Complex Analysis HW13

1. (a) ① $\forall$ compact set $K\subset\mathbb{C}$, $\exists\{f_{n_k}\}\subset\{f_n\}$ s.t. $f_{n_k}\rightrightarrows f$ on K.

So $f'_{n_k}\rightrightarrows f'$ on K $\Rightarrow$ $\{f'_n\}$ is normal.

② Take $K=\overline{B(0,R)}\subset\mathbb{C}$, $C=\partial B(0,2R)$. Since $\{f_n\}$ is uniformly bounded on C, ~~take M max~~

let $M=\sup_{n,z\in C}|f_n(z)|$, then $|f'_n(z)|\le\frac{1}{2\pi}\int_C\left|\frac{f(\xi)}{(\xi-z)^2}d\xi\right|\le\frac{M(2R)}{(2R-|z|)^2}$ $(\forall z\in K)$.

So $\{f'_n\}$ is uniformly bounded on K $\Rightarrow$ normal.

(b) No. Let $f_n(z)=n(z^2-n)$, then $f_n\rightrightarrows\infty$ on any compact set.

However, $\rho(f'_n)=\frac{2|2n|}{1+|2nz|^2}$ is not bounded near $z=0$. By Marty thm, $\{f'_n\}$ is not normal.

2. (a) $\frac{f'}{f}$ is analytic on Ω. Ω simply connected $\Rightarrow$ $g(z)=\int_\gamma\frac{f'}{f}dz$ where $\gamma: z_0\to z$ is any smooth path from z_0 to z for Ω is path-connected.

Note that $g'=\frac{f'}{f}$, $(e^{-g}f)'=e^{-g}(f'-fg')=0$, $\exists c\in\mathbb{C}^\times$ s.t. $f=ce^g$ for Ω is connected.

Since $g(z_0)=0$, $c=f(z_0)=e^\alpha$. Let $h(z)=\alpha+g$, then $f=e^h$ and h is analytic.

(b) if $f=e^{g_1}=e^{g_2}$, then $g=g_1-g_2$ satisfies $e^g\equiv1$ $\Rightarrow$ $g\in2\pi i\mathbb{Z}$

Since g is analytic, $\mathbb{Z}$ is discrete and Ω is connected, $\exists k\in\mathbb{Z}$ s.t. $g_1-g_2=2\pi ik$.

3. Let $\mathcal{G}=\{g=\frac{f-1}{f+1}: f\in\mathcal{F}\}$. From $\operatorname{Re}f>0$ we know $\mathcal{G}$ is a family of analytic functions on Ω.

From $\operatorname{Re}f>1$ we know $|g|<1$ $(\forall g\in\mathcal{G})$ $\Rightarrow$ $\mathcal{G}$ is uniformly bounded on compact set ~~$\Rightarrow\mathcal{F}$ normal~~ $\Rightarrow\mathcal{G}$ normal

Since $\frac{2|f'|}{1+|f|^2}\le\frac{4|f'|}{1+|f|^2}=2|g'|$, $\mathcal{G}$ normal $\Rightarrow\{g'\in\mathcal{G}\}$ normal. $\mathcal{F}$ is normal w.r.t. chordal metric.

Claim: $\mathcal{F}$ is uniformly bounded on every compact subset of Ω $\Leftrightarrow$ $\exists w\in\Omega$ s.t. $\{f(w): f\in\mathcal{F}\}$ is bounded.

$\Rightarrow$: $\mathcal{F}$ is uniformly bounded near w $\Rightarrow$ $\{f(w): f\in\mathcal{F}\}$ is bounded.

$\Leftarrow$: Let $\mathcal{H}=\{h=e^{-f}: f\in\mathcal{F}\}$, then $\mathcal{H}$ is uniformly bounded on compact sets $\Rightarrow$ $\mathcal{H}$ is normal.

So $\forall$ compact $K\subset\Omega$, $\exists\{h_n\}\subset\mathcal{H}$ s.t. $h_n\rightrightarrows h$ on K for some analytic function h.

Because $\{f(w): f\in\mathcal{F}\}$ is bounded, $\{h_n(w)\}$ is ~~bounded~~ contained in an open subset of $\mathbb{C}^\times$.

Hence $h(w)\neq0$, say $h(w)=e^\alpha$.

Since e^z is Lipschitz continuous on K, $\exists\{f_{n_k}\}\subset\{f_n\}$ s.t. $f_{n_k}\rightrightarrows\log h+2\pi iN$ for an $N\in\mathbb{Z}$.

N exists for $\mathcal{F}$ is normal.

Thus $\mathcal{F}$ is uniformly bounded ~~on every~~ on K.

4. Note that $\{f_n\}$ is normal.

$\forall$ compact $K \subset \Omega$, $\exists \{f_{n_k}\} \subset \{f_n\}$ s.t. $f_{n_k} \rightrightarrows f$ on K.

If $f_n \not\rightrightarrows f$, then $\exists \{f_{n_j}\} \subset \{f_n\}$ s.t. $\sup_{z\in K} |f_{n_j}(z) - f(z)| > \varepsilon$. for some $\varepsilon > 0$.

Since $\{f_{n_j}\}$ is normal, $\exists \{f_{n_{j_i}}\} \subset \{f_{n_j}\}$ s.t. $f_{n_{j_i}} \rightrightarrows g$ on K.

Let $a \in A$ be an accumulation point of A in Ω, then we may enlarge K s.t. $a \in K^\circ$.

Take $\{a_n\} \subset A$ s.t. $a_n \to a$, then $f(a_n) = g(a_n) \Rightarrow f = g$ on K.

However, $\sup_{z\in K} |f_{n_j}(z) - f(z)| > \varepsilon \Rightarrow g \neq f$, a contradiction.

Thus $f_n \rightrightarrows f$ on K.

5. Let $g(z) = \overline{f(\bar{z})}$, then g is analytic on Ω.

Since g is bijective, $g(z_0) = \overline{f(\bar{z_0})} = 0$, $g'(z_0) = \overline{f'(\bar{z_0})} > 0$, by Riemann Mapping Theorem, $g = f$.

If $\exists\, w_1, w_2 \in \Omega^+$ s.t. $\operatorname{Re} f(w_1) > 0$, $\operatorname{Re} f(w_2) < 0$, let γ be any smooth curve connecting w_1, w_2 on Ω^+,

then by continuous $\exists\, w_3$ on γ s.t. $f(w_3) \in \mathbb{R}$.

However, $f(\bar{w_3}) = \overline{f(w_3)} = f(w_3)$ implies $w_3 = \bar{w_3} \Rightarrow w_3 \in \mathbb{R}$, a contradiction.

Thus $f(\Omega^+) \subset \{\operatorname{Im} z < 0\}$ or $f(\Omega^+) \subset \{\operatorname{Im} z > 0\}$.

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