****

LIN150117

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

January 2, 2025

Contents

Theorem 0.1. *If $B \in \mathbb{R}^{n\times n}$ satisfying $||B|| < 1$, then $I + B$ invertible and*

$$||(I+B)^{-1}||_2 \leqslant \frac{1}{1-||B||}$$

Cholesky transformation

Doolottle Decomposition

Condition number

$$\mathcal{K}_2(A)^2 = \operatorname{cond}(A^T A) \text{ if } A \text{ full of column rank}$$

$$\operatorname{cond}(A) \geqslant \frac{|\lambda_1|}{|\lambda_n|} \text{ equality holds if } A \text{ is symmetric matrix}$$

$$||A||_2^2 = \rho(A^T A) = ||A^T A||_2$$

Theorem 0.2. *If $\det A \neq 0$, then*

$$\min_{|A+\delta A|=0} \frac{||\delta A||_2}{||A||_2} = \frac{1}{\operatorname{cond}(A)_2}$$

Moore-Penrose pseudoinverse

Theorem 0.3. *For the least squrare equation of $Ax = b$,*

$$\frac{||\delta x||_2}{||x||_2} \leqslant \mathcal{K}_x(A) \cdot \frac{||A\delta x||_2}{||Ax||_2}$$

Hauseholder transiformation

Givens transoformation

$$R_k(B) = -\ln ||B^k||^{\frac{1}{k}}$$

$$R(B) = -\ln \rho(B)$$

Jacobian iteration

Gauss-Seidel iteration

Theorem 0.4.

Theorem 0.5. *By Steepest Descent Algorithm,*

$$||x^k - x^*||_A \leqslant \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)^k ||x^0 - x^*||_A$$

Theorem 0.6. *In conjugated gradient method,*

$$P^{(k)} = r^{(k)} + \beta_{k-1} P^{(k-1)}$$

with

$$\beta_{k-1} = -\frac{(r^{(k)}, AP^{(k-1)})}{(P^{(k-1)}, AP^{(k-1)})}, r^{(k)} = b - Ax^{(k)}$$

And the iteration

$$x^{(k+1)} = x^{(k)} + \alpha_k P^{(k)}$$

where

$$\alpha_k = \arg\min_{\alpha\in\mathbb{R}} \varphi(x^{(k)} + \alpha P^{(k)}) = \frac{(r^{(k)}, P^{(k)})}{(P^{(k)}, AP^{(k)})}$$

This iteration satsifies

$$x^{(k)} = \arg \min_{x - x^{(0)} \in \mathrm{Span}\{P^{(0)}, \cdots, P^{(k-1)}\}} \varphi(x)$$

Theorem 0.7. $(r^{(i)}, r^{(j)}) = 0, i \neq j$

$(AP^{(i)}, P^{(j)}) = 0, i \neq j$

$(r^{(j)}, P^{(i)}) = 0, i < j$

$$\text{Span}\{r^0, \cdots, r^{(k)}\} = \text{Span}\{P^{(0)}, \cdots, P^{(k)}\} = \text{Span}\{r^{(0)}, Ar^{(0)}, \cdots, A^k r^{(0)}\}$$

Theorem 0.8 (Arnoldi Elimination)**.**

$$\begin{aligned}
Aq_k &= Q_k h_k + \beta_k q_{k+1} \\
h_k &= Q_k^T A q_k \\
\beta_k &= \|Aq_k - Q_k h_k\| \\
q_{k+1} &= \frac{Aq_k - Q_k h_k}{\beta_k}
\end{aligned}$$

Then

$$AQ_k = Q_k H_k + h_{k+1,k} q_{k+1} e_k^T$$

and

$$AQ_n = Q_n H_n,\ Q_n^T A Q_n = H_n$$

Theorem 0.9 (Lanczos Elimination)**.** *For A symmetric,*

$$AQ_k = Q_k T_k + \beta_k q_{k+1} e_k^T$$

where T_k symmetric three triangular matrix.

Theorem 0.10 (Non-symmetric Lanczos Elimination)**.** $\omega_i v_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$

$$\begin{aligned}
\beta_j v_{j+1} &= Av_j - \alpha_j v_j - \gamma_{j-1} v_{j-1} \\
\gamma_j \omega_{j+!} &= A^T \omega_j - \alpha_j \omega_j - \beta_{j-1} \omega_{j-1}
\end{aligned}$$

Take inner product we obain $\alpha_j = \omega_j^T A v_j$. And $\beta_j = \sqrt{|\eta_j|}$ for $\eta_j = \tilde{v}_{j+1}^T \tilde{\omega}_{j+1}$.

It can be derived from

$$AV_k = V_k T_k + \beta_k v_{k+1} e_k^T$$

$$A^T W_k = W_k T_k^T + \gamma_k \omega_{k+1} e_k^T$$

$$V_k^T W_k = T_k, v_{k+1}^T W_k = 0, \omega_{k+1}^T V_k = 0, \omega_{k+1}^T v_{k+1} = 1$$

$$\mathcal{K}_k(A, v_1) = \mathrm{Span}\{v_1, \cdots, v_k\}$$

$$\mathcal{K}_k(A^T, \omega_1) = \mathrm{Span}\{\omega_1, \cdots, \omega_k\}$$

Theorem 0.11 (Conjugate Gradient Method). *It follows from the fact that* $(b - Ax_k) \perp \mathcal{K}_k(A, b)$ *if and only if* $x_k = \arg\min\{\|x - x_*\|_A : x \in \mathcal{K}_k(A, b)\}$.

Take $q_1 = \dfrac{b}{\|b\|_2}$. *Then Lanzos method implies*

$$AQ_k = Q_k T_k + \beta_k q_{k+1} e_k^T$$

For $x_k = Q_k y_k$, *it is equivalent to calculate*

$$T_k y_k = \beta_1$$

T_k *symmetric positive definite, then*

$$T_k = L_k D_k L_k^T$$

where

$$L_k = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \gamma_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_{k-1} & 1 \end{pmatrix},\ D_k = \mathrm{diag}\{\delta_1, \cdots, \delta_k\}$$

Then $x_{k+1} = \tilde{P}_{k+1} z_{k+1} = x_k + \zeta_{k+1} \tilde{p}_{k+1}$ *where* $\tilde{P}_k = Q_k L_k^{-T}$.

Theorem 0.12 (MINRES method). $A \in \mathbb{R}^{n \times n}$ *symmetric, by Lanzos elimination,*

$$AQ_k = Q_{k+1} \hat{T}_k$$

where

$$T_k = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{k-1} \\ & & \beta_{k-1} & \alpha_k \\ & & & \beta_k \end{pmatrix}$$

For Givens transformation $G_k G_{k-1} \cdots G_1 \hat{T}_k = \begin{pmatrix} R_k \\ 0 \end{pmatrix}$.

Then suffices to find y *s.t.*

$$y = \arg\min \|R_k y - t_k\|$$

Theorem 0.13. *If* $\varphi \in C^p$, $\varphi(x^*) = \varphi'(x^*) = \varphi''(x^*) = \cdots = \varphi^{(p-1)}(x^*) = 0$, *then*

$$\lim_{k \to \infty} \frac{e_{k+1}}{e_k^p} = \frac{\varphi^{(p)}(x^*)}{p!}$$

Theorem 0.14 (Steflensen iteration method). $\psi(x) = x - \dfrac{(\varphi(x) - x)^2}{\varphi(\varphi(x)) - 2\varphi(x) + x}$. *Then fixed points of* $\psi(x)$ *are that of* φ. *If* φ *converges of order 1, then* ψ *converges for order 2. If* φ *converges of order* $p > 1$, *then* ψ *converges of order* $2p - 1$. *All under the condition that* $\varphi \in C^{p+1}$.

Theorem 0.15. $f(x^*) = 0, f'(x^*) \neq 0$. $f \in C^2$ *locally. Then Newton's iteration* $\varphi(x) =$

$x - \dfrac{f(x)}{f'(x)}$ *converges of order 2 locally. And*

$$\lim_{k\to\infty} \frac{x_{k+1} - x^*}{(x_k - x^*)^2} = \frac{f''(x^*)}{2f'(x^*)}$$

For $f(x) = (x - x^*)^m g(x)$*,*

$$\varphi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{(x - x^*)g(x)}{mg(x) + (x - x^*)g'(x)}$$

$$\varphi'(x) = 1 - \frac{1}{m}$$

So Newton's method converges linearly.

For $\varphi(x) = x - \dfrac{mf(x)}{f'(x)}$*,* $\varphi'(x^*) = 0$*. It converges of at least two.*

Or $\mu(x) = \dfrac{f(x)}{f'(x)}$*. It ha simple root* x^**. Use Newton's method.*

Theorem 0.16 (GeXian)**.** $f'(x_k) \simeq \dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$.

If on interval $\triangle = [x^* - \delta, x^* + \delta]$ $f'(x) \neq 0$ *and* $f \in C^2(\triangle)$*.*

$M\delta < 1$ *where* $M = \dfrac{\max_{x\in\triangle} |f''(x)|}{2\min_{x\in\triangle} |f'(x)|}$.

Then if $x_0, x_1 \in \triangle$*, it converges of order* $\dfrac{1}{2}(1 + \sqrt{5})$*.*

Theorem 0.17. *Iteration converges locally if* $\rho(\Psi'(x^*)) = \sigma < 1$*.*

Theorem 0.18. *Newton's method* $x^{k+1} = x^k - (F'(x^k))^{-1}F(x^k)$ *converges of order* > 1 *if* $F'(x)$ *invertible and continuous. If* $\|F'(x) - F'(x^*)\| \leqslant \gamma\|x - x^*\|$*,* $\forall x \in S$*, then* $\{x^k\}$ *converges of at least order 2.*

Theorem 0.19. $x^{k+1} = x^k - A_k^{-1}F(x^k)$ *where*

$$\triangle A_k = \arg\min_{A\in Q} \|A\|_F$$

under the Frobenius norm and $Q = \{A \in \mathbb{R}^{n\times n} : L(A_k + A)p^k = q^k, p^k = x^{k+1} - x^k, q^k = F(x^{k+1}) - F(x^k)\}$.

Lemma 0.20. $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{p\times p}$, $X \in \mathbb{C}^{n\times p}$ *satisfies*

$$AX = XB, \operatorname{rank}(X) = p$$

Then there exists unitary matrix $Q \in \mathbb{C}^{n\times n}$

$$Q^H AQ = T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

and $\sigma(T_{11}) = \sigma(A) \cap \sigma(B)$.

Theorem 0.21 (Schur decomposition)**.** $A \in \mathbb{C}^{n\times n}$*, then there exists unitary matrix* $Q \in Cbb^{n\times n}$ *s.t.* $Q^H AQ = U$.

Theorem 0.22 (Real Schur decomposition)**.** $A \in \mathbb{R}^{n\times n}$*. Then there exists* $Q \in \mathbb{R}^{n\times n}$ *s.t.*

$$Q^T AQ = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ & R_{22} & \cdots & R_{2m} \\ & & \ddots & \vdots \\ & & & R_{mm} \end{pmatrix}$$

Theorem 0.23. μ *is an eigenvalue of* $A + E \in \mathbb{C}^{n\times n}$*. If* $X^{-1}AX$ *diagonal, then*

$$\min_{\lambda\in\sigma(A)} |\lambda - \mu| \leqslant \|X^{-1}\|_p \cdot \|X\|_p \cdot \|E\|_p$$

Condition number w.r.t. λ of A: $\dfrac{1}{|y^H x|}$

Theorem 0.24. *If A, E symmetric in $\mathbb{R}^{n\times n}$. Eigenvalues of $A, E, A+E$ are*

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$$

$$\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_n$$

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$$

Then $\lambda_i + \nu_n \leqslant \mu_i \leqslant \lambda_i + \nu_1$.

Proof use Rayeligh quotient.

List of Theorems