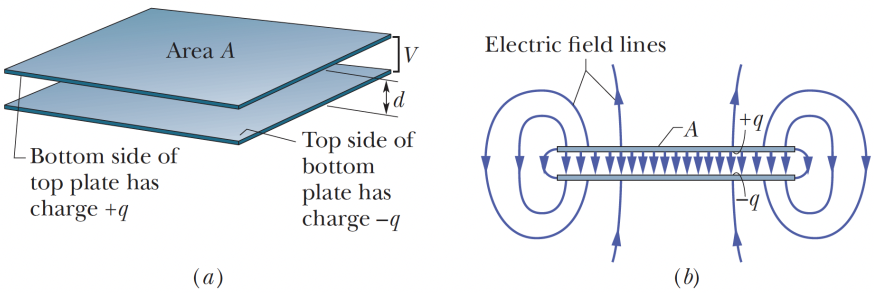
## 2.4 Capacitance, Current and Resistance

### 2.4.1 Capacitance

Two conductors, isolated electrically from each other and from their surroundings, form a capacitor. When the capacitor is charged, the charges on the conductors have the same magnitude of charges but opposite signs. No matter what their geometry, flat or not, people call these conductors *plates*. The following figure shows a conventional arrangement, called a *parallel-plate capacitor*, consisting of two parallel conducting plates of area $A$ separated by a distance $d$.




In principle, capacitors can be of all kinds of geometries, and we will study some special setups below. In this class, we always assume that no material medium is present in the region between the plates; in general, the space between the plates of a capacitor is filled with a *dielectric*, i.e., an insulating material such as glass or plastic.

When a capacitor is charged, its plates have charges of equal magnitudes but opposite signs $\pm q$. Then, we refer to the charge of a capacitor as $q$, the absolute value of these charges on the plates. Because the plates are conductors, they are equipotential surfaces; all points on a plate are at the same electric potential. Moreover, there is a potential difference between the two plates, whose absolute value is denoted by $V$. For capacitors, $q$ and $V$ are proportional to each other:

$$q = CV,$$

where the proportionality constant $C$ is called the *capacitance of the capacitor*. Its value depends only on the geometry of the plates and not on $q$ or $V$. The capacitance is a measure of how much charge can be stored on the plates to produce a certain potential difference between them: the greater the capacitance, the more charge can be stored in the capacitor. The SI unit of capacitance is farad (F), i.e., Coulomb per Volt (C/V).

We now look at some examples of capacitors with special geometries and calculate their capacitance. In general, the calculation of capacitance consists of the following steps:

1. Assume a virtual charge $q$ on the plates.

2. Calculate the electric field between the two plates in terms of $q$ (by using Coulomb's law or Gauss' law).

3. Calculate the potential difference $V$ between the plates by doing a path integral of the electric field from the positive plate to the negative one.

4. Calculate $C = q/V$.

**Example** (Parallel-Plate Capacitor)**.** *Consider a parallel-plate capacitor, where each plate is of area $A$ and the two plates are separated by $d$. Suppose the two plates are large enough compared to $d$. Find its capacitance.*

**Solution**: The electric field between the two plates is given by (2.1.11) pointing from the positive plate to the negative plate, where the charge density is $\sigma = q/A$. Then, the potential difference is

$$V = \frac{\sigma}{\varepsilon_0} d = \frac{qd}{\varepsilon_0 a}.$$

Hence, the capacitance is

$$C = \frac{q}{V} = \frac{\varepsilon_0 A}{d}. \tag{2.4.1}$$

**Example** (Cylindrical Capacitor)**.** *Consider a cylindrical capacitor of length $L$ formed by two coaxial cylinders of inner radius $a$ and outer radius $b$. Suppose $L \gg b$ so that we can neglect the fringing of the electric field that occurs at the ends of the cylinders. Find its capacitance.*

**Solution**: Using Gauss' law, we find that the electric field between the two plates is given by (2.3.7) pointing from the positive plate to the negative plate, where the charge density is $\lambda = q/L$. Then, the potential difference is

$$V = \int_a^b \frac{q}{2\pi\epsilon_0 L r} \mathrm{d}r = \frac{q}{2\pi\epsilon_0 L} \log\frac{b}{a}.$$

Hence, the capacitance is

$$C = \frac{q}{V} = \frac{2\pi\epsilon_0 L}{\log(b/a)}.$$

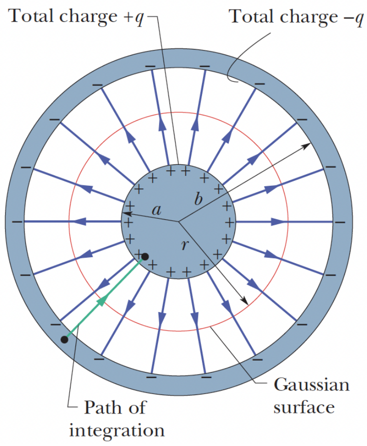



Figure 2.14: A cross-section of a long cylindrical capacitor, or a capacitor consisting of two concentric spherical shells.

**Example** (Spherical Capacitor)**.** *Consider a capacitor that consists of two concentric spherical shells, of inner radius $a$ and outer radius $b$. Find its capacitance.*

**Solution**: Using Gauss' law or Proposition 2.1.1, the electric field between the two plates is

$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

pointing from the positive plate to the negative plate. Then, the potential difference is

$$V = \int_a^b \frac{q}{4\pi\varepsilon_0 r^2}\mathrm{d}r = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Hence, the capacitance is

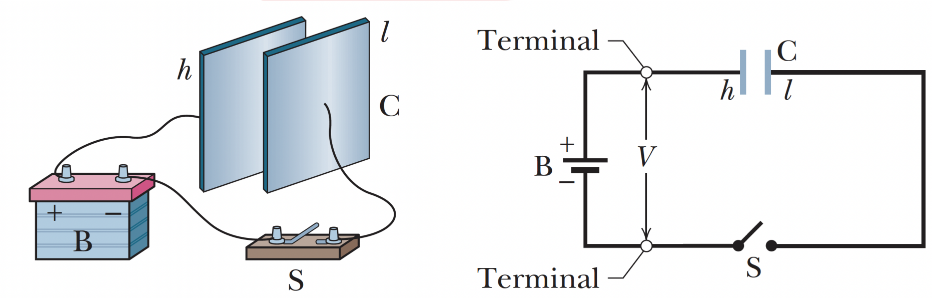
$$C = \frac{q}{V} = \frac{4\pi\epsilon_0 ab}{b-a}.$$

We can assign a capacitance to a single isolated spherical conductor of radius $a$ by assuming that the other plate is a conducting sphere of $\infty$ radius. Let $b \to \infty$ in the above equation, we get the capacitance of a single isolated spherical conductor as

$$C = 4\pi\epsilon_0 a.$$

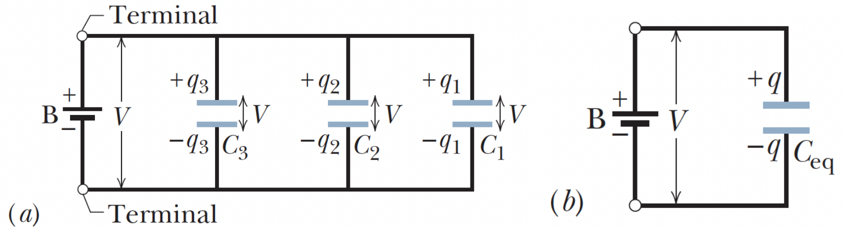
### 2.4.2 Capacitors in Circuits

To charge a capacitor, we place it in an electric circuit with a battery. In the following figure, a battery B, a switch S, an uncharged capacitor C, and interconnecting wires form a circuit. The battery maintains a potential difference $V$ between its terminals.




When the switch is closed, electrically connecting those wires, the circuit is complete and electrons are driven through the wires by an electric field that the battery sets up in the wires. The field drives electrons from the capacitor plate $h$ to the positive terminal of the battery, so plate $h$ will become positively charged. The field drives just as many electrons from the negative terminal of the battery to the capacitor plate $l$, so plate $l$ will become negatively charged just as much as plate $h$ becomes positively charged. Initially, when the plates are uncharged, the potential difference between them is zero. As the plates become oppositely charged, that potential difference increases until it equals the potential difference $V$ between the terminals of the battery. Then, plate $h$ and the positive terminal of the battery are at the same potential, and there is no longer an electric field in the wire between them. Similarly, plate $l$ and the negative terminal reach the same potential. Thus, with the field zero, there is no further drive of electrons. The capacitor is then said to be fully charged, with a potential difference $V$ and charge $q = CV$.

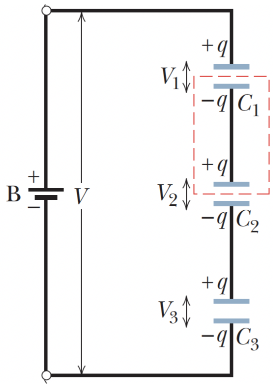
When there is a combination of capacitors in a circuit, we can sometimes replace that combination with an equivalent capacitor—that is, a single capacitor that has the same capacitance as the actual combination of capacitors.




The above figure shows an electric circuit in which three capacitors are connected in parallel to battery B. Here, each capacitor has the same potential difference $V$, and the total charge $q$ stored on the capacitors is the sum of the charges stored on all the capacitors. Thus, capacitors connected in parallel can be replaced with an equivalent capacitor that has the same total charge $q$ and the same potential difference $V$ as the actual capacitors. In general, consider $n$ capacitors connected in parallel, each with capacitance $C_i$, $i = 1, \ldots, n$. Then, the total charge is $q = \sum_{i=1}^{n} C_i V$, which gives the equivalent capacitance

$$C_{\text{eq}} = \frac{q}{V} = \sum_{i=1}^{n} C_i. \tag{2.4.2}$$

The following figure shows three capacitors connected in series to battery B. Here, the potential differences that exist across the capacitors in the series produce identical charges $q$ on them. Thus, capacitors that are connected in series can be replaced with an equivalent capacitor that has the same charge $q$ and the same total potential difference $V$ as the actual series capacitors.



In general, consider $n$ capacitors connected in parallel, each with capacitance $C_i$, $i = 1, \ldots, n$. Then, the total potential difference is $V = \sum_{i=1}^{n} q/C_i$, and hence the equivalent capacitance satisfies that

$$\frac{1}{C_{\text{eq}}} = \frac{V}{q} = \sum_{i=1}^{n} \frac{1}{C_i}. \tag{2.4.3}$$

### 2.4.3 Energy Stored in an Electric Field

Work must be done by an external battery to charge a capacitor, at the expense of its stored chemical energy. We visualize the work as being stored as electric potential energy in the electric field between the plates. We now evaluate this energy. Suppose that, at a given instant, a charge $q'$ has been transferred from one plate of a capacitor to the other. The potential difference $V'$ between the plates at that instant is equal to $q'/C$. Then, if an extra increment of charge $\mathrm{d}q'$ is transferred, the increment of work required will be

$$\mathrm{d}W = V'\mathrm{d}q' = \frac{q'}{C}\mathrm{d}q'.$$

The work required to bring the total capacitor charge up to a final value $q$ is

$$W = \int_0^q \frac{q'}{C}\mathrm{d}q' = \frac{q^2}{2C}.$$

Hence, the potential energy $U$ stored in the capacitor is given by

$$U = \frac{q^2}{2C} = \frac{1}{2}CV^2. \tag{2.4.4}$$

Consider a parallel-plate capacitor, where each plate is of area $A$ and the two plates are separated by $d$. Neglecting fringing, the electric field has the same value at all points between the plates. Thus, the energy density $u$, that is, the potential energy per unit volume between the plates, should also be uniform. Using (2.4.4) and (2.4.1), we obtain that

$$u = \frac{CV^2}{2Ad} = \frac{1}{2}\varepsilon_0 \left(\frac{V}{d}\right)^2.$$

Furthermore, the electric field between the plates is $E = V/d$. Hence, we get the following formula for the electric energy density

$$u = \frac{1}{2}\varepsilon_0 E^2.$$

Although we derived this result for the special case of an electric field of a parallel plate capacitor, it holds for any electric field. If an electric field $\vec{E}$ exists at any point in space, then at each point $\vec{r}$, there is an electric potential energy with a density (amount per unit volume) given by

$$u(\vec{r}) = \frac{1}{2}\varepsilon_0 |\vec{E}(\vec{r})|^2.$$

### 2.4.4 Electric Current

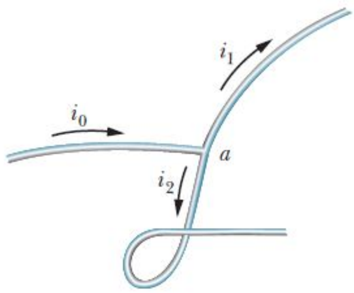
Simply speaking, an *electric current* is a stream of moving charges. In this section, we focus on steady currents of conduction electrons moving through metallic conductors such as copper wires. Recall that an isolated conducting loop, regardless of whether it has an excess charge, is all at the same potential. Hence, no net electric force acts on the conduction electrons and there is no current. If we insert a battery in the loop, the conducting loop is no longer at a single potential. Electric fields act inside the material making up the loop, exerting forces on the conduction electrons, causing them to move and thus establishing a current. After a very short time, the electron flow reaches a constant value and the current is in its steady state.

Take a section of a conductor, part of a conducting loop in which current has been established. If charge $\mathrm{d}q$ passes through a cross-section of the conductor in time $\mathrm{d}t$, then the current $i$ through that cross-section is defined as

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}.$$

As discussed before, the SI unit for current is ampere (A) or Coulomb per second (C/s), which is an SI base unit. Note that current is a scalar because both charge and time are scalars. Yet, we often represent a current with an arrow to indicate that charge is moving. More precisely, we will draw the current arrows in the direction in which *positively charged particles* would be forced to move through the loop by the electric field. Such positive charge carriers would move away from the positive battery terminal and toward the negative terminal. However, in applications, the charge carriers are often electrons and thus are negatively charged. The electric field forces them to move in the direction opposite the current arrows, from the negative terminal to the positive terminal. Then, the assumed motion of positive charge carriers in one direction has the same effect as the actual motion of negative charge carriers in the opposite direction.

We remark that the arrows of currents do not mean vectors, so they do not satisfy vector addition. Current arrows show only a direction (or sense) of flow along a conductor, not a direction in space. Due to the conservation of charges, there is also a "conservation of currents". For example, the following picture shows a conductor with current $i_0$ splitting at a junction into two branches:




Then, the magnitudes of the currents in the branches must add to yield the magnitude of the current in the original conductor, i.e., $i_0 = i_1 + i_2$. In more general circuits, we have the following Kirchhoff's junction rule (or Kirchhoff's current law).

**Physics law 10** (Kirchhoff's junction rule)**.** *The sum of the currents entering any junction must be equal to the sum of the currents leaving that junction.*

Sometimes, we want to take a localized view and study the flow of charge through a cross-section of the conductor at a particular point. To describe this flow, we can use the current density $\vec{J}$, which has the same direction as the velocity of the moving charges if they are positive and the opposite direction if they are negative. The magnitude $J$ is equal to the current per unit area through that element. Then, the total current through the surface is

$$i = \int \vec{J} \cdot \mathrm{d}\vec{A},$$

where $\mathrm{d}\vec{A}$ is the area vector of the element. If the current is uniform across the surface and parallel to $\mathrm{d}\vec{A}$, then $\vec{J}$ is also uniform and parallel to $\mathrm{d}\vec{A}$, and we have

$$i = JA \;\Leftrightarrow\; J = \frac{i}{A},$$

where $A$ is the total area of the surface. The SI unit for current density is $\mathrm{A/m^2}$.

### 2.4.5 Resistance and Resistivity

If we apply a potential between the ends of different conductors, different currents result depending on their *resistances*. We determine the resistance between any two points of a conductor by applying a potential difference $V$ between those points and measuring the current $i$ that results. The resistance $R$ is then defined as

$$R = \frac{V}{i}.$$

Its value depends only on the material and geomerty of the conductor and not on $V$ or $i$. The SI unit for resistance is ohm ($\Omega$) or volt per ampere (V/A). A conductor whose function in a circuit is to provide a specified resistance is called a resistor. Given a voltage difference $V$ between the two ends of the resistor, it generated a current $i = V/R$. On the other hand, a current $i$ across the resistor corresponds to a voltage difference $V = iR$.

We sometimes wish to take a localized view and focus on the electric field $\vec{E}$ at a point in a resistive material. Then, instead of dealing with the current $i$ through the resistor, we deal with the current density $\vec{J}$ at the point in question. Instead of the resistance $R$ of an object, we deal with the resistivity $\rho$ of the material:

$$\rho = \frac{E}{J}.$$

The SI units of $\rho$ is $(\mathrm{V/m})/(\mathrm{A/m^2}) = (\mathrm{V/A}) \cdot \mathrm{m} = \Omega \cdot \mathrm{m}$. We can rewrite the above equation into a more general vector form

$$\vec{J} = \frac{\vec{E}}{\rho}. \tag{2.4.5}$$

Compared to resistivity, people speak of the conductivity $\sigma$ of a material more often, which is simply the reciprocal of its resistivity,

$$\sigma = \rho^{-1}.$$

The definition of $\sigma$ allows us to write (2.4.5) into a slightly simpler form:

$$\vec{J} = \sigma \vec{E}. \tag{2.4.6}$$

Note that *resistance is a property of an object*, while *resistivity is a property of a material*. We now derive the relation between the resistivity of a material such as copper and the resistance of a length of wire made of that material. Let $A$ be the cross-sectional area of the wire and $L$ be its length. Apply a potential difference $V$ between its ends. Suppose the electric field and the current density are constant for all points within the wire. Then, they have values $E = V/L$ and $J = i/A$, which give that

$$\rho = \frac{E}{J} = \frac{V}{i}\frac{A}{L} = R\frac{A}{L} \;\Leftrightarrow\; R = \frac{\rho L}{A}.$$

This relation can be applied only to a homogeneous isotropic conductor of uniform cross-section.

As discussed above, a resistor is a conductor with a specified resistance $R$. In particular, $R$ is unchanged no matter what the magnitude and direction (polarity) of the applied potential difference are. In this case, we say that the resistor satisfies the **Ohm's Law**, which asserts that

*the current through a device is always proportional to the potential difference applied to it.*

Although, for historical reasons, the term "law" is used here, this assertion is correct only in certain situations. For example, the left device of Figure 2.15 obeys Ohm's law, while the right device of Figure 2.15—a semiconducting $pn$ junction diode—does not. For the $pn$ junction diode, current can exist only when the polarity of $V$ is positive and the applied potential difference is more than about 1.5 V. When current does exist, the relation between $i$ and $V$ is also not linear.

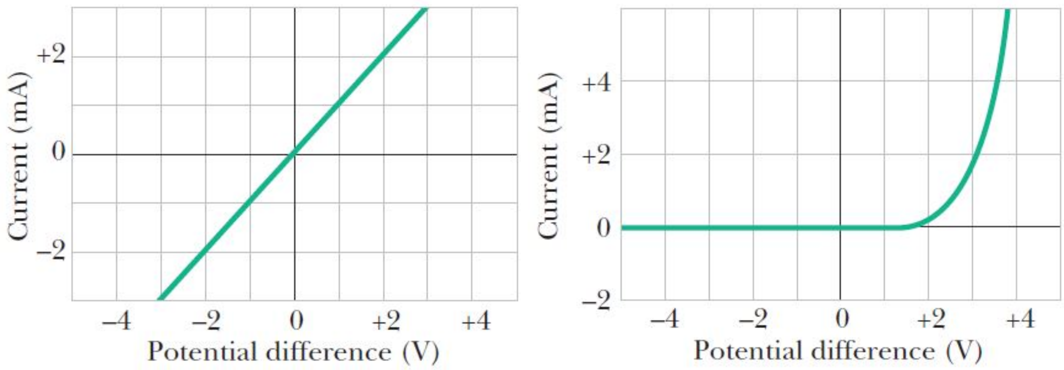



Figure 2.15: Left plot: current $i$ versus applied potential difference $V$ when the device is a resistor. Right plot: $i$ versus $V$ when the device is a semiconducting $pn$ junction diode.

*Remark.* In middle school physics, it is often contended that Ohm's law states that $V = iR$. However, this is not true—this equation is just the defining equation for resistance, and it applies to all conducting devices, whether they obey Ohm's law or not. The essence of Ohm's law is that $i$ is linear with respect to $V$, that is, $R$ is independent of $V$.

When there is a steady current $i$ across a resistor $R$, the amount of charge $\mathrm{d}q$ that moves between its ends in the time interval $\mathrm{d}t$ is equal to $\mathrm{d}q = i\mathrm{d}t$. This charge $\mathrm{d}q$ moves through a potential decrease of magnitude $V$, and thus its electric potential energy decreases in magnitude by

$$\mathrm{d}U = \mathrm{d}qV = iV\mathrm{d}t.$$

The principle of conservation of energy tells that the decrease in electric potential energy must be accompanied by a transfer of energy to some other form. The power $P$ associated with that transfer is

$$P = \frac{\mathrm{d}U}{\mathrm{d}t} = iV. \tag{2.4.7}$$

Note that the SI unit of $P$ is $\mathrm{V} \cdot \mathrm{A} = (\mathrm{J/C}) \cdot (\mathrm{C/s}) = \mathrm{J/s} = \mathrm{W}$, as it should be. Using $V = iR$, we can rewrite (2.4.7) as

$$P = iR^2 = \frac{V^2}{R},$$

which gives the rate of electrical energy dissipation due to resistance. Given a fixed current, to lower the electrical energy dissipation, it is desired to have as small resistance as possible. In particular, when the phenomenon of **superconductivity** occurs, the resistivity of the material drops to zero and there is no electrical energy dissipation. The study of the mechanisms for superconductivity remains one of the central topics in modern theoretical physics, and the search for high-temperature (or even room-temperature) superconductors is arguably one of the most important challenges among physicists.

### 2.4.6 Electric Circuits

In this subsection, we study the physics of electric circuits which are closed loops consisting of resistors, batteries, capacitors, and conducting wires between them, whose resistances are negligible. We restrict our attention to circuits through which charge flows in one direction, called direct current (DC) circuits.

Given an electric circuit, we want to know the voltage (or potential) at each point and the current in each segment of conducting wire. In principle, no matter how complicated an electric circuit is, it can always be solved by using Kirchhoff's junction rule (or Kirchhoff's current law) in Physics law 10 and the following Kirchhoff's loop rule (or Kirchhoff's voltage law).

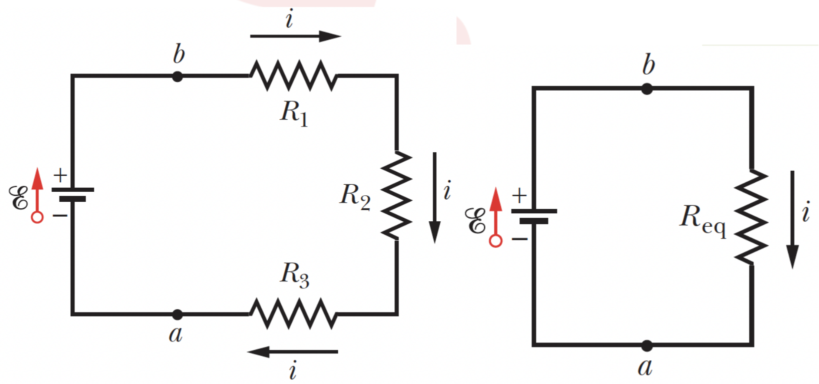
**Physics law 11** (Kirchhoff's loop rule)**.** *The algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.*

When we apply Kirchhoff's voltage laws, we start at a point, say $o$, and mentally walk clockwise or counterclockwise around the circuit until we are back at $o$, keeping track of potential changes as we move. During the walk, we deal with resistors, batteries, and capacitors in the following ways.

- For a move through resistance $R$ in the direction of the current, the change in potential is $-iR$, while in the opposite direction, it is $+iR$.
- A battery maintains its positive terminal at a higher electric potential than the negative terminal. People usually call the potential increase from the $-$ terminal to $+$ terminal as the emf (i.e., electromotive force) $\mathscr{E}$, meaning that it supplies the energy for the motion of electrons via the work it does. For a move through a battery in the direction of the emf arrow (i.e., from $-$ terminal to $+$ terminal), the change in potential is $+\mathscr{E}$; in the opposite direction, it is $-\mathscr{E}$.
- A capacitor is an insulator, i.e., it has $\infty$ resistance. Hence, there is no current across the capacitor. For a move across the capacitor with a potential difference $V$, the change in potential is $+V$ from the negative plate to the positive one; in the opposite direction, it is $-V$.

In applications of Kirchhoff's loop and junction rules, we need to solve a system of linear equations involving voltages at various points and currents along some wire segments. In some cases, the circuit structure can be greatly simplified if we use equivalent resistances for resistors in series or in parallel.

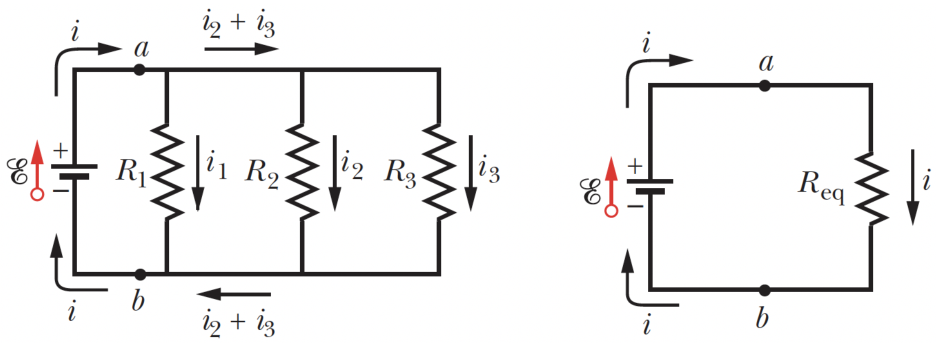
**Resistors in series.** The left figure shows an electric circuit in which three resistances are connected in series to an ideal battery with emf $\mathscr{E}$.




All three resistances have identical currents $i$, while the sum of the potential differences across the resistances is equal to $\mathscr{E}$. Thus, resistances connected in series can be replaced with an equivalent resistance $R_{\text{eq}}$ (the right figure) that has the same current $i$ and the same total potential difference $\mathscr{E}$ as the actual resistances. In general, consider $n$ capacitors connected in series, each with resistance $R_i$, $i = 1, \ldots, n$. Then, by the loop rule, we have

$$\mathscr{E} - \sum_{k=1}^{n} iR_k = 0 \;\Rightarrow\; R_{\text{eq}} = \frac{\mathscr{E}}{i} = \sum_{k=1}^{n} R_k. \tag{2.4.8}$$

**Resistors in parallel.** The left figure shows three resistances connected in parallel to an ideal battery with emf $\mathscr{E}$.




All three resistances have the same potential difference across them, producing a current through each. Thus, resistances connected in parallel can be replaced with an equivalent resistance $R_{\text{eq}}$ (the right figure) that has the same potential difference and the same total current as the actual resistances. In general, consider $n$ capacitors connected in series, each with resistance $R_i$, $i = 1, \ldots, n$. Then, applying the loop rule to the $n$ loop, each of which consists of the battery and a resistor, we get that

$$\mathscr{E} = i_k R_k, \quad k = 1, \ldots, n.$$

By the junction rule, the total current of the equivalent resistance is

$$i = \sum_{k=1}^{n} i_k.$$

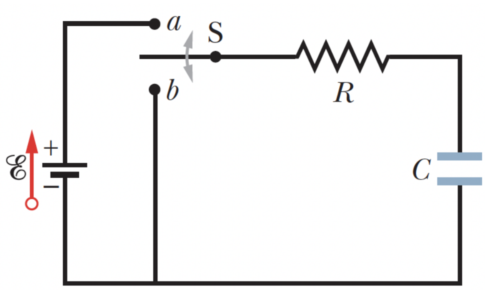
From the above two equations, we can derive the equivalent resistance as

$$R_{\text{eq}} = \frac{\mathscr{E}}{i} \Rightarrow \frac{1}{R_{\text{eq}}} = \sum_{k=1}^{n} \frac{1}{R_k}. \tag{2.4.9}$$

You can compare (2.4.8) and (2.4.9) for resistance with (2.4.2) and (2.4.3) for capacitance.

### 2.4.7 RC Circuits

In this subsection, we consider the following RC series circuit consisting of a capacitor $C$, an ideal battery of emf $\mathscr{E}$, and a resistance $R$:



When switch $S$ is closed on $a$, the capacitor is charged through the resistor. When the switch is afterward closed on $b$, the capacitor discharges through the resistor.

We know that as soon as the circuit is complete, charge begins to flow (current exists) between a capacitor plate and a battery terminal on each side of the capacitor until the potential difference between the two capacitor plates equals $\mathscr{E}$ across the battery. Then, we say that the system reaches equilibrium, and the equilibrium charge on the fully charged capacitor is equal to $C\mathscr{E}$. Here, we are interested in the charging process, which is a dynamic process. In particular, we want to know how the charge $q(t)$ on the capacitor plates, the potential difference $V(t)$ across the capacitor, and the current $i(t)$ in the circuit vary with time during the charging process.

Using the loop rule, we find that

$$\mathscr{E} - i(t)R - \frac{q(t)}{C} = 0.$$

Since $i = \mathrm{d}q/\mathrm{d}t$, we can rewrite the above equation as

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = \mathscr{E},$$

with initial condition $q(0) = 0$. It is easy to solve this equation and get

$$q(t) = C\mathscr{E}\left(1 - e^{-\frac{t}{RC}}\right).$$

From $q(t)$, we immediately obtain that

$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{\mathscr{E}}{R}e^{-\frac{t}{RC}}, \quad V(t) = \frac{q(t)}{C} = \mathscr{E}\left(1 - e^{-\frac{t}{RC}}\right).$$

As expected, as $t \to \infty$, $V(t) \to \mathscr{E}$, $i(t) \to 0$, and $q(t) \to C\mathscr{E}$.

Assume now that the capacitor has been fully charged. At a new time $t = 0$, switch $S$ is thrown from $a$ to $b$ so that the capacitor can discharge through resistance $R$. How does the discharging process behave? Similar as above, we obtain the following differential equation of $q(t)$:

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0,$$

with initial condition $q(0) = q_0 = C\mathscr{E}$. The solution to this equation is

$$q(t) = q_0 e^{-\frac{t}{RC}}.$$

Hence, $q$ decreases exponentially to zero. The current is then given by

$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} = -\frac{q_0}{RC}e^{-\frac{t}{RC}},$$

i.e., the current also decreases exponentially to zero.