Tsinghua University

DISCRETE OPTIMISTIC

Fall 2024

Homework 1

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- **Collaborators:** I finish this homework by myself.

Problem 1. (a) When OPT $\geq c$, assume with $\frac{1}{T}$ algorithm A outputs a solution of value at least s. $T \in O(poly(n))$ Run algorithm A for $T \cdot n$ iterations. Then with $(1-\frac{1}{T})^{Tn} < e^{-n}$ probability, the algorithm A outputs a solution of value less than s.

So with at least $1-e^{-n}$ probability, the algorithm A outputs a solution of value at least s.

(b)

$$s = \mathbb{E}[outputs] \leq \Pr[outputs \geq s - \frac{1}{n^a}] \cdot poly(n) + (1 - \Pr[outputs \geq s - \frac{1}{n^a}]) \cdot (s - \frac{1}{n^a})$$

Then

$$\Pr[outputs \geq s - \frac{1}{n^a}] \geq \frac{\frac{1}{n^a}}{poly(n) - s + \frac{1}{n^a}} = \frac{1}{n^a(poly(n)-s)+1}$$

Here we end the proof.

Problem 2.

Problem 3. (a) Let $k = c$, $U = \{1, 2, \cdots, c\}^q$ where q large enough. Introduce

$$S_{i,b} = \{e \in U : e_i = b\},\ i \in [q], b \in [c]$$

Choose $S_{i,t}, 1 \leq t \leq c$ and the coverage is 1.

$x^*_{i,b} = \frac{1}{q}$ will also achieves coverage 1.

Then we cover each $j \in U$ with probability

$$1 - (1 - \frac{1}{c})^c$$

So the expected coverage of rounding is

$$1 - (1 - \frac{1}{c})^c$$

AS c large enough, the expected coverage of rounding is $1 - \frac{1}{e}$.

(b) With instance $k = c$, $U = \{0, 1\}^q$, $n = 2^q$ and

$$S_{i,b} = \{e \in U : e_i = b\},\ i \in [q], b = 0, 1$$

The LP solution $x^*_{i,b} = \frac{1}{q}$.
$\alpha x^*_{i,b} = \frac{(1-\epsilon)\ln n}{q} = (1-\epsilon)\ln 2 < \ln 2$.
Then

$$\Pr[j \text{ is covered}] = 1 - (1-\alpha x^*_{i,b})^q < 1 - (1-\ln 2)^q < 1 - (2^{-1.5})^{\log_2 n} = 1 - n^{-3/2}$$

So

$$\Pr[U \text{ is all covered}] < (1-n^{-3/2})^n < 1 - n^{-\frac{1}{2}+\epsilon}$$

as n large enough. So the randomized rounding algorithm may not be able to find a feasible solutionwith probability at least $n^{-\frac{1}{2}+\epsilon}$.

Problem 4. (a)

(b) No, since the rounding algorithm gives a solution with expected value large than $(1-\frac{1}{e})$LP. So

$$\mathrm{OPT} \geq (1-\frac{1}{e})\mathrm{LP}$$

always holds.