Tsinghua University

Numerical Analysis

Fall 2024

Homework 7

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Problem 1. 1、注意到

$$E(1) = E(x^1) = E(x^2) = 0, E(x^3) = \frac{2}{9} \neq 0$$

所以代数精度是 2

2、

$$E(1) = E(x^1) = E(x^2) = E(x^3) = 0, E(x^4) \neq 0$$

所以代数精度是 3

Problem 2. 1、解如下方程组

$$\begin{aligned} 2 &= C_0 + C_1 + C_2 \\ 2 &= C_1 + 2C_2 \\ \frac{8}{3} &= C_1 + 4C_2 \end{aligned}$$

则

$$C_0 = \frac{1}{3}, C_1 = \frac{4}{3}, C_2 = \frac{1}{3}$$

有代数精度 2

2、解如下方程组

$$\begin{aligned} 1 &= \frac{1}{2} + C_1 \\ \frac{1}{2} &= \frac{1}{2}x_0 + C_1 x_1 \\ \frac{1}{3} &= \frac{1}{2}x_0^2 + C_1 x_1^2 \end{aligned}$$

解得

$$C_1 = \frac{1}{2}, x_0 = \frac{1 + \frac{\sqrt{3}}{3}}{2}, x_1 = \frac{1 - \frac{\sqrt{3}}{3}}{2}$$

Problem 3.

$$f[0,0,1,x](x-1)x^2 = f(x) - f(1) + x(x-1)f'(0) - (x^2-1)(f(1) - f(0))$$

故

$$\int_0^1 f(x) - x^2 f(1) + x(x-1)f'(0) + (x^2-1)f(0)\,\mathrm{d}x = \int_0^1 f[0,1,1,x](x-1)^2 x\,\mathrm{d}x = f'''(\zeta)\int_0^1 (x-1)^2\,\mathrm{d}x = \frac{1}{3}f'''(\zeta)$$

从而可设

$$C_0 = \frac{2}{3}, C_1 = \frac{1}{3}, B_0 = \frac{1}{6}, k = \frac{1}{3}$$

Problem 4. 考虑 partition $a, a + \frac{h}{2}, a + \frac{3}{2}h, \cdots, a + \frac{1}{2}h + (n-1)h$，则由于 f Riemann 可积，

$$\lim_{h\to 0}\sum_{k=1}^{n-1} h \cdot f(a + kh) + \frac{h}{2}f(a) + \frac{h}{2}f(b) = \int_a^b f$$

Problem 5.

$$\int_a^b \left[\int_c^d f(x,y)\,\mathrm{d}y\right]\mathrm{d}x$$
$$\approx \int_a^b \frac{d-c}{6}\left[f(x,c)+f(x,d)+4f(x,\frac{c+d}{2})\right]\mathrm{d}x$$
$$\approx \frac{b-a}{6}\cdot\frac{d-c}{6}$$
$$\cdot\left[f(a,c)+f(b,c)+4f(\frac{a+b}{2},c)+f(a,d)+f(b,d)+4f(\frac{a+b}{2},d)+4f(a,\frac{c+d}{2})+4f(b,\frac{c+d}{2})+16f(\frac{a+b}{2},\frac{c+d}{2})\right]$$

余项

$$R(x,y)=-\frac{d-c}{6}\cdot\frac{(b-a)^5}{2880}[f^{(4)}(\eta_1,c)+f^{(4)}(\eta_2,d)+f^{(4)}(\eta_3,\frac{d+c}{2})]$$

Problem 6. 关于权函数 $\rho(x)=\frac{1}{\sqrt{1-x^2}}$ 的正交序列多项式为 Chebyshev 多项式。由 Gauss-Chebyshev 求积公式可得

$$\int_{-1}^1 \rho(x)x^2\,\mathrm{d}x = A_0x_0^2 + A_1x_1^2$$

其中，x_0, x_1 为 2 次 Chebyshev 多项式的根 $\pm\frac{\sqrt{2}}{2}$

$$A_0=-\frac{\sqrt{2}}{2}\int_{-1}^1 \rho(x)(x-\frac{\sqrt{2}}{2})\,\mathrm{d}x, A_1=\frac{\sqrt{2}}{2}\int_{-1}^1 \rho(x)(x+\frac{\sqrt{2}}{2})$$

故

$$\frac{1}{2}(A_0+A_1)=\frac{1}{2}\int_{-1}^1\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\frac{\pi}{2}$$

Problem 7. 由 Vandermonde 行列式的可逆性，线性方程

$$\sum_{i=0}^n r_ix_i^k=\int_a^b x^k\,\mathrm{d}x,\, 0\le k\le n$$

有唯一解，且由于 $1, x, \cdots, x^n$ 为 P_n 的一组基，故

$$\sum_{i=0}^n r_ip(x_i)=\int_a^b p(x)\,\mathrm{d}x, \forall p\in P_n(x)$$

成立