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Physics-0 Lecture Notes

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Chapter 1

Mechanics

1.1 Kinematics: Velocity, Acceleration, Inertial Frame

1.1.1 Point particle and reference frame

Physics is the subject that describes nature, and many phenomena in nature are ultimately boiled down to the motion of some objects. Therefore, a very important question in physics is "How do we describe the motion of something?" To answer this question, we need first to ask what is the "something" we would like to describe its motion. If something is a dog, then its motion could be very complicated. It might run straight along some direction or it might bite its own tail and spin around. To simplify our task, let us consider the motion of an idealized object called a *point particle*. Its defining feature is that it lacks spatial extension; hence, it has zero volume. The motion of a point particle is significantly simplified than the motion of a dog because it has no internal motion like spin; hence, we could describe the motion of a point particle by the change of its position with time.

Definition 1 (Point particle)**.** *A point particle is an idealization of particles in physics. It has no spatial extension, and its motion is completely specified by the time evolution of its position.*

On the other hand, if all parts of an object move in the same way, then it could be effectively treated as a point particle. For example, a small ball falling from a high altitude can be approximated by a point particle.

We are still not yet to answer the question. To describe or measure the motion of a point particle, we need to first decide to what the motion of a point particle is referenced. For simplicity, we could choose to describe the motion of a point particle A referenced to another point particle O. Next, we introduce a Cartesian coordinate system where the origin is chosen to be the position of the particle O, which is called a *reference frame* of the particle O. For example, consider the point particle A moving on a two-dimensional plane as shown in Figure 1.1. At time $t = 0$, the particle A starts at the coordinate $(x, y) = (-\frac{1}{2}, \frac{3}{2})$ and moves along the red curve and ends at the coordinate $(x, y) = (\frac{9}{2}, -\frac{1}{2})$ at time $t = t_*$.

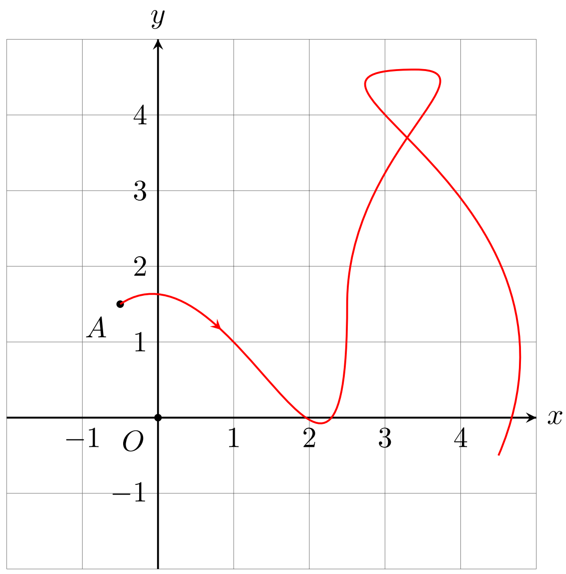

Figure 1.1: The motion of a point particle A in the reference frame of the point particle O.

The motion of the particle A can be described by the coordinates of the reference frame as functions of the time t, i.e. $(x(t), y(t))$. From time t_1 to t_2, the point particle A travels from (x_1, y_1) to (x_2, y_2). The *displacement* of the particle A at t_1 and t_2 is

$$(\Delta x, \Delta y) = (x_2 - x_1, y_2 - y_1)\,, \tag{1.1.1}$$

and the distance is given by

$$d = \sqrt{\Delta x^2 + \Delta y^2}\,. \tag{1.1.2}$$

If displacement of the point particle A from t_1 to t_2 is $(\Delta x, \Delta y)$ and from t_2 to t_3 is $(\Delta x', \Delta y')$, then the displacement from t_1 to t_3 is

$$(\Delta x + \Delta x', \Delta y + \Delta y')\,. \tag{1.1.3}$$

The coordinate (x, y) itself can also be understood as the displacement between the particle O and the particle A.

In the example we just studied, the motion of the point particle A confines on a two-dimensional plane, and we used the coordinates (x, y) to describe its motion. More generally, a particle moving in d dimensions can be described by the Cartesian coordinates $(x_1, x_2, \cdots, x_d)$. It is convenient to introduce the concept of a *vector*. A vector quantity is a physical quantity with both *magnitude and direction*.

Definition 2 (Vector)**.** *A d-dimensional vector $\vec{v}$ is a collection of d numbers as $\vec{v} = (v_1, v_2, \cdots, v_d)$ equipped with the operations: vector addition and scalar multiplication.*

- *Vector addition: Given two d-dimensional vectors $\vec{v} = (v_1, \cdots, v_d)$ and $\vec{v}' = (v_1', \cdots, v_d')$, the sum $\vec{v} + \vec{v}'$ is a d-dimensional vector defined by*

$$\vec{v} + \vec{v}' = (v_1 + v_1', \cdots, v_n + v_d')\,. \tag{1.1.4}$$

- *Scalar multiplication: Given a number a and a d-dimensional vector $\vec{v} = (v_1, \cdots, v_d)$, the product $a\vec{v}$ is defined by*

$$a\vec{v} = (av_1, \cdots, av_d)\,. \tag{1.1.5}$$

Let us look at some examples in two dimensions. The position of a particle is a vector $\vec{r} = (x, y)$. Using vector addition and scalar multiplication, we can write the displacement also as a vector as

$$\Delta\vec{r} = \vec{r}_2 + (-1)\vec{r}_1 = (x_2 - x_1, y_2 - y_1)\,. \tag{1.1.6}$$

The discussion around (1.1.2) says that the total displacement from t_1 to t_3 is the vector addition of the displacement from t_1 to t_2 and the displacement from t_2 to t_3.

Definition 3 (Inner product and norm)**.** *The inner product $\vec{u} \cdot \vec{v}$ of two n-dimensional vectors $\vec{u} = (u_1, \cdots, u_n)$ and $\vec{v} = (v_1, \cdots, v_n)$ is defined by*

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots u_d v_d\,. \tag{1.1.7}$$

The norm $|\vec{v}|$ of a vector $\vec{v}$ is defined by the square root of the inner product of $\vec{v}$ with itself, i.e.

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}\,. \tag{1.1.8}$$

The norm of a displacement gives the distance,

$$d = |\Delta\vec{r}|\,, \tag{1.1.9}$$

and (1.1.2) is an example in two dimensions.

So far, in our description of the motion of a point particle, we have ignored the units. We measure each physical quantity in its own units, by comparison with a standard, which corresponds to exactly 1 unit of the quantity. The unit for time is *second* with the symbol s and for length is *meter* with the symbol m. The corresponding standards are given by[1]

Definition 4 (Second)**.** *One second is the time taken by 9 192 631 770 oscillations of the light (of a specified wavelength) emitted by a cesium-133 atom.*

[1] The following two definitions are literally taken from [**Robert Resnick**; **David Halliday**; **Jearl Walker - Fundamentals of Physics**].

Definition 5 (Meter)**.** *The meter is the length of the path traveled by light in a vacuum during a time interval of 1/299 792 458 of a second.*

For example, when we said the time period is 135 s, we mean that during that time period, the light emitted by a cesium-133 atom has oscillated 135×9192631770 times. When we say that two points are 4.3 m apart, we mean that it takes $4.3 \times 1/299792458$ for the light to travel from one point to the other.

1.1.2 Velocity and acceleration

When people want to have some idea of the motion of something, the most common question people ask is "how fast does something move?". Velocity is one of the most important characteristics of the motion of a point particle. Consider a time interval from t to $t + \Delta t$. Suppose the particle A moves from $\vec{r}_1$ to $\vec{r}_2$, the *average velocity* of the particle A in the time interval $\Delta t = t_2 - t_1$ is

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{\Delta t} \,. \tag{1.1.10}$$

The unit of the average velocity is m/s. Given the average velocity, we can compute the displacement as

$$\Delta \vec{r} = \vec{v}_{\text{avg}} \Delta t \,, \tag{1.1.11}$$

and the distance as

$$d = |\vec{v}_{\text{avg}}| \Delta t \,. \tag{1.1.12}$$

We can see that the average velocity $\vec{v}_{\text{avg}}$ is associated with the time period Δt, and only depends on the initial and final positions of the particle. However, one usually would like to know the velocity of an object at a particular instance in time. We define the *instantaneous velocity* as given by the limit

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \,. \tag{1.1.13}$$

In terms of the components $\vec{v} = (v_x, v_y)$, we have

$$v_x(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \,, \quad v_y(t) = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} \,. \tag{1.1.14}$$

The operation we performed on the coordinate $x(t)$ in (1.1.13) to get the instantaneous velocity $v_x(t)$ is an example of a derivative. The derivative of a function $f(x)$ with respect to the variable x, denoted by $\frac{df(x)}{dx}$, is defined by the limit

$$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{f(x + \epsilon) - f(x)}{\epsilon} \,. \tag{1.1.15}$$

We have

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} \,. \tag{1.1.16}$$

We would not give the mathematical definitions for the limit "lim" in (1.1.13). Instead, let us try to understand the meaning of the instance velocity by looking at the following example. Consider a point particle moving along the x-direction, and we want to measure its instantaneous velocity at $t = 2.7\,\mathrm{s}$. To this end, we measure the position of this point particle at a sequence of time instances. The result of the measurements is listed in Table 1.1.

t (s)	2.7	2.7001	2.701	2.71	2.8
x (m)	0.968583	0.967734	0.959588	0.830246	−0.844328

Table 1.1: The data from measuring the positions of a point particle.

Now, we could approximate the instantaneous velocity by the average velocity of the particle in the time interval $[2.7\,\mathrm{s}, 2.8\,\mathrm{s}]$, and obtain[2]

$$v(2.7\,\mathrm{s}) \approx v_{\mathrm{avg}} = -18.129\,\mathrm{m/s}\,. \tag{1.1.17}$$

We can get better approximations of the instantaneous velocity by using smaller time intervals $[2.7\,\mathrm{s}, 2.71\,\mathrm{s}]$, $[2.7\,\mathrm{s}, 2.701\,\mathrm{s}]$, and $[2.7\,\mathrm{s}, 2.7001\,\mathrm{s}]$. The average velocities associated with these time intervals are

$$v(2.7\,\mathrm{s}) \approx v_{\mathrm{avg}} = \begin{cases} -13.834\,\mathrm{m/s} & \text{for time interval } [2.7\,\mathrm{s}, 2.71\,\mathrm{s}]\,, \\ -8.996\,\mathrm{m/s} & \text{for time interval } [2.7\,\mathrm{s}, 2.701\,\mathrm{s}]\,, \\ -8.494\,\mathrm{m/s} & \text{for time interval } [2.7\,\mathrm{s}, 2.7001\,\mathrm{s}]\,. \end{cases} \tag{1.1.18}$$

When we use smaller and smaller intervals, we obtain better and better approximations of the instantaneous velocity. The equation (1.1.13) means that the instantaneous velocity equals the average velocity associated with an interval of an infinitesimal length ϵ. Of course, the "interval of an infinitesimal length" could only be achieved ideally, and in practice, one could only obtain approximations of the instantaneous velocity, but when the interval becomes smaller and smaller, the approximation becomes better and better. In fact, the data in table 1.1 come from the particle trajectory $x(t) = \sin(2\pi t^2)$, and we have the instantaneous velocity $v(2.7\,\mathrm{s}) = -8.43785\,\mathrm{m/s}$.

We can also understand the process of getting a better and better approximation of the instantaneous velocity pictorially. In Figure 1.2, we draw the motion of a point particle in the t-x plane. The instantaneous velocity at $t = t_0$ can be computed by the limit

$$v(t_0) = \lim_{n\to\infty} \frac{x_n - x_0}{t_n - t_0}\,, \tag{1.1.19}$$

where t_n approaches t_0 for n going to infinity. From the figure, we can also see that the instantaneous velocity is the slope of the trajectory on the t-x plane.

[2] We drop the subscript x when considering particles moving in one dimension.

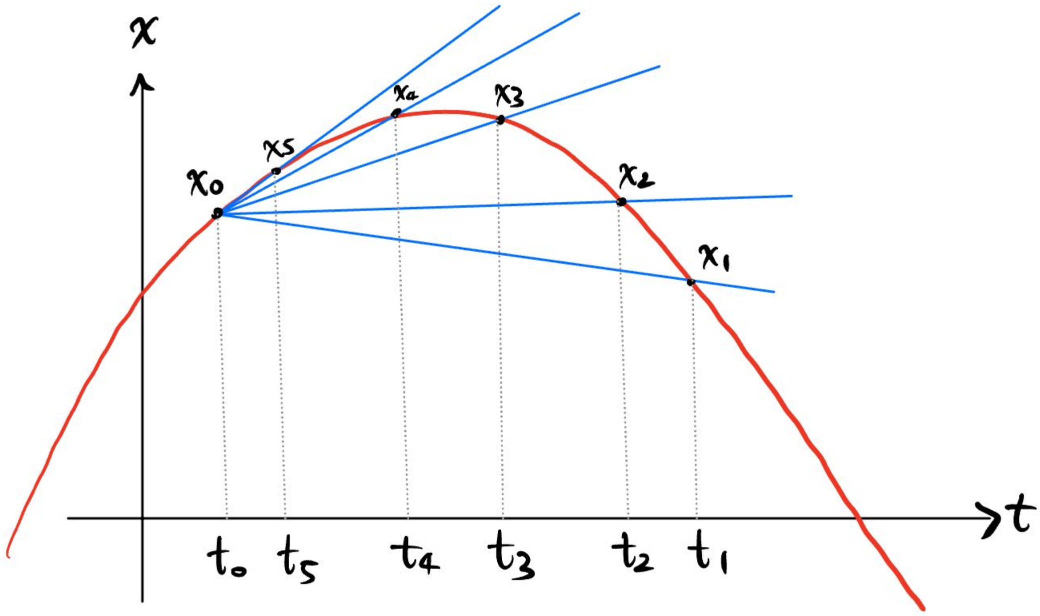

Figure 1.2: The trajectory of a point particle on the t-x plane.

Given the instantaneous velocity $v(t)$, one could integrate it to get the displacement of the particle,

$$\Delta\vec{r}(t_2, t_1) = \vec{r}(t_2) - \vec{r}(t_1) = \int_{t_1}^{t_2} \vec{v}(t)dt\,. \tag{1.1.20}$$

In a very similar fashion, we define the instantaneous acceleration as

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}\,. \tag{1.1.21}$$

1.1.3 Inertial frame and Galilean transformation

In Section 1.1.1, we learned that to describe the motion of a point particle A we need first to pick a reference frame, and we chose the reference frame associated with the point particle O. (O is always at the origin.) Our description of the motion of A heavily depends on the reference frame, namely the motion of O. If we instead choose a reference frame associated with a different point particle B, then our description of the motion of A would be completely different. This is of course not satisfactory. Hence, we would like to have a more universal way to describe the motion of A. This lead to the following two definitions.

Definition 6 (Free particle)**.** *A free particle is a point particle that receives no influence from any other objects.*

Definition 7 (Inertial frame)**.** *An inertial frame is a frame that is referenced to a free particle.*

Let us try to understand the above definitions using some analogies. Say you have a case in court. In a normal situation, you would prefer the jury of your case to be just, i.e. not influenced by any other people outside the jury. The point particle of our reference frame is like the jury, and a free particle is like a just jury that we are seeking to have. Therefore, we would like always to use inertial frames to describe the motion of objects.

Inertial frames are not unique. Every free particle defines an inertial frame. We would like to have a way to compare our descriptions of the same motion but using different inertial frames. Consider two free particles O, O', and a point particle A. Both reference frames of O and O' are inertial frames. As we will see in the next section, Newton's first law implies that a free particle in an inertial frame has a constant velocity. Let us denote the constant velocity of O' in the inertial frame of O by $\vec{v}$.

The motion of the point particle A in the inertial frame of O is given by the position vector $\vec{r}$, and in the inertial frame of O' is given by the position vector $\vec{r}'$. The position vectors $\vec{r}$ and $\vec{r}'$ are related by

$$\vec{r}' = \vec{r} - \vec{v}t\,. \tag{1.1.22}$$

The above relation between the coordinates of the two inertial frames is called the *Galilean transformation.*

There are other more basic coordinate transformations. First, when the two free particles O and O' are separated by a displacement vector $\vec{d}$ but with no relative velocity, the position vectors $\vec{r}$ and $\vec{r}'$ are related by

$$\vec{r}' = \vec{r} + \vec{d}\,. \tag{1.1.23}$$

When the clocks of the two inertial frames differ by time s, the time coordinates t and t' are related by

$$t' = t + s\,. \tag{1.1.24}$$

The transformation from $(t, \vec{r})$ to $(t', \vec{r}')$ is called a *translation.* Next, we can consider the transformation that keeps the inertial frame of O but rotates the coordinates, i.e.

$$\vec{r}' = R \cdot \vec{r}\,, \tag{1.1.25}$$

where R is an orthogonal matrix ($RR^T = I$). This transformation is called a *rotation.* The *Galilean group* is the group that contains the compositions of Galilean transformation, translations, and rotations.

Exercise (Galilean group)**.**

1. *Work out the transformation rule of a generic element in the Galilean group.*
2. *Derive the composition rule of two generic elements.*
3. *Show that the Galilean group is a group, i.e. it obeys all the axioms of a group.*

1.1.4 Linear Motions, Circular Motion, Parabolic Motion

We will introduce three simple motions of a point particle. We will use Newton's notation for differentiation,

$$\dot{x}(t) = \frac{dx(t)}{dt}\,, \quad \ddot{x}(t) = \frac{d^2x(t)}{dt^2}\,. \tag{1.1.26}$$

Linear motions: A linear motion is a motion of a point particle along a straight line. Let x be the coordinate along the straight line. A linear motion can be described by the function $x(t)$. Let us give two examples of linear motions.

1. **Constant velocity:** The linear motion of a particle with a constant velocity v is given by

   $$x(t) = x_0 + vt\,. \tag{1.1.27}$$

   We compute the instantaneous velocity and acceleration by taking the first and second derivatives

   $$\dot{x}(t) = v\,, \quad \ddot{x}(t) = 0\,. \tag{1.1.28}$$

   We have verified that the particle is moving at a constant velocity v without acceleration.

2. **Constant acceleration:** All objects near Earth's surface when neglecting the contact or non-contact effects from the air or other objects (except Earth) move downwards with a constant acceleration, called the *free-fall acceleration*, denoted by g. We will learn in later sections that such acceleration is caused by the gravitational attraction force between the object and Earth. For now, we just give its value

   $$g = 9.8\,\mathrm{m/s}^2\,. \tag{1.1.29}$$

   The motion of a small ball with a free-fall acceleration is given by[3]

   $$y(t) = y_0 + vt - \frac{1}{2}gt^2\,. \tag{1.1.30}$$

   The first and second derivatives are

   $$\dot{y}(t) = v - gt\,, \quad \ddot{y}(t) = -g\,. \tag{1.1.31}$$

   We have verified that the particle is moving at a constant acceleration $-g$, and the minus sign is because the acceleration is pointing downward.

   Let us look at the position and velocity of the particle at three different important time instances. First, at the time $t = 0$, the position and velocity of the ball are $y = y_0$ and $\dot{y} = v$. We assume that the velocity v is positive. At $t = \frac{v}{g}$, the ball reaches the maximum height $y = y_0 + \frac{v^2}{2g}$ and has zero velocity. At $t = \frac{v}{g}$, the ball is back to the starting height $y = y_0$

[3] We changed our coordinate from x to y for the vertical direction.

with an opposite velocity $\dot{y} = -v$. The trajectory of the ball on the t-x plane is drawn in Figure 1.3. It is a parabola.

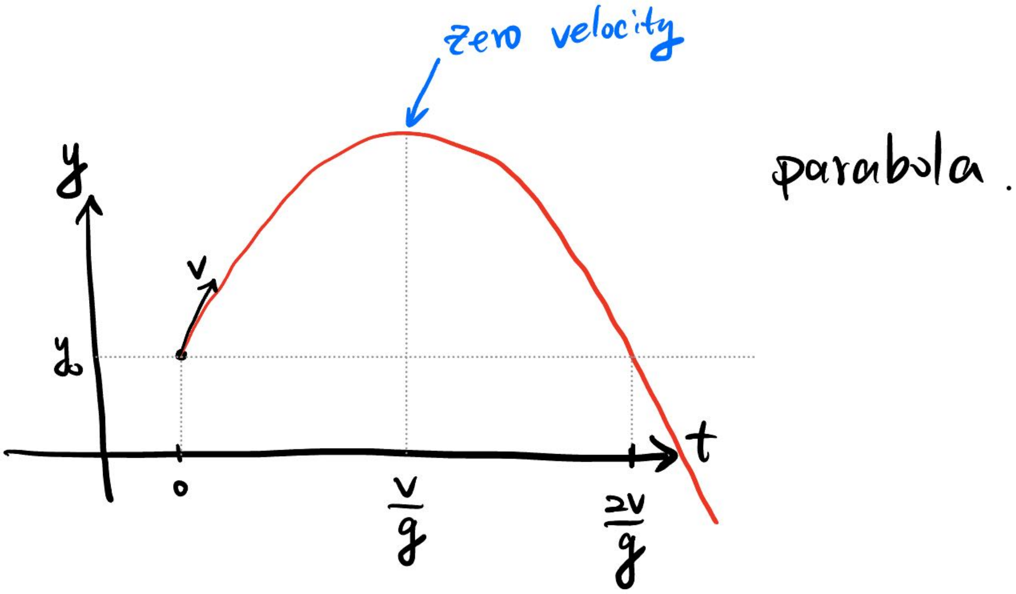

Figure 1.3: Trajectory of a free-falling object near Earth's surface on the t-x plane.

Circular motion: A point particle moving along a circle in two dimensions is called a circular motion, as shown in Figure 1.4. To describe a circular motion, it is convenient to change from the Cartesian coordinates (x, y) to the polar coordinate (r, θ) with the coordinate transformation:

$$x = r\cos\theta\,, \quad y = r\sin\theta\,. \tag{1.1.32}$$

The inverse transformation is

$$r = \sqrt{x^2 + y^2} = |\vec{r}|\,, \quad \theta = \arctan\frac{y}{x}\,. \tag{1.1.33}$$

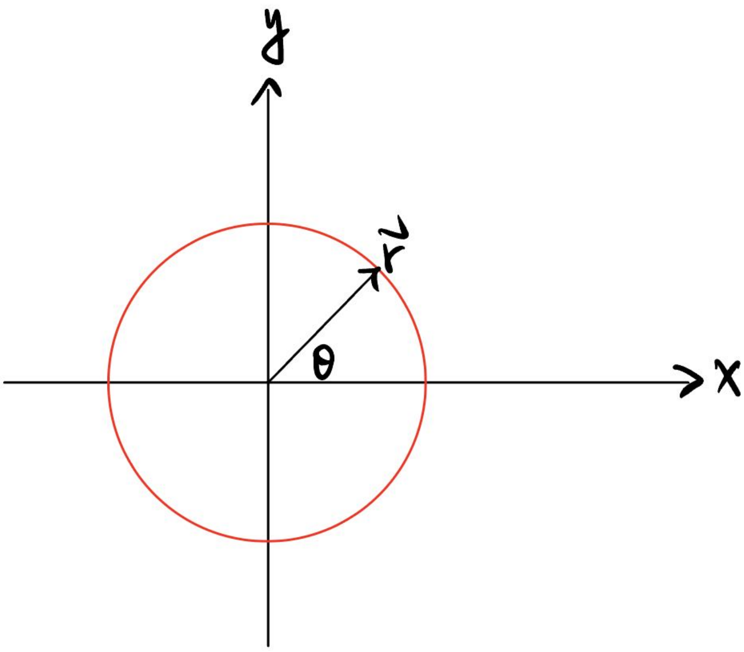

Figure 1.4: The particle is confined on a circle for the circular motion.

For circular motions, the radius r is a constant, and the angle θ is a function of the time t, i.e. the function $\theta(t)$ describes a circular motion. The first derivative of θ is called the *angular velocity*,

$$\frac{d\theta(t)}{dt} = \omega(t)\,. \tag{1.1.34}$$

Let us consider a circular motion with a constant angular velocity ω, described by

$$\theta(t) = \theta_0 + \omega t\,. \tag{1.1.35}$$

Using the coordinate transformation (1.1.32), we find

$$\vec{r} = (r\cos(\omega t), r\sin(\omega t))\,. \tag{1.1.36}$$

The velocity and acceleration are

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} = (-r\omega\sin(\omega t), r\omega\cos(\omega t))\,, \\ \vec{a} &= \frac{d^2\vec{r}}{dt^2} = (-r\omega^2\cos(\omega t), -r\omega^2\sin(\omega t))\,. \end{aligned} \tag{1.1.37}$$

We see that the acceleration vector $\vec{a}$ is proportional to the coordinate $\vec{r}$, and pointing towards the origin, i.e.

$$\vec{a} = -\omega^2\vec{r}. \tag{1.1.38}$$

This acceleration is called *centripetal acceleration*, which is a very important characteristic of circular motions. Taking the norms of the velocity and the acceleration, we find

$$|\vec{v}| = r\omega\,, \quad |\vec{a}| = r\omega^2\,, \tag{1.1.39}$$

which gives the relation

$$|\vec{a}| = \frac{|\vec{v}|^2}{r}\,. \tag{1.1.40}$$

The period of the circular motion is

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{|\vec{v}|}\,. \tag{1.1.41}$$

Parabolic Motion: Consider a point particle near Earth's surface. It has free-fall acceleration in the vertical direction and no acceleration in the horizontal direction. However, the motion along the horizontal direction is not completely trivial, since there could be a non-zero constant horizontal velocity. The most general such motion is described by

$$x(t) = x_0 + v_x t\,, \quad y(t) = y_0 + v_y t - \frac{1}{2}gt^2\,. \tag{1.1.42}$$

The first and second derivatives are

$$\begin{aligned} \dot{x}(t) &= v_x\,, & \dot{y}(t) &= v_y - gt\,, \\ \ddot{x}(t) &= 0\,, & \ddot{y}(t) &= -g\,. \end{aligned} \tag{1.1.43}$$

We have verified that there is a constant horizontal velocity and a constant vertical acceleration. We can eliminate the time t from the equations (1.1.42) and find

$$y - y_0 = \frac{v_y}{v_x}(x - x_0) - \frac{1}{2}g\left(\frac{x - x_0}{v_x}\right)^2\,, \tag{1.1.44}$$

which is a parabola on the two-plane.

Let us again look at some special time instances. First, at $t = 0$, the initial position and velocity are $(x, y) = (x_0, y_0)$ and $(\dot{x}, \dot{y}) = (v_x, v_y)$. At $t = \frac{v_y}{g}$, the ball reaches the maximum height at position $(x, y) = (x_0 + \frac{v_x v_y}{g}, y_0 + \frac{v_y^2}{2g})$, and the velocity is pointing in the x-direction as $(\dot{x}, \dot{y}) = (v_x, 0)$. At $t = \frac{2v_y}{g}$, the ball is back to the initial position $(x, y) = (x_0, y_0)$ with the velocity $(v_x, -v_y)$.