Homework 1

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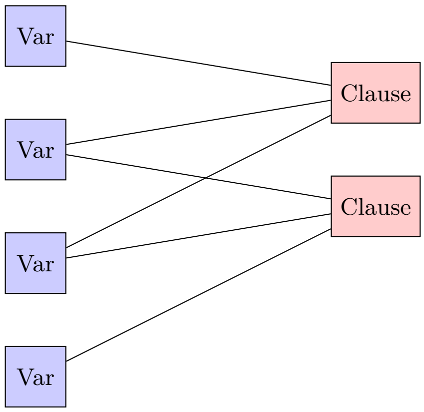
- **Collaborators:** I finish this homework by myself.

Problem 1. (a) Reduce from the instance of MAX-E3SAT-6.



Variables x_i have $\sigma(x_i) \in \{0,1\}$ and Clauses $c_i = x^1_{j_i} \wedge x^2_{j_i} \wedge x^3_{j_i}$ have $\sigma(c_i) \in [7]$ to represent the state of c_i.

Therefore, constraint is naturally induced.

In the instance of MAX-E3SAT, the radio of $|U|$ and $|V|$ is 2. So this is a regular Label-Cover Game for $K = 2, L = 7$ and $|V| = 2|U|$.

In the lecture we have proved that this is an instance of $\mathrm{MAX} - \mathrm{LC}_{1,1-\epsilon}$ for some ϵ.

So $\mathrm{MAX} - \mathrm{LC}_{1,1-\epsilon}$ is NP-Hard.

(b) We actually can construct another graph induced by (a).

We add $\bar{x}_i$ to the graph in (a) and add the induced constraints from c_i contains variable x_i to $\bar{x}_i$.

Here the Label-Cover Game is regular and symmetric.

Then for the $\mathrm{MAX} - \mathrm{E3SAT} - 6_{1,1-\epsilon}$ instance, the completeness is trivial.

Now we prove the soundness. That's because, if $\mathrm{OPT}_{\mathrm{MAX-E3SAT-6}} \le 1 - \epsilon$, consider any $\sigma : U \to \{0,1\}, V \to [7]$. At least $(1-\epsilon)|V|$ clauses are not satisfied by $\sigma|_U$. For each clause, there exists at least one variable $x_i/\bar{x}_i$ such that do not satisfy the constraint.

So Verifier rejects with probability at least $(1-\epsilon)|V|/2|V| = (1-\epsilon)/2$. So the soundness property holds if we set $\epsilon' = \frac{1+\epsilon}{2}$.

So we prove that $\mathrm{GAP} - \mathrm{LC}(K,L)_{1,1-\epsilon}$ is NP-Hard for some ϵ and K, L even if the graph is regular and symmetric.

By Raz' Paralled Repetition Theorem, we can reduce an instance of $\mathrm{GAP}-\mathrm{LC}(K,L)_{1,\delta}$ to the instance of $\mathrm{GAP}-\mathrm{LC}_{1,\exp(-\Omega(\frac{\delta^3 t}{\log t}))}$. Therefore, we finally prove that for any $\eta > 0$, there exists K, L such that $\mathrm{GAP}-\mathrm{LP}(K,L)_{1,\eta}$ is NP-Hard.

Problem 2. (a) For a regular Label-Cover problem $G = (U, V, E)$ that every veritce in U matches k vertices in V, $|U| = |V| = n$, consider the k-uniform hypergraph $H = (V', E')$ where $V' = E$ and k-tuples are all $[(u, v_1), (u, v_2), \cdots, (u, v_k)]$ for $(u, v_i) \in E$. $[L']$ now represents the value of (u, v_i), *i.e.* $[L'] = [L] \times [K]$. $[K] = [k + 1]$.

The maps are defined as: For the labeling $\sigma : [V] \to [L] \times [K]$, $\sigma(u, v_i) = (l, k)$. If $\pi_{(u,v_i)}(k) = l$ is matching in Labek-Cover problem, then we let $\pi_e^i(\sigma(u, v_i)) = k + 1$. Otherwise, if (l, k) does not satisfy the constraint, then we let $\pi_e^i(\sigma(l, k)) = i$.

So the constraint is weakly satisfied iff at least two edges in the k-tuples are satisfied in the constraint before. Also, the constraint is strongly satisfied iff all edges in the k-tuples are satisfied.

Completeness is trivial since if there is some label in the Label-Cover Game satisfy all constraint, then it can be naturally induced in the hypergraph.

Soundness is because: Assume OPT $\geq \epsilon$ in k-ary-Consistent-Labeling problem. Then we choose all edges (u_i, v_j) that are satisfied in the Label-Cover Game, denoted as S. There are at least $2\epsilon n$ edges. Now we label each u_i, v_j one by one.

Since the graph G is regular, at most $2k - 1$ edges in S have common vertice with an edge in S.

So each time we choose an arbitrary $e = (u, v) \in S$, label it with the label in k-ary-Consistent-Labeling and then we remove those edges in S who intersects with e.

In the end, for those vertices that have not been labeled yet, label it randomly.

Then at least $\frac{2\epsilon n}{2k} = \frac{\epsilon}{k} n$ edges are satisfied in Label-Cover-Game.

Therefore OPT $\geq \frac{\epsilon}{k}$ for Label-Cover Game.

As a result, if OPT $\leq \eta$ in Label-Cover Game, then OPT $\leq k\eta$ in k-ary-Consistent-Labeling problem.

Since $\mathrm{MAX}-\mathrm{LC}_{1,\eta}$ is NP-Hard, to distingish instance with strong value 1 and weak value less than $k\eta$ k-ary-Consistent-Labeling problem is NP-Hard $\forall \eta > 0$.

Here we end the proof.

(b)

Problem 3. Consider all values $d(r, v) \pmod{\frac{1}{2}}$. They divide $[0, \frac{1}{2})$ into $|V| + 1$ pieces of interval.(including the interval $[v, v]$ if exists) If we choose θ in each interval, edges that will be removed are the same, so the cost is the same.

As a result, we can try θ in each interval and find the minimum cost. This will be less than 2OPT.

Problem 4. (a) If a connected component has diameter at most k in the $(10, 0.1, 1, 1)$-expandar G, we prove that it has at most 10^k vertices.

By induction, $k = 1$ is trivial. Assume $k - 1$ holds for it. Assume subgraph G' has the maxmimum number of vertices. There isn't any vertex in G' that has distance less than $k - 1$ with each vertex in G' and also connects with other vertex u outside. Otherwise, u can be added to g', which causes contradiction with the maximum property. Then for k, any vertex in the graph with diameter $k-1$ has degree 10 so at most 10^k vertices are connected to the graph. Since any vertex beyond G' has distance larger than k with some vertices in G' as we proved before, the expanded graph has at most 10^k vertices.

So each connected component has at most $10^{1/2\log_{10} n} = n^{1/2}$ vertices in this problem. As n large enough, $n^{1/2} < 0.1n$. For those connected components $S_1, \cdots, S_k$, removed edges are

$$|\partial S_1 \cup \partial S_2 \cup \cdots \partial S_k| = \frac{1}{2}\sum_{t=1}^{k} |\partial S_t| \geq \frac{1.01}{2}\sum_{t=1}^{k} |S_t| > 0.5n$$

So we must have deleted $\Omega(n)$ edges.

Now we set the pair (s_i, t_i) to be all (u, v) where $u, v \in G$ and distance between u and v is k.

Then for any possible connected component in multicut, vertices u, v in it have distance is less than k.

For a $(10, 0.1, 1.1)$-expandar graph, by (a) we removed at least $\frac{1}{2}n$ if $k = \frac{1}{2}\log_{10} n$.

However, in LP case, we can set $x_e = \frac{1}{k}$ for any edge e. Then the cost will be

$$\frac{1}{k} \cdot |E| = \frac{5}{k}|V| = \frac{5n}{k}$$

So the integral gap is $\Omega(\log n)$.

Problem 5. (a)

$$\begin{aligned}
\mathbb{E}(\text{cut value}) &= \sum_{(i,j)\in E} \omega_{ij} \cdot \frac{\arccos\langle v_i, v_j\rangle}{\pi} \\
&= \sum_{(i,j)\in E} \omega_{ij} - \sum_{(i,j)\in E} \frac{\frac{\pi}{2} + \arcsin\langle v_i, v_j\rangle}{\pi} \\
&\geq \sum_{(i,j)\in E} \omega_{ij} - \beta \cdot \sum_{(i,j)\in E} \omega_{ij}\sqrt{\frac{1+\langle v_i, v_j\rangle}{2}} \\
&\geq \sum_{(i,j)\in E} \omega_{ij} - \beta \left(\sum_{(i,j)\in E} \omega_{ij}\frac{1+\langle v_i, v_j\rangle}{2}\right)^{1/2} \left(\sum_{(i,j)\in E} \omega_{ij}\right)^{1/2} \\
&= 1 - \beta(1-\text{SDP})^{1/2} \\
&\geq 1 - \beta(1-\text{OPT})^{1/2}
\end{aligned}$$

where $\beta = \sup_{\alpha\in(-1,1)} \frac{\frac{\pi}{2}+\arcsin\alpha}{\sqrt{1+\alpha}} < +\infty$ by L'hospital rule.

Therefore, it is a $1-\epsilon$ vs. $1-\beta\sqrt{\epsilon}$ search algorithm.

(b)

Similar to max-cut. If we set $\mathbb{F}_2 = \{\pm 1\}$, then

$$\frac{1-bx_ix_j}{2} = \begin{cases} 1 & x_i \oplus x_j = b \\ 0 & x_i \oplus x_j \neq b \end{cases}$$

where $1 \oplus 1 = -1 \oplus -1 = -1, 1 \oplus -1 = 1 \oplus 1 = 1$.

So the problem is to maximize the objective

$$\sum_{(i,j)\in E} \omega_{ij} \frac{1 - b_{ij} x_i x_j}{2}$$

Similarly, we set the SDP relaxation:

$$\min \sum_{(i,j)\in E} \omega_{ij} \frac{1 - b_{ij} \langle v_i, v_j \rangle}{2}$$

conditioned on $\|v_i\| = 1$.

After finding a minimum, we design a randomize algorithm as follows:

Uniformly sample $\vec{r} \sim S^{n-1}$.

Set $x_i = \operatorname{sgn} \langle \vec{r}, \vec{v_i} \rangle$.

Then

$$\begin{aligned}
\mathbb{E}(\text{cut value}) &= \sum_{(i,j)\in E} \omega_{ij} \cdot \frac{\arccos b_{ij} \langle v_i, v_j \rangle}{\pi} \\
&= \sum_{(i,j)\in E} \omega_{ij} - \sum_{(i,j)\in E} \frac{\frac{\pi}{2} + \arcsin b_{ij} \langle v_i, v_j \rangle}{\pi} \\
&\geq \sum_{(i,j)\in E} \omega_{ij} - \beta \cdot \sum_{(i,j)\in E} \omega_{ij} \sqrt{\frac{1 + b_{ij} \langle v_i, v_j \rangle}{2}} \\
&\geq \sum_{(i,j)\in E} \omega_{ij} - \beta \left(\sum_{(i,j)\in E} \omega_{ij} \frac{1 + b_{ij} \langle v_i, v_j \rangle}{2} \right)^{1/2} \left(\sum_{(i,j)\in E} \omega_{ij} \right)^{1/2} \\
&= 1 - \beta (1 - \text{SDP})^{1/2} \\
&\geq 1 - \beta (1 - \text{OPT})^{1/2}
\end{aligned}$$

where $\beta = \sup_{\alpha\in(-1,1)} \frac{\frac{\pi}{2} + \arcsin \alpha}{\sqrt{1+\alpha}} < +\infty$ by L'hospital rule.

Therefore, it is a $1 - \epsilon$ vs. $1 - \beta\sqrt{\epsilon}$ search algorithm.

Problem 6. For an arbitrary graph $G = (V, E)$, $\epsilon > 0$, WLOG we assme $\sum_{(i,j)\in E} \omega(i,j) = 1$. Let $d_i = \max_{(i,j)\in E} \left\lceil \frac{\omega(v_i, v_j)}{\epsilon} \right\rceil$. We can construct a graph $G' = (V', E')$ with $|V'| \leq \lceil \frac{1}{\epsilon} \rceil |V|$ by split each v_i into $v_{i,1}, v_{i,2}, \cdots, v_{i,d_i}$ and if $(i,j) \in E$, connects $v_{i,t}$ and $v_{j,t}$ equipped with weight ϵ.

Add another $|V'|^2 - |V'|$ vertices to the graph G' and we can use the algorithm to find an α-approximating solution for G' in $f(|V'|^2, \frac{|E'|}{|V'|^2}) = \mathrm{p}oly(|V'|^2) = \mathrm{p}oly(|V'|)$.

Use an algorithm to strengthen this solution:

for $\exists (i,t) \quad s.t. \sum_{j\in[|V|]} \epsilon \cdot \mathbf{1}[((i,t),(j,t)) \in E', x_{i,t} x_{j,t} < 0] < \sum_{j\in[|V'|]} \epsilon \cdot \mathbf{1}[((i,t),(j,t)) \in E', x_{i,t} x_{j,t} > 0]$ **do**

  $x_i \leftarrow -x_i$

end for

Since each time in the loop will be larger than the previous result of at least ϵ, the algorithm will eventually terminate in $O(|E'|/\epsilon)$.

Now for each $j, t, t \geq 2$, we have

$$\sum_{i\in[|V|],\omega(i,j)\geq t} \frac{1-x_{j,t}x_{i,1}}{2} \geq \sum_{i\in[|V|],\omega(i,j)\geq t} \frac{1+x_{j,t}x_{i,1}}{2} \Rightarrow x_{j,t} \sum_{i\in[|V|],\omega(i,j)\geq t} x_{i,1} \leq 0$$

Now we use another algorithm to find a feasible solution in G as follows: set $x_i = x_{i,1}$ for each vertex v_i, *i.e.* we let v_i be in the same set of $v_{i,1}$.

This result will not be less than the previous result. That's because, suffices to prove that

$$\sum_{((i,t),(j,k))\in E'} \frac{1-x_{i,1}x_{j,1}}{2} \geq \sum_{((i,t),(j,k))\in E'} \frac{1-x_{i,t}x_{j,k}}{2}$$

which is equivalent to

$$\sum_{((i,t),(j,k))\in E'} x_{i,t}x_{j,k} \geq \sum_{((i,t),(j,k))\in E'} x_{i,1}x_{j,1}$$

$$\Leftrightarrow \sum_{(i,j)\in E} x_{i,1} \sum_{t=1}^{\omega(i,j)} x_{j,t} \geq \sum_{i,j} x_{i,1} \sum_{t=1}^{\omega(i,j)} x_{j,1}$$

$$\Leftrightarrow \sum_{(j,t)} (x_{j,t} - x_{j,1}) \sum_{i:\omega(i,j)\geq t} x_{i,1} \geq 0$$

Problem 7. hyperplane cuts $\frac{\alpha}{\pi}$ edges in G_d with angle α.

Then totally, hyperplane cuts

$$\frac{\int_{\arccos\rho^*}^{\pi} \sin^{d-2}\alpha \frac{\alpha}{\pi}\,\mathrm{d}\alpha}{\int_{\arccos\rho^*}^{\pi} \sin^{d-2}\alpha\,\mathrm{d}\alpha} < \frac{\int_{\arccos\rho^*}^{\pi} (\pi-\alpha)^{(d-2)/2} \frac{\alpha}{\pi}\,\mathrm{d}\alpha}{\int_{\arccos\rho^*}^{\pi} (\pi-\alpha)^{(d-2)/2}\,\mathrm{d}\alpha} = \frac{\arccos\rho^*}{\pi} + O(\frac{1}{d})$$

The first inequality is because $\dfrac{\sin\alpha}{\sin\beta} > \dfrac{\sqrt{\pi-\alpha}}{\sqrt{\pi-\beta}}$ if $\alpha < \beta$. Thus the probability of α in the left is less than the probability of β in the right if $\alpha < \beta$.

Problem 8. $f : \{\pm 1\}^n \to \mathbb{R}$ is a linear combination of function $f : \{\pm 1\}^n \to \{\pm 1\}$, which can be written in the form of linear combination of Fourier base functions:

$$f(x) = \sum_{S\subset[n]} \hat{f}(S)\chi_S(x)$$

where $\chi_S(x) = \prod_{i\in S} x_i$ is a multilinear polynomial.

So it is expressible as a multilinear polynomial.

The uniqueness is because, if there is some multilinear polynoimal g such that $g(x) = f(x), \forall x \in \{\pm 1\}$. Then using Parserval's Theorem we obtain that

$$\sum_{S\subset[n]} (f \hat{-} g)(S)^2 = \mathbb{E}_{\vec{x}\sim\{\pm1\}^n}(f(\vec{x}) - g(\vec{x}))^2 = 0$$

So $f-g=\sum_{S\subset[n]}(\widehat{f-g})(S)\chi_S=0$.

Problem 9.

$$\langle f,g\rangle=\left\langle\sum_{S\subset[n]}\hat{f}(S)\chi_S,\sum_{S\subset[n]}\hat{f}(S)\chi_S\right\rangle=\sum_{S_1,S_2\subset[n]}\hat{f}(S_1)\hat{g}(S_2)\left\langle\chi_{S_1},\chi_{S_2}\right\rangle=\sum_{S\subset[n]}\hat{f}(S)\hat{g}(S)$$

However, if we let $f=\chi_{\{x\}},g=\chi_{\{y\}},h=\chi_{\{x,y\}}$, then

$$\mathbb{E}_{\vec{t}}\chi_{\{x\}}(t)\chi_{\{y\}}(t)\chi_{\{x,y\}}(t)=\mathbb{E}_{\vec{t}}t_x^2t_y^2$$

But

$$\hat{f}(S)\hat{g}(S)\hat{h}(S)\equiv 0,\,\forall S\subset[n]$$

due to they are Fourier basis.