Tsinghua University

Numerical Analysis

Fall 2024

Homework 1

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- **Collaborators:** I finish this homework by myself.

Problem 1. It is exactly Theorem 2.4.7

Let $T_n(x) = \cos(n \arccos(X))$ be the Chebyshev polynomial of degree n.

Then

$$L_n f(x) = \frac{1}{n^2} T_n^2(x) \sum_{i=1}^{n} f(x_i) \frac{1 - x x_i}{(x - x_i)^2}$$

which impilies that $L_n f$ is a monotone linear operator.

By the monotone linear operator theorem, it suffices to prove it for $f = 1, \varphi_t$.

Since $L_n 1 = 1$,

$$|L_n \varphi_t(t)| = \left| \frac{1}{n^2} T_n^2(t) \sum_{i=1}^{n} (t - x_i)^2 \frac{1 - t x_i}{(t - x_i)^2} \right| \leq \frac{1}{n^2} \cdot 2n = \frac{2}{n} \to 0$$

Problem 2. By the result in lecture last term, it suffices to prove

$$\min_{p_k \in P_k} \max_{\lambda_1 \leq \lambda \leq \lambda_n} \leq 2 \left(\frac{\sqrt{\lambda_n} - \sqrt{\lambda_1}}{\sqrt{\lambda_n} + \sqrt{\lambda_1}} \right)^k$$

For $p_0(x) = \dfrac{T_k(\frac{\lambda_n + \lambda_1 - 2x}{\lambda_n - \lambda_1})}{T_k(\frac{\lambda_n + \lambda_1}{\lambda_n - \lambda_1})}$, note that $x_k = \dfrac{\lambda_n(1 - \cos\frac{n\pi}{k}) + \lambda_1(1 + \cos\frac{n\pi}{k})}{2}$ takes its optimal value. By Chebyshev theorem, $p_0(x)$ is the unique solution of the optimal solution, with the minimal value,

$$\frac{1}{T_k(\frac{\lambda_n + \lambda_1}{\lambda_n - \lambda_1})}$$

Moreover, $T_k(x) = \frac{1}{2}((x + \sqrt{x^1 - 1})^n + (x - \sqrt{x^2 - 1})^n)$, hence

$$\begin{aligned} T_k(\frac{\lambda_n + \lambda_1}{\lambda_n - \lambda_1}) &= \frac{1}{2}((\frac{\sqrt{\lambda_n} - \sqrt{\lambda_1}}{\sqrt{\lambda_n} + \sqrt{\lambda_1}})^k + (\frac{\sqrt{\lambda_n} + \sqrt{\lambda_1}}{\sqrt{\lambda_n} - \sqrt{\lambda_1}})^k) \\ &\geq \frac{1}{2}(\frac{\sqrt{\lambda_n} + \sqrt{\lambda_1}}{\sqrt{\lambda_n} - \sqrt{\lambda_1}})^k \end{aligned}$$

Here we end the proof.

Problem 3. That's because, $\bar{Q}_i(x)$, $0 \leq i \leq n$ is an orthogonormal basis of polynomial space of degree n, $\mathbb{P}_n$. So

$$p(x) = \sum_{i=0}^{n} \left\langle p, \bar{Q}_i \right\rangle \bar{Q}_i(x) = \int_a^b p(t) \bar{Q}_i(t) \omega(t) \, \mathrm{d}t \bar{Q}_i(x) = \int_a^b p(t) K_n(t, x) \omega(t) \, \mathrm{d}t$$

Problem 4. If $\lim\limits_{n\to\infty}\|L_nf-f\|_\infty=0$, obviously it holds for $f=1,\sin x,\cos x$.

Conversely, if it holds for $f=1,\sin x,\cos x$, then for any $f\in \mathrm{Span}\{1,\sin x,\cos x\}$, $\lim\limits_{n\to\infty}\|L_nf-f\|_\infty=0$. Then let $\varphi_t(x)=1-\cos(x-t)=1-\cos x\cos t-\sin t\sin x$

$$\begin{aligned}|(L_n\varphi_t)(t)|&=|(L_n\cdot 1)(t)-\sin t(L_n\sin x)(t)-\cos t(L_n\cos x)(t)-(1-\cos(t-t))|\\&\le\|(L_n\cdot 1)-1\|_\infty+|\sin t|\cdot\|(L_n\sin x)-\sin x\|_\infty+|cost|\cdot\|(L_n\cos x)-\cos x\|_\infty\\&\rightrightarrows 0\end{aligned}$$

uniformly converges.

Now $\forall\epsilon>0$, $\exists\delta>0$ such that $\forall|x-y|\le\arcsin\delta$, $|f(x)-f(y)|<\epsilon$.

Then if $|x-t|<\arccos\delta$, $|f(x)-f(t)|<\delta$.

If $|x-t|\ge\arccos\delta$, then $|f(x)-f(t)|\le 2\|f\|_\infty\le 2\|f\|_\infty\cdot\frac{\varphi_t(x)}{1-\delta}\le\alpha\varphi_t(x)$ with $\alpha=\dfrac{2\|f\|_\infty}{1-\delta}$.

Then $\forall x\in[-\pi,\pi]$,

$$-(\epsilon\cdot 1+\alpha\varphi_t(x))\le f(x)-f(t)\le\epsilon\cdot 1+\alpha\varphi_t(x)$$

By monotonity,

$$-(\epsilon\cdot(L_n1)+\alpha(L_n\varphi_t)(t))\le(L_nf)(t)-f(t)(L_n\cdot 1)(t)\le\epsilon\cdot(L_n1)+\alpha(L_n\varphi_t)(t)$$

Therefore

$$\|L_nf-(L_n\cdot 1)f\|_\infty\le\epsilon\|L_n1\|+\alpha\|L_n\varphi_t(t)\|$$

Since $L_n\varphi_t(t)\rightrightarrows 0$ for all t, letting $n\to\infty,\epsilon\to 0$, we have

$$\lim_{x\to\infty}\|L_nf-(L_n\cdot 1)f\|_\infty=0$$

Noticed that $\lim\limits_{n\to\infty}\|L_n1-1\|_\infty=0$, this causes

$$\lim_{n\to\infty}\|L_nf-f\|_\infty=0$$

Problem 5. $s_n\sin x=\sin x,s_n\cos x=\cos x$. So $G_n\sin x=\sin x,G_n\cos x=\cos x,G_n\cdot 1=1$ remains. By problem 4,

$$\lim_{n\to\infty}\|G_nf-f\|_\infty=0$$