Complex Analysis

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October 30, 2024

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2 Complex Functions

2.1 Analytic functions and rational functions

2.1.1 Harmonic function

Definition 2.1 (Cauchy-Riemann equation)**.**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 2.2 (Harmonic function)**.** A function u is **harmonic** if it satisfied **Laplace equation** $\triangle u = 0$.

If two harmonic function u and v satisfies Cauchy-Riemann equations, then we say that v is **conjugate harmonic function of** $u \Rightarrow u$ is conjugate harmonic of $-v$.

2.1.2 Polynomials and rational function

The polynomial $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ is analytic in $\mathbb{C}$.

We will prove the fundamental theorem of algebra

Theorem 2.3 (Fundamental Theorem of Algebra)**.** *Every polynomial with degree $n > 0$ has at least one point.*

Theorem 2.4 (Gauss-Lucus theorem)**.** *The smallest convex polygon that contain the zeros of P also contains the zeros of P'.*

Proof. Only need to check.

We can get this equation.

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{j=1}^{n} \frac{1}{\alpha - \alpha_j} = 0 \Rightarrow \sum_{j=1}^{n} \frac{\overline{\alpha - \alpha_j}}{|\alpha - \alpha_j|^2}$$

Hence α is linearly represented by α_j. $\square$

Proposition 2.5. *Let P and Q be two polynomial with no common zeros. Then the rational function $R(z) = \dfrac{P(z)}{Q(z)}$ is analytic away from the zeros of Q.*
*The zeros of Q are called **poles** of R, and the **order of a pole** is equal to the order of the corresponding zero of Q.*

We often view R as a function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. $R_1(z) := R(\frac{1}{z})$.

If $R_1(0) = 0$, the order of the zero at ∞ (of R) is the order of the zero of $R_1(z)$ at $z = 0$.

If $R_1(0) = \infty$, the order of the pole at ∞ (of R) is the order of the pole of $R_1(z)$ at $z = 0$.

Suppose

$$R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, a_n \neq 0, b_m \neq 0$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$$

By discussing m and n, we can infer the situation of $R(z)$ at ∞.
By adding the order of poles and zeros at ∞, we can get the following theorem.

Theorem 2.6. *The total number of zeros and poles of a rational function are the same.*

Remark 2.7. This common number is called the **order of the rational function**.

Corollary 2.8. *Suppose a rational function R has order p. Then every equation $R(z) = a$ has exactly p roots.*

Proof. $\hat{R}(z) = R(z) - a$ has the same poles as R. $\square$

A rational function of order 1 is a **linear fraction** $R(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$
Such fraction is often called **Möbius transformation**

Every rational function has a representation by **partial fractions**.

- If R has a pole at ∞. Then we can write

$$R(z) = G(z) + H(z) \tag{$*$}$$

where G is a polynomial without constant term, and H is finite at ∞.

The degree of G is the order of the pole of R at ∞. G is called the **singular part** of R at ∞.

- Let the distinct finite poles of R be $\beta_1, \cdots, \beta_k$. Let $R_j(\psi) = R(\beta_j + \frac{1}{\psi})$. Then R_j is a rational function with a pole at ∞. As in $(*)$, we can write

$$R_j = G_j + H_j$$

with H_j finite at ∞. Then

$$R(z) = G_j(\frac{1}{z-\beta_j}) + H(\frac{1}{z-\beta_j})$$

with G_j is a polynomial in $\frac{1}{z-\beta_j}$ without constant term called the **singular point** of R at β_j.

- Let $F(z) = R(z) - G(z) - \sum\limits_{j=1}^{k} G_j(\frac{1}{z-\beta_j})$.
Then F is a rational function which can only have poles among β_j, ∞

Since by our construction, F is finite at every $\beta_j, 1 \leqslant j \leqslant k$ and ∞.

So F is a constant.

In particular, $R(z) = G(z) + \sum\limits_{j=1}^{k} G_j(\frac{1}{z-\beta_j}) + c$.

2.2 Power Series

2.2.1 Power series

Theorem 2.9 (Abel's theorem). *If $\sum a_n$ converges, then $f(z) = \sum a_n z^n \to f(1)$ as $z \to 1$ in such a way that $\frac{|1-z|}{1-|z|}$ remains bounded.*

2.3 Exponential, Trigonometric and logorithmic functions

2.3.1 Exponential and Trigonometric function

The **exponential function** is defined as the solution if the differential equation

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1 \end{cases}$$

We denote $e^z = \exp z = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$.

The **trigonometric function** are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

2.3.2 Logorithmic Functions

The **logorithmic function** ln is defined by $z = \ln w$ is a root of the equation $e^z = w$.

For $w \neq 0$, we write $z = x + iy$, then

$$e^{x+iy} = w \Leftrightarrow \begin{cases} e^x = |w| \\ e^{iy} = \dfrac{w}{|w|} \end{cases}$$

The first equation has a unique solution $x = \ln |w|$.

The second equation $e^{iy} = \frac{w}{|w|}$ has a unique solution $y_0 \in [0, 2\pi)$.

If we write $w = re^{i\theta}$, then $x = \ln w, y = \theta = \arg w$.

Thus, for $w \neq 0$, we have

$$\ln w = \ln |w| + i \arg w$$

The function $\ln$ is actually not single-valued. But we can define a single-valued function Ln

We define

$$a^b = \exp(b \ln a)$$

We will prove Ln is analytic in $\mathbb{C} - (-\infty, 0]$ but not continuous in $(-\infty, 0]$.

Ln is the principal branch of the logithm.

3 Conformal Mappings

3.1 Basic topology

3.1.1 Connectedness

Theorem 3.1. *A nonempty open set in $\mathbb{C}$ is connected iff any two of its points can be joined by a polygon which lies in the set.(i.e. Connectedness is equivalent to Path Connectedness)*

An nonempty connected subset is called a **region**

3.1.2 Compactness

Definition 3.2. A set X is **totally bounded** if $\forall \varepsilon > 0$, X can be covered by finitely many balls of radius ε

Theorem 3.3. *A set is compact iff it is complete and totally bounded.*

Theorem 3.4. *A subset $X \subset$ is compact iff every infinite sequence of X has a limit point in X.*

3.1.3 Continuous Functions

Theorem 3.5. *Continous function maps connected space to connected space.*

Theorem 3.6. *Continous function maps compact space to compact space.*

3.2 Conformality, geometric consequences of the existence of a derivative

3.2.1 Arcs and closed curves

The equation of an **arc** r in $\mathbb{C}$ can be represented by one of the terms

- $x = x(t), y = y(t)$, $\alpha \leqslant t \leqslant \beta$, x, y are continuous at t
- $z(t) = x(t) + iy(t)$, $\alpha \leqslant t \leqslant \beta$.
- The continuous mapping $\gamma : [\alpha, \beta] \to \mathbb{C}$.

For a non-decreasing function $\varphi : [\alpha, \beta] \to [\alpha, \beta]$, $z = z(\varphi(t)), \alpha' \leqslant \tau \leqslant \beta'$ is **change of parameter** of $z(t)$.

The change is **reversible** iff φ is strictly increasing.

If γ is differentiable, then call γ a **curve**.

γ is **simple** , or a **Jordan curve**, if γ is injective.

γ is **closed curve** if $\gamma(0) = \gamma(1)$.

3.2.2 Analytic Functions in Regions

A function f is analytic on an arbitrary set A if it is the restriction to A of a function which is analytic in some open set containing A.

Theorem 3.7. *An analytic function in a region(i.e. open and connected) Ω whose derivative is 0 must reduce to a constant. The same hold if the real part, the imaginary part, the modulus, or the argument is constant.*

3.2.3 Conformal Mappings

Suppose $f : \Omega \to \mathbb{C}$ is analytic in Ω. $r_1 = z_1(t), r_2 = z_2(t)$, $\alpha \leqslant t \leqslant \beta$.

$z_0 = z_1(t_0) = z_2(t_0'), z_1'(t_0) \neq 0, z_2'(\hat{t_0}) \neq 0, \alpha < t_0, \hat{t_0} < \beta$.

$f'(z_0) \neq 0, w_1(t) = f(z_1(t_0)), w_2 = f(z_2(\hat{(t_0)}))$

$\Gamma_1 = \{w_1(t) | \alpha \leqslant t \leqslant \beta\}, \Gamma_2 = \{w_2(t) | \alpha \leqslant t \leqslant \beta\}$

Then

$$w_1'(t) = f'(z_1(t))z_1'(t)$$
$$w_2'(t) = f'(z_2(\hat{t}))z_2'(\hat{t})$$

$\Rightarrow$

$$w_1'(t_0) \neq 0, w_2'(t_0) \neq 0$$
$$\arg w_1'(t_0) = \arg f'(z_1(t_0))z_1'(t_0)$$
$$\arg w_2'(t_0) = \arg f'(z_2(\hat{t_0}))z_2'(\hat{t_0})$$

So the "angle" $\arg w_1'(t_0) - \arg w_2'(\hat{t_0} = \arg z_1(t_0) - \arg z_2(\hat{t_0})$ remains the same.

Now we give the definition.

Definition 3.8. $w = f(z)$ is said to be **conformal** in Ω if f is analytic in Ω and $f'(z) \neq 0$ for $\forall z \in \Omega$.

Easy to prove that linear change of scale at z_0 is independent of the direction. *i.e.* $|f'(z_0)| = \lim\limits_{z \to z_0} \frac{\delta\sigma}{\delta s}$

3.2.4 Length and Area

The **length** of a differentiable arc γ with the equation $z(t) = x(t) + iy(t)$, $a \leqslant t \leqslant b$

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \mathrm{d}t = \int_a^b |z'(t)| \mathrm{d}t$$

For $\Gamma = f(\gamma)$ where f conformal mapping.

Then

$$L(\Gamma) = \int_a^b |f'(z(t))| \cdot |z'(t)| \mathrm{d}t$$

The **area** of $E \subset \mathbb{R}$ is $A(E) = \int\int_E \mathrm{d}x\mathrm{d}y$

Then by the differentiable functional transformation, the area $\hat{E} = f(E)$ is

$$A(\hat{E}) = \int\int_E |u_x v_y - u_y v_x| \mathrm{d}x\mathrm{d}y$$

If f is the conformal mapping of an open set containing E, then by Caucht-Riemann equation

$$A(\hat{E}) = \int\int_E |f'(z)|^2 \mathrm{d}x\mathrm{d}y$$

3.3 Möbius Transformation

Recall that a **Möbius transformation** is a function of the form

$$w = s(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Then it has an inverse $z = S^{-1}(w) = \dfrac{dw - b}{-cw + a}$.
We may define $S(\infty) = \lim_{z\to\infty} S(z) = \frac{a}{c}$, $S(\frac{-d}{c}) = \infty$
With these definition, $S : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a topological mapping. Here one may use the chordal metric to define the topology.

$$S'(z) = \frac{ad - bc}{(cz + d)^2}$$

Then S is conformal in $\hat{\mathbb{C}} - \{-\frac{d}{c}, \infty\}$.

$w = z + \alpha$ is called a **parallel translation**.

$w = kz$ with $|k| = 1$ is a **rotation**.

$w = kz$ with $k > 0$ is a **homothetic transformation**.

$x = \frac{1}{z}$ is called an **inversion**.

Proposition 3.9. *Every Möbius transformation is a composition of the above four operations.*

3.3.1 Cross ratio

For three distinct points $z_2, z_3, z_4 \in \hat{\mathbb{C}}$,we can find a Möbius transformation S such that $S(z_2) = 0, S(z_3) = 1, S(z_4) = \infty$.

Lemma 3.10. *The Möbius transformation satisfying the above conditions is unique.*

The **cross ratio** (z_1, z_2, z_3, z_4) is the image z_1 under the Möbius transformation which maps z_2 to 1, z_3 to 0 and z_4 to ∞.

Theorem 3.11. *If $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ are distinct, and T is any Möbius transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Lemma 3.12. *Let T be a Möbius transformation, $T(\mathbb{R})$ is either a circle or a straight line.*

Theorem 3.13. *The cross ratio* (z_1, z_2, z_3, z_4) *is real iff the four points lie on a circle or a straight line.*

Remark 3.14. One may prove the theorem by elementary geometry

Theorem 3.15. *A Möbius transformation maps circles into circles.*

3.3.2 Symmetry

Suppose T is a Möbius transformation which maps $\hat{\mathbb{R}}$ onto a circle C.
We say that $w = Tz$ and $w^* = T\bar{z}$ are **symmetric** w.r.t. C.

Remark 3.16. This definition is independent of T. Suppose S is another Möbius transformation which maps $\hat{\mathbb{R}}$ onto C, then $S^{-1}T$ maps $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$, and this $S^{-1}w = S^{-1}Tz$ and $S^{-1}w^* = S^{-1}T\bar{z}$ are conjugate.

The points z and z^* are **symmetric w.r.t C through** z_1, z_2, z_3 iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.

This can be another definition.

Note that only the points on C are symmetric to themselves.

The mapping $z \mapsto z^*$ is 1-1 and is called **reflection** w.r.t. C.

Geometric Meaning of Symmetry

Case1: C is a straight line. We may assume $z_3 = \infty$.

z, z^* are symmetric w.r.t. C if and only if

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Then

$$|z^* - z_2| = |z - z_2|, \quad \forall z_2 \in C \text{ and } z_2 \neq \infty$$

$$\operatorname{Im} \frac{z^* - z_2}{z_1 - z_2} = \operatorname{Im} \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

So C is the bisecting normal of the segment between z and z^*.

Case2: C is the circle $|z - a| = R$.

Then for $\forall$ distinct $z_1, z_2, z_3 \in \mathbb{C}$, $\overline{(z, z_1, z_2, z_3)} = \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)}$
$= (\bar{z} - \bar{a}, \bar{z}_1 - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}) = (\bar{z} - \bar{a}, \dfrac{R^2}{z_1 - a}, \dfrac{R^2}{z_2 - a}, \dfrac{R^2}{z_3 - a}) = (\dfrac{R^2}{\bar{z} - \bar{a}}, z_1 - a, z_2 -$
$a, z_3 - a)$
$= (\dfrac{R^2}{\bar{z} - \bar{a}}, z_1, z_2, z_3)$.
Then the symmetric point of z w.r.t. C is

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

or

$$(z^* - a)(\bar{z} - \bar{a}) = R^2$$

$\Rightarrow$

$$\begin{cases} |z^* - a| \cdot |z - a| = R^2 \\ \dfrac{z^* - a}{z - a} = \dfrac{(z^* - a)(\bar{z} - \bar{a})}{|z - a|^2} > 0 \end{cases}$$

Theorem 3.17 (The Symmetric principle)**.** *If a Möbius transformation maps a circle C_1 onto a circle C_2, then it transforms any pair of symmetric points w.r.t. C_1 into a pair of symmetric points w.r.t. C_2.*

Proof. Case1: $C_1 = \hat{\mathbb{R}}$. Let T be the Möbius transformation which maps $\hat{\mathbb{R}}$ onto C_2. $\forall z \in \mathbb{C}$, by definition, $w = Tz$ and $w^* = T\bar{z}$ are symmetric w.r.t. C_2.

Case2: C_1 is a general circle. Let $T : C_1 \to C_2$ and $S : \mathbb{R} \to C_2$ be Möbius transformation.

Suppose w, w^* are symmetric w.r.t. C_1. Then there exists z *s.t.* $w = Sz, w^* =$

$S\bar{z}$.

Then we can find $Tw = TSz, Tw^* = TS\bar{z}$ are symmetric w.r.t. C_2 since $TS : \hat{\mathbb{R}} \to C_2$ □

Remark 3.18. (1). The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w, z_2 \mapsto w_2, z_3 \mapsto w_3$ where $z_1, z_2, z_3 \in C_1, w_1, w_2, w_3 \in C_2$ is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3,)$$

(2). The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w_1$, $z_2 \mapsto w_2$ where $z_1 \in C_1, z_2 \notin C_1, w_1 \in C_2, w_2 \notin C_2$ is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

3.3.3 Steiner Circles, circular net

For $S(z) = \dfrac{az+b}{cz+d}$, $S'(z) = \dfrac{ad-bc}{(cz+d)^2}$.

A point $z \notin$ a circle C is said to on the **right(left, resp.)** of C if $\operatorname{Im}(z, z_1, z_2, z_3) > 0$($\operatorname{Im}(z, z_1, z_2, z_3) < 0$)

Remark 3.19.

(1). This agrees with everyday use since $(i, 1, 0, \infty) = i$

(2). This distinct between left and right is the same for all triples, while the meaning may be reversed.

(If $C = \hat{\mathbb{R}}$, then $(z, z_1, z_2, z_3) = \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R} \Rightarrow \operatorname{Im}(z, z_1, z_2, z_3) = \dfrac{ad-bc}{|cz+d|^2}\operatorname{Im}(z)$)

(3). We can define an absolute positive orientation of all finite circles by requiring that ∞ should be lie to the right of the oriented circles.

Consider a Möbius transformation of the form

$$w = k \cdot \frac{z-a}{z-b}$$

Here, $z = a \mapsto w = 0, z = b \mapsto w = \infty$.

Then circles through a, b maps to straight line through $0, \infty$.

The concentric circle about the origin, $|w| = \rho$, correspond to circles with the equation

$$\left|\frac{z-a}{z-b}\right| = \frac{\rho}{|k|}$$

These are the circles of **Apollonius** with limit points a and b.

Denote by C_1 the circles through a, b and C_2 the circles of Apollonius with these limit points. The configuration formed by all the circles C_1 and C_2 is called the **Steiner circles**(or **circular net**)

Theorem 3.20.

(a) There is exactly one C_1 and one C_2 through each point in $\hat{\mathbb{C}} \backslash \{a, b\}$

(b) Every C_1 meets every C_2 under right angle.

(c) Reflection in a C_1 transforms every C_2 into itself and every C_1 into another C_1.

(d) The limit points a, b are symmetric w.r.t. each C_2, but not w.r.t. other circles.

Proof. If the limit points are $0, \infty$, those properties are trivial in the w-plane. The general case follows since all properties are invariant under Möbius transformations. $\square$

4 Elementary Conformal mapping

Example 4.1. $w = z^\alpha$ where $\alpha > 0$.

Let $S(u_1, u_2)$ with $0 < \varphi_2 - \varphi_1 \leqslant 2\pi$ be $\{z \in \mathbb{C} : z \neq 0, \varphi_1 < \arg(z) < \varphi_2\}$ where $\arg(z)$ can be chosen as any value of it.

Then $S(\varphi_1, \varphi_2)$ is a region.

In this region, a unique value of $w = z^\alpha$ is defined by $\arg w = \alpha \arg z$.

This function is analytic with $\dfrac{\mathrm{d}w}{\mathrm{d}z} = \alpha \dfrac{w}{z}$.

This function is $1-1$ only if $\alpha(\varphi_2 - \varphi_1) \leqslant 2\pi$.

Example 4.2. $w = e^z$ maps $\{z \in \mathbb{C} : -\dfrac{\pi}{2} < \mathrm{Im}(z) < \dfrac{\pi}{2}\}$ onto $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$

Example 4.3. $w = \dfrac{z-1}{z+1}$ maps $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ onto $\{ww \in \mathbb{C} : |w| < 1\}$

Example 4.4.

$$\mathbb{C}\backslash[-1,1] \xrightarrow{z_1 = \frac{z+1}{z-1}} \mathbb{C}\backslash(-\infty, 0] \xrightarrow{z_2 = \sqrt{z_1}} \{\mathrm{Re}(z_2) > 0\} \xrightarrow{w = \frac{z_2 - 1}{z_2 + 1}} \{w \in \mathbb{C} : |w| < 1\} \tag{4.1}$$

4.1 Elementary Riemann surfaces

Example 4.5. $w = z^n$, $n \in \mathbb{Z}_+$ and $n > 1$.

There is a 1-1 correspondence between each angle $\dfrac{(k-1)2\pi}{n} < \arg z < \dfrac{k \cdot 2\pi}{n}, k = 1, 2, \cdots, n$ and while w-plane except for the positive real axis.

Example 4.6. $w = e^z$. This function maps each parallel strip $(k-1)2\pi < \mathrm{Im}\, z < k \cdot 2\pi, k \in \mathbb{Z}$ onto a sheet with a cut along the positive axis.

5 Complex Integration

5.1 Fundamental Theorems

5.1.1 Line integral and rectifiable arcs

Let $f(t) = u(t) + iv(t)$ be a complex-valued defined on $t \in [a, b] \subset \mathbb{R}$ where u, v are real-valued functions. If f is continuous on $[a, b]$, we may define the **integral**

$$\int_a^b f(t)\mathrm{d}t := \int_a^b u(t)\mathrm{d}t + i\int_a^b v(t)\mathrm{d}t$$

Let γ be a piecewise differential arc in $\mathbb{C}$ with the equation $z = z(t), a \leqslant t \leqslant b$. If f is continuous on γ, then $f(z(t))$ is continuous on $[a, b]$, and we define

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t \tag{5.1}$$

The integral defined in 5.1 is independent of the parametrization of γ. Suppose that anther parametrization of γ is $\gamma : (\alpha, \beta) \to \mathbb{C}, \tau \mapsto z(t(\tau))$, where $t : (\alpha, \beta) \to (a, b), \tau \mapsto t(\tau)$ is piecewise differentiable. Then we have

$$\int_a^b f(z(t))z'(t)\mathrm{d}t = \int_\alpha^\beta f(z(t(\tau)))z'(t(\tau))t'(\tau)\mathrm{d}t = \int_\alpha^\beta f(z(t(\tau)))\frac{\mathrm{d}z(t(\tau))}{\mathrm{d}\tau}\mathrm{d}\tau \tag{5.2}$$

For an arc γ with equation $z = z(t), a \leqslant t \leqslant b$, we define $-\gamma$ by $z = z(-t), -b \leqslant t \leqslant a$.

Then we have

$$\int_{-\gamma} f(z)\mathrm{d}z = \int_{-b}^{-a} f(z(-t))\frac{\mathrm{d}z(-t)}{\mathrm{d}t}\mathrm{d}t$$

$$
\begin{aligned}
&= -\int_{-a}^{-b} f(z(-t))z'(-t)\mathrm{d}t \\
&= -\int_{a}^{b} f(z(\tau))z'(\tau)\mathrm{d}\tau \\
&= -\int_{\gamma} f(z)\mathrm{d}z
\end{aligned}
$$

So we have those properties:

Proposition 5.1.

(a) $\displaystyle\int_{-\gamma} f(z)\mathrm{d}z = -\int_{\gamma} \mathrm{d}z$

(b) *Let f and g be two continuous functions on the piecewise differentiable arc γ, then*

$$\int_{\gamma} (\lambda_1 f + \lambda_2 g)\mathrm{d}z = \lambda_1 \int_{\gamma} f\mathrm{d}z + \lambda_2 \int_{\gamma} g\mathrm{d}z, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

(c) *If γ can be subdivided into two pieces differentiable arcs γ_1 and γ_2, and f is continuous on γ_1 , then*

$$\int_{\gamma} f\mathrm{d}z = \int_{\gamma_1} f\mathrm{d}z + \int_{\gamma_2} f\mathrm{d}z$$

(d) (c) *implies that the integral of a closed curve doesn't depend on the starting point on the curve*

Example 5.2. Evaluate $\displaystyle\int_{\gamma} \frac{1}{z-a}\mathrm{d}z$ where γ is the circle centered at $a \in \mathbb{C}$ with radius R.

Let $z = z(t) = a + Re^{it}$. Then the integral is $2\pi i$

5.1.2 The fundamental theorem of Calculus for integrals in $\mathbb{C}$

The line integral w.r.t. $\bar{z}$ is defined by

$$\int_{\gamma} f(z)\overline{\mathrm{d}z} = \overline{\int_{\gamma} \overline{f(z)}\mathrm{d}z}$$

With this notation, line integrals w.r.t. $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ can be defined by

$$\int_\gamma f(z)\mathrm{d}x = \frac{1}{2}[\int_\gamma f(z)\mathrm{d}z + \int_\gamma f(z)\overline{\mathrm{d}z}]$$

$$\int_\gamma f(z)\mathrm{d}y = \frac{1}{2i}[\int_\gamma f(z)\mathrm{d}z - \int_\gamma f(z)\overline{\mathrm{d}z}]$$

if we write $f(z) = \mu + i\nu$, we have

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}x + i\int_\gamma f(z)\mathrm{d}y = \int_\gamma (\mu\mathrm{d}x - \nu\mathrm{d}y) + i\int_\gamma (\nu\mathrm{d}x + \mu\mathrm{d}y)$$

Remark 5.3. It is followed by the intuition. We can view the integration as the multiplication between f and $\mathrm{d}z$.

The integral w.r.t. **arc length** is defined by

$$\int_\gamma f(z)|\mathrm{d}z| = \int_a^b f(z(t))|z'(t)|\mathrm{d}t$$

This integral is again independent of the parametrization. It is easy to check

$$\int_{-\gamma} f(z)|\mathrm{d}z| = \int_\gamma f(z)|\mathrm{d}z|$$

Now we define **length** of a curve γ: $L(\gamma) = \int_\gamma |\mathrm{d}z|$

We have the inequality:

$$\left|\int_\gamma f\mathrm{d}z\right| \leqslant \int_\gamma |f|\cdot|\mathrm{d}z| \leqslant L(\gamma)\cdot\sup_{z\leqslant\gamma}|f(z)|$$

The length of an arc γ ($z = z(t)$) can also be defined as the least upper bound of all sums

$$\sum_{i=1}^{n} |z(t_i) - z(t_{i-1})|$$

where $a = t_0 < t_1 < \cdots < t_n = b$ If this least upper bound is finite, we say that the arc is **rectifiable**

It is easy to show that piecewise differentiable arcs are rectifiable.

The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_\gamma f(z)\mathrm{d}z = \lim \sum_{k=1}^{n} f(z(\psi_k))[z(t_k) - z(t_{k-1})]$$

Theorem 5.4. *Let $\Omega \subset \mathbb{C}$ be a region, and P, Q two (possibly complex-valued) functions that are continuous on Ω, γ closed curve. The integral $\int_\gamma p(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$ depends only on the end point of γ iff there exists a function $U(x,y)$ on Ω with $\dfrac{\partial U}{\partial x} = P, \dfrac{\partial U}{\partial y} = Q$.*

Proof. "$\Leftarrow$": If such a U exists, then

$$\int_\gamma P\mathrm{d}x + Q\mathrm{d}y = \int_\gamma \frac{\partial U}{\partial x}\mathrm{d}x + \frac{\partial U}{\partial y}\mathrm{d}y = \int_\gamma \frac{\mathrm{d}U}{\mathrm{d}t}\mathrm{d}t = U(\gamma(b)) - U(\gamma(a))$$

"$\Rightarrow$": Fix a point $(x_0, y_0) \in \Omega$. We define $U(x,y) = \int_\gamma P\mathrm{d}x + Q\mathrm{d}y$ where γ is any curve between (x_0, y_0) and (x, y). Easy to check that it is true. $\square$

Theorem 5.5 (Fundamental theorem of Calculus for integrals on $\mathbb{C}$)**.** *Let f be continuous on a region Ω containing γ. $\int_\gamma f\mathrm{d}z$ depends on the endpoints iff f is the derivative of an analytic function F in Ω.*

Remark 5.6. We will prove $\displaystyle\int_\gamma f\mathrm{d}z = F(\omega_2) - F(\omega_1)$ where γ begins at ω_1 and ends at ω_2.

Proof. Transform the line integration into the composition of two real integration.

$\square$

Corollary 5.7. *If F is analytic on Ω with $F' = f$, and γ is a closed curve in Ω, then $\int_\gamma f \mathrm{d}z = 0$. Conversely if f is continuous on Ω and $\int_\gamma f \mathrm{d}z = 0$ for any closed curve in Ω, then f is the derivative of an analytic function F in Ω.*

5.1.3 Cauchy's theorem for a rectangle

There is some notes in this section:

R is the rectangle in $\mathbb{C}$, $R = \{x + iy \in \mathbb{C} : a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$. And ∂R is boundary curve oriented in the counterclockwise direction.

Theorem 5.8 (Cauch's theorem for a rectangle)**.** *If f is analytic on an open set which contains R, then $\displaystyle\int_{\partial R} f(z)\mathrm{d}z = 0$*

Proof. For $\forall$ rectangle $\tilde{R}$ inside R, we define $Z(\tilde{R}) = \displaystyle\int_{\partial \tilde{R}} f(z)\mathrm{d}z$. Then $Z(R) = Z(R_1) + Z(R_2)$ if R is divided into Z_1, Z_2.

Since we can divide R into four equal rectangles, and find a rectangle with $|Z(R^{(1)})| \geqslant \frac{1}{4}|Z(R)|$. Then repeat the above steps and we obtain a sequence of nested rectangles $R \supset R^{(1)} \supset \cdots$ with the property

$$Z(R^{(n)}) \geqslant \frac{1}{4}|Z(R^{(n-1)})| \geqslant \cdots \geqslant \frac{1}{4^n}Z(R) \tag{5.3}$$

$\forall \delta > 0$, $\exists n \in \mathbb{N}$ *s.t.* $R^{(n)} \subset \{z \in \mathbb{C} : |z - z_0| < \delta\}, \forall n \geqslant N$, where z_0 is the limit of $R^{(n)}$ as $n \to \infty$.

f is analytic in $R \Rightarrow \forall \varepsilon$, $\exists \delta > 0$ *s.t.*

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon, \forall z \text{ with } |z - z_0| < \delta \tag{5.4}$$

We assume that δ satisfies both conditions. We have

$$Z(R^{(n)}) = \int_{\partial R^{(n)}} f(z)\mathrm{d}z = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)]\mathrm{d}z$$

$$\Rightarrow |Z(R^{(n)})| \leqslant \varepsilon \int_{\partial R^{(n)}} |z - z_0| \mathrm{d}z \text{ by } 5.4$$

Let d_n be the length of diagonal of $R^{(n)}$, L_n be the length of its perimeter. Then $|z - z_0| \leqslant d_n$, $\forall z \in \partial R^{(n)}$.

$\Rightarrow |Z(R^{(n)})| \leqslant \varepsilon d_n L_n = \varepsilon \dfrac{D}{2^n} \cdot \dfrac{L}{2^n}$ where D, L are the diameter and perimeter of R.

$\Rightarrow |Z(R)| \overset{5.3}{\leqslant} 4^n |Z(R^{(n)})| \leqslant \varepsilon DL \Rightarrow Z(R) = 0$ since ε is arbitary. □

We will next prove the following stronger theorem:

Theorem 5.9 (stronger version of Cauchy's theorem for a rectangle)**.** *Let f be analytic on $R' = R \backslash \{\psi_1, \cdots, \psi_m\}, m \in \mathbb{N}$. If $\lim\limits_{z \to \psi_j} (z - \psi_j) f(z) = 0, \forall 1 \leqslant j \leqslant m$, then* $\displaystyle\int_{\partial R} f(z) \mathrm{d}z = 0.$

Proof. WLOG, we may assume f is not analytic at only one point $\psi \in R$. If we put psi into a small rectangle S_0, then the previous theorem tells us $\int_{\partial R} f(z) \mathrm{d}z = \int_{\partial S_0} f(z) \mathrm{d}z$.

$\forall \varepsilon > 0$, we may choose S_0 small enough such that $|f(z)| \leqslant \dfrac{\varepsilon}{|z - \varepsilon}$, $\forall z \in \partial S_0$

$\Rightarrow | \displaystyle\int_{\partial R} f(z) \mathrm{d}z \leqslant \varepsilon \int_{\partial S_0} \frac{|\mathrm{d}z|}{|z - \psi|} \leqslant \varepsilon \frac{1}{\frac{l}{2}} \cdot 4l = 8\varepsilon$

$\Rightarrow \int_{\partial R} f(z) \mathrm{d}z = 0$ since ε is arbitrary. □

5.1.4 Cauchy's Theorem for a disk

$\Delta := \{z \in \mathbb{C} : |z - z_0| < R\}$ where $R > 0$.

Theorem 5.10 (Cauchy's Theorem for a disk)**.** *If f is analytic in an open disk Δ, then $\int_\gamma f(z) \mathrm{d}z = 0$ for closed curve γ in Δ.*

Proof. Suppose the center of Δ is $z_0 = x_0 + iy_0$, $z = x + iy$. We define

$$F(z) = \int_\gamma f(z) \mathrm{d}z$$

where γ is the horizontal line segment from z_0 to (x, y_0) added with vertical line segment from (x, y_0) to z. We have

$$\frac{\partial F}{\partial y} = \lim_{\delta y \to 0} \frac{F(x, y+\delta y) - F(x,y)}{\delta y} = \lim_{\delta y \to 0} \frac{1}{\delta y} \int_{\delta\gamma} f(z)\mathrm{d}z = if(z) \tag{5.5}$$

By Cauchy' theorem on rectangles, one has $F(z) = -\int_{\tilde{\gamma}} f(z)\mathrm{d}z$, where $\tilde{\gamma}$ is the vertical line segment from z_0 to (x_0, y) added with horizontal line segment from (x_0, y) to z.

Similarly, $\dfrac{\partial F}{\partial x} = f(z)$.

$\Rightarrow \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} \Rightarrow F$ is analytic in Δ with derivative f. By Fundamental Theorem 5.5 of Calulus $\Rightarrow \int_\gamma f(z)\mathrm{d}z = 0$ for $\forall$ closed curve in Δ. □

Here is a stronger version.

Theorem 5.11 (stronger version of Cauchy's Theorem for a disk)**.** *Let f be analytic in a region $\Delta' = \Delta \backslash \{\psi_1, \cdots, \psi_m\}$ with $m \in \mathbb{N}$. If f satisfies $\lim\limits_{z \to \psi_j} (z - \psi_j) f(z) = 0, \forall 1 \leqslant j \leqslant m$, then $\displaystyle\int_\gamma f(z)\mathrm{d}z = 0, \forall \gamma$ closed in Δ'*

Proof. It is similar to the above proof.

For the case no ψ_j lies on $x = x_0$ and $y = y_0$, we can find a similar curve γ with last segment is a vertical one. Let $F(z) = \int_\gamma f(z)\mathrm{d}z$. And continue the process of proof of the previous theorem.

For the case that $\exists\ \psi_j$ lies on the lines $x = x_0, y = y_0$, we actually can move the center to another point *s.t.* no ψ_j lies on the lines $x = x_0', y = y_0'$. □

5.2 Cauchy's integral formula

5.2.1 Index of a point with resect to a closed curve

Lemma 5.12. *If the piecewise differentiable closed curve γ does not pass through $z \in \mathbb{C}$, then the value of the integral $\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}$ is a multiple of $2\pi i$.*

Proof. $\gamma : \zeta = \zeta(t), \alpha \leqslant t \leqslant \beta$. $h(t) = \int_\alpha^t \frac{\zeta'(x)}{\zeta(s)-z} \mathrm{d}s$.

$z \in \gamma \Rightarrow h$ is defined and continuous on $[\alpha, \beta]$. For all t *s.t.* $\zeta'(t)$ is continuous, we have

$$h'(t) = \frac{\zeta'(t)}{\zeta(t) - z} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-h(t)}(\zeta(t) - z)\right] = 0$$

So $e^{-h(t)}(\zeta(t) - z)$ is constant on $[\alpha, \beta]$.

Then $e^{h(t)} = \dfrac{\zeta(t) - z}{\zeta(\alpha) - z} \Rightarrow e^{h(\beta)} = 1 \Rightarrow h(\beta) \in \{2k\pi i : k \in \mathbb{Z}\}$. □

The **index of the point** z w.r.t. the closed curve γ is the number

$$n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}$$

n is also called the **winding number**.

Theorem 5.13. *Let γ be a piecewise differentiable closed curve. The function $z \mapsto n(\gamma, z)$ is constant on each connected set of $\mathbb{C}\backslash\gamma$, and zero if this set is unbounded.*

Proof. Define $f : \mathbb{C}\backslash\gamma \to \gamma, z \mapsto n(\gamma, z) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}$.

Then

$$|f(z) - f(z_0)| = \frac{1}{2\pi}\left|\int_\gamma \frac{z - z_0}{(\zeta - z)(\zeta - z_0)}\mathrm{d}\zeta\right| \leqslant \frac{|z - z_0|}{2\pi}\int_\gamma \frac{1}{|\zeta - z| \cdot |\zeta - z_0|}|\mathrm{d}\zeta|$$

$\Rightarrow f$ is continuous on each open connected set of $\mathbb{C}\backslash\gamma$. Let Ω be any open connected set of $\mathbb{C}\backslash\gamma$. We have $f(\Omega)$ is connected $\xRightarrow{f(\Omega)\subset\mathbb{Z}}$ $f(\Omega)$ contains at most one point $\Rightarrow f$ is constant on Ω.

If $|z|$ is sufficient large, $\exists$ a disk of radius R, $B(0, R)$, *s.t.* $\gamma \subset B(0, R)$ but $z \notin B(0, R)$. Cauchy's theorem for a disk 5.10 tells us that $f(z) = n(\gamma, z) = 0$. So it is zero if this set is unbounded. □

Lemma 5.14. *Let z_1, z_2 be two points on a closed curve γ and $0 \notin \gamma$.*

Suppose z_1 in the lower half space and z_2 in upper half space. If $\gamma_1 \cap \{(x,0) : x \leqslant 0\} = \varnothing$, and $\gamma_2 \cap \{(x,0) : x \geqslant 0\} = \varnothing$, then $n(\gamma, 0) = 1$.

Remark 5.15. One method to prove this lemma is to create two segment from z_i to the point in the unit circle. By divide the curve into two parts, we can easily remove the part of previous curve by using the theorem 5.13, since 0 is in the unbounded set.

In this proof, we can find that Theorem 5.13 is such powerful that we can change any curve to a more simple curve easily!

5.2.2 Cauchy's integral formula

Theorem 5.16 (Cauch's integral formula)**.** *Suppose that f is analytic in an open disk $\triangle$, and let γ be a closed curve in $\triangle$. For $\forall z \notin \gamma$,*

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

where $n(\gamma, z)$ is the index of z w.r.t. γ.

Proof. If $z \notin \triangle$, The both sides of the equation is 0.

So we may assume $z \in \triangle$ and $z \notin \gamma$. Define $F : \triangle \backslash \{z\} \to \mathbb{C}, \zeta \mapsto \dfrac{f(\zeta) - f(z)}{\zeta - z}$.

Then F is analytic in $\triangle \backslash \{z\}$, and $\lim\limits_{\zeta \to z} (\zeta - z) F(zeta)$.

By Cauchy's Theorem 5.9 $\Rightarrow \int_\gamma F(\zeta) \mathrm{d}\zeta = 0 \Rightarrow \displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = f(z) \int_\gamma \frac{1}{\zeta - z} \mathrm{d}\zeta = f(z) \cdot 2\pi i \cdot n(\gamma, z)$ □

Remark 5.17. This proof let us find that for a good-enough function, its integral over a closed curve is a constant.

The theorem still holds if f is analytic except at a finite number of ζ_j *s.t.*

$$\lim_{\zeta \to \zeta_j} (\zeta - \zeta_j) f(\zeta) = 0$$

and $z \neq \zeta_j$ for each j, since Cauchy's theorem is still applicable.

Theorem 5.18 (The mean value property for analytic functions). *f is analytic in a region Ω which contain $\overline{B(z,R)}$. Then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+Re^{i\theta})\mathrm{d}t$$

Proof. The previous theorem 5.16 $\Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta - z| = R}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta \xlongequal{\zeta = z+Re^{it}} \frac{1}{2\pi}\int_0^{2\pi} f(z+Re^{it})\mathrm{d}t$$

□

If f is analytic in an open disk $\triangle$, and γ is a closed curve in $\triangle$. And $n(\gamma, z) = 1$. Then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

This is usually referred to as **Cauchy's integral formula**

5.2.3 Higher derivatives

Lemma 5.19. *Let $\Omega \subset \mathbb{C}$ be a region and γ be an arc in Ω. If φ is continuous on γ, then the function*

$$F_n(z) := \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n}\mathrm{d}\zeta$$

is analytic in each of the regions $\Omega \backslash \gamma$, and its derivative is $F_n'(z) = nF_{n+1}(z)$

Proof. We prove it by induction.

The lemma is true if $n = 0$: $F_0(z) = \int_\gamma \varphi(\zeta)\mathrm{d}\zeta$ and $F_0'(z) = 0 = 0 \cdot F_1(z)$.

We suppose that the lemma holds for $n-1$ with $n \in \mathbb{N}$: $\forall$ continuous φ on γ, F_{n-1} is analytic in $\Omega\backslash\gamma$ and $F_{n-1}'(z) = (n-1)F_n(z), \forall z \in \Omega\backslash\gamma$.

Fix $z_0 \in \Omega \backslash \gamma$. For $\forall z \in B(z_0, \frac{\delta}{2})$, with $B(z_0, \delta) \subset \Omega \backslash \gamma$, we have $|\zeta - z| > \frac{delta}{2}$, $\forall \zeta \in \gamma$.

For $\forall$ continuous φ on γ,

$$\begin{aligned} F_n(z) - F_n(z_0) &= \int_\gamma \frac{\varphi(\zeta)(\zeta - z + z - z_0)}{(\zeta - z)^n(\zeta - z_0)} \mathrm{d}\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n} \mathrm{d}\zeta \\ &= \left[\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} \mathrm{d}\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^{n-1}(\zeta - z)} \mathrm{d}\zeta \right] \\ &+ (z - z_0) \int_\gamma \frac{\varphi(\zeta) \mathrm{d}\zeta}{(\zeta - z)^n(\zeta - z_0)} \end{aligned}$$

Let $\psi(\zeta) = \dfrac{\psi(\zeta)}{\zeta - z_0}$, which is continuous except γ.

Using the induction condition to ψ, we can finish the proof. □

Theorem 5.20. *An analytic function on a region Ω has derivatives of all orders which are analytic in Ω. More precisely, $\forall z_0 \in \Omega$, choose $B(z, \delta) \subset \Omega$ and a circle $C \subset B(z_0, \delta)$ with center z_0. For $\forall$ z in the interior of C, Cauchy's integral formula gives*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Then the previous lemma implies $f'(z) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta$ is analytic in the interior of C. More generally, for $\forall n \in \mathbb{N}$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta \tag{5.6}$$

5.2.4 Consequences of Cauchy

Theorem 5.21 (Morera's Theorem). *If f is continuous in a region Ω, and if $\int_\gamma f(z) \mathrm{d}z = 0$ for $\forall$ closed curve γ in Ω. Then f is analytic in Ω.*

Proof. We proved in Corollary 5.7 that under the hypothesis of theorem, $f = F'$

where F is analytic in Ω. The last theorem $\Rightarrow$ f is analytic. □

Suppose f is analytic in a disk, $\overline{B(z_0, R)}$, and bounded on the circle γ given by $|z - z_0| = R$. Then $\forall z \in \gamma$, $|f(z)| \leqslant M$ for some $M \geqslant 0$. By (5.6),

$$|f^{(n)}(z)| \leqslant \frac{n!}{2\pi} \int_C \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} |\mathrm{d}\zeta| \leqslant \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = MR^{-n} n! \tag{5.7}$$

This inequality is known as **Cauchy's estimate**.

Theorem 5.22 (Liouville's Theorem)**.** *A bounded entire function (i.e. analytic in $\mathbb{C}$) is constant.*

Proof. Suppose $|f(z)| \leqslant M$, $\forall z \in \mathbb{C}$. Cauchy's estimate $\Rightarrow$

$$|f'(z)| \leqslant \frac{M}{R}, \forall z \in \mathbb{C}, \forall R > 0 \tag{5.8}$$

□

$\xRightarrow{R \to \infty}$ $f'(z) = 0$ for $z \in \mathbb{C} \Rightarrow f = 0$.

Theorem 5.23 (Fundamental Theorem for Algebra)**.** *Every polynomial of degree $n \geqslant 1$ has n roots.*

Proof. It suffices to prove it has at least one root.

Suppose $P(z) = a_n z^n + \cdots a_1 z + a_0$ with $a_0 \neq 0$ does not have a root.

Then $f(z) := \frac{1}{P(z)}$ is an entire function. As $z \to \infty$, $\lim\limits_{|z| \to \infty} \dfrac{|P(z)|}{|z|^n} = |a_n| \Rightarrow \lim\limits_{|z| \to \infty} \dfrac{1}{|P(z)|} = 0$.

So f is bounded. By Liouville's Theorem, f is a constant. Where $f = f(\infty) = 0$. That causes contradiction. □

Theorem 5.24 (Power series). *If f is analytic in a region Ω which contains a closed disk $\overline{B(z_0, R)}$, then f has a power series expansion at z_0,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad \forall z \in B(z_0, R) \tag{5.9}$$

Proof. $\forall z \in B(z_0, R)$, $\forall \zeta$ with $|\zeta - z_0| = R$.

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\ &= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\ &= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \end{aligned} \tag{5.10}$$

This series converges uniformly in ζ with $|\zeta - z_0| = R$.

For $\forall z \in B(z, R)$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta - z| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \\ &\overset{\text{uniformly}}{=\!=\!=} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \cdot (z - z_0)^n \\ &\overset{(5.6)}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \end{aligned} \tag{5.11}$$

$\square$

5.3 Local properties of analytic functions

5.3.1 Removable Singularities and Taylor's Theorem

We remarked that Cauchy's integral formula holds if f is analytic except at a finite number of point ζ_j *s.t.* $\lim\limits_{\zeta\to\zeta_j}(\zeta-\zeta_j)f(\zeta)=0$. We will prove f can be extended to an analytic function in $\triangle$. In other word, ζ_j are **removable singularities**.

Theorem 5.25 (Riemann's Removable Singularities Theorem)**.** *Suppose that f is analytic in the region $\Omega'=\Omega\backslash\{\zeta_0\}$ where Ω is also a region. Then there exists an analytic function in Ω which coincides with f in Ω' if and only if $\lim\limits_{z\to\zeta_0}(z-\zeta_0)f(z)=0$.*

Proof. The uniqueness and "$\Rightarrow$" part is trivial since the extended function is continuous at ψ_0.

"$\Leftarrow$": Cauchy's integral formula $\Rightarrow$

$$f(z)=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta,\ \forall z\in\triangle \text{ and } z\neq\zeta_0 \tag{5.12}$$

Lemma 5.19 $\Rightarrow$ the RHS of the last equation 5.12 is analytic in $z\in\triangle$. Then

$$\hat{f}(z)=\begin{cases} f(z), & z\neq\zeta_0 \\ \dfrac{1}{2\pi i}\displaystyle\int_C\frac{f(\zeta)}{\zeta-\zeta_0}\mathrm{d}\zeta, & z=\zeta_0\end{cases} \tag{5.13}$$

is analytic in Ω. □

We apply Theorem 5.25 to the function $F(z)=\dfrac{f(z)-f(\zeta)}{z-\zeta}$, where f is analytic in a region Ω. Note that

$$\lim_{z\to\zeta_0}(z-\zeta)F(z)=0,\ \lim_{z\to\zeta}F(z)=f'(\zeta) \tag{5.14}$$

Theorem 5.25 $\Rightarrow \exists$ analytic function f_1 on Ω *s.t.*

$$f_1(z) = \begin{cases} F(z), & z \neq \zeta_0 \\ f'(\zeta), z = \zeta_0 \end{cases} \tag{5.15}$$

we may thus write $f(z) = f(\zeta) + (z-\zeta)f_1(z)$.

Repeating this process for f_1, we get an analytic function f_2 on Ω *s.t.*

$$f_1(z) = f_1(\zeta) + (z-\zeta)f_2(z) \tag{5.16}$$

where

$$f_2(z) = \begin{cases} \dfrac{f_1(z) - f_1(\zeta)}{z-\zeta}, & z \neq \zeta \\ f_2'(\zeta), & z = \zeta \end{cases} \tag{5.17}$$

Continuing the recursion, we have the general form

$$f_{n-1}(z) = f_{n-1}(\zeta) + (z-\zeta)f_n(z) \tag{5.18}$$

$\Rightarrow$

$$f(z) = f(\zeta) + (z-\zeta)f_1(\zeta) + \cdots + (z-\zeta)^{n-1}f_n(\zeta) + (z-\zeta)^n f_n(z) \tag{5.19}$$

Differentiating n times and setting $z = \zeta \Rightarrow f^{(n)}(\zeta) = n! f_n(\zeta)$

We just prove **Taylor's Theorem**

Theorem 5.26 (Taylor's Theorem). *If f is analytic in a region Ω, $\zeta \in \Omega$, then we have*

$$f(z) = f(\zeta) + (z-\zeta)f'(\zeta) + \cdot + \frac{f^{(n-1)}(\zeta)}{(n-1)!}(z-\zeta)^{n-1} + f_n(z)(z-\zeta)^n \tag{5.20}$$

where f_n is analytic in Ω. Moreover,

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega-\zeta)^n(\omega-z)}\mathrm{d}\omega \tag{5.21}$$

where C is a circle in Ω s.t. its interior $\triangle$ is also in Ω and $\zeta, z \in \triangle$

Proof. It suffices to prove the second part.

Cauchy's integral formula $\Rightarrow f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\omega)}{\omega-z}\mathrm{d}\omega$, $\forall z \in \triangle$.

For $f_n(z)$, we substitute the expression from (5.20). The first term is

$$\frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega-\zeta)^n(\omega-z)}\mathrm{d}\omega \tag{5.22}$$

The remaining terms have the following form, except for constant factors:

$$g_k(\zeta) = \int_C \frac{1}{(\omega-\zeta)^n(\omega-z)}\mathrm{d}\omega,\ 1 \leqslant k \leqslant n \tag{5.23}$$

The lemma 5.19 applies to $\varphi(\omega) = \dfrac{1}{\omega-z}$, $g_k'(\zeta) = kg_{k-1}(\zeta), k \in \mathbb{N}, \forall\zeta \in \triangle$. So

$$\begin{aligned} g_1(\zeta) &= \int_C \frac{1}{(\omega-\zeta)(\omega-z)}\mathrm{d}\omega \\ &= \frac{1}{\zeta-z}\left[\int_C \frac{1}{\omega-\zeta}\mathrm{d}\omega - \int_C \frac{1}{\omega-z}\mathrm{d}\omega\right] \\ &= \frac{1}{\omega-z}[2\pi i - 2\pi i] = 0 \end{aligned} \tag{5.24}$$

So $g_k(z) = 0$, $\forall k \in \mathbb{N}, \forall z \in \triangle$. □

5.3.2 Zeros and poles

Theorem 5.27. *If f is analytic in a region Ω and $\exists a \in \Omega$ s.t. $f^{(n)}(a) = 0$ for $\forall n \in \mathbb{N}\cup\{0\}$, then $f \equiv 0$ in Ω.*

Proof. Let $B(a,R)$ be the disk s.t. $\overline{B(a,R)} \subset \Omega$. Let $C = \partial B(0,R)$.

Taylor's theorem $\Rightarrow f(z) = (z-a)^n f_n(z)$ with

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega-a)^n(\omega-z)}\mathrm{d}\omega,\ \forall n\in\mathbb{N}\cup\{0\}, \forall z\in B(a,R) \tag{5.25}$$

Let $M = \max_{z\in C}|f(z)|$.

$$\begin{aligned}
\Rightarrow&|f_n(z)| \leqslant \frac{1}{2\pi}\cdot\frac{M}{R^n(R-|z-a|)}\cdot 2\pi R\\
\Rightarrow&|f(z)| \leqslant \frac{|z-a|^n}{R^n}\cdot\frac{MR}{R-|z-a|} \to 0 \text{ as } n\to\infty,\ \forall z\in B(0,R)\\
\Rightarrow&f(z) = 0,\ \forall z\in B(0,R)
\end{aligned}$$

Now define

$$\begin{aligned}
E_1 &= \left\{z\in\Omega | f^{(n)}(z) = 0, \forall n\in\mathbb{N}\cup\{0\}\right\}\\
E_2 &= \Omega\backslash E_1 = \left\{z\in\Omega | f^{(n)}(z) \neq 0, \text{ for some } n\in\mathbb{N}\cup\{0\}\right\}
\end{aligned}$$

We just proved E_1 is open. E_2 is open because $f^{(n)}$ is continuous in Ω for $\forall n\in\mathbb{N}\cup\{0\}$. Ω is a region $\Rightarrow$ either $R_1 = \varnothing$ or $R_2 = \varnothing$.

The assumption of the theorem $\Rightarrow E_1 \neq \varnothing \Rightarrow E_1 = \Omega$. □

Let f be analytic in Ω which is not identically zero, $f(a) = 0$ for some $a\in\Omega$. The previous theorem implies $\exists$ first $N\in\mathbb{N}$ *s.t.* $f^{(N)}(a)\neq 0$. Taylor's theorem implies that $f(a) = (z-a)^N f_N(z)$ where f_N is analytic and $f_N(a)\neq 0$. We say that a is a **zero of order** N of f.

f_N is continuous $\Rightarrow \exists\delta > 0$ *s.t.* $f(z)\neq 0$ for $\forall z\in B(a,\delta)\backslash\{0\}$.

So we have just proved an important result: Zeros of analytic functions are isolated, or equivalently, we have a famous theorem:

Theorem 5.28 (Identity Theorem)**.** *If f and g are analytic in a region ω, and $f = g$ on a set which has an accumulation point in Ω, then $f(z) = g(z)$.*

Corollary 5.29.

(1) *If $f \equiv 0$ in a subregion of Ω and f is analytic in Ω, then $f \equiv 0$ in Ω.*

(2) *If f is analytic in Ω and vanishes on an arc in Ω which doesn't reduce to a point, then $f \equiv 0$ in Ω.*

If f is analytic in a neighborhood of a, but perhaps not at a itself, then a is called an **isolated singularity** of f.

If $\lim_{z \to a} f(z) = \infty$, then a is said to be a **pole** of f, and we set $f(a) = \infty$. Continuity implies $\exists \delta > 0 \quad s.t.\ f(z) \neq 0$ for $\forall z \in B(0, \delta) \backslash \{a\}$. Thus, $g(z) = \dfrac{1}{f(z)}$ is analytic in $B(a, \delta) \backslash \{a\}$. $\lim_{z \to a} (z - a) g(z) = 0 \Rightarrow a$ is a removable singularity of g, and g has an analytic extension with $g(a) = 0$. $g \not\equiv 0 \Rightarrow a$ is a zero of g with finite order. The **order of the pole** of f at a is the order N of the zero of g at a.

We can write

$$f(z) = (z - a)^{-N} f_N(z),\ \forall z \in B(a, \delta) \backslash \{a\} \tag{5.26}$$

where f_N is analytic and nonzero in a neighborhood of a.

Definition 5.30. A function which is analytic in a region Ω excpet for (isolated) poles is called a **meromorphic function**.

Example 5.31. If f and g are analytic in Ω and $g \not\equiv 0$, then $\frac{f}{g}$ is a meromorphic function in Ω. (See the Identity Theorem 5.28)

Remark 5.32. The sum, the product and quotient (if denominator is not always zero) of two meromorphic functions are meromorphic.

If f has a pole of order N at a, then $(z - a)^N f(z)$ is analytic at a, and Taylor's theorem 5.26 implies

$$(z - a)^N f(z) = b_N + b_{N-1}(z - a) + \cdots + b_1 (z - a)^{N-1} + \varphi(z) \cdot (z - a)^N \tag{5.27}$$

where φ is analytic at a.

$$\Rightarrow f(z) = b_N(z-a)^{-N} + b_{N-1}(z-a)^{-(N-1)} + \cdots + b_1(z-a)^{-1} + \varphi(z),\ \forall z \neq a. \quad (5.28)$$

Theorem 5.33. *If f is analytic in a neighborhood of a, but perhaps not at a itself, then exactly one of the following 3 cases ocuurs:*

(i) $f \equiv 0$ *in this neighborhood.*

(ii) $\exists$ *integer* $N \in \mathbb{Z}$ *s.t.* $\lim_{z\to a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > N \\ \infty, & \alpha < N \end{cases}$

(iii) neither $\lim_{z\to a} |z-a|^\alpha \cdot |f(z)| = 0$ *for any* $\alpha \in \mathbb{R}$ *nor* $\lim_{z\to a} |z-a|^\alpha \cdot |f(z)| = \infty$ *for any* $\alpha \in \mathbb{R}$

Proof.

① If $\lim_{z\to a} |z-a|^\alpha \cdot |f(z)| = 0$ for $\forall \alpha \in \mathbb{R}$, then $\lim_{z\to a} |z-a|^m \cdot |f(z)| = 0$ for $\forall$ integer $m > \alpha$.

$\Rightarrow (z-a)^m f(z)$ has a removable singularity at a and vanishes at $z = a$

$\Rightarrow$ Either $f \equiv 0$ in $B(a,\delta)\backslash\{a\}$, which is case (i), or $(z-a)^m f(\alpha)$ has a zero of finite order k at $a \Rightarrow \lim_{z\to a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > m-k \\ \infty, & \alpha < m-k \end{cases}$

② If $\lim_{z\to a} |z-a|^\alpha |f(z)| = \infty$ for some $\alpha \in \mathbb{R}$, then $\lim_{z\to a} |z-a|^n \cdot |f(z)| = \infty$ for $\forall$ integer $n < \alpha$.

$\Rightarrow (z-a)^n f(z)$ has a pole of finite order l at a

$\Rightarrow \lim_{z\to a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > n+l \\ \infty, & \alpha < n+l \end{cases}$ $\square$

Remark 5.34. In case (ii), N may be called the **algebraic order** of f at a. $N > 0$ if a is a pole, $N < 0$ if a is a zero, and $N = 0$ if f is analytic at a and $f(a) \neq 0$. The order is always an integer, there is no analytic function which tends to 0 or ∞, like a fractional power of $|z-a|$.

In some sense, three cases depends on whether $\lim\limits_{z\to a}(z-a)^N f(z)$ converges for some N.

In case (iii), the point a is an **essential isolated singularity**.

Example 5.35. $f(z) = \exp(\frac{1}{z})$ has an essential isolated singularity $z = 0$.

Theorem 5.36 (Weierstrass). *An analytic function comes arbitarily close to any complex value in every neighborhood of an essential singularity. Or equivalently, the codomain of f on every neighborhood of an essential singularity is dense in $\mathbb{C}$.*

Proof. Suppose the statement is false.

$\exists A \in \mathbb{C}$, $\delta > 0$ and $\varepsilon > 0$ *s.t.*

$$|f(z) - A| > \delta, \ \forall z \text{ with } 0 < |z-a| < \varepsilon \tag{5.29}$$

$\Rightarrow \lim\limits_{z\to a} |z-a|^\alpha \cdot |f(z) - A| = \infty$ for $\forall \alpha < 0$. $\Rightarrow a$ is not an essential singularity of $f(z) - A$.

The previous theorem $\Rightarrow \exists\ \beta \in \mathbb{R}$ *s.t.* $\lim\limits_{z\to a} |z-a|^\beta \cdot |f(z) - A| = 0$, and we may choose $\beta > 0$.

Then $\lim\limits_{z\to a} |z-a|^\beta \cdot |A| = 0 \Rightarrow \lim\limits_{z\to a} |z-a|^\beta \cdot |f(z)| = 0$ by the triangular inequality.

So a is not an essential singularity of f, which causes contradiction!

So the statement has to be true. □

Remark 5.37. If f is analytic in $|z| > R$. We treat ∞ as an isolated singularity. Removable singularity, pole or essential singularity of f at ∞ is defined according to $g(z) = f(\frac{1}{z})$ at $z = 0$.

5.3.3 The Local Mappings

Theorem 5.38 (The Argument Principle)**.** *Let f be analytic in a disk $\triangle$ s.t. f does not vanish identically. Let z_j be the zeros of f, each zero being counted as many times as its order indicates. For every closed curve γ in $\triangle$ which does not pass through a zero, we have*

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z \tag{5.30}$$

where the sum has only a finite number of terms with nonzero value.

Proof.

Case I: f has exactly n zeros $z_1, \cdots, z_n$.

By repeated application of Taylor' Theorem 5.26, we can write

$$f(z) = (z - z_1)(z - z_2)\cdots(z - z_n)g(z),\ z \in \triangle \tag{5.31}$$

where g is analytic in $\triangle$ and $g(z) \neq 0$ for $\forall z \in \triangle$. $\Rightarrow$

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)},\ \forall z \in \triangle \text{ and } z \neq z_j \tag{5.32}$$

Cauchy' Theorem 5.10 $\Rightarrow$

$$\int_\gamma \frac{g'(z)}{g(z)}\mathrm{d}z = 0 \Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_{j=1}^{n} n(\gamma, z_j) \tag{5.33}$$

Case II: f has infinitely many zeros in $\triangle$. Then γ is inside a concentric disk $\triangle'$ smaller than $\triangle$.

$f \not\equiv 0 \Rightarrow$ There is only a finite number of zeros in $\triangle'$.

So we can apply (5.33) to the disk $\triangle' \Rightarrow$ (5.30) holds since $n(\gamma, z_j) = 0$ if $z \notin \triangle'$. $\square$

Remark 5.39.

- The function $\omega = f(z)$ maps γ onto a closed curve Γ in the ω-plane, and we have
$$\int_{\Gamma} \frac{\mathrm{d}\omega}{\omega} = \int_{\gamma} \frac{f'(z)}{f(z)} \mathrm{d}z \tag{5.34}$$
Then (5.30) can be interpreted as $n(\Gamma, 0) = \sum_j n(\gamma, z_j)$.

- The most useful application of the theorem is to the case when γ is a circle (or more generally a simple closed curve). So that
$n(\gamma, z) = \begin{cases} 1, & z \text{ is inside } \gamma \\ 0, & z \text{ is outside } \gamma \end{cases}$ Then (5.30) yields a formula for the total number of zeros enclosed by γ.

Let $a \in \mathbb{C}$. Apply the previous theorem to $f(z) - a$

$$\sum_j n(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - a} \mathrm{d}z$$

where $z_j(a)$ are zeros of $f - a$ (or roots of $f(z) = a$), and γ is a closed curve in $\triangle$ which doesn't pass $z_j(a) \Rightarrow$

$$n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$$

If a and b are in the same region determined by Γ, then $n(\Gamma, a) = n(\Gamma, b) \Rightarrow$

$$\sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b)) \tag{5.35}$$

If γ is a circle, then f takes the values a and b equally many times inside γ.

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