Tsinghua University

Numerical Analysis

Fall 2024

Homework 1

Lin Zejin April 16, 2025

First, V has dimension $2N+3$.

Actualy, for $u \in V$, it should satisfy

$$u(x_i) = u(x_{i+1}), u'(x_i) = u'(x_{i+1}), u''(x_i) = u''(x_{i+1})$$

There are already $3N-3$ linear equations.

u is composed by N polynoimal, so it is in the polynoimal space with free elements $5N$.

Hence the dimension of V is $5N - 3N + 3 = 2N + 3$.

Now for $f_i \in V$, denote $f_{i,j}$ to be $f_i|_{[x_j,x_{j+1}]}$.

We construct basis V such that the length of $\text{supp}(f_i) \leq \frac{3}{N}$.

First, take a funtion $f_0^j(x)$ with support in $[x_0, x_3]$ such that

$$f_{i,0}(x_1) = f_{i,1}(x_1), f'_{i,0}(x_1) = f'_{i,1}(x_1), f''_{i,0}(x_1) = f''_{i,1}(x_1)$$

$$f_{i,1}(x_2) = f_{i,2}(x_2), f'_{i,1}(x_2) = f'_{i,2}(x_2), f''_{i,1}(x_2) = f''_{i,2}(x_2)$$

$$f_{i,2}(x_3) = 0, f'_{i,2}(x_3) = 0, f''_{i,2}(x_3) = 0$$

Noticed that There are 9 linear equations but the free degree is 15, so we can find 5 linearly independent functions, which are linearly independent in $[x_0, x_1]$.

Similarly, we find a function $f_i^(x)$ with support in $[x_j, x_{j+3}]$, $1 \leq j \leq N-4$, such that

$$f_{i,j}(x_j) = 0, f'_{i,j}(x_j) = 0, f''_{i,j}(x_j) = 0$$

$$f_{i,j}(x_{j+1}) = f_{i,j+1}(x_{j+1}), f'_{i,j}(x_{j+1}) = f'_{i,j+1}(x_{j+1}), f''_{i,j}(x_{j+1}) = f''_{i,j+1}(x_{j+1})$$

$$f_{i,j+1}(x_{j+2}) = f_{i,j+2}(x_{j+2}), f'_{i,j+1}(x_{j+2}) = f'_{i,j+2}(x_{j+2}), f''_{i,j+1}(x_{j+2}) = f''_{i,j+2}(x_{j+2})$$

$$f_{i,j+2}(x_{j+3}) = 0, f'_{i,j+2}(x_{j+3}) = 0, f''_{i,j+2}(x_{j+3}) = 0$$

There are 12 linear equations but the free degree is 15. So we can find 2 functions which are linearly independent in $[x_j, x_{j+1}]$.

Finally, for functions with support in $[x_{N-3}, x_N]$,

$$f_{i,N-3}(x_{N-3}) = 0, f'_{i,N-3}(x_{N-3}) = 0, f''_{i,N-3}(x_{N-3}) = 0$$

$$f_{i,N-3}(x_{N-2}) = f_{i,N-2}(x_{N-2}), f'_{i,N-3}(x_{N-2}) = f'_{i,N-2}(x_{N-2}), f''_{i,N-3}(x_{N-2}) = f''_{i,N-2}(x_{N-2})$$

$$f_{i,N-2}(x_{N-1}) = f_{i,N-1}(x_{N-1}), f'_{i,N-2}(x_{N-1}) = f'_{i,N-1}(x_{N-1}), f''_{i,N-2}(x_{N-1}) = f''_{i,N-1}(x_{N-1})$$

There are 9 linear equations but the free degree is 15. So we can find 6 linearly independent functions.

We obtain $5 + 2(N-4) + 6 = 2N + 3$ functions. Now we suffice to prove they are linearly independent.

Actually, if there is some linear combination such that

$$\sum_{i=1}^{2N-3} a_i f_i = 0$$

Then since there are only 5 functions with support on $[x_0, x_1]$, which is linearly independent $\Rightarrow a_i = 0$ for those functions.

Similarly, by induction, there are only 2 functions with support on $[x_j, x_{j+1}]$, which is linearly independent $\Rightarrow a_i = 0$ for those functions.

Therefore, the linear combination is actually the linear combination of those functions with support on $[x_{N-3}, x_N]$. But they are linearly independent. So $a_i = 0, \forall i$.

So we have found those $2N - 3$ linearly independent functions, which forms a basis.