Complex Analysis

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2 Complex Functions

2.1 Analytic functions and rational functions

2.1.1 Harmonic function

Definition 2.1 (Cauchy-Riemann equation).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 2.2 (Harmonic function). A function u is **harmonic** if it satisfied **Laplace equation** $\triangle u = 0$.

If two harmonic function u and v satisfies Cauchy-Riemann equations, then we say that v is **conjugate harmonic function of** $u \Rightarrow u$ is conjugate harmonic of $-v$.

2.1.2 Polynomials and rational function

The polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ is analytic in $\mathbb{C}$.

We will prove the fundamental theorem of algebra

Theorem 2.3 (Fundamental Theorem of Algebra). *Every polynomial with degree $n > 0$ has at least one point.*

Theorem 2.4 (Gauss-Lucus theorem). *The smallest convex polygon that contain the zeros of P also contains the zeros of P'.*

Proof. Only need to check.

We can get this equation.

$$\frac{P'(\alpha)}{P(\alpha)} = \sum_{j=1}^{n} \frac{1}{\alpha - \alpha_j} = 0 \Rightarrow \sum_{j=1}^{n} \frac{\overline{\alpha - \alpha_j}}{|\alpha - \alpha_j|^2}$$

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Hence α is linearly represented by α_j. $\qquad\square$

Proposition 2.5. *Let P and Q be two polynomial with no common zeros. Then the rational function $R(z) = \dfrac{P(z)}{Q(z)}$ is analytic away from the zeros of Q.*

*The zeros of Q are called **poles** of R, and the **order of a pole** is equal to the order of the corresponding zero of Q.*

We often view R as a function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. $R_1(z) := R(\frac{1}{z})$.

If $R_1(0) = 0$, the order of the zero at ∞ (of R) is the order of the zero of $R_1(z)$ at $z = 0$.

If $R_1(0) = \infty$, the order of the pole at ∞ (of R) is the order of the pole of $R_1(z)$ at $z = 0$.

Suppose

$$R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, a_n \neq 0, b_m \neq 0$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$$

By discussing m and n, we can infer the situation of $R(z)$ at ∞.

By adding the order of poles and zeros at ∞, we can get the following theorem.

Theorem 2.6. *The total number of zeros and poles of a rational function are the same.*

Remark 2.7. This common number is called the **order of the rational function**.

Corollary 2.8. *Suppose a rational function R has order p. Then every equation $R(z) = a$ has exactly p roots.*

Proof. $\hat{R}(z) = R(z) - a$ has the same poles as R. $\qquad\square$

A rational function of order 1 is a **linear fraction** $R(z) = \frac{az+b}{cz+d}, ad - bc \neq 0$

Such fraction is often called **Möbius transformation**

Every rational function has a representation by **partial fractions**.

- If R has a pole at ∞. Then we can write

$$R(z) = G(z) + H(z) \qquad (*)$$

  where G is a polynomial without constant term, and H is finite at ∞.

  The degree of G is the order of the pole of R at ∞. G is called the **singular part** of R at ∞.

- Let the distinct finite poles of R be $\beta_1, \cdots, \beta_k$. Let $R_j(\psi) = R(\beta_j + \frac{1}{\psi})$. Then R_j is a rational function with a pole at ∞. As in $(*)$, we can write

$$R_j = G_j + H_j$$

  with H_j finite at ∞. Then

$$R(z) = G_j\left(\frac{1}{z - \beta_j}\right) + H\left(\frac{1}{z - \beta_j}\right)$$

  with G_j is a polynomial in $\frac{1}{z-\beta_j}$ without constant term called the **singular point** of R at β_j.

- Let $F(z) = R(z) - G(z) - \sum\limits_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right)$.

  Then F is a rational function which can only have poles among β_j, ∞

  Since by our construction, F is finite at every $\beta_j, 1 \leqslant j \leqslant k$ and ∞.

  So F is a constant.

  In particular, $R(z) = G(z) + \sum\limits_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right) + c$.

2.2 Power Series

2.2.1 Power series

Theorem 2.9 (Abel's theorem). *If $\sum a_n$ converges, then $f(z) = \sum a_n z^n \to f(1)$ as $z \to 1$ in such a way that $\dfrac{|1 - z|}{1 - |z|}$ remains bounded.*

2.3 Exponential, Trigonometric and logarithmic functions

2.3.1 Exponential and Trigonometric function

The **exponential function** is defined as the solution if the differential equation

$$
\begin{cases}
f'(z) = f(z) \\
f(0) = 1
\end{cases}
$$

We denote $e^z = \exp z = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$.

The **trigonometric function** are defined by

$$
\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}
$$

2.3.2 Logarithmic Functions

The **logarithmic function** ln is defined by $z = \ln w$ is a root of the equation $e^z = w$.

For $w \neq 0$, we write $z = x + iy$, then

$$
e^{x+iy} = w \Leftrightarrow
\begin{cases}
e^x = |w| \\
e^{iy} = \dfrac{w}{|w|}
\end{cases}
$$

The first equation has a unique solution $x = \ln |w|$.

The second equation $e^{iy} = \dfrac{w}{|w|}$ has a unique solution $y_0 \in [0, 2\pi)$.

If we write $w = re^{i\theta}$, then $x = \ln w, y = \theta = \arg w$.

Thus, for $w \neq 0$, we have

$$\ln w = \ln |w| + i \arg w$$

The function $\ln$ is actually not single-valued. But we can define a single-valued function Ln

We define

$$a^b = \exp(b \ln a)$$

We will prove Ln is analytic in $\mathbb{C} - (-\infty, 0]$ but not continuous in $(-\infty, 0]$.

Ln is the principal branch of the logithm.

3   Conformal Mappings

3.1   Basic topology

3.1.1   Connectedness

Theorem 3.1. *A nonempty open set in $\mathbb{C}$ is connected iff any two of its points can be joined by a polygon which lies in the set, i.e. Connectedness is equivalent to Path Connectedness*

An nonempty connected subset is called a **region**

3.1.2 Compactness

Definition 3.2. A set X is **totally bounded** if $\forall \varepsilon > 0$, X can be covered by finitely many balls of radius ε

Theorem 3.3. *A set is compact iff it is complete and totally bounded.*

Theorem 3.4. *A subset $X \subset$ is compact iff every infinite sequence of X has a limit point in X.*

3.1.3 Continuous Functions

Theorem 3.5. *Continous function maps connected space to connected space.*

Theorem 3.6. *Continous function maps compact space to compact space.*

3.2 Conformality, geometric consequences of the existence of a derivative

3.2.1 Arcs and closed curves

The equation of an **arc** r in $\mathbb{C}$ can be represented by one of the terms

- $x = x(t), y = y(t), \alpha \leqslant t \leqslant \beta, x, y$ are continuous at t

- $z(t) = x(t) + iy(t), \alpha \leqslant t \leqslant \beta$.

- The continuous mapping $\gamma : [\alpha, \beta] \to \mathbb{C}$.

For a non-decreasing function $\varphi : [\alpha, \beta] \to [\alpha, \beta]$, $z = z(\varphi(t)), \alpha' \leqslant \tau \leqslant \beta'$ is **change of parameter** of $z(t)$.

The change is **reversible** iff φ is strictly increasing.

If γ is differentiable, then call γ a **curve**.

γ is **simple** , or a **Jordan curve**, if γ is injective.

γ is **closed curve** if $\gamma(0) = \gamma(1)$.

3.2.2 Analytic Functions in Regions

A function f is analytic on an arbitrary set A if it is the restriction to A of a function which is analytic in some open set containing A.

Theorem 3.7. *An analytic function in a region(i.e. open and connected) Ω whose derivative is 0 must reduce to a constant. The same hold if the real part, the imaginary part, the modulus, or the argument is constant.*

3.2.3 Conformal Mappings

Suppose $f : \Omega \to \mathbb{C}$ is analytic in Ω. $r_1 = z_1(t), r_2 = z_2(t), \alpha \leqslant t \leqslant \beta$.
$z_0 = z_1(t_0) = z_2(t_0'), z_1'(t_0) \neq 0, z_2'(\hat{t}_0) \neq 0, \alpha < t_0, \hat{t}_0 < \beta$.
$f'(z_0) \neq 0, w_1(t) = f(z_1(t_0)), w_2 = f(z_2((t_0)))$
$\Gamma_1 = \{w_1(t) | \alpha \leqslant t \leqslant \beta\}, \Gamma_2 = \{w_2(t) | \alpha \leqslant t \leqslant \beta\}$

Then

$$w_1'(t) = f'(z_1(t))z_1'(t)$$
$$w_2'(t) = f'(z_2(\hat{t}))z_2'(\hat{t})$$

$\Rightarrow$

$$w_1'(t_0) \neq 0, w_2'(t_0) \neq 0$$
$$\arg w_1'(t_0) = \arg f'(z_1(t_0))z_1'(t_0)$$
$$\arg w_2'(t_0) = \arg f'(z_2(\hat{t}_0))z_2'(\hat{t}_0)$$

So the "angle" $\arg w_1'(t_0) - \arg w_2'(\hat{t}_0 = \arg z_1(t_0) - \arg z_2(\hat{t}_0)$ remains the same. Now we give the definition.

Definition 3.8. $w = f(z)$ is said to be **conformal** in Ω if f is analytic in Ω and $f'(z) \neq 0$ for $\forall z \in \Omega$.

Easy to prove that linear change of scale at z_0 is independent of the direction.
i.e. $|f'(z_0)| = \lim\limits_{z \to z_0} \frac{\delta\sigma}{\delta s}$

3.2.4 Length and Area

The **length** of a differentiable arc γ with the equation $z(t) = x(t) + iy(t)$, $a \leqslant t \leqslant b$

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t = \int_a^b |z'(t)|\mathrm{d}t$$

For $\Gamma = f(\gamma)$ where f conformal mapping.
Then

$$L(\Gamma) = \int_a^b |f'(z(t))| \cdot |z'(t)|\mathrm{d}t$$

The **area** of $E \subset \mathbb{R}$ is $A(E) = \int\int_E \mathrm{d}x\mathrm{d}y$

Then by the differentiable functional transformation, the area $\hat{E} = f(E)$ is

$$A(\hat{E}) = \int\int_E |u_x v_y - u_y v_x|\mathrm{d}x\mathrm{d}y$$

If f is the conformal mapping of an open set containing E, then by Caucht-Riemann equation

$$A(\hat{E}) = \int\int_E |f'(z)|^2\mathrm{d}x\mathrm{d}y$$

3.3 Möbius Transformation

Recall that a **Möbius transformation** is a function of the form

$$w = s(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Then it has an inverse $z = S^{-1}(w) = \dfrac{dw - b}{-cw + a}$.

We may define $S(\infty) = \lim\limits_{z \to \infty} S(z) = \frac{a}{c}$, $S(\frac{-d}{c}) = \infty$

With these definition, $S : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a topological mapping. Here one may use the chordal metric to define the topology.

$$S'(z) = \frac{ad - bc}{(cz + d)^2}$$

Then S is conformal in $\hat{\mathbb{C}} - \{-\frac{d}{c}, \infty\}$.

$w = z + \alpha$ is called a **parallel translation**.

$\quad$ $w = kz$ with $|k| = 1$ is a **rotation**.

$\quad$ $w = kz$ with $k > 0$ is a **homothetic transformation**.

$\quad$ $x = \frac{1}{z}$ is called an **inversion**.

Proposition 3.9. *Every Möbius transformation is a composition of the above four operations.*

3.3.1 Cross ratio

For three distinct points $z_2, z_3, z_4 \in \hat{\mathbb{C}}$,we can find a Möbius transformation S such that $S(z_2) = 0, S(z_3) = 1, S(z_4) = \infty$.

Lemma 3.10. *The Möbius transformation satisfying the above conditions is unique.*

The **cross ratio** (z_1, z_2, z_3, z_4) is the image z_1 under the Möbius transformation which maps z_2 to 1, z_3 to 0 and z_4 to ∞.

Theorem 3.11. *If $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ are distinct, and T is any Möbius transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Lemma 3.12. *Let T be a Möbius transformation, $T(\mathbb{R})$ is either a circle or a straight line.*

Theorem 3.13. *The cross ratio (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle or a straight line.*

Remark 3.14. One may prove the theorem by elementary geometry

Theorem 3.15. *A Möbius transformation maps circles into circles.*

3.3.2 Symmetry

Suppose T is a Möbius transformation which maps $\hat{\mathbb{R}}$ onto a circle C. We say that $w = Tz$ and $w^* = T\bar{z}$ are **symmetric** w.r.t. C.

Remark 3.16. This definition is independent of T. Suppose S is another Möbius transformation which maps $\hat{\mathbb{R}}$ onto C, then $S^{-1}T$ maps $\hat{\mathbb{R}}$ to $\hat{\mathbb{R}}$, and this $S^{-1}w = S^{-1}Tz$ and $S^{-1}w^* = S^{-1}T\bar{z}$ are conjugate.

The points z and z^* are **symmetric w.r.t C through** z_1, z_2, z_3 iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.

This can be another definition.

Note that only the points on C are symmetric to themselves.

The mapping $z \mapsto z^*$ is 1-1 and is called **reflection** w.r.t. C.

Geometric Meaning of Symmetry

Case1: C is a straight line. We may assume $z_3 = \infty$.

z, z^* are symmetric w.r.t. C if and only if

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Then

$$|z^* - z_2| = |z - z_2|, \quad \forall z_2 \in C \text{ and } z_2 \neq \infty$$

$$\text{Im}\,\frac{z^* - z_2}{z_1 - z_2} = \text{Im}\,\frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

So C is the bisecting normal of the segment between z and z^*.

Case2: C is the circle $|z - a| = R$.

Then for $\forall$ distinct $z_1, z_2, z_3 \in \mathbb{C}$, $\overline{(z, z_1, z_2, z_3)} = \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)}$

$= (\bar{z} - \bar{a}, \bar{z}_1 - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}) = (\bar{z} - \bar{a}, \dfrac{R^2}{z_1 - a}, \dfrac{R^2}{z_2 - a}, \dfrac{R^2}{z_3 - a}) = (\dfrac{R^2}{\bar{z} - \bar{a}}, z_1 - a, z_2 -$

$a, z_3 - a)$

$= (\dfrac{R^2}{\bar{z} - \bar{a}}, z_1, z_2, z_3).$

Then the symmetric point of z w.r.t. C is

$$z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$$

or

$$(z^* - a)(\bar{z} - \bar{a}) = R^2$$

$\Rightarrow$

$$\begin{cases} |z^* - a| \cdot |z - a| = R^2 \\ \dfrac{z^* - a}{z - a} = \dfrac{(z^* - a)(\bar{z} - \bar{a})}{|z - a|^2} > 0 \end{cases}$$

Theorem 3.17 (The Symmetric principle). *If a Möbius transformation maps a circle C_1 onto a circle C_2, then it transforms any pair of symmetric points w.r.t. C_1 into a pair of symmetric points w.r.t. C_2.*

Proof. Case1: $C_1 = \hat{\mathbb{R}}$. Let T be the Möbius transformation which maps $\hat{\mathbb{R}}$ onto C_2. $\forall z \in \mathbb{C}$, by definition, $w = Tz$ and $w^* = T\bar{z}$ are symmetric w.r.t. C_2.

Case2: C_1 is a general circle. Let $T : C_1 \to C_2$ and $S : \mathbb{R} \to C_2$ be Möbius transformation.

Suppose w, w^* are symmetric w.r.t. C_1. Then there exists z s.t. $w = Sz, w^* = S\bar{z}$.

Then we can find $Tw = TSz, Tw^* = TS\bar{z}$ are symmetric w.r.t. C_2 since $TS : \hat{\mathbb{R}} \to C_2$ $\qquad\qquad\square$

Remark 3.18. (1). The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w, z_2 \mapsto w_2, z_3 \mapsto w_3$ where $z_1, z_2, z_3 \in C_1$, $w_1, w_2, w_3 \in C_2$ is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3,)$$

(2). The Möbius transformation from C_1 to C_2 satisfies $z_1 \mapsto w_1$, $z_2 \mapsto w_2$ where $z_1 \in C_1, z_2 \notin C_1, w_1 \in C_2, w_2 \notin C_2$ is given by

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

3.3.3 Steiner Circles, circular net

For $S(z) = \dfrac{az + b}{cz + d}$, $S'(z) = \dfrac{ad - bc}{(cz + d)^2}$.

A point $z \notin$ a circle C is said to on the **right(left, resp.)** of C if $\mathrm{Im}(z, z_1, z_2, z_3) > 0 (\mathrm{Im}(z, z_1, z_2, z_3) < 0)$

Remark 3.19.

(1). This agrees with everyday use since $(i, 1, 0, \infty) = i$

(2). This distinct between left and right is the same for all triples, while the meaning may be reversed.

(If $C = \hat{\mathbb{R}}$, then $(z, z_1, z_2, z_3) = \dfrac{az + b}{cz + d}$ with $a, b, c, d \in \mathbb{R} \Rightarrow \mathrm{Im}(z, z_1, z_2, z_3) = \dfrac{ad - bc}{|cz + d|^2} \mathrm{Im}(z)$)

(3). We can define an absolute positive orientation of all finite circles by requiring that ∞ should be lie to the right of the oriented circles.

Consider a Möbius transformation of the form

$$w = k \cdot \dfrac{z - a}{z - b}$$

Here, $z = a \mapsto w = 0, z = b \mapsto w = \infty$.

Then circles through a, b maps to straight line through $0, \infty$.

The concentric circle about the origin, $|w| = \rho$, correspond to circles with the equation

$$\left| \frac{z - a}{z - b} \right| = \frac{\rho}{|k|}$$

These are the circles of **Apollonius** with limit points a and b.

Denote by C_1 the circles through a, b and C_2 the circles of Apollonius with these limit points. The configuration formed by all the circles C_1 and C_2 is called the **Steiner circles**(or **circular net**)

Theorem 3.20.

   (a) *There is exactly one C_1 and one C_2 through each point in $\hat{\mathbb{C}} \backslash \{a, b\}$*

   (b) *Every C_1 meets every C_2 under right angle.*

   (c) *Reflection in a C_1 transforms every C_2 into itself and every C_1 into another C_1.*

   (d) *The limit points a, b are symmetric w.r.t. each C_2, but not w.r.t. other circles.*

Proof. If the limit points are $0, \infty$, those properties are trivial in the w-plane. The general case follows since all properties are invariant under Möbius transformations.   □

4 Elementary Conformal mapping

Example 4.1. $w = z^\alpha$ where $\alpha > 0$.

Let $S(u_1, u_2)$ with $0 < \varphi_2 - \varphi_1 \leqslant 2\pi$ be $\{z \in \mathbb{C} : z \neq 0, \varphi_1 < \arg(z) < \varphi_2\}$ where $\arg(z)$ can be chosen as any value of it.

Then $S(\varphi_1, \varphi_2)$ is a region.

In this region, a unique value of $w = z^\alpha$ is defined by $\arg w = \alpha \arg z$.

This function is analytic with $\dfrac{\mathrm{d}w}{\mathrm{d}z} = \alpha \dfrac{w}{z}$.

This function is $1 - 1$ only if $\alpha(\varphi_2 - \varphi_1) \leqslant 2\pi$.

Example 4.2. $w = e^z$ maps $\{z \in \mathbb{C} : -\dfrac{\pi}{2} < \mathrm{Im}(z) < \dfrac{\pi}{2}\}$ onto $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$

Example 4.3. $w = \dfrac{z - 1}{z + 1}$ maps $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ onto $\{ww \in \mathbb{C} : |w| < 1\}$

Example 4.4.

$$\mathbb{C}\backslash[-1, 1] \xrightarrow{z_1 = \frac{z+1}{z-1}} \mathbb{C}\backslash(-\infty, 0] \xrightarrow{z_2 = \sqrt{z_1}} \{\mathrm{Re}(z_2) > 0\} \xrightarrow{w = \frac{z_2 - 1}{z_2 + 1}} \{w \in \mathbb{C} : |w| < 1\} \quad (4.1)$$

4.1 Elementary Riemann surfaces

Example 4.5. $w = z^n$, $n \in \mathbb{Z}_+$ and $n > 1$.

There is a 1-1 correspondence between each angle $\dfrac{(k - 1)2\pi}{n} < \arg z < \dfrac{k \cdot 2\pi}{n}$, $k = 1, 2, \cdots, n$ and while w-plane except for the positive real axis.

Example 4.6. $w = e^z$. This function maps each parallel strip $(k - 1)2\pi < \mathrm{Im}\, z < k \cdot 2\pi$, $k \in \mathbb{Z}$ onto a sheet with a cut along the positive axis.

5 Complex Integration

5.1 Fundamental Theorems

5.1.1 Line integral and rectifiable arcs

Let $f(t) = u(t) + iv(t)$ be a complex-valued defined on $t \in [a, b] \subset \mathbb{R}$ where u, v are real-valued functions. If f is continuous on $[a, b]$, we may define the **integral**

$$\int_a^b f(t) \mathrm{d}t := \int_a^b u(t) \mathrm{d}t + i \int_a^b v(t) \mathrm{d}t$$

Let γ be a piecewise differential arc in $\mathbb{C}$ with the equation $z = z(t), a \leqslant t \leqslant b$. If f is continuous on γ, then $f(z(t))$ is continuous on $[a, b]$, and we define

$$\int_\gamma f(z) \mathrm{d}z = \int_a^b f(z(t)) z'(t) \mathrm{d}t \tag{5.1}$$

The integral defined in 5.1 is independent of the parametrization of γ. Suppose that anther parametrization of γ is $\gamma : (\alpha, \beta) \to \mathbb{C}, \tau \mapsto z(t(\tau))$, where $t : (\alpha, \beta) \to (a, b), \tau \mapsto t(\tau)$ is piecewise differentiable. Then we have

$$\int_a^b f(z(t)) z'(t) \mathrm{d}t = \int_\alpha^\beta f(z(t(\tau))) z'(t(\tau)) t'(\tau) \mathrm{d}t = \int_\alpha^\beta f(z(t(\tau))) \frac{\mathrm{d}z(t(\tau))}{\mathrm{d}\tau} \mathrm{d}\tau \tag{5.2}$$

For an arc γ with equation $z = z(t), a \leqslant t \leqslant b$, we define $-\gamma$ by $z = z(-t), -b \leqslant t \leqslant a$.

Then we have

$$\int_{-\gamma} f(z) \mathrm{d}z = \int_{-b}^{-a} f(z(-t)) \frac{\mathrm{d}z(-t)}{\mathrm{d}t} \mathrm{d}t$$

$$= -\int_{-a}^{-b} f(z(-t))z'(-t)\mathrm{d}t$$

$$= -\int_{a}^{b} f(z(\tau))z'(\tau)\mathrm{d}\tau$$

$$= -\int_{\gamma} f(z)\mathrm{d}z$$

So we have those properties:

Proposition 5.1.

(a) $\int_{-\gamma} f(z)\mathrm{d}z = -\int_{\gamma} \mathrm{d}z$

(b) *Let f and g be two continuous functions on the piecewise differentiable arc γ, then*

$$\int_{\gamma} (\lambda_1 f + \lambda_2 g)\mathrm{d}z = \lambda_1 \int_{\gamma} f\mathrm{d}z + \lambda_2 \int_{\gamma} g\mathrm{d}z, \forall \lambda_1, \lambda_2 \in \mathbb{C}$$

(c) *If γ can be subdivided into two pieces differentiable arcs γ_1 and γ_2, and f is continuous on γ_1 , then*

$$\int_{\gamma} f\mathrm{d}z = \int_{\gamma_1} f\mathrm{d}z + \int_{\gamma_2} f\mathrm{d}z$$

(d) (c) *implies that the integral of a closed curve doesn't depend on the starting point on the curve*

Example 5.2. Evaluate $\int_{\gamma} \dfrac{1}{z-a}\mathrm{d}z$ where γ is the circle centered at $a \in \mathbb{C}$ with radius R.

Let $z = z(t) = a + Re^{it}$. Then the integral is $2\pi i$

5.1.2 The fundamental theorem of Calculus for integrals in $\mathbb{C}$

The line integral w.r.t. $\bar{z}$ is defined by

$$\int_{\gamma} f(z)\overline{\mathrm{d}z} = \overline{\int_{\gamma} \overline{f(z)}\mathrm{d}z}$$

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With this notation, line integrals w.r.t. $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ can be defined by

$$\int_\gamma f(z)\mathrm{d}x = \frac{1}{2}[\int_\gamma f(z)\mathrm{d}z + \int_\gamma f(z)\overline{\mathrm{d}z}]$$

$$\int_\gamma f(z)\mathrm{d}y = \frac{1}{2i}[\int_\gamma f(z)\mathrm{d}z - \int_\gamma f(z)\overline{\mathrm{d}z}]$$

if we write $f(z) = \mu + i\nu$, we have

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}x + i\int_\gamma f(z)\mathrm{d}y = \int_\gamma (\mu\mathrm{d}x - \nu\mathrm{d}y) + i\int_\gamma (\nu\mathrm{d}x + \mu\mathrm{d}y)$$

Remark 5.3. It is followed by the intuition. We can view the integration as the multiplication between f and $\mathrm{d}z$.

The integral w.r.t. **arc length** is defined by

$$\int_\gamma f(z)|\mathrm{d}z| = \int_a^b f(z(t))|z'(t)|\mathrm{d}t$$

This integral is again independent of the parametrization. It is easy to check

$$\int_{-\gamma} f(z)|\mathrm{d}z| = \int_\gamma f(z)|\mathrm{d}z|$$

Now we define **length** of a curve γ: $L(\gamma) = \int_\gamma |\mathrm{d}z|$

We have the inequality:

$$\left|\int_\gamma f\mathrm{d}z\right| \leq \int_\gamma |f| \cdot |\mathrm{d}z| \leq L(\gamma) \cdot \sup_{z \leq \gamma} |f(z)|$$

The length of an arc γ ($z = z(t)$) can also be defined as the least upper bound of

all sums

$$\sum_{i=1}^{n} |z(t_i) - z(t_{i-1})|$$

where $a = t_0 < t_1 < \cdots < t_n = b$ If this least upper bound is finite, we say that the arc is **rectifiable**

It is easy to show that piecewise differentiable arcs are rectifiable.

The integral of a continuous function f on a rectifiable arc may be defined as

$$\int_{\gamma} f(z)\mathrm{d}z = \lim \sum_{k=1}^{n} f(z(\psi_k))[z(t_k) - z(t_{k-1})]$$

Theorem 5.4. *Let $\Omega \subset \mathbb{C}$ be a region, and P, Q two (possibly complex-valued) functions that are continuous on Ω, γ closed curve. The integral $\int_{\gamma} P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$ depends only on the end point of γ iff there exists a function $U(x,y)$ on Ω with $\dfrac{\partial U}{\partial x} = P, \dfrac{\partial U}{\partial y} = Q$.*

Proof. "$\Leftarrow$": If such a U exists, then

$$\int_{\gamma} P\mathrm{d}x + Q\mathrm{d}y = \int_{\gamma} \frac{\partial U}{\partial x}\mathrm{d}x + \frac{\partial U}{\partial y}\mathrm{d}y = \int_{\gamma} \frac{\mathrm{d}U}{\mathrm{d}t}\mathrm{d}t = U(\gamma(b)) - U(\gamma(a))$$

"$\Rightarrow$": Fix a point $(x_0, y_0) \in \Omega$. We define $U(x, y) = \int_{\gamma} P\mathrm{d}x + Q\mathrm{d}y$ where γ is any curve between (x_0, y_0) and (x, y). Easy to check that it is true. $\qquad \square$

Theorem 5.5 (Fundamental theorem of Calculus for integrals on $\mathbb{C}$). *Let f be continuous on a region Ω containing γ. $\int_{\gamma} f\mathrm{d}z$ depends on the endpoints iff f is the derivative of an analytic function F in Ω.*

Remark 5.6. We will prove $\displaystyle\int_{\gamma} f\mathrm{d}z = F(\omega_2) - F(\omega_1)$ where γ begins at ω_1 and ends at ω_2.

Proof. Transform the line integration into the composition of two real integration.

$\qquad \square$

Corollary 5.7. *If F is analytic on Ω with $F' = f$, and γ is a closed curve in Ω, then $\int_\gamma f \mathrm{d}z = 0$. Conversely if f is continuous on Ω and $\int_\gamma f \mathrm{d}z = 0$ for any closed curve in Ω, then f is the derivative of an analytic function F in Ω.*

5.1.3 Cauchy's theorem for a rectangle

There are some notes in this section:

R is the rectangle in $\mathbb{C}$, $R = \{x + iy \in \mathbb{C} : a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$. And ∂R is boundary curve oriented in the counterclockwise direction.

Theorem 5.8 (Cauchy's theorem for a rectangle). *If f is analytic on an open set which contains R, then $\int_{\partial R} f(z)\mathrm{d}z = 0$*

Proof. For $\forall$ rectangle $\tilde{R}$ inside R, we define $Z(\tilde{R}) = \int_{\partial \tilde{R}} f(z)\mathrm{d}z$. Then $Z(R) = Z(R_1) + Z(R_2)$ if R is divided into Z_1, Z_2.

Since we can divide R into four equal rectangles, and find a rectangle with $|Z(R^{(1)})| \geqslant \frac{1}{4}|Z(R)|$. Then repeat the above steps and we obtain a sequence of nested rectangles $R \supset R^{(1)} \supset \cdots$ with the property

$$Z(R^{(n)}) \geqslant \frac{1}{4}|Z(R^{(n-1)})| \geqslant \cdots \geqslant \frac{1}{4^n}Z(R) \tag{5.3}$$

$\forall \delta > 0, \exists n \in \mathbb{N}$ *s.t.* $R^{(n)} \subset \{z \in \mathbb{C} : |z - z_0| < \delta\}, \forall n \geqslant N$, where z_0 is the limit of $R^{(n)}$ as $n \to \infty$.

f is analytic in $R \Rightarrow \forall \varepsilon, \exists \delta > 0$ *s.t.*

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon, \forall z \text{ with } |z - z_0| < \delta \tag{5.4}$$

We assume that δ satisfies both conditions. We have

$$Z(R^{(n)}) = \int_{\partial R^{(n)}} f(z)\mathrm{d}z = \int_{\partial R^{(n)}} [f(z) - f(z_0) - (z - z_0)f'(z_0)]\mathrm{d}z$$

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$$\Rightarrow |Z(R^{(n)})| \leqslant \varepsilon \int_{\partial R^{(n)}} |z - z_0| \mathrm{d}z \text{ by } 5.4$$

Let d_n be the length of diagonal of $R^{(n)}$, L_n be the length of its perimeter. Then $|z - z_0| \leqslant d_n, \forall z \in \partial R^{(n)}$.

$$\Rightarrow |Z(R^{(n)})| \leqslant \varepsilon d_n L_n = \varepsilon \frac{D}{2^n} \cdot \frac{L}{2^n} \text{ where } D, L \text{ are the diameter and perimeter of}$$
R.

$$\Rightarrow |Z(R)| \overset{5.3}{\leqslant} 4^n |Z(R^{(n)})| \leqslant \varepsilon DL \Rightarrow Z(R) = 0 \text{ since } \varepsilon \text{ is arbitrary.} \qquad \square$$

We will next prove the following stronger theorem:

Theorem 5.9 (stronger version of Cauchy's theorem for a rectangle). *Let f be analytic on $R' = R\backslash\{\psi_1, \cdots, \psi_m\}, m \in \mathbb{N}$. If $\lim_{z \to \psi_j} (z - \psi_j) f(z) = 0, \forall 1 \leqslant j \leqslant m$, then*
$$\int_{\partial R} f(z) \mathrm{d}z = 0.$$

Proof. WLOG, we may assume f is not analytic at only one point $\psi \in R$. If we put *psi* into a small rectangle S_0, then the previous theorem tells us $\int_{\partial R} f(z) \mathrm{d}z = \int_{\partial S_0} f(z) \mathrm{d}z$.

$\forall \varepsilon > 0$, we may choose S_0 small enough such that $|f(z)| \leqslant \dfrac{\varepsilon}{|z - \varepsilon|}, \forall z \in \partial S_0$

$$\Rightarrow |\int_{\partial R} f(z) \mathrm{d}z \leqslant \varepsilon \int_{\partial S_0} \frac{|\mathrm{d}z|}{|z - \psi|} \leqslant \varepsilon \frac{1}{\frac{l}{2}} \cdot 4l = 8\varepsilon$$
$$\Rightarrow \int_{\partial R} f(z) \mathrm{d}z = 0 \text{ since } \varepsilon \text{ is arbitrary.} \qquad \square$$

5.1.4 Cauchy's Theorem for a disk

$\Delta := \{z \in \mathbb{C} : |z - z_0| < R\}$ where $R > 0$.

Theorem 5.10 (Cauchy's Theorem for a disk). *If f is analytic in an open disk Δ, then $\int_\gamma f(z) \mathrm{d}z = 0$ for closed curve γ in Δ.*

Proof. Suppose the center of Δ is $z_0 = x_0 + iy_0$, $z = x + iy$. We define

$$F(z) = \int_\gamma f(z) \mathrm{d}z$$

where γ is the horizontal line segment from z_0 to (x, y_0) added with vertical line segment from (x, y_0) to z. We have

$$\frac{\partial F}{\partial y} = \lim_{\delta y \to 0} \frac{F(x, y + \delta y) - F(x, y)}{\delta y} = \lim_{\delta y \to 0} \frac{1}{\delta y} \int_{\delta \gamma} f(z) \mathrm{d}z = if(z) \qquad (5.5)$$

By Cauchy' theorem on rectangles, one has $F(z) = -\int_{\tilde\gamma} f(z)\mathrm{d}z$, where $\tilde\gamma$ is the vertical line segment from z_0 to (x_0, y) added with horizontal line segment from (x_0, y) to z.

Similarly, $\dfrac{\partial F}{\partial x} = f(z)$.

$\Rightarrow \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} \Rightarrow F$ is analytic in Δ with derivative f. By Fundamental Theorem 5.5 of Calulus $\Rightarrow \int_\gamma f(z)\mathrm{d}z = 0$ for $\forall$ closed curve in Δ. $\qquad \square$

Here is a stronger version.

Theorem 5.11 (stronger version of Cauchy's Theorem for a disk)**.** *Let f be analytic in a region $\Delta' = \Delta \setminus \{\psi_1, \cdots, \psi_m\}$ with $m \in \mathbb{N}$. If f satisfies $\lim\limits_{z \to \psi_j} (z - \psi_j) f(z) = 0, \forall 1 \leqslant j \leqslant m$, then $\displaystyle\int_\gamma f(z)\mathrm{d}z = 0, \forall \gamma$ closed in Δ'*

Proof. It is similar to the above proof.

For the case no ψ_j lies on $x = x_0$ and $y = y_0$, we can find a similar curve γ with last segment is a vertical one. Let $F(z) = \int_\gamma f(z)\mathrm{d}z$. And continue the process of proof of the previous theorem.

For the case that $\exists \psi_j$ lies on the lines $x = x_0, y = y_0$, we actually can move the center to another point *s.t.* no ψ_j lies on the lines $x = x_0', y = y_0'$. $\qquad \square$

5.2 Cauchy's integral formula

5.2.1 Index of a point with resect to a closed curve

Lemma 5.12. *If the piecewise differentiable closed curve γ does not pass through $z \in \mathbb{C}$, then the value of the integral $\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}$ is a multiple of $2\pi i$.*

Proof. $\gamma : \zeta = \zeta(t), \alpha \leqslant t \leqslant \beta.$ $h(t) = \int_\alpha^t \frac{\zeta'(s)}{\zeta(s)-z} ds.$

$z \in \gamma \Rightarrow h$ is defined and continuous on $[\alpha, \beta]$. For all t *s.t.* $\zeta'(t)$ is continuous, we have

$$h'(t) = \frac{\zeta'(t)}{\zeta(t) - z} \Rightarrow \frac{d}{dt} \left[e^{-h(t)}(\zeta(t) - z) \right] = 0$$

So $e^{-h(t)}(\zeta(t) - z)$ is constant on $[\alpha, \beta]$.

Then $e^{h(t)} = \frac{\zeta(t) - z}{\zeta(\alpha) - z} \Rightarrow e^{h(\beta)} = 1 \Rightarrow h(\beta) \in \{2k\pi i : k \in \mathbb{Z}\}.$ $\qquad \square$

The **index of the point** z w.r.t. the closed curve γ is the number

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}$$

n is also called the **winding number**.

Theorem 5.13. *Let γ be a piecewise differentiable closed curve. The function $z \mapsto n(\gamma, z)$ is constant on each connected set of $\mathbb{C} \backslash \gamma$, and zero if this set is unbounded.*

Proof. Define $f : \mathbb{C} \backslash \gamma \to \gamma, z \mapsto n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}.$

Then

$$|f(z) - f(z_0)| = \frac{1}{2\pi} \left| \int_\gamma \frac{z - z_0}{(\zeta - z)(\zeta - z_0)} d\zeta \right| \leqslant \frac{|z - z_0|}{2\pi} \int_\gamma \frac{1}{|\zeta - z| \cdot |\zeta - z_0|} |d\zeta|$$

$\Rightarrow f$ is continuous on each open connected set of $\mathbb{C} \backslash \gamma$. Let Ω be any open connected set of $\mathbb{C} \backslash \gamma$. We have $f(\Omega)$ is connected $\xrightarrow{f(\Omega) \subset \mathbb{Z}} f(\Omega)$ contains at most one point $\Rightarrow f$ is constant on Ω.

If $|z|$ is sufficient large, $\exists$ a disk of radius R, $B(0, R)$, *s.t.* $\gamma \subset B(0, R)$ but $z \notin B(0, R)$. Cauchy's theorem for a disk 5.10 tells us that $f(z) = n(\gamma, z) = 0$. So it is zero if this set is unbounded. $\qquad \square$

Lemma 5.14. *Let z_1, z_2 be two points on a closed curve γ and $0 \notin \gamma$.*

Suppose z_1 in the lower half space and z_2 in upper half space. If $\gamma_1 \cap \{(x,0) : x \leqslant 0\} = \varnothing$, and $\gamma_2 \cap \{(x,0) : x \geqslant 0\} = \varnothing$, then $n(\gamma, 0) = 1$.

Remark 5.15. One method to prove this lemma is to create two segment from z_i to the point in the unit circle. By divide the curve into two parts, we can easily remove the part of previous curve by using the theorem 5.13, since 0 is in the unbounded set.

In this proof, we can find that Theorem 5.13 is such powerful that we can change any curve to a more simple curve easily!

5.2.2 Cauchy's integral formula

Theorem 5.16 (Cauch's integral formula)**.** *Suppose that f is analytic in an open disk $\triangle$, and let γ be a closed curve in $\triangle$. For $\forall z \notin \gamma$,*

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

where $n(\gamma, z)$ is the index of z w.r.t. γ.

Proof. If $z \notin \triangle$, The both sides of the equation is 0.

So we may assume $z \in \triangle$ and $z \notin \gamma$. Define $F : \triangle\backslash\{z\} \to \mathbb{C}, \zeta \mapsto \dfrac{f(\zeta) - f(z)}{\zeta - z}$.

Then F is analytic in $\triangle\backslash\{z\}$, and $\lim\limits_{\zeta \to z}(\zeta - z)F(\zeta)$.

By Cauchy's Theorem 5.9 $\Rightarrow \int_\gamma F(\zeta)\mathrm{d}\zeta = 0 \Rightarrow \displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = f(z)\int_\gamma \frac{1}{\zeta - z}\mathrm{d}\zeta = f(z) \cdot 2\pi i \cdot n(\gamma, z)$ $\qquad \square$

Remark 5.17. This proof let us find that for a good-enough function, its integral over a closed curve is a constant.

The theorem still holds if f is analytic except at a finite number of ζ_j s.t.

$$\lim_{\zeta \to \zeta_j}(\zeta - \zeta_j)f(\zeta) = 0$$

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and $z \neq \zeta_j$ for each j, since Cauchy's theorem is still applicable.

Theorem 5.18 (The mean value property for analytic functions). *f is analytic in a region Ω which contain $\overline{B(z, R)}$. Then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\theta}) dt$$

Proof. The previous theorem 5.16 $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z} d\zeta \xrightarrow{\zeta = z + Re^{it}} \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{it}) dt$$

$\square$

If f is analytic in an open disk $\triangle$, and γ is a closed curve in $\triangle$. And $n(\gamma, z) = 1$. Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

This is usually referred to as **Cauchy's integral formula**

5.2.3 Higher derivatives

Lemma 5.19. *Let $\Omega \subset \mathbb{C}$ be a region and γ be an arc in Ω. If φ is continuous on γ, then the function*

$$F_n(z) := \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n} d\zeta$$

is analytic in each of the regions $\Omega \backslash \gamma$, and its derivative is $F_n'(z) = nF_{n+1}(z)$

Proof. We prove it by induction.

The lemma is true if $n = 0$: $F_0(z) = \int_\gamma \varphi(\zeta) d\zeta$ and $F_0'(z) = 0 = 0 \cdot F_1(z)$.

We suppose that the lemma holds for $n - 1$ with $n \in \mathbb{N}$: $\forall$ continuous φ on γ, F_{n-1} is analytic in $\Omega \backslash \gamma$ and $F_{n-1}'(z) = (n-1)F_n(z), \forall z \in \Omega \backslash \gamma$.

Fix $z_0 \in \Omega \backslash \gamma$. For $\forall z \in B(z_0, \frac{\delta}{2})$, with $B(z_0, \delta) \subset \Omega \backslash \gamma$, we have $|\zeta - z| > \frac{\delta}{2}$, $\forall \zeta \in \gamma$. For $\forall$ continuous φ on γ,

$$
\begin{aligned}
F_n(z) - F_n(z_0) &= \int_\gamma \frac{\varphi(\zeta)(\zeta - z + z - z_0)}{(\zeta - z)^n(\zeta - z_0)} d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n} d\zeta \\
&= \left[\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^{n-1}(\zeta - z)} d\zeta \right] \\
&\quad + (z - z_0) \int_\gamma \frac{\varphi(\zeta) d\zeta}{(\zeta - z)^n(\zeta - z_0)}
\end{aligned}
$$

Let $\psi(\zeta) = \dfrac{\psi(\zeta)}{\zeta - z_0}$, which is continuous except γ.

Using the induction condition to ψ, we can finish the proof. $\qquad \square$

Theorem 5.20. *An analytic function on a region Ω has derivatives of all orders which are analytic in Ω. More precisely, $\forall z_0 \in \Omega$, choose $B(z, \delta) \subset \Omega$ and a circle $C \subset B(z_0, \delta)$ with center z_0. For $\forall z$ in the interior of C, Cauchy's integral formula gives*

$$
f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta
$$

Then the previous lemma implies $f'(z) = \dfrac{1}{2\pi i} \int_C \dfrac{f(\zeta)}{(\zeta - z)^2} d\zeta$ is analytic in the interior of C. More generally, for $\forall n \in \mathbb{N}$,

$$
f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{5.6}
$$

5.2.4 Consequences of Cauchy

Theorem 5.21 (Morera's Theorem). *If f is continuous in a region Ω, and if $\int_\gamma f(z) dz = 0$ for $\forall$ closed curve γ in Ω. Then f is analytic in Ω.*

Proof. We proved in Corollary 5.7 that under the hypothesis of theorem, $f = F'$ where F is analytic in Ω. The last theorem $\Rightarrow f$ is analytic. $\qquad \square$

Suppose f is analytic in a disk, $\overline{B(z_0, R)}$, and bounded on the circle γ given by $|z - z_0| = R$. Then $\forall z \in \gamma$, $|f(z)| \leqslant M$ for some $M \geqslant 0$. By (5.6),

$$|f^{(n)}(z)| \leqslant \frac{n!}{2\pi} \int_C \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} |\mathrm{d}\zeta| \leqslant \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = MR^{-n} n! \qquad (5.7)$$

This inequality is known as **Cauchy's estimate**.

Theorem 5.22 (Liouville's Theorem). *A bounded entire function (i.e. analytic in $\mathbb{C}$) is constant.*

Proof. Suppose $|f(z)| \leqslant M$, $\forall z \in \mathbb{C}$. Cauchy's estimate $\Rightarrow$

$$b|f'(z)| \leqslant \frac{M}{R}, \quad \forall z \in \mathbb{C}, \forall R > 0 \qquad (5.8)$$

$\square$

$\xRightarrow{R \to \infty} f'(z) = 0$ for $z \in \mathbb{C} \Rightarrow f = 0$.

Theorem 5.23 (Fundamental Theorem for Algebra). *Every polynomial of degree $n \geqslant 1$ has n roots.*

Proof. It suffices to prove it has at least one root.

Suppose $P(z) = a_n z^n + \cdots a_1 z + a_0$ with $a_0 \neq 0$ does not have a root.

Then $f(z) := \frac{1}{P(z)}$ is an entire function. As $z \to \infty$, $\lim_{|z| \to \infty} \frac{|P(z)|}{|z|^n} = |a_n| \Rightarrow$ $\lim_{|z| \to \infty} \frac{1}{|P(z)|} = 0$.

So f is bounded. By Liouville's Theorem, f is a constant. Where $f = f(\infty) = 0$. That causes contradiction. $\square$

Theorem 5.24 (Power series). *If f is analytic in a region Ω which contains a closed disk $\overline{B(z_0, R)}$, then f has a power series expansion at z_0,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \quad \forall z \in B(z_0, R) \qquad (5.9)$$

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Proof. $\forall z \in B(z_0, R)$, $\forall \zeta$ with $|\zeta - z_0| = R$.

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\
&= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \\
&= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}
\end{aligned} \tag{5.10}$$

This series converges uniformly in ζ with $|\zeta - z_0| = R$.

For $\forall z \in B(z, R)$,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi i} \int_{|\zeta - z| = R} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta \\
&\xlongequal{\text{uniformly}} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \cdot (z - z_0)^n \\
&\overset{(5.6)}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\end{aligned} \tag{5.11}$$

$\square$

5.3 Local properties of analytic functions

5.3.1 Removable Singularities and Taylor's Theorem

We remarked that Cauchy's integral formula holds if f is analytic except at a finite number of point ζ_j *s.t.* $\lim_{\zeta \to \zeta_j} (\zeta - \zeta_j) f(\zeta) = 0$. We will prove f can be extended to an analytic function in $\triangle$. In other word, ζ_j are **removable singularities**.

Theorem 5.25 (Riemann's Removable Singularities Theorem)**.** *Suppose that f is*

analytic in the region $\Omega' = \Omega\backslash\{\zeta_0\}$ where Ω is also a region. Then there exists an analytic function in Ω which coincides with f in Ω' if and only if $\lim\limits_{z\to\zeta_0}(z-\zeta_0)f(z) = 0$.

Proof. The uniqueness and "$\Rightarrow$" part is trivial since the extended function is continuous at ψ_0.

"$\Leftarrow$": Cauchy's integral formula $\Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta, \ \forall z \in \triangle \text{ and } z \neq \zeta_0 \tag{5.12}$$

Lemma 5.19 $\Rightarrow$ the RHS of the last equation 5.12 is analytic in $z \in \triangle$. Then

$$\hat{f}(z) = \begin{cases} f(z), & z \neq \zeta_0 \\ \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(\zeta)}{\zeta - \zeta_0}\mathrm{d}\zeta, z = \zeta_0 \end{cases} \tag{5.13}$$

is analytic in Ω. $\qquad\qquad\square$

We apply Theorem 5.25 to the function $F(z) = \dfrac{f(z) - f(\zeta)}{z - \zeta}$, where f is analytic in a region Ω. Note that

$$\lim_{z\to\zeta}(z - \zeta)F(z) = 0, \ \lim_{z\to\zeta}F(z) = f'(\zeta) \tag{5.14}$$

Theorem 5.25 $\Rightarrow \exists$ analytic function f_1 on Ω *s.t.*

$$f_1(z) = \begin{cases} F(z), & z \neq \zeta_0 \\ f'(\zeta), z = \zeta_0 \end{cases} \tag{5.15}$$

we may thus write $f(z) = f(\zeta) + (z - \zeta)f_1(z)$.

Repeating this process for f_1, we get an analytic function f_2 on Ω *s.t.*

$$f_1(z) = f_1(\zeta) + (z - \zeta)f_2(z) \tag{5.16}$$

where

$$f_2(z) = \begin{cases} \dfrac{f_1(z) - f_1(\zeta)}{z - \zeta}, & z \neq \zeta \\ f_2'(\zeta), & z = \zeta \end{cases} \tag{5.17}$$

Continuing the recursion, we have the general form

$$f_{n-1}(z) = f_{n-1}(\zeta) + (z - \zeta)f_n(z) \tag{5.18}$$

$\Rightarrow$

$$f(z) = f(\zeta) + (z - \zeta)f_1(\zeta) + \cdots + (z - \zeta)^{n-1}f_n(\zeta) + (z - \zeta)^n f_n(z) \tag{5.19}$$

Differentiating n times and setting $z = \zeta \Rightarrow f^{(n)}(\zeta) = n!f_n(\zeta)$

We just prove **Taylor's Theorem**

Theorem 5.26 (Taylor's Theorem). *If f is analytic in a region Ω, $\zeta \in \Omega$, then we have*

$$f(z) = f(\zeta) + (z - \zeta)f'(\zeta) + \cdot + \frac{f^{(n-1)}(\zeta)}{(n-1)!}(z - \zeta)^{n-1} + f_n(z)(z - \zeta)^n \tag{5.20}$$

where f_n is analytic in Ω. Moreover,

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - \zeta)^n(\omega - z)} d\omega \tag{5.21}$$

where C is a circle in Ω s.t. its interior $\triangle$ is also in Ω and $\zeta, z \in \triangle$

Proof. It suffices to prove the second part.

Cauchy's integral formula $\Rightarrow f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\omega)}{\omega - z} d\omega$, $\forall z \in \triangle$.

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For $f_n(z)$, we substitute the expression from (5.20). The first term is

$$\frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega - \zeta)^n(\omega - z)}\mathrm{d}\omega \tag{5.22}$$

The remaining terms have the following form, except for constant factors:

$$g_k(\zeta) = \int_C \frac{1}{(\omega - \zeta)^n(\omega - z)}\mathrm{d}\omega, \ 1 \leqslant k \leqslant n \tag{5.23}$$

The lemma 5.19 applies to $\varphi(\omega) = \dfrac{1}{\omega - z}$, $g'_k(\zeta) = kg_{k-1}(\zeta), k \in \mathbb{N}, \forall \zeta \in \triangle$. So

$$\begin{aligned}
g_1(\zeta) &= \int_C \frac{1}{(\omega - \zeta)(\omega - z)}\mathrm{d}\omega \\
&= \frac{1}{\zeta - z}\left[\int_C \frac{1}{\omega - \zeta}\mathrm{d}\omega - \int_C \frac{1}{\omega - z}\mathrm{d}\omega\right] \\
&= \frac{1}{\omega - z}[2\pi i - 2\pi i] = 0
\end{aligned} \tag{5.24}$$

So $g_k(z) = 0, \forall k \in \mathbb{N}, \forall z \in \triangle$. $\qquad\square$

5.3.2 Zeros and poles

Theorem 5.27. *If f is analytic in a region Ω and $\exists a \in \Omega$ s.t. $f^{(n)}(a) = 0$ for $\forall n \in \mathbb{N} \cup \{0\}$, then $f \equiv 0$ in Ω.*

Proof. Let $B(a, R)$ be the disk s.t. $\overline{B(a, R)} \subset \Omega$. Let $C = \partial B(0, R)$.

Taylor's theorem $\Rightarrow f(z) = (z - a)^n f_n(z)$ with

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega - a)^n(\omega - z)}\mathrm{d}\omega, \ \forall n \in \mathbb{N} \cup \{0\}, \forall z \in B(a, R) \tag{5.25}$$

Let $M = \max\limits_{z \in C} |f(z)|$.

$$\Rightarrow |f_n(z)| \leqslant \frac{1}{2\pi} \cdot \frac{M}{R^n(R - |z - a|)} \cdot 2\pi R$$

$$\Rightarrow |f(z)| \leqslant \frac{|z - a|^n}{R^n} \cdot \frac{MR}{R - |z - a|} \to 0 \text{ as } n \to \infty, \ \forall z \in B(0, R)$$

$$\Rightarrow f(z) = 0, \ \forall z \in B(0, R)$$

Now define

$$E_1 = \left\{ z \in \Omega \mid f^{(n)}(z) = 0, \forall n \in \mathbb{N} \cup \{0\} \right\}$$

$$E_2 = \Omega \backslash E_1 = \left\{ z \in \Omega \mid f^{(n)}(z) \neq 0, \text{ for some } n \in \mathbb{N} \cup \{0\} \right\}$$

We just proved E_1 is open. E_2 is open because $f^{(n)}$ is continuous in Ω for $\forall n \in \mathbb{N} \cup \{0\}$. Ω is a region $\Rightarrow$ either $R_1 = \varnothing$ or $R_2 = \varnothing$.

The assumption of the theorem $\Rightarrow E_1 \neq \varnothing \Rightarrow E_1 = \Omega$. $\qquad\qquad\square$

Let f be analytic in Ω which is not identically zero, $f(a) = 0$ for some $a \in \Omega$. The previous theorem implies $\exists$ first $N \in \mathbb{N}$ s.t. $f^{(N)}(a) \neq 0$. Taylor's theorem implies that $f(a) = (z - a)^N f_N(z)$ where f_N is analytic and $f_N(a) \neq 0$. We say that a is a **zero of order** N of f.

f_N is continuous $\Rightarrow \exists \delta > 0$ s.t. $f(z) \neq 0$ for $\forall z \in B(a, \delta) \backslash \{0\}$.

So we have just proved an important result: Zeros of analytic functions are isolated, or equivalently, we have a famous theorem:

Theorem 5.28 (Identity Theorem). *If f and g are analytic in a region ω, and $f = g$ on a set which has an accumulation point in Ω, then $f(z) = g(z)$.*

Corollary 5.29.

(1) If $f \equiv 0$ in a subregion of Ω and f is analytic in Ω, then $f \equiv 0$ in Ω.

(2) If f is analytic in Ω and vanishes on an arc in Ω which doesn't reduce to a point, then $f \equiv 0$ in Ω.

If f is analytic in a neighborhood of a, but perhaps not at a itself, then a is called an **isolated singularity** of f.

If $\lim_{z \to a} f(z) = \infty$, then a is said to be a **pole** of f, and we set $f(a) = \infty$. Continuity implies $\exists \delta > 0$ *s.t.* $f(z) \neq 0$ for $\forall z \in B(0, \delta) \backslash \{a\}$. Thus, $g(z) = \dfrac{1}{f(z)}$ is analytic in $B(a, \delta) \backslash \{a\}$. $\lim_{z \to a} (z - a) g(z) = 0 \Rightarrow a$ is a removable singularity of g, and g has an analytic extension with $g(a) = 0$. $g \not\equiv 0 \Rightarrow a$ is a zero of g with finite order. The **order of the pole** of f at a is the order N of the zero of g at a.

We can write

$$f(z) = (z - a)^{-N} f_N(z), \ \forall z \in B(a, \delta) \backslash \{a\} \tag{5.26}$$

where f_N is analytic and nonzero in a neighborhood of a.

Definition 5.30. A function which is analytic in a region Ω except for (isolated) poles is called a **meromorphic function**.

Example 5.31. If f and g are analytic in Ω and $g \not\equiv 0$, then $\frac{f}{g}$ is a meromorphic function in Ω. (See the Identity Theorem 5.28)

Remark 5.32. The sum, the product and quotient (if denominator is not always zero) of two meromorphic functions are meromorphic.

If f has a pole of order N at a, then $(z - a)^N f(z)$ is analytic at a, and Taylor's theorem 5.26 implies

$$(z - a)^N f(z) = b_N + b_{N-1}(z - a) + \cdots + b_1(z - a)^{N-1} + \varphi(z) \cdot (z - a)^N \tag{5.27}$$

where φ is analytic at a.

$$\Rightarrow f(z) = b_N(z-a)^{-N} + b_{N-1}(z-a)^{-(N-1)} + \cdots + b_1(z-a)^{-1} + \varphi(z), \ \forall z \neq a. \quad (5.28)$$

Theorem 5.33. *If f is analytic in a neighborhood of a, but perhaps not at a itself, then exactly one of the following 3 cases occurs:*

(i) *$f \equiv 0$ in this neighborhood.*

(ii) *$\exists$ integer $N \in \mathbb{Z}$ s.t. $\displaystyle\lim_{z \to a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > N \\ \infty, & \alpha < N \end{cases}$*

(iii) *neither $\displaystyle\lim_{z \to a} |z-a|^\alpha \cdot |f(z)| = 0$ for any $\alpha \in \mathbb{R}$ nor $\displaystyle\lim_{z \to a} |z-a|^\alpha \cdot |f(z)| = \infty$ for any $\alpha \in \mathbb{R}$*

Proof.

① If $\displaystyle\lim_{z \to a} |z-a|^\alpha \cdot |f(z)| = 0$ for $\forall \alpha \in \mathbb{R}$, then $\displaystyle\lim_{z \to a} |z-a|^m \cdot |f(z)| = 0$ for $\forall$ integer $m > \alpha$.

$\Rightarrow (z-a)^m f(z)$ has a removable singularity at a and vanishes at $z = a$

$\Rightarrow$ Either $f \equiv 0$ in $B(a, \delta) \backslash \{a\}$, which is case (i), or $(z-a)^m f(\alpha)$ has a zero of finite order k at $a \Rightarrow \displaystyle\lim_{z \to a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > m - k \\ \infty, & \alpha < m - k \end{cases}$

② If $\displaystyle\lim_{z \to a} |z-a|^\alpha |f(z)| = \infty$ for some $\alpha \in \mathbb{R}$, then $\displaystyle\lim_{z \to a} |z-a|^n \cdot |f(z)| = \infty$ for $\forall$ integer $n < \alpha$.

$\Rightarrow (z-a)^n f(z)$ has a pole of finite order l at a

$\Rightarrow \displaystyle\lim_{z \to a} |z-a|^\alpha \cdot |f(z)| = \begin{cases} 0, & \alpha > n + l \\ \infty, & \alpha < n + l \end{cases}$ $\qquad\qquad \square$

Remark 5.34. In case (ii), N may be called the **algebraic order** of f at a. $N > 0$ if a is a pole, $N < 0$ if a is a zero, and $N = 0$ if f is analytic at a and $f(a) \neq 0$. The order is always an integer, there is no analytic function which tends to 0 or ∞, like a fractional power of $|z - a|$.

In some sense, three cases depends on whether $\lim\limits_{z \to a}(z - a)^N f(z)$ converges for some N.

In case (iii), the point a is an **essential isolated singularity**.

Example 5.35. $f(z) = \exp(\frac{1}{z})$ has an essential isolated singularity $z = 0$.

Theorem 5.36 (Weierstrass). *An analytic function comes arbitarily close to any complex value in every neighborhood of an essential singularity. Or equivalently, the codomain of f on every neighborhood of an essential singularity is dense in $\mathbb{C}$.*

Proof. Suppose the statement is false.

$\exists A \in \mathbb{C}, \delta > 0$ and $\varepsilon > 0$ s.t.

$$|f(z) - A| > \delta, \ \forall z \text{ with } 0 < |z - a| < \varepsilon \tag{5.29}$$

$\Rightarrow \lim\limits_{z \to a} |z - a|^\alpha \cdot |f(z) - A| = \infty$ for $\forall \alpha < 0$. $\Rightarrow a$ is not an essential singularity of $f(z) - A$.

The previous theorem $\Rightarrow \exists\, \beta \in \mathbb{R}$ s.t. $\lim\limits_{z \to a} |z - a|^\beta \cdot |f(z) - A| = 0$, and we may choose $\beta > 0$.

Then $\lim\limits_{z \to a} |z - a|^\beta \cdot |A| = 0 \Rightarrow \lim\limits_{z \to a} |z - a|^\beta \cdot |f(z)| = 0$ by the triangular inequality.
So a is not an essential singularity of f, which causes contradiction!

So the statement has to be true. □

Remark 5.37. If f is analytic in $|z| > R$. We treat ∞ as an isolated singularity. Removable singularity, pole or essential singularity of f at ∞ is defined according to $g(z) = f(\frac{1}{z})$ at $z = 0$.

5.3.3 The Local Mappings

Theorem 5.38 (The Argument Principle). *Let f be analytic in a disk $\triangle$ s.t. f does not vanish identically. Let z_j be the zeros of f, each zero being counted as many times as **its order indicates**. For every closed curve γ in $\triangle$ which does not pass through a zero, we have*

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z \tag{5.30}$$

where the sum has only a finite number of terms with nonzero value.

Proof.

Case I: f has exactly n zeros $z_1, \cdots, z_n$.

By repeated application of Taylor' Theorem 5.26, we can write

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)g(z), \ z \in \triangle \tag{5.31}$$

where g is analytic in $\triangle$ and $g(z) \neq 0$ for $\forall z \in \triangle$. $\Rightarrow$

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}, \ \forall z \in \triangle \text{ and } z \neq z_j \tag{5.32}$$

Cauchy' Theorem 5.10 $\Rightarrow$

$$\int_\gamma \frac{g'(z)}{g(z)} \mathrm{d}z = 0 \Rightarrow \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \sum_{j=1}^n n(\gamma, z_j) \tag{5.33}$$

Case II: f has infinitely many zeros in $\triangle$. Then γ is inside a concentric disk $\triangle'$ smaller than $\triangle$.

$f \not\equiv 0 \Rightarrow$ There is only a finite number of zeros in $\triangle'$.

So we can apply (5.33) to the disk $\triangle' \Rightarrow$ (5.30) holds since $n(\gamma, z_j) = 0$ if $z \notin \triangle'$. $\qquad\square$

Remark 5.39.

- The function $\omega = f(z)$ maps γ onto a closed curve Γ in the ω-plane, and we have

$$\int_\Gamma \frac{d\omega}{\omega} = \int_\gamma \frac{f'(z)}{f(z)} dz \tag{5.34}$$

Then (5.30) can be interpreted as $n(\Gamma, 0) = \sum_j n(\gamma, z_j)$.

- The most useful application of the theorem is to the case when γ is a circle (or more generally a simple closed curve). So that

$$n(\gamma, z) = \begin{cases} 1, & z \text{ is inside } \gamma \\ 0, & z \text{ is outside } \gamma \end{cases} \quad \text{Then (5.30) yields a formula for the total num-}$$

ber of zeros enclosed by γ.

Let $a \in \mathbb{C}$. Apply the previous theorem to $f(z) - a$

$$\sum_j n(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} dz$$

where $z_j(a)$ are zeros of $f - a$ (or roots of $f(z) = a$), and γ is a closed curve in $\triangle$ which doesn't pass $z_j(a) \Rightarrow$

$$n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$$

If a and b are in the same region determined by Γ, then $n(\Gamma, a) = n(\Gamma, b) \Rightarrow$

$$\sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b)) \tag{5.35}$$

If γ is a circle, then f takes the values a and b equally many times inside γ, counted as many times as their orders indicate.

We have the equation that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - a} \mathrm{d}z = n(\Gamma, a) = n(\Gamma, b)$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{\mathrm{d}\omega}{\omega - b} = \frac{1}{2\pi i} \frac{f'(z)\mathrm{d}z}{f(z) - b} \tag{5.36}$$

$$= \mathrm{card}\{z \text{ inside } \gamma : f(z) = b\}$$

Theorem 5.40. *Suppose f is analytic at z_0, and $f(z) - \omega_0$ has a zero of order $N \in \mathbb{N}$ at z_0. Then for $\forall \varepsilon > 0$ sufficiently small, $\exists \delta > 0$ s.t. for $\forall a$ with $|a - \omega_0| < \delta$, the equation $f(z) = a$ has exactly N roots in the disk $|z - z_0| < \varepsilon$*

Proof. We choose $\varepsilon > 0$ s.t.

(1) f is analytic in $|z - z_0| \leqslant \varepsilon$

(2) z_0 is the only zero of $f(z) - \omega_0$ in this disk.

(3) $f'(z) \neq 0$ for $\forall z$ with $0 < |z - z_0| < \varepsilon$

Let γ be the circle $|z - z_0| < \varepsilon$ and $\Gamma = f \circ \gamma$.

$\omega_0 \notin \Gamma \Rightarrow \exists \delta > 0$ s.t. $B(\omega_0, \delta) \cap \Gamma = \varnothing$.

The consequence of the argument principle 5.38, *i.e.* (5.36) $\Rightarrow f$ takes all values $a \in B(\omega_0, \delta)$ the same number of times N inside γ, since $f(z) = \omega_0$ has exactly N coiciding roots inside γ.

(3) $\Rightarrow$ all roots $f(z) = a$ with $a \in B(\omega_0, \delta) \backslash \{\omega_0\}$ are simple $\qquad \square$

Corollary 5.41 (open mapping theorem). *A nonconstant analytic function maps open sets onto open sets.*

Proof. The previous theorem $\Rightarrow \forall \varepsilon > 0$, $f(B(z_0, \varepsilon)) \supset B(\omega_0, \delta)$ $\qquad \square$

Corollary 5.42. *If f is analytic at z_0 with $f'(z_0) \neq 0$. It maps a neighborhood of z_0 conformally and topologically onto a region.*

Proof. This is the case $N = 0$. The previous theorem $\Rightarrow$ There is 1-1 corresponding between the disk $|\omega - \omega_0| < \delta$ and an open subset of $|z - z_0| < \varepsilon$. The open mapping theorem 5.41 $\Rightarrow f^{-1}$ is continuous $\Rightarrow f$ is a topological map. And f is conformal on $|z - z_0| < \varepsilon$ $\qquad\qquad\square$

Remark 5.43. Under the assumption of Corollary 5.42, f^{-1} is continuous $\Rightarrow f^{-1}$ is analytic $\Rightarrow f^{-1}$ is conformal map.

If $f : \Omega \to \mathbb{C}$ is 1-1 and analytic, Theorem 5.40 can hold only with $N = 1 \Rightarrow$ $f'(z) \neq 0$ for $\forall z \in \mathbb{C}$. So this condition is stronger than the conformal condition.

5.3.4 The Maximum Principle

Theorem 5.44 (The maximum principle). *If f is analytic and nonconstant in a region Ω, then its modules $|f|$ has no maximum in Ω.*

Proof. $\forall z_0 \in \Omega$, the open mapping theorem 5.41 $\Rightarrow \exists$ an open disk $|\omega - f(z_0)| < \delta$ contained in $F(\Omega)$. In this disk, $\exists \omega$ *s.t.* $|\omega| > |f(z_0)| \Rightarrow |f(z_0)|$ is not the maximum of $|f|$. $\qquad\qquad\square$

Theorem 5.45 (The maximum principle). *If f is defined and continuous on a closed bounded set E and analytic in the interior of E, then the maximum of $|f|$ on E is assumed on the boundary of E.*

Remark 5.46. The maximum principle can also be proved by the mean value theorem 5.18 for analytic functions.

Theorem 5.47 (Schwarz Lemma). *If f is analytic in the disk $|z| < 1$ and satisfies $f(0) = 0$, $|f(z)| \leq 1$, $\forall z \in B(0, 1)$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Furthermore, if $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ where $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. We define $g(z) = \begin{cases} \dfrac{f(z)}{z}, & z \neq 0, z \in B(0,1) \\ f'(0), & z = 0 \end{cases}$.

Then g is analytic with $g'(0) = \dfrac{f'(0)}{2}$ using Taylor series (5.20).

The maximum principle implies that $|g(z)| \leqslant \dfrac{1}{r}, \forall z \in \overline{B(0,r)}$ where $0 < r < 1$.

Setting $r \to 1$, we get $|g(z)| \leqslant 1, \forall |z| < 1$.

If $|f(z)| = |z|$ for some $z \neq 0$, or $|f'(0)| = 1$, then $|g| = 1$ attains its maximum at some interior points. By maximum principle, g has to be a constant. $\qquad \square$

Remark 5.48. For a general analytic function $f : B(0, R) \to B(0, M), z_0 \mapsto w_0$.

Let $T(z) = \dfrac{\frac{z}{R} - \frac{z_0}{R}}{1 - \frac{\bar{z}_0}{R} \cdot \frac{z}{R}}$

$S(\omega) = \dfrac{\frac{\omega}{M} - \frac{\omega_0}{M}}{1 - \frac{\bar{\omega}_0}{M} \cdot \frac{\omega}{M}}$.

Then $S \circ f \circ T^{-1}$ satisfies $S \circ f \circ T^{-1}(0) = 0$ and $|S \circ f \circ T^{-1}(z)| \leqslant 1 \overset{Schwarz \, lemma}{\implies}$

$|S \circ f \circ T^{-1}(\zeta)| \leqslant |\zeta|$.

$\Rightarrow |S \circ f(z)| \leqslant |T(z)| \Rightarrow$

$$\left| \frac{M(f(z) - \omega_0)}{M^2 - \bar{\omega}_0 f(z)} \right| \leqslant \left| \frac{R(z - z_0)}{R^2 - \bar{z}_0 z} \right|, \forall z \in B(0, R)$$

5.4   The General Form of Cauchy's Theorem

5.4.1   Chains and Cycles

Let $\Omega \subset \mathbb{C}$ be open. Let $\gamma_j : [\alpha_j, \beta_j] \to \Omega$ be piecewise continuously differentiable curves in Ω. The sum $\gamma_1 + \gamma_2 + \cdots + \gamma_N$, which need not be a curve is called a **chain**. The **integral** of a continuous f in Ω along this chain is defined by

$$\int_{\gamma_1 + \gamma_2 + \cdots + \gamma_N} f = \sum_{j=1}^{N} \int_{\gamma_j} f. \tag{5.37}$$

Two chains are **identical** if they yield the same line integrals for all function f.

A chain is a **cycle** if it can be represented as a finite sum of closed curves.

5.4.2 Simple connectivity and homology

A region is **simply connected** if its complement w.r.t. $\hat{\mathbb{C}}$ is connected.

Example 5.49. A disk, a half plane, a parallel strip are simply connected.

$\mathbb{C}\backslash\overline{B(0,1)}$ is not simply connected since its complement w.r.t. $\hat{\mathbb{C}}$ consists of $\overline{B(0,1)}$ and ∞.

Theorem 5.50. *A region $\Omega \subset \mathbb{C}$ is simply connected iff $n(\gamma, z) = 0$ for all cycles γ in Ω and all points $z \notin \Omega$.*

Proof. "$\Rightarrow$": $\forall$ cycle $\gamma \subset \Omega$, $\hat{\mathbb{C}}\backslash\Omega$ must be in one of the regions in $\hat{\mathbb{C}}\backslash\gamma$ since $\hat{\mathbb{C}}\backslash\Omega$ is connected.

$\infty \in \hat{\mathbb{C}}\backslash\Omega \Rightarrow \mathbb{C}\backslash\Omega$ is in the unbounded region of $\mathbb{C}\backslash\gamma$. By theorem 5.13 $n(\gamma, z) = 0$, $\forall z \in \mathbb{C}\backslash\Omega$.

"$\Leftarrow$": Suppose Ω is not simply connected, *i.e.*, $\hat{C}\backslash\Omega$ is not connected. Let $\hat{\mathbb{C}}\backslash\Omega = A \sqcup B$ with A, B disjoint closed sets.

Suppose that $\infty \in B$. Then A is the bounded set. δ is defined to be the distance between A and B. The $\delta > 0$. Cover A with a net of squares Ω of side less than $\dfrac{\delta}{\sqrt{2}}$.

Suppose $z_0 \in A$ lies at the center of a square cycle $\gamma := \sum\limits_{Q:Q\cap A\neq 0} \partial\Omega$.

z_0 is only in one of these squares $\Rightarrow n(\gamma, z_0) = 1$.

Since sides of squares are less than $\dfrac{\delta}{\sqrt{2}}$, $\gamma \cap B \neq \varnothing$.

$\gamma \cap A = \varnothing$ after cancellations of the multiple sides.

$\Rightarrow \gamma \in \Omega$ with $n(\gamma, z_0) = 1$. That's a contradiction. $\qquad\qquad\square$

A cycle γ in an open set Ω is said to be **homologous to zero** w.r.t. Ω if $n(\gamma, z) = 0$ for $\forall z \in \mathbb{C}\backslash\Omega$.

In symbols, we write $\gamma \sim 0 (\mathrm{mod}\, \Omega)$. So $\gamma_1 \sim \gamma_2$ means $\gamma_1 - \gamma_2 \sim 0 (\mathrm{mod}\, \Omega)$.

5.4.3 The general form of Cauchy's theorem

Theorem 5.51 (General form of Cauchy's theorem)**.** *If f is analytic in an open set Ω, then $\int_\gamma f(z)\mathrm{d}z = 0$ for $\forall$ cycle γ which is homologous to zero in Ω.*

In combination with the theorem 5.50 in the previous section, we have

Corollary 5.52. *If f is analytic in a simply connected region Ω, then $\int_\gamma f(z)\mathrm{d}z = 0$ for all cycles γ in Ω.*

In combination with the fundamental theorem 5.5 of Calculus for integrals in $\mathbb{C}$, we have

Corollary 5.53. *If f is analytic in a simply connected region Ω, then $\exists$ an analytic function F in Ω s.t. $F'(z) = f(z)$ for $\forall z \in \Omega$.*

Corollary 5.54. *If f is analytic in a simply connected region Ω and $f(z) \neq 0$ for $\forall z \in \Omega$, then it is possible to define single-valued analytic branches of $\ln f(z)$ and $\sqrt[n]{f(z)}$ in Ω*

Proof. $\dfrac{f'(z)}{f(z)}$ is analytic in Ω $\overset{Corollary 5.53}{\Longrightarrow}$ $\exists$ an analytic function F s.t. $F'(z) = \dfrac{f'(z)}{f(z)}$, $\forall z \in \Omega$.

$\Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}z}\left[f(z)e^{-F(z)}\right] = 0, \forall z \in \Omega \Rightarrow f(z) = C \cdot e^{F(z)}$ for some $C \in \mathbb{C}\backslash\{0\}$.

Choose $z_0 \in \Omega$ and one of the infinite values of $\ln f(z_0)$.

$\Rightarrow \exp\left[F(z) - F(z_0) + \ln f(z_0)\right] = \dfrac{f(z)}{C} \cdot e^{-F(z_0)} = f(z), \forall z \in \Omega.$

We may define $\ln f(z) = F(z) - F(z_0) + \ln f(z_0)$, $\sqrt[n]{f(z)} = \exp\left[\dfrac{1}{n}\ln f(z)\right]$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof of Cauchy's Theorem 5.51. Let γ be a cycle in Ω satisfying $\gamma \sim 0 \mod \Omega$. The theorem 5.13 implies that

$$E := \{z \in \mathbb{C}\backslash\gamma : n(\gamma, z) = 0\} \text{ is open}$$

We define $g : \Omega \times \Omega \to \mathbb{C}$ by

$$
g(z, \zeta) := \begin{cases} \dfrac{f(z) - f(\zeta)}{z - \zeta}, & z \neq \zeta \\[2mm] f'(z), & z = \zeta \end{cases} \tag{5.38}
$$

Taylor's theorem implies g is continuous in $(z, \zeta) \in \Omega \times \Omega$. For $\forall \zeta_0 \in \Omega$, $g(z, \zeta_0)$ is analytic in Ω since $\lim\limits_{z \to \zeta_0} (z - \zeta_0) g(z, \zeta_0) = 0$.

Define $h(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_\gamma g(z, \zeta) \mathrm{d}\zeta, & z \in \Omega \\[3mm] \dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{f(z)}{z - \zeta} \mathrm{d}\zeta, & z \in E \end{cases}$. $\gamma \sim 0 \Rightarrow n(\gamma, z) = 0, \forall z \in \mathbb{C} \backslash \Omega \Rightarrow$

$C \backslash \Omega \subset E \Rightarrow \Omega \cup E = \mathbb{C}$. So h is defined on $\mathbb{C}$.

These two expressions are equal on $\Omega \cap E$ since $n(\gamma, z) = 0, \forall z \in \Omega \cap E$.

Lemma 5.19 implies that h is analytic in E.

The last exercise in Homework 6 $\Rightarrow h$ is analytic on $\Omega \Rightarrow h$ is entire.

$n(\gamma, z) = 0$ if $|z|$ is sufficiently large $\Rightarrow z \in E$ if $|z|$ large enough.

f is bounded on $\gamma \Rightarrow h(z) \to 0$ as $|z| \to \infty \Rightarrow h$ is bounded and thus $h \equiv 0$. By Liouville's Theorem 5.22,

$\dfrac{1}{2\pi i} \displaystyle\int_\gamma g(z, \zeta) \mathrm{d}\zeta = 0, \forall z \in \Omega \backslash \gamma$. Then

$$
n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta, \ \forall z \in \Omega \backslash \gamma \tag{5.39}
$$

Equation 5.39 is the **generalized version of Cauchy's integral formula**.

Let $z_0 \in \Omega \backslash \gamma$. Define $h_1(z) = (z - z_0) f(z)$. Then h_1 analytic and

$$
\int_\gamma f(z) \mathrm{d}z = \int_\gamma \frac{h_1(z)}{z - z_0} \mathrm{d}z \stackrel{(5.39)}{=} 2\pi i \cdot n(\gamma, z_0) \cdot h_1(z_0) = 0 \tag{5.40}
$$

$\square$

5.5 The Calculus of Residues

5.5.1 The Residue Theorem

Suppose f is analytic in a region Ω except for the isolated singularity at a. Consider a circle C centered at a and contained in Ω. The **residue** of f at a is defined by

$$\operatorname{Res}_{z=a} f(z) := \frac{1}{2\pi i} \int_C f(z)\mathrm{d}z \tag{5.41}$$

It is independent of choice of circle followed from the general Cauchy's theorem 5.51.

Now suppose f is analytic in a region Ω except for finitely many singularities a_j. Let γ be cycle in $\Omega' = \Omega \backslash \{a_1, \cdots, a_n\}$ which is homologous to zero w.r.t. Ω. Then

$$\gamma \sim \sum_{j=1}^{N} n(\gamma, a_j) C_j \quad \mod \Omega' \tag{5.42}$$

where C_j is any circle centered at a_j and contained in Ω'.

The general Cauchy's theorem 5.51 implies

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{N} n(\gamma, a_j) \int_{C_j} f(z)\mathrm{d}z \tag{5.43}$$

So $\dfrac{1}{2\pi i} \displaystyle\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{N} n(\gamma, a_j) \operatorname{Res}_{z=a_j} f(z)$.

We just proved the residue theorem under the assumption that there are only a finite number of singularities

Theorem 5.55 (The Residue Theorem). *Let f be analytic except for countably many*

isolated singularities a_j in a region Ω. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{N} n(\gamma, a_j)\mathrm{Res}_{z=a_j} f(z) \tag{5.44}$$

for any circle γ which is homologous to zero in Ω and does not pass through any of a_j.

Proof. We already proved the case when number of singularities is finite. For the general case,

it is enough to prove that $n(\gamma, a_j) = 0$ except for a finite number of a_j.

Let $E := \{z \in \mathbb{C}\backslash\gamma : n(\gamma, z) = 0\}$.

Then E is open by theorem 5.13 and contains all points outside of a large circle. $\Rightarrow E^c$ is compact. So E^c contains a finite number of the isolated points $a_j \Rightarrow n(\gamma, a_j) \neq 0$ only for a finite number of a_j. $\qquad\square$

Remark 5.56.

(1) In the applications it is often the case that each $n(\gamma, a_j) \in \{0, 1\}$.

(2) When f has essential singularity, there is usually no simple method to compute residues.

(3) If f has a pole of order N at a, we proved in §3.2 that

$$(z-a)^N f(z) = b_N + b_{N-1}(z-a) + \cdots + b_1(z-a)^{N-1} + \varphi(z)(z-a)^N, \; z \neq a \tag{5.45}$$

where $\varphi(z)$ is analytic at a and $b_N \neq 0$. So we have

$$\mathrm{Res}_{z=a} f(z) = b_1 = \frac{1}{(N-1)!} \cdot \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} \left[(z-a)^N f(z) \right] \tag{5.46}$$

This is because when the term $b_1(z-a)^{-1}$ is omitted, the remainder of the RHS of (5.46) is a derivative.

In particular, if $f(z) = \frac{g(z)}{h(z)}$, h has a simple zero at a and $g(a) \neq 0$, then

$$\text{Res}_{z=a} f(z) = \lim_{z \to a} \left[\frac{g(z)}{h(z)} (z - a) \right] = \lim_{z \to a} \frac{g(z)}{\frac{h(z)-h(a)}{h-a}} = \frac{g(a)}{h'(a)} \tag{5.47}$$

Example 5.57. Compute $\int_{|z|=1} \frac{e^{iz}}{z^3} dz$.

Solution. The only pole is at $z = 0$ with order 3. The residue theorem 5.55 implies:

$$\int_{|z|=1} \frac{e^{iz}}{z^3} dz = 2\pi i \text{Res}_{z=0} f(z) = 2\pi i \frac{1}{2!} \frac{d^2}{dz^2} \left[z^3 \cdot \frac{e^{iz}}{z^3} \right] \Big|_{z=0} = -\pi i \tag{5.48}$$

Or one can use Taylor's series (5.20)

$$\int_{|z|=1} \frac{e^{iz}}{z^3} dz = \int_{|z|=1} \frac{1 + iz + +\frac{(iz)^2}{2} + \cdots}{z^3} dz = -\pi i \tag{5.49}$$

$\square$

5.5.2 The Argument Principle

Theorem 5.58 (The Argument Principle). *If f is meromorphic in a region Ω with zeros a_j and poles b_k. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k) \tag{5.50}$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of zeros and poles. The sums in (5.50) are finite, and multiple zeros and poles have to be repeated as many times as their order indicates.

Proof. We assume that f has a finite number of zeros and poles, and denote that number by K.

Let N_j be the order of the zero or pole of f at $z_j \in \{a_1, a_2, \cdots, b_1, b_2, \cdots\}$.

Define $\tilde{N}_j :=$
$$
\begin{cases}
N_j, & z_j \text{ is a zero} \\
-N_j, & z_j \text{ is a pole}
\end{cases}
$$

Let $g(z) = f(z) \cdot \prod_{j=1}^{K} (z - z_j)^{-\tilde{N}_j}$. Then g only has removable singularities in Ω, and we can view it as analytic in Ω. Moreover, $g(z) \neq 0$ for $\forall z \in \Omega$.

$f(z) = g(z) \cdot \prod_{j=1}^{K} (z - z_j)^{\tilde{N}_j}$ implies that

$$
\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{j=1}^{N} \frac{\tilde{N}_j}{(z - z_j)}, \quad \forall z \neq z_j \tag{5.51}
$$

Then

$$
\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} \mathrm{d}z + \sum_{j=1}^{K} \frac{1}{2\pi i} \int_\gamma \frac{\tilde{N}_j}{z - z_j} \mathrm{d}z
$$

$$
= \sum_{j=1}^{K} \tilde{N}_j \cdot n(\gamma, z_j)
$$

$$
= \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)
$$

If f has infinite number of zeros or poles, the proof is the same as that of the residue theorem. *i.e.* $n(\gamma, z) \neq= 0$ for finite many z zeros or poles. $\qquad\square$

Theorem 5.59 (Rouchē's Theorem). *Let γ be a cycle which is homologous to zero in a region Ω s.t. $n(\gamma, z) \in \{0, 1\}, \forall z \in \Omega \backslash \gamma$.*

Suppose f, g are analytic in Ω, $|f(z) - g(z)| < |f(z)|, \forall z \in \gamma$. Then f and g have the same number of zeros enclosed by γ.

Proof. First we have $f(z) \neq 0, g(z) \neq 0$ for $z \in \gamma$.

Let $\psi(z) = \frac{g(z)}{f(z)}, z \in \gamma$. Then $|\psi(z) - 1| < 1, \forall z \in \gamma$. For $\Gamma = \psi(\gamma)$

$$
\int_\gamma \frac{\psi'(z)}{\psi(z)} \mathrm{d}z = \int_\Gamma \frac{\mathrm{d}\omega}{\omega} = 2\pi i \cdot n(\Gamma, 0) = 0
$$

since 0 is in the unbounded connected component of $\mathbb{C}\backslash\Gamma$.

The argument principle implies that $0 = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{\psi'(z)}{\psi(z)}\mathrm{d}z$ is equal to the difference of number of zeros of g and f. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The argument principle can be generalized to

Theorem 5.60 (The Argument Principle)**.** *Under the hypothesis of the argument principle 5.58, and if h is analytic in Ω, then we have*

$$\frac{1}{2\pi i}\int_\gamma h(z)\frac{f'(z)}{f(z)}\mathrm{d}z = \sum_j n(\gamma,a_j)h(a_j) - \sum_k n(\gamma,b_k)h(b_k) \qquad (5.52)$$

Remark 5.61. In §5.3.3, we proved Theorem 5.40 that if f is analytic at z_0, and $f(z) - \omega_0$ has zero of order N at z_0, then for ε small enough, there exists $\delta > 0$ s.t. $\forall\omega$ with $|\omega - \omega_0| < \delta$, $f(z) = \omega$ has exactly N roots $z_j(\omega)$ in the disk $|z - z_0| < \varepsilon$. If we apply (5.52) with $h(z) = z$, we get

$$\sum_{j=1}^N z_j(\omega) = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} z\frac{f'(z)}{f(z) - \omega}\mathrm{d}z, \ \forall\omega \in B(\omega_0,\delta) \qquad (5.53)$$

For $N = 1$, the inverse function $f^{-1}(\omega)$ can thus be represented by

$$f^{-1}(\omega) = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} z\frac{f'(z)}{f(z) - \omega}\mathrm{d}z, \ \forall\omega \in B(\omega_0,\delta) \qquad (5.54)$$

If we apply (5.52) with $h(z) = z^m$, we get

$$\sum_{j=1}^N z_j^m(\omega) = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} \frac{z^m f'(z)}{f(z) - \omega}\mathrm{d}z, \ \forall\omega \in B(\omega_0,\delta) \qquad (5.55)$$

5.5.3 Evaluation of Definite integrals

① All integrals of the form $\int_0^{2\pi} R(\cos\theta, \sin\theta)\mathrm{d}\theta$, where the integrand is a rational function of $\cos\theta$ and $\sin\theta$. The substitution $z = e^{i\theta}$ transform it into the line integral

$$\int_{|z|=1} R(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i})\frac{\mathrm{d}z}{iz}$$

It remains to determine the residues which correspond to the poles of the integrand inside $\{z : |z| < 1\}$.

Example 5.62. Compute $\int_0^\pi \dfrac{\mathrm{d}\theta}{a+\cos\theta}, a > 1$.

$$\int_0^\pi \frac{\mathrm{d}\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta}$$

$$\overset{z=e^{i\theta}}{} a + \frac{z+z^{-1}}{2}\cdot\frac{\mathrm{d}z}{iz} = \frac{1}{i}\int_{|z|=1} \frac{1}{z^2+2az+z}\mathrm{d}z$$

$$= \frac{1}{i}\int_{|z|=1} \frac{1}{\left[z-(-a+\sqrt{a^2-1})\right]\cdot\left[z-(-a-\sqrt{a^2-1})\right]}\mathrm{d}z$$

Note that $|-a+\sqrt{a^2-1}| = \frac{1}{|a+\sqrt{a^2-1}|} < 1$ and $|-a-\sqrt{a^2-1}| > 1$.

Residue Theorem implies that

$$\int_0^\pi \frac{\mathrm{d}\theta}{a+\cos\theta} = \frac{1}{i}\cdot 2\pi i \operatorname{Res}_{z=-a+\sqrt{a^2-1}} f(z)$$

$$= 2\pi \cdot \frac{1}{-a+\sqrt{a^2-1}-(-a-\sqrt{a^2-1})}$$

$$= \frac{\pi}{\sqrt{a^2-1}}$$

② An integral of the form $\int_{-\infty}^{\infty} R(x)\mathrm{d}x$ converges if and only if in the rational function R, the degree of denominator $\geqslant$ the degree of numerator+2 and has no pole lies in $\mathbb{R}$.

Consider this semicircle γ. If ρ is large enough, γ encloses all poles of R in the

$$A(-\rho, 0) \qquad\qquad B(\rho, 0)$$
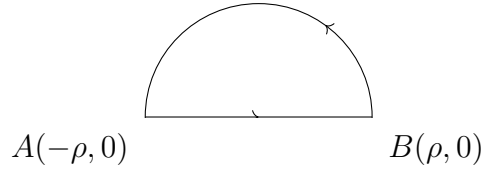
upper half-plane. It is easy to see that

$$\lim_{\rho \to \infty} \int_{z=\rho e^{it}, 0 \leqslant t \leqslant \pi} R(z)\mathrm{d}z = 0$$

So we have $\displaystyle\int_{-\infty}^{+\infty} R(x)\mathrm{d}x = 2\pi i \sum_{y>0} \mathrm{Res}_{x+iy} R(z)$.

③

(a) The same method can be applied to $\displaystyle\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x$, where the rational function has a zero of at least two at ∞. Then $|e^{iz}| = e^{-y} \geqslant 1$ in the upper-half plane. So

$$\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x = 2\pi i \sum_{y>0} \mathrm{Res}_{x+iy} R(z)e^{iz}$$

(b) We now consider the case that R has only a simple zero at ∞ and no pole on $\mathbb{R}$.

$$(0, Y)$$

$$(-x_1, 0) \qquad\qquad (x_2, 0)$$
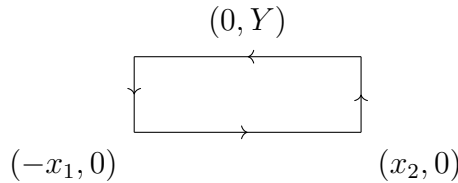
There exists $M > 0$ and $C > 0$ $\;$ s.t. this rectangle all poles of R in the upper half-plane if $x_1 > M, x_2 > M$, and $Y > M$. $|zR(z)| \leqslant C$ if $|z| \geqslant M$.

$$\left| \int_{\text{right vertical line}} R(z)e^{iz}\mathrm{d}z \right| \leqslant \int_0^Y \frac{C}{|z|} e^{-y}\mathrm{d}y \leqslant \frac{C}{x_2} \int_0^Y e^{-y}\mathrm{d}y \leqslant \frac{C}{x_2}$$

Similarly,

$$\left| \int_{\text{left vertical line}} R(z)e^{iz}dz \right| \leqslant \frac{C}{x_1}$$

$$\left| \int_{\text{upper horizontal line}} R(z)e^{iz}dz \right| \leqslant \int_{-x_1}^{x_2} \frac{C}{|z|}e^{-Y}dx \leqslant \frac{Ce^{-Y}}{Y} \int_{-x_1}^{x_2} dx = \frac{Ce^{-Y}(x_1 + x_2)}{Y}$$

Fix x_1 and x_2, setting $Y \to \infty$. Then

$$\left| \int_{-x_1}^{x_2} R(x)e^{ix}dx - 2\pi i \sum_{y>0} \text{Res}_{x+iy} R(z)e^{iz} \right| \leqslant C \left(\frac{1}{x_1} + \frac{1}{x_2} \right)$$

So

$$\int_{-x_1}^{x_2} R(x)e^{ix}dx = 2\pi i \sum_{y>0} \text{Res}_{x+iy} R(z)e^{iz}$$

(c) R has only a single zero at ∞ and a simple pole at 0. Suppose that $R(z)e^{iz} = \frac{B}{z} + \varphi(z)$ where φ is analytic at 0.
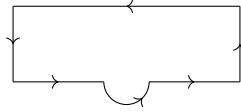
Then it is easy to use this curve to prove that

$$\lim_{\delta \to 0^+} \left[\int_{-\infty}^{-\delta} R(x)e^{ix}dx + \int_{\delta}^{\infty} R(x)e^{ix}dx \right] = 2\pi i \left[\sum_{y>0} \text{Res}_{x+iy} R(z)e^{iz} + \frac{B}{2} \right]$$

Denote this integral as P.V. $\left[\int_{-\infty}^{\infty} R(x)e^{ix}dx \right]$, called **Cauchy principle value of the integral**.

Example 5.63.

$$\text{P.V.} \left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx \right) = 2\pi i \cdot \frac{1}{2} = \pi i$$

$$= \text{P.V.} \left(\int_{-\infty}^{\infty} \frac{\cos x}{x}dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x}dx \right)$$

$$= \text{P.V.} \left(\int_{-\infty}^{\infty} \frac{\cos x}{x} \mathrm{d}x \right) + i \int_{-\infty}^{\infty} \frac{\sin x}{x} \mathrm{d}x$$

$$= i \cdot 2 \int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x$$

So we obtain $\int_{0}^{\infty} \frac{\sin x}{x} \mathrm{d}x = \frac{\pi}{2}$

④ Calculate $\int_{0}^{\infty} x^{\alpha} R(x) \mathrm{d}x$, where $\alpha \in (0, 1)$, $R(z)$ has a zero of order larger than 2 at ∞, and at most a simple pole at 0. Then

$$\int_{0}^{\infty} x^{\alpha} R(x) \mathrm{d}x \overset{x=t^2}{=} 2 \int_{0}^{\infty} t^{2\alpha+1} R(t^2) \mathrm{d}t \tag{5.56}$$

$f(x) = z^{2\alpha}$ is analytic in $\mathbb{C} \backslash \{iy : y \leqslant 0\}$ if we require $\arg f(x) \in (-\pi\alpha, 3\pi\alpha)$.

$A(-\rho, 0)$ $\qquad\qquad$ $B(\rho, 0)$
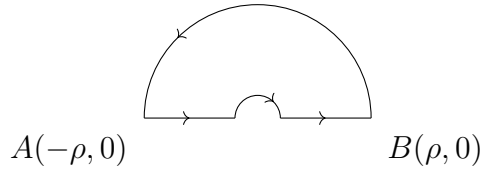
Applying residue theorem 5.55 to $z^{2\alpha+1} R(z^2)$ and take limits we have

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) \mathrm{d}z = 2\pi i \sum_{y>0} \mathrm{Res}_{x+iy} z^{2\alpha+1} R(z^2)$$

And

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) \mathrm{d}z = \int_{0}^{\infty} z^{2\alpha+1} R(z^2) \mathrm{d}z + \int_{0}^{\infty} (-z)^{2\alpha+1} R(z^2) \mathrm{d}z$$

$$= (1 - e^{2\alpha\pi i})(1 - e^{2\alpha\pi i}) \int_{0}^{\infty} z^{2\alpha+1} R(z^2) \mathrm{d}z$$

So $\int_{0}^{\infty} x^{\alpha} R(x) = \frac{2}{1 - e^{2\alpha\pi i}} \cdot 2\pi i \cdot \sum_{y>0} \mathrm{Res}_{x+iy} z^{2\alpha+1} R(z^2)$.

Example 5.64. Compute $\int_{0}^{\infty} \frac{x^{\frac{1}{2}}}{1+x^2} \mathrm{d}x$.

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\mathrm{d}x = 2\int_0^\infty \frac{t^2}{1+t^4} = \int_{-\infty}^\infty \frac{t^2}{1+t^4}\mathrm{d}t$$

Take $f(z) = \dfrac{z^2}{1+z^4}$ and apply Residue Theorem 5.55 to f, we have

$$\int_{-\infty}^\infty \frac{t^2}{1+t^4} = \int_{-\infty}^\infty f(z)\mathrm{d}z = 2\pi i \sum_{y>0} \mathrm{Res}_{x+yi} f(z) = 2\pi i[\mathrm{Res}_{\exp(\frac{i\pi}{4})}f + \mathrm{Res}_{\exp(\frac{3i\pi}{4})}f] = \frac{\sqrt{2}\pi}{2}$$

5.6 Harmonic Functions

5.6.1 Definition and basis properties

A real-valued function $u(z) = u(x,y)$ in a region Ω is **harmonic** if it is in C^2 and satisfying the Laplace's equation

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5.57}$$

We already know that if $f(z) = u(x,y) + iv(x,y)$ is analytic in Ω, then u and v satisfy the Cauchy-Riemann equations, and are therefore harmonic in Ω.

If u is harmonic in Ω, then $f(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ is analytic in Ω. This is because, for $U := \dfrac{\partial u}{\partial x}, V = -\dfrac{\partial u}{\partial y}$.

$$\begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial V}{\partial y} \\[4mm] \dfrac{\partial U}{\partial y} = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial V}{\partial x} \end{cases} \tag{5.58}$$

We may write the differential

$$f\,\mathrm{d}z = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(\mathrm{d}x + i\mathrm{d}z) = \left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + i\left(\frac{\partial u}{\partial x}\mathrm{d}y - \frac{\partial u}{\partial y}\mathrm{d}x\right) \quad (5.59)$$

In this expression, the real part is $\mathrm{d}u = \dfrac{\partial u}{\partial x}\mathrm{d}x + \dfrac{\partial u}{\partial y}\mathrm{d}y$. And if u has a conjugate harmonic function v, then the imaginary part is

$$\mathrm{d}v = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y = -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y$$

In general, however, there is no (single-valued) conjugate function. We thus define

$$^*\mathrm{d}u := -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y \quad (5.60)$$

and call $^*\mathrm{d}u$ the **conjugate differential of** $\mathrm{d}u$. We may write (5.59) as

$$f\,\mathrm{d}z = \mathrm{d}u + i^*\mathrm{d}u \quad (5.61)$$

Lemma 5.65. *Let γ be a cycle in a region Ω s.t. $\gamma \sim 0 \mod \Omega$. Then*

$$\int_\gamma {}^*\mathrm{d}u = 0 \quad (5.62)$$

Proof. (5.61) implies $\displaystyle\int_\gamma f(z)\mathrm{d}z = \int_\gamma \mathrm{d}u + i\int_\gamma {}^*\mathrm{d}u$.

Cauchy's Theorem 5.51 implies $\int_\gamma f(z)\mathrm{d}z = 0$. And $\int_\gamma \mathrm{d}u = 0$ since $\mathrm{d}u$ is an exact diffenrential.

Hence, $\int_\gamma {}^*\mathrm{d}u = 0$. $\qquad\qquad\qquad\square$

Theorem 5.66. *If Ω is simply connected and u is harmonic in Ω, then u has a (single-valued) conjugate function v which uniquely determined up to additive constant.*

Proof. The last lemma 5.65 and theorem 5.4 imply that there is a (single-valued)

function v *s.t.* $* \, du = dv$ *i.e.*

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

So v is a conjugate function of u. (Notice that we use the property of simply connection that every cycle in Ω is homologous to zero)

If v_1 and v_2 are two such harmonic functions, then $f_1 = u + iv_1$, $f_2 = u + iv_2$ are both analytic in Ω. So $f_1 - f_2 = i(v_1 - v_2)$ is analytic in Ω. The open mapping theorem 5.41 implies $f_1 - f_2$ is a constant. □

Remark 5.67. We see that the open mapping theorem 5.41 has such power that it gives a way to prove an analytic function with some closed property is constant.

Remark 5.68. The condition on simply connectness can not be removed. For instance, $u(z) = \ln|z|$ is harmonic in $\mathbb{C}\backslash\{0\}$, but it cannot be written as the real part of an analytic function since $\ln|z| = \operatorname{Re} \ln z$

5.6.2 The Mean-value Property

Theorem 5.69 (Mean-value Property)**.** *Let u be harmonic in a region Ω. If $\overline{B(z, R)} \subset \Omega$, then*

$$u(z) = \int_0^{2\pi} u(z_0 + Re^{i\theta}) d\theta \tag{5.63}$$

Proof. The previous theorem 5.66 implies that u has a conjugate function v on $\overline{B(z_0, R)}$. Consider the analytic function $f = u + iv$. The Cauchy integral formula 5.39 shows

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta \tag{5.64}$$

This theorem follows by taking the real part of the equation. □

Theorem 5.70. *If u is harmonic in Ω, and $\{z \in \mathbb{C} : 0 < R_1 \leqslant |z - z_0| \leqslant R_2\} \subset \Omega$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta = \alpha \ln r + \beta \, r \in [R_1, R_2] \tag{5.65}$$

where α and β are constants

Proof. In polar coordinate (r, θ),

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2} = r^{-1} \frac{\partial}{\partial r}\left(r \cdot \frac{\partial}{\partial r}\right) + r^{-2} \frac{\partial^2}{\partial \theta^2}$$

Let $U(r) = \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$. $z \mapsto u(z_0 + z)$ is harmonic. Then

$$\triangle U(r) = \frac{1}{2\pi} \int_0^{2\pi} \triangle u(z_0 + re^{i\theta}) d\theta = 0 \tag{5.66}$$

Therefore, $\dfrac{\partial}{\partial r}\left(r\dfrac{\partial U}{\partial r}\right) = 0$. Therefore, $U(r) = \alpha \ln \gamma + \beta$. $\qquad \square$

Theorem 5.71 (Maximal Principle of Harmonic Function). *A nonconstant harmonic function has neither a maximum nor a minimum in its region of definition.*

Proof. Suppose u attains a maximum at $z_0 \in \Omega$. $\exists R > 0$ s.t. $B(z_0, R) \subset \Omega$. Suppose $\exists a \in B(z_0, R)$ s.t. $u(a) < u(z_0) = M$.

The mean-value property implies

$$M = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta < M$$

by continuity. This causes a contradiction.

So u is a constant in $B(z_0, R)$.

Then for every z_1 in the region, since we can find a series of disk such that the center of the disk is in the previous disk and z_0 is the center of the first disk, z_1 is in the last disk.

Then by the property above, $u(z_0) = u(z_1)$. So u is a constant, which causes a contradiction! $\qquad\square$

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List of Theorems