Contents

Analysis Summary

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Theorem 6.5.1. *Suppose* $f, g : [a, w) \to \mathbb{R}, \quad f, g \in \mathscr{R}$ *on* $[a, b]$ *for* $\forall b \in [a, w)$. *Suppose the improper intergral* $\int_a^w f(x)\,\mathrm{d}x$ *and* $\int_a^w g(x)\,\mathrm{d}x$ *are defined. Then*

a) *if* $f \in \mathscr{R}([a, w])$, *the values of* $\int_a^w f(x)\,\mathrm{d}x$ *are the same as a proper or improper integral*

b) $\forall \lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 f + \lambda_2 g$ *is integrable in the improper sense on* $[a, w)$ *and*

$$\int_a^w \lambda_1 f + \lambda_2 g\,\mathrm{d}x = \lambda_1 \int_a^w f(x)\,\mathrm{d}x + \lambda_2 \int_a^w g(x)\,\mathrm{d}x$$

c) $\forall c \in [a, w)$,

$$\int_a^w f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^w f(x)\,\mathrm{d}x$$

d) *if* $\varphi : [A, U) \to [a, w)$ *is strictly increasing continuous function with* $\varphi(A) = a$ *and* $\varphi(y) \to w$ *as* $y \to U, y \in [A, U)$ *,and* $\varphi' \in (R)$ *on* $[A, B]$ *for* $\forall B \in [A, U)$ *,Then* $f(\varphi)\varphi'$ *is integrable in the improper sense on* $[A, U)$ *and*

$$\int_a^w f(x)\,\mathrm{d}x = \int_A^U f(\varphi(y))\varphi'(y)\,\mathrm{d}y$$

e) *Newton-Leibniz formula is similiar to improper integral*

Theorem 6.5.2 (Abel-Dirichlet test For the convergence of an integral)**.** *Let* $f, g : [A, w) \to \mathbb{R}, f \in \mathscr{R}$ *on* $[a, b]$, $g \in \mathscr{R}$ *on* $[a, b]$, $\forall b \in [a, w)$. *Suppose that* g *is monotonic, Then* $\int_a^w fg\,\mathrm{d}x$ *converges if one of the following pairs of conditions holds:*

$$\begin{cases} 1) \quad \int_a^w f(x)\,\mathrm{d}x \quad \textit{converges} \\ 2) \quad g \quad \textit{bounded on } [a, w) \end{cases}$$

or

$$\begin{cases} 1') \quad F(b) = \int_a^b f(x)\,\mathrm{d}x \quad \textit{is bounded on } [a, w) \\ 2') \quad \lim\limits_{x \to \infty} g(x) = 0 \end{cases}$$

6.6 Rectifiable Curves

Definition 6.6.1. *A continuous mapping $\gamma : [a, b] \to \mathbb{R}^k$ is called a* <u>*curve*</u> *in* $\mathbb{R}^k$.

If γ is one to one.gamma is called an <u>*arc*</u>.

If $\gamma(a) = \gamma(b)$, $\quad \gamma$ is said to be a <u>*closed curve*</u>

For a partition $P = \{x_0 = a, x_1, x_2, \dots, x_n = b\}$, we define

$$\Lambda(P, \gamma) := \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

We define the <u>legth</u> of γ as

$$\Lambda(\gamma) := \sup_P \Lambda(P, \gamma)$$

If $\Lambda(\gamma) < \infty$, we say that γ is <u>rectifiable</u>.

Theorem 6.6.1. *If γ' is continuous on $[a, b]$,then γ is rectifiable and*

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t$$

Remark 1. *Let $\phi : [c, d] \to [a, b]$ be continuous bijection. Let $\gamma_2 := \gamma_1(\phi(t)), \quad t \in [c, d]$. Then γ_2 is rectifiable iff γ_1 is. and γ_2 and γ_1 have the same lenth.*

7 Sequence and series of function

We say that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges to f pointwise on E if the sequence of numbers $f_n(x)$ converges for each $x \in E$
Then we can define a function f by

$$f(x) := \lim_{n\to\infty} f_n(x), \quad x \in E$$

f is called the limit function of f_n.

Similiarly, if $\sum f_n(x)$ converges for each $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in E$$

the function f is called the sum of the series of $\sum f_n$

7.1 Uniformly Convergence

Theorem 7.1.1. *We define* $M_n := \sup_{x \in E} |f_n(x) - f(x)|$. *Then* $f_n \to f$ *uniformly on* E *iff* $M_n \to 0$ *as* $n \to \infty$.

Theorem 7.1.2 (Cauchy criterion for uniform convergence)**.** *The sequence of functions* $\{f_n\}$, *defined on* E, *converges uniformly on* E *iff for* $\forall \epsilon > 0, \exists N \in \mathbb{N}$ *s.t.*

$$|f_n(x) - f_m(x)| < \epsilon, \quad \forall x \in E, \forall n, m \geqslant N$$

Theorem 7.1.3 (Weiersress M-test)**.** *Suppose* $\{f_n\}$ *is a sequence of functions defined on E, and Suppose*

$$|f_n(x)| \leqslant M_n, \forall x \in E, \forall n \in \mathbb{N}$$

The $\sum f_n$ *converges uniformly on* E *if* $\sum M_n$ *converges;*

7.2 Uniform Convergence and Continuity

Theorem 7.2.1. *Suppose* $f_n \to f$ *uniformly on* E *in a metric space. Let* x *be a limit point of* E *and Suppose that* $\lim_{t \to x} f_n(t) = A_n, \forall n \in \mathbb{N}$. *Then* $\{A_n\}$ *converges, and*

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$$

i.e.

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

Theorem 7.2.2. *If* $\{f_n\}$ *is a sequence of continuous functions on* E, *and if* $f_n \to f$ *uniformly on* E, *then* f *is continuous on* E

Example 1. *Let* $f_n(x) = nx(1-x)^n, \quad x \in [0,1]$,*Then we can know the convergence is not uniformly with the previous theorem.*

Theorem 7.2.3. *Suppose* K *is compact, and*

(a) $\{f_n\}$ *is a sequence of continuous functions on* K.

(b) $\{f_n\}$ *converges pointwise to a continuous function* f *on* K.

(c) $f_n(x) \geqslant f_{n+1}(x)$, $\forall x \in K, n \in \mathbb{N}$.

Then $f_n \to f$ *uniformly on* K.

Example 2. *The compactness is really needed here.*
$f_n(x) = \frac{1}{nx+1}, x \in (0,1)$, $f_n(x) \downarrow 0$. *But the convergence is not uniform since* $f(\frac{1}{n}) = \frac{1}{2}$

Let X be a metric space, and let $\mathscr{C}(X)$ denote the set of all complex-valued, continuous, bounded functions with domain X. We associate each $f \in \mathscr{C}(X)$ its supremum norm as

$$||f|| := \sup_{x \in X} |f(x)|$$

It is easy to check it's a well-defined norm. And we define $d(f,g) = ||f - g||$

Theorem 7.2.4. *The metric* d *makes* $\mathscr{C}(X)$ *into a complete metric space.*

7.3 Uniform Convergence and Integration

Theorem 7.3.1. *Let* α *be increasing on* $[a,b]$. *Suppose* $f_n \in \mathscr{R}(\alpha)$ *on* $[a,b]$ *for* $n \in \mathbb{N}$, *and suppose* $f_n \to f$ *uniformly on* $[a,b]$. *Then* $f \in \mathscr{R}(\alpha)$ *on* $[a,b]$, *and*

$$\int_a^b f \,\mathrm{d}\alpha = \int_a^b \lim_{n\to\infty} f_n \,\mathrm{d}\alpha = \lim_{n\to\infty} \int_a^b f_n \,\mathrm{d}\alpha$$

Corollary 1. *If* $f_n \in \mathscr{R}(\alpha)$ *on* $[a,b]$ *and* $\sum\limits_{j=1}^{n} f_j(x)$ *converges uniformly on* $[a,b]$ *to* f, *then*

$$\int_a^b f \,\mathrm{d}\alpha = \sum_{j=1}^{\infty} \int_a^b f_j \,\mathrm{d}\alpha$$

7.4 Uniform Convergence and Differentiation

Theorem 7.4.1. *Suppose* $\{f_n\}$ *is a sequence of differentiable functions on* $[a,b]$, *and* $\{f_n(x)\}$ *convergence for some* $x_0 \in [a,b]$. *If* $\{f_n'\}$ *convergence uniformly on* $[a,b]$ *then* $\{f_n\}$ *converges uniformly on* $[a,b]$, *to a function* f, *and*

$$f'(x) = \frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x) = \lim_{n\to\infty} f_n'(x)$$

Proof. $\forall \epsilon > 0$, choose $N \in \mathbb{N}$ s.t.

$$\begin{cases} |f_n(x_0) - f_m(x_0)| < \epsilon, \, \forall n, m \geqslant N \\ |f_n'(t) - f_m'(t)| < \epsilon, \, \forall n, m \geqslant N \end{cases}$$

MVT implies that exists ξ between t and x, such that

$$|f_n(x) - f_m(x) - (f_n(t) - f_m(t))| = |f_n'(\xi) - f_m'(\xi)||t - x| < \epsilon|x - t| < \epsilon(b - a) \quad (*)$$

$$\begin{aligned} \Rightarrow f_n(x) - f_m(x) &\leqslant |f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\ &< \epsilon(b - a) + \epsilon, \quad \forall x \in [a, b], \forall n, m, \in N \end{aligned}$$

$\Rightarrow \{f_n\}$ converges uniformly on $[a, b]$.

Define $f(x) := \lim\limits_{n\to\infty} f_n(x)$, $\forall x \in [a, b]$.

For fixed $x \in [a, b]$, define $g_n(t) := \frac{f_n(t) - f_n(x)}{t - x}, g(t) = \frac{f(t) - f(x)}{t - x}$, $\forall t \in [a, b], t \neq x$

$$\begin{aligned} (*) &\Rightarrow |g_n(t) - g_m(t)| < \epsilon, \forall t \in [a, b], t \neq x, \forall n, m \geqslant N && (1) \\ &\Rightarrow \{g_n\} \text{ converges uniformly on } [a, b] - \{x\} && (2) \end{aligned}$$

Note that $\lim\limits_{t\to x} g_n(t) = f_n'(x)$.

1st theorem in 7.2 implies

$$f'(x) = \lim_{t\to x} g(t) = \lim_{t\to x} \lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} \lim_{t\to x} g_n(t) = \lim_{n\to\infty} f_n'(x)$$

$\square$

7.5 Equicontinuous of Families of Functions

We say that $\{f_n\}$ is <u>pointwise bounded</u> on E if the sequence $\{f_n(x)\}$ is bounded for each $x \in E$.

We say that $\{f_n\}$ is <u>uniformly bounded</u> on E if there exists $M \in \mathbb{R}$ s.t.

$$|f_n(x)| \leqslant M, \, \forall x \in E, \forall n \in \mathbb{N}$$

Example 3. *Let* $f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}, x \in [0, 1], n \in \mathbb{N}$, *then* $\{f_n\}$ *is uniformly bounded on* $[0, 1]$, *but no subsequence of it can converge uniformly on* $[0, 1]$.

A family $\mathscr{F}$ of complex-values functions f defined on E in a metric space X is said to be equicontinuous on E if $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \epsilon, \forall x, y \in E \quad with \quad d(x,y) < \delta, \forall f \in \mathscr{F}$$

Remark 2. *Each member of an equicontinuous family is uniformly continuous*

Theorem 7.5.1. *If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{F_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for $\forall x \in E$.*

Proof. Let $E := \{x_i : i \in \mathbb{N}\}$. Consider the array $\{f_{ij}\}$ such that $\lim\limits_{j\to\infty} f_{nj}(x_n)$ exists for each $n \in \mathbb{N}$ and $\{f_{(i+1)j}\}$ is the subsequence of $\{f_{ij}\}$
then consider the diagonal of the array $f_{11}, f_{22}, \cdots$.
We have $\lim\limits_{j\to\infty} f_{jj}(x_i)$ exists for each $x_i \in E$ $\square$

Theorem 7.5.2. *If K is a compact metric space, and if $f_n \in \mathscr{C}(K)$ for $\forall n \in \mathbb{N}$, and $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K*

Theorem 7.5.3 (Arzela-Ascoli Theorem)**.** *If K is compact, $f_n \in \mathscr{C}(K)$ for $\forall n \in \mathbb{N}$, and $\{f_n\}$ is pointwise bounded and equicontinuous on K. Then*

(a) $\{f_n\}$ is uniformly bounded on K

(b) $\{f_n\}$ contains a uniformly convergent subsequence

Proof. $\forall \epsilon > 0$, $\{f_n\}$ equicontinuous $\Rightarrow \exists \delta > 0$ s.t.

$$|f_n(x) - f_n(y)| < \epsilon, \quad \forall x \in K, \forall y \in N_\delta(x), \forall n \in \mathbb{N} \tag{$*$}$$

$$\begin{aligned}
K \subset \bigcup_{x\in K} N_\delta(x) &\xRightarrow{K compact} \exists x_1, \cdots, x_n,\ K \subset \bigcup_{j=1}^{m} N_\delta(x_j) \\
&\Rightarrow \forall t \in K,\ \exists x_j \text{ with } j \in \{1, 2, \cdots, m\} \text{ s.t. } t \in N_\delta(x_j), \\
&\quad \text{and thus } |f_n(t) - f_n(x_j)| < \epsilon,\ \forall n \in \mathbb{N}. \\
&\Rightarrow \sup_{t\in K, n\in\mathbb{N}} |f_n(t)| \leqslant \sup_{1\leqslant j\leqslant m} \sup_{n\in\mathbb{N}} |f_n(x_j)| + \epsilon < \infty \text{ since } \{f_n\} \text{ is pointwise bounded} \\
&\Rightarrow \{f_n\} \text{ is uniformly bounded on } K
\end{aligned}$$

Let E be a countable dense subset of K (see Exercise 25 of Chapter 2)

The first theorem in this section $\Rightarrow \{f_n\}$ has a subsequence $\{f_{n_k}\}$ s.t. $\{f_{n_k}(x)\}$ converges for $\forall x \in E$.

E is dense in $K \Rightarrow K \subset \bigcup\limits_{x\in E} N_\delta(x) \xRightarrow{K compact} \exists x_1, \cdots, x_m \in E$ s.t. $K \subset \bigcup\limits_{j=1}^{m} N_\delta(x_j)$.

We define $g_k := f_{n_k}$. Then $\{g_k(x)\}$ converges for $\forall x \in E \Rightarrow \exists N \in \mathbb{N}$ s.t.

$$|g_p(x_j) - g_q(x_j)| < \epsilon \quad \forall p \geqslant N, q \geqslant N, j \in \{1, 2 \cdots, m\}$$

For $\forall t \in K$, $\exists j_0 \in \{1, 2, \cdots, m\}$ s.t. $t \in N_\delta(x_{j_0}) \xRightarrow{(*)}$

$$|g_p(t) - g_p(x_{j_0})| < \epsilon \quad \text{for } \forall p \in \mathbb{N}$$

$\Rightarrow |g_p(t) - g_q(t)| \leqslant |g_p(t) - g_p(x_{j_0})| + |g_p(x_{j_0}) - g_q(x_{j_0})| + |g_q(x_{j_0}) - g_q(t)| < 3\epsilon$, $\forall p \geqslant N, q \geqslant N, \forall t \in K$

$\Rightarrow \{g_k\} = \{f_{n_k}\}$ converges uniformly on K.

□

7.6 The Stone-Weiertrass Theorem

Theorem 7.6.1. *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials $\{P_n\}$ such that*

$$\lim_{n\to\infty} P_n(x) = f(x) \text{ uniformly on } [a, b]$$

Proof. WLOG, let $[a, b] = [0, 1]$, $f(0) = f(1) = 0$. And $f(x) = 0$, $\forall x \in \mathbb{R}\backslash[0, 1] \Rightarrow$ f is uniformly continuous on $\mathbb{R}$.

$$Q_n(x) := c_n(1 - x^2)^n,\ x \in [-1, 1]$$

where c_n satisfies $\int_{-1}^{1} Q_n(x)\,\mathrm{d}x = 1$, $\forall n \in \mathbb{N}$.

$$\int_{-1}^{1} (1-x^2)^n \,\mathrm{d}x = 2\int_0^1 (1-x^2)^n \,\mathrm{d}x \geqslant 2\int_0^{\frac{1}{\sqrt{n}}} (1-x^2)^n \,\mathrm{d}x \geqslant 2\int_0^{\frac{1}{\sqrt{n}}} (1-nx^2) \,\mathrm{d}x = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}, n \in \mathbb{N}$$

$\Rightarrow c_n < \sqrt{n}, n \in \mathbb{N}$.

$\Rightarrow$ For $\forall \delta \in (0, 1)$, $Q_n(x) \leqslant \sqrt{n}(1 - \delta^2)^n$, $\forall x$ with $|x| \in [\delta, 1]$.

$\Rightarrow Q_n(x) \to 0$ uniformly in $[-1, -\delta] \cup [\delta, 1]$

Let $P_n(x) := \int_{-1}^{1} f(x+t)Q_n(t)\,\mathrm{d}t, x \in [0,1]$

Then $P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)\,\mathrm{d}t = \int_0^1 f(t)Q(t-x)\,\mathrm{d}t$ is a polynomial in x, which is real if f is real.

$\forall \epsilon > 0$, $\exists \delta \in (0,1)$ s.t. $|f(y) - f(x)| < \frac{\epsilon}{2}$ whenever $|y - x| < \delta$.

Then

$$\begin{aligned}
|P_n(x) - f(x)| &= |\int_{-1}^{1} f(x+t)Q_n(t)\,\mathrm{d}t - \int_{-1}^{1} f(x)Q_n(x)\,\mathrm{d}t| \\
&\leqslant \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)\,\mathrm{d}t \\
&\leqslant 2M\int_{-1}^{-\delta} Q_n(t)\,\mathrm{d}t + \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,\mathrm{d}t + 2M\int_{\delta}^{1} Q_n(t)\,\mathrm{d}t \qquad (M := sup|f(x)|) \\
&\leqslant 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} \\
&< \epsilon \text{ for all large } n
\end{aligned}$$

$\Rightarrow P_n \to f$ uniformly on $[0,1]$. □

7.7 Continuous Nowhere Differentiable Functions

Theorem 7.7.1. *There exists a real continuous function on $\mathbb{R}$, which is nowhere differentiable.*

Proof. Let $\varphi(x) := |x|, x \in [-1,1]$.

Extend φ to all $x \in \mathbb{R}$ by $\varphi(x+2) = \varphi(x), x \in \mathbb{R}$.

Then $|\varphi(x) - \varphi(y)| \leqslant |x-y|$, $\forall x, y \in \mathbb{R} \quad (*)$.

Define $f(x) := \sum\limits_{n=0}^{\infty} (\frac{3}{4})^n \varphi(4^n x), x \in \mathbb{R}$.

$||\phi(x)|| \leqslant 1 \xRightarrow{M-text}$ the last series converges uniformly on $\mathbb{R} \xRightarrow{Thm7.2.2} f$ is continuous on $\mathbb{R}$

Fix $x \in \mathbb{R}$, choose

$$\delta_m := \begin{cases} \frac{1}{2}4^{-m}, & [4^m x, 4^m(x + \frac{1}{2}4^{-m})] \cap \mathbb{Z} = \phi \\ -\frac{1}{2}4^{-m}, & (4^m(x - \frac{1}{2}4^{-m}), 4^m x) \cap \mathbb{Z} = \phi \end{cases}$$

Now Define

$$\gamma_n := \frac{\phi(4^n(x+\delta_m)) - \phi(4^n x)}{\delta_m}, \, n \in \mathbb{N}$$

Then $\gamma_n = 0$ if $n > m$ (since $4^n \delta_m \in 2\mathbb{Z}$), and $|\gamma_n| \leqslant 4^n$ if $0 \leqslant n \leqslant m$ by $(*)$, and $|\gamma_m| = 4^m$.

$$\begin{aligned}
\Rightarrow |\frac{f(x+\delta_m) - f(x)}{\delta_m}| &= |\sum_{n=0}^{\infty} (\frac{3}{4})^n \gamma_n| \\
&= |\sum_{n=0}^{m} (\frac{3}{4})^n \gamma_n| \\
&\geqslant (\frac{3}{4})^m |\gamma_m| - \sum_{n=0}^{m-1} (\frac{3}{4})^n |\gamma_n| \\
&\geqslant 3^m - \sum_{n=0}^{m-1} 3^n \\
&= \frac{3^m + 1}{2} \to \infty \quad as\, m \to \infty.
\end{aligned}$$

Note that $\delta_m \to 0$ as $m \to 0$, $\Rightarrow$ f is not differentiable at x. $\square$

8 Some Special Functions

8.1 Power Series

Functions which are represented by power series, i.e. $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, are called analytic functions.

We shall restrict to real values of x.

WLOG, we shall often take $a = 0$.

Theorem 8.1.1. *Suppose the series* $\sum_{n=0}^{\infty} c_n x^n (**)$. *converges for* $|x| < R$, *and define* $f(x) := \sum_{n=0}^{\infty} c_n x^n$, $|x| < R$. *Then* $(**)$ *converges uniformly on* $[-R+\epsilon, R-\epsilon]$ *for* $\forall \epsilon > 0$. f *is continuous and differentiable on* $(-R, R)$ *and*

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1},\ |x| < R \qquad (***)$$

Proof. Fix $\epsilon \in (0, R)$, we have $|c_n x^n| \leqslant |c_n (R-\epsilon)^n| \quad \forall |x| < R - \epsilon$. By the root test, each power series converges absolutely in the interior of its interval of convergence, i.e. $|\sum |c_n (R-\epsilon)^n|$ converges.

M-test $\Rightarrow$ $(**)$ converges uniformly on $[-R+\epsilon, R-\epsilon]$.

$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1 \Rightarrow \limsup\limits_{n\to\infty}(n|c_n|)^{\frac{1}{n}} = \limsup\limits_{n\to\infty}(|c_n|)^{\frac{1}{n}} \Rightarrow$ the series $(**)$ and $(***)$ have the same interval of convergence.

$\Rightarrow$ $(***)$ converges uniformly on $[-R+\epsilon, R-\epsilon]$.

theorem 7.4.1 tells us $(***)$ holds if $|x| < R-\epsilon$,

$\Rightarrow$ $(***)$ holds for $\forall |x| < R$ since ϵ is arbitrary.

f is continuous because it is differentiable. □

Corollary 2. *Under the hypotheres of the previous theorem, f has derivatives of all orders in $(-R, R)$:*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k}$$

In particular, $f^{(k)}(0) = k!c_k$.

Example 4. *Let $f(x) = e^{-\frac{1}{x^2}}, x \neq 0$, $f(0) = 0$. Then $f^{(k)}(0) = 0$ fir $\forall k \in \mathbb{N}$. So f cannot be expanded in a power series about $x = 0$.*

Theorem 8.1.2 (Abel's theorem)**.** *Let $f(x) := \sum\limits_{n=0}^{\infty} c_n x^n, x \in (-1, 1)$, and suppose $\sum c_n$ converges. Then $\lim\limits_{x\to 1^-} f(x) = \sum\limits_{n=0}^{\infty} c_n$.*

Proof. Let $A_n := \sum\limits_{k=0}^{n} c_k, n \in \mathbb{N}$, and $A_{-1} = 0$. Then

$$\sum_{n=0}^{m} c_n x^n = \sum_{n=0}^{m}(A_n - A_{n-1})x^n = (1-x)\sum_{n=0}^{m} A_n x^n + A_m x^{m+1}$$

$\xRightarrow{m\to\infty} f(x) = (1-x)\sum\limits_{n=0}^{\infty} A_n x^n$, $|x| < 1$.

Suppose $A := \lim_{n\to\infty} A_n \Rightarrow$ Fix $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $|A_n - A| < \frac{\epsilon}{2}, \forall n \geqslant N$.

$$
\begin{aligned}
\Rightarrow |f(x) - A| &= |(1-x)\sum_{n=0}^{\infty} A_n x^n - (1-x)\sum_{n=0}^{\infty} A x^n| \\
&= |(1-x)\sum_{n=0}^{\infty}(A_n - A)x^n| \\
&\leqslant |1-x|\sum_{n=0}^{N}|A_n - A||x|^n + |1-x|\frac{\epsilon}{2}\sum_{n=N}^{\infty}|x|^n \\
&\leqslant |1-x|\sum_{n=0}^{N}|A_n - A||x|^n + \frac{\epsilon}{2}\frac{|1-x|}{1-|x|},\ |x| < 1 \\
&< \epsilon \quad \text{ for some } \delta > 0 \text{ and } \forall x \in (1-\delta, 1)
\end{aligned}
$$

$\Rightarrow \lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} c_n$ □

As an application, we prove:

Theorem 8.1.3. *If the series $\sum a_n, \sum b_n, \sum c_n$ converge to A, B, C, and if $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$, then $C = AB$.*

Proof. Let $f(x) := \sum_{n=0}^{\infty} a_n x^n, \quad g(x) := \sum_{n=0}^{\infty} b_n x^n, \quad h(x) := \sum_{n=0}^{\infty} c_n x^n$, $x \in [0,1]$. These series converge absolutely if $x \in [0,1) \Rightarrow$

$$
f(x) \cdots g(x) = g(x), \quad x \in [0,1)
$$

The previous theorem $\Rightarrow \lim_{x\to 1^-} f(x) = A$, $\lim_{x\to 1^-} g(x) = B$, $\lim_{x\to 1^-} h(x) = C$
$\Rightarrow AB = C$ □

Theorem 8.1.4 (Fubini's theorem for infinite series)**.** *Given a double sequence $\{a_{ij}\}, i, j \in \mathbb{N}$. Suppose that*

$$
\sum_{j=1}^{\infty} |a_{ij}| = b_i, \forall i \in \mathbb{N} \textit{ and } \sum_{i=1}^{\infty} b_i \textit{ converges}
$$

Then

$$
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}
$$

Proof. Let $E := \{x_0, x_1, x_2, \cdots\}$ and suppose $\lim_{n\to\infty} x_n = x_0$. Define

$$f_i(x_0) := \sum_{j=1}^{\infty} a_{ij},\ \forall i \in \mathbb{N} \qquad f_i(x_n) := \sum_{j=1}^{n} a_{ij},\ \forall i, j \in \mathbb{N}, \qquad g(x) := \sum_{i=1}^{\infty} f_i(x),\ \forall x \in E$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} < \infty \Rightarrow\ f_i \text{ is continuous at } x_0,\ \forall i \in \mathbb{N}.$$

$$|f_i(x)| \leqslant b_i,\ \forall x \in E \xRightarrow{M-test} \sum_{i=1}^{\infty} f_i(x) \text{ converges uniformly on } E$$

use 1st theorem in 7.2 we know that g is continuous at x_0.

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} &= \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) \\
&= \lim_{n\to\infty} g(x_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(x_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} \\
&= \lim_{n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{\infty} a_{ij}
\end{aligned}$$

□

Theorem 8.1.5. *Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and the series converges in $|x| < R$. If $a \in (-R, R)$, then f can be expanded in a power series about $x = a$ which converges in $|x-a| < R - |a|$, and $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$, $|x-a| < R - |a|$*

Proof.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} c_n[(x-a)+a]^n \\
&= \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(x-a)^k \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \binom{n}{k} c_n a^{n-k}(x-a)^k \qquad \text{(previous theorem)} \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad \text{(Corollary in this section)}
\end{aligned}$$

□

Theorem 8.1.6. *Suppose the series $\sum a_n x^n$ and $\sum b_n x^n$ converge in $(-R, R)$. Let E be the set of all x in $(-R, R)$ s.t. $\sum\limits_{n=0}^{\infty} a_n x^n = \sum\limits_{n=0}^{\infty} b_n x^n$. If E has a limit point in $(-R, R)$, then $a_n = b_n$ for $\forall n \in N \cup \{0\}$. Hence $E = (-R, R)$.*

Proof. **Claim**. Let A be a subset of a metric space X and X is connected. If A is both open and closed, then $A = \phi$ or $A = X$. (Cause $X = A \cup A^c$)
Let $f(x) := \sum\limits_{n=0}^{\infty} (a_n - b_n) x^n, \qquad x \in (-R, R)$.
Then $E := \{x \in (-R, R) : f(x) = 0\}$
1st theorem in §8.1 implies f is continuous in $(-R, R) \Rightarrow E$ is closed (relative to $(-R, R)$)
We prove in Homework 1 that E' is closed. We will prove E' is open. Then with the claim we know $E = E' = (-R, R)$.
Let $x_0 \in E'$, the previous theorem $\Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n, \quad |x - x_0| < R - |x_0|$$

We Claim that $d_n = 0$ for $\forall n \in \mathbb{N} \cup \{0\}$. Otherwise, let k be smallest nonnegative integer s.t. $d_k \neq 0$. Then $f(x) = (x - x_0)^k g(x)$, where $g(x) = \sum\limits_{m=0}^{\infty} d_{m+k} (x - x_0)^m$
1st theorem in 8.1 implies g is continuous at x_0, and $g(x_0) = d_k \neq 0$.
$\Rightarrow \exists \delta > 0$ s.t. $g(x) \neq 0$, $\forall x \in (x_0 - \delta, x_0 + \delta)$
$\Rightarrow f(x) \neq 0$ for $\forall x \in (x_0 - \delta, x_0 + \delta) \backslash \{x_0\}$
$\Rightarrow x_0 \notin E'$, which is not correct.
That way, we can know $f(x) = 0$, whenever $|x - x_0| < R - |x_0|$. So there exists a NBHD of x_0 is contained by E'. Then E' is open.

□

Remark 3. *The proof use the continuity of power series functions.*

8.2 Fourier Series

A trigonometric polynomial is a finite sum of the form

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx), \quad \forall x \in \mathbb{R}$$

where $a_0, a_1, \cdots, a_N, b_1, \cdots, b_N \in \mathbb{C}$. Equivalently,

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}, \quad x \in \mathbb{R} \tag{$*$}$$

with $a_0 = c_0, a_n = c_n + c_{-n}, b_n = c_n - c_{-n}$.

It is easy to see

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \, \mathrm{d}x = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{Z} \backslash \{0\} \end{cases}$$

$$\Rightarrow c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} \, \mathrm{d}x, \qquad m = -N, -N+1, \cdots, N \tag{$**$}$$

Remark 4. *f is real* $\Leftrightarrow f(x) = \overline{f(x)} \Leftrightarrow \sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} \overline{c_n} e^{-inx} \Leftrightarrow \sum_{n=-N}^{N} (c_n - \overline{c_{-n}}) e^{inx} = 0 \Leftrightarrow c_n = \overline{c_{-n}}$ *for* $\forall\, n = 0, 1, \cdots, N$.

A trigonometric series is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad x \in \mathbb{R} \tag{$***$}$$

whose Nth partial sum is defined to be $(*)$.

If $f \in \mathscr{R}$ on $[-\pi, \pi]$, the numbers $c_m, m \in \mathbb{Z}$ defined by $(**)$ are called Fourier coefficients of f, and the series $(***)$ formed with these coefficients is called the Fourier series of f.

Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a sequence of complex functions on $[a, b]$ s.t.

$$\int_a^b \phi_n(x) \overline{\phi_m(x)} \, \mathrm{d}x = 0, \quad \forall n \neq m$$

Then $\{\phi_n\}$ is said to be an orthogonal system of functions on $[a, b]$.

If, in addition, $\int_a^b |\phi_n(x)|^2 \, \mathrm{d}x = 1$, then it is called orthonormal

Example 5. $\{\frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in \mathbb{Z}}$ *form an orthonormal system on* $[-\pi, \pi]$.

If $\{\phi_n\}$ is orthonormal on $[a, b]$ and if

$$c_n = \int_a^b f(t) \overline{\phi_n(t)} \, \mathrm{d}t, \quad \forall n \in \mathbb{N}$$

We called c_n the nth Fourier coefficients of f relative to $\{\phi_n\}$, we write

$$f(x) \backsim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

and call this series the Fourier series of f relative to $\{\phi_n\}$.

Theorem 8.2.1. *Let $\{\phi_n\}$ be orthonormal on $[a, b]$. Let $S_n(x) = \sum\limits_{m=1}^{n} c_m \phi_m(x)$ be the nth partial sum of the Fourier series of f with $f \in \mathscr{R}$ on $[a, b]$, and Suppose*

$$t_n(x) = \sum_{m=1}^{n} d_m \phi_m(x)$$

Then $\int_a^b |f - S_n|^2 \, \mathrm{d}x \leqslant \int_a^b |f - t_n|^2 \, \mathrm{d}x$.
and equality holds iff $d_m = c_m, m \in \mathbb{N}, \forall m \in \mathbb{N}, m \leqslant N$

Remark 5. *The theorem says, among all functions t_n, s_n gives the best possible mean square approximation to f.*

Proof.

$$\begin{aligned}
\int_a^b |f - t_n|^2 \, \mathrm{d}x &= \int_a^b |f|^2 \, \mathrm{d}x + \int_a^b |t_n|^2 \, \mathrm{d}x - \int_a^b f \overline{t_n} \, \mathrm{d}x - \int_a^b \overline{f} t_n \, \mathrm{d}x \\
&= \int_a^b |f|^2 \, \mathrm{d}x + \sum_{m=1}^{n} \sum_{j=1}^{n} \int_a^b d_m \phi_m \overline{d_j \phi_j} \, \mathrm{d}x - \sum_{m=1}^{n} \overline{d_m} \int_a^b f \overline{\phi_m} \, \mathrm{d}x - \sum_{m=1}^{n} d_m \int_a^b \overline{f} \phi_m \, \mathrm{d}x \\
&= \int_a^b |f|^2 \, \mathrm{d}x + \sum_{m=1}^{n} |d_m|^2 - \sum_{m=1}^{n} (\overline{d_m} c_m + d_m \overline{c_m}) \\
&= \int_a^b |f|^2 \, \mathrm{d}x - \sum_{m=1}^{n} |c_m|^2 + \sum_{m=1}^{n} |d_m - c_m|^2. \qquad (\square)
\end{aligned}$$

which is minimized if and only if $d_m = c_m, m = 1, \cdots, n$. $\square$

Let $d_m = c_m$ in $(\square)$,we get

$$\int_a^b |S_n|^2 \, \mathrm{d}x = \sum_{m=1}^{n} |c_m|^2 = \int_a^b |f|^2 \, \mathrm{d}x - \int_a^b |f - S_n|^2 \, \mathrm{d}x \leqslant \int_a^b |f|^2 \, \mathrm{d}x \quad (\square\square)$$

Setting $n \to \infty$ in the last inequality, we obtain

Theorem 8.2.2 (Bessel's inequality)**.** *If $\{\phi_n\}$ is orthonormal on $[a,b]$ and $f \in \mathscr{R}$ on $[a,b]$, and if*

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

Then

$$\sum_{n=1}^{\infty} |c_n|^2 \leqslant \int_a^b |f(x)|^2 \, \mathrm{d}x$$

In particular, $\lim_{n\to\infty} c_n = 0$.

For the rest of the section, we only consider the trigonometric system. For $f \in \mathscr{R}$ on $[-\pi,\pi]$ and has period 2π. Then the orthonormal system is $\{\frac{1}{\sqrt{2\pi}}e^{inx}\}_{n\in\mathbb{Z}}$.

$$(\square\square) \Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n|^2 \, \mathrm{d}x = \sum_{m=-n}^{n} |c_n|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 \, \mathrm{d}x \qquad (\triangle)$$

We define the <u>Dirichlet Kernel</u>

$$D_N(x) := \sum_{n=-N}^{N} e^{inx} = \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{i}x} = \frac{\sin{(N+\frac{1}{2})x}}{\sin\frac{x}{2}}$$

Then

$$\begin{aligned} S_N(x) &= \sum_{n=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int} \, \mathrm{d}t e^{inx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_N(x-t) \, \mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) D_N(t) \, \mathrm{d}t \qquad (\square\square\square) \end{aligned}$$

Theorem 8.2.3. *If, for some x, $\exists \delta > 0$ and $M < \infty$ s.t.*

$$|f(x+t) - f(x)| \leqslant M|t|, \forall t \in (-\delta, \delta)$$

then $\lim_{N\to\infty} S_N(x) = f(x)$.

Proof. Define

$$g(t) := \begin{cases} \dfrac{f(x-t) - f(t)}{\sin\frac{t}{2}}, & 0 < |t| \leqslant \pi \\ 0 \qquad\qquad\qquad , & t = 0 \end{cases}$$

By the definition of D_N, $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,\mathrm{d}t = 1$

$$
\begin{aligned}
(\square\square\square) \Rightarrow 2\pi(S_N(x) - f(x)) &= \int_{-\pi}^{\pi} [f(x-t) - f(x)] D_N(t)\,\mathrm{d}t \\
&= \int_{-\pi}^{\pi} g(t) \sin[(N + \frac{1}{2})t]\,\mathrm{d}t \\
&= \int_{-\pi}^{\pi} [g(t)\cos\frac{t}{2}] \sin Nt\,\mathrm{d}t + \int_{-\pi}^{\pi} [g(t)\sin\frac{t}{2}] \cos Nt\,\mathrm{d}t
\end{aligned}
$$

$$
|f(x+t) - f(x)| \leqslant M|t| \Rightarrow \limsup_{t\to 0} |g(t)| \leqslant \limsup_{t\to 0} \frac{M|t|}{|\sin\frac{t}{2}|} = 2M
$$

$$
\Rightarrow g(t)\cos(\frac{t}{2}) \quad \textit{and} \quad g(t)\sin(\frac{t}{2}) \in \mathscr{R} \quad \textit{on} \quad [-\pi, \pi]
$$

Bessel'inequality

$$
\Rightarrow \lim_{N\in\infty} \int_{-\pi}^{\pi} h(t)\sin(Nt)\,\mathrm{d}t = \lim_{N\to\infty} \int_{-\pi}^{\pi} h(t)\cos(Nt)\,\mathrm{d}t = 0,\ \forall h \in \mathscr{R}\ \textit{on}\ [-\pi, \pi]
$$

$$
\Rightarrow \lim_{N\to\infty} S_N(x) = f(x) \qquad \square
$$

Corollary 3. (1) *If* $f(x) = 0$ *for* $\forall x \in (a, b)$*, then* $\lim_{N\to\infty} S_N(x) = 0$ *for* $\forall x \in (a, b)$*.*

(2) *If* $f(t) = g(t)$ *for* $\forall t$ *in some NBHD of* x*, then* $S_N(f; x) - S_N(g; x) = S_N(f - g; x) \to 0$ *as* $N \to \infty$

Theorem 8.2.4. *If* f *is continuous (with period* 2π*) and if* $\epsilon > 0$*, then there is a trigonometric polynomial* P *s.t.* $|P(x) - f(x)| < \epsilon$ *for* $\forall x \in \mathbb{R}$*.*

The proof is given by homework.

Theorem 8.2.5 (Parseval's theorem)**.** *Suppose* f *and* g *are Riemann-integrable on* $[-\pi, \pi]$ *with period* 2π*, and*

$$
f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad g(x) \sim \sum_{n=-\infty}^{\infty} d_n e^{inx}
$$

Then

a) $\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f; x)|\,\mathrm{d}x = 0,$

b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} c_n\overline{d_n}$,

c) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} |c_n|^2$

Proof. Fix $\epsilon > 0$, EX5 of HW7 $\Rightarrow \exists$ a continuous 2π-periodic function h s.t.

$$||f-h||_2 := [\int_{-\pi}^{\pi} |f(x)-h(x)|^2\,\mathrm{d}x] < \epsilon$$

The previous theorem $\Rightarrow \exists$ trigonometric polynomial P s.t. $|h(x)-P(x)| < \frac{\epsilon}{\sqrt{2\pi}}$, $\forall x \in \mathbb{R}$

$\Rightarrow ||h-P||_2 < \epsilon$

Suppose P has degree N_0, the 1st theorem in this section $\Rightarrow ||h-S_N(h)||_2 \leqslant ||h-P||_2 < \epsilon$, $\forall N \geqslant N_0$.

$$\begin{aligned}
(\triangle) &\Rightarrow ||S_N(h)-S_N(f)||_2 = ||S_N(h-f)||_2 \leqslant ||h-f||_2 < \epsilon \\
&\Rightarrow ||f-S_N(f)||_2 \leqslant ||f-h||_2 + ||h-S_N(h)||_2 + ||S_N(h)-S_N(f)||_2 < 3\epsilon,\ \forall N \geqslant N_0 \\
&\Rightarrow \lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f-S_N(f)|^2\,\mathrm{d}x = 0 \qquad (\star)
\end{aligned}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S_N(f)\overline{g}\,\mathrm{d}x = \sum_{n=-N}^{N} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\overline{g(x)}\,\mathrm{d}x = \sum_{n=-N}^{N} c_n\overline{d_n}$$

$$\begin{aligned}
\Rightarrow \left|\int_{-\pi}^{\pi} f\overline{g}\,\mathrm{d}x - \int_{-\pi}^{\pi} S_N(f)\overline{g}\,\mathrm{d}x\right| &\leqslant \int_{-\pi}^{\pi} |f-S_N(f)|\cdot|g|\,\mathrm{d}x \\
&\overset{C-S}{\leqslant} [\int_{-\pi}^{\pi} |f-S_N(f)|^2\,\mathrm{d}x \cdot \int_{-\pi}^{\pi} |g|^2\,\mathrm{d}x]^{\frac{1}{2}} \\
&\to 0 \text{ as } N\to\infty \text{ by } (\star)
\end{aligned}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{1}{2\pi}\lim_{N\to\infty}\int_{-\pi}^{\pi} S_N(f)\overline{g}\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} c_n\overline{d_n}.$$

Setting $f=g$, we get $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} |c_n|^2$ $\square$

8.3 The Gamma Function

$$\Gamma(x) := \int_0^{\infty} t^{x-1}e^{-t}\,\mathrm{d}t,\ x \in (0,\infty)$$

Note that the integral converges for $x \in (0,\infty)$.

Theorem 8.3.1. *(a)* $\Gamma(x+1) = x\Gamma(x)$, $x \in (0,\infty)$

(b) $\Gamma(n+1) = n!$, $n \in \mathbb{N}$

(c) $\ln\Gamma$ *is convex on* $(0,\infty)$.

Proof. We only prove (c)
Let $p \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned}
\Gamma(\frac{x}{p} + \frac{y}{q}) &= \int_0^\infty t^{\frac{x}{p}+\frac{y}{q}-1} e^{-t}\,\mathrm{d}t \\
&= \int_0^\infty (t^{\frac{x}{p}-\frac{1}{p}} e^{-\frac{t}{p}})(t^{\frac{y}{q}-\frac{1}{q}} e^{-\frac{t}{q}})\,\mathrm{d}x \\
&\leqslant [\Gamma(x)]^{\frac{1}{p}} [\Gamma(y)]^{\frac{1}{q}} \quad \text{(Holder's inequality)} \\
&\Rightarrow \ln\Gamma \text{ is convex on } (0,\infty).
\end{aligned}$$

□

Theorem 8.3.2. *If* $f : (0,\infty) \to (0,\infty)$ *satisfies:*

(a) $f(x+1) = xf(x)$, $\forall x \in (0,\infty)$

(b) f(1)=1

(c) $\ln f$ *is convex on* $(0,\infty)$

then $f(x) = \Gamma(x)$

Proof. Γ satisfies (a), (b) and (c). So it is enough to prove that $f(x)$ is unique determined by (a), (b) and (c) for $\forall x > 0$.
Actually, it's enough to prove this for $\forall x \in (0,1)$ as we use (a) and (b). Let $\varphi(x) = \ln f(x)$, $x > 0$

$$\begin{aligned}
\varphi \text{ convex } &\Rightarrow \ln n = \frac{\varphi(n+1) - \varphi(n)}{(n+1) - n} \leqslant \frac{\varphi(n+1+x) - \varphi(n+1)}{(n+1+x) - (n+1)} \leqslant \frac{\varphi(n+2) - \varphi(n+1)}{(n+2) - (n+1)} = \ln(n+1) \\
&\Rightarrow \varphi(x) = \lim_{n\to\infty} \ln[\frac{n! n^x}{x(x+1)\cdots(x+n)}] \\
&\Rightarrow \varphi(x) \text{ is unique determined on } (0,1)
\end{aligned}$$

□

Corollary 4. $\Gamma(x) = \lim_{n\to\infty} \ln[\frac{n!n^x}{x(x+1)\cdots(x+n)}]$, $x > 0$

Theorem 8.3.3. $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $\forall x, y > 0$

Proof. Fix $y > 0$, $b(1,y) = \int_0^1 (1-t)^{y-1}\,\mathrm{d}t = \frac{1}{y}$, and

$$B(\frac{x_1}{p} + \frac{x_2}{q}, y) \overset{Holder}{\leqslant} [B(x_1,y)]^{\frac{1}{p}}[B(x_2,y)]^{\frac{1}{q}}, \quad \forall x_1, x_2 > 0, \frac{1}{p} + \frac{1}{q} = 1, p > 1$$

$$\Rightarrow \ln B(\cdot, y) \text{ is convex on } (0,\infty)$$

$$B(x+1,y) = \int_0^1 (\frac{t}{1-t})^x (1-t)^{x+y-1}\,\mathrm{d}t \xlongequal{I.B.P} B(x,y), \quad \forall x > 0$$

$$\Rightarrow f(x) := \frac{\Gamma(x+y)}{\Gamma(y)} B(x,y) \text{ satisfies (a), (b) and (c) of the previous theorem}$$

$$\Rightarrow f(x) = \Gamma(x), \forall x > 0$$

$\square$

Some applications:

1) If we set $t = \sin^2\theta$ in the beta function, we get

$$2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2x-1}\theta\,\mathrm{d}\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \forall x, y > 0$$

Let $x = y = \frac{1}{2} \Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$

2) $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \xlongequal{t=s^2} 2\int_0^\infty s^{2x-1}e^{-s^2}\,\mathrm{d}s$, $x > 0$
Setting $x = \frac{1}{2} \Rightarrow \int_0^\infty e^{-s^2}\,\mathrm{d}s = \frac{\sqrt{\pi}}{2}$

3) The 2nd theorem in this section $\Rightarrow$ Duplication formula.

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2})$$

Stirling formula. $\displaystyle\lim_{x\to\infty} \frac{\Gamma(x+1)}{(\frac{x}{e})^x\sqrt{2\pi x}} = 1$

Proof.

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t \xlongequal{t=x(1+u)} \int_{-1}^\infty e^{-x(1+u)}x\,\mathrm{d}u = x^{x+1}e^{-x}\int_{-1}^\infty [(1+u)e^{-u}]^x\,\mathrm{d}u, \ x > 0 \tag{$\square$}$$

We define $h(u)$ s.t. $h(0)=1$, and $(1+u)e^{-u}=e^{-\frac{u^2}{2}h(u)}, u\in(-1,\infty), u\neq 0$

$$\Rightarrow h(u)=\frac{2}{u^2}[u-\ln(1+u)],\ \forall u\in(-1,\infty), u\neq 0$$
$$\Rightarrow h'(u)=2u^{-3}[2\ln(1+u)-u-\frac{u}{1+u}]<0,\ \forall u\in(-1,\infty) \qquad (h'(0)=0)$$
$\Rightarrow$ h is continuous, $h(u)$ decrease strictly from ∞ to 0 as u increases from -1 to ∞

$$(\square)\Rightarrow \Gamma(x+1)=x^{x+1}e^{-x}\int_{-1}^{\int} e^{-\frac{u^2x}{2}h(u)}\,\mathrm{d}u \overset{u=s\sqrt{\frac{2}{x}}}{=\!=\!=} x^x e^{-x}\sqrt{2x}\int_{-\sqrt{\frac{x}{2}}}^{\infty}\psi_x(s)\,\mathrm{d}s,\ \forall x>0 \tag{$\square\square$}$$

where

$$\psi_x(s):=\begin{cases} e^{-s^2h(s\sqrt{\frac{2}{x}})}, & -\sqrt{\dfrac{x}{2}}<s<\infty \\ 0 & ,\ s\leqslant -\sqrt{\dfrac{x}{2}} \end{cases}$$

It is easy to check:

① $\forall$ fixed $s\in\mathbb{R}$, $\lim\limits_{x\to\infty}\psi_x(s)=e^{-s^2}$.

② $\psi_x(s)\to e^{-s^2}$ uniformly on $[-M,M]$ as $x\to\infty$ for $\forall M\in\mathbb{R}$.

③ If $s<0$, then $0\leqslant\psi_x(s)\leqslant e^{-s^2}$, $\forall x>0$.

④ If $s>0$, then $0\leqslant\psi_x(s)\leqslant\psi_1(s)$, $\forall x>1$.

⑤ $\int_0^\infty\psi_1(s)\,\mathrm{d}s=\int_0^\infty e^{-s^2h(s\sqrt{2})}\,\mathrm{d}s=\int_0^\infty(1+s\sqrt{2})e^{-\sqrt{2}s}\,\mathrm{d}s<\infty$

EX6 on HW8 $\Rightarrow \lim\limits_{x\to\infty}\int_{-\infty}^{\infty}\psi_x(s)\,\mathrm{d}s=\int_{-\infty}^{\infty}e^{-s^2}\,\mathrm{d}s=\sqrt{\pi}$.

$$(\square\square)\Rightarrow \lim_{x\to\infty}\frac{\Gamma(x+1)}{(\frac{x}{e})^x\sqrt{2\pi x}}=1$$

$\square$

8.4 A Probabilistic Proof of the Weierstrass Theorem

Theorem 8.4.1. *If $f:[0,1]\to\mathbb{R}$ is continuous, there exists a sequence of polynomials $\{P_n\}$ such that*

$$\lim_{n\to\infty}P_n(x)=f(x) \textit{ uniformly on } [0,1]$$

Ingredients: $X \overset{d}{=}$ binomial distribution with parameter $n \in \mathbb{N}$ and $x \in [0,1]$
$X = X_1 + \cdots + X_n$ where X_j's are independent and $\mathbb{P}(X_j = 1) = x = 1 - \mathbb{P}(X_j = 0)$

Then $\mathbb{P}(X = j) = \binom{n}{j} x^j (1-x)^{n-j}, 0 \leqslant j \leqslant n$.
$\Rightarrow$ mean of X: $\mathbb{E}X := \sum\limits_{j=0}^{n} j\mathbb{P}(X = j) = nx$
and variance of X: $Var(X) := \mathbb{E}(X - \mathbb{E}X)^2 = nx(1-x)$.

Markov's Inequality: Y is a random with $Y \geqslant 0$, then $\mathbb{P}(Y \geqslant a) \leqslant \dfrac{\mathbb{E}Y}{a}$ for $a > 0$

Proof. $F_Y(y) := \mathbb{P}(Y \leqslant y)$, then

$$\mathbb{E}Y = \int_0^\infty y \, \mathrm{d}F_Y(y) \geqslant \int_a^\infty \, \mathrm{d}F_Y(y) \geqslant a \int_a^\infty \, \mathrm{d}F_Y(y) = a\mathbb{P}(Y \geqslant a) \qquad \square$$

Thus

$$\mathbb{P}(|X - \mathbb{E}X| \geqslant k\sqrt{Var(X)}) = \mathbb{P}(|X - \mathbb{E}X|^2 \geqslant k^2 Var(X)) \leqslant \frac{\mathbb{E}|X - \mathbb{E}X|^2}{k^2 Var(X)} = \frac{1}{k^2}$$

This is usually called $Chebyshev' inequality$.

Proof. Let $Y_n := f(\frac{X}{n}), n \in \mathbb{N}$.
Then

$$\mathbb{E}Y = \mathbb{E}f(\frac{X}{n}) = \sum_{j=0}^{n} f(\frac{j}{n})\mathbb{P}(X = j) = \sum_{j=0}^{n} f(\frac{j}{n})\binom{n}{j} x^j (1-x)^{n-j} := B_n(f, x)$$

B_n is called Bernstein polynomial.
We will prove $B_n \to f$ uniformly on $[0,1]$ as $n \to \infty$.
Let $M := \sup\limits_{0 \leqslant x \leqslant 1} |f(x)|$
f is continuous on $[0,1] \Rightarrow \forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \frac{\epsilon}{2}$, $\forall |x - y| < \delta$.

We choose $K \in \mathbb{N}$ s.t. $\frac{2M}{k^2} < 2\epsilon$, and choose $N \in \mathbb{N}$ s.t. $\frac{K}{2\sqrt{N}} < \delta$. Then

$$\begin{aligned}
|B_n(f;x) - f(x)| &= |\sum_{j=0}^{n}[f(\frac{j}{n}) - f(x)]\binom{n}{j}x^j(1-x)^{n-j}| \\
&\leqslant \sum_{j=0}^{n}|f(\frac{j}{n}) - f(x)|\binom{n}{j}x^j(1-x)^{n-j} \\
&< \sum_{|\frac{j}{n}-x|<\frac{k}{2\sqrt{n}}}|f(\frac{j}{n}) - f(x)|\binom{n}{j}x^j(1-x)^{n-j} \\
&+ \sum_{|\frac{j}{n}-x|\geqslant\frac{k}{2\sqrt{n}}}|f(\frac{j}{n}) - f(x)|\binom{n}{j}x^j(1-x)^{n-j} \\
&< \frac{\epsilon}{2}\sum_{j=0}^{n}\binom{n}{j}x^j(1-x)^{n-j} + 2M\cdot\mathbb{P}(|\frac{X}{n} - x| \geqslant \frac{k}{2\sqrt{n}})
\end{aligned}$$

$$\mathbb{P}(|\frac{X}{n} - x| \geqslant \frac{k}{2\sqrt{n}}) = \mathbb{P}(|X - nx| \geqslant \frac{k\sqrt{n}}{2}) \leqslant \mathbb{P}(|X - nx| \geqslant k\sqrt{nx(1-x)}) \leqslant \frac{1}{k^2}$$

$$\Rightarrow |B_n(f;x) - f(x)| < \frac{\epsilon}{2} + 2M\cdot\frac{1}{k^2} < \epsilon, \quad \forall x \in [0,1], \forall n \geqslant N$$

$$\Rightarrow B_n(f;x) \rightrightarrows f(x) \text{ on } [0,1] \text{ as } n \to \infty$$

□

8.5 Stone's Generalization of the Weierstrass Theorem

Corollary 5 (of the Weierstrass theorem)**.** *For every interval* $[-a,a]$ *there is a sequence of real polynomials* P_n *s.t.*

$$P_n(0) = 0 \text{ and } \lim_{n\to\infty} P_n(x) = |x| \text{ uniformly on } [-a,a]$$

A family $\mathscr{A}$ of complex functions defined on a set E is said to be an algebra if

(i) $f + g \in \mathscr{A}$

(ii) $fg \in \mathscr{A}$

(iii) $cf \in \mathscr{A}$ for $\forall f, g \in \mathscr{A}$

If (iii) only holds for $c \in \mathbb{R}$, the $\mathscr{A}$ is an algebra of real functions.

$\mathscr{A}$ is said to be uniformly closed if: $f_n \in \mathscr{A}$ and $f_n \rightrightarrows f$ on $E \Rightarrow f \in \mathscr{A}$.

Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequence of members of $\mathscr{A}$. i.e. $\mathscr{B} = \mathscr{A} \cup \mathscr{A}'$ with $d(f,g) := ||f-g|| = \sup_{x \in E} |f(x) - g(x)|$. Then $\mathscr{B}$ is called the uniform closure of $\mathscr{A}$.

Example 6. *The set of all polynomials is an algebra.*

Weierstrass theorem $\Leftrightarrow$ the set of continuous functions on $[a,b]$ = the uniform closure of the set of polynomials on $[a,b]$.

Theorem 8.5.1. *Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.*

Let $\mathscr{A}$ be a family of functions on E. $\mathscr{A}$ is said to be separate points on E if $\forall x_1 \neq x_2 \in E$, $\exists f \in \mathscr{A}$, $f(x_1) \neq f(x_2)$

We say that $\mathscr{A}$ vanishes at no point of $\forall x \in E$, $\exists g \in \mathscr{A}$ s.t. $g(x) \neq 0$.

Theorem 8.5.2. *Suppose $\mathscr{A}$ is an algebra of functions on E, separate points on E, and vanishes at no point of E. Suppose $x_1 \neq x_2 \in E$ and C_1, C_2 are constants ($C_1, C_2 \in \mathbb{R}$ if $\mathscr{A}$) is a real algebra. Then $\exists f \in \mathscr{A}$ s.t.*

$$f(x_1) = C_1, \quad f(x_2) = C_2$$

Proof. $\exists g, h, k \in \mathscr{A}$ s.t.

$$g(x_1) \neq g(x_2),\ h(x_1) \neq 0,\ k(x_2) \neq 0$$

Let $u(x) := g(x)k(x) - g(x_1)k(x), \quad v(x) := g(x)h(x) - g(x_2)h(x), x \in E$

$$\Rightarrow u \in \mathscr{A} \text{ and } v \in \mathscr{A},\ u(x_1) = v(x_2) = 0, \quad u(x_2) \neq 0, v(x_1) \neq 0$$

$$\Rightarrow f(x) := \frac{C_1 v(x)}{v(x_1)} + \frac{C u(x)}{u(x_2)},\ x \in E \text{ satisfies } f(x_1) = C_1, f(x_2) = C_2 \qquad \square$$

Theorem 8.5.3 (Stone-Weierstrass Theorem)**.** *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanished at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.*

Proof. Claim 1. If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.

Let $a := \sup_{x \in K} |f(x)|$

For $\forall \epsilon > 0$, the corollary of this section $\Rightarrow \exists c_1, \cdots, c_n \in \mathbb{R}$ s.t.

$$|\sum_{j=1}^{n} c_j y^j - |y|| < \epsilon, \quad \forall y \in [-a, a] \tag{$*$}$$

The first theorem in this section $\Rightarrow \mathscr{B}$ is an algebra $\Rightarrow g := \sum_{j=1}^{n} c_j f^j \in \mathscr{B}$.

$(*) \Rightarrow |g(x) - |f(x)|| < \epsilon, \quad \forall x \in K$.

$\mathscr{B}$ is uniformly closed $\Rightarrow |f| \in \mathscr{B}$

Claim 2. $f \in \mathscr{B}, g \in \mathscr{B} \Rightarrow \max(f, g) \in \mathscr{B}$ and $min(f, g) \in \mathscr{B}$

This follows from Claim 1 and

$$\begin{cases} \max(f, g) = \dfrac{f+g}{2} + \dfrac{|f-g|}{2} \\ \min(f, g) = \dfrac{f+g}{2} - \dfrac{|f-g|}{2} \end{cases}$$

Claim 3. $\forall f : K \to \mathbb{R}$ continuous, $\forall x \in K$, $\forall \epsilon > 0$, $\exists g_x \in \mathscr{B}$ s.t.

$$g_x(x) = f(x), \qquad g_x(t) > f(t) - \epsilon, \forall t \in K$$

$\forall y \in K$, the previous theorem $\Rightarrow \exists h_y \in \mathscr{A} \subset \mathscr{B}$ s.t. $h_y(x) = f(x), h_y(y) = f(y)$. h_y continuous $\Rightarrow J_y := \{t \in K : h_y(t) > f(t) - \epsilon\}$ is open and containing x and y

$\Rightarrow K \subset \bigcup_{y \in K \setminus \{x\}} J_y \xRightarrow{K\ compact} \exists y_1, \cdots, y_n$ s.t. $K \subset \bigcup_{j=1}^{n} J_{y_j}$

Let $g_x := \max(h_{y_1}, \cdots, h_{y_n})$. Then $g_x(t) > f(t) - \epsilon$, $\forall t \in K$, $g_x(x) = f(x)$

Claim 4. $\forall f : K \to \mathbb{R}$ continuous, $\forall \epsilon > 0$, $\exists h \in \mathscr{B}$ s.t. $|h(x) - f(x)| < \epsilon$, $\forall x \in K$.

For $\forall x \in K$, let g_x be function constructed in Claim 3.

g_x and f continuous $\Rightarrow V_x := \{t \in K : g_x(t) < f(x) + \epsilon\}$ is open and containing x.

$\Rightarrow K \subset \bigcup_{x \in K} V_x \xRightarrow{K\ compact} \exists x_1, \cdots, x_n$ s.t. $K \subset \bigcup_{j=1}^{n} V_{x_j}$.

Let $h := min(g_{x_1}, \cdots, g_{x_n})$. Then $h(t) < f(t) + \epsilon$, $\forall t \in K$.

Claim 2 $\Rightarrow h \in \mathscr{B}$. $\square$

Example 7. *Let $K : \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, and let $\mathscr{A}$ be the algebra of all functions of the form $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$, $\theta \in [0, 2\pi)$*
Then $\mathscr{A}$ separates points on K and vanishes at no point of K by considering the function $f(e^{i\theta}) = e^{i\theta}$. For $\forall f \in \mathscr{A}$, we have

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} \, \mathrm{d}\theta = 0$$

Let $\mathscr{B}$ be the uniform closure of $\mathscr{A}$. Then $\exists f_n \in \mathscr{A}$, $f_n \rightrightarrows g$, for $g \in \mathscr{B}$. Thus $\int_0^{2\pi} g(e^{i\theta}) e^{i\theta} \, \mathrm{d}\theta = 0$ for $g \in \mathscr{B}$.
$\Rightarrow$ the continuous function $h(e^{i\theta}) = e^{-i\theta} \notin \mathscr{B}$.

So for complex algebra, we need an extra condition: $\mathscr{A}$ is <u>self-adjoint</u>, if

$$\forall f \in \mathscr{A}, \overline{f} \in \mathscr{A} \text{ where } \overline{f}(x) = \overline{f(x)}$$

Theorem 8.5.4. *Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanished at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K.*

Proof. Let $\mathscr{A}_\mathbb{R}$ be the set of all real functions on K which belong to $\mathscr{A}$.

$$\forall f \in \mathscr{A} \Rightarrow f = u + iv \text{ where } u \text{ and } v \text{ are real.}$$
$$\Rightarrow u = \frac{1}{2}(f + \overline{f}) \in \mathscr{A}_\mathbb{R} \text{ since } \mathscr{A} \text{ is self-adjoint.}$$

Easy to check that $\mathscr{A}_\mathbb{R}$ separates points on K with Theorem 8.5.2.
$\forall x_0 \in K \Rightarrow \exists g \in \mathscr{A}$ s.t. $g(x_0) \neq 0$. Let $f(x) = \overline{g(x)} g(x)$, $x \in K$. $\Rightarrow f(x_0) \neq 0$ and $f(x) \in \mathscr{A}_\mathbb{R}$.
$\Rightarrow \mathscr{A}_\mathbb{R}$ vanishes at no point of K The Stone-Weierstrass $\Rightarrow \forall$ continuous $f : K \to \mathbb{R}$ lies in the uniform closure of $\mathscr{A}_\mathbb{R} \Rightarrow f \in \mathscr{B}$.
So for $\forall$ continuous $g : K \to \mathbb{C}$, $\operatorname{Re} g, \operatorname{Im} g \in \mathscr{B}$. $\Rightarrow g \in \mathscr{B}$. $\square$

9 Functions of Several Variables

9.1 Linear Transformations

A nonempty set $X \subset \mathbb{R}^n$ is a vector space if $c_1\vec{x} + c_2\vec{y} \in X$ for $\forall \vec{x}, \vec{y} \in X$ and $\forall c_1, c_2 \in \mathbb{R}$.

Note that if $B = \{\vec{x_1}, \cdots, \vec{x_r}\}$ is a basis of X, then $\forall \vec{x} \in X$ ha s a unique representation of the form $\vec{x} = \sum\limits_{j=1}^{r} c_j\vec{x_j}$. The numbers $c_1, c_2, \cdots, c_r$ are called coordinates of $\vec{x}$ w.r.t. B.

We called $\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$ the standard basis of $\mathbb{R}^n$, where $\vec{e_j} = (0, \cdots, 1, \cdots, 0)$.

Theorem 9.1.1. *Let $r \in \mathbb{N}$. If a vector space X is spanned by a set of r vectors, then* $\dim X \leqslant r$.

Theorem 9.1.2. *Suppose X is a vector space, and* $\dim X = n$. *Then*

(a) A set E of n vectors in X spans X iff E is independent.

(b) X has a basis, and every basis consists of n vectors.

(c) If $1 \leqslant r \leqslant n$ and $\{\vec{y_1}, \vec{y_2}, \cdots, \vec{y_r}\}$ is a independent set in X, then X has a basis contained $\{\vec{y_1}, \cdots, \vec{y_r}\}$.

A mapping $A : X \to Y$ is said to be a linear transformations (or linear operator) if $A(\lambda_1\vec{x_1} + \lambda_2\vec{x_2}) = \lambda_1 A(\vec{x_1}) + \lambda_2 A(\vec{x_2})$, $\forall \vec{x_1}, \vec{x_2} \in X, \forall \lambda_1, \lambda_2 \in \mathbb{R}$.

If A is a linear operator on X satisfying one-to-one and maps X onto X, we say that A is invertible.

Theorem 9.1.3. *A linear operator A on a finite-dimensional vector space X is 1-1 iff the range of A is all of X.*

Let $L(X, Y)$ be the set of all linear transformations of the vector space X into the vector space Y. We usually write $L(X)$ for $L(X, Y)$.

If $A_1, A_2 \in L(X, Y)$, we define

$$(c_1A_1 + c_2A_2)\vec{x} := c_1A_1\vec{x} + c_2A_2\vec{x}$$

If X, Y, Z are vector spaces, and if $A \in L(X,Y)$ and $B \in L(Y,Z)$, we define their product BA by

$$(BA)\vec{x} := B(A\vec{x}), \forall \vec{x} \in X$$

For $\forall A \in L(\mathbb{R}^n, \mathbb{R}^m)$, we define the norm $||A||$ of A by

$$||A|| := \sup_{\vec{x}:|\vec{x}|\leqslant 1} |A\vec{x}|$$

Theorem 9.1.4. *(a) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$,then A is uniformly continuous and thus $||A|| < \infty$.*

(b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $c \in \mathbb{R}$, then $||A+B|| \leqslant ||A|| + ||B||, ||cA|| = |c| \cdot ||A||$. Hence, $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space with $d(A,B) := ||A-B||$

(c) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then $||BA|| \leqslant ||B|| \cdot ||A||$

Theorem 9.1.5. *Let Ω be the set of all invertible linear operator on $\mathbb{R}$*

(a) If $A \in \Omega, B \in L(\mathbb{R}^n)$ and $||B-A|| \cdot ||A^{-1}|| < 1$, then $B \in \Omega$.

(b) Ω is an open subset of $L(\mathbb{R}^n)$, and the mapping $\Omega \to \Omega, A \mapsto A^{-1}$ is continuous

(a) just use the 3rd theorem.

Suppose $\{\vec{x_1}, \cdots, \vec{x_n}\}$ is a basis of X, $\{\vec{y_1}, \cdots, \vec{y_m}\}$ is a basis of Y. Then $\forall A \in L(X,Y)$ determines a set of numbers a_{ij} s.t.

$$A\vec{x_j} = \sum_{i=1}^{m} a_{ij}\vec{y_i}, \qquad 1 \leqslant j \leqslant n \tag{$*$}$$

It is convenient to visulize these numbers on an $m \times n$ matrix:

$$[A] = [a_{ij}]$$

Then we find that there is a natural 1-1 correspondence between $L(X,Y)$ and the set of all $m \times n$ real matrices

and $[BA] = [B] \cdot [A]$

Suppose $\{x_i\}$ and $\{y_i\}$ are standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$. Then

$$|A\vec{x}|^2 \leqslant |\vec{x}|^2 \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij})$$

$$\Rightarrow ||A|| \leqslant (\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij})^{\frac{1}{2}}$$

So we just proved

Theorem 9.1.6. *If S is a metric space and $a_{11}, \cdots, a_{mn}$ are real continuous functions on S, and if , for $\forall p \in S$, A_p is the linear transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$ whose matrix has entries $a_{ij}(p)$, then mapping $S \to L(\mathbb{R}^n, \mathbb{R}^m), p \mapsto A_p$ is continuous*

9.2 Differentiation

Suppose E is an open set in $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$, and $\vec{x} \in E$. If there is an $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{\vec{h}\to\vec{0}} \frac{|f(\vec{x}+\vec{h}) - f(\vec{x}) - A\vec{h}|}{|\vec{h}|} = 0 \tag{$*$}$$

then we say that f is <u>differentiable</u> at $\vec{x}$, and we write $f'(\vec{x}) = A$.
If f is differentiable at each $\vec{x} \in E$, we say f is differentiable in E.

Theorem 9.2.1. *In the above definition, if $(*)$ holds for A_1 and A_2, then $A_1 = A_2$.*

Proof. Easy to know:

$$\lim_{\vec{h}\to 0} \frac{|A_1\vec{h} - A_2\vec{h}|}{|\vec{h}|} = 0 \Rightarrow A_1 = A_2$$

□

Remark 6. (a) *$(*)$ can be rewritten in the form*

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = f'(\vec{x})\vec{h} + r(\vec{h}) \tag{$**$}$$

where the remainder $r(\vec{h})$ satisfies $\lim_{\vec{h}\to\vec{0}} \frac{|r(\vec{h})|}{|\vec{h}|} = 0$

(b) *The derivative defined by $(*)$ or $(**)$ is often called the <u>differential</u> of f at $\vec{x}$, or the <u>total derivative</u> of f at $\vec{x}$.*

(c) *$(**) \Rightarrow f$ is continuous at any point where f is differentiable.*

(*d*) *If f is differentiable in E, then f' is a function which maps E into $L(\mathbb{R}^n, \mathbb{R}^m)$.*

Example 8. *If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\vec{x} \in \mathbb{R}^n$, then $A'(\vec{x}) = A$.*

Theorem 9.2.2 (chain rule)**.** *Suppose $E \subset \mathbb{R}^n$ open, $f : E \to \mathbb{R}^m$, f is differentiable at $\vec{x_0} \in E$, g maps an open set containing $f(E)$ into $\mathbb{R}^k$, and g is differentiable at $f(\vec{x_0})$, Then the mapping $F : E \to \mathbb{R}^k, \vec{x} \mapsto g(f(\vec{x}))$ is differentiable at $\vec{x_0}$ and*

$$F'(\vec{x_0}) = g'(f(\vec{x_0}))f'(\vec{x_0})$$

Let $\{e_1, \vec{\cdots} \vec{e_n}\}$ and $\{\vec{u_1}, \cdots, \vec{u_m}\}$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.
For $\forall f : E \to \mathbb{R}^m$, the components of f are the real functions $f_1, \cdots, f_m$ defined by $f_i(\vec{x}) := f(\vec{x}) \cdot \vec{u_i}$, $1 \leqslant i \leqslant m$.
For $\vec{x} \in E$, $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$, we define

$$\frac{\partial f_i}{\partial x_j}(\vec{x}) := \lim_{t \to 0} \frac{f_i(\vec{x} + t\vec{e_j}) - f_i(\vec{x})}{t}$$

provided the limit exists. $\frac{\partial f_i}{\partial x_j}$ is the derivative of f_i w.r.t x_j. Keeping the other variables fixed. $\frac{\partial f_i}{\partial x_j}$ is called a partial derivative.

Theorem 9.2.3. *Suppose $f : E \to \mathbb{E}^m$, and f is differentiable at $\vec{x} \in E$. Then the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist, and*

$$f'(\vec{x})(\vec{e_j}) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(\vec{x})\vec{u_i} \tag{$\star$}$$

Proof. f is differentiable at $\vec{x_0} \Rightarrow \lim_{h \to \vec{0}} \frac{|f(\vec{x} + \vec{h}) - f(\vec{x}) - f'(\vec{x})\vec{h}|}{|\vec{h}|} = 0$

$$\begin{aligned}
&\Rightarrow \lim_{t \to 0} \frac{|f(\vec{x} + t\vec{e_j}) - f(\vec{x}) - f'(\vec{x})t\vec{e_j}|}{t} = 0 \\
&\Rightarrow \lim_{t \to 0} \frac{|f_i(\vec{x} + t\vec{e_j}) - f_i(\vec{x}) - tf'(\vec{x})\vec{e_j} \cdot \vec{u_i}|}{t} = 0 \\
&\Rightarrow \frac{\partial f_i}{\partial x_j}(\vec{x}) = \lim_{t \to 0} \frac{f_i(\vec{x} + t\vec{e_j}) - f_i(\vec{x})}{t} = f'(x)(\vec{e_j}) \cdot \vec{u_i}
\end{aligned}$$

$\square$

Remark 7. *(a) Let $[f'(\vec{x})]$ be the matrix which represents $f'(\vec{x})$ w.r.t. our standard bases. Then*

$$[f'(\vec{x})] = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \cdots & \frac{\partial f_1}{\partial x_m}(\vec{x}) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{x}) & \cdots & \frac{\partial f_n}{\partial x_m}(\vec{x}) \end{pmatrix}$$

(b) If $\vec{h} = \sum_{j=1}^{n} h_j \vec{e_j} \in \mathbb{R}^n$, then $(\star) \Rightarrow f'(\vec{x})\vec{h} = [f'(\vec{x})] \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$

Let $\gamma : (a, b) \to E$ be differentiable in (a, b), and $f : E \to \mathbb{R}$ be differentiable in E. Define $g(t) = f(\gamma(t))$.

Chain rule $\Rightarrow g'(t) = f'(\gamma(t))\gamma'(t)$, $t \in (a, b)$.

w.r.t. the standard basis $\{\vec{e_i}, \cdots, \vec{e_n}\}$ of $\mathbb{R}^n$, $[\gamma'(t)] = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \vdots \\ \gamma_n'(t) \end{pmatrix}$

For $\forall \vec{x} \in E$, $[f'(\vec{x})] = \left(\frac{\partial f}{\partial x_1}(\vec{x}), \cdots, \frac{\partial f}{\partial x_n}(\vec{x}) \right)$

$$\Rightarrow g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma(t))\gamma_i'(t) \tag{$\triangle$}$$

The <u>gradient</u> of f at $\vec{x} \in E$ is defined by $\nabla f(\vec{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})\vec{e_i}$.

So $(\triangle)$ can be written in the form

$$g'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) \tag{$\triangle\triangle$}$$

Fix an $\vec{x} \in E$, Let $\vec{u} \in \mathbb{R}^n$ be a unit vector, and specialize γ s.t. $\gamma(t) = \vec{x} + t\vec{u}$, $t \in \mathbb{R}$. Then $\gamma'(t) = \vec{u}$ for $\forall t \in \mathbb{R}$. $(\triangle\triangle) \Rightarrow g'(0) = \nabla f(\vec{x})\vec{u}$.

$$\Rightarrow \lim_{t \to 0} \frac{f(\vec{x} + t\vec{u}) - f(\vec{x})}{t} = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \nabla f(\vec{x})\vec{u} \tag{$\triangle\triangle\triangle$}$$

The limit $\lim_{t \to 0} \frac{f(\vec{x} + t\vec{u}) - f(\vec{x})}{t}$ is called the <u>directional derivative</u> of f at $\vec{x}$, in the direction of the unit vector $\vec{u}$, and may be denoted by $\nabla_{\vec{u}} f(\vec{x})$ or $D_{\vec{u}} f(\vec{x})$

If $\vec{u} = \sum_{i=1}^{n} u_i \vec{e_i}$, $\Rightarrow \nabla_{\vec{u}} f(\vec{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) u_i$

Theorem 9.2.4. *Suppose $f : E \to \mathbb{R}^m$ is differentiable in E where $E \subset \mathbb{R}^n$ is convex and open, and $\exists M \in \mathbb{R}$ s.t. $||f'(x)|| \leqslant M$, $\forall \vec{x} \in E$. Then*

$$|f(\vec{b}) - f(\vec{a})| \leqslant M|\vec{b} - \vec{a}|, \, \forall \vec{a}, \vec{b} \in E$$

Proof. Define $\gamma(t) = (1-t)\vec{a} + t\vec{b}$
E is convex $\Rightarrow \gamma(t) \in E$ for $\forall t \in [0, 1]$.
Let $g(t) = f(\gamma(t))$.
Then $g'(t) = f'(\gamma(t))\gamma'(t) = f'(\gamma(t))(\vec{b} - \vec{a})$
$\Rightarrow |g'(t)| \leqslant ||f'(\gamma(t))|| \cdot |\vec{b} - \vec{a}| \leqslant M \cdot |\vec{b} - \vec{a}|, \forall t \in [0, 1]$.
The last theorem in §5.4 (Weak MVT for vector-valued functions) $\Rightarrow$
$|g(1) - g(0)| \leqslant M|\vec{b} - \vec{a}|$. □

Corollary 6. *If, in addition, $f'(\vec{x}) = 0$ for $\forall \vec{x} \in E$, then f is constant.*

A differentiable mapping $f : E \to \mathbb{R}^m$ is continuously differentiable in E if $f' : E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.
If so, we say that f is a $\mathscr{C}'$-mapping or that $f \in \mathscr{C}'(E)$.

Theorem 9.2.5. *Suppose $f : E \to \mathbb{R}^m$ where $E \subset \mathbb{R}^n$ is open. Then $f \in \mathscr{C}'(E)$ iff $\frac{\partial f}{\partial x_j}$ exists and are continuous on E for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$.*

Proof. $' \Rightarrow'$ Recall $||f'(\vec{y}) - f'(\vec{x})|| = \sup\limits_{|\vec{z}|=1} |[f'(\vec{y}) - f'(\vec{x})]\vec{z}|$
We already proved in the 3rd theorem that $\frac{\partial f}{\partial x_j}$ exist.
Taking $\vec{z} = \vec{e_j}$, we get

$$||f'(\vec{y}) - f'(\vec{x})|| \geqslant |[f'(\vec{y}) - f'(\vec{x})]\vec{e_j}| = \left\{ \sum_{i=1}^{m} [\frac{\partial f_i}{\partial x_j}(\vec{y}) - \frac{\partial f_i}{\partial x_j}(\vec{x})]^2 \right\}^{\frac{1}{2}} \geqslant |\frac{\partial f_i}{\partial x_j}(\vec{y}) - \frac{\partial f_i}{\partial x_j}(\vec{x})|$$

$\Rightarrow$ $\frac{\partial f_i}{\partial x_j}$ is continuous on E for $\forall i \leqslant m, 1 \leqslant j \leqslant n$
$' \Leftarrow'$ It is enough to prove:

$$\forall \vec{x} \in E, \lim_{\vec{h} \to \vec{0}} \frac{|f(\vec{x} + \vec{h}) - f(\vec{x}) - f'(\vec{x})\vec{h}|}{|\vec{h}|} = 0 \qquad (*)$$

Where

$$[f'(\vec{x})] = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\vec{x}) & \cdots & \dfrac{\partial f_1}{\partial x_m}(\vec{x}) \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(\vec{x}) & \cdots & \dfrac{\partial f_n}{\partial x_m}(\vec{x}) \end{pmatrix}$$

Note that the continuity of f' in E follows the last theorem in 9.1.

$(*)$ follows if we can prove it for each component.

We now fix i.

$\frac{\partial f_i}{\partial x_j}$ are continuous at $\vec{x} \Rightarrow \forall \epsilon > 0, \exists r > 0$ s.t.

$$\left| \frac{\partial f_i}{\partial x_j}(\vec{y}) - \frac{\partial f_i}{\partial x_j}(\vec{x}) \right| < \frac{\epsilon}{n}, \forall |\vec{y} - \vec{x}| < r, 1 \leqslant j \leqslant n \qquad (**)$$

Then

$$\begin{aligned} |f_i(\vec{x} + \vec{h}) - f_i(\vec{x}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\vec{x}) h_j| &\leqslant \sum_{t=1}^{n} |f_i(\vec{x} + \sum_{j=1}^{t} h_j \vec{e_j}) - f_i(\vec{x} + \sum_{j=1}^{t-1} h_j \vec{e_j}) - \frac{\partial f_i}{\partial x_t}(\vec{x}) h_t| \\ &\overset{MVT}{=\!=\!=} \sum_{t=1}^{n} \left| \frac{\partial f_i}{\partial x_t}(\vec{x} + \sum_{j=1}^{t-1} h_j \vec{e_j} + \theta_t h_t \vec{e_t}) \cdot h_n - \frac{\partial f_i}{\partial x_t}(\vec{x}) h_t \right| \\ &\overset{(**)}{\leqslant} \frac{\epsilon}{n} [\sum_{t=1}^{n} |h_t|] \\ &\leqslant \epsilon |h| \end{aligned}$$

□

9.3 The Contraction Principle

Let X be a metric space with metric d. If $\varphi : X \to X$ and $\exists c \in [0, 1)$ s.t.

$$d(\varphi(x), \varphi(y)) \leqslant c\, d(x, y), \quad \forall x, y \in X$$

then φ is said to be a <u>contraction</u> of X into X.

Theorem 9.3.1 (Banach fixed point theorem)**.** *If X is a complete metric space, and if φ is a contraction of X into X, then there exists one and only one $x \in X$ s.t. $\varphi(x) = x$.*

Proof. The proof is given in Mid-exam. □

9.4 The Inverse Function Theorem

Theorem 9.4.1 (the inverse function theorem)**.** *Suppose $E \subset \mathbb{R}^n$ is open, and $f : E \to \mathbb{R}^n$, $f \in \mathscr{C}'(E)$, $f'(\vec{a})$ is invertible for some $\vec{a} \in E$, and $\vec{b} = f(\vec{a})$. Then*

(a) *$\exists$ open sets $U, V \subset \mathbb{R}^n$ s.t. $\vec{a} \in U, \vec{b} \in V$, $f : U \to V$ is 1-1 and onto.*

(b) *if g is the inverse of f, defined on V by $g(f(\vec{x})) = \vec{x}$, $\forall \vec{x} \in V$, then $g \in \mathscr{C}'(V)$.*

Remark 8. *The theorem says: the system of n equation:*

$$y_i = f_i(x_1, \cdots, x_n), \quad 1 \leqslant i \leqslant n$$

can be solved for $x_1, \cdots, x_n$ in terms of $y_1, \cdots, y_n$, if we restrict $\vec{x}$ and $\vec{y}$ to small enough NBHDs of $\vec{a}$ and $\vec{b}$.

Proof. Let $A := f'(\vec{a})$, and choose λ s.t.

$$2\lambda ||A^{-1}|| = 1 \tag{$*$}$$

f' is continuous at $\vec{a} \Rightarrow \exists$ open ball $U \subset E$ centered at $\vec{a}$ s.t.

$$||f'(\vec{x}) - A|| < \lambda, \quad \forall \vec{x} \in U \tag{$**$}$$

For each $\vec{y} \in \mathbb{R}^n$, we define a function φ by

$$\varphi(\vec{x}) = \vec{x} + A^{-1}(\vec{y} - f(\vec{x})), \quad \forall \vec{x} \in E \tag{$***$}$$

Then $f(\vec{x}) = \vec{y}$ iff $\vec{x}$ is a fixed point of φ. And

$$\varphi'(\vec{x}) = I - A^{-1} f'(\vec{x}) = A^{-1}(A - f'(\vec{x}))$$

$$(*)(**) \Rightarrow ||\varphi(\vec{x})|| < \frac{1}{2}, \quad \forall \vec{x} \in U$$

4th theorem in 9.2 $\Rightarrow$

$$|\varphi(\vec{x_1}) - \varphi(\vec{x_2})| \leqslant \frac{1}{2}|\vec{x_1} - \vec{x_2}|, \forall \vec{x_1}, \vec{x_2} \in U \tag{$****$}$$

The uniqueness part of the fixed point theorem $\Rightarrow \varphi$ has at most one fixed point in U.

$\Rightarrow f(\vec{x}) = \vec{y}$ for at most one $\vec{x} \in U \Rightarrow f$ is 1-1 in U.

Let $V := f(U)$ and pick $\vec{y_0} \in V$. Then $\vec{y_0} = f(\vec{x_0})$ for some $\vec{x_0} \in U$.

Let $B := N_r(\vec{x_0})$ with $r > 0$ s.t. $\overline{B} \subset U$

For $\forall \vec{y} \in \mathbb{R}$ with $|\vec{y} - \vec{y_0}| < \lambda r$, we will prove $\vec{y} \in V$ (Thus V is open).

$$|\varphi(\vec{x_0}) - \vec{x_0}| \xlongequal{(***)} |A^{-1}(\vec{y} - f(\vec{x_0}))| = |A^{-1}(\vec{y} - \vec{y_0})| < ||A^{-1}||\lambda r \overset{(*)}{=} \frac{r}{2}$$

For $\forall \vec{x} \in \overline{B}$, $|\varphi(x) - \vec{x_0}| \leqslant |\varphi(\vec{x}) - \varphi(\vec{x_0})| + |\varphi(\vec{x_0}) - \vec{x_0}| \overset{(****)}{<} \frac{1}{2}|\vec{x} - \vec{x_0}| + \frac{r}{2} \leqslant r$.

$\Rightarrow \varphi(\vec{x}) \in B \Rightarrow \varphi$ is a contraction of $\overline{B}$ into $\overline{B}$.

The fixed point theorem $\Rightarrow \varphi$ has a fixed point $\Rightarrow \vec{y} \in f(\overline{B}) \subset f(U) = V$

Now let $\vec{y} \in V$ and $\vec{y} + \vec{k} \in V$.

$\Rightarrow \exists \vec{x} \in U, \vec{x} + \vec{h} \in U$ s.t. $\vec{y} = f(\vec{x}), \vec{y} + \vec{k} = f(\vec{x} + \vec{h})$

$(***) \Rightarrow \varphi(\vec{x} + \vec{h}) - \varphi(\vec{x}) = \vec{h} - A^{-1}\vec{k}$

$$\begin{aligned}(****) &\Rightarrow |\vec{h} - A^{-1}\vec{k}| < \frac{1}{2}|\vec{h}| \\ &\Rightarrow |A^{-1}\vec{k}| \geqslant |\vec{h}| - |\vec{h} - A^{-1}\vec{k}| > \frac{1}{2}|\vec{h}| \\ &\Rightarrow |\vec{h}| \leqslant 2|A^{-1}\vec{k}| \leqslant 2||A^{-1}|| \cdot ||\vec{k}|| = \lambda^{-1}|\vec{k}| \qquad (\square)\end{aligned}$$

With Theorem 9.1.5, $f'(\vec{x})$ has an inverse, say T.

$$\Rightarrow \frac{|g(\vec{y} + \vec{k}) - g(\vec{y}) - T\vec{k}|}{|\vec{k}|} \overset{(\square)}{\leqslant} \frac{||T||}{\lambda} \cdot \frac{|f(\vec{x} + \vec{h}) - f(\vec{x}) - f'(\vec{x})\vec{h}|}{|\vec{h}|} \qquad (\square\square)$$

$\Rightarrow g'(\vec{y}) = T$ is continuous in V with 5th theorem in section 9.1. $\square$

An immediate consequence of part (a) of the previous theorem is

Theorem 9.4.2. *If $f : E \to \mathbb{R}^n$ and $f \in \mathscr{C}'(E)$, and $f'(\vec{x})$ is invertible for $\forall \vec{x} \in E$, then $f(W)$ is an open subset of $\mathbb{R}^n$ for every open set $W \subset E$. i.e. f is an open mapping of E into $\mathbb{R}^n$.*

9.5 The Implicit Function Theorem

If $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \cdots, y_m) \in \mathbb{R}^m$, we write $(\vec{x}, \vec{y})$ for the point

$$(x_1, \cdots, x_n, y_1, \cdots, y_m) \in \mathbb{R}^{n+m}$$

$\forall A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ can be split into two linear tranformations A_x and A_y, defined by

$$A_x\vec{h} := A(\vec{h}, 0),\ A_y\vec{k} = A(0, \vec{k}),\ \forall \vec{h} \in \mathbb{R}^n, k \in \mathbb{R}^m$$

Then $A_x \in L(\mathbb{R}^n)$, $A_Y \in L(\mathbb{R}^m, \mathbb{R}^n)$, and

$$A(\vec{h}, \vec{k}) = A_x\vec{h} + A_y\vec{k} \tag{$*$}$$

Theorem 9.5.1. *If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and if A_x is invertible, then for $\forall \vec{k} \in \mathbb{R}^m$, $\exists$ a unique $\vec{h} \in \mathbb{R}^n$ s.t. $A(\vec{h}, \vec{k}) = \vec{0}$. This $\vec{h}$ can be computed from $\vec{k}$ by*

$$\vec{h} = -(A_x)^{-1}A_y\vec{k}$$

Theorem 9.5.2 (the implicit function theorem)**.** *Let f be a $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^n$ s.t. $f(\vec{a}, \vec{b}) = \vec{0}$ for some $(\vec{a}, \vec{b}) \in E$. Let $A := f'(\vec{a}, \vec{b})$ and assume that A_x is invertible. Then $\exists$ open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$, with $(\vec{a}, \vec{b}) \in U$ and $\vec{b} \in W$, having the following property.*

For $\forall \vec{y} \in W$, $\exists$ a unit $\vec{x}$ s.t. $(\vec{x}, \vec{y}) \in U$ and $f(\vec{x}, \vec{y}) = \vec{0}$. If this $\vec{x}$ is defined to be $g(\vec{y})$; then $g \in \mathscr{C}'(W)$, $g(\vec{b}) = \vec{a}$ and $g'(\vec{b}) = -(A_x)^{-1}A_y$

Remark 9.

$$f(\vec{x}, \vec{y}) = \vec{0} \Leftrightarrow \begin{cases} f_1(\vec{x}, \vec{y}) = \vec{0} \\ f_2(\vec{x}, \vec{y}) = \vec{0} \\ \qquad \vdots \\ f_n(\vec{x}, \vec{y}) = \vec{0} \end{cases} \tag{$\Box$}$$

$$A_x \text{ is invertible} \Leftrightarrow \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix} \text{ evaluated at } (\vec{a}, \vec{b}) \text{ is invertible}$$

If $(\Box)$ holds when $\vec{x} = \vec{a}$ and $\vec{y} = \vec{b}$, the theorem say that $(\Box)$ can be solved for $x_1, \cdots, x_n$ in terms of $y_1, \cdots, y_m$, for $\forall \vec{y}$ in a NBHD of $\vec{b}$, and that these solutions are continuously differentiable functions of $\vec{y}$.

Proof. Define F by $F(\vec{x}, \vec{y}) = (f(\vec{x}, \vec{y}), \vec{y})$, $\forall (\vec{x}, \vec{y}) \in E$.
Then F is a $\mathscr{C}'$-mapping of E into $\mathbb{R}^{n+m}$. We will prove $F'(\vec{a}, \vec{b})$ is invertible.

$\Rightarrow F'(\vec{a},\vec{b}) \in L(\mathbb{R}^{n+m})$ which maps $(\vec{h},\vec{k})$ to $(A(\vec{h},\vec{k}),\vec{k})$.

And $(A(\vec{h},\vec{k}),\vec{k}) = \vec{0} \Rightarrow \vec{h} = \vec{k} = \vec{0}$ with the previous theorem.

Then $F'(\vec{a},\vec{b})$ is 1-1 $\Rightarrow F'(\vec{a},\vec{b})$ is invertible.

The inverse function theorem applied to $F \Rightarrow \exists$ open sets $U, V \subset \mathbb{R}^{n+m}$ with $(\vec{a},\vec{b}) \in U, (\vec{0},\vec{b}) \in V$ s.t. $F : U \to V$ is 1-1 and onto

Define $W := \{\vec{y} \in \mathbb{R}^m : (\vec{0},\vec{y}) \in V\}$. Then $\vec{b} \in W$. V is open $\Rightarrow W$ is open.

For $\forall \vec{y} \in W$, $(\vec{0},\vec{y}) = F(\vec{x},\vec{y})$ for some $(\vec{x},\vec{y}) \in U$.

Suppose that with the same $\vec{y}$, $\exists(\vec{x'},\vec{y}) \in U$ s.t. $f(\vec{x'},\vec{y}) = \vec{0}$ which means $F(\vec{x'},\vec{y}) = F(\vec{x},\vec{y}) = \vec{0}$ with contraction to that F is 1-1.

For the second part of the theorem, for $\forall \vec{y} \in W$, define $g(\vec{y})$ s.t.

$$(g(\vec{y}),\vec{y}) \in U \text{ and } f(g(\vec{y}),\vec{y}) = \vec{0} \tag{$\square\square$}$$

Then

$$F(g(\vec{y}),\vec{y}) = (\vec{0},\vec{y}), \forall \vec{y} \in W \tag{$***$}$$

Let $G : W \to U$ be the inverse of F, then the inverse function $\Rightarrow G \in \mathscr{C}'(V)$.

$(***) \Rightarrow (g(\vec{y}),\vec{y}) = G(0,\vec{y}), \forall \vec{y} \in W$

$\Rightarrow g \in \mathscr{C}'(W)$ since $G \in \mathscr{C}'(V)$. Now to compute $g'(\vec{b})$, define $\Phi(\vec{y}) := (g(\vec{y}),\vec{y})$, $\forall \vec{y} \in W$.

Then $\Phi'(\vec{y})\vec{k} = (g'(\vec{y})\vec{k},\vec{k})$, for $\forall \vec{y} \in W, \vec{k} \in \mathbb{R}^m$.

$f(\Phi(\vec{y})) = \vec{0}, \forall \vec{y} \in W \Rightarrow f'(\Phi(\vec{y}))\Phi'(\vec{y}) = 0$.

$\Phi(\vec{b}) = (\vec{a},\vec{b})$ and $f'(\Phi(\vec{b})) = A \Rightarrow A\Phi'(\vec{b}) = 0$.

i.e.

$$A_x g'(\vec{b})\vec{k} + A_y\vec{k} = \vec{0}, \forall \vec{k} \in \mathbb{R}^m \tag{$\triangle$}$$

.

With A_x is invertible we know that $g'(\vec{b}) = -A_x^{-1}A_y$ $\square$

Remark 10. $(\triangle) \Leftrightarrow$

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\vec{a},\vec{b})\frac{\partial g_j}{\partial y_k}(\vec{b}) = -\frac{\partial f_i}{\partial y_k}(\vec{a},\vec{b}), \quad 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant m$$

Example 9. *Take $n = 2, m = 3$, and consider the mapping $f : \mathbb{R}^5 \to \mathbb{R}^2$ given by*

$$\begin{cases} f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3 \\ f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos(x_1) - 6x_1 + 2y_1 - y_3 \end{cases}$$

$\vec{a} = (0, 1), \vec{b} = (3, 2, 7)$, *then $f(\vec{a}, \vec{b}) = \vec{0}$. We can use the previous theorem there*

9.6 The Rank Theorem

For $\forall A \in L(X, Y)$, the null space of A, $\mathscr{N}(A)$ is the set of all $\vec{x} \in X$ s.t. $A\vec{x} = \vec{0}$.
It is clear that $\mathscr{N}(A)$ is a vector space in X.
The range of A, $\mathscr{R}(A)$, is a vector space in Y.
The rank of A is the dimension of $\mathscr{R}(A)$.
An operator $P \in L(X)$ is said to be a projection in X if $P^2 = P$.

Proposition 1. (a) *If P is a projection in X, then every $\vec{x} \in X$ has a unique representation of the form $\vec{x} = \vec{x_1} + \vec{x_2}$.*
where $x_1 \in \mathscr{P}, \vec{x_2} \in \mathscr{N}(P)$.

(b) *If X is finite dimensional vector space and X_1 is a vector space in X, then $\exists$ a projection in X with $\mathscr{R}(P) = X$.*

(c) *The level sets of F in U are the images under H of the flat level sets of Φ in V. They are "$(n - r)$-dimension surface" in U since* $\dim \mathscr{N}(A) = n - r$.

Theorem 9.6.1 (The rank theorem)**.** *Suppose $m, n, r \in \mathbb{N} \cup 0$, $m \geqslant r, n \geqslant r$. F is $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and $F'(\vec{x})$ has rank r for $\forall \vec{x} \in E$. Fix $\vec{a} \in E$, and let $A := F'(\vec{a})$. Let $Y_1 = \mathscr{R}(A)$, P be a projection in $\mathbb{R}^m$ whose range is Y_1, $Y_2 := \mathscr{N}(P)$. Then $\exists$ open sets U and V in $\mathbb{R}^n$, with $\vec{a} \in U, U \subset E$, and $\exists$ 1-1 $\mathscr{C}'$-mapping H of V onto U (whose inverse is also of class $\mathscr{C}'$) s.t.*

$$F(H(\vec{x})) = A\vec{x} + \varphi(A\vec{x}), \forall \vec{x} \in V \tag{$\star$}$$

where φ is a $\mathscr{C}'$-mapping of the open set $A(V) \subset Y_1$ into Y_2.

Remark 11.

(a) If $\vec{y} \in F(U)$ then $\vec{y} = F(H(\vec{x}))$ for some $\vec{x} \in V$.

$(\star) \Rightarrow P\vec{y} = A\vec{x}$
$\Rightarrow \vec{y} = P\vec{y} + \varphi(P\vec{y}), \forall \vec{y} \in F(U)$
$\Rightarrow P$ restricted to $F(U)$ is 1-1 mapping of $F(U)$ onto $A(V)$.
$\Rightarrow F(U)$ is an "r-dimensional surface" with precisely one point "over" each point of $A(V)$

(b) If $\Phi(\vec{x}) = F(H(\vec{x}))$, $(\star) \Rightarrow$ the level set of Φ are precisely the level set of A in V. There are "flat" since they are intersections with V of translates of the vector space $\mathscr{N}(A)$

9.7 Determinants

If $(j_1, \cdots, j_n)$ is an ordered n-tuple of integers. Define

$$s(j_1, \cdots, j_n) := \prod_{p<q} sgn(j_q - j_p) \tag{$*$}$$

Where

$$sgn(x) \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$$

Let $[A]$ be the matrix of a linear operator A on $\mathbb{R}^n$, relative to the standard basis $\{\vec{e_1}, \cdots, \vec{e_n}\}$, with entries $a(i,j)$ in the ith row and jth column. The determinant of $[A]$ is defined by

$$det[A] := \sum s(j_1, \cdots, j_n) a(1, j_1) \cdots a(n, j_n) \tag{$**$}$$

where the sum is over all ordered n-tuples of integers $(j_1, \cdots, j_n)$ with $1, \leqslant j_r \leqslant n$.

The column vectors $\vec{x_j}$ of A are

$$\vec{x_j} = \sum_{i=1}^{n} a(i,j)\vec{e_i}, \quad 1 \leqslant j \leqslant n \tag{$***$}$$

$$det(\vec{x_1}, \cdots, \vec{x_n}) = det[A]$$

Theorem 9.7.1. (a) $\det(I) = 1$

(b) det *is a linear function of each column vector* $\vec{x_j}$ *if the others are fixed.*

(c) *If* $[A]_1$ *is obtained from* $[A]$ *by interchanging two columns* $\det[A]_1 = -\det[A]$.

(d) *If* $[A]$ *has two equal columns, then* $\det[A] = 0$

Theorem 9.7.2. *If* $[A]$ *and* $[B]$ *are* $n \times n$ *matrices, then* $\det([B][A]) = \det[B]\det[A]$

Theorem 9.7.3. *A linear operator* A *on* $\mathbb{R}^n$ *is invertible iff* $\det[A] \neq 0$.

Remark 12. *Suppose* $\{\vec{e_1}, \cdots, \vec{e_n}\}$ *and* $\{\vec{u_1}, \cdots, \vec{u_n}\}$ *are bases in* $\mathbb{R}^n$. $\forall A \in L(\mathbb{R}^n)$ *determinant of matrices* $[A]$ *and* $[A]_u$ *is the same.*

If $f : E \to \mathbb{R}^n$ is differentiable at $\vec{x} \in E$, the determinant of the linear operator $f'(\vec{x})$ is called the Jacobian of f at $\vec{x}$. In symbols $J_f(\vec{x}) = \det f'(\vec{x})$.
If $(y_1, \cdots, y_n) = f(x_1, \cdots, x_n)$, we also use the notation $\dfrac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}$ for $J_f(\vec{x})$.

9.8 Derivatives of Higher Order

Suppose $f : E \to \mathbb{R}$ has partial derivatives $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$. If the functions $\frac{\partial f}{\partial x_j}$ are also differentiable, then second-order partial derivative of f are defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j}), \quad i, j = 1, 2, \cdots, n$$

If all these functions $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are continuous in E, we say that f is of class $\mathscr{C}''$ in E, or that $f \in \mathscr{C}''(E)$.
A mapping $f : E \to \mathbb{R}^m$ is said to be of class $\mathscr{C}''$ if each component of f is of class $\mathscr{C}''$.
WLOG. We state the next two theorems for real functions of two variables.

Theorem 9.8.1. *Suppose* $E \subset \mathbb{R}^2$, $f : E \to \mathbb{R}$, *and* $\frac{\partial f}{\partial x_i}$ *and* $\frac{\partial^2 f}{\partial x_i \partial x_j}$ *exists at each point in* E. *Suppose* $Q \subset E$ *is a closed rectangle with vertices* $(a, b), (a + h, b), (a, b + k), (a + h, b + k)$ *where* $h \neq 0, k \neq 0$, *write*

$$\Delta(f, Q) := f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$

Then $\exists$ a point (x,y) in the interior of Q s.t. $\Delta(f,Q) = hk\frac{\partial^2 f}{\partial x_2 \partial x_1}(x,y)$

Proof. Define $u(t) := f(t, b+k) - f(t,b)$. Then

$$\begin{aligned}
\Delta(f,Q) &= u(a+h) - u(a) \overset{MVT}{=\!=\!=} hu'(x) \quad \text{where } x \text{ is between } a \text{ and } a+h \\
&= h[\frac{\partial f}{\partial x_1}(x, b+k) - \frac{\partial f}{\partial x_1}(x,b)] \\
&\overset{MVT}{=} hk\frac{\partial^2 f}{\partial x_2 \partial x_1}(x,y) \quad \text{where } y \text{ is between } b \text{ and } b+k
\end{aligned}$$

$\square$

Theorem 9.8.2. *Suppose $f : E \to \mathbb{R}$, $\frac{\partial^2 f}{\partial x_2 \partial x_1}$ $\frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_1}$ exist at each point of E, and $\frac{\partial^2 f}{\partial x_2 \partial x_1}$ is continuous at $(a,b) \in E$. Then $\frac{\partial^2 f}{\partial x_1 \partial x_2}$ exists at (a,b) and*

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(a,b) = \frac{\partial^2 f}{\partial x_2 \partial x_1}$$

Corollary 7. *If $f \in \mathscr{C}''(E)$, then $\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_2}$ on E.*

Proof. Let $A := \frac{\partial^2 f}{\partial x_2 \partial x_1}(a,b)$. For $\forall \epsilon > 0$, we may choose $|h|$ and $|k|$ small s.t.

$$\left| A - \frac{\partial^2 f}{\partial x_2 \partial x_1}(x,y) \right| < \epsilon \quad \forall (x,y) \in Q$$

The previous theorem $\Rightarrow \left| \frac{\Delta(f,Q)}{hk} - A \right| < \epsilon$ i.e.

$$\left| \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)}{hk} - A \right| < \epsilon$$

Fix $h \neq 0$, and let $k \to 0$, then

$$\left| \frac{\frac{\partial f}{\partial x_2}(a+h, b) - \frac{\partial f}{\partial x_2}(a,b)}{h} - A \right| \leqslant \epsilon$$

Let $h \to 0, \epsilon \to 0, \Rightarrow \frac{\partial^2 f}{\partial x_1 \partial x_2}(a,b) = A$ $\square$

remark. The proof focus on the range of $\Delta(f,Q)$.

9.9 Differentiation of Integrals

In this section, we study: under what conditions on φ can one prove that

$$\frac{d}{dt}\int_a^b \varphi(x,t)\,\mathrm{d}x = \int_a^b \frac{\partial\varphi}{\partial t}(x,t)\,\mathrm{d}x$$

Theorem 9.9.1. *Suppose*

(*a*) $\varphi(x,t)$ *is defined for* $a \leqslant x \leqslant b, c \leqslant t \leqslant d$;

(*b*) α *is increasing on* $[a,b]$;

(*c*) $\varphi(\cdot,t) \in \mathscr{R}(\alpha)$ *for* $\forall t \in [c,d]$.

(*d*) $\exists s \in (c,d)$ *s.t. for* $\forall \epsilon > 0$, $\exists \delta > 0$ *s.t.* $|\frac{\partial\varphi}{\partial t}(x,t) - \frac{\partial\varphi}{\partial t}(x,s)| < \epsilon, \forall x \in [a,b], t \in N_\delta(s)$

Define $f(t) := \int_a^b \varphi(x,t)\,\mathrm{d}\alpha(x)$. *Then* $\frac{\partial\varphi}{\partial t}(\cdot,s) \in \mathscr{R}(\alpha)$, $f'(s)$ *exists, and*

$$f'(s) := \int_a^b \frac{\partial\varphi}{\partial t}(x,s)\,\mathrm{d}\alpha(x)$$

Proof. Define

$$\psi(x,t) := \frac{\varphi(x,t)-\varphi(x,s)}{t-s}, \quad 0 < |t-s| < \delta$$

$$MVT \Rightarrow \psi(x,t) = \frac{\partial\varphi}{\partial t}(x,u(x,t)), \quad \text{where } u(x,t) \text{ is between } t \text{ and } s$$

$$(d) \Rightarrow |\psi(x,t) - \frac{\partial\varphi}{\partial t}(x,s)| < \epsilon \tag{$\triangle$}$$

Note that

$$\frac{f(t)-f(s)}{t-s} = \int_a^b \psi(x,t)\,\mathrm{d}\alpha(x) \tag{$\triangle\triangle$}$$

$(\triangle) \Rightarrow \psi(\cdot,t) \to \frac{\partial\varphi}{\partial t}(\cdot,s)$ uniformly on $[a,b]$ as $t \to s$. $(\triangle\triangle)$ and the theorem in §7.3 $\Rightarrow \frac{\partial\varphi}{\partial t}(\cdot,s) \in \mathscr{R}(\alpha)$, $f'(s)$ exists, and

$$f'(s) := \int_a^b \frac{\partial\varphi}{\partial t}(x,s)\,\mathrm{d}\alpha(x)$$

□

Remark 13. *One may prove analogues of the previous theorem with* $(-\infty,\infty)$ *in place of* $[a,b]$.