Algebra-1 Note

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Contents

1 Rings and Fields

Definition 1.0.1 (Group)**.** *A group is a set G with an operation $\cdot : G \times G \to G$ such that:*

(1) *(associative law)* $\forall x, y, z \in G, (x \cdot y) \cdot z = x \cdot (y \cdot z)$

(2) *There is an **identity** element* 1 *such that* $1 \cdot x = x \cdot 1 = x, \forall x \in G$

(3) *For any $x \in G$, $\exists$ an **inverse** x^{-1} such that* $x \cdot x^{-1} = x^{-1} \cdot x = 1$

*A group G is called **Abelian/commutative**, if* $\forall x, y \in G, x \cdot y = y \cdot x$

Definition 1.0.2 (Ring)**.** *A ring is a set with two operations $+$ and $\cdot$ such that for $x, y, z \in \mathbb{R}$*

(1) *(commutative law)* $x + y = y + x$

(2) *(associative law)* $(x + y) + z = x + (y + z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$

(3) *(distribution law)* $(x+y)\cdot z = x\cdot z + y\cdot z, \quad z\cdot(x+y) = z\cdot x + z\cdot y$

Moreover, there is a ***zerojjj element*** 0 *and* ***identity element*** $1 \neq 0$ *s.t.* $\forall x \in \mathbb{R}$

(4) $x+0 = 0+x = x = x\cdot 1 = 1\cdot x$

(5) $\exists$ ***opposite*** $-x$ *of* x *such that* $-x+x = 0 = x+(-x)$

$x \in R$ *is called* ***invertible*** *or a* ***unit*** *if* $\exists x^{-1} \in R$ *s.t.* $x^{-1}\cdot x = x\cdot x^{-1} = 1$. *Then* x^{-1} *is called the* ***inverse*** *of* x.
A ring $(R,+,\cdot)$ *is called* ***commutative*** *if* $\forall x,y \in R, xy = yx$ *(say* x ***commutes*** y*)*

Remark 1. *If* $xy = 1$*, then* y *is called a* ***right inverse*** *of* x

Proposition 1 (Cancellation law)**.** *If* $x,y,z \in R, x+y = x+z$*, then* $y = z$. *If* x *is unit,* $x\cdot y = x\cdot z$*, then* $y = z$.

Definition 1.0.3 (Field)**.** *A commutative ring is a* ***field*** *if every nonzero element is a unit*

Remark 2. *Columns of* $C = A\cdot B$ *is combination of columns of* A. *Rows of* $C = C\cdot B$ *is combination of rows of* B.

Definition 1.0.4 (Elementary row operations)**.** *Type 1: Interchange* r_i *with* r_j, $i \neq j$

Type 2: For a unit $u \in \mathbb{R}$*, replace* r_i *with* $u\cdot r_i$.

Type 3: Replace r_i *with* $r_i + cr_j$, $i \neq j, c \in \mathbb{R}$

There are three elementary matrices not mentioned and they are invertible. We write it as $E_{i,j}, E_{i,u}, E_{i,j,c}$ and

$$E_{i,j}^{-1} = E_{i,j},\ E_{i,u}^{-1} = E_{i,u^{-1}},\ E_{i,j,c}^{-1} = E_{i,j,-c}$$

Remark 3. *Elementary row operations on* A *is multiply on the left elementary matrices.*

Definition 1.0.5 (Transpose)**.** *$A = (a_{i,j})$ of $m \times n$ define the* ***transpose*** *of A: $A^T = (a_{j,i})$*

1.1 Gauss elimination

Given an equation $A\vec{x} = \vec{b}$, call $(A|\vec{b})$ the **augment matrix**

Definition 1.1.1. *A matrix is called* ***reduced row echelon*** *matrix if*

1. *If $r_i = 0$, then $r_j = 0$ for $j > i$;*
2. *If $r_i \neq 0$, then the left-most nonzero entry is 1 (let $a_{i,k_i} = 1$), called a* ***pivot****;*
3. *the pivot of r_i is strictly on the right of the pivot of r_{i-1};*

Definition 1.1.2. *A system of linear equations of form of $a_{j,k_i} = 0$ for all $j < i$ is called a* ***reduced row echelon system****;*
Call $x_{k_i}, \cdots, x_{k_r}$ ***principal unknowns****, the others* ***free unknowns***

Proposition 2.
1. *Any system of linear equations can be reduced to be a r.r.e system, preserving the set of solutions.*
2. *Any matrix can be reduced to a r.r.e matrix by row denoting operations.*

Proposition 3. *$A \in M_n(K)$, TAFE:*

1. *A is invertible*
2. *A can be reduced to I_n by row elementary operations*
3. *A is a product of elementary matrices.*

Proposition 4. *$A \in M_n(K)$, TAFE:*

1. *A is invertible*
2. *$Ax = b$ has a unique solution for any b*
3. *$Ax = b$ has a unique solution.*

2 Determinant

Definition 2.0.1. *A **permutation** of* $\{1, \cdots, n\}$ *is a bijective map* $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$ *denote* $(\sigma(1), \cdots, \sigma(n))$.
*A **transposition** is a permutation* $\tau_{i,j}, i \neq j$ *such that* $\tau_{i,j}(i) = j, \tau_{i,j}(j) = i, \tau_{i,j} = k, \forall k \neq i, j$

denote: $S_n = \{\text{permutation of}\{1, 2, \cdots, n\}\}$ is called **symmetric group**

Proposition 5. *Every permutation is a product of transposition.*

Definition 2.0.2. $f(x_{\sigma(1)}, \cdots, f(x_{\sigma(n)})) = (-1)^m f(x_1, \cdots, x_n)$
$(-1)^m$ *is called the **sign** of* σ*, denoted by* $sign(\sigma)$
If $sign(\sigma) = 1$*, then sigma is called an even permutation.*
If $sign(\sigma) = -1$*, then* σ *is called an odd permutation.*

We can define the determinant with three properties.

Proposition 6. *R commutative,* $A, B \in M_n(R) \Rightarrow \det(A \cdot B) = \det(A)\det(B)$.

Proposition 7. $\det A = \det A^t$

Proposition 8. $\det A = \sum_{(t_1, t_2, \cdots, t_n)} sign(t_1, \cdots, t_n) a_{1,t_1, \cdots, n, t_n}$

3 vector space

Definition 3.0.1. *A **vector space** over* k *is an Abelian group* $(V, +)$ *together with a **scalar multiplication*** $k \times V \to V, (c, v) \mapsto c \cdot v$ *such that:*

(1) $1 \cdot v = v$ *for* $1 \in k$*, any* $v \in V$

(2) *(associative law)* $(ab) \cdot v = a \cdot (b \cdot v)$

(3) *(distributive law)*

$(a + b) \cdot v = a \cdot v + b \cdot v$

$a \cdot (u + v) = a \cdot u + a \cdot v$

for all $a, b \in k, u, v \in V$