Complex Analysis HW4

1. (a) $\lambda = (z_1, z_2, z_3, z_4)$ means $(z_1, z_2, z_3, z_4) \mapsto (\lambda, 1, 0, \infty)$

By definition, $\lambda = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} \Rightarrow (z_i, z_j, z_k, z_l)$ depends on how the (z_j, z_k, z_l) maps to $\{1, 0, \infty\}$, i.e.

$\lambda = (1234) = (2143) = (3412) = (4321)$

$1-\lambda = (1324) = (2413) = (3142) = (4231)$

$\frac{1}{\lambda} = (1243) = (2134) = (3421) = (4312)$

$\frac{1}{1-\lambda} = (1342) = (2431) = (3124) = (4213)$

$\frac{\lambda-1}{\lambda} = (1423) = (2314) = (3241) = (4132)$

$\frac{\lambda}{\lambda-1} = (1432) = (2341) = (3214) = (4123)$

(b) By (a), $(1234) = \lambda$, $(4123) = \frac{\lambda}{\lambda-1} \Rightarrow \begin{cases} |z_2 - z_3| |z_1 - z_4| = \frac{1}{|\lambda|} |z_1 - z_3| \cdot |z_2 - z_4| \\ |z_1 - z_2| \cdot |z_3 - z_4| = \left|\frac{\lambda-1}{\lambda}\right| |z_1 - z_3| \cdot |z_2 - z_4| \end{cases}$

Since $z_1 \cdots z_4$ are distinct, $\lambda \neq 0$. $z_1 \cdots z_4$ consecutive (on the same circle), $z_2 \mapsto 1$, $z_3 \mapsto 0$, $z_4 \mapsto \infty$, we have

$z_1 \mapsto \lambda > 1$ $\quad$ $\overset{z_4}{\infty}$ —— $\overset{z_3}{0}$ —— $\overset{z_2}{1}$ —— $\overset{z_1}{\lambda}$ $\Rightarrow \lambda > 0$, $\frac{\lambda-1}{\lambda} > 0$

Hence $|z_2 - z_3| \cdot |z_1 - z_4| + |z_1 - z_2| \cdot |z_3 - z_4| = |z_1 - z_3| \cdot |z_2 - z_4|$

2. If given $z_1 \mapsto 1$, $z_2 \mapsto -1$, $z_3 \mapsto k$, $z_4 \mapsto -k$, then let $T = -k \frac{(k+1)z - (k-1)}{(k+1)z + k - 1}$, we have $T1 = -1$, $T0 = k$, $T\infty = -k$. Denote $\lambda = (z_1, z_2, z_3, z_4)$, then TS maps z_2, z_3, z_4 to $-1, k, -k$

$\Rightarrow TSz_1 = 1$. Hence $\lambda = (z_1, \cdots z_4) = (TSz_1, \cdots, TSz_4) = (1, -1, k, -k) = \left(\frac{1-k}{1+k}\right)^2$

$\Rightarrow k = \frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}$ or $\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$. Here $S: z_2, z_3, z_4 \mapsto 1, 0, \infty$.

If we only know $\{z_1, \dots, z_4\} \mapsto \{\pm 1, \pm k\}$, then by 1.(a) there are 6 choices of such λ.

3. Assume T maps $\mathbb{R}$ to some circle, and $f: w = Tz \mapsto w^* = T\bar{z}$ be the reflection, then $\forall$ circle C on $\mathbb{C}$, $T^{-1}(C)$ is also a circle $\Rightarrow \overline{T^{-1}(C)}$ is a circle.

So $f(C) = T(\overline{T^{-1}(C)})$ is a circle.

4. ① By HW2.6, $T = Re^{i\theta} \frac{z}{R} = e^{i\theta} z$, or $T = Re^{i\theta} \frac{R}{z} = R^2 \frac{e^{i\theta}}{z}$, or $T = Re^{i\theta} \frac{\frac{z}{R} - \alpha}{\bar{\alpha}\frac{z}{R} - 1} = \frac{Rz - \alpha R^2}{\bar{\alpha} z - R} e^{i\theta}$ $(\alpha \in \mathbb{C}$

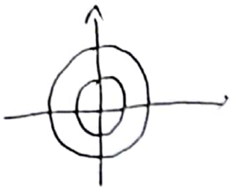
② Assume $T = \frac{az+b}{cz+d}$, if $a = 0$, then $T = R^2 \frac{e^{i\theta}}{z}$; if $c = 0$, then $T = e^{i\theta} z$.

If $a, c \neq 0$, then $T(\frac{-b}{a}) = 0$, $T(\frac{-d}{c}) = \infty$ is a pair of symmetric points $\Rightarrow \frac{-b}{a}$, $\frac{-d}{c}$ are symmetric points.

Let $\alpha = \frac{-b}{a}$, then $\frac{-d}{c} = \frac{R^2}{\bar{\alpha}} \Rightarrow T = \frac{a\bar{\alpha}}{c} \frac{z - \alpha}{\bar{\alpha} z - R^2}$. Take $z = R$, then $|Tz| = R \Rightarrow \left|\frac{a\bar{\alpha}}{c}\right| = R^2$

Hence $T = R^2 e^{i\theta} \frac{z - \alpha}{\bar{\alpha} z - R^2}$. $(\alpha \in \mathbb{C})$

5. We may assume the circles has center $z=0$ and radii $R>r$.



Let T be the Möbius transformation, and TC_R, TC_r has center y,

then $R'=|Tz-y|$ on $|z|=R$, $r'=|Tz-y|$ on $|z|=r$.

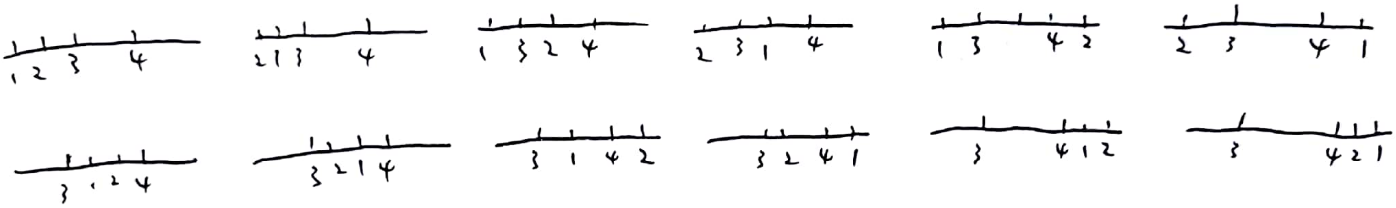
Let $S=T-y$. then by Ex 4 we have $S=\frac{R'}{R}e^{i\theta}z=\frac{r'}{r}e^{i\theta'}z$, or $S=e^{i\theta}R'\frac{R}{z}=e^{i\theta'}r'\frac{r}{z}$

or $S=R'e^{i\theta}\dfrac{\frac{z}{R}-\alpha}{\bar{\alpha}\frac{z}{R}-1}=r'e^{i\theta'}\dfrac{\frac{z}{r}-\alpha'}{\bar{\alpha}'\frac{z}{r}-1}$. in all case the ratios are the same.

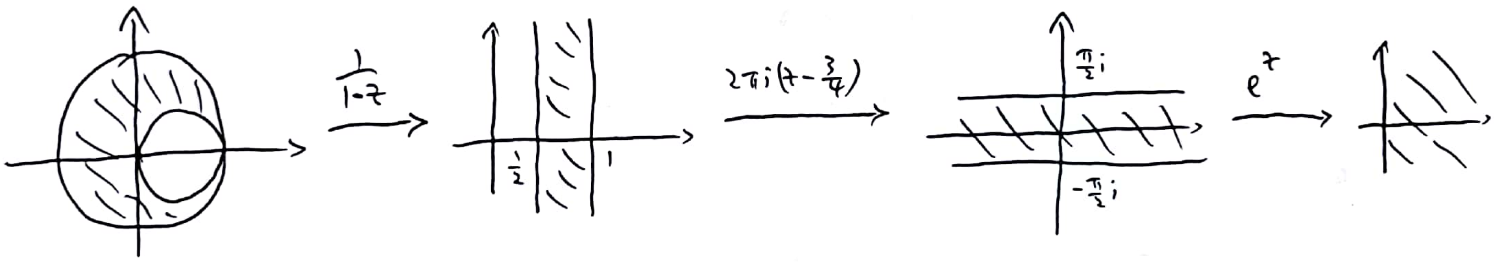
6. We can find a Möbius function that maps the circle to $\mathbb{R}$.

Since Möbius function doesn't change the orientation, we can assume $z_i\in\mathbb{R}$.

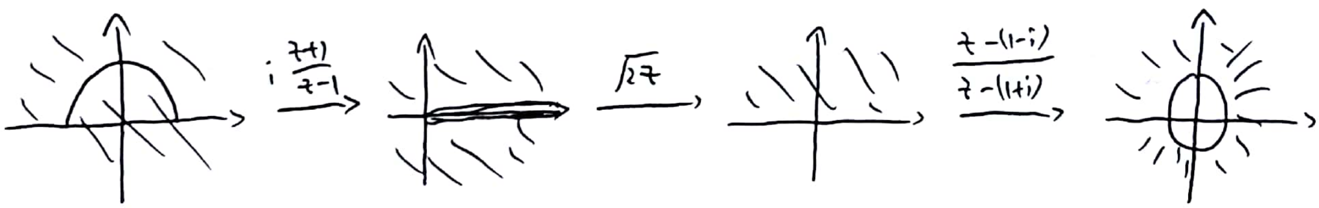
By definition $(z_1,z_2,z_3,z_4)=\frac{z_1-z_3}{z_1-z_4}\cdot\frac{z_2-z_4}{z_2-z_3}$, and it's easy to check that (z_1,z_3,z_4), (z_2,z_3,z_4) ~~has~~ have the same orientation $\Leftrightarrow(z_1,z_2,z_3,z_4)>0$ from the following 12 cases:



7.



8.



$\infty \longmapsto i \longmapsto 1+i \longmapsto \infty$