

**Homework 7**

Lin Zejin

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**Problem 1.**

$$\begin{aligned} y(x+h) - y(x) - hf(x+h, y+hf(x, y)) &= hy'(x) + \frac{1}{2}h^2y''(x) - h(f(x, y) + h\frac{\partial f}{\partial x} + hf(x, y)\frac{\partial f}{\partial y}) + O(h^3) \\ &= \frac{h^2}{2}y''(x) - h^2y''(x) + O(h^3) \\ &= -\frac{h^2}{2}y''(x) + O(h^3) \end{aligned}$$

**Problem 2.**

$$k_1 = \frac{dy}{dx}$$

$$\begin{aligned} k_2 &= f(x + c_2h, y + ha_{21}k_1) \\ &= f(x, y) + hc_2\frac{\partial f}{\partial x} + k_1ha_{21}\frac{\partial f}{\partial y} + \left( c_2^2h^2\frac{\partial^2 f}{\partial x^2} + a_{21}^2k_1^2h^2\frac{\partial^2 f}{\partial y^2} + 2c_2a_{21}h^2\frac{\partial^2 f}{\partial x\partial y} \right) + O(h^3) \end{aligned}$$

$$\begin{aligned} k_3 &= f(x + c_3h, y + ha_{31}k_1 + ha_{32}k_2) \\ &= f(x + c_3h, y + ha_{31}k_1) + ha_{32}k_2\frac{\partial f}{\partial y} + \frac{h^2a_{32}^2k_2^2}{2}\frac{\partial^2 f}{\partial y^2} + O(h^3) \\ &= f(x, y) + hc_3\frac{\partial f}{\partial x} + k_1ha_{31}\frac{\partial f}{\partial y} + \left( c_3^2h^2\frac{\partial^2 f}{\partial x^2} + a_{31}^2k_1^2h^2\frac{\partial^2 f}{\partial y^2} + 2c_3a_{31}h^2\frac{\partial^2 f}{\partial x\partial y} \right) \\ &\quad + ha_{32}k_2\frac{\partial f}{\partial y} + \frac{h^2a_{32}^2k_2^2}{2}\frac{\partial^2 f}{\partial y^2} + O(h^3) \end{aligned}$$

where  $\tilde{k}_2 = k_2/h$

Noticed that

$$y'''(x) = \left( \frac{\partial f}{\partial x} + y'(x)\frac{\partial f}{\partial y} \right)' = \frac{\partial^2 f}{\partial x^2} + 2y'\frac{\partial^2 f}{\partial x\partial y} + (y')^2\frac{\partial^2 f}{\partial y^2} + y''\frac{\partial f}{\partial y}$$

Then compared each term,  $b_1k_1 + b_2k_2 + b_3k_3 = y' + \frac{1}{2}hy'' + \frac{1}{6}h^2y''' + O(h^3)$  if

$$\begin{aligned} b_1 + b_2 + b_3 &= 1 \\ b_2c_2 + b_3c_3 &= \frac{1}{2} \\ b_2c_2^2 + b_3c_3^2 &= \frac{1}{3} \\ b_3c_2a_{32} &= \frac{1}{6} \end{aligned}$$

**Problem 3.**

$$\begin{aligned} T_{n+3} &= y(x_{n+3}) + \alpha(y(x_{n+2}) - y(x_{n+1})) - y(x_n) - \frac{1}{2}(3 + \alpha)h[f(x_{n+2}, y(x_{n+2})) + f(x_{n+1}, y(x_{n+1}))] \\ &= \end{aligned}$$