Tsinghua University

Discrete Optimistic

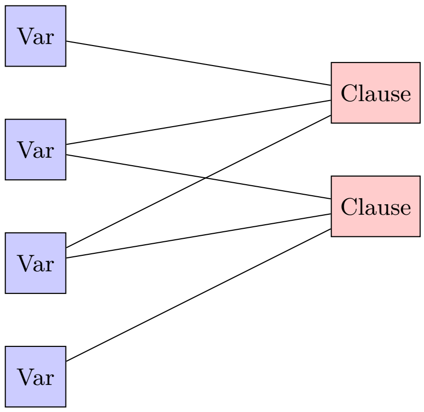
Fall 2024

**Homework 1**

Lin Zejin May 15, 2025

- **Collaborators:** I finish this homework by myself.

**Problem 1.** (a) Reduce from the instance of MAX-E3SAT-6.




Variables $x_i$ have $\sigma(x_i) \in \{0,1\}$ and Clauses $c_i = x_{j_i}^1 \wedge x_{j_i}^2 \wedge x_{j_i}^3$ have $\sigma(c_i) \in [7]$ to represent the state of $c_i$.

Therefore, constraint is naturally induced.

In the instance of MAX-E3SAT, the radio of $|U|$ and $|V|$ is 2. So this is a regular Label-Cover Game for $K = 2, L = 7$ and $|V| = 2|U|$.

In the lecture we have proved that this is an instance of $\mathrm{MAX-LC}_{1,1-\epsilon}$ for some $\epsilon$.

So $\mathrm{MAX-LC}_{1,1-\epsilon}$ is NP-Hard.

(b) We actually can construct another graph induced by (a).

We add $\bar{x}_i$ to the graph in (a) and add the induced constraints from $c_i$ contains variable $x_i$ to $\bar{x}_i$.

Here the Label-Cover Game is regular and symmetric.

Then for the $\mathrm{MAX-E3SAT-6}_{1,1-\epsilon}$ instance, the completeness is trivial.

Now we prove the soundness. That's because, if $\mathrm{OPT}_{\mathrm{MAX-E3SAT-6}} \leq 1-\epsilon$, consider any $\sigma : U \to \{0,1\}, V \to [7]$. At least $(1-\epsilon)|V|$ clauses are not satisfied by $\sigma|_U$. For each clause, there exists at least one variable $x_i/\bar{x}_i$ such that do not satisfy the constraint.

So Verifier rejects with probability at least $(1-\epsilon)|V|/2|V| = (1-\epsilon)/2$. So the soundness property holds if we set $\epsilon' = \frac{1+\epsilon}{2}$.

So we prove that $\mathrm{GAP-LC}(K,L)_{1,1-\epsilon}$ is NP-Hard for some $\epsilon$ and $K, L$ even if the graph is regular and symmetric.

By Raz' Paralled Repetition Theorem, we can reduce an instance of $\mathrm{GAP}-\mathrm{LC}(K,L)_{1,\delta}$ to the instance of $\mathrm{GAP}-\mathrm{LC}_{1,\exp(-\Omega(\frac{\delta^3 t}{\log t}))}$. Therefore, we finally prove that for any $\eta > 0$, there exists $K, L$ such that $\mathrm{GAP}-\mathrm{LP}(K,L)_{1,\eta}$ is NP-Hard.

**Problem 2.** (a) For a regular Label-Cover problem $G = (U, V, E)$ that every veritce in $U$ matches $k$ vertices in $V$, $|U| = |V| = n$ , consider the $k$-uniform hypergraph $H = (V', E')$ where $V' = E$ and $k$-tuples are all $[(u, v_1), (u, v_2), \cdots, (u, v_k)]$ for $(u, v_i) \in E$. $[L']$ now represents the value of $(u, v_i)$, *i.e.* $[L'] = [L] \times [K]$. $[K] = [k + 1]$.

The maps are defined as: For the labeling $\sigma : [V] \to [L] \times [K]$, $\sigma(u, v_i) = (l, k)$. If $\pi_{(u,v_i)}(k) = l$ is matching in Labek-Cover problem, then we let $\pi_e^i(\sigma(u, v_i)) = k + 1$. Otherwise, if $(l, k)$ does not satisfy the constraint, then we let $\pi_e^i(\sigma(l, k)) = i$.

So the constraint is weakly satisfied iff at least two edges in the $k$-tuples are satisfied in the constraint before. Also, the constraint is strongly satisfied iff all edges in the $k$-tuples are satisfied.

Completeness is trivial since if there is some label in the Label-Cover Game satisfy all constraint, then it can be naturally induced in the hypergraph.

Soundness is because: Assume $\mathrm{OPT} \ge \epsilon$ in $k$-ary-Consistent-Labeling problem. Then we choose all edges $(u_i, v_j)$ that are satisfied in the Label-Cover Game, denoted as $S$. There are at least $2\epsilon n$ edges. Now we label each $u_i, v_j$ one by one.

Since the graph $G$ is regular, at most $2k - 1$ edges in $S$ have common vertice with an edge in $S$.

So each time we choose an arbitrary $e = (u, v) \in S$, label it with the label in $k$-ary-Consistent-Labeling and then we remove those edges in $S$ who intersects with $e$.

In the end, for those vertices that have not been labeled yet, label it randomly.

Then at least $\frac{2\epsilon n}{2k} = \frac{\epsilon}{k} n$ edges are satisfied in Label-Cover-Game.

Therefore $\mathrm{OPT} \ge \frac{\epsilon}{k}$ for Label-Cover Game.

As a result, if $\mathrm{OPT} \le \eta$ in Label-Cover Game, then $\mathrm{OPT} \le k\eta$ in $k$-ary-Consistent-Labeling problem.

Since $\mathrm{MAX}-\mathrm{LC}_{1,\eta}$ is NP-Hard, to distingish instance with strong value 1 and weak value less than $k\eta$ $k$-ary-Consistent-Labeling problem is NP-Hard $\forall \eta > 0$.

Here we end the proof.

(b)

**Problem 3.** Consider all values $d(r, v) \pmod{\frac{1}{2}}$. They divide $[0, \frac{1}{2})$ into $|V| + 1$ pieces of interval.(including the interval $[v, v]$ if exists) If we choose $\theta$ in each interval, edges that will be removed are the same, so the cost is the same.

As a result, we can try $\theta$ in each interval and find the minimum cost. This will be less than 2OPT.

**Problem 4.** (a) If a connected component has diameter at most $k$ in the $(10, 0.1, 1, 1)$-expandar $G$, we prove that it has at most $10^k$ vertices.

By induction, $k = 1$ is trivial. Assume $k - 1$ holds for it. Assume subgraph $G'$ has the maxmimum number of vertices. There isn't any vertex in $G'$ that has distance less than $k - 1$ with each vertex in $G'$ and also connects with other vertex $u$ outside. Otherwise, $u$ can be added to $g'$, which causes contradiction with the maximum property. Then for $k$, any vertex in the graph with diameter $k-1$ has degree 10 so at most $10^k$ vertices are connected to the graph. Since any vertex beyond $G'$ has distance larger than $k$ with some vertices in $G'$ as we proved before, the expanded graph has at most $10^k$ vertices.

So each connected component has at most $10^{1/2\log_{10} n} = n^{1/2}$ vertices in this problem. As $n$ large enough, $n^{1/2} < 0.1n$. For those connected components $S_1, \cdots, S_k$, removed edges are

$$|\partial S_1 \cup \partial S_2 \cup \cdots \partial S_k| = \frac{1}{2}\sum_{t=1}^{k} |\partial S_t| \geq \frac{1.01}{2}\sum_{t=1}^{k} |S_t| > 0.5n$$

So we must have deleted $\Omega(n)$ edges.

Now we set the pair $(s_i, t_i)$ to be all $(u, v)$ where $u, v \in G$ and distance between $u$ and $v$ is $k$.

Then for any possible connected component in multicut, vertices $u, v$ in it have distance is less than $k$.

For a $(10, 0.1, 1.1)$-expandar graph, by (a) we removed at least $\frac{1}{2}n$ if $k = \frac{1}{2}\log_{10} n$.

However, in LP case, we can set $x_e = \frac{1}{k}$ for any edge $e$. Then the cost will be

$$\frac{1}{k} \cdot |E| = \frac{5}{k}|V| = \frac{5n}{k}$$

So the integral gap is $\Omega(\log n)$.

**Problem 5.**

**Problem 6.**

**Problem 7.**

**Problem 8.** $f : \{\pm 1\}^n \to \mathbb{R}$ is a linear combination of function $f : \{\pm 1\}^n \to \{\pm 1\}$, which can be written in the form of linear combination of Fourier base functions:

$$f(x) = \sum_{S \subset [n]} \hat{f}(S)\chi_S(x)$$

where $\chi_S(x) = \prod_{i \in S} x_i$ is a multilinear polynomial.

So it is expressible as a multilinear polynomial.

The uniqueness is because, if there is some multilinear polynoimal $g$ such that $g(x) = f(x), \forall x \in \{\pm 1\}$. Then using Parserval's Theorem we obtain that

$$\sum_{S \subset [n]} (f \hat{-} g)(S)^2 = \mathbb{E}_{\vec{x} \sim \{\pm 1\}^n}(f(\vec{x}) - g(\vec{x}))^2 = 0$$

So $f - g = \sum_{S \subset [n]} (f \hat{-} g)(S)\chi_S = 0$.

**Problem 9.**

$$\langle f, g\rangle = \left\langle \sum_{S\subset[n]} \hat{f}(S)\chi_S, \sum_{S\subset[n]} \hat{f}(S)\chi_S \right\rangle = \sum_{S_1,S_2\subset[n]} \hat{f}(S_1)\hat{g}(S_2)\left\langle \chi_{S_1}, \chi_{S_2}\right\rangle = \sum_{S\subset[n]} \hat{f}(S)\hat{g}(S)$$

However, if we let $f = \chi_{\{x\}}, g = \chi_{\{y\}}, h = \chi_{\{x,y\}}$, then

$$\mathbb{E}_{\vec{t}}\chi_{\{x\}}(t)\chi_{\{y\}}(t)\chi_{\{x,y\}}(t) = \mathbb{E}_{\vec{t}}t_x^2t_y^2$$

But

$$\hat{f}(S)\hat{g}(S)\hat{h}(S) \equiv 0, \forall S \subset [n]$$

due to they are Fourier basis.