Differential Geometry

LIN150117

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

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1 Smooth Manifold

Theorem 1.1 (Kervaire). *$\exists$ 1 10-dimensional topological manifold without smooth manifold.*

Theorem 1.2 (Milnor). *$\exists$ a smooth manifold M s.t. $M \cong S^7$ but not in diffeomorphism meaning.*

Theorem 1.3 (Kervaire-Milnor). *$\exists$ 28 smooth structures (up to orientation preserving diffeomorphism) on S^7*

Theorem 1.4 (Morse-Birg). *On S^7. If $n \leqslant 3$, then any n-dimensional topological manifold M has a unique smooth structure up to diffeomorphism.*

Theorem 1.5 (Stallings). *If $n \neq 4$, then $\exists$ a unique smooth structure on $\mathbb{R}^n$ up to diffeomorphism.*

Theorem 1.6 (Donaldson-Freedom-Gompf-Faubes). *$\exists$ uncountable smooth structures on $\mathbb{R}^4$ up to diffeomorphism.*

Our motivation for studying manifold is to study the space of solution for equations.

Question 1. Given $f : \mathbb{R}^n \to \mathbb{R}$ smooth, $q \in \mathbb{R}^n$, when is $f^{-1}(q)$ is a smooth manifold?

For $f : U \to \mathbb{R}^n$ smooth, U open in $\mathbb{R}^m$, the differential of f at $p \in U$ denoted as $\mathrm{d}f(p)$.

Theorem 1.7 (Implicit function theorem). *If $p \in U$ is a regular point of $f : U \to \mathbb{R}^n$. Then there exists*

- *An open neighbourhood V of p in U*

- *An open subset V' of $\mathbb{R}^m$*
- *A diffeomorphism $\varphi : V \to V'$ such that $P \circ \varphi = f$ where P is the projection from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

In other words, near a regular point, we can do local coordinate change to turn f into the projection.

Corollary 1.8. *If q is a regular value of $f : U \to \mathbb{R}^n$ then $f^{-1}(q)$ is a smooth manifold.*

Theorem 1.9 (Sard). *If $f : U \to \mathbb{R}^n$ is a smooth map, then the set of critical values of f has measure 0.*

Corollary 1.10. *If $f : U \to \mathbb{R}^n$ is smooth and $m < n$ then $f(U)$ has measure 0.*

1.1 Lie groups and homogeneous spaces

Theorem 1.11 (Carton). *Let H be a closed subgroup of Lie group G. Then H is a Lie group. More precisely, H is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also, G/H is a smooth manifold*

Theorem 1.12. *p is always a diffeomorphism.*

Therefore, we have this proposition

Proposition 1.13. *M is a homogeneous space $\Leftrightarrow M = G/H$ for some closed subgroup H.*

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to $\mathbb{RP}^2, S^2, T^2$ and Klein bottle.

Theorem 1.14 (Whithead). *Any smooth manifold has a triangulation.*

Theorem 1.15 (Poincare-Hopf). *G is compact Lie group $\Rightarrow \chi(G) = 0$.*

Theorem 1.16 (Mostow2005). *M is a compact homogeneous space $\Rightarrow \chi(M) \geqslant 0$.*

1.2 Bump Function and Partition of Unity

Theorem 1.17 (Urysohn smooth version)**.** *Given M, closed disjoint A, B, $\exists$ smooth $f : M \to [0,1]$ s.t. $f|_A = 0, f|_B = 1$.*

Theorem 1.18 (Tietze)**.** *Given M, closed A, smooth $f : A \to \mathbb{R}^n$, there exists smooth $\hat{f} : M \to \mathbb{R}^n$ s.t. $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

Lemma 1.19. *Let U be a neighbourhood of $p \in M$. Then $\exists$ smooth $\sigma : M \to [0,1]$ s.t.*

1. *$\sigma \equiv 1$ near p*
2. *Supp $\sigma \subset U$*

*Such σ is called a **bump function** at p, supported in U.*

Proposition 1.20. *Given compact $K \subset U$ and open neighbourhood U of K, $\exists$ a smooth $g : M \to [0, +\infty)$ s.t. $g|_K \equiv 1$ and $Supp\, g \subset U$.*

Theorem 1.21. *Any topological manifold has an exhaust.*

Given two open covers $\mathcal{U}, \mathcal{V}$, we say $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if $\forall U_\alpha \in U$, $\exists V_\beta \in \mathcal{V}$ s.t. $V_\beta \subset U_\alpha$.

We say a space X is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact.(The proof is hard)

Proposition 1.22. *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a topological manifold M. Then there exists countable open covers $\mathcal{W} = \{W_i\}$, $\mathcal{V} = \{V_i\}$ s.t.*

- *For any i, $\overline{V_i}$ is compact and $\overline{V_i} \subset W_i$*

- *$\mathcal{W}$ is locally finite.*
- *$\mathcal{W}$ is a refinement of $\mathcal{U}$.*

As a corollary, we have any topological manifold is paracompact.

Theorem 1.23 (Existence of P.O.U)**.** *For any open cover $\mathcal{U}$ of smooth M, $\exists$ a P.O.U subordinate to $\mathcal{U}$*

Theorem 1.24 (Whitney approximation theorem)**.** *Given any smooth M, any closed A and any continuous $f : M \to \mathbb{R}$, $\delta : M \to (0, +\infty)$. Suppose f is smooth on A. Then $\exists g : M \to \mathbb{R}$ smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M$, $|g(p) - f(p)| < \delta(p)$.

2 Tangent space and tangent vectors

2.1 Tangent Space

Given $p \in M$, consider the set $C_p^\infty(M) = \{\text{smooth function } V \to \mathbb{R}\}/_\sim$ where $f_1 \sim f_2$ if and only if $\exists$ neighbourhood U of p, $f_1|_U = f_2|_U$.

$C_p^\infty(M)$ is the space of **genus of smooth function** near p.

A **partial-derivative** of p is a $\mathbb{R}$-linear map $D : C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

Proposition 2.1. *For $M = U \subset \mathbb{R}^n$ open. We have $\{\frac{\partial}{\partial x_i}\}$ is a basis for T_pU.*

Proposition 2.2.

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leqslant i \leqslant n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

M, N smooth manifolds, $C^\infty(N, M) = \{\text{smooth } F : N \to M\}$.

Given $F \in C^\infty(N, M)$, F induces $F^* : C^\infty_{F(p)}(M) \to C^\infty_p(N)$, $f \mapsto f \circ F$.

By taking dual, we get

$$F_* : T_pN \to T_{F(p)}M$$

we also write F_* as $F_{*,p}$, call it the **differential** of F at p.

where

$$F_*(\frac{\partial}{\partial x^i}|_p) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

Proposition 2.3. *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_pN \to T_{G \circ F(p)}W$$

Proposition 2.4. *Given vector bundle $f : E \to B$, the fiber $f^{-1}(p)$ has a structure of a vector space.*

Given vector bundles $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$, a bundle map consists of $(\hat{f}, f)$ *s.t.*

- $\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$ commutes.
- $\forall b \in B$, $\hat{f} : \pi_1^{-1}(b) \to \pi_2^{-1}(f(b))$ is linear.

If $\hat{f}, f$ are diffeomorphisms, then we call $(\hat{f}, f)$ an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

Theorem 2.5. *If G is a Lie group, then TG is trivial.*

Proposition 2.6 (Adams, 1960s). *TS^n is trivial if and only if $n = 0, 1, 3, 7$.*

Proposition 2.7. 1. *Given any $F \in C^\infty(M, N)$, $F_* : TM \to TN$ is a bundle map.*

2. *TS^n is isomorphic to the following bundle:*

$$B = s^n \qquad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

2.2 Vector Field, Curves and Flows

Given any $f : \mathbb{R}^n \to \mathbb{R}$, define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leqslant i \leqslant n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

Theorem 2.8 (Poincare-Hopf). *For closed M, M has a nowhere vanishing vector field if and only if $\chi(M) = 0$.*

So S^n has a nowhere vanishing vector field if and only if n is odd.

Theorem 2.9 (MaoQiu). *S^2 has no no-where vanishing vector field.*

So We cannot smooth out all the hairs on a ball.

Given a vector field $X = \{X_p\}_{p \in M}$, a curve $\gamma : (a, b) \to M$ is called an **integral curve** of X if $\gamma'(t) = X_{\gamma(t)}$, $\forall t \in (a, b)$, where $\gamma'(t) = \gamma_*(\dfrac{\partial}{\partial t}) \in T_{\gamma(t)}M$.

We say γ is maximal if the domain cannot be extended to a larger interval. Denote the set of all smooth vector fields on M by $\mathfrak{X}M$

Recall that γ is maximal if it's domain can not be extended to a large open interval.

In a local chart $(U, x^1, \cdots, x^n)$, $X|_U = \sum_{i=1}^{n} a^i \partial x^i$. Then γ is an integral curve if and only if $(\gamma^i)'(t) = a^i(\gamma(t))$, $\forall 1 \leqslant i \leqslant n$, where $\gamma^i = x^i \circ \gamma : (a, b) \to \mathbb{R}$.

And in this case the initial value condition: $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$.

Locally, solving integral curve starting at p is equivalent to solving ODE with initial value $p^1, \cdots, p^n$. By existence and uniqueness of solutions of ODE, we have

Theorem 2.10 (Fundamental theorem of integral curve). *Let $X \in \mathfrak{X}M$, $p \in M$, then:*

(1) (Uniqueness) Given any two integral curves $\gamma_1, \gamma_2 : (a, b) \to M$, then we have:

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a, b) \Rightarrow \gamma_1 = \gamma_2$$

(2) there exists a unique max integral curve $\gamma : (a(p), b(p)) \to M$ starting at p.

(3) (integral curve smoothly depend on initial values) $\exists$ Nbh U of p, $\varepsilon > 0$, and smooth $\varphi : (-\varepsilon, \varepsilon) \times U \to M \quad$ s.t. $\forall q \in U$, $\varphi_\varepsilon := \varphi(-, q) : (-\varepsilon, \varepsilon) \to M$ is an integral curve starting at p.

*we call such φ a local **flow** generated by X.*

If such global flow exists, then we say X is **complete**.

Given $X \in \mathfrak{X}M$, we define $\mathrm{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$.

Theorem 2.11. *If a vector field X is compactly supported, then X is complete.*

Corollary 2.12. *Any vector field on closed manifold is complete.*

Lemma 2.13 (Escaping lemma). *Suppose $\gamma : (a, b) \to M$ is a max integral curve, with $(a, b) \neq \mathbb{R}$. Then $\nexists$ compact $K \subset M \quad$ s.t. $\gamma(a, b) \subset K$*

A smooth $\varphi : \mathbb{R} \times M \to M$ is called an **one-parameter transformation group** if

(1) $\varphi_0 := \varphi(0, -) = \mathrm{id}_M$

(2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}$. In particular, $\varphi_s^{-1} = \varphi_{-s}$.

Theorem 2.14. *$\varphi \in C^\infty(\mathbb{R} \times M, M)$, then φ is an one-parameter transformation group if and only if φ is the global flow generated by some $X \in \mathfrak{X}M$*

Lemma 2.15 (Translation lemma). *If $\gamma : (a,b) \to M$ is an integral curve for some $X \in \mathfrak{X}M$, then $\forall s \in \mathbb{R}$, $\gamma(- + s) : (a-s, b-s) \to M$ is also an integral curve for X.*

Lemma 2.16. *Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$ provided that $s, t, s+t \in (-\varepsilon, \varepsilon), p, \varphi_r(p) \in U$.*

Lemma 2.17. *Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$ i.e. any vector field is invariant under its flow.*

Time dependent vector field is a smooth map $X : \mathbb{R} \times M \to TM$ *s.t.* $X_{(t,p)} \in T_pM$.

A smooth curve $\gamma(a,b) \to M$ is the **integral curve** for X if $\gamma'(t) = X_{(t,\gamma(t))}$.

In local chart, solving γ is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness: γ_1, γ_2 are both integral curves for X, $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define $\mathrm{Supp}X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$.

Then X is compactly supported, then X is complete(*i.e.* a global flow $\varphi : \mathbb{R} \times M \to M$)

But something is not true for time dependent vector field:

- translation lemma is not true.
- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

2.3 Another definition of vector field

A derivation on M is a $\mathbb{R}$-linear map $C^\infty(M) \xrightarrow{D} C^\infty(M)$ that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

Theorem 2.18. *We have a bijection:*

$$\rho : \mathfrak{T}M \xrightarrow{1:1} \{\textit{derivation on } M\}$$
$$X \mapsto D_X : f \mapsto X(f)$$

Lemma 2.19. *$D_p : \mathfrak{T}_pM \to \mathbb{R}$-linear map $C^\infty(M) \to \mathbb{R}$ s.t. $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$ is an isomorphism of vector spaces.*

Lemma 2.20. *Given a vector field(not necessarily smooth) $X = \{X_p\}_{p \in M}$, X is smooth $\Leftrightarrow \forall f \in C^\infty(M), X(f)$ is smooth.*

Theorem 2.21. *The map $\rho : \mathfrak{T}M \to \{\textit{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$ is well-defined and bijective.*

3 Lie group, Lie algebra and Lie bracket

3.1 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned}\{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\}\\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \quad \textit{s.t.}\end{aligned}$$

$$D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\}$$

$$\begin{aligned}\{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\}\end{aligned}$$

Notation 3.1. We will identify $X \in \mathfrak{X}M$ with its derivation $D_x : C^\infty(M) \to C^\infty(M)$. So a vector field is just a $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M)$ *s.t.* $X(fg) = fX(g) + X(f)g$.

Theorem 3.2. *For any $X, Y \in \mathfrak{X}M$, $[X, Y] \in \mathfrak{X}M$*

So What is the geometric meaning of $[X, Y]$? Non commutatiy of flows.

Fact 3.3. *Given $X, Y \in \mathfrak{X}M$, we say X, Y are commutative vector field if $[X, Y] = 0$*

X, Y are commuative iff for any local flows $\varphi^X : (-\varepsilon, \varepsilon) \times U \to M$, $\varphi^Y : (-\varepsilon, \varepsilon) \times U \to M$ we have $\varphi_s^X \circ \varphi_t^T = \varphi_t^Y \circ \varphi_s^X$

Proposition 3.4 (Calculation of $[V, W]$ using local charts)**.** *Chart $(U, x^1, \cdots, x^n)$, $V, W \in \mathfrak{X}M$, $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$, $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$. Then*

$$\begin{aligned}[V, W]|_U &= \sum_{i=1}^n (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n (\sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i} \\ &= \sum_{1 \leqslant i,j \leqslant n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i}\end{aligned}$$

Proposition 3.5 (Properties of Lie bracket)**.**

(a) Natuality under push-forword.

Given any $F \in \mathrm{Diff}(M, N)$*,* $V \in \mathfrak{X}M, W \in \mathfrak{X}M$*, we have* $[F_*V, F_*W] = F_*[V, W]$.

(b) $\mathbb{R}$*-linearity* $\forall a, b \in \mathbb{R}$

$$[aX + bV, W] = a[X, W] + b[V, W]$$
$$[W, aX + bV] = b[W, X] + a[W, V]$$

(c) anti-symmetric $[V, W] = -[W, V]$

(d) Jacobi identity

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

(f) Leibuniz rule

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

Note 1. In general, given $V \in \mathfrak{X}M$ and $F \in C^\infty(M, N)$. There may not exist $W \in \mathfrak{X}M$ *s.t.* V, W are F-related. Even such W exists, it may not be unique.

However, if F is a diffeomorphism, given any V, $\exists$ unique W *s.t.* V and W are F-related. Actually, $W_p = F_* V_{F^{-1}(p)}$.

Such W is called **push forward** of V along F, denoted by F_*V, only defined when F is a diffeomorphism.

Lemma 3.6. $\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N,\ F \in C^\infty(M, N)$*. Then* W *is* F*-related to* V *iff* $\forall f \in C^\infty(N), V(f \circ F) = W(f) \circ F \in C^\infty(M)$

Proposition 3.7. *Given $V_0, V_1 \in \mathfrak{X}M$, $W_0, W_1 \in \mathfrak{X}N$, $F \in C^\infty(M, N)$, W_i is F-related to V_i, $i = 0, 1 \Rightarrow [W_0, W_1]$ is F-related to $[V_0, V_1]$*

Corollary 3.8 (Naturality of Lie bracket). *Given any $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, we have $[F_*V, F_*W] = F_*[V, W]$*

The rest of Proposition 3.5 is easy to check if it is viewed as a mapping $C^\infty(M) \to C^\infty(M)$.

3.2 Lie algebra of a Lie group

For G Lie group, $\forall g \in G$ we have diffeomorphism

$$l^g : G \to G, h \mapsto gh$$
$$r^g : G \to G, h \mapsto hg$$

We say $X \in \mathfrak{X}G$ is **left invariant** if $l^g_*(X) = X$, $\forall g \in G$. Similarly, X is **right invariant** if $r^g_*(X) = X$.

Proposition 3.9. *X, Y are left/right invariant $\Rightarrow [X, Y]$ is left/right invariant.*

So we can find a natural Lie algebra of G:

$$\mathrm{Lie}(G) := \{\text{left invariant vector fields on } G\}, \text{with } [-,-] \text{ restricted from } \mathfrak{X}G$$

Theorem 3.10. *Given any $V \in T_eG$, $\exists$ unique left invariant $\hat{V} \in \mathfrak{X}G \quad s.t.\ \hat{V}_e = V$.*

Corollary 3.11. *$\mathrm{Lie}(G) \cong T_eG$ as vector spaces.*

Theorem 3.12. *$\forall A, B \in \mathrm{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$, $[A, B] = AB - BA$.*

Lemma 3.13. *$\forall A \in \mathrm{gl}(n,\mathbb{R})$, the left invariant vector field $\hat{A}$ is complete and generated the flow $\varphi_t : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2t^2}{2!} + \cdots)$*

Similarly, for $G = \mathrm{GL}(n,\mathbb{C}), \mathrm{Lie}(G) = \mathrm{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$, we have $[A,B] = AB - BA$.

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group G, its universal cover $\hat{G}$ is simply-connected with $\pi^{-1}(G) \subset Z(\hat{G})$.

What is the meaning of Lie bracket. There is a fact about it:

Fact 3.14. *G is connected Lie group. G is abelian iff $[-,-] = 0$ on $\mathrm{Lie}(G)$*

3.3 Morphisms between Lie group and Lie algebras

A smooth map $F : G \to H$ between two Lie group is called a **morphism** if $F(gh) = F(g)F(h)$.

A linear map $L : g \to h$ between Lie algebra is called a **morphism** if $L[u,v] = [Lu, Lv]$.

Proposition 3.15. *Let $F : G \to H$ be a morphism of Lie groups. Then $F_{e,*} : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a morphism of Lie algebra.*

4 Vector Field

4.1 Canonical form of a field

Recall that $V \in \mathfrak{X}M$, $p \in M$ is called a **regular point** if $V_p \neq 0$, and is called a **singular point** if $V_p = 0$.

Theorem 4.1 (Canonical Form Theorem). *Let p be a regular point of V. Then $\exists$ local chart $(U, x^1, \cdots, x^n)$ around p s.t. $V|_U = \partial x^1$*

4.2 Lie derivative of vector field

$V, W \in \mathfrak{X}M$, $\mathcal{L}_V W$= is the directional derivative of W in the direction of V.

Theorem 4.2. *TFAE:*

1 V, W commutes.

2 W is invariant under the flow generated by V, i.e. $\theta_{t,*}(W_p) = W_{\theta_t(p)}$

3 The flow for V, W commutes, i.e. $\theta_t \circ \eta_s = \eta_s \circ \theta_t$ whenever either side is defined or equivalently, whose the domain is compatible.

Lemma 4.3. *Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}N$. Then W is F-related to V if and only if $\forall t \in \mathbb{R}$, $\eta_t \circ F = F \circ \theta_t$ on the domain of θ_t, which means*

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \big\downarrow{\scriptstyle \theta_t} & & \big\downarrow{\scriptstyle \eta_t} \\ M & \longrightarrow & N \end{array} \quad \textit{commutes.}$$

4.2.1 Canonical form of commuting vector field

Theorem 4.4. *Given $V_1, \cdots, V_k \in \mathfrak{X}M$, s.t.*

1) $[V_i, V_j] = 0$, $\forall i, j$.

2) $V_{1,p}, V_{2,p}, \cdots, V_{k,p}$ linearly independent at some $p \in M$

Then $\exists$ local chart $(U, x^1, \cdots, x^n)$ around p s.t. $V_i|_U = \dfrac{\partial}{\partial x^i}$, $\forall 1 \leqslant i \leqslant k$

We prove it using the inverse function theorem.

4.3 The constant rank theorem

$F \in C^\infty(M, N)$, $p \in M$. The **rank** of F at p is

$$\begin{aligned} \mathrm{rank}_p F &:= \mathrm{rank}(F_{p,*} : T_pM \to T_{F(p)}N) \\ &= \mathrm{rank}\left(\frac{\partial F^i(p)}{\partial x^j}\right)_{i,j} \end{aligned}$$

We say F has **constant rank** k near p if $\exists$ Nbh U of p *s.t.* $\mathrm{rank}_q F = k$, $\forall q \in U$

Proposition 4.5.

$$\mathrm{rank}_q(F) \leqslant \min(\dim(M), \dim(N))$$

Theorem 4.6 (The constant rank theorem)**.** *Suppose $F : M \to N$ has constant rank k near $p \in M$, then $\exists$ local charts $U \xrightarrow[\cong]{\varphi} \mathbb{R}^m$ around p, $V \xrightarrow[\cong]{\psi} \mathbb{R}^n$ around $F(p)$ s.t.*

$$\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \textit{ is given by } (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^k, 0, \cdots, 0)$$

Lemma 4.7. *X is compact, Y Hausdorff, then $F : X \to Y$ is proper.*

Proposition 4.8. *$F \in C^\infty(M, N)$ is an injective immersion, and F is proper. Then F is an embedding.*

Theorem 4.9 (Sard)**.** *Singular value has measure 0.*

Theorem 4.10. *M is an embedded submanifold of N if and only if $\forall p \in M \subset N$, $\exists$ local chart $(U, x^1, \cdots, x^n)$ around p of N s.t. $M \cap U = \{(x^1, \cdots, x^m, 0, \cdots, 0)\}$*

Theorem 4.11. *$F \in C^\infty(M, N)$, q is a regular valur of F. Then $F^{-1}(q)$ is an embedded submanifold of M. And*

$$\forall p \in F^{-1}(q), T_pF^{-1}(q) = \ker(F_{p,*} : T_pM \to T_{F(p)}N)$$

Denote

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) | A + A^T = 0\}$$

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) | A + A^* = 0\}$$

$$\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) | \mathrm{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \mathrm{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) | \mathrm{tr} A = 0\}$$

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