Complex Analysis HW2

1. $\frac{\partial u}{\partial x} = \frac{2xy^3}{(x^2+y^2)^2}$ $\frac{\partial u}{\partial y} = \frac{x^4-x^2y^2}{(x^2+y^2)^2}$ $\frac{\partial v}{\partial x} = \frac{y^4-x^2y^2}{(x^2+y^2)^2}$ $\frac{\partial v}{\partial y} = \frac{2x^3y}{(x^2+y^2)^2}$ $\Rightarrow$ satisfies CR at $z=0$.

However, $\lim_{(x,0)\to(0,0)} \frac{f(z)-f(0)}{|z|} = 0$, $\lim_{(x,x)\to(0,0)} \frac{f(z)-f(0)}{\sqrt{2}x} = \frac{1}{2\sqrt{2}} \neq 0 \Rightarrow f$ is not analytic at $z=0$.

2. $\Delta u = 6ax+2by+2cx+6dy=0 \Rightarrow \begin{cases} c=-3a \\ b=-3d \end{cases}$, i.e. $u = ax^3-3dx^2y-3axy^2+dy^3$ $(a,d\in\mathbb{R})$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v = 3ax^2y-3dxy^2-ay^3+A(x)$ $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \Rightarrow A(x) = dx^3+C$

$\Rightarrow v = dx^3+3ax^2y-3dxy^2-ay^3+C$ $(a,d,C\in\mathbb{R})$

3. If $|f|\equiv 0$, then $f=0$. If $|f|=c>0$, let $f=u+iv$, then $u^2+v^2=c$. Apply $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ we get

$\begin{cases} u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0 \\ u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0 \end{cases}$. By CR, $\begin{cases} u^2\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y}=0 \\ uv\frac{\partial u}{\partial y}+v^2\frac{\partial u}{\partial x}=0 \end{cases} \Rightarrow (u^2+v^2)\frac{\partial u}{\partial x}=0 \Rightarrow \frac{\partial u}{\partial x}=0$.

Similarly we get $\frac{\partial u}{\partial y}=0=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y} \Rightarrow u,v$ are constant $\Rightarrow f\equiv C$.

4. Let $f=u+iv$, then $\overline{f(\bar z)} = u(x,-y)-iv(x,-y) := U(x,y)+iV(x,y)$. Clearly U,V are continuously differentiable and $\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}$, $\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x} \Rightarrow \overline{f(\bar z)}$ is analytic. The converse is also true ($g(z)=\overline{f(\bar z)}$, $\overline{g(\bar z)}=f(z)$).

5. (a) Let $R = Q(z)\sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}$, it's easy to check that $R(\alpha_k)=P(\alpha_k)$ and $\deg R<n \Rightarrow R=P$.

Alternatively, consider the laurent expension of $\frac{P}{Q} = T(z)+\sum_{k=1}^{n}\frac{b_k}{z-\alpha_k}$ where $b_k\in\mathbb{C}$, $T(z)\in\mathbb{C}[z]$.

Let $z\to\infty$ we get $T(z)=0$. Since $\lim_{z\to\alpha_k}\frac{P(z)(z-\alpha_k)}{Q(z)} = \frac{P(\alpha_k)}{Q'(\alpha_k)}$, $b_k = \frac{P(\alpha_k)}{Q'(\alpha_k)} \Rightarrow \frac{P(z)}{Q(z)} = \sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}$

Note that $\{\alpha_k\}$ are distinct, $Q'(\alpha_k)\neq 0$.

(b) Existence: $P(z) = \sum_{k=1}^{n}\frac{c_kQ(z)}{Q'(\alpha_k)(z-\alpha_k)}$, where $Q(z)=\prod_{k=1}^{n}(z-\alpha_k)$

Uniqueness: If P^* is another polynomial, then $P^*(z) = \sum_{k=1}^{n}\frac{P^*(\alpha_k)Q(z)}{Q'(\alpha_k)(z-\alpha_k)} = \sum_{k=1}^{n}\frac{c_kQ(z)}{Q'(\alpha_k)(z-\alpha_k)} = P(z)$.

6. Assume $R(z)\overline{R(z)}=1$ $(\forall |z|=1)$, then $R(z)\overline{R(\frac{1}{\bar z})}=1=R(z)\bar R(\frac{1}{z})$ $(\forall |z|=1)$.

Note that $\bar R(\frac{1}{z})$ is also a rational function, in fact $R(z)\bar R(\frac{1}{z})=1$ for almost all $z\in\mathbb{C}$.

If $w\neq 0$ is a zero of R, then $\frac{1}{\bar w}$ is a pole of R with the same degree.

Write $R=\frac{z^nP}{Q}$, where $(P,Q)=1$ and $z\nmid PQ$, then $P=a\prod_{k=1}^{m}(z-\alpha_k) \Rightarrow Q=b\prod_{k=1}^{m}(\bar\alpha_k z-1)$ $(\alpha_k\neq 0)$

By HW1.1, $\left|\frac{z-\alpha_k}{\bar\alpha_k z-1}\right|=1$ $(\forall |z|=1)$ if $|\alpha_k|\neq 1$. Thus $R = c\prod_{k=1}^{m}\frac{z-\alpha_k}{\bar\alpha_k z-1}z^n$, $|c|=1$, $|\alpha_k|\neq 1$, $\alpha_k\neq 0$, $n\in\mathbb{Z}$.

Alternatively, by multipling suitably z^n and $\frac{z-\alpha_k}{\bar\alpha_k z-1}$ on R, we may assume R has no zeros or poles in $\{|z|\le 1\}$.

Note that the condition $|R(z)|=1$ on $|z|=1$ implies $|\alpha_k|\neq 1$. Apply the Maximum Modulus Principle to R and $\frac{1}{R}$ we know $|R|\equiv 1$, i.e. $R=cz^n\prod_{k=1}^{m}\frac{z-\alpha_k}{\bar\alpha_k z-1}$ with $|c|=1$.

7. (a) Let $w=\frac{z}{1+z}$, then $\sum w^n$ converges $\Leftrightarrow |w|<1 \Leftrightarrow |z|<|1+z| \Leftrightarrow \operatorname{Re} z > \frac{-1}{2}$.

(b) $z=0$ converges. If $|z|>1$, then $\left|\frac{z^n}{1+z^{2n}}\right| \le \frac{1}{|z|^n-1} \le \frac{2}{|z|^n}$ for large n. If $0<|z|<1$, then $\left|\frac{z^n}{1+z^{2n}}\right| \le \frac{1}{|z^{-1}|^n-1} \le \frac{2}{|z^{-1}|^n}$ for large n. So $\sum_{n=1}^{\infty}\frac{z^n}{1+z^{2n}}$ converges for $|z|\neq 1$.

If $|z|=1$, then $\left|\frac{z^n}{1+z^{2n}}\right| \ge \frac{1}{1+1} = \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}\frac{z^n}{1+z^{2n}}$ diverges.