Homework 1

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- **Collaborators:** I finish this homework by myself.

Problem 1. *Proof of Lemma 2.5.3.* $B_1 = \begin{cases} 1 & x \leq 0 \\ 0 & x \geq 1 \\ 1-x & 0 < x < 1 \end{cases}$.

Then $B_2 = \begin{cases} 1 & x \leq -\frac{1}{2} \\ 0 & x \geq \frac{3}{2} \\ -\frac{x^2}{2} - \frac{1}{2}x + \frac{7}{8} & -\frac{1}{2} < x < \frac{1}{2} \\ \frac{1}{2}(\frac{3}{2} - x)^2 & \frac{1}{2} < x < \frac{3}{2} \end{cases}$

So $B_3 = \begin{cases} 0 & x \leq -1 \\ -\dfrac{t^3 + 3\,t^2 - 3\,t - 5}{6} & -1 < x < 0 \\ \dfrac{t^3}{3} - \dfrac{1}{2}t^2 - \dfrac{1}{2}t + \dfrac{5}{6} & 0 < x < 1 \\ \dfrac{(2-t)^3}{6} & 1 < x < 2 \\ 1 & x \geq 2 \\ 3 & -1 < x < 2 \end{cases}$ with B_2

So $B_3(x), B_3(x-1), B_3(x-2)$ linearly independent. $\qquad\square$

Proof of Lemma 2.5.4. If

$$\sum_{k=-1}^{n+1} \lambda_k B_3\left(\frac{x-a-kh}{h}\right) = 0$$

Noticed that

$$B_3\left(\frac{x-a-kh}{h}\right) = B_3\left(\frac{x-a}{h} - k\right)$$

So on each interval $(a+jh, a+(j+1)h)$, it will be

$$\sum_{k=-1}^{n+1} \lambda_k B_3\left(j-k+\frac{x-jh-a}{h}\right) = \lambda_{j-1}B_3\left(-1+\frac{x-jh-a}{h}\right) + \lambda_j B_3\left(0+\frac{x-jh-a}{h}\right) + \lambda_{j+1}B_3\left(1+\frac{x-jh-a}{h}\right) = 0$$

where $\dfrac{x-jh-a}{h} \in (0,1)$

So we have matrix equation

$$
\begin{pmatrix}
B_3(-1+t) & B_3(t) & B_3(1+t)\cdots & \cdots & \cdots & 0 & \\
0 & B_3(-1+t) & B_3(t) & B_3(1+t)\cdots & \cdots & 0 & \\
0 & 0 & B_3(-1+t) & B_3(t) & B_3(1+t) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \\
0 & 0 & 0 & \cdots & B_3(-1+t) & B_3(t) & B_3(1+t) \\
0 & 0 & 0 & 0 & \cdots & B_3(-1+t) & B_3(t) \\
0 & 0 & 0 & 0 & \cdots & 0 & B_3(-1+t)
\end{pmatrix}
\begin{pmatrix}
\lambda_{-1} \\ \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \\ \lambda_n
\end{pmatrix} = 0
$$

on $(0,1)$ By lemma 2.5.3, we have $\lambda_{-1} = \lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. So they are linearly independent. $\qquad\square$

Proof of lemma 2.6.2. We have proved in lemma 2.6.1 that

$$
\sum |c_i| \le \frac{\|P\|_Y}{\theta - \Omega(\delta)}
$$

Then

$$
\|f - P\|_X \le \max_{y \in Y, x \in X} |f(x) - f(y)| + |f(y) - P(y)| + |P(y) - P(x)|
$$

$$
\le \omega(\delta, f) + \|f - P\|_Y + \sum |c_i| \Omega(\delta)
$$

$$
\le \omega(\delta, f) + \|f - P\|_Y + \frac{\|P\|_Y \Omega(\delta)}{\theta - \Omega(\delta)}
$$

As $\Omega(\delta) \to 0$, $\frac{2}{\theta}$ is what we need. $\theta > 0$ since g_i linearly independent. $\qquad\square$

Problem 2. Use straight line to connect

$$
(x_0, 1), (x_1, 2), \cdots, (x_n, n+1)
$$

It forms $L(x)$.

Then $B_3(L_3(x) - k)$, $k = 0, 1, 2, \cdots, n+2$ linearly independent in S_3^n and whose discontinuities are among

$x_0, x_1, \cdots, x_n$.

It is B-spline for $x_0 < x_1 < \cdots < x_n$.

Problem 3. For f odd.

Consider the optimal approximation polynomial p on $[-1, 1]$. Since $f(x) = -f(-x)$, so $-p(-x)$ is also the optimal approximation polynomial of $-f(-x) = f(x)$ on $[-1, 0] \Rightarrow$ By the uniqueness, $p(x) = -p(-x)$ *i.e.* $p(x)$ is odd.

For f is even

Consider the optimal approximation polynomial p on $[-1, 1]$. Since $f(x) = f(-x)$, so $p(-x)$ is also the optimal approximation polynomial of $f(-x) = f(x)$ on $[-1, 0] \Rightarrow$ By the uniqueness, $p(x) = p(-x)$ i.e. $p(x)$ is even.

Problem 4. For $s(x_j) = m_j$, $s(x)$ can be expressed as

$$s(x) = m_j \alpha_j(x) + m_{j+1} \alpha_{j+1}(x) + y_j \beta_j(x) + y_{j+1} \beta_{j+1}(x)$$

on (x_j, x_{j+1}), where α_i, β_j is the Hermite polynomial of degree 3.

Use $s''(x_j^+) = s''(x_j^-)$ and we have

$$\frac{\mu_j}{h_j} m_{j+1} + \left(\frac{\lambda_j}{h_{j-1}} - \frac{\mu_j}{h_j}\right) m_j - \frac{\lambda_j}{h_{j-1}} m_{j-1} = d_j$$

where

$$d_j = \frac{1}{3}\left(\lambda_j y_{j-1} + 2y_j + \mu_j y_{j+1}\right), \lambda_j = \frac{h_j}{h_{j-1} + h_j}, \mu_j = \frac{h_{j-1}}{h_{j-1} + h_j}, h_j = x_{j+1} - x_j$$

The boundary condition $s(x_0) = m_0 = a, s(x_n) = m_n = b$ contributes to

$$\begin{pmatrix} \frac{\lambda_1}{h_0} - \frac{\mu_1}{h_1} & \frac{\mu_1}{h_1} & 0 & \cdots & 0 & 0 \\ -\frac{\lambda_2}{h_1} & \frac{\lambda_2}{h_1} - \frac{\mu_2}{h_2} & \frac{\mu_2}{h_2} & \cdots & 0 & 0 \\ 0 & -\frac{\lambda_3}{h_2} & \frac{\lambda_3}{h_2} - \frac{\mu_3}{h_3} & \frac{\mu_3}{h_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\frac{\lambda_{n-1}}{h_{n-2}} & \frac{\lambda_{n-1}}{h_{n-2}} - \frac{\mu_{n-1}}{h_{n-1}} \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{n-1} \end{pmatrix} = \begin{pmatrix} d_1 + \frac{\lambda_1}{h_0} a \\ d_2 \\ \vdots \\ d_{n-1} - \frac{\mu_{n-1}}{h_{n-1}} b \end{pmatrix}$$

The boundary condition $s''(x_0) = y_0', s''(x_n) = y_n'$ contributes to

$$-2y_0 - y_1 + \frac{3}{h_0} m_1 - \frac{3}{h_0} m_0 = \frac{1}{2} y_0' h_0$$

So we have

$$\begin{pmatrix} -\frac{3}{h_0} & \frac{3}{h_0} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\lambda_1}{h_0} & \frac{\lambda_1}{h_0} - \frac{\mu_1}{h_1} & \frac{\mu_1}{h_1} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{\lambda_2}{h_1} & \frac{\lambda_2}{h_1} - \frac{\mu_2}{h_2} & \frac{\mu_2}{h_2} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{\lambda_3}{h_2} & \frac{\lambda_3}{h_2} - \frac{\mu_3}{h_2} & \frac{\mu_3}{h_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -\frac{\lambda_{n-1}}{h_{n-2}} & \frac{\lambda_{n-1}}{h_{n-2}} - \frac{\mu_{n-1}}{h_{n-1}} \\ 0 & 0 & \cdots & 0 & 0 & -\frac{3}{h_{n-1}} & \frac{3}{h_{n-1}} \end{pmatrix} \cdot \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \\ m_{n-1} \\ m_n \end{pmatrix} = \begin{pmatrix} 2y_0 + y_1 + \frac{1}{2} y_0' h_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ 2y_n + y_{n-1} + \frac{1}{2} y_n' h_{n-1} \end{pmatrix}$$

And the period boudary condtion $s(x_0) = s(x_n), s'(x_0) = s'(x_n), s''(x_0) = s''(x_n)$ contributes to

$$m_0 = m_n, \frac{\mu_j}{h_j} m_{j+1} + \left(\frac{\lambda_j}{h_{j-1}} - \frac{\mu_j}{h_j}\right) m_j - \frac{\lambda_j}{h_{j-1}} m_{j-1} = d_j$$

for $j = 0$, $\mu_{-1} = \mu_n$.

So

$$
\begin{pmatrix}
\frac{\lambda_1}{h_0} - \frac{\mu_1}{h_1} & \frac{\mu_1}{h_1} & 0 & \cdots & 0 & -\frac{\lambda_1}{h_0} \\
-\frac{\lambda_2}{h_1} & \frac{\lambda_2}{h_1} - \frac{\mu_2}{h_2} & \frac{\mu_2}{h_2} & \cdots & 0 & 0 \\
0 & -\frac{\lambda_3}{h2} & \frac{\lambda_3}{h_2} - \frac{\mu_3}{h_3} & \frac{\mu_3}{h_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\mu_{n-1}}{h_{n-1}} & 0 & \cdots & 0 & -\frac{\lambda_{n-1}}{h_{n-2}} & \frac{\lambda_{n-1}}{h_{n-2}} - \frac{\mu_{n-1}}{h_{n-1}}
\end{pmatrix}
\cdot
\begin{pmatrix}
m_1 \\ m_2 \\ \vdots \\ m_{n-1}
\end{pmatrix}
=
\begin{pmatrix}
d_1 \\ d_2 \\ \vdots \\ d_{n-1}
\end{pmatrix}
$$

In short, we discuss about three different boundary condition in this problem.

Problem 5. The boundary condition is

$$
2m_0 + m_1 = 3f[x_0, x_1] - \frac{1}{2}f''(x_0)h
$$

So combined with the equation in Lemma 2.5.2, we have

$$
Am = d
$$

where

$$
A =
\begin{pmatrix}
2 & 1 & & & & & \\
\lambda_1 & 2 & \mu_1 & & & & \\
0 & \lambda_2 & 2 & \mu_2 & & & \\
\vdots & \vdots & \vdots & & \vdots & & \\
0 & 0 & 0 & \lambda_{n-1} & 2 & \mu_{n-1} \\
0 & 0 & 0 & 0 & 1 & 2
\end{pmatrix}
$$

$$
d =
\begin{pmatrix}
d_0 \\ d_1 \\ \vdots \\ d_n
\end{pmatrix}
$$

$d_j = 3\lambda_j f[x_{j-1}, x_j] + 3\mu_j f[x_j, x_{j+1}]$, $1 \le j \le n-1$. $d_0 = 3f[x_0, x_1] - \frac{1}{2}f''(x_0)h$, $d_n = 3f[x_{n-1}, x_n] - \frac{1}{2}f''(x_n)h$.

Let $q = [m_0 - f_0', m_1 - f_1', \cdots, m_{n-1} - f_{n-1}']^T$, then

$$
Aq = c
$$

where $c = [c_j]^T$,

$$
c_j = d_j - \lambda_j f'(x_{j-1}) - 2f'(x_j) - \mu_j f'(x_{j+1}), 1 \le j \le n-1
$$

$$
c_0 = d_0 - 2f'(x_0) - f'(x_1), c_n = d_n - f'(x_{n-1}) - 2f'(x_n)
$$

4

Similar to Lemma 2.5.2, we can prove that

$$\|q\|_\infty \le \|A^{-1}\| \cdot \|c\| \le \|c\|_\infty$$

And we have proved in Lemma 2.5.2 that

$$|c_j| \le \frac{1}{24}h^3\|f^{(4)}\|_\infty, 1 \le j \le n-1$$

Now suffices to prove it for c_0, c_n.

By Talor's equation

$$
\begin{aligned}
c_0 &= 3 \cdot \frac{f(x_1) - f(x_0)}{h_0} - 2f'(x_0) - f'(x_1) - \frac{1}{2}f''(x_0)h_0 \\
&= 3\left(f'(x_0) + \frac{1}{2}f''(x_0)h_0 + \frac{1}{6}f'''(x_0)h_0^2 + \frac{1}{6h_0}\int_{x_0}^{x_1}(x_1-v)^3 f^{(4)}(v)\,\mathrm{d}v\right) \\
&\quad - 2f'(x_0) - \left(f'(x_0) + f''(x_0)h_0 + \frac{1}{2}f'''(x_0)h_0^2 + \frac{1}{2}\int_{x_0}^{x_1}(x_1-v)^2 f^{(4)}(v)\,\mathrm{d}v\right) - \frac{1}{2}f''(x_0)h_0 \\
&= \frac{1}{2}\int_{x_0}^{x_1}\left[\frac{1}{h_0}(x_1-v)^3 - (x_1-v)^2\right]f^{(4)}(v)\,\mathrm{d}v \\
&\le \frac{1}{2}\|f^{(4)}(v)\|_\infty \cdot \frac{1}{h_0}|\int_0^{h_0}\tau^3 - h_0\tau^2\,\mathrm{d}\tau| \\
&= \frac{1}{24}\|f^{(4)}(v)\|_\infty
\end{aligned}
$$

By symmetry, c_n also satisfy the same bound.

So Lemma 2.5.2 still holds. Hence Theorem 2.5.1 still holds.   *i.e.*

$$\|s - f\|_\infty \le \frac{5}{384}h^4\|f^{(4)}\|_\infty$$

Problem 6. Let $\varphi(x)$ be the Hermite polynomial of degree 3 w.r.t f,   *i.e.*

$$\varphi(x) = f(x_j)\alpha_j(x) + f(x_{j+1})\alpha_{j+1}(x) + f'(x_j)\beta_j(x) + f'(x_{j+1})\beta_{j+1}(x)$$

Then we have $\varphi(x) - f(x) = \dfrac{f^{(4)}(\xi)}{24}\omega^2(x)$, where $\omega(x) = (x - x_j)(x - x_{j+1})$.

Note that Lemma 2.5.2 tells us $|m_j - f'_j| \le \dfrac{1}{24}h^3\|f^{(4)}\|_\infty$ Since

$$s(x) - \varphi(x) = (m_j - f'_j)\beta_j(x) + (m_{j+1} - f'_{j+1})\beta_{j+1}(x)$$

we have

$$|s'(x) - f'(x)| \leq |s'(x) - \varphi'(x)| + |\varphi'(x) - f'(x)|$$
$$\leq \frac{1}{24}h^3\|f^{(4)}\|_\infty A + \frac{1}{24}\|f^{(4)}\|_\infty B$$
$$\leq \frac{1}{6}h^3\|f^{(4)}\|_\infty$$

$$|s''(x) - f''(x)| \leq |s''(x) - \varphi''(x)| + |\varphi''(x) - f''(x)|$$
$$\leq \frac{1}{24}h^3\|f^{(4)}\|_\infty A' + \frac{1}{24}\|f^{(4)}\|_\infty B'$$
$$\leq \frac{2}{3}h^3\|f^{(4)}\|_\infty$$

where

$$A = \max|\beta_j'(x)| + |\beta_{j+1}'(x)| \leq 1,\ B = \max|\omega(x)\omega'(x)| \leq 3h^3$$

$$A' = \max|\beta_j''(x)| + |\beta_{j+1}''(x)| \leq \frac{4}{h}\ B' = \max|(\omega^2(x))''| \leq 12h^2$$

So $\|s' - f'\| \leq \frac{1}{6}h^3\|f^{(4)}\|_\infty$, $\|s'' - f''\| \leq \frac{2}{3}h^3\|f^{(4)}\|_\infty$