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PULSAR TIMING MEASUREMENTS AND THE SEARCH FOR GRAVITATIONAL WAVES

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ABSTRACT

Pulse arrival time measurements of pulsars may be used to search for gravitational waves with periods on the order of 1 to 10 years and dimensionless amplitudes $\sim 10^{-11}$. The analysis of published data on pulsar regularity sets an upper limit to the energy density of a stochastic background of gravitational waves, with periods ~ 1 year, which is comparable to the closure density of the universe.

Subject headings: cosmology — gravitation — pulsars — relativity

I. INTRODUCTION

The most quiet and regular pulsars are astronomical clocks of amazing precision. The arrival time of a pulse a year hence may sometimes be predicted with an uncertainty of only 200 μs. Such precision provides the means of detecting ultralong-period (~ 3 years) gravitational waves of dimensionless amplitude $\gtrsim 10^{-11}$. Sazhin (1978) discusses the possibility of observing the effects of the gravitational waves from a close binary along the line of sight to a more distant pulsar, but he concludes that for no pulsar-binary pair is the alignment likely to be close enough to detect an effect. In this paper we consider other possible effects of gravitational waves on pulsar measurements.

A gravitational wave incident upon either a pulsar or upon the Earth changes the measured frequency and appears then as an anomalous residual in the pulse arrival time, as we discuss in § II. In § III we show that the current observations of pulsars already set an upper limit of 1.3×10^{-29} g cm^{-3} for the energy density of a stochastic background of gravitational waves of possible cosmological origin. And finally in § IV we discuss the orders of magnitude of various sources of ultralong-wavelength gravitational waves.

II. THEORY

The effect of a gravitational wave upon the measured frequency of a pulsar is easily found by a slight variation of a calculation by Estabrook and Wahlquist (1975). Only the results are presented below.

As a weak gravitational wave passes through the solar system, the metric is given by

$$ds^2 = -dt^2 + (1 + A_+)dx^2 + (1 - A_+)dy^2 + dz^2 + 2A_\times dxdy\,. \tag{1}$$

The spatial coordinates form a nearly Cartesian coordinate system centered upon the barycenter of the solar system. The dimensionless quantities A_+ and $A_\times$ are both assumed to be much less than 1 and are functions of $t - z$ with both $A_+(t - z < 0) = A_\times(t - z < 0) = 0$. Thus the wave is a plane wave traveling in the positive z direction with A_+ and $A_\times$ the wave amplitudes in the two polarizations.

For simplicity we assume that there is a pulsar of constant frequency, ν_0, with direction cosines α, β, and γ in the x, y, and z directions. As the wave goes by the solar system, the measured frequency $\nu(t)$ varies slightly and is given by

$$[\nu_0 - \nu(t)]/\nu_0 = [(\alpha^2 - \beta^2)A_+(t) + 2\alpha\beta A_\times(t)]/2(1 + \gamma)\,; \tag{2}$$

terms of second and higher orders of A_+ and $A_\times$ have been discarded. This small variation in the measured frequency will cause an anomalous residual, $R(t)$, in the pulse arrival time; hence

$$R(t) = \int_0^t dt[\nu_0 - \nu(t)]/\nu_0\,, \tag{3}$$

where $R(t)$ is measured in units of seconds.

To characterize a gravitational wave coming from an unknown direction it is necessary to monitor $R(t)$ for at least three pulsars not coplanar with the solar system. Each essentially gives the projection of the metric perturbation onto a plane perpendicular to the direction toward the pulsar. However, a slight ambiguity remains: the forward and backward directions of the wave cannot be distinguished.

III. UPPER LIMIT OF A STOCHASTIC BACKGROUND

A useful measure of the sensitivity of a gravitational wave detector is the upper limit which it can place on a stochastic background. Generally the residual of a pulsar is not predominantly (if at all) caused by gravitational waves; however, the energy density found below is an interesting upper limit.

For much of the past decade the young, noisy pulsars such as the Crab have garnered most of the observational and theoretical attention. Hence it is difficult to use only published data and analyses to search for the effects of gravitational waves. Nonetheless, from the thesis of Helfand (1977) it is clear that at least a few pulsars (1919+21, 1933+16, and 2016+28 among them) have been observed over a period of at least 2000 days and have shown an rms residual of less than or equal to 0.15 ms. We use this number as the standard below.

The effective energy density of a gravitational wave (cf. Misner, Thorne, and Wheeler 1973) is

$$\rho = (32\pi G)^{-1}\langle h_{ij,0}h_{ij,0}\rangle \,, \tag{4}$$

where h_{ij} is the metric perturbation, $\langle\ \rangle$ denotes an average over time, and summations over i and j are implied. For a gravitational wave of the form of equation (1) this is

$$\rho = (16\pi G)^{-1}\langle \dot{A}_+{}^2(t) + \dot{A}_\times{}^2(t)\rangle \,. \tag{5}$$

The background is assumed to consist of a distribution of stationary random waves which is isotropic and of both polarizations with different directions and polarizations uncorrelated. The quantities A_+ and $A_\times$ have the same spectral features by assumption, and for either polarization we may generally write

$$A(t) = \oiint \mathscr{A}(t, \theta, \varphi)d\Omega \,. \tag{6}$$

It follows then that

$$\langle \dot{A}^2\rangle = \oiint T^{-1}\int_0^T \dot{\mathscr{A}}^2(t, \theta, \varphi)dtd\Omega \tag{7}$$

for some long time T. And further

$$\langle \dot{A}^2\rangle = 4\pi\langle \dot{\mathscr{A}}^2(t, \theta, \varphi)\rangle \tag{8}$$

for any choice of θ and φ. Finally equations (5) and (8) combine to give the energy density of the gravitational waves in terms of a time average of the gravitational waves from only one direction and one polarization:

$$\rho = (2G)^{-1}\langle \dot{\mathscr{A}}^2\rangle \,. \tag{9}$$

In principle the spectral density of $\mathscr{A}(t)$ may be determined from a measurement of the anomalous residual, $R(t)$, of a perfect pulsar. Equations (2), (3), and (6) imply that

$$R(t) = \int_0^t \oiint \left[\frac{\alpha^2 - \beta^2}{2(1+\gamma)}\mathscr{A}_+(t, \theta, \varphi) + \frac{\alpha\beta}{(1+\gamma)}\mathscr{A}_\times(t, \theta, \varphi)\right]d\Omega dt + \text{similar terms} \,, \tag{10}$$

where "similar terms" are added to account for the presence of the gravitational waves at the pulsar, and these additional terms have spectral features identical with the first two. If the pulsar is monitored for some time T, then the autocorrelation function of the residual is

$$\phi_T(\tau) \equiv T^{-1}\int_{-T/2}^{T/2} R(t)R(t+\tau)dt = 4\Theta T^{-1}\int_{-T/2}^{T/2}\left[\int_0^t \mathscr{A}(t')dt'\right]\left[\int_0^{t+\tau}\mathscr{A}(t'')dt''\right]dt \,, \tag{11}$$

where

$$\Theta \equiv \oiint \left[\frac{\alpha^2 - \beta^2}{2(1+\gamma)}\right]^2 d\Omega = \oiint \left(\frac{\alpha\beta}{1+\gamma}\right)^2 d\Omega = \frac{2\pi}{3} \,. \tag{12}$$

Therefore the autocorrelation functions of $R(t)$ and of $2\Theta^{1/2}\int \mathscr{A}(t)dt$ are identical and so are their spectral densities. It now follows from equation (9) that the anomalous residual of a perfect pulsar gives the average energy density of a random background of gravitational waves by

$$\rho = 3(16\pi G)^{-1}\langle \ddot{R}^2(t)\rangle \,. \tag{13}$$

The time average of $\ddot{R}^2$ is found from the autocorrelation function; the quantity $\phi_T(\tau)$ may be Fourier analyzed to determine the spectral density,

$$\phi_T(\tau) = \sum_{n=-\infty}^{\infty} |r_n|^2 \exp(in2\pi f_0\tau) \,, \tag{14}$$

where $f_0 = T^{-1}$. The mean square residual is hence

$$\langle R^2(t)\rangle = \phi_T(0) = \sum_{n=-\infty}^{\infty} |r_n|^2 . \tag{15}$$

And from equation (13) an upper limit to ρ is

$$\rho = 3(16\pi G)^{-1} \sum_{n=-\infty}^{\infty} (n2\pi f_0)^4 |r_n|^2 . \tag{16}$$

We now assume that the gravitational wave energy spectrum is flat, centered on some frequency f, and has a bandwidth also equal to f. It follows that $d\rho/df$ is a constant and

$$\rho = f d\rho/df . \tag{17}$$

If the residual were caused solely by the gravitational waves, then such an energy spectrum would imply that

$$|r_n|^2 = (8\pi G\rho/3)/(n2\pi f_0)^4 fT \tag{18}$$

for all n such that

$$\tfrac{1}{2}f \le nf_0 \le \tfrac{3}{2}f , \tag{19}$$

and $|r_n|^2$ is zero otherwise. In the limit that fT is large, such a spectral density gives a mean square residual of

$$\langle R^2(t)\rangle = \frac{208}{243}\frac{G\rho}{\pi^3 f^4} . \tag{20}$$

The measured value of 2.3×10^{-8} s^2 for $\langle R^2\rangle$ thus gives an energy density upper limit of

$$\rho = 1.28 \times 10^{-29}(\text{g cm}^{-3})(f_{\text{yr}})^4 , \tag{21}$$

where f_{yr} is the frequency, in units of cycles per year, of the center and the width of the energy spectrum. For comparison the density needed for closure of the universe in the standard models of cosmology is approximately

$$\rho_{\text{closure}} = 2 \times 10^{-29}(\text{g cm}^{-3})h^2 , \tag{22}$$

where h is Hubble's constant measured in units of 100 km s^{-1} Mpc^{-1}. Hence the measurement of pulsar residuals easily puts interesting limits on the cosmological role of a background of ultralong-wavelength gravitational waves.

A more detailed analysis of the data will clearly lower this upper limit. Helfand mentions that for PSR 1919+21, as an example, the small residual is primarily the result of measurement uncertainty. It is likely that in this case a careful noise analysis will show a large high-frequency component, and hence the upper limit on ρ for low frequencies should come down.

Another means of improving the analysis is to monitor two pulsars nearby in the sky. The cross-correlation of their residuals then distinguishes some event on a pulsar from one on Earth as the cause of a residual.

IV. THE POSSIBLE OBSERVATION OF INDIVIDUAL SOURCES

Pulsar analysis is most appropriate for detecting a large-amplitude, long-period gravitational wave. The close encounter of two supermassive black holes at a cosmological distance might generate such a wave, but of course the existence of black holes and their interactions are matters of speculation. We take the point of view that many questions concerning black holes will not be answered until gravitational wave astronomy becomes a reality; hence, in this section we derive the consequences of a supermassive black hole interaction without regard to the likelihood of such an event.

Generally the sources of gravitational waves can be divided into two classes depending upon whether or not the quantity A in equation (1) ultimately returns to zero. A permanent change in A implies a permanent change in the measured frequency of a pulsar and, from equation (3), a residual which grows linearly in time. A source of this nature is the flyby and gravitational deflection of two massive objects, i.e., gravitational bremsstrahlung. This effect has been studied by a number of workers (cf. Kovács and Thorne 1978). In the slow-motion, weak-field limit the magnitude of the change in A is

$$\Delta A \sim 4m_1 m_2/br , \tag{23}$$

where m_1 and m_2 are the masses of the objects, b is the impact parameter, and r is the distance from the encounter to the Earth. For supermassive objects at a cosmological distance this is

$$\Delta A \sim 10^{-14}\frac{m_1 m_2}{(10^{10}\,M_\odot)^2}\left(\frac{0.1\text{ lt-yr}}{b}\right)\left(\frac{10^{10}\text{ lt-yr}}{r}\right) . \tag{24}$$

For b any bigger than 0.1 lt-yr the pulsar frequency changes over a time scale long enough that the change does not appear discrete.

How small a frequency change can be measured? Manchester and Taylor (1974) report a change of one part in 10^{10} for PSR 1508+55; and it seems, judging from their Figure 1, that they might have detected a change an order of magnitude smaller than this. But PSR 1508+55 has an rms residual about 10 times that of, say, PSR 1919+21. We conclude that with the present instrumentation a change in the frequency of a quiet pulsar by one part in 10^{12} might not escape observation.

This number might be improved by a slight variation in the methods of data analysis. A gravitational wave changes the period of a pulsar but not any of the period derivatives by a significant amount. This effect could easily be taken into account when fitting the pulse arrival time to a polynomial in the determination of the pulsar parameters.

The second class of sources—those where A returns to its initial value—are more difficult to detect. The frequency change is transient. And although the residual as a function of time might in principle show a net change, the present methods of data analysis would tend to hide such an effect. The collision of two black holes is an example of this second class. The analyses of Smarr (1979) and Detweiler (1979) show that even for this quite relativistic situation the dimensional analysis of Zel'dovich and Novikov (1971) is essentially correct; the amplitude of the gravitational wave is

$$A_{\max} \sim m_1 m_2 / r(m_1 + m_2)\,, \tag{25}$$

and the time scale for the variation of A is

$$\tau \sim m_1 + m_2\,. \tag{26}$$

For the equal-mass case these are

$$A_{\max} \sim 10^{-13}\left(\frac{M}{10^{10}\,M_\odot}\right)\left(\frac{10^{10}\text{ lt-yr}}{r}\right) \tag{27}$$

and

$$\tau \sim 10^5\text{ s}\left(\frac{M}{10^{10}\,M_\odot}\right) = 1.16\text{ days}\left(\frac{M}{10^{10}\,M_\odot}\right). \tag{28}$$

Together these give a residual of

$$R \sim \tau A = 10^{-8}\text{ s}\left(\frac{M}{10^{10}\,M_\odot}\right)^2\left(\frac{10^{10}\text{ lt-yr}}{r}\right) \tag{29}$$

from equation (3).

Finally we consider a periodic source: two supermassive black holes of mass M in a circular orbit of radius R_0 about one another. The Landau and Lifshitz (1975) quadrupole moment formalism implies that

$$A \sim 5 \times 10^{-14}\left(\frac{200M}{R_0}\right)\left(\frac{M}{10^{10}\,M_\odot}\right)\left(\frac{10^{10}\text{ lt-yr}}{r}\right). \tag{30}$$

And the frequency of the radiation is

$$\omega = 2 \times 10^{-8}\text{ s}^{-1}\left(\frac{200M}{R_0}\right)^{3/2}\left(\frac{10^{10}\,M_\odot}{M}\right), \tag{31}$$

which gives a sinusoidal residual of amplitude

$$R \sim 2 \times 10^{-6}\text{ s}\left(\frac{R_0}{200M}\right)^{1/2}\left(\frac{M}{10^{10}\,M_\odot}\right)^2\left(\frac{10^{10}\text{ lt-yr}}{r}\right). \tag{32}$$

V. CONCLUSIONS

A comparison of the results of § IV with the rms residual of 150 ms for, say, PSR 1919+21 leads to the conclusion that pulsar measurements miss detection by at least two or three orders of magnitude even for a rather exotic class of sources. Nonetheless this difference is less than that for any other method of gravitational wave detection which is in operation today or is planned to be implemented in the near future (see Weiss 1979). Furthermore, the upper limit from § III for the energy density of a stochastic background (comparable to the closure density of the universe) is lower than that based on any other method of detection.

All of the data used in this paper came from a cursory reading of the literature. But pulsar measurements have primarily been taken for the purpose of studying pulse emission mechanisms, proper motion, crust quakes, and other effects associated with pulsars. In that context the most interesting pulsars are those which show relatively great variations of the frequency. It seems clear that more observations of the quiet pulsars in conjunction with the analysis of data with gravitational wave detection in mind must improve the sensitivity.

Under some circumstances it is possible to differentiate with certainty the effects on the residual caused by a gravitational wave from those caused by some pulsar phenomenon. For example, the cross-correlation of the signals from a number of pulsars could determine that an anomalous residual was produced by an event in the solar system rather than on the pulsar. Or, in addition, Detweiler (1980) finds that the fundamental quadrupole mode of oscillation of a nonrotating black hole has a frequency of $0.38M^{-1}$ and a damping time of $0.089M$. Hence, a residual which appears as a damped sinusoid with a product of frequency to damping time of 0.034 could be associated with a black hole event with a large amount of confidence.

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REFERENCES

Detweiler, S. 1979, in *Sources of Gravitational Radiation*, ed. L. Smarr (Cambridge: Cambridge University Press).
———. 1980, in preparation.
Estabrook, F. B., and Wahlquist, H. D. 1975, *Gen. Rel. Grav.*, **6**, 439.
Helfand, D. J. 1977, unpublished dissertation, University of Massachusetts.
Kovács, S. J., Jr., and Thorne, K. S. 1978, *Ap. J.*, **224**, 62.
Landau, L. D., and Lifshitz, E. M. 1975, *Classical Theory of Fields* (New York: Pergamon).
Manchester, R. N., and Taylor, J. H. 1974, *Ap. J. (Letters)*, **191**, L63.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1973, *Gravitation* (San Francisco: Freeman).
Sazhin, M. V. 1978, *Soviet Astr.—AJ*, **22**, 36.
Smarr, L. 1979, in *Sources of Gravitational Radiation*, ed. L. Smarr (Cambridge: Cambridge University Press).
Weiss, R. 1979, in *Sources of Gravitational Radiation*, ed. L. Smarr (Cambridge: Cambridge University Press).
Zel'dovich, Ya. B., and Novikov, I. D. 1971, *Stars and Relativity* (Chicago: University of Chicago Press).

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