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LIN150117

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Contents

Theorem 0.1. *If $B \in \mathbb{R}^{n\times n}$ satisfying $||B|| < 1$, then $I + B$ invertible and*

$$||(I+B)^{-1}||_2 \leqslant \frac{1}{1-||B||}$$

Cholesky transformation

Doolottle Decomposition

Condition number

$$\mathcal{K}_2(A)^2 = \mathrm{cond}(A^TA) \text{ if } A \text{ full of column rank}$$

$$\mathrm{cond}(A) \geqslant \frac{|\lambda_1|}{|\lambda_n|} \text{ equality holds if } A \text{ is symmetric matrix}$$

$$||A||_2^2 = \rho(A^TA) = ||A^TA||_2$$

Theorem 0.2. *If $\det A \neq 0$, then*

$$\min_{|A+\delta A|=0} \frac{||\delta A||_2}{||A||_2} = \frac{1}{\mathrm{cond}(A)_2}$$

Moore-Penrose pseudoinverse

Theorem 0.3. *For the least squrare equation of $Ax = b$,*

$$\frac{||\delta x||_2}{||x||_2} \leqslant \mathcal{K}_x(A) \cdot \frac{||A\delta x||_2}{||Ax||_2}$$

Hauseholder transiformation

Givens transoformation

$$R_k(B) = -\ln ||B^k||^{\frac{1}{k}}$$

$$R(B) = -\ln \rho(B)$$

Jacobian iteration

Gauss-Seidel iteration

Theorem 0.4.

Theorem 0.5. *By Steepest Descent Algorithm,*

$$||x^k - x^*||_A \leqslant \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)^k ||x^0 - x^*||_A$$

Theorem 0.6. *In conjugated gradient method,*

$$P^{(k)} = r^{(k)} + \beta_{k-1} P^{(k-1)}$$

with

$$\beta_{k-1} = -\frac{(r^{(k)}, AP^{(k-1)})}{(P^{(k-1)}, AP^{(k-1)})}, r^{(k)} = b - Ax^{(k)}$$

And the iteration

$$x^{(k+1)} = x^{(k)} + \alpha_k P^{(k)}$$

where

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}} \varphi(x^{(k)} + \alpha P^{(k)}) = \frac{(r^{(k)}, P^{(k)})}{(P^{(k)}, AP^{(k)})}$$

This iteration satsifies

$$x^{(k)} = \arg \min_{x - x^{(0)} \in \mathrm{Span}\{P^{(0)}, \cdots, P^{(k-1)}\}} \varphi(x)$$

Theorem 0.7. $(r^{(i)}, r^{(j)}) = 0, i \neq j$

$(AP^{(i)}, P^{(j)}) = 0, i \neq j$

$(r^{(j)}, P^{(i)}) = 0, i < j$

$$\mathrm{Span}\{r^0, \cdots, r^{(k)}\} = \mathrm{Span}\{P^{(0)}, \cdots, P^{(k)}\} = \mathrm{Span}\{r^{(0)}, Ar^{(0)}, \cdots, A^k r^{(0)}\}$$

List of Theorems