside the paper) at constant velocity v as in Figure 2.25. Suppose the resistance of the loop is R. Find the induced current, the force we act on the loop, the rate of work we do on the loop, and the rate of energy dissipation in the loop.*

Solution: We choose the coordinate axes in Figure 2.25 such that $\vec{v} = v\vec{e}_x$ is along x direction, and $\vec{B}$ is along the $-z$ direction, i.e., $\vec{B} = -B\vec{e}_z$.

As we move the loop to the right, the portion of its area within the magnetic field decreases, so the flux through the loop also decreases. The magnetic flux is $\Phi_B = BLx$, which is changing with rate

$$\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = BL\frac{\mathrm{d}x}{\mathrm{d}t} = -BLv,$$