3. Newton 迭代法和割线法

求解方程 $f(x)=0$，设已有近似值 x_k

由 Taylor 展开得

$$0=f(x^*)\approx f(x_k)+f'(x_k)(x^*-x_k)$$

由此解得 x^* 的近似值作为 x_{k+1}

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)} \qquad (6.3.1)$$

相应的迭代函数为

$$\varphi(x)=x-\frac{f(x)}{f'(x)}$$

定理 6.3.1. 设 $f(x^*)=0$，$f'(x^*)\neq 0$，且 f 在 x^* 的邻域上有二阶连续导数，则 Newton 迭代法局部收敛到 x^* 且至少二阶收敛

$$\lim_{k\to\infty}\frac{x_{k+1}-x^*}{(x_k-x^*)^2}=\frac{f''(x^*)}{2f'(x^*)}$$

证明：由于 f 有二阶连续导数所以 φ' 存在且连续，且有

$$\varphi(x^*)=x^*,\quad \varphi'(x^*)=0$$

所以 Newton 迭代局部收敛

由 Taylor 展开可得

$$0=f(x^*)=f(x_k)+f'(x_k)(x^*-x_k)+\frac{f''(\xi_k)}{2}(x^*-x_k)^2$$

$$0=f(x_k)+f'(x_k)(x_{k+1}-x_k)$$

$$\Rightarrow \quad \frac{x_{k+1}-x^*}{(x_k-x^*)^2} = \frac{f''(\xi_k)}{2f'(x_k)}$$

$$\Rightarrow \quad \lim_{k\to\infty} \frac{x_{k+1}-x^*}{(x_k-x^*)^2} = \frac{f''(x^*)}{2f'(x^*)}$$

· 重根情形

$$f(x) = (x-x^*)^m g(x)$$

$m>1$, $g(x^*) \neq 0$, g 有二阶导数.

$$\varphi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{(x-x^*)g(x)}{mg(x)+(x-x^*)g'(x)}$$

$$\varphi'(x^*) = 1 - \frac{1}{m}$$

此时 Newton 法局部线性收敛.

方法 1. $\varphi(x) = x - \frac{mf(x)}{f'(x)}$

$$\varphi(x^*) = x^*, \qquad \varphi'(x^*) = 0.$$

则迭代法

$$x_{k+1} = x_k - \frac{mf(x_k)}{f'(x_k)}$$

至少二阶收敛.

方法2. 令 $\mu(x) = \dfrac{f(x)}{f'(x)}$

$$\mu(x) = (x-x^*)\frac{g(x)}{mg(x)+(x-x^*)g'(x)}$$

则 x^* 是 $\mu(x)=0$ 的单根. 使用 Newton 法得

$$\varphi(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

由此得到的迭代法至少二阶收敛.

- 割线法

采用差商近似导数 即

$$f'(x_k) \approx \frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}$$

Newton 法变为

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k)-f(x_{k-1})}$$

定理 5.3.2. 设 $f(x^*)=0$, 在区间 $\Delta = [x^*-\delta, x^*+\delta]$ 上 $f'(x) \neq 0$ 且 $f \in C^2(\Delta)$, 设 $M\delta < 1$ 其中

$$M = \frac{\max\limits_{x\in\Delta} |f''(x)|}{2\min\limits_{x\in\Delta} |f'(x)|}$$

则当 $x_0, x_1 \in \Delta$ 时, 割线法按 $P = \frac{1}{2}(1+\sqrt{5})$ 阶收敛到

x^*.

证明：设 $x_{k-1}, x_k \in \Delta$，令

$$P_1(x) = f(x_k)\frac{x-x_{k-1}}{x_k-x_{k-1}} + f(x_{k-1})\frac{x-x_k}{x_{k-1}-x_k}$$

$$R(x) = P_1(x) - f(x), \quad E(t) = R(t) - \frac{R(x^*)}{(x^*-x_{k-1})(x^*-x_k)}(t-x_{k-1})(t-x_k)$$

则有

$$E(x^*) = 0, \quad E(x_{k-1}) = R(x_{k-1}) = 0, \quad E(x_k) = R(x_k) = 0$$

根据微分中值定理有 存在 ξ 在 x^*, x_{k-1}, x_k 之间

有

$$E''(\xi) = -f''(\xi) - \frac{2R(x^*)}{(x^*-x_{k-1})(x^*-x_k)} = 0$$

$$\Rightarrow \quad R(x^*) = -\frac{1}{2}f''(\xi)(x^*-x_{k-1})(x^*-x_k)$$

$$\Rightarrow \quad P_1(x^*) = -\frac{1}{2}f''(\xi)(x^*-x_{k-1})(x^*-x_k)$$

另一方面

$$P_1(x^*) = P_1(x^*) - P_1(x_{k+1})$$

$$= \frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}(x^*-x_{k+1})$$

$$= f'(\eta)(x^*-x_{k+1})$$

其中 η 在 x_k, x_{k-1} 之间

$$\Rightarrow \quad x^*-x_{k+1} = -\frac{1}{2}\frac{f''(\xi)}{f'(\eta)}(x^*-x_{k-1})(x^*-x_k)$$

$$\Rightarrow \quad e_{k+1} = \left|\frac{f''(\xi)}{2f'(\eta)}\right| e_k\, e_{k-1}$$

$$\le M\, e_k\, e_{k-1} \le M\delta\cdot\delta < \delta$$

$\Rightarrow \quad x_{k+1} \in \Delta$ 且

$$e_k \le (M\delta)^k\,\delta \to 0$$

$$\Rightarrow \quad \lim_{k\to\infty} x_k = x^*$$

关于收敛阶，仅给出启发式分析

当 $k\to+\infty$ 时有

$$e_{k+1} = C\, e_k\, e_{k-1}, \qquad C = \left|\frac{f''(x^*)}{2f'(x^*)}\right|$$

令 $E_k = C\, e_k$ 则有

$$E_{k+1} = E_k\, E_{k-1}$$

$$\Rightarrow \quad P_{k+1} = P_k + P_{k-1}, \qquad P_k = \ln E_k$$

$$\Rightarrow \quad P_k = C_1\lambda_1^k + C_2\lambda_2^k$$

$$\lambda_1 = \frac{1+\sqrt5}{2}, \qquad \lambda_2 = \frac{1-\sqrt5}{2}$$

$$\Rightarrow \quad e_k \approx e^{C_1\lambda_1^k}$$

$\Rightarrow$ 收敛阶数为 $\lambda_1 = \frac{1+\sqrt5}{2}$