Complex Analysis HW9

1. let $f(z)=z^4-6z+3$, $g(z)=-6z$, $h(z)=z^4$, then

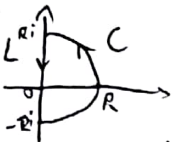
On $|z|=1$, $|f(z)-g(z)|=|z^4+3|\le 4<6=|g(z)| \Rightarrow f(z)$ has a root in $\{|z|<1\}$.

On $|z|=2$, $|f(z)-h(z)|=|-6z+3|\le 15<16=|h(z)| \Rightarrow f(z)$ has 4 roots in $\{|z|<2\}$.

Note that on $|z|=1$, $|f(z)|\ge 6|z|-|3+z^4|\ge 6-4=2$, $f(z)$ has no root on $|z|=1$.

So f has 3 roots in $\{1<|z|<2\}$.

2. ① let $f(z)=z^4+8z^3+3z^2+8z+3$, $R>0$ large enough s.t. all zeros of f lies in $\{|z|<R\}$.

Consider 

, by argument principle, $\frac{1}{2\pi i}\int_{C+L}\frac{f'}{f}dz=\#\{\text{zeros in this region}\}$.

$$\int_C \frac{f'}{f}dz=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{f'(Re^{i\theta})}{f(Re^{i\theta})}Rie^{i\theta}\,d\theta \Rightarrow \lim_{R\to\infty}\int_C\frac{f'}{f}dz=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4i\,d\theta=4\pi i$$

Note that $f(iy)=y^4-3y^2+3+(8y-8y^3)i$, $y^4-3y^2+3=(y^2-\frac{3}{2})^2+\frac{3}{4}\ge\frac{3}{4}>0$, log is well-defined on $\{f(iy)\}_{y\in\mathbb{R}}$

$$\int_L\frac{f'}{f}dz=-\int_{-R}^{R}\frac{f'(iy)}{f(iy)}i\,dy=-\int_{-R}^{R}d\log f(iy)=-\log\frac{f(iR)}{f(-iR)}\to 0 \text{ as } R\to\infty.$$

Thus $\#\{\text{zeros of } f \text{ on } \{\operatorname{Re}z>0\}\}=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{C+L}\frac{f'}{f}dz=\frac{4\pi i}{2\pi i}=2$.

② let $g(z)=z^4+3z^2+3$, then g has two roots on $\{\operatorname{Re}z>0\}$, two on $\{\operatorname{Re}z<0\}$.

On L, $|f-g|=|8y^3-8y|<|y^4-3y^2+3+(8y-8y^3)i|=|f(z)|$

On C, for R large enough, $|f(z)-g(z)|=|8z^3+8z|\le 8R^3+8R<R^4-8R^3-3R^2-8R-3\le|f(z)|$

By Rouché thm, f has 2 roots on $\{\operatorname{Re}z>0\}$ when $R\to\infty$.

3. (a) poles at $z=\pm1$ with order $=2$

$$\operatorname*{Res}_{z=1}f(z)=\frac{d}{dz}(z-1)^2f(z)\Big|_{z=1}=\frac{d}{dz}\frac{1}{(z+1)^2}\Big|_{z=1}=\frac{-1}{4}$$

$$\operatorname*{Res}_{z=-1}f(z)=\frac{d}{dz}(z+1)^2f(z)\Big|_{z=-1}=\frac{d}{dz}\frac{1}{(z-1)^2}\Big|_{z=-1}=\frac{1}{4}$$

(b) poles at $z=n\pi$ with order $=2$ $(n\in\mathbb{Z})$.

$\operatorname*{Res}_{z=n\pi}f(z)=\frac{d}{dz}\frac{(z-n\pi)^2}{\sin^2 z}\Big|_{z=n\pi}=0$. Note that f is periodic with period $=\pi$, we may only calculate the residue at $z=0$. It's 0 since f is even.

(c) ① $m=n=0$, $f=1$ has no pole

② $m=0, n\ge1$, f has a pole at $z=1$ with order n. $\operatorname*{Res}_{z=1}f(z)=\frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-1)^nf(z)\Big|_{z=1}=\begin{cases}1 & n=1\\ 0 & n\ge2\end{cases}$

③ $n=0, m\ge1$, f has a pole at $z=0$ with order m. $\operatorname*{Res}_{z=0}f(z)=\begin{cases}1 & m=1\\ 0 & m\ge2\end{cases}$

④ $m, n > 0$, f has poles at $0, 1$ with order m, n resp.

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} z^m f(z)\Big|_{z=0} = \binom{m+n-2}{m-1}$$

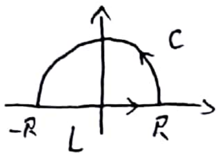
$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-1)^n f(z)\Big|_{z=1} = \frac{(-1)^n}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}\frac{1}{z^m}\Big|_{z=1} = -\binom{m+n-2}{n-1}.$$

4.(a) $$\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{2\,dx}{2a+1-\cos 2x} = \int_0^{\pi}\frac{dx}{2a+1-\cos x} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dx}{2a+1-\cos x} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dx}{2a+1-\frac{e^{ix}+e^{-ix}}{2}}$$

Let $z = e^{ix}$, then the integral equals: $i\int_{|z|=1}\frac{dz}{z^2-2(2a+1)z+1}$.

Note that $z^2-2(2a+1)z+1$ has 2 roots $2a+1 \pm 2\sqrt{a^2+a}$, and only $2a+1-2\sqrt{a^2+a}$ lies in $\{|z|<1\}$ for $a>1$.

So $$i\int_{|z|=1}\frac{dz}{z^2-2(2a+1)z+1} = i\cdot\frac{2\pi i}{(2a+1-2\sqrt{a^2+a})-(2a+1+2\sqrt{a^2+a})} = \frac{\pi}{2\sqrt{a^2+a}}.$$

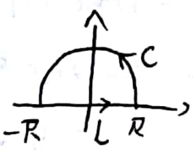
(b) Consider  with $R>3$. Note that $x^4+5x^2+6 = (x+\sqrt{2}i)(x-\sqrt{2}i)(x+\sqrt{3}i)(x-\sqrt{3}i)$,

$$\int_{C+L}\frac{z^2}{z^4+5z^2+6}\,dz = 2\pi i\left(\operatorname*{Res}_{z=\sqrt{2}i} f + \operatorname*{Res}_{z=\sqrt{3}i} f\right) = 2\pi i\left(\frac{\sqrt{2}}{2}i - \frac{\sqrt{3}}{2}i\right) = (\sqrt{3}-\sqrt{2})\pi.$$

Since $$\left|\int_C f\,dz\right| = \left|\int_0^{\pi} f(Re^{i\theta})\cdot Rie^{i\theta}\,d\theta\right| \le \int_0^{\pi}\frac{R^3}{R^4-5R^2-6}\,d\theta = \frac{\pi R^3}{R^4-5R^2-6} \to 0 \text{ as } R\to\infty,$$

we get $$\int_0^{\infty}\frac{x^2}{x^4+5x^2+6}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{x^4+5x^2+6}\,dx = \frac{\sqrt{3}-\sqrt{2}}{2}\pi.$$

(c) If $a=0$, the integral doesn't converge.

We may assume $a>0$ by symmetry. Consider  with $R>2|a|$.

$$\int_{C+L} f\,dz = 2\pi i\operatorname*{Res}_{z=ia} f(z) = 2\pi i\,\frac{1}{16a^3 i} = \frac{\pi}{8a^3}$$

Since $\left|\int_C f\,dz\right| \le \int_0^{\pi}\frac{R^3}{(R^2-a^2)^3}\,d\theta \to 0$ as $R\to\infty$, we get

$$\int_0^{\infty}\frac{x^2}{(x^2+a^2)^3}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^3}\,dx = \frac{\pi}{16a^3} \Rightarrow \int_0^{\infty}\frac{x^2}{(x^2+a^2)^3}\,dx = \frac{\pi}{16|a|^3} \text{ for all } a\in\mathbb{R}.$$

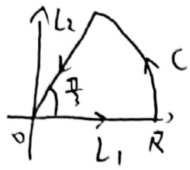
(d) Note that $\frac{x\sin x}{x^2+a^2} = \operatorname{Im}\left(\frac{xe^{ix}}{x^2+a^2}\right)$ $(x\in\mathbb{R})$, we consider $\int_0^{\infty}\frac{xe^{ix}}{x^2+a^2}\,dx$, and $a>0$.

Let  with $R>2|a|$, then $\int_{C+L} f\,dz = 2\pi i\operatorname*{Res}_{z=ai} f(z) = 2\pi i\cdot\frac{e^{-a}}{2} = \pi i e^{-a}$

Since $$\left|\int_C f\,dz\right| \le \int_0^{\pi}\left|\frac{R^2 e^{iRe^{i\theta}}}{R^2-a^2}\right|d\theta = \int_0^{\pi}\frac{R^2 e^{-R\sin\theta}}{R^2-a^2}\,d\theta \le 2\int_0^{\pi} e^{-R\sin\theta}\,d\theta \to 0 \text{ as } R\to\infty$$

we get $$\int_0^{\infty}\frac{x\sin x}{x^2+a^2} = \frac{1}{2}\operatorname{Im}\int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-a} \Rightarrow \int_0^{\infty}\frac{x\sin x}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-|a|}\ (\forall a\in\mathbb{R}).$$

(e) let $t=x^{\frac{1}{3}}$, then $\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx = 3\int_0^\infty \frac{t^3}{t^6+1}dt$. let $f(z)=\frac{z^3}{1+z^6}$.

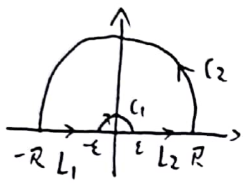
Consider 

with $R>3$, then $\int_{C+L_2+L_1} f\,dz = 2\pi i \operatorname{Res}_{z=e^{i\frac{\pi}{6}}} f(z) = 2\pi i\cdot\frac{1-\sqrt{3}i}{12}$

Since $\int_{L_2} f\,dz = \int_R^0 \frac{(e^{i\frac{\pi}{3}}t)^3}{(e^{i\frac{\pi}{3}}t)^6+1} e^{i\frac{\pi}{3}}dt = \frac{1+\sqrt{3}i}{2}\int_0^R \frac{t^3}{1+t^6}dt$

$\left|\int_C f\,dz\right| \le \int_0^{\frac{\pi}{3}}\frac{R^4}{R^6-1}d\theta \to 0$ as $R\to\infty$, we get

$$\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx = 3\int_0^\infty\frac{t^3}{t^6+1}dt = \frac{3\cdot 2\pi i\,\frac{1-\sqrt{3}i}{12}}{1+\frac{1+\sqrt{3}i}{2}} = \frac{\pi}{\sqrt{3}}$$

(f) Consider 

with $R>3$, $\varepsilon<\frac{1}{3}$. Take $\log z$ on $\{\operatorname{Arg} z\in(-\frac{\pi}{2},\frac{3}{2}\pi)\}$.

$$\int_{L_1+L_2+C_1+C_2} f\,dz = 2\pi i \operatorname{Res}_{z=i} f(z) = 2\pi i\cdot\frac{\pi}{4} = \frac{\pi^2}{2}i$$

Since $\left|\int_{C_2} f\,dz\right| \le \int_0^\pi \frac{(\log R+2\pi)R}{R^2-1}d\theta \to 0$ as $R\to\infty$

$\left|\int_{C_1} f\,dz\right| \le \int_0^\pi \frac{(|\log\varepsilon|+2\pi)\varepsilon}{1-\varepsilon^2}d\theta\to 0$ as $\varepsilon\to 0$

$$\int_{L_1} f\,dz = \int_{-R}^{-\varepsilon}\frac{\log x}{x^2+1}dx = \int_R^\varepsilon \frac{\log(-x)}{x^2+1}d(-x) = \int_\varepsilon^R\frac{\log x}{x^2+1}dx + \int_\varepsilon^R\frac{\pi i}{x^2+1}dx$$

$\int_{L_2} f\,dz = \int_\varepsilon^R \frac{\log x}{x^2+1}dx$, and $\int_0^\infty\frac{1}{x^2+1}dx=\frac{\pi}{2}$.

we get $2\int_0^\infty\frac{\log x}{x^2+1}dx + \frac{\pi}{2}\cdot\pi i = \frac{\pi^2}{2}i \Rightarrow \int_0^\infty \frac{\log x}{x^2+1}dx = 0$.

5. $\frac{1}{\pi}\iint_{|z|<1}\frac{f(z)}{(1-\bar{z}\zeta)^2}dx\,dy = \frac{1}{\pi}\int_0^1\int_0^{2\pi}\frac{f(re^{i\theta})}{(1-re^{-i\theta}\zeta)^2} r\,dr\,d\theta = \frac{1}{\pi}\int_0^1\int_{|z|=r}\frac{f(z)}{(1-\bar{z}\zeta)^2}\frac{r}{iz}dz\,dr$

$=\frac{1}{\pi i}\int_0^1\int_{|z|=r}\frac{f(z)rz}{(z-r^2\zeta)^2}dz\,dr = \frac{1}{\pi i}\int_0^1 2\pi i\operatorname{Res}_{z=r^2\zeta}\frac{zf(z)}{(z-r^2\zeta)^2}\, r\,dr$

$=2\int_0^1 \left(f'(r^2\zeta)r^2\zeta + f(r^2\zeta)\right) r\,dr = \int_0^1 d\, r^2 f(r^2\zeta) = f(\zeta)$.