清華大學 求真书院
Qiuzhen College, Tsinghua University

Physics-0 Lecture Notes

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Qiuzhen College, Tsinghua University
2023 Spring

1.5 Harmonic Oscillators, Pendulum

Harmonic oscillators are important in physics and engineering because they are the simplest type of oscillating system, and many real-world systems can be modeled as harmonic oscillators. Examples of harmonic oscillators include mass-spring systems, pendulums, and electric circuits. Understanding the behavior of harmonic oscillators can help us understand more complex systems, as well as develop technologies such as clocks, musical instruments, and sensors. Additionally, quantum harmonic oscillators are an important concept in quantum mechanics, and the study of harmonic oscillators plays a crucial role in the development of quantum field theory. Overall, studying harmonic oscillators provides fundamental insights into the behavior of physical systems and their applications in various fields.

1.5.1 Simple harmonic motion

The classic harmonic oscillator is a fundamental concept in physics that describes a particle with mass m coupled to a spring. The force acting on the particle is given by Hooke's law

$$\vec{F} = -k\vec{x}, \tag{1.5.1}$$

where $\vec{x}$ is the position vector of the particle with respect to the equilibrium point, and k is the spring constant.

According to the Newton's second law $\vec{F} = m\vec{a}$, the equation of motion for the particle is given by $-k\vec{x} = m\ddot{\vec{x}}$, where $\vec{a} = \ddot{\vec{x}}$ is the acceleration of the particle. This is a second-order differential equation that governs the motion of the oscillator. To simplify the equation of motion, we define the angular frequency of the oscillator as

$$\omega = \sqrt{\frac{k}{m}}. \tag{1.5.2}$$

The equation of motion for the particle now becomes

$$\ddot{\vec{x}}(t) = -\omega^2 \vec{x}(t), \tag{1.5.3}$$

which is a well-known differential equation that describes simple harmonic motion.

For simplicity, let us consider a particle that is constrained to move along a one-dimensional line. The generic solution of the differential equation for this particle is

$$x(t) = x_m \cos(\omega t + \phi), \tag{1.5.4}$$

where x_m is the amplitude, which is the maximum value of $x(t)$, and ϕ is the initial phase at time zero. The argument of the cosine function is called the phase. These terminologies are summarized in Figure 1.24. One can also obtain these parameters from the x-t plot. For instance, a negative shift in the initial phase will cause the cosine curve to shift towards the right (see Figure 1.25).

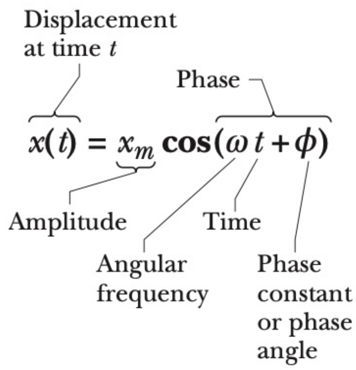

Figure 1.24

The motion of the harmonic oscillator $x(t)$ is periodic, which means that if we change time t to $t+\frac{2\pi}{\omega}$, $x(t)$ remains the same. Thus, $T=\frac{2\pi}{\omega}$ is the period of the motion. The period T is related to the frequency f and angular frequency ω as:

$$\omega = \frac{2\pi}{T} = 2\pi f. \tag{1.5.5}$$

Here, f is the frequency in Hertz ($\mathrm{Hz}=\mathrm{s}^{-1}$) and ω is the angular frequency in radians per second. The period and frequency are reciprocals of each other, and they are both related to the oscillation rate of the oscillator.

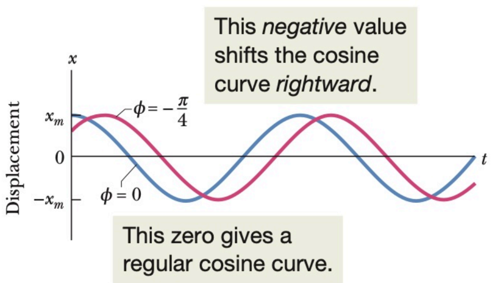

Figure 1.25

The velocity of a simple harmonic oscillator is given by the derivative of its displacement over time, which is expressed as:

$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\omega x_m \sin(\omega t+\phi) = \omega x_m \cos\left(\omega t + \phi + \frac{\pi}{2}\right). \tag{1.5.6}$$

The velocity amplitude is given by $v_m = \omega x_m$. And the phase constant is shifted by $\pi/2$ compared to $x(t)$. The acceleration of the oscillator can be calculated as the derivative of its velocity over time, which is expressed as:

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\omega^2 x_m \cos(\omega t+\phi) = \omega^2 x_m \cos(\omega t+\phi+\pi). \tag{1.5.7}$$

The acceleration amplitude is given by $a_m = \omega^2 x_m$ and the phase is shifted by π compared to $x(t)$. From these expressions, one can check easily that

$$a(t) = -\omega^2 x(t), \tag{1.5.8}$$

which demonstrates that the acceleration of the simple harmonic oscillator is directly proportional to its displacement, and is in the opposite direction to it. This result is required exactly by Newton's second law and Hooke's law.

Let us now consider the energies of the harmonic oscillator. By definition $-\mathrm{d}U(x)/\mathrm{d}x = F(x)$, the potential energy of the system is given by:

$$U(t) = \frac{1}{2}kx(t)^2 = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi). \tag{1.5.9}$$

Similarly, the kinetic energy of the system is given by:

$$K(t) = \frac{1}{2}mv(t)^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi). \tag{1.5.10}$$

Both of these energies oscillate with time, as they depend on the position or velocity of the particle. The period of this oscillation is $\pi/\omega = T/2$. However, the total energy, which is the sum of the potential and kinetic energies, is conserved and remains constant:

$$E(t) = U(t) + K(t) = \frac{1}{2}kx_m^2 = \text{const.} \tag{1.5.11}$$

This total energy is equal to the potential energy when the displacement of the particle reaches its maximum, since the velocity, and hence the kinetic energy, at that point is zero.

1.5.2 Damped harmonic motion

Damped harmonic motion is a type of oscillation that occurs in real physical systems. It involves an object oscillating on a spring, but unlike the idealized simple harmonic motion, the system experiences *damping* due to various factors such as internal friction and air resistance. As a result, the amplitude of oscillation decreases with time, and the system eventually comes to rest. This decay in amplitude is a consequence of the energy of the system being dissipated into thermal energy.

To account for the energy loss due to friction or other dissipative effects, we introduce a damping force $\vec{F}_d$ proportional to the velocity $\vec{v}$ of the system. In the case of *linear viscous damping*, the force is given by

$$\vec{F}_d = -b\vec{v}, \tag{1.5.12}$$

where b is a positive constant known as the damping coefficient. The damping force acts in the opposite direction of the motion and causes the system to slow down. A larger value of b corresponds to a stronger damping force, resulting in a faster decay of the oscillation amplitude. Note that this linear assumption is valid only for small velocities. The generic damping force is very complicated. It depends on the velocity, the properties of the medium, the geometry of the system, and many other factors.

With the damping force, we can write Newton's second law for the system as:

$$-bv - kx = ma. \tag{1.5.13}$$

By substituting $\mathrm{d}x/\mathrm{d}t$ for v and $\mathrm{d}^2x/\mathrm{d}t^2$ for a, we get the following second-order linear differential equation:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0. \tag{1.5.14}$$

The standard procedure of solving the above equation is by assuming $x(t) = e^{\lambda t}$, we obtain an algebraic equation $m\lambda^2 + b\lambda + k = 0$. Depending on the sign of $b^2 - 4mk$, we have three types of damped harmonic motion solutions with quite different physical behaviors. Figure 1.26 illustrates the position versus time plots for three types of damped harmonic motions.

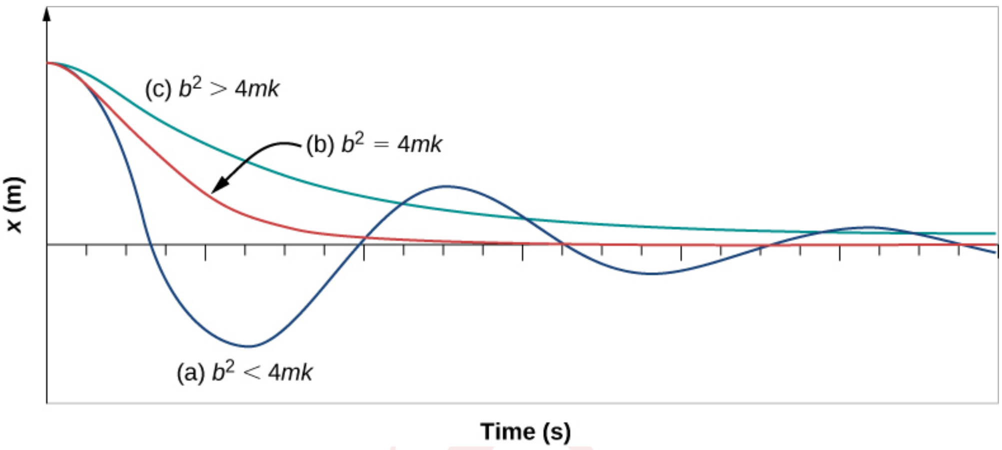

Figure 1.26: Three types of damped harmonic motions. (a) Underdamped; (b) Critically damped; (c) Overdamped.

1. **Underdamped** ($b < \sqrt{4mk}$ or $b^2 - 4mk < 0$): λ has two complex roots $\lambda_{1/2} = -\frac{b}{2m} \pm i\omega'$, where

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} < \omega = \sqrt{\frac{k}{m}} \tag{1.5.15}$$

is the angular frequency of the damped oscillator. The general solution can be written as

$$x(t) = e^{-\frac{b}{2m}t}(c_1 \cos\omega' t + c_2 \sin\omega' t) = x_m e^{-\frac{b}{2m}t}\cos(\omega' t + \phi). \tag{1.5.16}$$

Here, x_m is the amplitude of the motion. The system oscillates (at reduced frequency ω' compared to the undamped case ω) with the amplitude gradually decreasing to zero due to the exponential factor $e^{-\frac{b}{2m}t}$.

Since the amplitude of the underdamped oscillator can be understood as $x_m e^{-\frac{b}{2m}t}$, the total energy of the oscillator can be approximated by

$$E(t) \approx \frac{1}{2} k x_m^2 e^{-\frac{b}{m}t}, \tag{1.5.17}$$

which decreases exponentially with time. The energy loss is attributed to internal energy in the medium that is responsible for damping.

If the damping is absent ($b = 0$), then $\omega' = \omega = \sqrt{k/m}$, which is the angular frequency of an undamped oscillator. And the solution to the differential equation reduces to the solution of the undamped oscillator.

2. **Overdamped** ($b > \sqrt{4mk}$ or $b^2 - 4mk > 0$): λ has two real and distinct roots $\lambda_{1/2} = -\frac{b}{2m} \pm \omega'$ with

$$\omega' = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}. \tag{1.5.18}$$

The general solution can be written as the sum of two exponentially decaying ($\lambda_{1/2} < 0$) functions

$$x(t) = c_1 e^{-|\lambda_1| t} + c_2 e^{-|\lambda_2| t}, \tag{1.5.19}$$

where c_1 and c_2 are constants determined by initial conditions. Therefore, the overdamped system will return (exponentially decays) to equilibrium without any oscillating. The decay rate at long time is governed by $|\lambda_1| = -\lambda_1 = \frac{b}{2m} - \omega'$ with less absolute value:

$$x(t) \sim c_1 e^{-|\lambda_1| t}, \text{ as } t \to +\infty. \tag{1.5.20}$$

3. **Critically damped** ($b = \sqrt{4mk}$ or $b^2 - 4mk = 0$): λ has two identical roots $\lambda_{1/2} = \lambda = -\frac{b}{2m}$. The angular frequency is $\omega' = 0$. The general solution is

$$x(t) = (c_1 + c_2 t) e^{-|\lambda| t}, \tag{1.5.21}$$

where c_1 and c_2 are determined by initial conditions. The system returns to equilibrium as quickly as possible without oscillating (faster than the overdamped case).

1.5.3 Forced oscillations and resonance

In contrast to the simple harmonic oscillator and the damped oscillator, we may encounter a system where external force $F(t)$ acting on the particle are time-dependent. One example is a swing being pushed by a person with a periodic force. From Newton's second law, we obtain the equation of motion: $F(t) - bv(t) - kx(t) = ma(t)$. This is the second-order inhomogeneous linear differential equation

$$m \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + b \frac{\mathrm{d}x(t)}{\mathrm{d}t} + kx(t) = F(t), \tag{1.5.22}$$

where the time-dependent external force $F(t)$ is the inhomogeneous term.

By using Fourier transformation, a generic time-dependent force can be decomposed into several sine or cosine terms with different frequencies. For simplicity, let us consider a single-component periodic external force of the form:

$$F(t) = F_m \cos(\omega_F t). \tag{1.5.23}$$

The generic solution of the equation of motion, Eq. (1.5.22), is a superposition of a particular solution and the homogeneous solution of Eq.(1.5.14), as discussed in the previous subsection. We can assume the particular solution to be of the form

$$x(t) = A\cos(\omega_F t + \phi), \tag{1.5.24}$$

where the amplitude A and phase ϕ are unknown coefficients. This solution corresponds to a *steady state* that is periodic with respect to time. We can also consider it to be the solution in the limit of $t \to \infty$, since the damping term decays to zero in this limit. By substituting the trial solution Eq. (1.5.24) into Eq. (1.5.22), we have

$$A(k - m\omega_F^2)\cos(\omega_F t + \phi) - bA\omega_F \sin(\omega_F t + \phi) = F_m \cos(\omega_F t). \tag{1.5.25}$$

This equation can be rewritten as two equations in terms of A and ϕ by expanding $\cos(\omega_F t) = \cos(\omega_F t + \phi)\cos(\phi) + \sin(\omega_F t + \phi)\sin(\phi)$ on the right-hand side:

$$A(k - m\omega_F^2) = F_m \cos\phi, \tag{1.5.26}$$

$$-bA\omega_F = F_m \sin\phi, \tag{1.5.27}$$

So the amplitude and phase of the trial solution can be determined as

$$A = \frac{F_m}{\sqrt{m^2(\omega_F^2 - \omega^2)^2 + b^2\omega_F^2}}, \tag{1.5.28}$$

$$\phi = \arctan\frac{b\omega_F}{m(\omega_F^2 - \omega^2)}, \tag{1.5.29}$$

where $\omega = \sqrt{\frac{k}{m}}$ is the natural/intrinsic frequency of the oscillator (without damping or external force).

The steady-state solution of the forced oscillation problem, given by Eq. (1.5.24) in the limit $t \to \infty$, describes a harmonic oscillator with an angular frequency ω_F inherited from the external force. This frequency is NOT related to the natural/intrinsic frequency $\omega = \sqrt{\frac{k}{m}}$ of the oscillator. In fact, the steady-state solution oscillates at the same frequency as the driving force, regardless of the intrinsic parameters (m and k) of the oscillator and its initial conditions (see Figure 1.27).

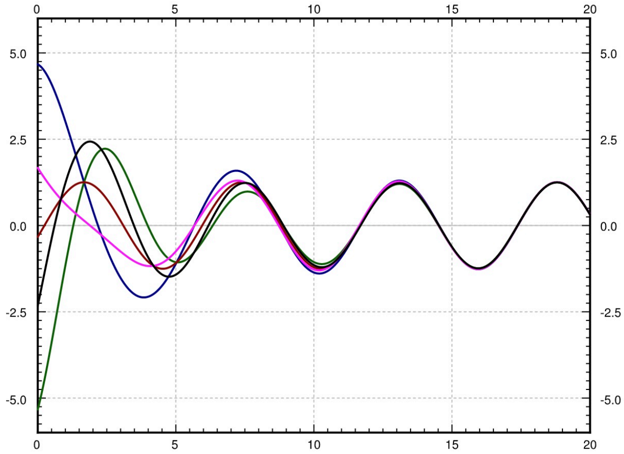

Figure 1.27: Forced oscillations with different initial conditions for $k = 1, m = 1, F_m = 1, b = 0.7$ and $\omega_F = \omega = 1.1$. All the curves converge to the steady oscillation Eq. (1.5.24) in the large t limit. The frequency of the motion also converges to the external frequency ω_F of the applied force.

One interesting phenomenon in the forced harmonic oscillator is resonance, where the amplitude of the steady-state motion is maximized. The amplitude A of the steady state in Eq.(1.5.28) depends on both the frequency ω_F of the external force and the natural frequency $\omega = \sqrt{\frac{k}{m}}$ of the oscillator (see Figure 1.31). From Eq.(1.5.28), the amplitude reaches its maximum value if the extrinsic and intrinsic frequencies coincide, i.e.,

$$\omega_F = \omega = \sqrt{\frac{k}{m}}, \tag{1.5.30}$$

which is independent of the damping coefficient b and initial conditions. This condition is called *resonance*.

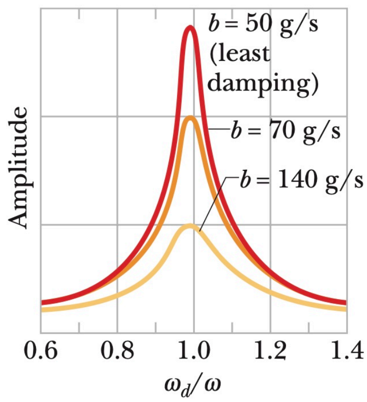

Figure 1.28: The amplitude A of the steady state as a function of frequency ratio ω_F/ω. The system exhibits resonance when the driving frequency is equal to the natural frequency of the oscillator, $\omega_F = \omega$.

1.5.4 Simple Pendulum

A pendulum is a physical system consisting of a mass (called the bob) m suspended from a fixed point by a string or rod with length L that is free to swing back and forth under the influence of gravity. Pendulums are studied in physics because they exhibit a simple harmonic motion in some limit.

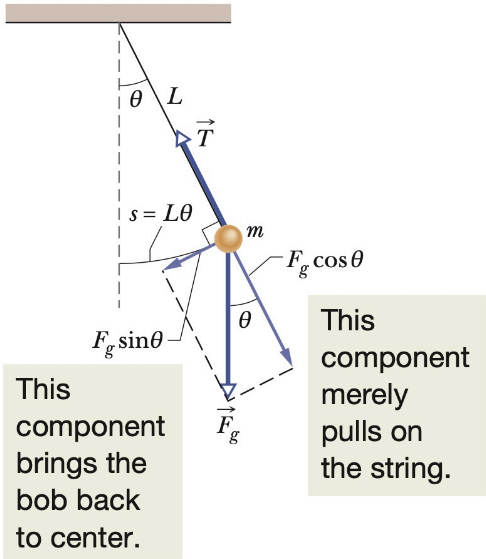

Figure 1.29: A simple pendulum.

From Newtons' second law, The motion of a pendulum is governed by the equation:

$$mg\cos\theta = T, \tag{1.5.31}$$

$$mg\sin\theta = mL\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}, \tag{1.5.32}$$

where θ is the angular displacement of the pendulum from its equilibrium position, and g is the acceleration due to gravity. The second equation is a nonlinear differential equation that is difficult to solve exactly.

In the case of small angles, where $\sin\theta \approx \theta$, the differential equation governing the motion of a simple pendulum can be simplified to:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\theta = 0, \tag{1.5.33}$$

which is exactly the linear differential equation Eq. (1.5.3) for a harmonic oscillator with a natural frequency of

$$\omega = \sqrt{\frac{g}{L}}. \tag{1.5.34}$$

The solution to this differential equation is:

$$\theta(t) = \theta_0 \cos(\omega t), \tag{1.5.35}$$

where θ_0 is the initial displacement of the pendulum from its equilibrium position. We can observe that the period of the pendulum, given by:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{1.5.36}$$

is independent of the mass m of the pendulum.

Similar to the harmonic oscillator of mass-spring system, the energy of a simple pendulum is transformed between kinetic energy and gravitational potential energy as the pendulum swings back and forth.

1.5.5 Double pendulum and chaos

The double pendulum is a classic example of a complex physical system that exhibits chaotic behavior. It consists of two pendulums attached to each other, with the second pendulum attached to the end of the first pendulum.

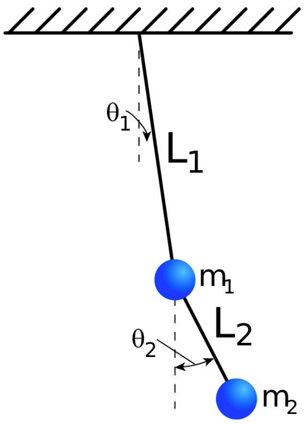

Figure 1.30: Double pendulum.

Consider a double pendulum consisting of two point masses m_1 and m_2, suspended by rigid, massless rods of length l_1 and l_2 respectively. The position of each pendulum is described by two angles θ_1 and θ_2. The position of each mass can be described by two coordinates: $x_1 = l_1 \sin\theta_1$ and $y_1 = -l_1 \cos\theta_1$ for mass m_1, and $x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$ and $y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$ for mass m_2. The velocities and accelerations of the masses are the time derivatives of these coordinates: $\dot{x}_i$, $\dot{y}_i$, $\ddot{x}_i$, and $\ddot{y}_i$, which can be expressed as functions of θ_i, $\dot{\theta}_i$, and $\ddot{\theta}_i$.

Using Newton's second law for each mass, we can write down the equations of motion:

$$m_1\ddot{x}_1 = -T_1 \sin\theta_1 + T_2 \sin\theta_2, \tag{1.5.37}$$

$$m_1\ddot{y}_1 = T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g, \tag{1.5.38}$$

$$m_2\ddot{x}_2 = -T_2 \sin\theta_2, \tag{1.5.39}$$

$$m_2\ddot{y}_2 = T_2 \cos\theta_2 - m_2 g, \tag{1.5.40}$$

where T_1 and T_2 are tensions of the two rods. By eliminating the tensions, we obtain a set of coupled, nonlinear differential equations for the two angles θ_1 and θ_2:

$$\ddot{\theta}_1 = \frac{-g(2m_1+m_2)\sin\theta_1 - m_2 g\sin(\theta_1-2\theta_2) - 2\sin(\theta_1-\theta_2)m_2(\dot{\theta_2}^2 l_2 + \dot{\theta_1}^2 l_1\cos(\theta_1-\theta_2))}{l_1(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}, \tag{1.5.41}$$

$$\ddot{\theta}_2 = \frac{2\sin(\theta_1-\theta_2)(\dot{\theta_1}^2 l_1(m_1+m_2) + g(m_1+m_2)\cos\theta_1 + \dot{\theta_2}^2 l_2 m_2\cos(\theta_1-\theta_2))}{l_2(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}, \tag{1.5.42}$$

These equations for the double pendulum are quite complex and difficult to solve analytically. However, the motion of the double pendulum can be simulated using numerical methods, such as the Runge-Kutta algorithm, which can accurately predict the motion of the double pendulum for a given set of initial conditions. You can find interactive simulations of double pendulum on this website.

One of the most fascinating features of the double pendulum is its chaotic behavior. Chaos is a phenomenon where a small change in the initial conditions of a system can lead to vastly different outcomes in its motion. In the case of the double pendulum, even the slightest differences in the initial positions or velocities of the pendulums can result in completely different trajectories.

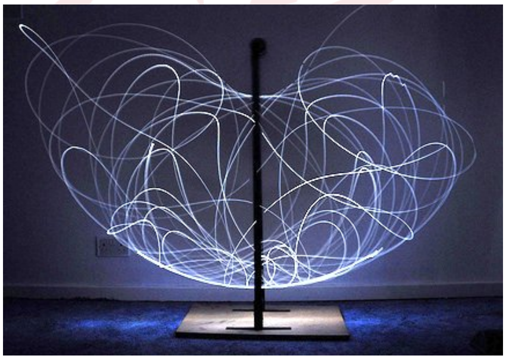

Figure 1.31: Long exposure of double pendulum exhibiting chaotic motion (tracked with an LED).

This characteristic of the double pendulum has significant implications. It means that it is practically impossible to predict the long-term motion of the system, since any small measurement error or uncertainty in the initial conditions can lead to dramatically different results. This is known as the "butterfly effect", as even the flapping of a butterfly's wings can potentially cause a hurricane on the other side of the world.