Complex Analysis HW 10

1、 let $f = \alpha + i\beta$, then $\Delta(u \circ f) = u_x \Delta\alpha + u_y \Delta\beta + u_{xx}\alpha_x + u_{yy}\beta_y + u_{xy}(\alpha_y + \beta_x) = 0$.

2、 ① $\exists\, \alpha, \beta \in \mathbb{C}$ s.t. $\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta = \alpha \log r + \beta$ with $0 < r < \rho$.

Since u is bounded near 0, $\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta$ is bounded when r is small. So $\alpha = 0$.

let $U(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta$, then $0 = \alpha = r\frac{\partial U}{\partial r} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial u(re^{i\theta})}{\partial r}\, r\,d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} (u_x r\cos\theta + u_y r\sin\theta)\,d\theta = \frac{1}{2\pi}\int_{|z|=r} u_x\,dy - u_y\,dx = \frac{1}{2\pi}\int_{|z|=r} {}^*du$.

Thus $\forall$ closed curve γ in $B(0,\rho)\setminus\{0\}$, $\int_\gamma {}^*du = 0 \Longrightarrow V(x,y) = \int_{(\frac{\rho}{2},0)}^{(x,y)} {}^*du$ is well-defined.

let $f = u + iv$, then f is analytic. Note that $\operatorname{Re} f$ is bounded, by HW7.4 f has a removable singularity at $z=0$. Extend f to $\tilde{f}$ on $B(0,\rho)$, define $\tilde{u} = \operatorname{Re}\tilde{f}$ we have $\tilde{u} = u$ on $B(0,\rho)\setminus\{0\}$.

② We may assume $\rho = 2$. Define $g(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u(e^{i\theta})\,d\theta$, then g is harmonic on $\mathbb{D}$ and $g(z) = u(z)$ on $\partial\mathbb{D}$, by Schwarz's theorem.

$\forall \varepsilon > 0$, let $h^+(z) = g(z) - u(z) + \varepsilon \log|z|$, then h^+ is harmonic on $\mathbb{D} - \{0\}$, and $h^+ = 0$ on $\partial\mathbb{D}$.

Since u, g are bounded in $\mathbb{D}$, $h^+(z) \to -\infty$ as $|z| \to 0$.

By Maximum Principle, $h^+ \le 0$ on $\mathbb{D} - \{0\}$. let $\varepsilon \to 0$, $g \le u$ on $\mathbb{D} - \{0\}$.

Similarly, define $h^- = g - u - \varepsilon \log|z|$ we can show that $g \ge u \Rightarrow g \equiv u$ on $\mathbb{D} - \{0\}$.

Thus g is an extension of u to $z = 0$.

3. (Hadamard Three-Circle Theorem)

Define $g(z) = \log|f(z)| + c\log|z|$ on $\{r_1 < |z| < r_2\} \setminus \{f^{-1}(0)\}$, then g is harmonic.

Note that $g \to -\infty$ as $z \to$ zero of f. by Maximum principle

$g(z) \le \max_{|z|=r_1,r_2} \{g(z)\} = \max\{\log M(r_1) + c\log r_1, \log M(r_2) + c\log r_2\}$

Take $c = \dfrac{\log M(r_2) - \log M(r_1)}{\log r_1 - \log r_2}$, then $\log M(r) + c\log r \le \log M(r_1) + c\log r_1$, which is

$\log M(r) \le \alpha \log M(r_1) + (1-\alpha)\log M(r_2)$. (The proof holds even if $M(r_1)M(r_2) = 0$).

4. let $T: \mathbb{D} \to \mathbb{H}$, $z \mapsto \frac{z+i}{iz+1}$, then $T: \partial\mathbb{D} \to \mathbb{R}$.

$P_U(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-\xi)^2+y^2} U(\xi)d\xi = \cancel{\frac{1}{\pi}\int_S \operatorname{Im}\frac{U \circ Tw}{Tw - z} dTw} = \frac{1}{\pi}\int_{\partial\mathbb{D}} \frac{y}{(x-Tw)^2+y^2} U(Tw)\, dTw$

$= \frac{1}{\pi}\int_0^{2\pi} \frac{y}{(x-Te^{i\theta})^2+y^2} U\cdot(Te^{i\theta})\, dTe^{i\theta} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|T^{-1}z|^2}{|e^{i\theta} - T^{-1}z|^2} V(\theta)\, d\theta$, where $V(\theta) := U(Te^{i\theta})$.

By Schwarz's theorem, $F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} V(\theta)\, d\theta$ is harmonic and $V(\theta) = F(e^{i\theta})$.

So $P_U(z) = F \circ T^{-1}(z)$ is also harmonic with $P_U = U$ on $\mathbb{R}$.

5. ① let $h_-(z) = u - P_u - \varepsilon \operatorname{Im}\sqrt{iz}$, where we define $\sqrt{z}$ on $\mathbb{C} - \mathbb{R}_{\ge 0}$. Then h_- is harmonic on $\mathbb{H}$.

Since u is bounded on $\mathbb{H}$, so is P_u. When $z \to \infty$ in $\mathbb{H}$, $\operatorname{Im}\sqrt{iz} \to +\infty$.

By Maximum Principle, $h_-(z) \le \max_{z\in\mathbb{R}} \{h_-(z)\}$.

$\forall x \in \mathbb{R}$, $u(x) = P_u(x)$ and $\operatorname{Im}\sqrt{ix} > 0 \Rightarrow h_-(z) \le 0$. let $\varepsilon \to 0$ we get $u \le P_u$.

Similarly consider $h_+ = u - P_u + \varepsilon \operatorname{Im}\sqrt{iz}$ we get $u \ge P_u \Rightarrow u = P_u$.

② let $v = u - P_u$, then $v = 0$ on $\mathbb{R}$. Define $\tilde{v}$ on $\mathbb{C}$ with $\tilde{v}(z) = \begin{cases} v(z) & z \in \mathbb{H} \\ -v(\bar z) & z \in \{\operatorname{Im} z < 0\} \end{cases}$, then by

Reflection Principle, $\tilde{v}$ is a harmonic function on $\mathbb{C}$, which is bounded.

We may find an entire function f s.t. $\tilde{v} = \operatorname{Re} f \Rightarrow f$ is constant $\Rightarrow \tilde{v}$ is constant.

$\tilde{v} = 0$ on $\mathbb{R} \Rightarrow \tilde{v} \equiv 0 \Rightarrow u = P_u$.

6. let $f = u + iv$, then by Schwarz's formula, $f(z) = \frac{1}{2\pi i}\int_{|w|=r} \frac{w+z}{w-z} \frac{u(w)}{w} dw$ $(\forall |z| < r)$.

Since $\lim_{z\to\infty} \frac{u}{z} = 0$, $\forall \varepsilon > 0$, $\exists R > 0$ s.t. $|u(z)| < \varepsilon|z|$ $(\forall |z| > R)$. Fix $z \in \mathbb{C}$, take $r > R + 3|z|$, then

$|f(z)| < \frac{1}{2\pi}\int_{|w|=r} \frac{|w+z|}{|w-z|} \varepsilon |dw| \le \frac{\varepsilon}{2\pi}\int_0^{2\pi} \frac{r+|z|}{r-|z|} r\, d\theta \le 2\varepsilon r$.

Hence $|f'(z)| \le \frac{1}{2\pi}\int_{|w|=r} \frac{|f(w)|}{|w-z|^2}|dw| \le \frac{1}{2\pi}\cdot 2\pi \cdot \frac{2\varepsilon r}{(r-|z|)^2} r = \frac{2\varepsilon}{(1-\frac{|z|}{r})^2}$.

let $r \to \infty$ we get $|f'(z)| \le 2\varepsilon$. let $\varepsilon \to 0$ we have $f'(z) = 0$ $(\forall z \in \mathbb{C})$.

Thus f is constant.

7. ① let $z^* = \bar{z}^{-1}$, then $z^* = z$ for $z \in \partial\mathbb{D}$ and $f(z) \in \partial\mathbb{D}$ for $z \in \partial\mathbb{D}$.

Note that f has finitely many zeros, by multiplying $\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)^{-1}$ where $\underset{|\alpha|<1}{f(\alpha)=0}$ we may assume f has no zero.

Define $\tilde{f}(z) = \begin{cases} f(z) & z \in \mathbb{D} \\ f(z^*)^* & z \in \mathbb{D}^* \end{cases}$, then by Reflection Principle, $\tilde{f}(z)$ is analytic on $\hat{\mathbb{C}}$ $\Rightarrow \tilde{f}$ is constant.

Hence f is rational.

② We may assume $f \neq 0$. By Maximum principle, $|f(z)| \leq 1$ $(\forall |z| \leq 1)$, $\left|\frac{1}{f(z)}\right| \leq 1$ $(\forall |z| \leq 1)$

$\Rightarrow |f(z)| = 1$ $(\forall |z| \leq 1)$. So f is a constant.

③ let $T: \mathbb{H} \to \mathbb{D}$, $z \mapsto \frac{z-i}{z+i}$, $g(z) := T^{-1} \circ f \circ T : \mathbb{H} \to \mathbb{C}$, then g is meromorphic on $\mathbb{H}$ with $g(\mathbb{R}) \subset \mathbb{R} \cup \{\infty\}$. By Reflection Principle, g extends to meromorphic function $\tilde{g}: \mathbb{C} \to \hat{\mathbb{C}}$.

let $S: \mathbb{H} \to \mathbb{D}$, $z \mapsto \frac{1-iz}{1+iz}$, $h(z) = T^{-1} \circ f \circ S : \mathbb{H} \to \mathbb{C}$, then h is meromorphic on $\mathbb{H}$ with

$h(\mathbb{R}) \subset \mathbb{R} \cup \{\infty\} \Rightarrow$ extends to $\tilde{h}: \mathbb{C} \to \hat{\mathbb{C}}$.

Note that $h(z) = g(\frac{1}{z})$ ~~for $z \neq 0$~~, g is meromorphic on $\hat{\mathbb{C}}$ $\Rightarrow g$ is rational $\Rightarrow f$ is rational.