E1

Proof. First, with Abel-Dirichlet test we know that $\int_0^\infty e^{-tx} f(x)\,\mathrm{d}x$ converges.
With the Newton-Leibniz formula, we know $\int_0^\infty te^{-tx}\,\mathrm{d}x = e^0 - \lim\limits_{x\to\infty} e^{-tx} = 1$
Then for $\epsilon > 0$, there exists $N > 0$ such that $\forall x > N$, $|f(x) - 1| < \epsilon$.
Then

$$\begin{aligned}
|t\int_0^\infty e^{-tx} f(x)\,\mathrm{d}x - 1| &= |t\int_0^\infty e^{-tx} f(x)\,\mathrm{d}x - t\int_0^\infty e^{-tx}\,\mathrm{d}x| \\
&= |t\int_0^\infty e^{-tx}(f(x) - 1)\,\mathrm{d}x| \\
&\leqslant |t\int_0^N e^{-tx}(f(x) - 1)\,\mathrm{d}x| + \epsilon|t\int_N^\infty e^{-tx}\,\mathrm{d}x|
\end{aligned}$$

Let $t \to 0^+$, then $|t\int_0^N e^{-tx}(f(x) - 1)\,\mathrm{d}x| \to 0$.
i.e. $\lim\limits_{t\to 0^+} |t\int_0^\infty e^{-tx} f(x)\,\mathrm{d}x - 1| \leqslant \lim\limits_{t\to 0^+} \epsilon|t\int_N^\infty e^{-tx}\,\mathrm{d}x|$.
Let $\epsilon \downarrow 0$, then $N \to \infty$, and $\epsilon|t\int_N^\infty e^{-tx}\,\mathrm{d}x| \to 0$.
$\Rightarrow \lim\limits_{t\to 0^+} |t\int_0^\infty e^{-tx} f(x)\,\mathrm{d}x - 1| = 0$.
i.e. $\lim\limits_{t\to 0^+} t\int_0^\infty e^{-tx} f(x)\,\mathrm{d}x = 1$ □

E2 (a)

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-imx}\,\mathrm{d}x = -\frac{1}{2i\pi m}(e^{-im\delta} - e^{im\delta}) = \frac{\sin m\delta}{\pi m},\ m \neq 0$$

(use Newton-Leibniz formula)
$c_0 = \dfrac{\delta}{\pi}$
(b) $2\sum\limits_{n=1}^\infty \dfrac{\sin(n\delta)}{n} = \sum\limits_{m=1}^\infty 2\pi c_m = \int_{-\delta}^{\delta} \sum\limits_{m=1}^\infty e^{-imx}\,\mathrm{d}x = \int_{-\delta}^{\delta} \dfrac{e^{-ix}}{1 - e^{-ix}}\,\mathrm{d}x = \dfrac{1}{i}\ln(1 - e^{-ix})\big|_{-\delta}^{\delta} = \pi - \delta$
(c)use the Parseval theorem

$$\begin{aligned}
\sum_{m=-\infty}^\infty c_m\overline{c_m} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{f(x)}\,\mathrm{d}x \\
&= \frac{\delta}{\pi}.
\end{aligned}$$

Then $\sum\limits_{m=1}^\infty \dfrac{\sin^2(m\delta)}{m^2} = \dfrac{\sum\limits_{m=-\infty}^\infty c_m^2 - c_0^2}{2}\pi^2 = \dfrac{\delta(\pi - \delta)}{2}$
(d)First, for $\dfrac{1}{x^2}$ is bounded and $\int_0^\infty sin^2 x\,\mathrm{d}x$ converges, we know $\int_0^\infty (\frac{\sin x}{x})^2\,\mathrm{d}x$ converges by Abel-Dirichlet test.

Notice that for $\epsilon > 0$, choose a partition $p = \{0, \epsilon, 2\epsilon, \cdots\}$ of $(0, \infty)$, and $\sum\limits_{n=1}^{\infty} \frac{\sin^2(n\epsilon)}{(n\epsilon)^2}(n\epsilon - (n-1)\epsilon) = \frac{n-\epsilon}{2}$. Let $\epsilon \downarrow 0$, then $\int_0^\infty (\frac{\sin x}{x})\,\mathrm{d}x = \frac{\pi}{2}$.

(e)Let $\delta = \frac{\pi}{2}$, then $\frac{\pi^2}{8} = \sum\limits_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n^2} = \sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

E3

Proof. Let $g(x) := x$, Then its Fourier coefficients

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{imx}\,\mathrm{d}x = \frac{1}{2\pi}\left.\frac{xe^{imx}}{im} + \frac{e^imx}{m^2}\right|_{-\pi}^{\pi} = \frac{e^{im\pi}}{im} \quad m \neq 0$$

$c_0 = 0$

Then use Parseval theorem we know that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|^2\,\mathrm{d}x = \frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} |c_n|^2 = \frac{1}{2}\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi}{6}$$

And with E2(c) we know:

$$\begin{aligned}
\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nx) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(1 - 2\sin^2(\frac{n|x|}{2})) \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} - \sum_{n=1}^{\infty} \frac{8\sin^2(n\frac{|x|}{2})}{n^2} \\
&= \frac{\pi^2}{3} + \frac{2\pi^2}{3} - 8\frac{\frac{|x|}{2}(\pi - \frac{|x|}{2})}{2} \\
&= \pi^2 - 2\pi|x| + x^2 \\
&= (\pi - |x|^2)
\end{aligned}$$

Then the Fourier coefficients of f

$$c_m = \begin{cases} \dfrac{2}{m^2} & m \neq 0 \\ \dfrac{\pi^2}{3} & m = 0 \end{cases}$$

Use the Parseval theorem we know that:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-\pi}^{\pi} \frac{1}{2\pi}(\pi - |x|)^4\,\mathrm{d}x = \frac{1}{5}\pi^4$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\sum\limits_{n=-\infty}^{\infty} |\frac{1}{2}c_n|^2 - |\frac{1}{2}c_0|^2}{2} = \frac{\pi^4}{90}$$

□

(a)

Proof.

$$\begin{aligned}
K_N(x) &= \frac{1}{N+1} \sum_{n=0}^{N} D_n(x) \\
&= \frac{1}{N+1} \frac{\sum\limits_{n=0}^{N} e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{\sin \frac{x}{2}} \\
&= \frac{1}{N+1} \frac{\sum\limits_{n=0}^{N} (e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x})(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}})}{\sin \frac{x}{2} \sin \frac{x}{2}} \\
&= \frac{1}{N+1} \frac{1 - e^{(N+1)x} - e^{-(N+1)x}}{2\sin^2 x} \\
&= \frac{1}{N+1} \frac{1 - \cos{(N+1)x}}{1 - \cos x}
\end{aligned}$$

As $\cos x \leqslant 1$, Then $K_N(x) \geqslant 0$. And

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) \,\mathrm{d}x = \sum_{n=0}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x) \,\mathrm{d}x = 1$$

$$K_n(x) \leqslant \frac{1}{N+1} \frac{1-(-1)}{1-\cos\delta} = \frac{1}{N+1} \frac{2}{1-\cos\delta} \qquad \text{As } 0 < \delta \leqslant |x| \leqslant \pi$$

□

(b)

Proof.

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1} \sum_{n=0}^{N} \sigma_n(x) \\
&= \frac{1}{N+1} \sum_{n=0}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) \,\mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \frac{1}{N+1} \sum_{n=0}^{N} D_n(t) \,\mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) \,\mathrm{d}t
\end{aligned}$$

□

(c)

Proof. As f is continuous on $[-\pi, \pi]$ with period 2π , Then f is uniformly continuous and bounded. Let $M := \max_{x \in [-\pi,\pi]} f(x)$
Then for $\epsilon > 0$, there exists $\frac{\pi}{2} > \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < 2\delta$. Then

$$\begin{aligned}
2\pi|\sigma_N(x) - f(x)| &= |\int_{-\pi}^{\pi}(f(x-t) - f(x))K_N(t)\,\mathrm{d}t| \\
&\leqslant |\int_{-\pi}^{-\delta}(f(x-t) - f(x))K_N(t)\,\mathrm{d}t| + |\int_{\delta}^{\pi}(f(x-t) - f(x))K_N(t)\,\mathrm{d}t| \\
&+ |\int_{-\delta}^{\delta}(f(x-t) - f(x))K_N(t)\,\mathrm{d}t| \\
&\leqslant 2M \times 2(\pi - \delta)\frac{1}{N+1}\frac{2}{1 - \cos\delta} + \epsilon|\int_{-\delta}^{\delta}K_N(t)\,\mathrm{d}t|
\end{aligned}$$

Then let N large enough, we have:

$$2\pi|\sigma_N(x) - f(x)| \leqslant 2\epsilon|\int_{-\delta}^{\delta}K_N(t)\,\mathrm{d}t| < 2\epsilon|\int_{-\pi}^{\pi}K_N(t)\,\mathrm{d}t|$$

The final inequality is because $K_N(t) > 0$, $\forall t \in \mathbb{R}$.
Then let $\epsilon \to 0$, we have $\lim_{n\to\infty} \sigma_n = f$ uniformly. □

(d)

Proof. For $\epsilon > 0$. $f(x+), f(x-)$,exists implies that there exists $\delta > 0$ s.t.

$$|f(x+) - f(y)| < \epsilon \quad \forall x < y < x + \delta$$

$$|f(x-) - f(y)| < \epsilon \quad \forall x - \delta < y < x$$

$$
\begin{aligned}
|4\pi\sigma_N(x) - 2\pi f(x+) - 2\pi f(x-)| &= |2\int_{-\pi}^{\pi} f(x-t)K_N(t)\,\mathrm{d}t - \int_{-\pi}^{\pi} f(x+)K_N(t)\,\mathrm{d}t \\
&\quad - \int_{-\pi}^{\pi} f(x-)K_N(t)\,\mathrm{d}t \\
&\leqslant |\int_{-\pi}^{-\delta} (2f(x-t) - f(x+) - f(x-))K_N(t)\,\mathrm{d}t \\
&\quad + \int_{\delta}^{\pi} (2f(x-t) - f(x+) - f(x-))K_N(t)\,\mathrm{d}t| \\
&\quad + 2|\int_{-\delta}^{0} (f(x-t) - f(x+))K_N(t)\,\mathrm{d}t| \\
&\quad + 2|\int_{0}^{\delta} (f(x-t) - f(x-))K_N(t)\,\mathrm{d}t| \quad (\text{for } K_N(t) - K_N(-t)) \\
&\leqslant +16M(\pi-\delta)\frac{1}{N+1}\frac{2}{1-\cos\delta} + 4\epsilon|\int_{-\pi}^{\pi} K_N(t)\,\mathrm{d}t|
\end{aligned}
$$

Then

$$\lim_{N\to\infty} |4\pi\sigma_N(x) - 2\pi f(x+) - 2\pi f(x-)| \leqslant 4\epsilon|\int_{-\pi}^{\pi} K_N(t)\,\mathrm{d}t|$$

Let $\epsilon \to 0^+$, Then $\lim\limits_{N\to\infty} \sigma_N(x) = \dfrac{f(x+) + f(x-)}{2}$

□

(e)

Proof. First, if $f(x) = e^{imx}$, $m \in \mathbb{Z}$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{im(x+n\alpha)} = e^{im(x+\alpha)} \lim_{N\to\infty} \frac{1}{N}\frac{e^{imN\alpha}-1}{e^{im\alpha}-1} = 0,\ m \neq 0$$

and $e^{im(x+n\alpha)} = 1$ for $m = 0$. Notice that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}\,\mathrm{d}x = \begin{cases} 1, & m = 0 \\ 0, & m \in \mathbb{Z}, m \neq 0 \end{cases}$$

Then equality holds for $f(x) = e^{imx}, m \in \mathbb{Z}$.
Then if we define X be the set of all the function f satisfying the equality. Easy to know that X is a linear subspace.
So The Nth partial sum $s_N = \sum\limits_{n=-N}^{N} c_n e^{inx} \in X$. Then $\sigma_N = \frac{s_0+s_1+\cdots,+s_N}{N+1} \in X$.
Then as $\sigma_N \to f$ uniformly.

Hence

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t &= \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_N\,\mathrm{d}x \\
&= \lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{M}\sum_{n=1}^{M}\sigma_N(x+n\alpha) \\
&= \lim_{M\to\infty}\frac{1}{M}\sum_{n=1}^{M}\lim_{N\to\infty}\sigma_N(x+n\alpha) \\
&= \lim_{M\to\infty}\frac{1}{M}\sum_{n=1}^{M} f(x+n\alpha)
\end{aligned}$$

□