

# Differential Geometry

LIN150117

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

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# 1 Smooth Manifold

**Definition 1.1** (Topological manifold). A space  $M$  is called a topological manifold if

- (1). locally Euclidean
- (2). Hausdorff
- (3). second countable

**Definition 1.2** (Smooth Manifold). A smooth structure is given by an equivalence class of smooth atlas  $\{(U_\alpha, \varphi_\alpha)\}$  s.t.  $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$  is smooth  $\forall \alpha, \beta$ .  $M = \cup U_\alpha$ .

A **smooth manifold** is a topological manifold with a smooth structure.

Define when a continuous map  $f : M_1 \rightarrow M_2$  is smooth if  $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$ , we have  $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$  is smooth.

**Definition 1.3.** Given  $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ . A homeomorphism  $f : M_1 \rightarrow M_2$  is called a diffeomorphism if  $f, f^{-1}$  is smooth.

In this case we say  $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$  are diffeomorphism.

**Theorem 1.4** (Kervaire).  $\exists$  1 10-dimensional topological manifold without smooth manifold.

**Theorem 1.5** (Milnor).  $\exists$  a smooth manifold  $M$  s.t.  $M \cong S^7$  but not in diffeomorphism meaning.

**Theorem 1.6** (Kervaire-Milnor).  $\exists$  28 smooth structures (up to orientation preserving diffeomorphism) on  $S^7$

**Theorem 1.7** (Morse-Birg). On  $S^7$ . If  $n \leq 3$ , then any  $n$ -dimensional topological manifold  $M$  has a unique smooth structure up to diffeomorphism.

**Theorem 1.8** (Stallings). *If  $n \neq 4$ , then  $\exists$  a unique smooth structure on  $\mathbb{R}^n$  up to diffeomorphism.*

**Theorem 1.9** (Donaldson-Freedman-Gompf-Faubes).  *$\exists$  uncountable smooth structures on  $\mathbb{R}^4$  up to diffeomorphism.*

**Definition 1.10** (topological manifold with boundary). A space  $M$  is called a topological manifold with boundary if

- (1).  $M$  is Hausdorff
- (2).  $M$  is second countable
- (3).  $\forall p \in M, \exists$  a neighbourhood  $U$  of  $p$  and a homeomorphism  $\varphi : U \rightarrow V$  where  $V$  is open in  $\mathbb{H}^n$

We say a manifold  $M$  is closed if  $M$  is compact and  $\partial M$  is empty.

Our motivation for studying manifold is to study the space of solution for equations.

**Question 1.** Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  smooth,  $q \in \mathbb{R}^n$ , when is  $f^{-1}(q)$  is a smooth manifold?

For  $f : U \rightarrow \mathbb{R}^n$  smooth,  $U$  open in  $\mathbb{R}^m$ , the differential of  $f$  at  $p \in U$  denoted as  $df(p)$ .

**Definition 1.11.** We say  $p \in U$  is a **regular point** of  $f$  if  $df(p)$  is surjective. Otherwise we say  $p \in U$  is a **critical point**.

A point  $q \in \mathbb{R}^n$  is called a **regular value** of  $f$  if  $\forall p \in f^{-1}(q)$ ,  $p$  is a regular point of  $f$ .

A point  $q \in \mathbb{R}^n$  is called a **critical value** of  $f$  if  $\forall p \in f^{-1}(q)$ ,  $p$  is a critical point of  $f$ .

**Theorem 1.12** (Implicit function theorem). *If  $p \in U$  is a regular point of  $f : U \rightarrow \mathbb{R}^n$ . Then there exists*

- *An open neighbourhood  $V$  of  $p$  in  $U$*
- *An open subset  $V'$  of  $\mathbb{R}^m$*
- *A diffeomorphism  $\varphi : V \rightarrow V'$  such that  $P \circ \varphi = f$  where  $P$  is the projection from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .*

*In other words, near a regular point, we can do local coordinate change to turn  $f$  into the projection.*

**Remark 1.13.** Inverse function theorem and Implicit function theorem gives a way to find the related from "a point" to "a beighbourhood"!

In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \dots, x_m) \in V' \mid (x_1, \dots, x_n) = f(p)\}$$

*i.e. if we set  $M = f^{-1}(f(p))$ , then  $(M \cap V, \varphi_p)$  is a chart that contains  $p$ .*

**Corollary 1.14.** *If  $q$  is a regular value of  $f : U \rightarrow \mathbb{R}^n$  then  $f^{-1}(q)$  is a smooth manifold.*

**Remark 1.15.** It suffices to show that the corresponding charts are compatible.

**Theorem 1.16** (Sard). *If  $f : U \rightarrow \mathbb{R}^n$  is a smooth map, then the set of critical values of  $f$  has measure 0.*

**Remark 1.17.** For a "generic"  $q$ ,  $f^{-1}(q)$  is a manifold of dimension  $m - n$ .

**Corollary 1.18.** *If  $f : U \rightarrow \mathbb{R}^n$  is smooth and  $m < n$  then  $f(U)$  has measure 0.*

## 1.1 Lie groups and homogeneous spaces

**Definition 1.19.** We say  $G$  is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map  $\cdot : G \times G \rightarrow G$  and the inverse map  $G \rightsquigarrow G$  is smooth.

**Example 1.20.**  $GL(n, \mathbb{R}) = \{n \times n \text{ matrices with non-zero determinant}\} \subset \mathbb{R}^{n \times n}$

$$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$$

$$SO(n) = \{A \in O(n) | \det A = 1\}$$

$$U(n) = \{A \in GL(n, \mathbb{C}) | A\bar{A}^T = I\}$$

$$SU(n) = \{A \in U(n) | \det A = 1\}$$

**Exercise 1.21.**

$$O(1) \cong S^0 \qquad SO(1) \cong \ast \qquad (1.1)$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \qquad (1.2)$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \qquad (1.3)$$

The last one is a diffeomorphism but do not preserve the multiplication, *i.e.* not an isomorphism of Lie group.

**Theorem 1.22 (Cartan).** *Let  $H$  be a closed subgroup of Lie group  $G$ . Then  $H$  is a Lie group. More precisely,  $H$  is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also,  $G/H$  is a smooth manifold*

**Definition 1.23.** Let  $M$  be a smooth manifold. We say  $M$  is a **homogeneous space** if  $\exists$  a Lie group  $G$  with a smooth transitive action  $\rho : G \times M \rightarrow M$ .

**Definition 1.24.** For  $M$  be a homogeneous space. The **isotropy** group of  $x \in M$  is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of  $G$

Given any  $x, x' \in M$ ,  $Iso(x) \cong Iso(x')$  because the group action is transitive.

Hence, we have a well-defined map

$$p : G/Iso(x) \rightarrow M \tag{1.4}$$

$$g \mapsto gx \tag{1.5}$$

**Theorem 1.25.**  *$p$  is always a diffeomorphism.*

Therefore, we have this proposition

**Proposition 1.26.**  *$M$  is a homogeneous space  $\Leftrightarrow M = G/H$  for some closed subgroup  $H$ .*

**Example 1.27.** If  $M = S^n$ , let  $G = SO(n+1)$ .

Then  $Iso(1, 0, \dots, 0) \cong SO(n)$ .

So  $S^n \cong SO(n+1)/(SO(n))$ .

Similarly, we can prove  $\mathbb{RP}^n \cong SO(n+1)/(O(n))$ ,  $\mathbb{CP}^n \cong SO(n+1)/(U(n))$

The isotropy  $k$  dimensional linear subspaces of  $\mathbb{R}^n$  can be  $O(k) \times O(n-k)$  if  $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to  $\mathbb{RP}^2$ ,  $S^2$ ,  $T^2$  and Klein bottle.

**Theorem 1.28** (Whithead). *Any smooth manifold has a triangulation.*

**Theorem 1.29** (Poincare-Hopf).  *$G$  is compact Lie group  $\Rightarrow \chi(G) = 0$ .*

**Theorem 1.30** (Mostow2005).  *$M$  is a compact homogeneous space  $\Rightarrow \chi(M) \geq 0$ .*

## 1.2 Bump Function and Partition of Unity

**Theorem 1.31** (Urysohn smooth version). *Given  $M$ , closed disjoint  $A, B$ ,  $\exists$  smooth  $f : M \rightarrow [0, 1]$  s.t.  $f|_A = 0, f|_B = 1$ .*

**Theorem 1.32** (Tietze). *Given  $M$ , closed  $A$ , smooth  $f : A \rightarrow \mathbb{R}^n$ , there exists smooth  $\hat{f} : M \rightarrow \mathbb{R}^n$  s.t.  $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

**Lemma 1.33.** *Let  $U$  be a neighbourhood of  $p \in M$ . Then  $\exists$  smooth  $\sigma : M \rightarrow [0, 1]$  s.t.*

(1).  $\sigma \equiv 1$  near  $p$

(2).  $\text{Supp } \sigma \subset U$

*Such  $\sigma$  is called a **bump function** at  $p$ , supported in  $U$ .*

**Definition 1.34.** An open cover of a space  $X$  is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

**Proposition 1.35.** *Given compact  $K \subset U$  and open neighbourhood  $U$  of  $K$ ,  $\exists$  a smooth  $g : M \rightarrow [0, +\infty)$  s.t.  $g|_K \equiv 1$  and  $\text{Supp } g \subset U$ .*

**Definition 1.36.** An **exhaust** of a space  $X$  is a sequence of open sets  $\{U_i\}$  s.t.

(1).  $X = \bigcup_{i=1}^{\infty} U_i$

(2).  $\overline{U_i}$  is compact and contained in  $U_{i+1}$

**Theorem 1.37.** *Any topological manifold has an exhaust.*

Given two open covers  $\mathcal{U}, \mathcal{V}$ , we say  $\mathcal{V}$  is a **refinement** of  $\mathcal{U}$  if  $\forall U_\alpha \in \mathcal{U}, \exists V_\beta \in \mathcal{V}$  s.t.  $V_\beta \subset U_\alpha$ .



We say a space  $X$  is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact. (The proof is hard)

**Proposition 1.38.** *Let  $\mathcal{U} = \{U_\alpha\}$  be an open cover of a topological manifold  $M$ . Then there exists countable open covers  $\mathcal{W} = \{W_i\}$ ,  $\mathcal{V} = \{V_i\}$  s.t.*

- *For any  $i$ ,  $\overline{V_i}$  is compact and  $\overline{V_i} \subset W_i$*
- *$\mathcal{W}$  is locally finite.*
- *$\mathcal{W}$  is a refinement of  $\mathcal{U}$ .*

As a corollary, we have any topological manifold is paracompact.

**Definition 1.39.** Given open cover  $\mathcal{U}$  of a smooth  $M$ , a partition of unity subordinate to  $\mathcal{U}$  is a collection of smooth functions  $\{\rho_\alpha : M \rightarrow [0, 1]\}_{\alpha \in \mathcal{A}}$  s.t.

- (1).  $\forall p \in M, \exists$  only finitely many  $\alpha \in \mathcal{A}$  s.t.  $p \in \text{Supp } \rho_\alpha$
- (2).  $\sum_{\alpha \in \mathcal{A}} \rho_\alpha(p) = 1$
- (3).  $\text{Supp } \rho_\alpha \subset U_\alpha$

**Theorem 1.40** (Existence of P.O.U). *For any open cover  $\mathcal{U}$  of smooth  $M$ ,  $\exists$  a P.O.U subordinate to  $\mathcal{U}$*

**Theorem 1.41** (Whitney approximation theorem). *Given any smooth  $M$ , any closed  $A$  and any continuous  $f : M \rightarrow \mathbb{R}$ ,  $\delta : M \rightarrow (0, +\infty)$ . Suppose  $f$  is smooth on  $A$ . Then  $\exists g : M \rightarrow \mathbb{R}$  smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M, |g(p) - f(p)| < \delta(p)$ .

## 2 Tangent space and tangent vectors

### 2.1 Tangent Space

Given  $p \in M$ , consider the set  $C_p^\infty(M) = \{\text{smooth function } V \rightarrow \mathbb{R}\} / \sim$  where  $f_1 \sim f_2$  if and only if  $\exists$  neighbourhood  $U$  of  $p$ ,  $f_1|_U = f_2|_U$ .

$C_p^\infty(M)$  is the space of **germs of smooth function** near  $p$ .

A **partial-derivative** of  $p$  is a  $\mathbb{R}$ -linear map  $D : C_p^\infty(M) \rightarrow \mathbb{R}$  that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

**Definition 2.1.** A **tangent vector** of  $M$  at  $p$  is a partial-derivative at  $p$ .

Define the **tangent space**  $T_p M = \{\text{all partial-derivative at } p\}$ , which is a  $\mathbb{R}$ -vector space.

**Proposition 2.2.** For  $M = U \subset \mathbb{R}^n$  open. We have  $\{\frac{\partial}{\partial x_i}\}$  is a basis for  $T_p U$ .

**Proposition 2.3.**

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leq i \leq n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

$M, N$  smooth manifolds,  $C^\infty(N, M) = \{\text{smooth } F : N \rightarrow M\}$ .

Given  $F \in C^\infty(N, M)$ ,  $F$  induces  $F^* : C_{F(p)}^\infty(M) \rightarrow C_p^\infty(N)$ ,  $f \mapsto f \circ F$ .

By taking dual, we get

$$F_* : T_p N \rightarrow T_{F(p)} M$$

we also write  $F_*$  as  $F_{*,p}$ , call it the **differential** of  $F$  at  $p$ .

where

$$F_*\left(\frac{\partial}{\partial x^i}|_p\right) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

**Proposition 2.4.** *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_p N \rightarrow T_{G \circ F(p)} W$$

**Definition 2.5.** A smooth **curve** is a smooth map  $\gamma : (a, b) \rightarrow M$ . We say  $\gamma$  starts at  $p$  if  $\gamma(0) = p$ . We define the **velocity** of  $\gamma$  at  $\gamma(0)$  as  $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)} M$

Take charts  $(U, x^1, \dots, x^n)$  about  $p$ , let  $\gamma^i = x^i \circ \gamma$ .

We say  $\gamma, \delta$  are **tangent** to each other at  $p$  if  $(\gamma^i)'(0) = (\delta^i)'(0)$ .

Now we can define

$$(T_p M)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\} / \sim$$

where  $\gamma \sim \delta$  iff they are tangent to each other.

Then these definition is more geometric.

**Lemma 2.6.** *Given  $F \in C^\infty(M, M)$ ,  $p \in N$ , the diagram commutes:*

$$\begin{array}{ccc} \gamma \in (T_p N)_{curve} & \xrightarrow{\cong} & T_p N \\ \downarrow & & \downarrow \\ F \circ \gamma \in (T_{F(p)} M)_{curve} & \xrightarrow{\cong} & T_{F(p)} M \end{array}$$

## 2.2 Tangent Bundle

Let  $(M, \mathcal{A})$  be a smooth manifold,  $TM = \bigcup_{p \in M} T_p M$ , called the **tangent bundle**

Now we want to define a natural topology and smooth structure on  $TM$ . Take any chart  $(U, \varphi) = (U, x^1, \dots, x^n) \in \mathcal{A}$ .

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \quad (2.1)$$

$$X \in T_p U \mapsto (\varphi(p), X^1, \dots, X^n) \quad (2.2)$$

where  $X = \sum X^i \frac{\partial}{\partial x^i} |_p$ .

Then pull back standard topology on  $\varphi(U) \times \mathbb{R}^n$  to a topology on  $TU$ .

$$\mathcal{B} = \{ \hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- $\mathcal{B}$  is a basis
- $\mathcal{B}$  generates a Hausdorff, second countable topology on  $TM$ .

So  $TM$  is a topological manifold covered by charts  $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$ .

Given  $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$ , the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi} \circ \hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \quad (2.3)$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \quad (2.4)$$

So  $\hat{\mathcal{A}}$  is a smooth atlas on  $TM$ , making  $TM$  into a smooth manifold.

**Definition 2.7** (vector bundle). Given a continuous map  $f : E \rightarrow B$ , we say  $f$  is a  $n$ -dimensional **vector bundle** if:  $\exists$  an open cover  $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$  of  $B$  and homeomorphisms  $\{f^{-1}(U_\alpha) \xrightarrow[\cong]{\rho_\alpha} U_\alpha \times \mathbb{R}^n\}$  s.t.

$$\begin{array}{ccc} f^{-1}(U_\alpha) & \xrightarrow{\rho_\alpha} & U_\alpha \times \mathbb{R}^n \\ \downarrow f & \swarrow \text{projection} & \\ U_\alpha & & \end{array} \quad \text{commutes for } \alpha \in I$$

- $\forall p \in U_\alpha \cap U_\beta$ , the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

is linear.

Call  $f^{-1}(p)$  the **fiber** over  $p$ .

**Proposition 2.8.** *Given vector bundle  $f : E \rightarrow B$ , the fiber  $f^{-1}(p)$  has a structure of a vector space.*

**Example 2.9** (Product bundle).  $E = \mathbb{R}^n \times B$

**Example 2.10** (Tautological bundle).

$$B = \mathbb{CP}^n = \{1\text{-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map  $(L, v) \mapsto L$

Given vector bundles  $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$ , a bundle map consists of  $(\hat{f}, f)$  s.t.

- $$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow \pi & & \downarrow \pi \\ B_1 & \xrightarrow{f} & B_2 \end{array} \text{ commutes.}$$
- $\forall b \in B, \hat{f} : \pi_1^{-1}(b) \rightarrow \pi_2^{-1}(f(b))$  is linear.

If  $\hat{f}, f$  are diffeomorphisms, then we call  $(\hat{f}, f)$  an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. A bundle is **trivial** if it has a trivialization.

**Example 2.11.**  $TS^1, TS^2$  are both trivial.

$$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$$

**Theorem 2.12.** *If  $G$  is a Lie group, then  $TG$  is trivial.*

*Proof.* For  $(x^1, x^2, \dots, x^n)$  is a basis of  $T_e G$  The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG, (g, c^1, \dots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \rightarrow G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism  $(l_g)_*$

□

**Proposition 2.13** (Adams, 1960s).  *$TS^n$  is trivial if and only if  $n = 0, 1, 3, 7$ .*

**Proposition 2.14.** (1). *Given any  $F \in C^\infty(M, N)$ ,  $F_* : TM \rightarrow TN$  is a bundle map.*

(2).  *$TS^n$  is isomorphic to the following bundle:*

$$B = S^n \quad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

**Definition 2.15** (smooth section). Given a smooth vector bundle  $\pi : E \rightarrow B$ , a **smooth section** is a smooth map  $S : B \rightarrow E$  s.t.  $\pi \circ S = id_B$ .

$$s_0 : B \rightarrow E, b \mapsto 0 \in 0\text{-vector in } \pi^{-1}b.$$

## 2.3 Vector Field, Curves and Flows

**Definition 2.16.** A (tangent) **vector field** is a smooth section of  $TM$ . i.e. a smooth map  $M \xrightarrow{X} TM$  s.t.  $X(p) \in T_p M, \forall p \in M$

Given any  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

**Example 2.17.**  $X = f^1 \partial x^1 + f^2 \partial x^2$  is a gradient field if and only if  $\frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}$

**Theorem 2.18** (Poincare-Hopf). *For closed  $M$ ,  $M$  has a nowhere vanishing vector field if and only if  $\chi(M) = 0$ .*

So  $S^n$  has a nowhere vanishing vector field if and only if  $n$  is odd.

**Theorem 2.19** (MaoQiu).  $S^2$  has no no-where vanishing vector field.

So We cannot smooth out all the hairs on a ball.

Given a vector field  $X = \{X_p\}_{p \in M}$ , a curve  $\gamma : (a, b) \rightarrow M$  is called an **integral curve** of  $X$  if  $\gamma'(t) = X_{\gamma(t)}$ ,  $\forall t \in (a, b)$ , where  $\gamma'(t) = \gamma_*\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M$ .

We say  $\gamma$  is maximal if the domain cannot be extended to a larger interval.

Denote the set of all smooth vector fields on  $M$  by  $\mathfrak{T}M$

Recall that  $\gamma$  is maximal if it's domain can not be extended to a large open interval.

In a local chart  $(U, x^1, \dots, x^n)$ ,  $X|_U = \sum_{i=1}^n a^i \partial x^i$ . Then  $\gamma$  is an integral curve if and only if  $(\gamma^i)'(t) = a^i(\gamma(t))$ ,  $\forall 1 \leq i \leq n$ , where  $\gamma^i = x^i \circ \gamma : (a, b) \rightarrow \mathbb{R}$ .

And in this case the initial value condition:  $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$ .

Locally, solving integral curve starting at  $p$  is equivalent to solving ODE with initial value  $p^1, \dots, p^n$ . By existence and uniqueness of solutions of ODE, we have

**Theorem 2.20** (Fundamental theorem of integral curve). Let  $X \in \mathfrak{T}M$ ,  $p \in M$ , then:

(1) (Uniqueness) Given any two integral curves  $\gamma_1, \gamma_2 : (a, b) \rightarrow M$ , then we have:

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a, b) \Rightarrow \gamma_1 = \gamma_2$$

(2) there exists a unique max integral curve  $\gamma : (a(p), b(p)) \rightarrow M$  starting at  $p$ .

(3) (integral curve smoothly depend on initial values)  $\exists$  Nbh  $U$  of  $p$ ,  $\epsilon > 0$ , and smooth  $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$  s.t.  $\forall q \in U$ ,  $\varphi_\epsilon := \varphi(-, q) : (-\epsilon, \epsilon) \rightarrow M$  is an integral curve starting at  $p$ .

we call such  $\varphi$  a local **flow** generated by  $X$ .

**Definition 2.21.** Given  $X \in \mathfrak{X}M$ , a global **flow** generated by  $X$  is a smooth map  $\varphi : \mathbb{R} \times M \rightarrow M$  s.t.  $\forall q \in M, \varphi_q := \varphi(-, q)$  is the maximal integral curve of  $X$  starting at  $q$ .

$$\Leftrightarrow \frac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s, p)}, \forall s \in \mathbb{R}, p \in M \text{ and } \varphi(0, p) = p, \forall p \in M.$$

If such global flow exists, then we say  $X$  is **complete**.

**Example 2.22.**

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$  is complete, where global flow  $\varphi : \mathbb{R} \times M \rightarrow M, \varphi(t, p) = p \cdot e^t$ .
- $X = x^2 \partial x$  is not complete. Max integral curve starting at 1 is given by  $\gamma(t) = \frac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$ .

Given  $X \in \mathfrak{X}M$ , we define  $\text{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$ .

**Theorem 2.23.** If a vector field  $X$  is compactly supported, then  $X$  is complete.

**Corollary 2.24.** Any vector field on closed manifold is complete.

**Lemma 2.25** (Escaping lemma). Suppose  $\gamma : (a, b) \rightarrow M$  is a max integral curve, with  $(a, b) \neq \mathbb{R}$ . Then  $\nexists$  compact  $K \subset M$  s.t.  $\gamma(a, b) \subset K$

*Proof.* Otherwise, suppose  $\gamma(a, b) \subset K$ . WLOG, we may assume  $b < +\infty$ .

Take  $(t_i) \rightarrow b$  from left. Then  $\gamma(t_i) \in K$ . After passing to subsequence, we may assume  $(\gamma(t_i)) \rightarrow p \in K$ .

Then  $\exists U$  Nbh of  $p$ , local flow  $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$ . Take  $n$  large enough s.t.  $b - t_n < \epsilon, \gamma(t_n) \in U$ . Then  $\gamma(- + t_n) : (a - t_n, b - t_n) \rightarrow M, \varphi(-, \gamma(t_n)) : (-\epsilon, \epsilon) \rightarrow M$  are both integral curves for  $X$  starting at  $\gamma(t_n)$ . By uniqueness, they coincide.

$$\text{Let } \hat{\gamma} : (a, t_n + \epsilon) \rightarrow M \text{ be defined by } \hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \epsilon) \end{cases}$$

Then  $\hat{\gamma}$  is an integral curve with larger domain, then  $\gamma$  contradiction with the maximality of  $\gamma$ . □



*Proof of 2.23.* Take any max integral curve  $\gamma : (a, b) \rightarrow M$ . Suppose  $(a, b) \neq \mathbb{R}$ . Then  $X_{\gamma(s)} \neq 0, \forall s$ . Otherwise, the constant map  $\mathbb{R} \rightarrow M, t \mapsto \gamma(s)$  is an integral curve with larger domain.

So  $\forall s, \gamma(s) \in \text{Supp} X \Rightarrow \gamma(a, b) \subset \text{Supp} X$  which is compact  $\Rightarrow (a, b) = \mathbb{R}$  by the lemma. This causes contradiction!  $\square$

A smooth  $\varphi : \mathbb{R} \times M \rightarrow M$  is called an **one-parameter transformation group** if

$$(1) \quad \varphi_0 := \varphi(0, -) = \text{id}_M$$

$$(2) \quad \varphi_s \circ \varphi_t = \varphi_{s+t} \text{ for all } s, t \in \mathbb{R}. \text{ In particular, } \varphi_s^{-1} = \varphi_{-s}.$$

**Theorem 2.26.**  $\varphi \in C^\infty(\mathbb{R} \times M, M)$ , then  $\varphi$  is an one-parameter transformation group if and only if  $\varphi$  is the global flow generated by some  $X \in \mathfrak{X}M$

**Lemma 2.27** (Translation lemma). If  $\gamma : (a, b) \rightarrow M$  is an integral curve for some  $X \in \mathfrak{X}M$ , then  $\forall s \in \mathbb{R}, \gamma(- + s) : (a - s, b - s) \rightarrow M$  is also an integral curve for  $X$ .

*Proof.* Let  $\iota = \gamma(- + s)$ . Then  $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$   $\square$

**Lemma 2.28.** Let  $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$  be a local flow for some  $X \in \mathfrak{X}M$ . Then  $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$  provided that  $s, t, s + t \in (-\epsilon, \epsilon), p, \varphi_r(p) \in U$ .

*Proof.*  $\gamma_p = \varphi(-, p)$  is an integral curve for  $X$ .

$\Rightarrow \gamma_p(- + s)$  is an integral curve for  $X$  starting at  $\gamma_p(s) = \varphi_s(p)$ . But  $\gamma_{\varphi_s(p)}$  is also an integral curve starting at  $\varphi_s(p)$ . Thus  $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$   $\square$

**Lemma 2.29.** Let  $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$  be a local flow for some  $X \in \mathfrak{X}M$ . Then  $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$  i.e. any vector field is invariant under its flow.

*Proof.* Take  $f \in C_{\varphi(p)}^{\infty}(M)$ .

$$\varphi_{s,*}(X_p)(f) = X_p(f \circ \varphi_s) \quad (2.5)$$

$$= \frac{d}{dt}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \quad (2.6)$$

$$= \frac{d}{dt}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \quad (2.7)$$

$$= X_{\varphi_s(p)}(f) \quad (2.8)$$

□

*Proof of 2.26.* " $\Leftarrow$ " is because the lemma  $\varphi_s \circ \varphi_r = \varphi_{s+r}$

" $\Rightarrow$ " Let  $X = \{X_p\}$  where  $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$ .

Leave it as an exercise.

□

**Time dependent** vector field is a smooth map  $X : \mathbb{R} \times M \rightarrow TM$  s.t.  $X_{(t,p)} \in T_p M$ .

A smooth curve  $\gamma(a, b) \rightarrow M$  is the **integral curve** for  $X$  if  $\gamma'(t) = X_{(t,\gamma(t))}$ .

In local chart, solving  $\gamma$  is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness:  $\gamma_1, \gamma_2$  are both integral curves for  $X$ ,  $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define  $\text{Supp} X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$ .

Then  $X$  is compactly supported, then  $X$  is complete( i.e. a global flow  $\varphi : \mathbb{R} \times M \rightarrow M$ )

But something is not true for time dependent vector field:

- translation lemma is not true.

- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

## 2.4 Another definition of vector field

A derivation on  $M$  is a  $\mathbb{R}$ -linear map  $C^\infty(M) \xrightarrow{D} C^\infty(M)$  that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

**Theorem 2.30.** *We have a bijection:*

$$\begin{aligned} \rho : \mathfrak{X}M &\xrightarrow{1:1} \{\text{derivation on } M\} \\ X &\mapsto D_X : f \mapsto X(f) \end{aligned}$$

**Lemma 2.31.**  $D_p : \mathfrak{X}_p M \rightarrow \mathbb{R}$ -linear map  $C^\infty(M) \rightarrow \mathbb{R}$  s.t.  $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$  is an isomorphism of vector spaces.

*Proof.* Leave it as an exercise. □

**Lemma 2.32.** *Given a vector field(not necessarily smooth)  $X = \{X_p\}_{p \in M}$ ,  $X$  is smooth  $\Leftrightarrow \forall f \in C^\infty(M)$ ,  $X(f)$  is smooth.*

*Proof.* " $\Leftarrow$ "  $\forall p \in M$ , take chart  $(U, x^1, x^2, \dots, x^n)$  around  $p$ .  $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$  :  $U \rightarrow \mathbb{R}$ , where  $f^i = X|_U(x^i)$ . Take  $\varphi : M \rightarrow [0, 1]$  s.t.  $\varphi \equiv 1$  near  $p$ ,  $\text{Supp} \varphi \subset U$ ,  $\varphi \cdot x^i \in C^\infty(M)$ .

Then  $X(\varphi \cdot x^i) = f^i$  near  $p$ . By assumption,  $f^i$  is smooth near  $p$ . So  $f^i$  is smooth, so  $X$  is smooth.

" $\Rightarrow$ " Similar. □

**Theorem 2.33.** *The map  $\rho : \mathfrak{T}M \rightarrow \{\text{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$  is well-defined and bijective.*

*Proof.*  $\rho$  is well-defined:  $X(f) \in C^\infty(M)$  by Lemma 2.32, and  $D_x(fg) = D_x(f)g + fD_x(g)$  since  $X$  is a point-derivation.

$\rho$  is injective:  $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$  as maps  $C^\infty(M)$  to  $\mathbb{R}$ . By Lemma 2.31, we have  $X_p = Y_p, \forall p$ . So  $X = Y$ .

$\rho$  is surjective: Given  $D : C^\infty(M) \rightarrow C^\infty(M)$ . Define  $D_p : C^\infty(M) \rightarrow \mathbb{R}$  by  $D_p(f) := D(f)(p)$  satisfies the Leibniz rule. By Lemma 2.31,  $D_p = D_{X_p}$  for some  $X_p \in T_p M$ . Define  $X = \{X_p\}_{p \in M}$ . Then  $X(f) = D(f), \forall f \in C^\infty(M)$ . By Lemma??,  $X$  is a smooth vector field.  $\square$

### 3 Lie group, Lie algebra and Lie bracket

#### 3.1 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned} \{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \text{ s.t.} \\ &\quad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\} \end{aligned}$$

$$\begin{aligned} \{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\} \end{aligned}$$

**Notation 3.1.** We will identify  $X \in \mathfrak{X}M$  with its derivation  $D_x : C^\infty(M) \rightarrow C^\infty(M)$ . So a vector field is just a  $\mathbb{R}$ -linear map  $X : C^\infty(M) \rightarrow C^\infty(M)$  s.t.  $X(fg) = fX(g) + X(f)g$ .

**Definition 3.2** (Lie bracket). Given two (smooth) vector field  $X, Y : C^\infty(M) \rightarrow C^\infty(M)$ , we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \rightarrow C^\infty(M)$$

**Theorem 3.3.** For any  $X, Y \in \mathfrak{X}M$ ,  $[X, Y] \in \mathfrak{X}M$

*Proof.* Easy to check that  $[X, Y]$  is linear.

By Leibuniz rule,

$$\begin{aligned} [X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\ &= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\ &= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\ &= [X, Y](f) \cdot g - f \cdot [X, Y](g) \end{aligned}$$

□

So What is the geometric meaning of  $[X, Y]$ ? Non commutativity of flows.

**Fact 3.4.** Given  $X, Y \in \mathfrak{X}M$ , we say  $X, Y$  are commutative vector field if  $[X, Y] = 0$

$X, Y$  are commutative iff for any local flows  $\varphi^X : (-\epsilon, \epsilon) \times U \rightarrow M$ ,  $\varphi^Y : (-\epsilon, \epsilon) \times U \rightarrow M$  we have  $\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X$

**Proposition 3.5** (Calculation of  $[V, W]$  using local charts). Chart  $(U, x^1, \dots, x^n)$ ,

$V, W \in \mathfrak{X}M$ ,  $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$ ,  $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$ . Then

$$\begin{aligned} [V, W]|_U &= \sum_{i=1}^n (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n \left( \sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} \\ &= \sum_{1 \leq i, j \leq n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i} \end{aligned}$$

**Example 3.6.**  $V = x\partial x + y\partial y$ ,  $W = -y\partial x + x\partial y$  commutes.

**Proposition 3.7** (Properties of Lie bracket).

(a) *Natuality under push-forward.*

Given any  $F \in \text{Diff}(M, N)$ ,  $V \in \mathfrak{X}M$ ,  $W \in \mathfrak{X}M$ , we have  $[F_*V, F_*W] = F_*[V, W]$ .

(b)  *$\mathbb{R}$ -linearity*  $\forall a, b \in \mathbb{R}$

$$[aX + bV, W] = a[X, W] + b[V, W]$$

$$[W, aX + bV] = b[W, X] + a[W, V]$$

(c) *anti-symmetric*  $[V, W] = -[W, V]$

(d) *Jacobi identity*

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

(f) *Leibniz rule*

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

**Definition 3.8.** Given  $F \in C^\infty(M, N)$ ,  $V \in \mathfrak{X}M$ ,  $W \in \mathfrak{X}N$ . We say  $W$  is  **$F$ -related** to  $V$  if  $\forall p \in M$ ,  $F_{p,*}(V_p) = W_{F(p)}$  where  $F_{p,*} : T_pM \rightarrow T_{F(p)}N$

**Example 3.9.**  $F : S^1 \rightarrow \mathbb{R}^2, \theta \mapsto (\cos \theta, \sin \theta)$ ,  $V = \partial \theta$ ,  $W = -y \partial x + x \partial y$ .

*Note.* In general, given  $V \in \mathfrak{X}M$  and  $F \in C^\infty(M, N)$ . There may not exist  $W \in \mathfrak{X}N$  s.t.  $V, W$  are  $F$ -related. Even such  $W$  exists, it may not be unique.

However, if  $F$  is a diffeomorphism, given any  $V$ ,  $\exists$  unique  $W$  s.t.  $V$  and  $W$  are  $F$ -related. Actually,  $W_p = F_* V_{F^{-1}(p)}$ .

Such  $W$  is called **push forward** of  $V$  along  $F$ , denoted by  $F_* V$ , only defined when  $F$  is a diffeomorphism.

**Lemma 3.10.**  $\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N, F \in C^\infty(M, N)$ . Then  $W$  is  $F$ -related to  $V$  iff  $\forall f \in C^\infty(N)$ ,  $V(f \circ F) = W(f) \circ F \in C^\infty(M)$

*Proof.* Check that  $F_{p,*}(V_p)(f) = W_{F(p)}(f)$ ,  $\forall f \in C^\infty(N)$  □

**Proposition 3.11.** Given  $V_0, V_1 \in \mathfrak{X}M$ ,  $W_0, W_1 \in \mathfrak{X}N$ ,  $F \in C^\infty(M, N)$ ,  $W_i$  is  $F$ -related to  $V_i$ ,  $i = 0, 1 \Rightarrow [W_0, W_1]$  is  $F$ -related to  $[V_0, V_1]$

**Corollary 3.12** (Naturality of Lie bracket). Given any  $F \in \text{Diff}(M, N)$ ,  $V \in \mathfrak{X}M$ ,  $W \in \mathfrak{X}N$ , we have  $[F_* V, F_* W] = F_* [V, W]$

The rest of Proposition 3.7 is easy to check if it is viewed as a mapping  $C^\infty(M) \rightarrow C^\infty(M)$ .

## 3.2 Lie algebra of a Lie group

**Definition 3.13.** A **Lie algebra**  $g$  is  $\mathbb{R}$ -linear space  $g$  with map  $[-, -] : g \times g \rightarrow g$  s.t. it is bilinear, anti-symmetric and satisfies the Jacobian identity.

Then  $(\mathfrak{X}M, [-, -])$  is an infinite dimensional Lie algebra.

For  $G$  Lie group,  $\forall g \in G$  we have diffeomorphism

$$l^g : G \rightarrow G, h \mapsto gh$$

$$r^g : G \rightarrow G, h \mapsto hg$$

We say  $X \in \mathfrak{X}G$  is **left invariant** if  $l_*^g(X) = X, \forall g \in G$ . Similarly,  $X$  is **right invariant** if  $r_*^g(X) = X$ .

**Proposition 3.14.**  $X, Y$  are left/right invariant  $\Rightarrow [X, Y]$  is left/right invariant.

*Proof.*  $l_*^g[X, Y] = [l_*^g X, l_*^g Y] = [X, Y]$  □

So we can find a natural Lie algebra of  $G$ :

$\text{Lie}(G) := \{\text{left invariant vector fields on } G\}$ , with  $[-, -]$  restricted from  $\mathfrak{X}G$

**Theorem 3.15.** Given any  $V \in T_e G, \exists$  unique left invariant  $\hat{V} \in \mathfrak{X}G$  s.t.  $\hat{V}_e = V$ .

**Corollary 3.16.**  $\text{Lie}(G) \cong T_e G$  as vector spaces.

*Proof of Theorem 3.15.*

**Uniqueness of  $\hat{V}$ :**  $\hat{V}_g = l_{e,*}^g(\hat{V}_e) = l_{e,*}^g(V)$ . So  $\hat{V}$  is determined by  $V$ .

**Existence of  $\hat{V}$ :** Let  $\hat{V} = \{\hat{V}_g\}_{g \in G}$  where  $\hat{V}_g = l_{e,*}^g(\hat{V}_e)$ .

$\hat{V}$  is left-invariant because

$$(l_*^h(\hat{V}))_g = l_{h^{-1}g,*}^h(\hat{V}_{h^{-1}g}) = l_{h^{-1}g,*}^h(l_{e,*}^{h^{-1}g}(V)) = l_{e,*}^g(V) = \hat{V}_g$$

$\hat{V}$  is smooth: Take any  $f \in C^\infty(G)$  suffices to show  $\hat{V}(f) \in C^\infty(G)$ .

Take any smooth  $\gamma : \mathbb{R} \rightarrow G$  s.t.  $\gamma(0) = e, \gamma'(0) = V$ . Then  $l^g \circ \gamma : \mathbb{R} \rightarrow G$  satisfies  $l^g \circ \gamma(0) = g, (l^g \circ \gamma)(0) = g, (l^g \circ \gamma)'(0) = l_{e,*}^g(V) = \hat{V}_g$



So

$$\hat{V}(f)(g) = \hat{V}_g(f) = \frac{d}{dt}f(l^g \circ \gamma(t))|_{t=0} = \frac{d}{dt}f(g \cdot \gamma(t))|_{t=0} \quad (3.1)$$

Consider the map

$$\begin{aligned} \hat{f} : G \times \mathbb{R} &\xrightarrow{\text{id} \times \gamma} G \times G && \xrightarrow{\quad} G \xrightarrow{f} \mathbb{R} \\ (g, t) &\mapsto (g, \gamma(t)) && \mapsto g \cdot \gamma(t) \mapsto f(g \cdot \gamma(t)) \end{aligned}$$

Then  $\hat{f}$  is smooth,  $\frac{\partial \hat{f}}{\partial t}|_{t=0} : G \rightarrow \mathbb{R}$  is smooth, but  $\frac{\partial \hat{f}}{\partial t}|_{t=0}(g) = \hat{V}(f)(g)$  by 3.1. So  $\hat{V}(f) \in C^\infty(G)$ .  $\square$

**Example 3.17.**  $G = \text{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^2$ .

$$\mathfrak{gl}(n, \mathbb{R}) = \text{Lie}(\text{GL}(n, \mathbb{R})) = T_I \text{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$$

**Theorem 3.18.**  $\forall A, B \in \mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}), [A, B] = AB - BA$ .

**Remark 3.19.** This theorem shows that the Lie bracket viewed as the Lie algebra and matrix are the same. In some sense, it means the Lie bracket defined in three sets  $\mathfrak{gl}(n, \mathbb{R}) = T_I \text{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$  can commute with those corresponding, or equivalently, are just the same.

**Lemma 3.20.**  $\forall A \in \mathfrak{gl}(n, \mathbb{R})$ , the left invariant vector field  $\hat{A}$  is complete and generated the flow  $\varphi_t : \text{GL}(n, \mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2 t^2}{2!} + \dots)$

*Proof.*

$$\hat{A}_g = g \cdot A \in T_g G = M_n(\mathbb{R})$$

$$\frac{\partial}{\partial t} \varphi_t(g) = \frac{\partial}{\partial t} (ge^{At}) = ge^{At} A = A_{g \cdot e^{At}} = \hat{A}_{\varphi_t(g)}$$

$\square$

*Proof of Theorem 3.18.* Take  $A, B \in \mathfrak{gl}(n, \mathbb{R})$ . Want to show  $[\hat{A}, \hat{B}]_I = AB - BA$ .

Pick  $f \in C_I^\infty(G)$ , need to show  $A(\hat{B}(f)) - B(\hat{A}(f)) = (AB - BA)(f)$

Further Simplification: Just need to focus on  $f = x^{ij}$ , where  $x^{ij} : \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}, E \mapsto (E - I)_{ij}$ .

Such  $f$  satisfies  $f(I + -)$  is  $\mathbb{R}$ -linear.

Recall that Given  $W \in \mathfrak{Z}M$ ,  $W(f)(p) = \frac{d}{dt}f(\varphi_t^W(p))|_{t=0}$ .

So  $\hat{B}(f)(g) = \frac{d}{dt}f(ge^{tB})|_{t=0}$ .

So

$$A(\hat{B}(f)) = \frac{d}{dt}(\hat{B}(f)(e^{As}))|_{s=0} = \frac{d^2}{dsdt}f(I + sA + tB + \frac{s^2}{2}A^2 + stAB + \frac{t^2}{2}B^2 + \dots)|_{s=t=0}$$

Similarly,

$$B(\hat{A}(f)) = \frac{d^2}{dsdt}f(I + sA + tB + \frac{s^2}{2}A^2 + stBA + \frac{t^2}{2}B^2 + \dots)|_{s=t=0}$$

So  $A(\hat{B}(f)) - B(\hat{A}(f)) = f(I + (AB - BA)) = (AB - BA)(f)$  since  $f$  is  $\mathbb{R}$ -linear.  $\square$

Similarly, for  $G = \text{GL}(n, \mathbb{C})$ ,  $\text{Lie}(G) = \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$ , we have  $[A, B] = AB - BA$ .

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group  $G$ , its universal cover  $\hat{G}$  is simply-connected with  $\pi^{-1}(G) \subset Z(\hat{G})$ .

What is the meaning of Lie bracket. There is a fact about it:

**Fact 3.21.**  $G$  is connected Lie group.  $G$  is abelian iff  $[-, -] = 0$  on  $\text{Lie}(G)$

### 3.3 Morphisms between Lie group and Lie algebras

A smooth map  $F : G \rightarrow H$  between two Lie group is called a **morphism** if  $F(gh) = F(g)F(h)$ .

A linear map  $L : g \rightarrow h$  between Lie algebra is called a **morphism** if  $L[u, v] = [Lu, Lv]$ .

**Proposition 3.22.** *Let  $F : G \rightarrow H$  be a morphism of Lie groups. Then  $F_{e,*} : \text{Lie}(G) \rightarrow \text{Lie}(H)$  is a morphism of Lie algebra.*

*Proof.*  $V_0, V_1 \in \text{Lie}(G) = T_e G$ ,  $W_i = F_{e,*}(V_i) \in \text{Lie}(H) = T_e H$ . Let  $\hat{V}, \hat{W}$  be left-invariant vector fields.

*Claim.*  $\hat{W}_i$  is  $F$ -compatible with  $\hat{V}_i$  for  $i = 0, 1$ .

*Proof of Claim.*  $\forall g \in G$ ,  $F_*(\hat{V}_g) = F_*(l^g_*(V)) = (F \circ l^g)_*(V) = (l^{F(g)} \circ F)_*(V) = l^{F(g)}_*(W) = \hat{W}_{F(g)}$  □

So  $[\hat{W}_0, \hat{W}_1]$  is  $F$ -compatible with  $[\hat{V}_0, \hat{V}_1]$ . In particular,  $[W_0, W_1] = F_*([V_0, V_1])$ . □

## 4 Differential form

### 4.1 Canonical form of a field

Recall that  $V \in \mathfrak{T}M$ ,  $p \in M$  is called a **regular point** if  $V_p \neq 0$ , and is called a **singular point** if  $V_p = 0$ .

**Theorem 4.1** (Canonical Form Theorem). *Let  $p$  be a regular point of  $V$ . Then  $\exists$  local chart  $(U, x^1, \dots, x^n)$  around  $p$  s.t.  $V|_U = \partial x^1$*

*Proof.* This is a local problem. We may assume  $M \subset \mathbb{R}^n$  open. We may also assume  $p = 0$ ,  $V_0 = \partial r^1|_0$  where  $r^i$  coordinate function.

Let  $\varphi : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)^n \rightarrow M$  be the local flow of  $V$ .

Define  $\psi : (-\epsilon, \epsilon)^n \rightarrow M$  by  $\psi(t, r^2, \dots, r^n) = \varphi(t, (0, r^2, \dots, r^n))$ . Then  $\psi(-, r^2, \dots, r^n)$  is an integral curve for  $V$ . Therefore,  $\psi_*(\partial t) = V$ .

At  $\vec{0}$ , we have  $\psi_{\vec{0},*}(\partial t) = V_{\vec{0}} = \partial r^1$ ,  $\psi_{\vec{0},*}(\partial r^i) = \partial r^i$ .

So  $\psi_{*,\vec{0}} : T_{\vec{0}}(-\epsilon, \epsilon)^n \rightarrow T_{\vec{0}}M$  is an isomorphism.

By the inverse function theorem,  $\exists U' \subset (-\epsilon, \epsilon)^n$ ,  $U \subset M$  s.t.  $\psi|_{U'} : U' \rightarrow U$  is a diffeomorphism.

Then  $(U, (\psi|_{U'})^{-1})$  is the local chart what we need. □

**Remark 4.2.** Regular point in a vector field is simple, as we can view it in the standard chart locally. However, behavior of  $V$  at a singular point can be complicated. For example, for  $f(x, y) = x^2 - y^2$ ,  $\nabla f = 2x\partial x - 2y\partial y$ ,  $g : \mathbb{C} \rightarrow C, z \mapsto z^n$ , they behave differently at  $\vec{0}$ .

## 4.2 Lie derivative of vector field

$V, W \in \mathfrak{X}M$ ,  $\mathcal{L}_V W$  is the directional derivative of  $W$  in the direction of  $V$ .

**Definition 4.3.** The **Lie derivative**  $\mathcal{L}_V W \in \mathfrak{X}M$  is defined as follows:  $\forall p \in M$ , let  $\{\theta_t : U \rightarrow M\}_{t \in (-\epsilon, \epsilon)}$  be the local flow for  $V$ . Then

$$(\mathcal{L}_V W)_p = \lim_{t \rightarrow 0} \frac{(\theta_{-t})_* W_{\theta_t(p)} - W_p}{t}$$

**Remark 4.4.** This definition is actually a difference between  $T_{\theta_t(p)}$  and  $T_p$ , which need pullback.

**Lemma 4.5.**  $\mathcal{L}_V W$  is well-defined and smooth.

*Proof.* For  $p \in M$ , take local chart  $(U, x^1, \dots, x^n)$ . Let  $\theta : (-\epsilon, \epsilon) \times U \rightarrow M$  be the flow of  $V$ . Take  $J_0 \subset (-\epsilon, \epsilon)$ ,  $U_0 \subset U$ . Let  $\theta^i = x^i \circ \theta : J_0 \times U_0 \rightarrow \mathbb{R}$ ,  $W|_U = \sum_{i=1}^n W^i \partial x^i$ .

Under the basis  $\{\partial x^i\}$ ,  $(\theta_{-t})_* : T_{\theta_t(p)}M \rightarrow T_pM$  is represented by

$$\left( \frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j} \right)_{i,j}$$

So  $(\theta_{-t})_* W_{\theta_t(x)} = \sum_{i,j} \frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j} W^j(\theta(t, x)) \cdot \partial x^i$  is smooth in  $t, x$ . So

$$(\mathcal{L}_V W)_x = \frac{\partial((\theta_{-t})_*(W_{\theta_t(x)}))}{\partial t} \Big|_{t=0}$$

is well-defined and smooth. □

**Theorem 4.6.** For all  $V, W \in \mathfrak{T}M$ ,  $\mathcal{L}_V W = [V, W]$ .

*Proof.* For  $p$  is a regular point of  $V$ . By canonical form theorem 4.1,  $\exists$  local chart  $(U, x^1, \dots, x^n)$  around  $p$  s.t.  $V|_U = \partial x^1$ . Let  $W|_U = \sum_{i=1}^n W^i \partial x^i$ .

Then  $\theta_t(x_1, \dots, x_n) = (x_1 + t, x_2, \dots, x_n)$ . So

$$\mathcal{L}_V W|_U = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

.

$$[V, W]|_U = \sum_i V(W^i) \partial x^i - \sum_i W(V^i) \partial x^i = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

Then  $[V, W]|_U = \mathcal{L}_V W$ .

For  $p$  is a singular point but  $p \in \text{Supp}(V)$ . Then  $\exists p_i \rightarrow p$  s.t.  $V_{p_i} \neq 0$ . By the previous case  $(\mathcal{L}_V W)_{p_i} = [V, W]|_{p_i}$ . By continuity, We have  $(\mathcal{L}_V W)_p = [V, W]_p$ .

For  $p \notin \text{Supp}(V)$ ,  $\exists$  Nbd  $U$  of  $p$  s.t.  $V|_U = 0$ . Then  $\theta_t(q) = q$ . So

$$(\mathcal{L}_V W)|_U = 0 = [V, W]|_U$$

□

**Corollary 4.7.**

- $\mathcal{L}_V W$  is  $\mathbb{R}$ -linear with respect to  $V, W$ .
- $\mathcal{V}W = -\mathcal{L}_W V$ .
- $\mathcal{L}_V[W, X]$ .
- (Jacobian identity)  $\mathcal{L}_V[W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$ .
- (Jacobian identity)  $\mathcal{L}_{[V, W]}X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$ .
- $\mathcal{L}_V(fW) = (Vf) \cdot W + f\mathcal{L}_V W$
- Let  $F : M \rightarrow N$  be a diffeomorphism. Then  $F_*(\mathcal{L}_V W) = \mathcal{L}_{F_*(V)} F_*(W)$ .

### 4.3 Commuting vector fields

**Definition 4.8.** We say  $V, W \in \mathfrak{X}M$  **commutes** if  $[V, W] = 0$ .

**Theorem 4.9.** TFAE:

- 1  $V, W$  commutes.
- 2  $W$  is invariant under the flow generated by  $V$ , i.e.  $\theta_{t,*}(W_p) = W_{\theta_t(p)}$
- 3 The flow for  $V, W$  commutes, i.e.  $\theta_t \circ \eta_s = \eta_s \circ \theta_t$  whenever either side is defined or equivalently, whose the domain is compatible.

**Lemma 4.10.** Given  $F \in C^\infty(M, N)$ ,  $V \in \mathfrak{X}M, W \in \mathfrak{X}N$ . Then  $W$  is  $F$ -related to  $V$  if and only if  $\forall t \in \mathbb{R}, \eta_t \circ F = F \circ \theta_t$  on the domain of  $\theta_t$ , which means

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow \theta_t & & \downarrow \eta_t \\ M & \longrightarrow & N \end{array} \text{ commutes.}$$

*Proof.* " $\Rightarrow$ " Let  $\gamma = F \circ \theta^p : J \rightarrow N$  satisfies

$$\gamma'(t) = (F \circ \theta^p)'(t) = F_*((\theta^p)'(t)) = F_*(V_{\theta^p(t)}) = W_{F(\theta^p(t))} = W_{\gamma(t)}$$

So  $\gamma$  is an inetgral curve of  $W$  starting at  $\gamma(0) = F(p)$  i.e.  $F \circ \theta^p = \gamma(t) = \eta^{F(p)}(t)$   
i.e.  $F \circ \theta_t = \eta \circ F$ .

" $\Leftarrow$ " Suppose  $F \circ \theta_t = \eta \circ F$ . Then  $(F \circ \theta^p)(t) = \eta^{F(p)}(t)$ .

Then  $F_*V_p = F_*((\theta^p)'(0)) = (F \circ \theta^p)'(0) = (\eta^{F(p)})'(0) = W_{F(p)}$ . So  $W$  is  $F$ -related to  $V$ .  $\square$

*Proof of Theorem 4.9.*  $2 \Rightarrow 1$ :  $(\theta_{-t})_*(W_{\theta_t(p)}) = W_p$ . So

$$\mathcal{L}_V W = \lim_{t \rightarrow 0} \frac{(\theta_{-t})_*(W_{\theta_t(p)}) - W_p}{t} = 0$$

$1 \Rightarrow 2$ : Let  $X(t) = (\theta_{-t})_*(W_{\theta_t(p)})$ ,  $p \in M$ .

Want to show that  $X(t) = X_p$  for all  $t$ . Suffices to show  $\frac{d}{dt}|_{t=t_0} X(t) = 0$ .

For  $t_0 = 0$ ,  $\frac{d}{dt}|_{t=0} X(t) = (\mathcal{L}_V W)_p = 0$ .

In general, set  $s = t - t_0$ ,  $X(t) = (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))})$ . Then

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} X(t) &= \frac{d}{ds}|_s X(s + t_0) \\ &= \frac{d}{ds}|_s (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))}) \\ &= (\theta_{t_0})_* \frac{d}{ds}|_{s=0} (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))}) \\ &= (\theta_{t_0})_*(\mathcal{L}_V W)_{\theta_{t_0}(p)} \\ &= 0 \end{aligned}$$

$2 \Rightarrow 3$ . For simplicity, assume  $V, W$  are complete.  $F = \theta_s : M \rightarrow M$ . By 2,  $W$  is  $F$ -related to  $W$ . So by the lemma,

$$\begin{array}{ccc}
 M & \xrightarrow{F} & M \\
 \downarrow \theta_t & & \downarrow \eta_t \\
 M & \xrightarrow{F} & M
 \end{array} \text{ commutes.}$$

$\eta_t$  is flow for  $W$ . *i.e.*  $\theta_s \circ \eta_t = \eta \circ \theta_s$

3  $\Rightarrow$  2 is similar. The diagram commutes, so  $W$  is  $F$ -related to  $W$ . □



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