

# Differential Geometry

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# 1 Smooth Manifold

**Definition 1.1** (Topological manifold). A space  $M$  is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

**Definition 1.2** (Smooth Manifold). A smooth structure is given by an equivalence class of smooth atlas  $\{(U_\alpha, \varphi_\alpha)\}$  s.t.  $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$  is smooth  $\forall \alpha, \beta$ .  $M = \cup U_\alpha$ .

A **smooth manifold** is a topological manifold with a smooth structure.

Define when a continuous map  $f : M_1 \rightarrow M_2$  is smooth if  $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$ , we have  $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$  is smooth.

**Definition 1.3.** Given  $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ . A homeomorphism  $f : M_1 \rightarrow M_2$  is called a diffeomorphism if  $f, f^{-1}$  is smooth.

In this case we say  $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$  are diffeomorphism.

**Theorem 1.4** (Kervaire).  $\exists$  1 10-dimensional topological manifold without smooth manifold.

**Theorem 1.5** (Milnor).  $\exists$  a smooth manifold  $M$  s.t.  $M \cong S^7$  but not in diffeomorphism meaning.

**Theorem 1.6** (Kervaire-Milnor).  $\exists$  28 smooth structures (up to orientation preserving diffeomorphism) on  $S^7$

**Theorem 1.7** (Morse-Birg). On  $S^n$ . If  $n \leq 3$ , then any  $n$ -dimensional topological manifold  $M$  has a unique smooth structure up to diffeomorphism.

**Theorem 1.8** (Stallings). *If  $n \neq 4$ , then  $\exists$  a unique smooth structure on  $\mathbb{R}^n$  up to diffeomorphism.*

**Theorem 1.9** (Donaldson-Freedman-Gompf-Faubes).  *$\exists$  uncountable smooth structures on  $\mathbb{R}^4$  up to diffeomorphism.*

**Definition 1.10** (topological manifold with boundary). A space  $M$  is called a topological manifold with boundary if

1.  $M$  is Hausdorff
2.  $M$  is second countable
3.  $\forall p \in M, \exists$  a neighbourhood  $U$  of  $p$  and a homeomorphism  $\varphi : U \rightarrow V$  where  $V$  is open in  $\mathbb{H}^n$

We say a manifold  $M$  is closed if  $M$  is compact and  $\partial M$  is empty.

Our motivation for studying manifold is to study the space of solution for equations.

**Question 1.** Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  smooth,  $q \in \mathbb{R}^n$ , when is  $f^{-1}(q)$  is a smooth manifold?

For  $f : U \rightarrow \mathbb{R}^n$  smooth,  $U$  open in  $\mathbb{R}^m$ , the differential of  $f$  at  $p \in U$  denoted as  $df(p)$ .

**Definition 1.11.** We say  $p \in U$  is a **regular point** of  $f$  if  $df(p)$  is surjective. Otherwise we say  $p \in U$  is a **critical point**.

A point  $q \in \mathbb{R}^n$  is called a **regular value** of  $f$  if  $\forall p \in f^{-1}(q)$ ,  $p$  is a regular point of  $f$ .

A point  $q \in \mathbb{R}^n$  is called a **critical value** of  $f$  if  $\forall p \in f^{-1}(q)$ ,  $p$  is a critical point of  $f$ .

**Theorem 1.12** (Implicit function theorem). *If  $p \in U$  is a regular point of  $f : U \rightarrow \mathbb{R}^n$ . Then there exists*

- *An open neighbourhood  $V$  of  $p$  in  $U$*
- *An open subset  $V'$  of  $\mathbb{R}^m$*
- *A diffeomorphism  $\varphi : V \rightarrow V'$  such that  $P \circ \varphi = f$  where  $P$  is the projection from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ .*

*In other words, near a regular point, we can do local coordinate change to turn  $f$  into the projection.*

**Remark 1.13.** In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \dots, x_m) \in V' \mid (x_1, \dots, x_n) = f(p)\}$$

*i.e.* if we set  $M = f^{-1}(f(p))$ , then  $(M \cap V, \varphi_p)$  is a chart that contains  $p$ .

**Corollary 1.14.** *If  $q$  is a regular value of  $f : U \rightarrow \mathbb{R}^n$  then  $f^{-1}(q)$  is a smooth manifold.*

**Remark 1.15.** It suffices to show that the corresponding charts are compatible.

**Theorem 1.16** (Sard). *If  $f : U \rightarrow \mathbb{R}^n$  is a smooth map, then the set of critical values of  $f$  has measure 0.*

**Remark 1.17.** For a "generic"  $q$ ,  $f^{-1}(q)$  is a manifold of dimension  $m - n$ .

**Corollary 1.18.** *If  $f : U \rightarrow \mathbb{R}^n$  is smooth and  $m < n$  then  $f(U)$  has measure 0.*

## 1.1 Lie groups and homogeneous spaces

**Definition 1.19.** We say  $G$  is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map  $\cdot : G \times G \rightarrow G$  and the inverse map  $G \rightsquigarrow G$  is smooth.

**Example 1.20.**  $GL(n, \mathbb{R}) = \{n \times n \text{ matrices with non-zero determinant}\} \subset \mathbb{R}^{n \times n}$

$$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$$

$$SO(n) = \{A \in O(n) | \det A = 1\}$$

$$U(n) = \{A \in GL(n, \mathbb{C}) | A\bar{A}^T = I\}$$

$$SU(n) = \{A \in U(n) | \det A = 1\}$$

**Exercise 1.21.**

$$O(1) \cong S^2 \qquad SO(1) \cong * \qquad (1.1)$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \qquad (1.2)$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \qquad (1.3)$$

The last one is a diffeomorphism but do not preserve the multiplication, *i.e.* not an isomorphism of Lie group.

**Theorem 1.22** (Cartan). *Let  $H$  be a closed subgroup of Lie group  $G$ . Then  $H$  is a Lie group. More precisely,  $H$  is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also,  $G/H$  is a smooth manifold*

**Definition 1.23.** Let  $M$  be a smooth manifold. We say  $M$  is a **homogeneous space** if  $\exists$  a Lie group  $G$  with a smooth transitive action  $\rho : G \times M \rightarrow M$ .

**Definition 1.24.** For  $M$  be a homogeneous space. The **isotropy** group of  $x \in M$  is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of  $G$

Given any  $x, x' \in M$ ,  $Iso(x) \cong Iso(x')$  because the group action is transitive.

Hence, we have a well-defined map

$$p : G/Iso(x) \rightarrow M \qquad (1.4)$$

$$g \mapsto gx \tag{1.5}$$

**Theorem 1.25.**  *$p$  is always a diffeomorphism.*

Therefore, we have this proposition

**Proposition 1.26.**  *$M$  is a homogeneous space  $\Leftrightarrow M = G/H$  for some closed subgroup  $H$ .*

**Example 1.27.** If  $M = S^n$ , let  $G = SO(n + 1)$ .

Then  $ISO(1, 0, \dots, 0) \cong SO(n)$ .

So  $S^n \cong SO(n + 1)/(SO(n))$ .

Similarly, we can prove  $\mathbb{RP}^n \cong SO(n + 1)/(O(n))$ ,  $\mathbb{CP}^n \cong SO(n + 1)/(U(n))$

The isotropy  $k$  dimensional linear subspaces of  $\mathbb{R}^n$  can be  $O(k) \times O(n - k)$  if  $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to  $\mathbb{RP}^2, S^2, T^2$  and Klein bottle.

**Theorem 1.28** (Whithead). *Any smooth manifold has a triangulation.*

**Theorem 1.29** (Poincare-Hopf).  *$G$  is compact Lie group  $\Rightarrow \chi(G) = 0$ .*

**Theorem 1.30** (Mostow2005).  *$M$  is a compact homogeneous space  $\Rightarrow \chi(M) \geq 0$ .*

## 1.2 Bump Function and Partition of Unity

**Theorem 1.31** (Urysohn smooth version). *Given  $M$ , closed disjoint  $A, B$ ,  $\exists$  smooth  $f : M \rightarrow [0, 1]$  s.t.  $f|_A = 0, f|_B = 1$ .*

**Theorem 1.32** (Tietze). *Given  $M$ , closed  $A$ , smooth  $f : A \rightarrow \mathbb{R}^n$ , there exists smooth  $\hat{f} : M \rightarrow \mathbb{R}^n$  s.t.  $\hat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

**Lemma 1.33.** *Let  $U$  be a neighbourhood of  $p \in M$ . Then  $\exists$  smooth  $\sigma : M \rightarrow [0, 1]$  s.t.*

1.  $\sigma \equiv 1$  near  $p$

2.  $\text{Supp } \sigma \subset U$

Such  $\sigma$  is called a **bump function** at  $p$ , supported in  $U$ .

**Definition 1.34.** An open cover of a space  $X$  is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

**Proposition 1.35.** *Given compact  $K \subset U$  and open neighbourhood  $U$  of  $K$ ,  $\exists$  a smooth  $g : M \rightarrow [0, +\infty)$  s.t.  $g|_K \equiv 1$  and  $\text{Supp } g \subset U$ .*

**Definition 1.36.** An **exhaust** of a space  $X$  is a sequence of open sets  $\{U_i\}$  s.t.

1.  $X = \bigcup_{i=1}^{\infty} U_i$

2.  $\overline{U_i}$  is compact and contained in  $U_{i+1}$

**Theorem 1.37.** *Any topological manifold has an exhaust.*

Given two open covers  $\mathcal{U}, \mathcal{V}$ , we say  $\mathcal{V}$  is a **refinement** of  $\mathcal{U}$  if  $\forall U_\alpha \in \mathcal{U}, \exists V_\beta \in \mathcal{V}$  s.t.  $V_\beta \subset U_\alpha$ .

We say a space  $X$  is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact.(The proof is hard)

**Proposition 1.38.** *Let  $\mathcal{U} = \{U_\alpha\}$  be an open cover of a topological manifold  $M$ . Then there exists countable open covers  $\mathcal{W} = \{W_i\}, \mathcal{V} = \{V_i\}$  s.t.*

- For any  $i$ ,  $\overline{V_i}$  is compact and  $\overline{V_i} \subset W_i$

- $\mathcal{W}$  is locally finite.
- $\mathcal{W}$  is a refinement of  $\mathcal{U}$ .

As a corollary, we have any topological manifold is paracompact.

**Definition 1.39.** Given open cover  $\mathcal{U}$  of a smooth  $M$ , a partition of unity subordinate to  $\mathcal{U}$  is a collection of smooth functions  $\{\rho_\alpha : M \rightarrow [0, 1]\}_{\alpha \in \mathcal{A}}$  s.t.

1.  $\forall p \in M, \exists$  only finitely many  $\alpha \in \mathcal{A}$  s.t.  $p \in \text{Supp } \rho_\alpha$
2.  $\sum_{\alpha \in \mathcal{A}} \rho_\alpha(p) = 1$
3.  $\text{Supp } \rho_\alpha \subset U_\alpha$

**Theorem 1.40** (Existence of P.O.U). *For any open cover  $\mathcal{U}$  of smooth  $M$ ,  $\exists$  a P.O.U subordinate to  $\mathcal{U}$*

**Theorem 1.41** (Whitney approximation theorem). *Given any smooth  $M$ , any closed  $A$  and any continuous  $f : M \rightarrow \mathbb{R}$ ,  $\delta : M \rightarrow (0, +\infty)$ . Suppose  $f$  is smooth on  $A$ . Then  $\exists g : M \rightarrow \mathbb{R}$  smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M, |g(p) - f(p)| < \delta(p)$ .

## 2 Tangent space and tangent vectors

### 2.1 Tangent Space

Given  $p \in M$ , consider the set  $C_p^\infty(M) = \{\text{smooth function } V \rightarrow \mathbb{R}\} / \sim$  where  $f_1 \sim f_2$  if and only if  $\exists$  neighbourhood  $U$  of  $p$ ,  $f_1|_U = f_2|_U$ .

$C_p^\infty(M)$  is the space of **germs of smooth function** near  $p$ .



A **partial-derivative** of  $p$  is a  $\mathbb{R}$ -linear map  $D : C_p^\infty(M) \rightarrow \mathbb{R}$  that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

**Definition 2.1.** A **tangent vector** of  $M$  at  $p$  is a partial-derivative at  $p$ .

Define the **tangent space**  $T_p M = \{\text{all partial-derivative at } p\}$ , which is a  $\mathbb{R}$ -vector space.

**Proposition 2.2.** For  $M = U \subset \mathbb{R}^n$  open. We have  $\{\frac{\partial}{\partial x_i}\}$  is a basis for  $T_p U$ .

**Proposition 2.3.**

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leq i \leq n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

$M, N$  smooth manifolds,  $C^\infty(N, M) = \{\text{smooth } F : N \rightarrow M\}$ .

Given  $F \in C^\infty(N, M)$ ,  $F$  induces  $F^* : C_{F(p)}^\infty(M) \rightarrow C_p^\infty(N)$   $f \rightarrow f \circ F$ .

By taking dual, we get

$$F_* : T_p N \rightarrow T_{F(p)} M$$

we also write  $F_*$  as  $F_{*,p}$ , call it the **differential** of  $F$  at  $p$ .

where

$$F_*\left(\frac{\partial}{\partial x^i}|_p\right) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

**Proposition 2.4.** The differential satisfies the composition law.

$$(G \circ F)_* = G_* \circ F_* : T_p N \rightarrow T_{G \circ F(p)} W$$

**Definition 2.5.** A smooth **curve** is a smooth map  $\gamma : (a, b) \rightarrow M$ . We say  $\gamma$  starts at  $p$  if  $\gamma(0) = p$ . We define the **velocity** of  $\gamma$  at  $\gamma(0)$  as  $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)}M$

Take charts  $(U, x^1, \dots, x^n)$  about  $p$ , let  $\gamma^i = x^i \circ \gamma$ .

We say  $\gamma, \delta$  are **tangent** to each other at  $p$  if  $(\gamma^i)'(0) = (\delta^i)'(0)$ .

Now we can define

$$(T_p M)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\} / \sim$$

where  $\gamma \sim \delta$  iff they are tangent to each other.

Then these definition is more geometric.

**Lemma 2.6.** Given  $F \in C^\infty(M, M)$ ,  $p \in N$ , the diagram commutes:

$$\begin{array}{ccc} \gamma \in (T_p N)_{curve} & \xrightarrow{\cong} & T_p N \\ \downarrow & & \downarrow \\ F \circ \gamma \in (T_{F(p)} M)_{curve} & \xrightarrow{\cong} & T_{F(p)} M \end{array}$$

## 2.2 Tangent Bundle

Let  $(M, \mathcal{A})$  be a smooth manifold,  $TM = \bigcup_{p \in M} T_p M$ , called the **tangent bundle**

Now we want to define a natural topology and smooth structure on  $TM$ . Take any chart  $(U, \varphi) = (U, x^1, \dots, x^n) \in \mathcal{A}$ .

We have a map

$$\hat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \quad (2.1)$$

$$X \in T_p U \mapsto (\varphi(p), X^1, \dots, X^n) \quad (2.2)$$

where  $X = \sum X^i \frac{\partial}{\partial x^i} |_p$ .

Then pull back standard topology on  $\varphi(U) \times \mathbb{R}^n$  to a topology on  $TU$ .

$$\mathcal{B} = \{ \hat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- $\mathcal{B}$  is a basis
- $\mathcal{B}$  generates a Hausdorff, second countable topology on  $TM$ .

So  $TM$  is a topological manifold covered by charts  $\hat{\mathcal{A}} = \{(TU, \hat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$ .

Given  $(TU, \hat{\varphi}), (TV, \hat{\psi}) \in \hat{\mathcal{A}}$ , the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\hat{\psi} \circ \hat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \quad (2.3)$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \quad (2.4)$$

So  $\hat{\mathcal{A}}$  is a smooth atlas on  $TM$ , making  $TM$  into a smooth manifold.

**Definition 2.7** (vector bundle). Given a continuous map  $f : E \rightarrow B$ , we say  $f$  is a  $n$ -dimensional **vector bundle** if:  $\exists$  an open cover  $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$  of  $B$  and homeomorphisms  $\{f^{-1}(U_\alpha) \xrightarrow[\cong]{\rho_\alpha} U_\alpha \times \mathbb{R}^n\}$  s.t.

- $$\begin{array}{ccc} f^{-1}(U_\alpha) & \xrightarrow{\rho_\alpha} & U_\alpha \times \mathbb{R}^n \\ \downarrow f & \swarrow \text{projection} & \\ U_\alpha & & \end{array} \quad \text{commutes for } \alpha \in I$$
- $\forall p \in U_\alpha \cap U_\beta$ , the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

is linear.

Call  $f^{-1}(p)$  the **fiber** over  $p$ .

**Proposition 2.8.** *Given vector bundle  $f : E \rightarrow B$ , the fiber  $f^{-1}(p)$  has a structure of a vector space.*

**Example 2.9** (Product bundle).  $E = \mathbb{R}^n \times B$

**Example 2.10** (Tautological bundle).

$$B = \mathbb{CP}^n = \{1\text{-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map  $(L, v) \mapsto L$

Given vector bundles  $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$ , a bundle map consists of  $(\hat{f}, f)$  s.t.

- $$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow \pi & & \downarrow \pi \\ B_1 & \xrightarrow{f} & B_2 \end{array} \text{ commutes.}$$
- $\forall b \in B, \hat{f} : \pi_1^{-1}(b) \rightarrow \pi_2^{-1}(f(b))$  is linear.

If  $\hat{f}, f$  are diffeomorphisms, then we call  $(\hat{f}, f)$  an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. A bundle is **trivial** if it has a trivialization.

**Example 2.11.**  $TS^1, TS^2$  are both trivial.

$$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$$

**Theorem 2.12.** *If  $G$  is a Lie group, then  $TG$  is trivial.*

*Proof.* For  $(x^1, x^2, \dots, x^n)$  is a basis of  $T_e G$  The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG, (g, c^1, \dots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \rightarrow G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism  $(l_g)_*$

□

**Proposition 2.13** (Adams, 1960s).  $TS^n$  is trivial if and only if  $n = 0, 1, 3, 7$ .

**Proposition 2.14.** 1. Given any  $F \in C^\infty(M, N)$ ,  $F_* : TM \rightarrow TN$  is a bundle map.

2.  $TS^n$  is isomorphic to the following bundle:

$$B = S^n \quad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

**Definition 2.15** (smooth section). Given a smooth vector bundle  $\pi : E \rightarrow B$ , a **smooth section** is a smooth map  $S : B \rightarrow E$  s.t.  $\pi \circ S = id_B$ .

$$s_0 : B \rightarrow E, b \mapsto 0 \in 0\text{-vector in } \pi^{-1}b.$$

## 2.3 Vector Field, Curves and Flows

**Definition 2.16.** A (tangent) **vector field** is a smooth section of  $TM$ . i.e. a smooth map  $M \xrightarrow{X} TM$  s.t.  $X(p) \in T_p M, \forall p \in M$

Given any  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

**Example 2.17.**  $X = f^1 \partial x^1 + f^2 \partial x^2$  is a gradient field if and only if  $\frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}$

**Theorem 2.18** (Poincare-Hopf). For closed  $M$ ,  $M$  has a nowhere vanishing vector field if and only if  $\chi(M) = 0$ .

So  $S^n$  has a nowhere vanishing vector field if and only if  $n$  is odd.

**Theorem 2.19** (MaoQiu).  $S^2$  has no no-where vanishing vector field.

So We cannot smooth out all the hairs on a ball.

Given a vector field  $X = \{X_p\}_{p \in M}$ , a curve  $\gamma : (a, b) \rightarrow M$  is called an **integral curve** of  $X$  if  $\gamma'(t) = X_{\gamma(t)}, \forall t \in (a, b)$ , where  $\gamma'(t) = \gamma_*\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M$ .

We say  $\gamma$  is maximal if the domain cannot be extended to a larger interval.

Denote the set of all smooth vector fields on  $M$  by  $\mathfrak{T}M$

Recall that  $\gamma$  is maximal if it's domain can not be extended to a large open interval.

In a local chart  $(U, x^1, \dots, x^n)$ ,  $X|_U = \sum_{i=1}^n a^i \partial x^i$ . Then  $\gamma$  is an integral curve if and only if  $(\gamma^i)'(t) = a^i(\gamma(t)), \forall 1 \leq i \leq n$ , where  $\gamma^i = x^i \circ \gamma : (a, b) \rightarrow \mathbb{R}$ .

And in this case the initial value condition:  $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$ .

Locally, solving integral curve starting at  $p$  is equivalent to solving ODE with initial value  $p^1, \dots, p^n$ . By existence and uniqueness of solutions of ODE, we have

**Theorem 2.20** (Fundamental theorem of integral curve). Let  $X \in \mathfrak{T}M, p \in M$ , then:

(1) (Uniqueness) Given any two integral curves  $\gamma_1, \gamma_2 : (a, b) \rightarrow M$ , then we have:

$$\gamma_1(c) = \gamma_2(c) \text{ for some } c \in (a, b) \Rightarrow \gamma_1 = \gamma_2$$

(2) there exists a unique max integral curve  $\gamma : (a(p), b(p)) \rightarrow M$  starting at  $p$ .

(3) (integral curve smoothly depend on initial values)  $\exists$  Nbh  $U$  of  $p, \epsilon > 0$ , and smooth

$\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$  s.t.  $\forall q \in U, \varphi_\epsilon := \varphi(-, q) : (-\epsilon, \epsilon) \rightarrow M$  is an integral curve starting at  $p$ .

we call such  $\varphi$  a local **flow** generated by  $X$ .

**Definition 2.21.** Given  $X \in \mathfrak{X}M$ , a global **flow** generated by  $X$  is a smooth map  $\varphi : \mathbb{R} \times M \rightarrow M$  s.t.  $\forall q \in M, \varphi_q := \varphi(-, q)$  is the maximal integral curve of  $X$  starting at  $q$ .

$$\Leftrightarrow \frac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s, p)}, \forall s \in \mathbb{R}, p \in M \text{ and } \varphi(0, p) = p, \forall p \in M.$$

If such global flow exists, then we say  $X$  is complete.

**Example 2.22.**

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$  is complete, where global flow  $\varphi : \mathbb{R} \times M \rightarrow M, \varphi(t, p) = p \cdot e^t$ .
- $X = x^2 \partial x$  is not complete. Max integral curve starting at 1 is given by  $\gamma(t) = \frac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$ .

Given  $X \in \mathfrak{X}M$ , we define  $\text{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$ .

**Theorem 2.23.** If a vector field  $X$  is compactly supported, then  $X$  is complete.

**Corollary 2.24.** Any vector field on closed manifold is complete.

**Lemma 2.25** (Escaping lemma). Suppose  $\gamma : (a, b) \rightarrow M$  is a max integral curve, with  $(a, b) \neq \mathbb{R}$ . Then  $\nexists$  compact  $K \subset M$  s.t.  $\gamma(a, b) \subset K$

*Proof.* Otherwise, suppose  $\gamma(a, b) \subset K$ . WLOG, we may assume  $b < +\infty$ .

Take  $(t_i) \rightarrow b$  from left. Then  $\gamma(t_i) \in K$ . After passing to subsequence, we may assume  $(\gamma(t_i)) \rightarrow p \in K$ .

Then  $\exists U$  Nbh of  $p$ , local flow  $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$ . Take  $n$  large enough s.t.  $b - t_n < \epsilon, \gamma(t_n) \in U$ . Then  $\gamma(- + t_n) : (a - t_n, b - t_n) \rightarrow M, \varphi(-, \gamma(t_n)) : (-\epsilon, \epsilon) \rightarrow M$  are both integral curves for  $X$  starting at  $\gamma(t_n)$ . By uniqueness, they coincide.

$$\text{Let } \hat{\gamma} : (a, t_n + \epsilon) \rightarrow M \text{ be defined by } \hat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a, b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \epsilon) \end{cases}$$

Then  $\hat{\gamma}$  is an integral curve with larger domain, then  $\gamma$  contradiction with the maximality of  $\gamma$ . □

*Proof of 2.23.* Take any max integral curve  $\gamma : (a, b) \rightarrow M$ . Suppose  $(a, b) \neq \mathbb{R}$ . Then  $X_{\gamma(s)} \neq 0, \forall s$ . Otherwise, the constant map  $\mathbb{R} \rightarrow M, t \mapsto \gamma(s)$  is an integral curve with larger domain.

So  $\forall s, \gamma(s) \in \text{Supp} X \Rightarrow \gamma(a, b) \subset \text{Supp} X$  which is compact  $\Rightarrow (a, b) = \mathbb{R}$  by the lemma. This causes contradiction!  $\square$

A smooth  $\varphi : \mathbb{R} \times M \rightarrow M$  is called an **one-parameter transformation group** if

$$(1) \quad \varphi_0 := \varphi(0, -) = \text{id}_M$$

$$(2) \quad \varphi_s \circ \varphi_t = \varphi_{s+t} \text{ for all } s, t \in \mathbb{R}. \text{ In particular, } \varphi_s^{-1} = \varphi_{-s}.$$

**Theorem 2.26.**  $\varphi \in C^\infty(\mathbb{R} \times M, M)$ , then  $\varphi$  is an one-parameter transformation group if and only if  $\varphi$  is the global flow generated by some  $X \in \mathfrak{X}M$

**Lemma 2.27** (Translation lemma). If  $\gamma : (a, b) \rightarrow M$  is an integral curve for some  $X \in \mathfrak{X}M$ , then  $\forall s \in \mathbb{R}, \gamma(- + s) : (a - s, b - s) \rightarrow M$  is also an integral curve for  $X$ .

*Proof.* Let  $\iota = \gamma(- + s)$ . Then  $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$   $\square$

**Lemma 2.28.** Let  $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$  be a local flow for some  $X \in \mathfrak{X}M$ . Then  $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$  provided that  $s, t, s + t \in (-\epsilon, \epsilon), p, \varphi_r(p) \in U$ .

*Proof.*  $\gamma_p = \varphi(-, p)$  is an integral curve for  $X$ .

$\Rightarrow \gamma_p(- + s)$  is an integral curve for  $X$  starting at  $\gamma_p(s) = \varphi_s(p)$ . But  $\gamma_{\varphi_s(p)}$  is also an integral curve starting at  $\varphi_s(p)$ . Thus  $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$   $\square$

**Lemma 2.29.** Let  $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$  be a local flow for some  $X \in \mathfrak{X}M$ . Then  $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$  i.e. any vector field is invariant under its flow.



*Proof.* Take  $f \in C_{\varphi(p)}^\infty(M)$ .

$$\varphi(s, *) (X_p)(f) = X_p(f \circ \varphi_s) \quad (2.5)$$

$$= \frac{d}{dt}(f \circ \varphi_s(\varphi_t(p)))|_{t=0} \quad (2.6)$$

$$= \frac{d}{dt}(f \circ \varphi_t(\varphi_s(p)))|_{t=0} \quad (2.7)$$

$$= X_{\varphi_s(p)}(f) \quad (2.8)$$

□

*Proof of 2.26.* " $\Leftarrow$ " is because the lemma  $\varphi_s \circ \varphi_r = \varphi_{s+r}$

" $\Rightarrow$ " Let  $X = \{X_p\}$  where  $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$ .

Leave it as an exercise. □

**Time dependent** vector field is a smooth map  $X : \mathbb{R} \times M \rightarrow TM$  s.t.  $X_{(t,p)} \in T_p M$ .

A smooth curve  $\gamma(a, b) \rightarrow M$  is the **integral curve** for  $X$  if  $\gamma'(t) = X_{(t, \gamma(t))}$ .

In local chart, solving  $\gamma$  is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness:  $\gamma_1, \gamma_2$  are both integral curves for  $X$ ,  $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define  $\text{Supp} X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$ .

Then  $X$  is compactly supported, then  $X$  is complete( i.e. a global flow  $\varphi : \mathbb{R} \times M \rightarrow M$ )

But something is not true for time dependent vector field:

- translation lemma is not true.

- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

## 2.4 Another definition of vector field

A derivation on  $M$  is a  $\mathbb{R}$ -linear map  $C^\infty(M) \xrightarrow{D} C^\infty(M)$  that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

**Theorem 2.30.** *We have a bijection:*

$$\begin{aligned} \rho : \mathfrak{X}M &\xrightarrow{1:1} \{\text{derivation on } M\} \\ X &\mapsto D_X : f \mapsto X(f) \end{aligned}$$

**Lemma 2.31.**  $D_p : \mathfrak{X}_p M \rightarrow \mathbb{R}$ -linear map  $C^\infty(M) \rightarrow \mathbb{R}$  s.t.  $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$  is an isomorphism of vector spaces.

*Proof.* Leave it as an exercise. □

**Lemma 2.32.** *Given a vector field(not necessarily smooth)  $X = \{X_p\}_{p \in M}$ ,  $X$  is smooth  $\Leftrightarrow \forall f \in C^\infty(M)$ ,  $X(f)$  is smooth.*

*Proof.* " $\Leftarrow$ "  $\forall p \in M$ , take chart  $(U, x^1, x^2, \dots, x^n)$  around  $p$ .  $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$  :  $U \rightarrow \mathbb{R}$ , where  $f^i = X|_U(x^i)$ . Take  $\varphi : M \rightarrow [0, 1]$  s.t.  $\varphi \equiv 1$  near  $p$ ,  $\text{Supp} \varphi \subset U$ ,  $\varphi \cdot x^i \in C^\infty(M)$ .

Then  $X(\varphi \cdot x^i) = f^i$  near  $p$ . By assumption,  $f^i$  is smooth near  $p$ . So  $f^i$  is smooth, so  $X$  is smooth.

" $\Rightarrow$ " Similar. □

**Theorem 2.33.** *The map  $\rho : \mathfrak{X}M \rightarrow \{\text{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$  is well-defined and bijective.*

*Proof.*  $\rho$  is well-defined:  $X(f) \in C^\infty(M)$  by Lemma 2.32, and  $D_x(fg) = D_x(f)g + fD_x(g)$  since  $X$  is a point-derivation.

$\rho$  is injective:  $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$  as maps  $C^\infty(M)$  to  $\mathbb{R}$ . By Lemma 2.31, we have  $X_p = Y_p, \forall p$ . So  $X = Y$ .

$\rho$  is surjective: Given  $D : C^\infty(M) \rightarrow C^\infty(M)$ . Define  $D_p : C^\infty(M) \rightarrow \mathbb{R}$  by  $D_p(f) := D(f)(p)$  satisfies the Leibniz rule. By Lemma 2.31,  $D_p = D_{X_p}$  for some  $X_p \in T_pM$ . Define  $X = \{X_p\}_{p \in M}$ . Then  $X(f) = D(f), \forall f \in C^\infty(M)$ . By Lemma??,  $X$  is a smooth vector field.  $\square$

## 2.5 Lie bracket

In this section, we can actually find those identification:

$$\begin{aligned} \{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \text{ s.t.} \\ &\quad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\} \end{aligned}$$

$$\begin{aligned} \{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\} \end{aligned}$$

**Notation 2.34.** We will identify  $X \in \mathfrak{X}M$  with its derivation  $D_x : C^\infty(M) \rightarrow C^\infty(M)$ . So a vector field is just a  $\mathbb{R}$ -linear map  $X : C^\infty(M) \rightarrow C^\infty(M)$  s.t.  $X(fg) = fX(g) + X(f)g$ .

**Definition 2.35** (Lie bracket). Given two (smooth) vector field  $X, Y : C^\infty(M) \rightarrow C^\infty(M)$ , we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \rightarrow C^\infty(M)$$

**Theorem 2.36.** For any  $X, Y \in \mathfrak{X}M$ ,  $[X, Y] \in \mathfrak{X}M$

*Proof.* Easy to check that  $[X, Y]$  is linear.

By Leibniz rule,

$$\begin{aligned} [X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\ &= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\ &= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\ &= [X, Y](f) \cdot f \cdot [X, Y](g) \end{aligned}$$

□

So What is the geometric meaning of  $[X, Y]$ ? Non commutativity of flows.

**Fact 2.37.** Given  $X, Y \in \mathfrak{X}M$ , we say  $X, Y$  are commutative vector field if  $[X, Y] = 0$

$X, Y$  are commutative iff for any local flows  $\varphi^X : (-\epsilon, \epsilon) \times U \rightarrow M$ ,  $\varphi^Y : (-\epsilon, \epsilon) \times U \rightarrow M$  we have  $\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X$

**Proposition 2.38** (Calculation of  $[V, W]$  using local charts). Chart  $(U, x^1, \dots, x^n)$ ,

$V, W \in \mathfrak{X}M$ ,  $V|_U = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$ ,  $W|_U = \sum_{i=1}^n W^i \frac{\partial}{\partial x^i}$ . Then

$$\begin{aligned} [V, W]|_U &= \sum_{i=1}^n (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n \left( \sum_{j=1}^n V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} \end{aligned}$$

$$= \sum_{1 \leq i, j \leq n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i}$$

**Example 2.39.**  $V = x\partial x + y\partial y$ ,  $W = -y\partial x + x\partial y$  commutes.

**Proposition 2.40** (Properties of Lie bracket).

- *Natuality under push-forward.*

Given any  $F \in \text{Diff}(M, N)$ ,  $V \in \mathfrak{X}M$ ,  $W \in \mathfrak{X}M$ , we have  $[F_*V, F_*W] = F_*[V, W]$ .

- $\mathbb{R}$ -linearity  $\forall a, b \in \mathbb{R}$

$$[aX + bV, W] = a[X, W] + b[V, W]$$

$$[W, aX + bV] = b[W, X] + a[W, V]$$

- *anti-symmetric*  $[V, W] = -[W, V]$
- *Jacobi identity*

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$$

- *Leibniz rule*

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

**Definition 2.41.** Given  $F \in C^\infty(M, N)$ ,  $V \in \mathfrak{X}M$ ,  $W \in \mathfrak{X}N$ . We say  $W$  is  **$F$ -related** to  $V$  if  $\forall p \in M$ ,  $F_{p,*}(V_p) = W_{F(p)}$ ,  $F_{p,X} : T_pM \rightarrow T_{f(p)}N$

**Example 2.42.**  $F : S^1 \rightarrow \mathbb{R}^2$ ,  $\theta \mapsto (\cos \theta, \sin \theta)$ ,  $V = \partial \theta$ ,  $W = -y\partial x + x\partial y$ .

*note 1.* In general, given  $V \in \mathfrak{T}M$  and  $F \in C^\infty(M, N)$ . There may not exist  $W \in \mathfrak{T}M$  s.t.  $V, W$  are  $F$ -related. Even such  $W$  exists, it may not be unique.

However, if  $F$  is a diffeomorphism, given any  $V$ ,  $\exists$  unique  $W$  s.t.  $V$  and  $W$  are  $F$ -related. Actually,  $W_p = F_* V_{F^{-1}(p)}$ .

Such  $W$  is called **push forward** of  $V$  along  $F$ , denoted by  $F_* V$ , only defined when  $F$  is a diffeomorphism.

**Lemma 2.43.**  $\forall V \in \mathfrak{T}M, W \in \mathfrak{T}N, F \in C^\infty(M, N)$ . Then  $W$  is  $F$ -related to  $V$  iff  $\forall f \in C^\infty N, V(f \circ F) = W(f) \circ F \in C^\infty(M)$

*Proof.* Check that  $F_{p,*}(V_p)(f) = W_{F(p)}(f), \forall f \in C^\infty(N)$  □

**Proposition 2.44.** Given  $V_0, V_1 \in \mathfrak{T}M, W_0, W_1 \in \mathfrak{T}N, F \in C^\infty(M, N), W_i$  is  $F$ -related to  $V_i, i = 0, 1 \Rightarrow [W_0, W_1]$  is  $F$ -related to  $[V_0, V_1]$

*Proof.* □

**Corollary 2.45** (Naturality of Lie bracket). Given any  $F \in \text{Diff}(M, N), V \in \mathfrak{T}M, W \in \mathfrak{T}M$ , we have  $[F_* V, F_* W] = F_* [V, W]$

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