Complex Analysis HW 1

1. $|z-w|^2-|1-\bar z w|^2 = -(1-|z|^2)(1-|w|^2)$. So if $|z|=1$ or $|w|=1$ then $\left|\frac{z-w}{1-\bar z w}\right|=1$.

When $|z|=|w|=1$, $1-\bar z w\neq 0 \Rightarrow z\neq w$.

2. let $z=(z_1,\dots,z_n)$, $w=(w_1,\dots w_n)\in\mathbb{C}^n$, then

$$\sum_{j<k}|z_j\overline{w_k}-z_k\overline{w_j}|^2=\frac12\sum_{j,k}|z_j\overline{w_k}-z_k\overline{w_j}|^2=\frac12\sum_j\langle z_j\bar w-\overline{w_j}z,\ z_j\bar w-\overline{w_j}z\rangle$$

$$=\frac12\sum_j|z_j|^2|\bar w|^2+|w_j|^2|z|^2-\overline{z_jw_j}\langle\bar w,z\rangle-w_jz_j\langle z,\bar w\rangle$$

$$=\frac12\left(|\bar w|^2|z|^2+|w|^2|z|^2-\langle z,\bar w\rangle\langle\bar w,z\rangle-\langle\bar w,z\rangle\langle z,\bar w\rangle\right)$$

$$=|w|^2|z|^2-|\langle z,\bar w\rangle|^2=\sum_{j=1}^n|z_j|^2\sum_{j=1}^n|w_j|^2-\left|\sum_{j=1}^n z_jw_j\right|^2.$$

3. $|1-\bar z w|^2-|z-w|^2=(1-|z|^2)(1-|w|^2)>1 \Rightarrow \left|\frac{z-w}{1-\bar z w}\right|<1$

4. $\Rightarrow$: $2|c|=|z+a|+|z-a|\geq|z+a+a-z|=2|a|$.

$\Leftarrow$: If $a=0$, take $z=c$. if $a\neq0$, take $z=\left|\frac{c}{a}\right|a$

Note that $2|c|=|z+a|+|z-a|\geq 2|z| \Rightarrow |z|\leq|c|$, and $z=\begin{cases}c & a=0\\ \left|\frac{c}{a}\right|a & a\neq0\end{cases}$ s.t. $|z|=|c|$.

$(2|c|)^2=(|z+a|+|z-a|)^2\leq 2(|z+a|^2+|z-a|^2)=4(|z|^2+|a|^2)\Rightarrow |z|\geq\sqrt{|c|^2-|a|^2}$

the equality holds when $z=\begin{cases}c & a=0\\ \frac{ai}{|a|}\sqrt{|c|^2-|a|^2} & a\neq0\end{cases}$. Hence $|z|_{max}=|c|$, $|z|_{min}=\sqrt{|c|^2-|a|^2}$.

5. $\Rightarrow$: let ω be a primitive 3^{rd} root of unity, then $\exists\, z,r\in\mathbb{C}$, s.t. $z_i=z+r\omega^i$.

So $\sum z_i^2-\sum z_iz_j=\sum(z+r\omega^i)^2-\sum(z+r\omega^i)(z+r\omega^j)=0$.

$\Leftarrow$: Note that the equation holds for $(z_1,z_2,z_3)\to(z_1-z_1,z_2-z_1,z_3-z_1)$, we may assume $z_1=0$.

If $z_2=0$, then $z_3=0\Rightarrow$ degenerate equilateral triangle.

If $z_2\neq0$, let $z=\frac{z_3}{z_2}$ we get $1+z^2=z\Rightarrow z=e^{\pm\frac{\pi i}{3}}$, i.e. $z_3=z_2e^{\frac{\pm\pi i}{3}}\Rightarrow z_1,z_2,z_3$ are vertices of an equilateral triangle.

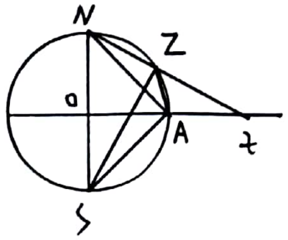
6. Consider the case $(w_1,w_2,0)$. Denote the center by w and the radius by R, then

$|w|^2=|w-w_1|^2=|w-w_2|^2=R^2$, i.e. $\begin{pmatrix}\overline{w_1} & w_1\\ \overline{w_2} & w_2\end{pmatrix}\begin{pmatrix}w\\ \bar w\end{pmatrix}=\begin{pmatrix}|w_1|^2\\|w_2|^2\end{pmatrix}\Rightarrow\begin{pmatrix}w\\ \bar w\end{pmatrix}=\frac{1}{\overline{w_1}w_2-w_1\overline{w_2}}\begin{pmatrix}w_2&-w_1\\-\overline{w_2}&\overline{w_1}\end{pmatrix}\begin{pmatrix}|w_1|^2\\|w_2|^2\end{pmatrix}$

Note that $(w_1,w_2,0)$ forms a triangle, $|\mathrm{Arg}\,w_1-\mathrm{Arg}\,w_2|=0$ or $\pi\Rightarrow \mathrm{Im}\,\overline{w_1}w_2\neq0$.

Now let $w_1=z_1-z_3$, $w_2=z_2-z_3$ we get $z=w+z_3$.

7.



Since $\angle NzA=\frac{\pi}{2}-\angle SNZ=\angle NSZ=\angle NAZ$, $\triangle NZA\sim\triangle NAz$.

So $NZ\cdot Nz=NA^2=2$. Similarly $NZ'\cdot Nz'=2$.

From $\angle ZNz=\angle Z'Nz'$ and $\frac{NZ}{NZ'}=\frac{Nz'}{Nz}$ we get $\triangle NZZ'\sim\triangle Nz'z$.

Hence $d(z,z')=ZZ'=|z-z'|\cdot\frac{NZ}{Nz'}=|z-z'|\frac{2}{Nz\cdot Nz'}=\frac{2|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}$.