THE THEORY OF GRAVITATIONAL RADIATION: AN INTRODUCTORY REVIEW*

KIP S. THORNE

California Institute of Technology, Pasadena, California
and
Institute for Theoretical Physics, University of California, Santa Barbara

TABLE OF CONTENTS

*Supported in part by the National Science Foundation [AST79-22012 and PHY77-27084].

1. INTRODUCTION AND OVERVIEW

1.1 The nature of these lectures

These lectures are an introduction to and a progress report on the effort to bring gravitational-wave theory into a form suitable for astrophysical studies — a form for use in the future, when waves have been detected and are being interpreted. I will not describe all aspects of this effort. Several of the most important aspects will be covered by other lecturers, elsewhere in this volume. These include the computation of waves from models of specific astrophysical sources (lectures of Eardley); the techniques of numerical relativity — our only way of computing waves from high-speed, strong-gravity, large-amplitude sources (lectures of York and Piran); and a full analysis of radiation reaction and other relativistic effects in binary systems such as the binary pulsar — our sole source today of quantitative observational data on the effects of gravitational waves (lectures of Damour).

My own lectures will provide a sort of framework for those of Eardley, York and Piran, and Damour: I shall present the mathematical description of gravitational waves in a form suitable for astrophysical applications (§2); I shall describe a variety of methods for computing the gravitational waves emitted by astrophysical sources (§3); I shall describe methods for analyzing the propagation of waves from their sources, through our lumpy universe, to earth (§2); and I shall describe methods of analyzing the interaction of gravitational waves with earth-based and solar-system-based detectors (§4). Here and there in my lectures I shall sketch derivations of the methods of analysis and of the formulas presented; but in most places I shall simply refer the reader to derivations elsewhere in the literature and/or pose the derivations as exercises for the reader.

1.2 What is a gravitational wave?

A gravitational wave is a ripple in the curvature of spacetime, which propagates with the speed of light (Fig. 1). In the real universe gravitational waves propagate on the back of a large-scale, slowly changing spacetime curvature created by the universe's lumpy, cosmological distribution of matter. The background curvature is characterized, semiquantitatively, by two length scales

$$\mathcal{R} \equiv \begin{pmatrix}\text{radius of curvature}\\ \text{of background spacetime}\end{pmatrix} \equiv \left|\begin{matrix}\text{typical component } R_{\hat\alpha\hat\beta\hat\gamma\hat\delta} \text{ of Rieman}\\ \text{tensor of background in a local Lorentz frame}\end{matrix}\right|^{-\frac{1}{2}},$$

$$\mathcal{L} \equiv \begin{pmatrix}\text{inhomogeneity scale of}\\ \text{background curvature}\end{pmatrix} \equiv \begin{pmatrix}\text{length scale on which}\\ R_{\hat\alpha\hat\beta\hat\gamma\hat\delta} \text{ varies}\end{pmatrix} \lesssim \mathcal{R}\,; \tag{1.1}$$

and the gravitational waves are characterized by one length scale

$$\lambda\!\!\!^{-} \equiv \begin{pmatrix}\text{reduced wavelength of}\\ \text{gravitational waves}\end{pmatrix} = \frac{1}{2\pi} \times (\text{wavelength } \lambda). \tag{1.2}$$

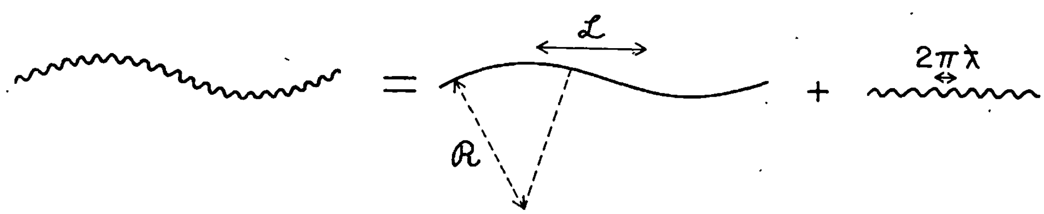

Fig. 1 A heuristic embedding diagram for the decomposition of curve spacetime into a background spacetime plus gravitational waves.

(Of course $\lambda\!\!\!\!-$, $\mathscr{L}$, and $\mathcal{R}$ are not precisely defined; they depend on one's choice of coordinates or reference frame. But in typical astrophysical situations there are preferred frames — e.g. the "asymptotic rest frame" of the source of the waves, or the "mean local rest frame" of nearby galaxies; and these permit $\lambda\!\!\!\!-$, $\mathscr{L}$, and $\mathcal{R}$ to be defined with adequate precision for astrophysical discussion.) The separation of spacetime curvature into a background part $R^{(B)}_{\alpha\beta\gamma\delta}$ and a wave part $R^{(W)}_{\alpha\beta\gamma\delta}$ depends critically on the inequality

$$\lambda\!\!\!\!- \ll \mathscr{L}. \tag{1.3}$$

The waves are the part that varies on the lengthscale $\lambda\!\!\!\!-$; the background is the part that varies on the scale $\mathscr{L}$; the separation is impossible if $\lambda\!\!\!\!- \sim \mathscr{L}$. See Figure 1.

In constructing the theory of gravitational waves one typically expands the equations of general relativity in powers of $\lambda\!\!\!\!-/\mathscr{L}$ and $\lambda\!\!\!\!-/\mathcal{R}$. In the real universe these expansions constitute perturbation theory of the background spacetime (these lectures and that of Yvonne Choquet). In an idealized universe consisting of a source surrounded by vacuum (so that $\mathscr{L} \equiv r$ = distance to source) these expansions constitute "asymptotic analyses of spacetime structure near future timelike infinity $\mathscr{I}^+$" (lectures of Martin Walker).

1.3 Regions of space around a source of gravitational waves

I shall characterize any source of gravitational waves, semiquantitatively, by the following length scales as measured in the source's "asymptotic rest frame".

$$L \equiv \begin{pmatrix}\text{size of}\\ \text{source}\end{pmatrix} = \begin{pmatrix}\text{radius of region inside which the stress-energy } T^{\alpha\beta}\\ \text{and all black-hole horizons are contained}\end{pmatrix},$$

$$2M \equiv \begin{pmatrix}\text{gravitational}\\ \text{radius of source}\end{pmatrix} = \begin{pmatrix}2\times\text{mass of source in}\\ \text{units where } G=c=1\end{pmatrix},$$

$$\lambda\!\!\!\!- \equiv (\text{reduced wavelength of the waves emitted}), \tag{1.4}$$

$$\left.\begin{aligned} r_I &\equiv (\text{inner radius of local wave zone})\\ r_O &\equiv (\text{outer radius of local wave zone})\end{aligned}\right\}(\text{see below}).$$

Corresponding to these length scales, I shall divide space around a source into the following regions (Fig. 2):

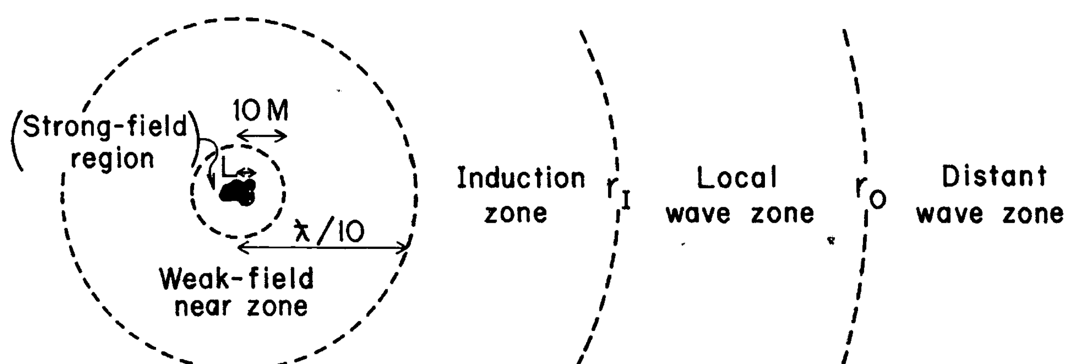

Fig. 2 · Regions of space around a source of gravitational waves.

$$\begin{array}{lll} \text{Source:} & r \lesssim & L, \\ \text{Strong-field region:} & r \lesssim & 10\,M \text{ if } 10\,M \gtrsim L, \\ & & \text{typically does not exist if } L \gg 10\,M, \\ \text{Weak-field near zone:} & L < & r,\ 10\,M < r < \lambdabar/10, \\ \text{Induction zone:} & L < & r,\ \lambdabar/10 < r < r_I \\ \text{Local wave zone:} & r_I < r < r_O & \\ \text{Distant wave zone:} & r_O < r. & \end{array} \tag{1.5}$$

Although Figure 2 suggests the lengthscale ordering $L < 10\,M < \lambdabar/10$, no such ordering will be assumed in these lectures. Thus, we might have $\lambdabar \gg L$ and M ("slow-motion source"), or $\lambdabar \ll L$ and M (high-frequency waves from some small piece of a big source; weak field near zone does not exist), or $\lambdabar \sim L$ or M; and we might have $L \gg M$ or perhaps $L \sim M$.

At radius r outside the source $(r > L)$ the background curvature due to the source has lengthscales

$$\mathcal{R}_s \simeq (r^3/M)^{\frac{1}{2}} \quad , \quad \mathcal{L}_s \simeq r. \tag{1.6}$$

Consequently, the dynamically changing part of the curvature can be regarded as "gravitational waves" (i.e. has $\lambdabar \ll \mathcal{L}$) only in the "wave zone" $r \gg \lambdabar$. I split the wave zone up into two parts, the local wave zone and the distant wave zone, so as to facilitate a clean separation of two mathematical problems: the generation of waves by the source, and the propagation of those waves through the lumpy, real universe to earth. The local wave zone $(r_I \lesssim r \lesssim r_O)$ will serve as a matching region for the two problems: the theory of wave generation will cover the local wave zone and all regions interior to it; the theory of wave propagation will cover the local wave zone and its exterior.

To facilitate the matching I shall choose r_I and r_O in such a manner that throughout the local wave zone the background curvature can be ignored and the background metric can thus be approximated as that of flat Minkowskii spacetime. More specifically, the inner edge of the local wave zone (r_I) is the location at which one or more of the following effects becomes important: (i) the waves cease to be waves and become a near-zone field, i.e., r becomes $\lesssim \lambdabar$; (ii) the gravitational pull of the source produces a significant red shift, i.e., r becomes $\sim 2M$ = (Schwarzschild radius of source); (iii) the background curvature produced by the source distorts the wave fronts and backscatters the waves significantly, i.e., $(r^3/M)^{1/2}$ becomes $\lesssim \lambdabar$; (iv) the outer limits of the source itself are encountered, i.e., r becomes $\lesssim L$ = (size of source). Thus, the inner edge of the local wave zone is given by

$$\begin{aligned} r_I &= \alpha \times \max\{\lambdabar, 2M, (M\lambdabar^2)^{1/3}, L\}\,, \\ \alpha &\equiv \begin{pmatrix} \text{some suitable number} \\ \text{large compared to unity} \end{pmatrix}. \end{aligned} \tag{1.7}$$

The outer edge of the local wave zone r_O is the location at which one or more of the following effects becomes important: (i) a significant phase shift has been produced by the "M/r" gravitational field of the source, i.e., $(M/\lambdabar) \times \ln(r/r_I)$ is no longer $\ll \pi$; (ii) the background curvature due to nearby masses or due to the external universe perturbs the propagation of the waves, i.e., r is no longer $\ll \mathcal{R}_u$ = (background radius of curvature of universe). Thus, the outer edge of the local wave zone is given by

$$r_O = \min[r_I \exp(\lambdabar/\beta M), \mathcal{R}_u/\gamma],$$

$$\beta,\gamma = \begin{pmatrix}\text{some suitable numbers} \\ \text{large compared to unity}\end{pmatrix}. \tag{1.8}$$

Of course, we require that our large numbers α,β,γ be adjusted so that the thickness of the local wave zone is very large compared to the reduced wavelength:

$$r_O - r_I \gg \lambdabar . \tag{1.9}$$

In complex situations the location of the local wave zone might not be obvious. Consider, for example, a neutron star passing very near a supermassive black hole. The tidal pull of the hole sets the neutron star into oscillation, and the star's oscillations produce gravitational waves [Mashhoon (1973); Turner (1977)]. If the hole is large enough, or if the star is far enough from it, there may exist a local wave zone around the star which does not also enclose the entire hole. Of greater interest — because more radiation will be produced — is the case where the star is very near the hole and the hole is small enough ($M_h \lesssim 100\ M_\odot$) to produce large-amplitude oscillations, and perhaps even disrupt the star. In this case, before the waves can escape the influence of the star, they get perturbed by the background curvature of the hole. One must then consider the entire star-hole system as the source, and construct a local wave zone that surrounds them both.

* * * * *

Exercise 1. Convince yourself that for all astrophysical sources except the big-bang singularity (e.g., for the neutron-star/black-hole source of the last paragraph) α, β, and γ can be so chosen as to make condition (1.9) true.

1.4 Organization of these lectures; Notation and conventions

Section 2 of these lectures will discuss the propagation of gravitational waves from the local wave zone out through our lumpy universe to the earth. Section 3 will discuss the generation of gravitational waves, including their propagation into the local wave zone where they can be matched onto the propagation theory of Section 2. Section 4 will discuss the detection of gravitational waves on earth and in the solar system.

My notation and conventions are those of Misner, Thorne, and Wheeler (1973) (cited henceforth as "MTW"): I use geometrized units ($c = G = 1$); Greek indices range from 0 to 3 (time and space), Latin indices from 1 to 3 (space only); the metric signature is +2; $\eta_{\alpha\beta} \equiv \mathrm{diag}(-1,1,1,1)$ is the Minkowskii metric; $\epsilon_{\alpha\beta\gamma\delta}$ and ϵ_{ijk} are the spacetime and space Levi-Civita tensors with $\epsilon_{0123} > 0$ in a right-hand-oriented basis; and the signs of the Riemann, Ricci, Einstein, and stress-energy tensors are given by

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \text{"}\Gamma\Gamma\text{"} - \text{"}\Gamma\Gamma\text{"} , \qquad R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta} ,$$

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} = 8\pi\, T_{\alpha\beta} , \qquad T^{00} > 0 . \tag{1.10}$$

Much of the viewpoint embodied in these lectures I have adopted or developed since 1972 when Misner, Wheeler and I completed MTW. However, many of the new aspects of my viewpoint are contained in my 1975 Erice lectures (Thorne 1977) and/or in a recent Reviews of Modern Physics article (Thorne 1980a; cited henceforth as "RMP").

2. THE PROPAGATION OF GRAVITATIONAL WAVES

2.1 Gravitational waves in metric theories of gravity: Description and propagation speed

Gravitational waves are not unique to Einstein's theory of gravity. They must exist in any theory which incorporates some sort of local Lorentz invariance into its gravitational laws. Many such theories have been invented; see, e.g., Will (1982) for examples and references.

Among the alternative theories of gravity there is a wide class — the so-called "metric theories" — whose members are so similar to general relativity that a discussion of their gravitational waves brings the waves of Einstein's theory into clearer perspective. Thus, I shall initiate my discussion of wave propagation within the framework of an arbitrary metric theory, and then shall specialize to Einstein's general relativity.

A metric theory of gravity is a theory (i) in which gravity is characterized, at least in part, by a 4-dimensional, symmetric spacetime metric $g_{\alpha\beta}$ of signature +2; and (ii) in which the Einstein equivalence principle is satisfied — i.e., all the nongravitational laws of physics take on their standard special relativistic forms in the local Lorentz frames of $g_{\alpha\beta}$ (aside from familiar complications of "curvature coupling"; chapter 16 of MTW).

Examples of metric theories are: general relativity [$g_{\alpha\beta}$ is the sole gravitational field]; the Dicke-Brans-Jordan theory (e.g., Dicke 1964) [contains a scalar gravitational field Φ in addition to $g_{\alpha\beta}$; matter generates Φ via a curved-spacetime wave equation; then Φ and the matter jointly generate $g_{\alpha\beta}$ via Einstein-like field equations]; and Rosen's (1973) theory [a "bimetric" theory with a flat metric $\eta_{\alpha\beta}$ in addition to the physical metric $g_{\alpha\beta}$; the matter generates $g_{\alpha\beta}$ via a flat-spacetime wave equation whose characteristics are null lines of $\eta_{\alpha\beta}$]. See Will (1982) for further details, references, and other examples.

The Einstein equivalence principle guarantees that in any metric theory, as in general relativity, freely falling test particles move along geodesics of $g_{\alpha\beta}$, and that the separation vector ξ^α between two nearby test particles (separation $\ll$ $\lambda\!\!\!^{-}$) is governed by the equation of geodesic deviation:

$$D^2\xi^\alpha/d\tau^2 + R^\alpha{}_{\beta\gamma\delta}\,u^\beta\,\xi^\gamma\,u^\delta = 0\,. \tag{2.1a}$$

Here u^α is the 4-velocity of one of the test particles; τ is proper time measured by that particle;

$$D^2\xi^\alpha/d\tau^2 \equiv (\xi^\alpha{}_{;\beta}\,u^\beta)_{;\gamma}\,u^\gamma \tag{2.1b}$$

is the relative acceleration of the particles; and $R^\alpha{}_{\beta\gamma\delta}$ is the Riemann curvature tensor associated with $g_{\alpha\beta}$. Throughout Sections 2 (wave propagation) and 3 (wave generation) I shall use geodesic deviation and the Riemann tensor to characterize the physical effects of gravitational waves. Only in Section 4 (wave detection) will I discuss other physical effects of waves.

The Riemann tensor $R_{\alpha\beta\gamma\delta}$ contains two parts: background curvature and wave curvature

$$R_{\alpha\beta\gamma\delta} = R^{(B)}_{\alpha\beta\gamma\delta} + R^{(W)}_{\alpha\beta\gamma\delta}\,. \tag{2.2}$$

As discussed in §1.2 $R^{(B)}_{\alpha\beta\gamma\delta}$ varies on a long lengthscale $\mathcal{L}$, while $R^{(W)}_{\alpha\beta\gamma\delta}$ varies on a short lengthscale $\lambda\!\!\!^{-}$. Consequently, if by $\langle\ \rangle$ we denote an average over spacetime regions somewhat larger than $\lambda\!\!\!^{-}$ but much smaller than $\mathcal{L}$ ("Brill-Hartle average";

Exercise 35.14 of MTW), then we can write

$$R^{(B)}_{\alpha\beta\gamma\delta} \equiv \langle R_{\alpha\beta\gamma\delta} \rangle \quad , \quad R^{(W)}_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - \langle R_{\alpha\beta\gamma\delta} \rangle \ . \tag{2.3a}$$

Similarly we can define the background metric, of which $R^{(B)}_{\alpha\beta\gamma\delta}$ is the Riemann tensor, by

$$g^{(B)}_{\alpha\beta} \equiv \langle g_{\alpha\beta} \rangle \ . \tag{2.3b}$$

[For a discussion of delicacies which require the use of "steady coordinates" in the averaging of $g_{\alpha\beta}$ see Isaacson (1968), or more briefly §35.13 of MTW.]

In general relativity and in the Dicke-Brans-Jordan theory gravitational waves propágating through vacuum are governed by the wave equation

$$R^{(W)}_{\alpha\beta\gamma\delta|\mu\nu}\, g^{\mu\nu}_{(B)} = 0 \ , \tag{2.4}$$

whereas in Rosen's theory they are governed by

$$R^{(W)}_{\alpha\beta\gamma\delta,\mu\nu}\, \eta^{\mu\nu} = 0 \ . \tag{2.5}$$

Here "$|$" denotes covariant derivative with respect to $g^{\mu\nu}_{(B)}$ while "," denotes covariant derivative with respect to the flat matric $\eta^{\mu\nu}$. These equations imply that in general relativity and in Dicke-Brans-Jordan theory gravitational waves propagate through vacuum with precisely the speed of light, $c_{GW} = c_{EM}$, but in Rosen's theory they propagate with a different speed, $c_{GW} \neq c_{EM}$. As a rough rule of thumb, whenever a theory of gravity possesses "prior geometry" such as a flat auxiliary metric (MTW, §17.6), it will have $c_{GW} \neq c_{EM}$; often when there is no prior geometry, $c_{GW} = c_{EM}$.

High-precision experiments to test $c_{GW} = c_{EM}$ will be possible if and when electromagnetic waves and gravitational waves are observed from outbursts in the same distant source. For example, for a supernova in the Virgo cluster of galaxies (about 4×10^7 light years from earth; distance at which several supernovae are seen each year) one can hope to discover the light outburst within one day (of retarded time) after the explosion is triggered by gravitational collapse. If gravitational waves from the collapse are observed, then a test is possible with precision

$$\left|\frac{\Delta c}{c}\right| = \left|\frac{c_{GW} - c_{EM}}{c}\right| \sim \frac{1 \text{ day}}{4 \times 10^7 \text{ yr}} \simeq 1 \times 10^{-10} \ . \tag{2.6}$$

Actually, there already exists strong observational evidence that gravitational waves do not propagate more slowly than light. If they did, then high-energy cosmic rays with speeds v in the range $c_{GW} < v < c_{EM}$ would emit gravitational Cerenkov radiation very efficiently and would be slowed quickly by gravitational radiation reaction to $v = c_{GW}$. Since cosmic rays are actually detected with v as large as $c_{EM} \times (1-10^{-18})$, c_{GW} presumably is no smaller than this. (For further details and for a discussion of whether we really understand gravitational Cerenkov radiation in alternative theories of gravity see Caves (1980); also earlier work by Aichelburg, Ecker, and Sexl (1971).

2.2 Plane waves on a flat background in metric theories with $c_{GW} = c_{EM}$

Henceforth I shall restrict attention either to metric theories that have $c_{GW} = c_{EM}$ always (e.g., general relativity and Dicke-Brans-Jordan); or, for theories like Rosen's, to regions of spacetime where c_{GW} happens to equal c_{EM}.

In this section and the next several, I shall make a further restriction to spacetime regions of size $\ll \mathcal{R}$ (but $\gg \lambda\!\!\!^{-}$). In such regions with good accuracy I can ignore the curvature of the background; i.e., I can and will introduce global Lorentz frames of the background metric, in which

$$g^{(B)}_{\alpha\beta} = \eta_{\alpha\beta} \, . \tag{2.7}$$

Far from their source gravitational waves will have wave fronts with radii of curvature large compared to $\lambda\!\!\!^{-}$, i.e., they will be locally plane fronted. Thus, with good accuracy I can and shall approximate $R^{(W)}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$ as precisely plane fronted; and by correctly orienting my spatial axes I shall make the waves propagate in the $x^3 = z$ direction. Since they propagate with the speed of light, the waves are then functions of $t - z$:

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(t-z) \, . \tag{2.8}$$

The analysis of such waves in arbitrary metric theories of gravity, as described below, is due to Eardley, Lee, Lightman, Wagoner, and Will (1973), cited henceforth as ELLWW. For greater detail see Eardley, Lee, and Lightman (1973).

2.2.1 Bianchi identities and dynamical degrees of freedom

Because the Riemann tensor of any metric theory is derivable from a metric $g_{\alpha\beta}$, it must satisfy the Bianchi identities $R_{\alpha\beta[\gamma\delta;\epsilon]} = 0$. For the plane-wave Riemann tensor (2.8) on a flat background (2.7) the total content of the Bianchi identities is

$$\begin{aligned}
&R_{\alpha\beta12,0} = 0 && \Rightarrow \; R_{\alpha\beta12} = 0 \; ; \\
&R_{\alpha\beta13,0} - R_{\alpha\beta10,3} = 0 && \Rightarrow \; R_{\alpha\beta13} = -R_{\alpha\beta10} \; , \\
&R_{\alpha\beta23,0} - R_{\alpha\beta20,3} = 0 && \Rightarrow \; R_{\alpha\beta23} = -R_{\alpha\beta20} \; .
\end{aligned} \tag{2.9}$$

Recalling the pair-wise symmetry $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$ we see from (2.9) that any purely spatial pair of indices (12 or 13 or 23) either vanishes or can be converted into a space-time pair (10 or 20 or 30). This means that the six quantities

$$R_{i0j0}(t-z) = R_{j0i0}(t-z) \tag{2.10}$$

are a complete set of independent components of our plane-wave Riemann tensor. All other components of Riemann can be expressed algebraically in terms of these.

In a general metric theory of gravity these six quantities represent six independent degrees of freedom of the gravitational field — i.e., six independent polarizations of a gravitational wave.

In the special case of general relativity a vacuum gravitational wave must have vanishing Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = 0 \tag{2.11}$$

(Einstein field equations). One can show easily that this reduces the number of independent degrees of freedom from six to two:

$$R_{x0x0} = -R_{y0y0} \qquad \text{and} \qquad R_{x0y0} = R_{y0x0} \, . \tag{2.12}$$

* * * * *

Exercise 2. Show that the Bianchi identities for a plane wave on a flat background imply equations (2.9) and that they, in turn, guarantee that R_{i0j0} are a complete set of independent components of the Riemann tensor.

Exercise 3. Show that the vacuum Einstein field equations (2.11) reduce the independent plane-wave components of Riemann to (2.12).

2.2.2 Local inertial frames (side remarks)

In the next section I shall use geodesic deviation to elucidate the physical nature of the six gravity-wave polarizations. But as a foundation for that discussion I must first remind you of the mathematical and physical details of local inertial frames (LIF); see, e.g., §§8.6, 11.6, and 13.6 of MTW.

Physically an LIF is the closest thing to a global inertial frame that an experimenter can construct. The central building block of an LIF is a freely falling test particle ("fiducial particle"), which carries with itself three orthogonally pointing gyroscopes. The experimenter attaches a Cartesian coordinate grid to the gyroscopes. Because of spacetime curvature, this grid cannot be precisely Cartesian; but deviations from Cartesion structure can be made second order in the spatial distance r from the fiducial particle:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + O(r^2\, R_{\mu\nu\rho\sigma}) \; . \tag{2.13}$$

From an experimental viewpoint the details of the $O(r^2\, R_{\mu\nu\rho\sigma})$ corrections often are unimportant. Those corrections actually produce geodesic deviation, if one calculates geodesics directly from $g_{\alpha\beta}$; but geodesic deviation is more clearly described as the effect of $R_{\mu\nu\rho\sigma}$ in the geodesic deviation equation (2.1a), which now reads for a test particle at spatial location $x^j = \xi^j$ = (separation from fiducial test particle) and, as always in geodesic deviation, with velocity $|dx^j/dt| \ll 1$:

$$d^2x^j/dt^2 = -R_{j0k0}\, x^k \; . \tag{2.14}$$

It is this "experimenter's version" of geodesic deviation to which I shall appeal in discussing gravitational waves.

* * * * *

Exercise 4. Show that in an LIF with metric (2.13) the fiducial particle (at rest at the spatial origin) moves along a geodesic. Show further that if $\xi^j = x^j$ is the separation vector between the fiducial test particle and another test particle, the equation of geodesic deviation (2.1a) takes on the form (2.14).

Exercise 5. One realization of an LIF is a "Fermi normal coordinate system" obtained by letting the spatial coordinate axes be spacelike geodesics that start out along the directions of the three gyroscopes. Show that in such a coordinate system

$$\begin{aligned} ds^2 = {} & -dt^2(1+R_{0\ell 0m}\,x^\ell x^m)dt^2 - \tfrac{4}{3}\,R_{0\ell 0m}\,x^\ell x^m\,dt\,dx^j \\ & +(\delta_{ij} - \tfrac{1}{3}\,R_{i\ell jm}\,x^\ell x^m)dx^i dx^j . \end{aligned} \tag{2.15}$$

For details see, e.g., §13.6 of MTW.

Exercise 6. Show that in the de Donder gauge of §3.1.3 below and in the vacuum of general relativity a mathematical realization of an LIF is

$$ds^2 = -dt^2(1 + R_{0\ell 0m}x^\ell x^m)dt^2 - \frac{4}{3} R_{0\ell jm}x^\ell x^m dt dx^j$$
$$+ \delta_{ij}(1 - R_{0\ell 0m}x^\ell x^m)dx^i dx^j \ . \tag{2.16}$$

(Note: neither this nor (2.15) requires any assumption of a plane-wave Riemann tensor.) For details see, e.g., Hartle and Thorne (1983).

2.2.3 Physical description of plane-wave polarizations

Consider a cloud of test particles surrounding a central, fiducial test particle. Initially the cloud resides in flat spacetime, all its particles are at rest with respect to each other, and its shape is precisely spherical with radius a. Then a gravitational wave hits and deforms the cloud. The deformations can be analyzed using the equation of geodesic deviation only if the cloud is small compared to the inhomogeneity scale of the Riemann tensor, $a \ll \lambda\!\!\!\bar{}$. Assume this to be the case, and analyze the cloud's deformations in the LIF of the fiducial particle:

$$d^2x^j/dt^2 = -R_{j0k0}(t)x^k. \tag{2.17}$$

Here $x^j(t)$ is the location, in the LIF, of some specific test particle at time t; and $R_{j0k0}(t-z)$ is evaluated at the fiducial particle's location $(x,y,z) = 0$. Consider, in turn, and with the help of Figure 3, the six independent polarizations of the wave (further details in ELLWW):

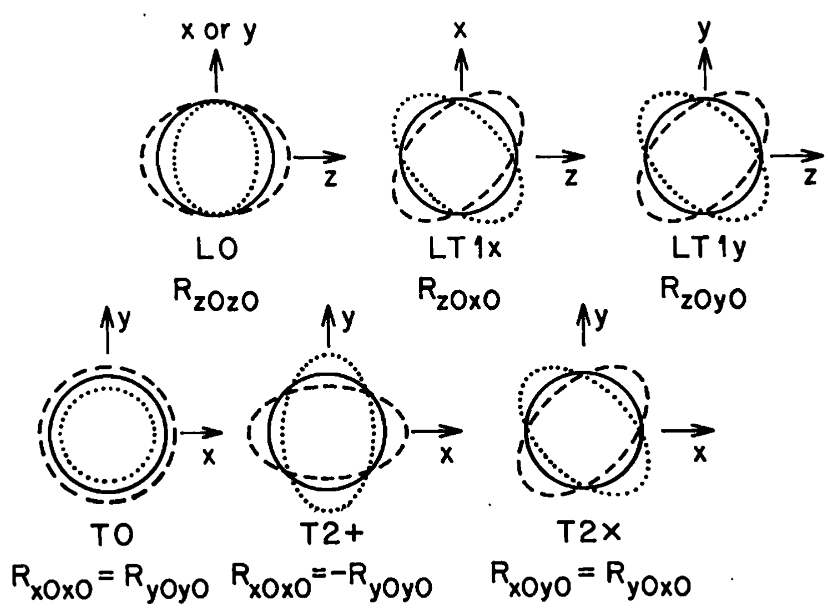

Fig. 3. The deformations of a sphere of test particles produced by gravitational waves with each of six polarizations. As the wave oscillates, the sphere (solid) first gets deformed in the manner shown dashed; then in the manner shown dotted.

R_{z0z0} produces deformations

$$\ddot{x} = 0, \ \ddot{y} = 0, \ \ddot{z} = -R_{z0z0}z. \qquad (2.18a)$$

As $R_{z0z0}(t)$ alternately oscillates negative and positive this expands and then squeezes the sphere longitudinally (i.e., along the direction of wave propagation, z), while leaving its transverse cross section unchanged. In this sense the wave is "purely longitudinal". At any moment of time the deformed sphere is invariant under rotations about the propagation direction $\vec{e}_z$. This means, in the language of "canonical field theory", that the wave has "spin zero" (or "helicity zero"). These properties are summarized by saying that the wave is "L0" (longitudinal and spin zero).

$R_{z0x0} = R_{x0z0}$ produces deformations

$$\ddot{x} = -R_{x0z0}z, \quad \ddot{y} = 0, \quad \ddot{z} = -R_{x0z0}x. \qquad (2.18b)$$

As $R_{x0z0}(t)$ oscillates this expands and then squeezes the sphere at a 45 degree angle to $\vec{e}_z$ in the "logitudinal-transverse" z-x plane, while leaving it undeformed in the y direction. At any moment the deformed sphere is invariant under a 360° rotation about the propagation direction, a property that it shares with the electric or magnetic field of an electromagnetic wave. Thus, this gravitational wave like an electromagnetic wave has spin one — but whereas the electromagnetic wave is "T1" (transverse, spin one), this gravitational wave is "LT1" (longitudinal-transverse, spin one).

$R_{z0y0} = R_{y0z0}$ produces deformations

$$\ddot{x} = 0, \quad \ddot{y} = -R_{y0z0}z, \quad \ddot{z} = -R_{y0z0}y, \qquad (2.18c)$$

and is thus also "LT1". It is the LT1 polarization orthogonal to R_{z0x0}.

A wave with $R_{x0x0} = R_{y0y0}$ and all other R_{j0k0} zero produces deformations

$$\ddot{x} = R_{x0x0}x, \quad \ddot{y} = R_{x0x0}y, \quad \ddot{z} = 0. \qquad (2.18d)$$

This wave alternately expands and compresses the sphere in the transverse plane while leaving it transversely circular and leaving it totally unchanged in the longitudinal direction. Thus, this wave is T0 ("transverse, spin-zero").

A wave with $R_{x0x0} = -R_{y0y0}$ and all other R_{j0k0} zero produces deformations

$$\ddot{x} = -R_{x0x0}x, \quad \ddot{y} = +R_{x0x0}y, \quad \ddot{z} = 0. \qquad (2.18e)$$

As $R_{x0x0}(t)$ oscillates this wave expands the sphere in the x direction and squeezes it in y, then expands it in y and squeezes it in x. Thus the deformations are purely transverse, and at any moment the deformed sphere is invariant under a 180° rotation about the propagation direction. The "spin" ("helicity") S of any wave which propagates with the speed of light is determined by the angle Θ of rotations about the propagation direction that leave all momentary physical effects of the wave unchanged:

$$S = 360^\circ/\Theta.$$

Thus, this wave has spin S = 2, i.e., it is a T2 ("transverse, spin 2) wave. The orientation of the polarization is identified as "+" (x and y; horizontal and vertical).

$R_{x0y0} = R_{y0x0}$ produces deformations

$$\ddot{x} = R_{x0y0}\,y, \quad \ddot{y} = R_{x0y0}\,x, \quad \ddot{z} = 0 \tag{2.18f}$$

which are also T2; but the orientation of the polarization is "×". This is the T2 wave orthogonal to "+".

That the spin 0 waves have just one polarization state while the spin 1 and spin 2 waves each have two orthogonal polarization states is familiar both from quantum mechanics and from canonical, classical field theory. It is a consequence of the fact that the waves propagate with the speed of light — i.e., their quanta have zero rest mass.

This familiar feature is deceptively reassuring. Actually, a nasty surprise awaits us if we try to check the fundamental tenet of canonical field theory that the spin of a wave must be Lorentz invariant. If we begin in one Lorentz frame with a gravitational wave that is pure T0 or pure T2, we will find it to be pure T0 or pure T2 in all other Lorentz frames. However, if we begin with a pure LT1 wave in one frame, we will find a mixture of LT1, T0, and T2 in other frames; and if we begin with pure L0 in one frame, we will find a mixture of L0, LT1, T0, and T2 in other frames. See Eardley, Lee, and Lightman (1973) for proofs.

This means that any metric theory of gravity possessing L0 or LT1 waves violates the tenets of canonical field theory and cannot be quantized by canonical methods, even in the weak-gravity limit. Rosen's theory is an example (in the special case $c_{GW} = c_{EM}$ to which our discussion applies; when $c_{GW} \neq c_{EM}$ there exist eight polarizations, not six [C. M. Caves, private communication]). By contrast, metric theories with purely transverse, speed-of-light gravity waves obey the canonical tenets and are quantizable by canonical means, at least in the weak-gravity limit. Examples are general relativity which has pure T2 waves (Exercise 3 above) and the Dicke-Brans-Jordan theory which has both T2 waves and T0 waves.

* * * * *

Exercise 7. Verify the claims made above about the behavior of polarization states under Lorentz transformations. (See Eardley, Lee, and Lightman 1973 for solution.)

2.3 Plane waves on a flat background in general relativity

2.3.1 The gravitational-wave field h^{TT}_{jk}

Specialize now and henceforth to general relativity. Then a plane wave on a flat background has precisely two orthogonal polarization states: T2+ and T2× (denoted simply "+" and "×" henceforth).

Choose a specific Lorentz frame of the flat background space, and in that frame define a "gravitational-wave field" h^{TT}_{jk} by

$$R_{j0k0}(t-z) = -\tfrac{1}{2}h^{TT}_{jk,tt} \tag{2.19}$$

with $h^{TT}_{jk} = 0$ before any waves ever arrive. Then the waves are fully characterized equally well by R_{j0k0} or by h^{TT}_{jk}. Note that the only nonzero components of h^{TT}_{jk} are

$$h^{TT}_{xx} = -h^{TT}_{yy} \equiv A_+(t-z), \qquad h^{TT}_{xy} = h^{TT}_{yx} = A_\times(t-z), \tag{2.20}$$

A_+ being the "amplitude function" for the + polarization state and $A_\times$ being that for the $\times$ state. Note that h^{TT}_{jk} is purely spatial, symmetric, transverse to the propagation direction $\vec{n} = \vec{e}_z$, and also <u>traceless</u> (hence the TT superscript)

$$h^{TT}_{jk} = h^{TT}_{kj}, \qquad h^{TT}_{jk}\, n^k = 0, \qquad \delta^{jk}\, h^{TT}_{jk} = 0. \tag{2.21}$$

Note further that the experimenter's version of the equation of geodesic deviation (eq. 2.17) can be integrated to give

$$\delta x^j = \tfrac{1}{2}\, h^{TT}_{jk}\, x^k \tag{2.22}$$

for the change in location, in an LIF, of a test particle initially at x^k. This equation suggests the common interpretation of h^{TT}_{jk} as a "dimensionless strain of space".

2.3.2 Behavior of h^{TT}_{jk} under Lorentz transformations

A single gravitational wave is described, in different Lorentz frames of the flat background, by different h^{TT}_{jk} fields, each one purely spatial and TT in its own frame. How are these gravitational-wave fields related to each other? I shall state the answer in this section, leaving the proof as an exercise for the reader.

Begin in some fiducial background Lorentz frame, in which h^{TT}_{jk} is described by equations (2.20) above. Define

$$\psi \equiv t - z = \text{"retarded time"}, \tag{2.23a}$$

regard ψ as a scalar field in spacetime (a sort of "phase function" for the wave), and regard the amplitude functions A_+ and $A_\times$ of equations (2.20) as scalar fields which are known functions of ψ. From the scalar field ψ construct the "propagation vector"

$$\vec{k} \equiv -\vec{\nabla}\psi, \quad \text{so} \quad k^0 = k^z = 1,\ \ k^x = k^y = 0 \text{ in fiducial frame.} \tag{2.23b}$$

This propagation vector and the basis vector $\vec{e}_x$, which determines the orientations of the + and $\times$ polarization states, together define a "fiducial 2-flat" (plane)

$$\overleftrightarrow{f} \equiv \vec{k} \wedge \vec{e}_x \equiv (\text{2-flat spanned by } \vec{k} \text{ and } \vec{e}_x). \tag{2.23c}$$

Now choose some other background Lorentz frame with 4-velocity $\vec{u}'$. In that frame define basis vectors

$$\vec{e}_{0'} \equiv \vec{u}', \quad \vec{e}_{z'} \equiv \begin{bmatrix}\text{unit vector obtained by projecting } \vec{k} \\ \text{orthogonal to } \vec{u}' \text{ and renormalizing}\end{bmatrix},$$

$$\vec{e}_{x'} \equiv [\text{unit vector lying in } \overleftrightarrow{f} \text{ and orthogonal to } \vec{u}'], \tag{2.24a}$$

$$\vec{e}_{y'} \equiv \begin{bmatrix}\text{unit vector such that } \vec{e}_{0'}, \vec{e}_{x'}, \vec{e}_{y'}, \vec{e}_{z'} \text{ are a right-hand} \\ \text{oriented, orthonormal frame}\end{bmatrix}.$$

Then in this basis the gravitational-wave field has components

$$h'^{TT}_{x'x'} = -h'^{TT}_{y'y'} = A_+(\psi), \quad h'^{TT}_{x'y'} = h'^{TT}_{y'x'} = A_\times(\psi), \tag{2.24b}$$

where A_+ and $A_\times$ are the <u>same</u> scalar fields as describe the waves in the fiducial frame. Note, however, that ψ is not $t'-z'$; rather, it differs from $t'-z'$ by the standard doppler-shift factor:

$$\psi = t-z = (\nu'/\nu)(t'-z'), \tag{2.24c}$$

$$\frac{\nu'}{\nu} = \left[\begin{matrix}\text{ratio of frequencies of photons, propagating in } \vec{k} \text{ direction,}\\ \text{as measured in the two reference frames}\end{matrix}\right].$$

Thus, under a Lorentz transformation the gravitational-wave frequencies get doppler shifted just like those of light, but the amplitude functions are left unchanged and the polarization directions change only by a projection that keeps them spatial (eq. 2.24a for $\vec{e}_x'$). The fact, that h^{TT}_{jk} is a "spin-2" quantity with amplitude functions A_+ and $A_\times$ that are Lorentz invariant is embodied in the statement that "$A_+ + iA_\times$ has spin-weight 2 and boost-weight 0" [language of Geroch, Held, and Penrose (1973); $i \equiv \sqrt{-1}$].

It is often convenient to define polarization tensors

$$\overleftrightarrow{e}'_+ \equiv \vec{e}_{x'} \otimes \vec{e}_{x'} - \vec{e}_{y'} \otimes \vec{e}_{y'} \quad , \quad \overleftrightarrow{e}'_\times \equiv \vec{e}_{x'} \otimes \vec{e}_{y'} + \vec{e}_{y'} \otimes \vec{e}_{x'} \, , \tag{2.24d}$$

and to rewrite equations (2.24b) as

$$\overleftrightarrow{h}'^{TT} = A_+(\psi)\overleftrightarrow{e}'_+ + A_\times(\psi)\overleftrightarrow{e}'_\times \, . \tag{2.24e}$$

Another useful relation is

$$R_{\alpha'\beta'\gamma'\delta'} = \tfrac{1}{2}\left(\ddot{h}'^{TT}_{\alpha'\delta'}k_{\beta'}k_{\gamma'} + \ddot{h}'^{TT}_{\beta'\gamma'}k_{\alpha'}k_{\delta'} - \ddot{h}'^{TT}_{\beta'\delta'}k_{\alpha'}k_{\gamma'} - \ddot{h}'^{TT}_{\alpha'\gamma'}k_{\beta'}k_{\delta'}\right), \tag{2.24f}$$

where $^{\bullet} = d/d\psi$.

* * * * *

<u>Exercise 8</u>. Show that in the fiducial frame of this section, equation (2.19) is equivalent to $R_{j0k0} = -(1/2)\ddot{h}^{TT}_{jk}k_0k_0$ where the dot means $d/d\psi$. Show further that in the fiducial frame the full Riemann tensor is given by equation (2.24f) without the primes. (Hint: use equations 2.9).

<u>Exercise 9</u>. Show that the Riemann tensor of (2.24f) with the primes and with $h'^{TT}_{\alpha'\gamma'}$ given by (2.24a,b) is obtained from that of Exercise 8 (no primes) by a standard Lorentz transformation. Convince yourself that this fully justifies the claimed behavior of h^{TT}_{jk} under a change of frames (eqs. 2.24a,b).

<u>2.3.3 Relationship of h^{TT}_{jk} to Bondi news function</u>

Consider gravitational waves propagating radially outward from a source, and approximate the background as flat. Introduce spherical polar coordinates and denote the associated orthonormal basis vectors by,

$$\vec{e}_{\hat{r}} = \partial/\partial r, \quad \vec{e}_{\hat{\theta}} = r^{-1}\partial/\partial\theta, \quad \vec{e}_{\hat{\varphi}} = (r\sin\theta)^{-1}\partial/\partial\varphi. \tag{2.25a}$$

Let $\vec{e}_{\hat{\theta}}$ be the fiducial direction (analog of $\vec{e}_x$ above) used in defining the polarization base states, so that

$$\overleftrightarrow{e}_+ = \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\varphi}} \otimes \vec{e}_{\hat{\varphi}} \quad , \quad \overleftrightarrow{e}_\times = \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\varphi}} + \vec{e}_{\hat{\varphi}} \otimes \vec{e}_{\hat{\theta}} \; ; \tag{2.25b}$$

$$h^{TT}_{jk} = A_+ \overleftrightarrow{e}_+ + A_\times \overleftrightarrow{e}_\times \, . \tag{2.25c}$$

Because the wave fronts are spherical (though very nearly plane on length scales $\lambdabar \ll r$), A_+ and $A_\times$ die out as $1/r$ (Exercise 14 below)

$$A_+ = r^{-1}\, F_+(\psi;\theta,\varphi), \quad A_\times = r^{-1}\, F_\times(\psi;\theta,\varphi), \quad \psi = t - r. \tag{2.25d}$$

In gravitational-wave studies near "future timelike infinity" $\mathscr{I}^+$, mathematical physicists often use instead of h^{TT}_{jk} a different description of the waves due to Bondi, van der Burg, and Metzner (1962) and to Sachs (1962): The role of the gravitational-wave amplitude is played by the "Bondi News Function"

$$N \equiv \tfrac{1}{2} \frac{\partial}{\partial t} (F_+ + i\, F_\times) = \tfrac{1}{2}\, r\, \frac{\partial}{\partial t} (A_+ + i\, A_\times), \tag{2.26a}$$

where $i \equiv \sqrt{-1}$. This complex News function has the advantage of depending only on angles θ, φ, and retarded time ψ ($1/r$ dependence factored out); but for this it pays the price of not being a scalar field. In the language of Geroch, Held, and Penrose, it has "boost weight 2", whereas $A_+ + iA_\times$ has "boost weight 0". It is conventional in the Bondi-Sachs formalism to introduce the complex vector

$$\vec{\bar{m}} = (1/\sqrt{2})\ (\vec{e}_{\hat\theta} - i\, \vec{e}_{\hat\varphi}). \tag{2.26b}$$

In terms of N and $\vec{\bar{m}}$ the gravitational-wave field is

$$\frac{\partial}{\partial t}\, h^{TT}_{jk} = \mathrm{Real}\left\{ \frac{4N}{r}\, \bar{m}_j \bar{m}_k \right\}. \tag{2.26c}$$

Several lecturers in this volume will use the Bondi-Sachs formalism (e.g., M. Walker, A. Ashtekar, and R. Isaacson).

2.4 Weak perturbations of curved spacetime in general relativity

2.4.1 Metric perturbations and Einstein field equations

Abandon, now, the approximation that the background spacetime is flat. As a foundation for discussing gravitational waves in curved spacetime, consider the general problem of linear perturbations of a curved background metric:

$$g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu} \, . \tag{2.27}$$

In analyzing the metric perturbations $h_{\mu\nu}$, I shall not make explicit the small dimensionless parameter that underlies the perturbation expansion. It might be $\lambdabar/\mathscr{L}$ (gravitational wave expansion, §2.4.2 below); it might be the dimensionless amplitude of pulsation of a neutron star, $\delta R/R$ (linear pulsation-theory expansion, Thorne and Campolattaro 1967); it might be the mass ratio m/M of a small body m falling into a Schwarzschild black hole M and generating gravitational waves as it falls (linear perturbations of Schwarzschild geometry; Davis et al. 1971). In the latter two cases, near the star and hole $\lambdabar/\mathscr{L}$ is not $\ll 1$; but nevertheless the linearized equations of this section are valid.

The perturbed Einstein field equations for $h_{\mu\nu}$ are expressed most conveniently in terms of the "trace-reversed" metric perturbation

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\, h\, g^{(B)}_{\mu\nu} \quad , \quad h \equiv h_{\mu\nu}\, g^{\mu\nu}_{(B)} \, . \tag{2.28}$$

A straightforward calculation (cf. §§35.13 and 35.14 of MTW) gives for the first-order perturbations of the field equations

$$\bar{h}_{\mu\nu|\alpha}{}^{\alpha} + g^{(B)}_{\mu\nu}\bar{h}^{\alpha\beta}{}_{|\beta\alpha} - 2\bar{h}_{\alpha(\mu|}{}^{\alpha}{}_{\nu)} + 2R^{(B)}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} - 2R^{(B)}_{\alpha(\mu}\bar{h}_{\nu)}{}^{\alpha} = -16\pi\delta T_{\mu\nu}. \qquad (2.29)$$

Here a slash "|" denotes covariant derivative with respect to $g^{(B)}_{\mu\nu}$; indices on $\bar{h}_{\alpha\beta}$ are raised and lowered with $g^{(B)}_{\mu\nu}$; $R^{(B)}_{\alpha\beta\gamma\delta}$ and $R^{(B)}_{\alpha\beta}$ are the Riemann and Ricci tensors of the background, and $\delta T_{\mu\nu}$ is the first-order perturbation of the stress-energy tensor.

The first-order perturbed field equation (2.29) can be used to study a wide variety of phenomena, including wave generation (§3.5 below), wave propagation on a curved vacuum background (§2.4.2), absorption and dispersion of waves due to interaction with matter (§2.4.3), and scattering of waves off background curvature and the resulting production of wave tails (§2.4.2).

2.4.2 Wave propagation on a curved vacuum background

Consider gravitational waves of reduced wavelength $\lambda\!\!\!^{-}$ propagating on a curved vacuum background with radius of curvature $\mathcal{R}$ and inhomogeneity scale $\mathcal{L}$. In keeping with the discussion in §1.2 assume $\lambda\!\!\!^{-} \ll \mathcal{R}$, but <u>for the moment do not assume</u> $\lambda\!\!\!^{-} \ll \mathcal{L}$. Then "vacuum" implies $\delta T_{\mu\nu} = 0$ in the field equations (2.29); and $\lambda\!\!\!^{-} \ll \mathcal{R}$ implies that the terms involving $R^{(B)}_{\alpha\mu\beta\nu}$ and $R^{(B)}_{\alpha\beta}$, which are of size $h/\mathcal{R}^2$, can be neglected compared to the first three terms, which are of size $h/\lambda\!\!\!^{-2}$. Simplify the resulting field equations further by an infinitesimal coordinate change ("gauge change")

$$x^{\alpha}_{new} = x^{\alpha}_{old} + \xi^{\alpha}, \qquad h^{new}_{\mu\nu} = h^{old}_{\mu\nu} - \xi_{\mu|\nu} - \xi_{\nu|\mu} \qquad (2.30)$$

so designed as to make

$$\bar{h}_{\mu}{}^{\alpha}{}_{|\alpha} = 0 \qquad (\text{"Lorentz gauge"}). \qquad (2.31a)$$

(See, e.g., §35.14 of MTW for discussion of such gauge changes.) The first-order field equations (2.29) then become a simple source-free wave equation in curved spacetime:

$$\bar{h}_{\mu\nu|\alpha}{}^{\alpha} = 0. \qquad (2.31b)$$

The Riemann curvature tensor associated with these waves

$$R^{(W)}_{\alpha\mu\beta\nu} = \tfrac{1}{2}\left(h_{\alpha\nu|\mu\beta} + h_{\mu\beta|\nu\alpha} - h_{\mu\nu|\alpha\beta} - h_{\alpha\beta|\mu\nu}\right) \qquad (2.32)$$

will also satisfy the wave equation

$$R^{(W)}_{\alpha\mu\beta\nu|\sigma}{}^{\sigma} = 0 \qquad (2.33)$$

(covariant derivatives "|" commute because $\lambda\!\!\!^{-} \ll \mathcal{R}$). Note that although we require $\lambda\!\!\!^{-} \ll \mathcal{L}$ in order to give a clear definition of "wave", we need not place any restriction on $\lambda\!\!\!^{-}/\mathcal{L}$ in order to derive the wave equation (2.31b).

The Lorentz gauge condition (2.31a) is preserved by any gauge change (2.30) whose generating function ξ_{α}, like $\bar{h}_{\mu\nu}$, satisfies the wave equation $\xi_{\alpha|\mu}{}^{\mu} = 0$. One of the four degrees of freedom in such a gauge change can be used to make $\bar{h}_{\mu\nu}$ trace-free everywhere

$$\bar{h}_{\alpha}{}^{\alpha} = 0 \quad \text{so} \quad \bar{h}_{\mu\nu} = h_{\mu\nu} \qquad (\text{"trace-free Lorentz gauge"}) \qquad (2.34)$$

(MTW exercise 35.13); and the other three degrees of freedom can be used locally (in a local inertial frame of the background $g^{(B)}_{\mu\nu}$) to guarantee that

$$h_{0\alpha} = 0, \quad h_{ij} = h^{TT}_{ij} \quad (\text{"local TT gauge"}), \tag{2.35}$$

where h^{TT}_{ij} is the gravitational-wave field defined, in the background LIF, by

$$R^{(W)}_{i0j0} \equiv -\tfrac{1}{2} h^{TT}_{ij,00} \;. \tag{2.36}$$

If the background is approximated as flat throughout the wave zone, then one can introduce a global inertial frame of $g^{(B)}_{\mu\nu}$ throughout the wave zone and one can impose the TT gauge globally. However, if the background is curved, a global TT gauge cannot exist (MTW exercise 35.13).

One often knows $h_{\alpha\beta}$ or $\bar{h}_{\alpha\beta}$ in a Lorentz but non-TT gauge and wants to compute its "gauge-invariant part" h^{TT}_{ij} in some LIF of the background. Such a computation is performed most easily by a "TT projection", which is mathematically equivalent to a gauge transformation (MTW Box 35.1): One identifies the propagation direction n_j in the LIF as the direction in which the wave is varying rapidly (on length scale $\lambda\!\!\!^{-}$). One then obtains h^{TT}_{ij} by discarding all parts of h_{ij} or $\bar{h}_{ij}$ along n_j and by then removing the trace:

$$h^{TT}_{ij} = P_{ia}h_{ab}P_{bj} - \tfrac{1}{2} P_{ij}P_{ab}h_{ab} = (\text{same expression with } h_{ab} \rightarrow \bar{h}_{ab}), \tag{2.37}$$

where $P_{ab} = \delta_{ab} + n_a n_b$. WARNING: This projection process gives the correct answer only in an LIF of the background and only if $h_{\mu\nu}$ is in a Lorentz gauge.

* * * * * *

Exercise 10. Show that the infinitesimal coordinate change (2.30) produces the claimed gauge change of $h_{\mu\nu}$. Show further that the Riemann tensor of the waves is correctly given by (2.32) in any gauge, and that this Riemann tensor is invariant under gauge changes (2.30).

Exercise 11. Show that a gauge change with $\xi_{\alpha|\mu}{}^{\mu} = 0$ can be used to make a Lorentz-gauge $h_{\mu\nu}$ trace-free globally (eq. 2.34) and TT locally (eq. 2.35). Show further that the TT projection process (2.37) produces the same result as this gauge transformation.

2.4.3 Absorption and dispersion of waves by matter

When electromagnetic waves propagate through matter (e.g., light through water, radio waves through the interplanetary medium), they are partially absorbed and partially scatter off charges; and the scattered and primary waves superpose in such a way as to change the propagation speed from that of light in vacuum ("Dispersion"). A typical model calculation of this absorption and dispersion involves electrons of charge e, mass m, and number density n, each bound to a "lattice point" by a 3-dimensional, isotropic, damped, harmonic-oscillator force:

$$\ddot{\mathbf{z}} + (1/\tau_*)\dot{\mathbf{z}} + \omega_o^{\,2}\mathbf{z} = (e/m)\mathbf{E} = -(e/m)\dot{\mathbf{A}}, \tag{2.38a}$$

where $\mathbf{A}$ is the vector potential in transverse Lorentz gauge and a dot denotes $\partial/\partial t$. These electrons produce a current density $\mathbf{J} = ne(d\mathbf{z}/dt)$, which enters into Maxwell's equations for wave propagation $\Box\mathbf{A} = -4\pi\mathbf{J}$ to give waves of the form $\mathbf{E} = \mathbf{E}_o \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$ with the dispersion relation (for weak dispersion)

$$\frac{\omega}{k} = (\text{phase speed}) = 1 - \frac{2\pi n e^2/m}{\omega_o^{\,2} - \omega^2 - i\omega/\tau_*} . \tag{2.38b}$$

This dispersion relation shows both absorption (imaginary part of ω/k) and dispersion (real part), and in real situations either or both can be very large.

When gravitational waves propagate through matter they should also suffer adsorption and dispersion. However, in real astrophysical situations the absorption and dispersion will be totally negligible, as the following model calculation shows. (For previous model calculations similar to this one see Szekeres 1971.)

The best absorbers or scatterers of gravitational waves that man has devised are Weber-type resonant-bar gravitational-wave detectors (§§4.1.2 and 4.1.4). On larger scales, a spherical self-gravitating body such as the earth or a neutron star is also a reasonably good absorber and scatterer (good compared to other kinds of objects such as interstellar gas). Consider, then, as idealized "medium" made of many solid spheres (spheres to avoid anisotropy of response to gravity waves), each of which has quadrupole vibration frequency ω_o, damping time (due to internal friction) τ_*, mass m and radius R. For ease of calculation (and because we only need order of magnitude estimates) ignore the self gravity and mutual gravitational interactions of the spheres, and place the spheres at rest in a flat background spacetime with number per unit volume n. Let h_{ij}^{TT} be the gravitational-wave field and require $\lambda\!\!\!^{-} > n^{-1/3} > R$. The waves' geodesic deviation force drives each sphere into quadrupolar oscillations with quadrupole moment $\mathcal{J}_{jk}$ satisfying the equation of motion (Exercise 22 in §4.1.4 below)

$$\ddot{\mathcal{J}}_{jk} + (1/\tau_*)\dot{\mathcal{J}}_{jk} + \omega_o^{\,2}\mathcal{J}_{jk} = (1/5)mR^2\ddot{h}_{jk}^{TT} \tag{2.39a}$$

(analog of the electromagnetic equation 2.38a). As a result of these oscillations each sphere reradiates. The wave equation for h_{jk}^{TT} with these reradiating sources (analog of $\Box\underset{\sim}{A} = -4\pi\underset{\sim}{j}$) is

$$\Box h_{jk}^{TT} \equiv \eta^{\alpha\beta}h_{jk,\alpha\beta}^{TT} = -8\pi n\ddot{\mathcal{J}}_{jk} \tag{2.39b}$$

(Exercise 12). By combining equations (2.39a,b) and assuming a wave of the form $h_{jk} \propto \exp(-i\omega t + i\underset{\sim}{k}\cdot\underset{\sim}{x})$ we obtain the gravitational-wave dispersion relation

$$\frac{\omega}{k} = (\text{phase speed}) = 1 - \frac{(4\pi/5)nmR^2\omega^2}{\omega_o^{\,2} - \omega^2 - i\omega/\tau_*} . \tag{2.39c}$$

To see that the absorption and dispersion are negligible, compare the length scale $\ell = |(1-\omega/k)\omega|^{-1}$ for substantial absorption or for a phase shift of $\sim \pi/2$ with the radius of curvature of spacetime produced by the scatterers (i.e., the maximum size that the scattering region can have without curling itself up into a closed universe), $\mathcal{R} = (nm)^{-1/2}$:

$$\frac{\ell}{\mathcal{R}} = \underbrace{\left|\frac{\omega_o^{\,2} - \omega^2 - i\omega/\tau_*}{(4\pi/5)\omega^2}\right|}_{\substack{\gtrsim 1 \text{ off resonance} \\ \sim(1/Q) \text{ on resonance}}} \frac{1}{\underbrace{(nR^3)^{1/2}}_{\lesssim 1}\ \underbrace{(m/R)^{1/2}}_{<1/\sqrt{2}}\ \underbrace{(\omega R)}_{<1}} . \tag{2.40}$$

Here $Q = 1/\omega\tau_*$ is the quality factor of a scatterer, $nR^3 \lesssim 1$ because the scatterers cannot be packed closer together than their own radii, $m/R < 1/2$ because a scatterer cannot be smaller than a black hole of the same mass, and $\omega R = R/\lambda\!\!\!^{-} < 1$

was required to permit a geodesic-deviation analysis (see above). In the most extreme of idealized universes $\ell/\mathcal{R}$ can be no smaller than unity off resonance (dispersion) and $1/Q$ on resonance (absorption); and such extreme values can be achieved only for neutron stars or black holes ($m/R \sim 1$) packed side by side ($nR^3 \sim 1$) with $R \sim \lambda\!\!\!^{-}$. In the real universe, $\ell/\mathcal{R}$ will always be $\ggg 1$; i.e., absorption and dispersion will be negligible regardless of what material the waves encounter and regardless of how far they propagate through it.*

For this reason, henceforth in discussing wave propagation through astrophysical matter (e.g., the interior of the Earth or Sun) I shall approximate $\bar{h}_{\mu\nu}|_{\alpha}{}^{\alpha} = -16\pi\delta T_{\mu\nu}$ by $\bar{h}_{\mu\nu}|_{\alpha}{}^{\alpha} = 0$. The matter will influence wave propagation only through the background curvature it produces (covariant derivative "$|$"), not through any direct scattering or absorption ($\delta T_{\mu\nu}$); see §2.6.1 below.

* * * * *

Exercise 12. For non-self-gravitating matter in flat spacetime and in Lorentz coordinates, show that $T^{\alpha\beta}{}_{,\beta} = 0$ implies $T^{jk} = (1/2)(\rho x^j x^k)_{,00} +$ (perfect spatial divergence), where ρ is mass density. Average this over a lattice of oscillating spheres with number density $n > \lambda\!\!\!^{-}{}^{-3}$ to get $T^{jk} = (1/2)n\ddot{I}_{jk}$, where $I_{jk} = \int \rho x^j x^k d^3x$ is the second moment of the mass distribution of each sphere. Passing gravitational waves excite the oscillations in accord with equation (2.39a) (result to be proved in Exercise 22). These oscillations involve no volume changes, so $\ddot{I}_{jk} = \ddot{\mathcal{I}}_{jk}$ = (trace-free part of $\ddot{I}_{jk}$); moreover, equation (2.39a) shows that $\mathcal{I}_{jk}$ is transverse and traceless. Show that this permits TT gauge to be imposed in the field equations (2.29) in the presence of the oscillating, reradiating spheres (usually it can be imposed only outside all sources), and that the resulting field equations reduce to (2.39b). Then derive the gravitational-wave dispersion relation (2.39c) and the estimate (2.40) of the effects of dispersion and absorption.

2.4.4 Scattering of waves off background curvature, and tails of waves

A self-gravitating body of mass m and size R will typically generate gravitational waves with reduced wavelength

$$\lambda\!\!\!^{-} \sim (R^3/m)^{\frac{1}{2}} \sim \mathcal{R}_s = \text{(radius of curvature of spacetime near source).} \qquad (2.41)$$

If the body has strong self gravity, $m/R \sim 1$ (neutron star or black hole), then $\lambda\!\!\!^{-} \sim \mathcal{R}_s$ in the innermost parts of the wave zone; and the curvature coupling terms must be retained in the first-order Einstein equations (2.29). These terms cause the waves to scatter off the background curvature; and repetitively backscattered waves superimposing on each other produce a gravity-wave "tail" that lingers near the source long after the primary waves have departed, dying out as $t^{-(2\ell+2)}$ for waves of multipole order ℓ. See, e.g., Price (1972) for a more detailed discussion, and Cunningham, Price, and Moncrief (1978) for an explicit example.

I regard these backscatterings and tails as part of the wave generation problem and as irrelevant to the problem of wave propagation. In fact, I have defined the inner edge of the "local wave zone" to be so located that throughout it, and throughout the wave propagation problem, $\lambda\!\!\!^{-} \ll \mathcal{R}$ and backscatter and tails can be ignored (eq. 1.7 and associated discussion).

---

*For description of a physically unrealistic but conceivable material in which dispersion is so strong that it actually reflects gravitational waves see Press (1979).

2.4.5 The stress-energy tensor for gravitational waves

Gravitational waves carry energy and momentum and can exchange them with matter, e.g., with a gravitational-wave detector. Isaacson (1968) (see also §35.15) has quantified this by examining nonlinear corrections to the wave-propagation equation (2.31b). In this section I shall sketch the main ideas of his analysis.

Consider a gravitational wave with $\lambdabar \ll \mathcal{L} \lesssim \mathcal{R}$, and expand the metric of the full spacetime in a perturbation series

$$\underset{1,\mathcal{L}}{g_{\mu\nu} = g_{\mu\nu}^{(B)}} + \underset{\alpha,\lambdabar}{h_{\mu\nu}} + \underset{\alpha^2,\lambdabar}{j_{\mu\nu}} + \cdots \quad . \tag{2.42a}$$

Below each term I have written the characteristic magnitudes $(1, \alpha, \alpha^2)$ of the metric components, and the lengthscales $(\mathcal{L}, \lambdabar)$ on which they vary in the most "steady" of coordinate systems. Note that $j_{\mu\nu}$ is a nonlinear correction to the propagating waves. By inserting this perturbation series into the standard expression (MTW eqs. 8.47-8.49) for the Einstein curvature tensor $G_{\mu\nu}$ in terms of $g_{\mu\nu}$ and its derivatives, and by grouping terms according to their magnitudes and their lengthscales of variation, one obtains

$$G_{\mu\nu} = \underset{\le 1/\mathcal{R}^2,\mathcal{L}}{G_{\mu\nu}^{(B)}} + \underset{\alpha/\lambdabar^2,\lambdabar}{G_{\mu\nu}^{(1)}(h)} + \underset{\alpha^2/\lambdabar^2,\lambdabar}{G_{\mu\nu}^{(2)}(h)} + \underset{\alpha^2/\lambdabar^2,\lambdabar}{G_{\mu\nu}^{(1)}(j)} + \cdots \quad . \tag{2.42b}$$

Here $G_{\mu\nu}^{(B)}$ is the Einstein tensor of the background metric $g_{\mu\nu}^{(B)}$; $G_{\mu\nu}^{(1)}$(h or j) is the linearized correction to $G_{\mu\nu}$ (MTW eq. 35.58a, trace-reversed); and $G_{\mu\nu}^{(2)}(h)$ is the quadratic correction (MTW eq. 35.58b), trace-reversed).

Isaacson splits the Einstein equations into three parts: a part which varies on scales $\mathcal{L}$ (obtained by averaging, "$\langle\ \rangle$", over a few wavelengths)

$$G_{\mu\nu}^{(B)} = 8\pi\left(T_{\mu\nu}^{(B)} + \langle T_{\mu\nu}^{(2)}\rangle + T_{\mu\nu}^{(W)}\right), \quad T_{\mu\nu}^{(W)} \equiv -(1/8\pi)\,\langle G_{\mu\nu}^{(2)}(h)\rangle\,; \tag{2.43a}$$

a part of magnitude $\alpha/\lambdabar^2$ which varies on scales $\lambdabar$ and averages to zero on larger scales

$$G_{\mu\nu}^{(1)}(h) = 8\pi\, T_{\mu\nu}^{(1)} \Longleftrightarrow \bar{h}_{\mu\nu|\alpha}{}^{\alpha} = -16\pi\, T_{\mu\nu}^{(1)} \text{ in Lorentz gauge;} \tag{2.43b}$$

and a part of magnitude $\alpha^2/\lambdabar^2$ which varies on scales $\lambdabar$ and averages to zero on larger scales

$$G_{\mu\nu}^{(1)}(j) = -G_{\mu\nu}^{(2)}(h) + \langle G_{\mu\nu}^{(2)}(h)\rangle + 8\pi\left(T_{\mu\nu}^{(2)} - \langle T_{\mu\nu}^{(2)}\rangle\right). \tag{2.43c}$$

Here $T_{\mu\nu}^{(B)}$ is the stress-energy tensor of the background; $T_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(2)}$ are its first- and second-order perturbations; $T_{\mu\nu}^{(W)} \equiv -(1/8\pi)\langle G_{\mu\nu}^{(2)}(h)\rangle$ is a stress-energy tensor associated with the gravitational waves; and the averaging $\langle\ \rangle$ can be performed in the most naive of manners if the coordinates are sufficiently "steady", but must be performed carefully, by Brill-Hartle techniques (MTW exercise 35.14), if they are not. The "smoothed" field equations (2.43a), together with the contracted Bianchi identities $G^{\mu\nu}_{(B)}{}_{|\nu} \equiv 0$, imply a conservation law for energy and momentum in the presence of gravitational waves:

$$T^{\mu\nu}_{(B)}|_\nu + \langle T^{\mu\nu}_{(2)}\rangle|_\nu + T^{\mu\nu}_{(W)}|_\nu = 0. \tag{2.44}$$

(Here and throughout this section indices are raised and lowered with $g^{(B)}_{\mu\nu}$.)

To understand the physics of the field equations (2.43) and conservation law (2.44), let us reconsider the propagation of waves through a cloud of spherical oscillators (§2.4.3). Equation (2.43b) is the wave equation (2.39b) for h, which we used to calculate the absorption and dispersion of the waves. In this wave equation $T^{(1)}_{\mu\nu}$ is the part of $T_{\mu\nu}$ that is linear in the oscillators' amplitude of motion; in Exercise 12 its spatial part (after averaging over scales $< \lambda\!\!\!^{-}$) was shown to be $T^{(1)}_{jk} = \frac{1}{2} n \ddot{\mathcal{J}}_{jk}$. Equation (2.43c) describes the generation of nonlinear corrections $j_{\mu\nu}$ to the propagating waves. In Lorentz gauge it takes the explicit form

$$j_{\mu\nu}|_\alpha{}^\alpha = \begin{pmatrix}\text{source terms quadratic in } h_{\alpha\beta} \text{ and in}\\ \text{the amplitude of oscillator motion}\end{pmatrix}. \tag{2.45}$$

In §2.6.2 we shall see that these nonlinear corrections, like absorption and dispersion, are negligible in realistic astrophysical circumstances. Equation (2.43a) describes the generation of smooth, background curvature by the stress-energy of the gravitational waves $T^{(W)}_{\mu\nu}$ and of the matter. Note that the waves contribute an amount of order $\mathcal{A}^2/\lambda\!\!\!^{-2}$ to the background curvature $1/\mathcal{R}^2$, and that therefore $1/\mathcal{R}^2 \gtrsim \mathcal{A}^2/\lambda\!\!\!^{-2}$; i.e.,

$$\mathcal{A} \lesssim \lambda\!\!\!^{-}/\mathcal{R}. \tag{2.46}$$

Since $\lambda\!\!\!^{-}/\mathcal{R} \ll 1$ for propagating waves, such waves must necessarily have dimensionless amplitudes $\mathcal{A} \ll 1$. If ever $\mathcal{A}$ were to become of order unity, the wave would cease to be separable from the background curvature; the two would become united as a dynamically vibrating spacetime curvature to which the theory of propagating gravitational waves cannot be applied. Equation (2.44) describes the exchange of energy and momentum between matter and waves. In this conservation law $T^{\mu\nu}_{(B)} = mnu^\mu u^\nu$ is the stress-energy of the unperturbed spheres (number density n, mass m, 4-velocity u^μ) and by itself has vanishing divergence. The term $\langle T^{\mu\nu}_{(2)}\rangle$ is the quadratic-order stress-energy associated with the spheres' oscillations, averaged spatially over a few wavelengths and temporally over a few periods of the waves. In an LIF of the oscillators the only nonzero components are the energy density $\langle T^{00}_{(2)}\rangle$ which includes the kinetic energy and the potential energy of oscillation and the thermal energy produced when the oscillations are damped by internal friction [$1/\tau_*$ term in oscillators' equation of motion (2.39a)], and a transverse stress of magnitude comparable to $\langle T^{00}_{(2)}\rangle$. Thus, for our idealized problem the conservation law (2.44) describes the absorption of gravitational-wave energy by the oscillators and the subsequent conversion of oscillation energy into heat.

The gravitational-wave stress-energy tensor $T^{(W)}_{\mu\nu}$ "lives in" the background spacetime and is manipulated using background-spacetime mathematics [e.g., covariant derivative "$|$" in the conservation law (2.44)]. Because of the averaging $\langle\ \rangle$ in its definition, $T^{(W)}_{\mu\nu}$ gives a well-defined localization of the waves' energy and momentum only on lengthscales somewhat larger than $\lambda\!\!\!^{-}$ (no way to say whether the energy is in the "crest" of a wave or in its "trough"); no more precise localization of gravitational energy is possible in general relativity. Like $R^{(W)}_{\alpha\beta\gamma\delta}$ and unlike $h_{\alpha\beta}$, the stress-energy tensor $T^{(W)}_{\alpha\beta}$ is gauge invariant. Explicit calculations (Isaacson 1968; MTW §35.15) give

$$\begin{aligned} T^{(W)}_{\mu\nu} &= \frac{1}{32\pi}\left\langle \bar{h}_{\alpha\beta}|_\mu\, \bar{h}^{\alpha\beta}|_\nu - \tfrac{1}{2}\bar{h}|_\mu \bar{h}|_\nu - 2\,\bar{h}^{\alpha\beta}|_\beta\, \bar{h}_{\alpha(\mu|\nu)}\right\rangle \quad \text{in any gauge} \\ &= \frac{1}{32\pi}\left\langle \bar{h}_{\alpha\beta}|_\mu\, \bar{h}^{\alpha\beta}|_\nu \right\rangle \quad \text{in trace-free Lorentz gauge} \end{aligned} \tag{2.47}$$

$$= \frac{1}{32\pi} \left\langle h^{TT}_{jk,\mu}\, h^{TT}_{jk,\nu} \right\rangle \quad \text{in LIF of any observer.}$$

2.5 Wave propagation in the geometric optics limit

2.5.1 Differential equations of geometric optics

Return now to the explicit problem of the propagation of gravitational waves from the local wave zone of a source out through the lumpy universe toward earth. Throughout the local wave zone, and almost everywhere in the universe, not only will $\lambda\!\!\!^{-}$ be very small compared to the background radius of curvature $\mathcal{R}$, but also it will be small compared to the scale $\mathcal{L}$ on which the background curvature varies

$$\lambda\!\!\!^{-} \ll \mathcal{L}. \tag{2.48}$$

Here, as in the above discussion of the waves' stress-energy, I shall assume that $\lambda\!\!\!^{-} \ll \mathcal{L}$; later (§2.6.1) I shall relax that assumption. Here I shall also assume that $\lambda\!\!\!^{-} \ll \mathcal{L}^{(W)} \equiv$ (radius of curvature of the wave fronts of the waves or any smaller scale length for transverse variation of the waves).

The assumptions $\lambda\!\!\!^{-} \ll \mathcal{L}$ and $\lambda\!\!\!^{-} \ll \mathcal{L}^{(W)}$ permit us to solve for the propagation using the techniques of geometric optics (e.g., MTW Exercise 35.15): Introduce trace-free Lorentz gauge everywhere, and ignore the effects of direct interaction between the propagating waves and matter (negligible absorption and dispersion). Then

$$h^{\alpha}{}_{\alpha} = 0, \qquad h^{\mu\alpha}{}_{|\alpha} = 0, \qquad h_{\mu\nu|\alpha}{}^{\alpha} = 0. \tag{2.49}$$

The solution of these gauge and propagation equations is a rapidly varying function of retarded time ψ and a slowly varying function of the other spacetime coordinates:

$$h_{\mu\nu} = h_{\mu\nu}(\underbrace{\psi}_{\text{variation on scale } \lambda\!\!\!^{-}},\ \underbrace{x^{\alpha}}_{\text{variation on scales } \mathcal{L}^{(W)},\ \mathcal{L},\ \mathcal{R}}) \tag{2.50}$$

As in the discussion of waves in flat spacetime (§2.3.2), define the propagation vector

$$\vec{k} \equiv -\vec{\nabla}\psi . \tag{2.51a}$$

Then, aside from fractional corrections of order $\lambda\!\!\!^{-}/\mathcal{L}^{(W)}$, $\lambda\!\!\!^{-}/\mathcal{L}$, $\lambda\!\!\!^{-}/\mathcal{R}$ the gauge and field equations (2.49) imply

$$k_{\alpha}k^{\alpha} = 0 \text{ and } k_{\beta|\alpha}k^{\alpha} = 0 \Longleftrightarrow \vec{k} \text{ is tangent to null geodesics ("rays")}, \tag{2.51b}$$

$$h_{\alpha}{}^{\alpha} = 0,\ h_{\alpha\beta}k^{\beta} = 0 \qquad \Longleftrightarrow h_{\alpha\beta} \text{ is trace free and orthogonal to } \vec{k}, \tag{2.51c}$$

$$h_{\mu\nu|\alpha}k^{\alpha} = -\tfrac{1}{2}(\vec{\nabla}\cdot\vec{k})h_{\mu\nu} \qquad \text{(propagation equation for } h_{\mu\nu}\text{)}. \tag{2.51d}$$

Had we been analyzing the propagation of electromagnetic waves rather than gravitational, our Lorentz gauge equations for the vector potential would have been

$$A^{\alpha}{}_{|\alpha} = 0, \qquad A_{\mu|\alpha}{}^{\alpha} = 0 \tag{2.52a}$$

(MTW eq. 16.5' with curvature coupling term removed because $\lambda\!\!\!^{-} \ll \mathcal{R}$) (cf. eq.

2.49); our geometric-optics ansatz would have been

$$A_\mu = A_\mu(\psi;\, x^\alpha) \tag{2.52b}$$

(cf. eq. 2.50); and in the geometric optics limit the gauge and wave equations (2.52a) would have reduced to

$$k_\alpha k^\alpha = 0, \qquad k_{\beta|\alpha}k^\alpha = 0, \qquad A_\alpha k^\alpha = 0, \qquad A_{\mu|\alpha}k^\alpha = -\tfrac{1}{2}(\vec{\nabla}\cdot\vec{k})A_\mu \tag{2.52c}$$

(cf. eqs. 2.51c,d).

* * * * *

Exercise 13. Show that in the geometric optics limit $\lambdabar \ll \mathcal{L} \lesssim \mathcal{R}$ and $\lambdabar \ll \mathcal{L}^{(W)}$, and with the geometric optics ansatz (2.50), the gravitational gauge and propagation equations (2.49) reduce to the geometric optics equations (2.51). Similarly show that for electromagnetic waves (2.52a) reduce to (2.52c).

2.5.2 Solution of geometric optics equations in local wave zone

In the local wave zone of the source introduce (flat-background) spherical coordinates (t,r,θ,φ). The waves propagate radially outward from the source along the null-geodesic rays

$$\psi = t-r,\ \theta,\ \varphi \text{ all constant}, \qquad k^0 = k^r = 1. \tag{2.53a}$$

Throughout the local wave zone introduce transverse basis vectors $\vec{e}_{\hat\theta} = r^{-1}\,\partial/\partial\theta$ and $\vec{e}_{\hat\varphi} = (r\sin\theta)^{-1}\,\partial/\partial\varphi$ and polarization tensors

$$\overleftrightarrow{e}_+ \equiv \vec{e}_{\hat\theta}\otimes\vec{e}_{\hat\theta} - \vec{e}_{\hat\varphi}\otimes\vec{e}_{\hat\varphi}, \qquad \overleftrightarrow{e}_\times \equiv \vec{e}_{\hat\theta}\otimes\vec{e}_{\hat\varphi} + \vec{e}_{\hat\varphi}\otimes\vec{e}_{\hat\theta}\ . \tag{2.53b}$$

Then it turns out (Exercise 14) that in TT gauge the general solution to the gauge and propagation equations (2.51c,d) is

$$\begin{aligned} \overleftrightarrow{h}^{TT} &= A_+(\psi;r,\theta,\varphi)\ \overleftrightarrow{e}_+ + A_\times(\psi;r,\theta,\varphi) \\ A_+ &= r^{-1}F_+(\psi;\theta,\varphi), \qquad A_\times = r^{-1}F_\times(\psi;\theta,\varphi); \end{aligned} \tag{2.53c}$$

and similarly for electromagnetic waves

$$\begin{aligned} \vec{A} &= A_{\hat\theta}(\psi;r,\theta,\varphi)\ \vec{e}_{\hat\theta} + A_{\hat\varphi}(\psi;r,\theta,\varphi)\ \vec{e}_{\hat\varphi} \\ A_{\hat\theta} &= r^{-1}F_{\hat\theta}(\psi;\theta,\varphi), \qquad A_{\hat\varphi} = r^{-1}F_{\hat\varphi}(\psi;\theta,\varphi). \end{aligned} \tag{2.54}$$

Stated in words: In polarization bases $\overleftrightarrow{e}_+$, $\overleftrightarrow{e}_\times$ and $\vec{e}_{\hat\theta}$, $\vec{e}_{\hat\varphi}$ which are parallel transported along the rays, the amplitude functions A_+, $A_\times$ of gravitational waves and $A_{\hat\theta}$, $A_{\hat\varphi}$ of electromagnetic waves die out as $1/r$ but otherwise are constant along the rays.

The precise forms of $F_+(\psi;\theta,\varphi) = rA_+$ and $F_\times(\psi;\theta,\varphi) = rA_\times$ are to be determined by solution of the wave generation problem (§3 below). The local-wave-zone waves (2.53) are then to be used as "starting conditions" for propagation out through the universe.

* * * * *

Exercise 14. Show that equations (2.53) are the general solution of the gravitational geometric optics equations (2.51) specialized to TT gauge, for waves propagating radially outward through the local wave zone of a source. Similarly show that equations (2.54) are the solution of the electromagnetic equations (2.52c).

2.5.3 Solution of geometric optics equations in distant wave zone

Suppose that the wave generation problem has been solved to give h_{jk}^{TT} in the form (2.52) throughout the local wave zone. These waves can then be propagated throughout the rest of the universe (assuming $\lambdabar \ll \mathcal{L}$ and $\lambdabar \ll \mathcal{L}^{(W)}$) using the following constructive method [solution of geometric optics equations (2.51)]:

First extend the radial null rays (2.53a) of the local wave zone out through the universe by solving the geodesic equation. Continue to label each ray by ψ,θ,φ and parametrize it by an affine parameter denoted r (and equal to the radial coordinate in the local wave zone):

$$\text{rays are } (\psi,\theta,\varphi) = \text{const}; \quad \vec{k} \equiv -\vec{\nabla}\psi = d/dr. \tag{2.55a}$$

Next, along each ray parallel propagate the fiducial basis vector $\vec{e}_{\hat{\theta}}$

$$\vec{\nabla}_{\vec{k}}\, \vec{e}_{\hat{\theta}} = 0 \text{ everywhere}, \quad \vec{e}_{\hat{\theta}} = r^{-1}\, \partial/\partial\theta \text{ in local wave zone}. \tag{2.55b}$$

Together $\vec{e}_{\hat{\theta}}$ and $\vec{k}$ form a fiducial 2-flat $\vec{k} \wedge \vec{e}_{\hat{\theta}}$ to be used below in defining the polarization of the waves. Next, propagate A_+ and $A_\times$ along each ray by solving the ordinary differential

$$(\partial A/\partial r)_{\psi,\theta,\varphi} = -\tfrac{1}{2}(\vec{\nabla}\cdot\vec{k})A, \quad A = [\text{expression (2.53c) in local wave zone}]. \tag{2.55c}$$

The resulting A_+, $A_\times$, and fiducial 2-flat $\vec{k} \wedge \vec{e}_{\hat{\theta}}$ determine the gravitational-wave field in the manner of §2.3.2: At any event in the distant wave zone introduce an observer with 4-velocity $\vec{u}$; introduce an orthonormal basis

$$\vec{e}_0 \equiv \vec{u}, \quad \vec{e}_z \equiv \begin{bmatrix}\text{unit vector obtained by projecting } \vec{k} \\ \text{orthogonal to } \vec{u} \text{ and renormalizing}\end{bmatrix},$$

$$\vec{e}_x \equiv [\text{unit vector lying in } \vec{k} \wedge \vec{e}_{\hat{\theta}} \text{ and orthogonal to } \vec{u}]. \tag{2.55d}$$

$$\vec{e}_y \equiv \begin{bmatrix}\text{unit vector such that } \vec{e}_0, \vec{e}_x, \vec{e}_y, \vec{e}_z \text{ are a right-hand oriented,} \\ \text{orthonormal frame}\end{bmatrix};$$

and introduce corresponding polarization tensors

$$\overleftrightarrow{e}_+ \equiv \vec{e}_x \otimes \vec{e}_x - \vec{e}_y \otimes \vec{e}_y, \quad \overleftrightarrow{e}_\times \equiv \vec{e}_x \otimes \vec{e}_y + \vec{e}_y \otimes \vec{e}_x. \tag{2.55e}$$

Then the gravitational-wave field in this LIF is

$$\overleftrightarrow{h}^{TT} = A_+ \overleftrightarrow{e}_+ + A_\times \overleftrightarrow{e}_\times; \tag{2.55f}$$

and the Riemann tensor and stress-energy tensor associated with the waves are

$$R^{(W)}_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\,(\ddot{h}^{TT}_{\alpha\delta}\, k_\beta k_\gamma + \ddot{h}^{TT}_{\beta\gamma}\, k_\alpha k_\delta - \ddot{h}^{TT}_{\beta\delta}\, k_\alpha k_\gamma - \ddot{h}^{TT}_{\alpha\gamma}\, k_\beta k_\delta),$$
$$T^{(W)}_{\alpha\beta} = \frac{1}{16\pi}\left\langle \dot{A}_+^{\,2} + \dot{A}_\times^{\,2}\right\rangle k_\alpha k_\beta\,, \quad \text{where } \dot{} \equiv \partial/\partial\psi. \tag{2.56}$$

For electromagnetic waves the geometric optics equations (2.52c) have a similar solution. In a basis $\vec{e}_{\hat{\theta}}$, $\vec{e}_{\hat{\varphi}}$ obtained by parallel transport (eqs. 2.55b) along the rays (eqs. 2.55a) the components of the vector potential, $A_{\hat{\theta}}$ and $A_{\hat{\varphi}}$, satisfy <u>identically the same propagation equation</u> (2.55c) as the gravitational-wave amplitude functions A_+ and $A_\times$. Moreover, an observer with 4-velocity $\vec{u}$ can always put the waves into purely spatial Lorentz gauge (no component of $\vec{A}$ along $\vec{u}$) by a gauge change, which produces

$$\vec{A}^{\,S} = A_{\hat{\theta}}\,\vec{e}_x + A_{\hat{\varphi}}\,\vec{e}_y \tag{2.57}$$

(analog of eq. 2.55f) with $\vec{e}_x$, $\vec{e}_y$ given by equations (2.55d). The electromagnetic field tensor, and the stress-energy tensor of the waves averaged over several wavelengths (analogs of eqs. 2.56) are

$$F_{\alpha\beta} = \dot{A}^S_\beta\, k_\alpha - \dot{A}^S_\alpha\, k_\beta,$$
$$\langle T_{\alpha\beta}\rangle = \frac{1}{4\pi}\left\langle A_{\hat{\theta}}^{\,2} + A_{\hat{\varphi}}^{\,2}\right\rangle k_\alpha k_\beta\,. \tag{2.58}$$

* * * * * *

Exercise 15. Show that equations (2.55) constitute a solution of the gravitational geometric optics equations (2.51), transformed locally to TT gauge. Show further that this solution joins smoothly onto the local-wave-zone solution (2.53), and that the Riemann tensor and stress-energy tensor of these waves have the form (2.56). Similarly show that if $A_{\hat{\theta}}$ and $A_{\hat{\varphi}}$ are propagated via (2.55c), then $A_{\hat{\theta}}\vec{e}_{\hat{\theta}} + A_{\hat{\varphi}}\vec{e}_{\hat{\varphi}}$ is a solution of the electromagnetic equations (2.52c); (2.57) is this same solution in another gauge; and (2.58) are the field tensor and averaged stress-energy tensor of the waves.

<u>2.5.4 Example: Propagation through a Friedmann universe</u>

As an example of the geometric optics solution for wave propagation consider, as the background spacetime, a closed Friedmann universe with metric

$$g^{(B)}_{\alpha\beta}\, dx^\alpha dx^\beta = a^2(\eta)\;[-d\eta^2 + d\chi^2 + \Sigma^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)]$$
$$\Sigma = :\;\; \chi \text{ for } k = 0,\;\; \sin\chi \text{ for } k = +1,\;\; \sinh\chi \text{ for } k = -1. \tag{2.59}$$

Here k is the curvature parameter (k = 0 for a spatially flat universe, k = +1 for a closed universe, k = -1 for an open universe; see, e.g., chapters 27-29 of MTW). Orient the coordinates so the source of the waves is at $\chi = 0$, and let the source be active (emit waves) at a coordinate time $\eta \simeq \eta_e$ when the expansion factor of the universe is $a = a_e$ ("e" for "emission"). The flat, spherical coordinates of the local wave zone, and the retarded time are

$$t = a_e(\eta-\eta_e),\quad r = a_e\chi,\quad \theta,\quad \varphi; \qquad \psi = t-r = a_e(\eta-\chi-\eta_e); \tag{2.60}$$

and the waves in the local wave zone are described by equations (2.53b,c).

Throughout the wave zone (local and distant) the rays and propagation vector of equations (2.55a) are

$$\eta - \chi = \eta_e + \psi/a_e, \qquad k^\eta = k^\chi = \left(\frac{\partial\eta}{\partial r}\right)_{\psi,\theta,\varphi} = \left(\frac{\partial\chi}{\partial r}\right)_{\psi,\theta,\varphi} = \frac{a_e}{a^2}; \tag{2.61a}$$

the parallel-propagated fiducial basis vector (eq. 2.55b) is

$$\vec{e}_{\hat\theta} = (1/a\Sigma)\,\partial/\partial\theta; \tag{2.61b}$$

and the transported gravity-wave and electromagnetic-wave amplitude functions (eq. 2.55c) are

$$A_J = \frac{F_J(\psi;\theta,\varphi)}{a\Sigma}\ ;\ J = + \text{ or } \times \text{ (gravity)},\ J = \hat\theta \text{ or } \hat\varphi \text{ (electromagnetism)}. \tag{2.61c}$$

If we approximate the earth as at rest in the Friedmann coordinate system at χ_o, θ_o, φ_o and we denote the present epoch by $\eta \simeq \eta_o$, $a = a_o$, then the basis vectors (2.53b) of the earth's LIF are

$$\vec{e}_0 = \frac{1}{a_o}\frac{\partial}{\partial\eta}\ ,\quad \vec{e}_x = \frac{1}{a_o\Sigma_o}\frac{\partial}{\partial\theta}\ ,\quad \vec{e}_y = \frac{1}{a_o\Sigma_o\sin\theta_o}\frac{\partial}{\partial\varphi}\ ,\quad \vec{e}_z = \frac{1}{a_o}\frac{\partial}{\partial\chi}\ ; \tag{2.61d}$$

and the gravitational-wave field as measured at earth (eqs. 2.53b,c) is

$$\overleftrightarrow{h}^{TT} = \frac{1}{a_o\Sigma_o}\left[F_+(\psi;\theta_o,\varphi_o)(\vec{e}_x\otimes\vec{e}_x - \vec{e}_y\otimes\vec{e}_y) + F_\times(\psi;\theta_o,\varphi_o)(\vec{e}_x\otimes\vec{e}_y + \vec{e}_y\otimes\vec{e}_x)\right]. \tag{2.61e}$$

The energy density in these waves as measured at earth (eq. 2.56) is

$$T^{(W)}_{00} = \frac{\langle \dot F_+^{\,2} + \dot F_\times^{\,2}\rangle}{4\cdot\underbrace{4\pi a_o^2\Sigma_o^2}_{\text{(surface area around source today)}}}\underbrace{\left(\frac{a_e}{a_o}\right)^2}_{(1+Z)^{-2}}\ , \qquad \cdot = \partial/\partial\psi\ , \tag{2.62}$$

where Z is the cosmological redshift of the source. Similarly, for electromagnetic waves

$$\vec{A}^S = (1/a_o\Sigma_o)[F_{\hat\theta}(\psi;\theta_o,\varphi_o)\,\vec{e}_x + F_{\hat\varphi}(\psi;\theta_o,\varphi_o)\,\vec{e}_y]\ , \tag{2.63a}$$

$$\langle T_{00}\rangle = \frac{\langle \dot F_{\hat\theta}^{\,2} + \dot F_{\hat\varphi}^{\,2}\rangle}{4\pi a_o^2\Sigma_o^2}\left(\frac{a_e}{a_o}\right)^2. \tag{2.63b}$$

Note that the factor $1/a_o\Sigma_o$, by which the amplitudes of the waves die out as they recede from the source, is given in terms of cosmological parameters by

$$\frac{1}{a_o\Sigma_o} \equiv \frac{1}{R} = \frac{H_o q_o^{\,2}(1+Z)}{-q_o + 1 + q_o Z + (q_o-1)(2q_oZ+1)^{1/2}}$$

$$\approx \frac{H_o}{Z}\left[1 + \tfrac{1}{2}(1+q_o)Z + O(Z^2)\right] \qquad \text{for} \qquad Z \ll 1 \tag{2.64}$$

$$\approx H_o q_o \qquad \text{for } Z \gg 1 \text{ and } Z \gg 1/q_o$$

(MTW eqs. 29.28-29.33). Here H_o is the Hubble expansion rate; q_o is the deceleration parameter of the universe; Z is the cosmological redshift of the source; and I have assumed zero cosmological constant. For formulas with nonzero cosmological constant see MTW eqs. (29.32).

* * * * *

Exercise 16. Show that for propagation through a Friedmann universe equations (2.55)-(2.58) become (2.59)-(2.63).

2.6 Deviations from geometric optics

I have already discussed in detail several ways that wave propagation can differ from geometric optics: absorption and dispersion by matter (§2.4.3; almost always negligible for gravitational waves), and scattering of waves off background curvature with resulting production of tails (§2.4.4; important primarily near source, but also if waves encounter a sufficiently compact body — e.g., a neutron star or black hole). In this section I shall describe two other nongeometric-optics effects: diffraction and nonlinear interactions of the wave with itself.

2.6.1 Diffraction

As gravitational and electromagnetic waves propagate through the universe, they occasionally encounter regions of enhanced spacetime curvature due to concentrations of matter (galaxies, stars, ...) which produce a breakdown in $\lambda\!\!\!^{-} \ll \mathcal{L}$ and/or in $\lambda\!\!\!^{-} \ll \mathcal{L}^{(W)}$ and a resulting breakdown in geometric-optics propagation. Such a breakdown is familiar from light propagation, where it is called "diffraction".

Consider, as an example, the propagation of waves through the neighborhood and interior of the sun (Fig. 4), and ignore absorption and dispersion by direct interaction with matter (justified for gravitational waves, §2.4.3; not justified for electromagnetic waves). As they pass near and through the sun, rays from a distant source are deflected and forced to cross each other; i.e., they are

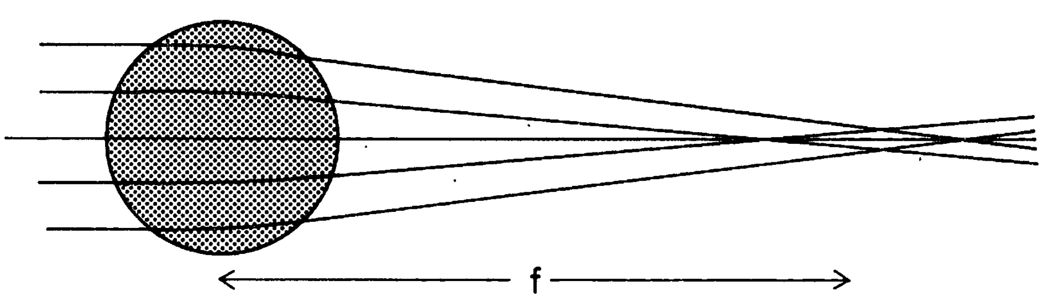

Fig. 4 The rays for geometric-optics wave propagation through the sun.

focussed gravitationally. The dominant source of deflection is the spacetime curvature of the solar core. It produces ray crossing ("caustics") along the optic axis at distances of order (and greater than) the "focal distance"

$$f \sim \frac{\mathcal{L}}{4M/\mathcal{L}} \simeq 20 \text{ AU} . \tag{2.65}$$

Here $\mathcal{L} \sim 10^5$ km is the inhomogeneity scale of the solar core, $M \sim 0.3\, M_\odot$ is the mass of the solar core, and the value 20 AU comes from detailed calculations with a detailed solar model (Cyranski and Lubkin 1974).

Geometric optics would predict infinite amplification of the waves at the caustics. However, geometric optics breaks down there because it also predicts $\mathcal{L}^{(W)} \to 0$. To understand the actual behavior of the waves near the caustics, think of the waves which get focussed by the solar core as a single wave packet that has transverse dimension $\Delta y \sim \mathcal{L}$ as it passes through the core. The uncertainty principle for waves ($\Delta y \Delta k_y \gtrsim 1$) forces this wave packet to spread in a nongeometric optics manner with a spreading angle

$$\theta_s \sim \Delta k_y / k_x \sim \lambda\!\!\!^{-}/\mathcal{L} . \tag{2.66}$$

This spreading is superimposed on the geometric-optics focussing, and it spreads out the highly focussed waves near the caustics over a lateral scale y_s

$$y_s \sim (\lambda\!\!\!^{-}/\mathcal{L}) f \sim (\lambda\!\!\!^{-}/4M)\mathcal{L} . \tag{2.67}$$

If $y_s \ll \mathcal{L}$ (i.e., if $\lambda\!\!\!^{-} \ll 4M$) there is substantial focussing: the wave energy density increases near the caustics by a factor $\sim (y_s/\mathcal{L})^2$ and the amplitude increases by $\sim y_s/\mathcal{L} \sim \lambda\!\!\!^{-}/4M$. The details of this regime are described by the laws of "Fresnel diffraction". On the other hand, if $y_s \gtrsim \mathcal{L}$ (i.e., if $\lambda\!\!\!^{-} \gtrsim 4M$) there is negligible focussing; and the little focussing that does occur is described by the laws of "Fraunhoffer diffraction". For full details see Bontz and Haugan (1981) and references therein.

For the case of the sun the dividing line between substantial focussing and little focussing is $\lambda\!\!\!^{-} \sim$ (gravitational radius of sun), i.e., (frequency) $\sim 10^4$ Hz. Since all strong sources of gravitational waves are expected to have $\lambda\!\!\!^{-} \gtrsim$ (gravitational radius of source) $\gtrsim$ (gravitational radius of Sun), i.e., (frequency) $\lesssim 10^4$ Hz, they all lie in the "little focussing regime" – a conclusion that bodes ill for any efforts to send gravitational-wave detectors on spacecraft to the orbit of Uranus in search of amplified gravitational waves; cf. Sonnabend (1979).

Far beyond the focal region the geometric optics approximation becomes valid again, except for a smearing of lateral structure of the waves over an angular scale $\sim \theta_s$. For example, ray crossing may produce multiple images of a gravitational-wave source in this region; and those images can be computed by geometric optics methods aside from θ_s-smearing.

2.6.2 Nonlinear effects in wave propagation

Once a gravitational wave has entered and passed through the local wave zone, its nonlinear interactions with itself are of no importance. To see this consider the idealized problem of a radially propagating, monochromatic wave in flat spacetime. At linear order, in spherical coordinates write the wave field as

$$h_{\hat\theta\hat\theta} = -h_{\hat\varphi\hat\varphi} = A_o(\theta,\varphi)\, \frac{\lambda\!\!\!^{-}}{r} \cos\left(\frac{t-r}{\lambda\!\!\!^{-}}\right) , \tag{2.68}$$

where hats denote components in an orthonormal, spherical basis. Note that the angular function A_o is the amplitude of the wave in the induction zone, where it is just barely starting to become a wave. For any realistic source $A_o \lesssim 1$; see eq. (3.55b) below.

As these waves propagate, their nonlinear interaction with themselves produces a correction $j_{\mu\nu}$ to $h_{\mu\nu}$. Like $h_{\mu\nu}$, $j_{\mu\nu}$ has the outgoing-wave form

$$j_{\hat{\mu}\hat{\nu}} = J_{\hat{\mu}\hat{\nu}}(r,\theta,\varphi)\ \cos[(t-r)/\lambdabar + \text{phase}]. \tag{2.69}$$

Equations (2.43c) and (2.45) describe the growth of this correction as it propagates. They have the form $j_{\mu\nu|\alpha}{}^{\alpha}$ = (source), which for $j_{\mu\nu}$ of the form (2.69) reduces to

$$\frac{1}{r\lambdabar}\frac{\partial}{\partial r}\left(r\ J_{\hat{\mu}\hat{\nu}}\right) = \alpha\left(\frac{A_o}{r}\right)^2 k_{\hat{\mu}}k_{\hat{\nu}} + O\left(\frac{A_o{}^2\lambdabar}{r^3}\right), \tag{2.70}$$

where α is a constant of order unity and $k_{\hat{r}} = -k_{\hat{0}} = 1$ is the propagation vector.

The leading, $1/r^2$ source term in (2.70) produces a rapidly growing correction

$$J_{\hat{\mu}\hat{\nu}} = \alpha A_o{}^2\,\frac{\lambdabar}{r}\,\ell n\left(\frac{r}{\lambdabar}\right)k_{\hat{\mu}}k_{\hat{\nu}}\ ; \tag{2.71}$$

but this correction is purely longitudinal, i.e., it has no transverse-traceless part, i.e., it is purely a gauge change. The $1/r^3$ source term in (2.70) produces corrections of negligible size:

$$J_{\hat{\mu}\hat{\nu}} \sim A_o{}^2\lambdabar^2/r^2 \ll A_o\lambdabar/r \sim h_{\hat{\mu}\hat{\nu}}\ . \tag{2.72}$$

Thus, the effects of nonlinearities are negligible as claimed.

* * * * *

Exercise 17. Use equations (35.58) of MTW to show that the wave equation (2.43c) for $j_{\mu\nu}$ reduces to (2.70). Show that the solution has the form (2.71), (2.72).

3 THE GENERATION OF GRAVITATIONAL WAVES

Turn now from wave propagation to wave generation. Elsewhere (Thorne 1977) I have given a rather thorough review of the theory of gravitational wave generation, including a variety of computational techniques valid for a variety of types of sources. In these lectures I shall focus almost entirely on computational techniques that involve multipole-moment decompositions. My discussion in large measure will be an overview of a long treatise on "multipole expansions of gravitational radiation" which I published recently in Reviews of Modern Physics (Thorne 1980a; cited henceforth as "RMP").

3.1 Foundations for multipole analyses

3.1.1 Multipole moments of a stationary system in linearized general relativity

I shall motivate my discussion of multipole moments by considering a stationary (time-independent), weakly gravitating system surrounded by vaccum and described using the linearized approximation to general relativity (MTW chapters 18 and 19). In a Cartesion coordinate system and in Lorentz gauge the Einstein field equations and gauge conditions are

$$\nabla^2\bar{h}^{00} = -16\pi\rho, \quad \nabla^2\bar{h}^{0j} = -16\pi\rho v^j, \quad \nabla^2\bar{h}^{jk} = -16\pi T^{jk},$$
$$\bar{h}^{0j}{}_{,j} = 0, \qquad \bar{h}^{ij}{}_{,j} = 0. \tag{3.1}$$

Here ∇^2 is the flat-space Laplacian, $\bar{h}^{\alpha\beta}$ is the trace-reversed metric perturbation, $\rho = T^{00}$ is the source's mass density, v^j is its velocity field, and T^{jk} is its stress tensor. These equations can be solved for $\bar{h}^{\alpha\beta}$ using the usual flat-space Green's function $-(1/4\pi)|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}$; and the resulting integrals can then be expanded in powers of $1/r$. By doing this and by then making gauge changes described in § VIII of RMP, one can bring the external gravitational field into the form

$$\bar{h}^{00} = \frac{4M}{r} + \frac{6}{r^3}\mathscr{I}_{jk}\,n_j n_k + \dots + \frac{4(2\ell-1)!!}{\ell!\,r^{\ell+1}}\underbrace{\mathscr{I}_{a_1\dots a_\ell}}_{\mathscr{I}_{A_\ell}}\underbrace{n_{a_1}\dots n_{a_\ell}}_{N_{A_\ell}} + \dots, \tag{3.2a}$$

$$\bar{h}^{0j} = \frac{2}{r^2}\epsilon_{jka}\,\mathcal{S}_k\,n_a + \dots + \frac{4\ell(2\ell-1)!!}{(\ell+1)!\,r^{\ell+1}}\epsilon_{jka_\ell}\,\mathcal{S}_{kA_{\ell-1}}\,N_{A_\ell} + \dots, \tag{3.2b}$$

$$\bar{h}^{ij} = 0. \tag{3.2c}$$

Here $r \equiv (\delta_{jk}x^j x^k)^{\frac{1}{2}}$ is radius, $n_j \equiv x^j/r$ is the unit radial vector, ϵ_{ijk} is the Levi-Civita tensor used to form cross products, $(2\ell-1)!!$ is the product $(2\ell-1)\cdot(2\ell-3)\cdots 1$, shorthand notations have been introduced for strings of indices $a_1\dots a_\ell \equiv A_\ell$ and for products of unit radial vectors $n_{a_1}\cdots n_{A_\ell} = N_{A_\ell}$, and spatial indices are moved up and down with impunity because the spatial coordinates are Cartesian. The "multipole moments" M, $\mathscr{I}_{A_\ell}$, $\mathcal{S}_{A_\ell}$ are given as integrals over the source by

$$M = (\text{mass}) = \int \rho\, d^3x, \tag{3.3a}$$

$$\mathscr{I}_{a_1\dots a_\ell} = \begin{pmatrix}\text{mass } \ell\text{-pole}\\ \text{moment}\end{pmatrix} = \left[\int (\rho + T^{jj})x^{a_1}\dots x^{a_\ell}\,d^3x\right]^{\mathrm{STF}}, \tag{3.3b}$$

$$\mathcal{S}_{a_1\dots a_\ell} = \begin{pmatrix}\text{current } \ell\text{-pole}\\ \text{moment}\end{pmatrix} = \left[\int (\epsilon_{a_\ell pq}x^p\rho v^q)x^{a_1}\dots x^{a_{\ell-1}}d^3x\right]^{\mathrm{STF}}. \tag{3.3c}$$

Here "STF" means "symmetric, trace-free part", i.e., "symmetrize and remove all traces"; cf. equation (2.2) of RMP. Note that the mass moments, which produce Newtonian-type gravitational accelerations $\underset{\sim}{g} = (1/4)\underset{\sim}{\nabla}\bar{h}^{00}$, are generated by $\rho + T^{jj}$ = (mass density + trace of stress tensor). For a description of a possible future experiment to verify the role of T^{jj} see §IV.D of Braginsky, Caves, and Thorne (1977).

For any realistic, weakly gravitating astrophysical source $T^{jj} \ll \rho$, so $\mathscr{I}_{a_1\dots a_\ell} = \mathscr{I}_{A_\ell}$ is the STF part of the ℓ'th moment of the mass density; and $\mathcal{S}_{A_\ell}$ is the STF part of the $(\ell-1)$'th moment of the angular momentum density (though I call it the "ℓ'th current moment"). Note that as in electromagnetism, so also here, the external gravitational field is fully characterized by just two families of moments: the "mass moments" $\mathscr{I}_{A_\ell}$ are analogs of electric moments, the "current moments" $\mathcal{S}_{A_\ell}$ are analogs of magnetic moments. In order of magnitude, for a source of mass M, size L, and characteristic internal velocity v,

$$|\mathcal{I}_{A_\ell}| \lesssim ML^\ell, \quad |\mathcal{S}_{A_\ell}| \lesssim MvL^\ell. \tag{3.4}$$

It is remarkable that, by an appropriate adjustment of gauge, $\bar{h}_{ij}$ can be made to vanish identically outside the source; and $\bar{h}_{00}$ is then determined fully by the mass moments while $\bar{h}_{0j}$ is determined fully by the current moments.

Note that the spatial coordinates of equations (3.2) have been "mass-centered" so the mass dipole moment $\mathcal{I}_j$ vanishes. I always mass-center my coordinates, thereby avoiding the issue of arbitrariness in the moments associated with arbitrariness in the origin of coordinates. Note further that the current dipole moment $\mathcal{S}_j$ is precisely the angular momentum of the source.

* * * * *

Exercise 18. Write down the solution of equations (3.1) using the Green's function $-(1/4\pi)|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}$, expanded in powers of $1/r$. Then specialize the discussion of §VIII of RMP to the stationary case and use its gauge changes to bring $\bar{h}^{\alpha\beta}$ into the form (3.2), (3.3).

3.1.2 Relation of STF tensors to spherical harmonics

The "STF" expansions (3.2) for $\bar{h}^{\alpha\beta}$ are mathematically equivalent to the more familiar expansions in terms of spherical harmonics $Y_{\ell m}(\theta,\varphi)$. The precise relationship between STF expansions and $Y_{\ell m}$ expansions is spelled out in §II of RMP. Here I shall describe only the flavor of that relationship.

Choose a specific value for the spherical-harmonic index ℓ. Then there are $2\ell+1$ linearly independent STF tensors of order ℓ ("STF-ℓ tensors"); and there are $2\ell+1$ linearly independent functions $Y_{\ell m}(\theta,\varphi)$. Moreover, the STF-$\ell$ tensors and the $Y_{\ell m}(\theta,\varphi)$ generate the <u>same</u> irreducible representation of the rotation group. Any scalar function $F(\theta,\varphi)$ can be expanded in two mathematically equivalent forms:

$$\begin{aligned} F(\theta,\varphi) &= \sum_{\ell,m} f_{\ell m}\, Y_{\ell m}(\theta,\varphi) \\ &= \sum_{\ell} \mathcal{F}_{A_\ell}\, N_{A_\ell}\ . \end{aligned} \tag{3.5}$$

In the first expansion the coefficients $f_{\ell m}$ are constant scalars, and the angular dependence is contained in the harmonics $Y_{\ell m}$. In the second expansion the coefficients $\mathcal{F}_{A_\ell}$ are constant STF-ℓ tensors, and the angular dependence is obtained by contracting the unit vectors $N_{A_\ell} \equiv n_{a_1}\ldots n_{a_\ell}$ into them.

STF expansions were widely used in the nineteenth century, before $Y_{\ell m}(\theta,\varphi)$ came into vogue; see, e.g., Kelvin and Tate (1879); Hobson (1931). In recent years they have been restored to common use by relativity theorists (e.g., Pirani 1964, RMP; Thorne 1981) because they are rather powerful when the spherical harmonics being manipulated are tensorial rather than scalar. In part, this power stems from the fact that the indices of $\mathcal{F}_{A_\ell}$ carry both angular dependence (implicitly) and tensorial component properties (explicitly), and carry them simultaneously. An example is the tensorial harmonic $\epsilon_{pqj}\,\mathcal{S}_{kqA_{\ell-2}}\,n_p\,N_{A_{\ell-2}}$. Harmonics of this form are second-rank tensors (two free indices j and k); they have harmonic order ℓ (ℓ indices on $\mathcal{S}$ implies these generate the same irreducible representation of the rotation group as do $Y_{\ell m}$); and they have parity $\pi=(-1)^{\ell+1}$ ("1" from ϵ_{pqj}, "ℓ" from $\mathcal{S}$).

3.1.3 The Einstein equations in de Donder (harmonic) gauge

In performing multipole decompositions of fully relativistic gravitational fields (the following sections) it is computationally powerful to work in de Donder (harmonic) gauge: Define the "gravitational field" $\bar{h}^{\alpha\beta}$ in terms of the "metric density" $\mathfrak{g}^{\alpha\beta}$ by

$$\mathfrak{g}^{\alpha\beta} \equiv (-g)^{\frac{1}{2}} g^{\alpha\beta} \equiv \eta^{\alpha\beta} - \bar{h}^{\alpha\beta}, \qquad g \equiv \det \|g_{\mu\nu}\| , \tag{3.6}$$

where $\eta^{\alpha\beta}$ is the Minkowski metric, diag(-1,1,1,1); and adjust the coordinates so as to impose the de Donder gauge conditions

$$\mathfrak{g}^{\alpha\beta}{}_{,\beta} = -\bar{h}^{\alpha\beta}{}_{,\beta} = 0. \tag{3.7}$$

Then the Einstein field equations take on the form (Landau and Lifshitz 1962, eq. 100.4; MTW eq. 20.21)

$$\underbrace{\mathfrak{g}^{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\mu\nu}}_{\text{characteristics: null rays of metric } g_{\alpha\beta}} = -16\pi(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{LL}) - \bar{h}^{\alpha\mu}{}_{,\nu}\bar{h}^{\beta\nu}{}_{,\mu} \tag{3.8}$$

or, equivalently

$$\underbrace{\eta^{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\mu\nu}}_{\text{characteristics: flat-spacetime rays}} = -16\pi(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{LL}) - \bar{h}^{\alpha\mu}{}_{,\nu}\bar{h}^{\beta\nu}{}_{,\mu} + \bar{h}^{\alpha\beta}{}_{,\mu\nu}\bar{h}^{\mu\nu} \equiv W^{\alpha\beta}. \tag{3.8'}$$

Here $t^{\alpha\beta}_{LL}$ is the Landau-Lifshitz pseudotensor (Landau and Lifshitz 1962, eq. 100.7; MTW eq. 20.22) which, in de Donder gauge, can be written as

$$\begin{aligned}16\pi(-g)t^{\alpha\beta}_{LL} = {} & \tfrac{1}{2}\,\mathfrak{g}^{\alpha\beta}\mathfrak{g}_{\lambda\mu}\,\bar{h}^{\lambda\nu}{}_{,\rho}\bar{h}^{\rho\mu}{}_{,\nu} + \mathfrak{g}_{\lambda\mu}\mathfrak{g}^{\nu\rho}\bar{h}^{\alpha\lambda}{}_{,\nu}\bar{h}^{\beta\mu}{}_{,\rho} \\ & - (\mathfrak{g}^{\alpha\lambda}\mathfrak{g}_{\mu\nu}\bar{h}^{\beta\nu}{}_{,\rho}\bar{h}^{\mu\rho}{}_{,\lambda} + \mathfrak{g}^{\beta\lambda}\mathfrak{g}_{\mu\nu}\bar{h}^{\alpha\nu}{}_{,\rho}\bar{h}^{\mu\rho}{}_{,\lambda}) \\ & + \tfrac{1}{8}\,(2\mathfrak{g}^{\alpha\lambda}\mathfrak{g}^{\beta\mu} - \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\lambda\mu})(2\mathfrak{g}_{\nu\rho}\mathfrak{g}_{\sigma\tau} - \mathfrak{g}_{\rho\sigma}\mathfrak{g}_{\nu\tau})\bar{h}^{\nu\tau}{}_{,\lambda}\bar{h}^{\rho\sigma}{}_{,\mu}\end{aligned} \tag{3.9}$$

where $\mathfrak{g}_{\alpha\beta} = (-g)^{-\frac{1}{2}}g_{\alpha\beta}$ is the inverse of $\mathfrak{g}^{\alpha\beta}$. The law of local conservation of energy and momentum $T^{\alpha\beta}{}_{;\beta} = 0$ can be written in terms of partial derivations as

$$\left[(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{LL})\right]_{,\beta} = 0 \tag{3.10}$$

(Landau and Lifshitz 1964, eq. 100.8; MTW eq. 20.23b).

The field equations (3.8) and (3.8') can be thought of as wave equations for $\bar{h}^{\alpha\beta}$ with source terms that include "gravitational stress-energy" (nonlinear terms in $\bar{h}^{\mu\nu}$). In the form (3.8) the wave operator is that of curved spacetime; its characteristics are the null rays of the curved spacetime metric $g_{\alpha\beta}$, and none of the source terms involve second derivatives of $\bar{h}^{\mu\nu}$. By contrast the form (3.8') involves a flat-spacetime wave operator; it is obtained from (3.8) by moving the second derivative term $\bar{h}^{\alpha\beta}{}_{,\mu\nu}\bar{h}^{\mu\nu}$ out of the wave operator and into the source.

The form (3.8'), with its flat-spacetime wave operator $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, has great computational advantages over (3.8): It can be solved (formally) for $\bar{h}^{\alpha\beta}$

using a flat-spacetime Green's function, whereas the (formal) solution of (3.8) requires a far more complicated curved-spacetime Green's function (cf. DeWitt and Brehme 1960); and its solution is naturally decomposed into spherical harmonics because spherical harmonics are eigenfunctions of the flat operator $\Box$ but not of the curve-spacetime wave operator (3.8).

On the other hand, the flat operator $\Box$ entails serious dangers: (i) It propagates gravitational waves with the wrong speed, thereby losing at linear order the "Coulomb" phase shift produced by the M/r field of the source, and then trying to correct for this loss at quadratic order with a term that diverges logarithmically in r far from the source. I avoid this danger by restricting my use of $\Box$ to the "wave generation problem", which is formulated entirely at radii $r \lesssim r_0$, and by using the correct curved-spacetime wave operator when studying "wave propagation" at radii $r > r_0$ [cf. the paragraph preceding eq. (1.8)]. (ii) The flat-operator field equations (3.8') produce divergences, due to the second-derivative source terms $\bar{h}^{\alpha\beta}{}_{,\mu\nu}\bar{h}^{\mu\nu}$, in calculations of the gravitational interactions of (idealized) point particles; see, e.g., Crowley and Thorne (1977). I avoid this danger in these lectures and in RMP by not using point-particle idealizations.

I believe, and hope, that <u>all</u> of my calculations with the flat-operator field equations (3.8') have been so designed as to avoid these and other pitfalls.

3.1.4 Multipole moments of a fully relativistic, stationary system

The de Donder formulation (3.8') of the Einstein field equations can be used to extend the linearized multipole analysis of §3.1.1 to fully relativistic, stationary systems. Full details are given in §X of RMP. Here I shall sketch the methods and summarize the results.

The key idea of the analysis is to construct, in de Donder gauge, the <u>general</u> external gravitational field of a fully relativistic, stationary (time-independent) system surrounded by vacuum. The de Donder coordinates are chosen to be stationary and asymptotically flat — i.e., to satisfy, in addition to equations (3.6)-(3.10), also

$$\bar{h}^{\alpha\beta}{}_{,0} = 0; \qquad \bar{h}^{\alpha\beta} \propto 1/r \quad \text{as} \quad r \equiv (\delta_{ij}x^i x^j)^{\frac{1}{2}} \to \infty . \tag{3.11}$$

For such a system the gauge conditions (3.7) and vacuum field equations (3.8') are

$$\bar{h}^{\alpha j}{}_{,j} = 0; \ \bar{h}^{\alpha\beta}{}_{,jj} = W^{\alpha\beta} = \begin{pmatrix} \text{expression of quadratic order and higher in} \\ \bar{h}^{\mu\nu} \text{ and its spatial derivatives, each term} \\ \text{containing precisely two spatial derivatives} \end{pmatrix}. \tag{3.12}$$

Here and throughout this section I use the notation of flat-space Cartesian coordinates in which the location of spatial indices, up or down, is of no importance. Equations (3.12) can be solved by a "nonlinearity expansion" in which terms of first order are linear in $\bar{h}^{\alpha\beta}$ (or, equivalently, linear in the gravitation constant $G = 1$), terms of second order are quadratic, etc. The first-order part of $\bar{h}^{\alpha\beta}$, denoted ${}_1\bar{h}^{\alpha\beta}$, satisfies the linearized equations ${}_1\bar{h}^{\alpha j}{}_{,j} = 0$ and ${}_1\bar{h}^{\alpha\beta}{}_{,jj} = 0$ and thus, with specialization of gauge and mass centering of coordinates, has the general linearized-theory form (3.2):

$$\begin{aligned} {}_1\bar{h}^{00} &= \frac{4M}{r} + \sum_{\ell=2}^{\infty} \frac{4(2\ell-1)!!}{\ell!\, r^{\ell+1}} \mathcal{I}_{A_\ell} N_{A_\ell} , \\ {}_1\bar{h}^{0j} &= \sum_{\ell=1}^{\infty} \frac{4\ell(2\ell-1)!!}{(\ell+1)!\, r^{\ell+1}} \epsilon_{jka_\ell} \mathcal{S}_{kA_{\ell-1}} N_{A_{\ell-1}} , \qquad \bar{h}^{ij} = 0. \end{aligned} \tag{3.13}$$

The quadratic-order part ${}_2\bar{h}^{\alpha\beta}$ satisfies

$$ {}_2\bar{h}^{\alpha j}{}_{,j} = 0, \qquad {}_2\bar{h}^{\alpha\beta}{}_{,jj} = \begin{pmatrix}\text{quadratic part of } W^{\alpha\beta} \\ \text{constructed from } {}_1\bar{h}^{\mu\nu}\end{pmatrix} . \tag{3.14} $$

It is straightforward, though tedious, to solve these equations for ${}_2\bar{h}^{\alpha\beta}$ and for higher-order corrections — the kind of task ideally suited for symbolic-manipulation software on a computer; cf. Appendix of Gürsel (1982). The full details are not of interest here, but the spherical-harmonic structure of the solution is of interest. That structure is dictated by the following properties of spherical harmonics: (i) Taking gradients and inverting Laplacians does not change the spherical-harmonic order of a term; and (ii) the product of two harmonics of order ℓ and ℓ' contains pieces of orders $\ell+\ell'$, $\ell+\ell'-1$, ..., $|\ell-\ell'|$. These properties plus the quadratic-order equations (3.14) and linear-order solutions (3.13) imply that the generic term in ${}_2\bar{h}^{\alpha\beta}$ has the form

$$ {}_2\bar{h}^{\alpha\beta} \sim \frac{\mathcal{M}_{A_\ell}}{r^{\ell+1}} \cdot \frac{\mathcal{M}_{B_{\ell'}}}{r^{\ell'+1}} \sim \frac{S_{\ell+\ell'} + S_{\ell+\ell'-1} + \dots + S_{|\ell-\ell'|}}{r^{\ell+\ell'+2}} . \tag{3.15} $$

Here $\mathcal{M}_{A_\ell} = (\mathcal{I}_{A_\ell}$ or $\mathcal{S}_{A_\ell})$, $\mathcal{M}_{B_{\ell'}} = (\mathcal{I}_{B_{\ell'}}$ or $\mathcal{S}_{B_{\ell'}})$, and $S_\ell \equiv$ (something unspecified that has harmonic order ℓ and is independent of r). The key feature of this generic term is that the power $\ell+\ell'+2$ of its radial dependence is larger by a factor 2 than the order of any of its harmonics. By an extension of this argument one sees that in ${}_n\bar{h}^{\alpha\beta}$ the generic term of order $1/r^k$ has harmonics of order k-n and smaller. Thus, the nonlinear parts of the solution add up to give

$$ {}_2\bar{h}^{\alpha\beta} + {}_3\bar{h}^{\alpha\beta} + \dots = \sum_{\ell=1}^{\infty} \frac{1}{r^{\ell+1}} \left(S_{\ell-1} + S_{\ell-2} + \dots + S_0\right). \tag{3.16} $$

By adding these to ${}_1\bar{h}^{\alpha\beta}$ and then computing the corresponding metric from (3.6) one obtains

$$ g_{00} = -1 + 2\frac{M}{r} - 2\frac{M^2}{r^2} + \sum_{\ell=2}^{\infty} \frac{1}{r^{\ell+1}} \left[\frac{2(2\ell-1)!!}{\ell!} \mathcal{I}_{A_\ell} N_{A_\ell} + S_{\ell-1} + \dots + S_0\right], \tag{3.17a} $$

$$ g_{0j} = \sum_{\ell=1}^{\infty} \frac{1}{r^{\ell+1}} \left[-\frac{4\ell(2\ell-1)!!}{(\ell+1)!} \epsilon_{jka_\ell} \mathcal{S}_{kA_{\ell-1}} N_{A_\ell} + S_{\ell-1} + \dots + S_0\right], \tag{3.17b} $$

$$ \begin{aligned} g_{ij} = {} & \delta_{ij}\left(1 + 2\frac{M}{r}\right) + \frac{M^2}{r^2}\left(\delta_{ij} + n_i n_j\right) \\ & + \sum_{\ell=2}^{\infty} \frac{1}{r^{\ell+1}} \left[\frac{2(2\ell-1)!!}{\ell!} \mathcal{I}_{A_\ell} N_{A_\ell} \delta_{ij} + S_{\ell-1} + \dots + S_0\right] . \end{aligned} \tag{3.17c} $$

Note the following features of this general, asymptotically flat, stationary, vacuum metric: (i) As in linearized theory, so also here, the metric is determined fully by two families of moments: the mass moments M, $\mathcal{I}_{ij}$, $\mathcal{I}_{ijk}$,...; and the current moments $\mathcal{S}_i$, $\mathcal{S}_{ij}$, $\mathcal{S}_{ijk}$,... . (ii) The mass dipole moment vanishes because I have insisted that the coordinates be mass centered. (iii) The moments are constant STF tensors that reside in the asymptotically flat region of spacetime (i.e., rigorously speaking, at spacelike infinity). (iv) In de Donder coordinates the mass ℓ-pole moment $\mathcal{I}_{A_\ell}$ can be "read off" the metric as the $1/r^{\ell+1}$, ℓ-harmonic-order part of g_{00}; and the current ℓ-pole moment $\mathcal{S}_{A_\ell}$ can similarly be read off g_{0j}.

It would be very unpleasant if one had to transform a metric to de Donder coordinates in order to compute its multipole moments. Fortunately, there are other ways of computing them. If one only wants to know the moments of order $\ell = 0, 1, 2, \ldots, \ell_{max}$, it is adequate to find coordinates where the metric has the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \sum_{\ell=0}^{\ell_{max}-1} r^{-(\ell+1)} \left[S_\ell + S_{\ell-1} + \ldots + S_0\right] + 0\left[r^{-(\ell_{max}+1)}\right] , \tag{3.18}$$

with the $1/r^2$ dipole of g_{00} vanishing. Such coordinates are called "Asymptotically Cartesian and Mass Centered to order ℓ_{max}-1" [ACMC — (ℓ_{max}-1)]. In them one can read off the first ℓ_{max} moments (both mass and current) by the same prescription as in de Donder coordinates, and one will obtain the same answers as one would in de Donder coordinates (RMP §XI). Alternatively, one can compute the moments by elegant techniques at spacelike infinity, due to Geroch (1970) and Hansen (1974). As Gürsel (1982) has shown, the Geroch-Hansen prescription gives the same moments as the above, aside from normalization:

$$\mathcal{I}_{A_\ell} = \frac{1}{(2\ell-1)!!} \mathcal{M}_{A_\ell} , \quad \mathcal{S}_{A_\ell} = \frac{(\ell+1)}{2\ell(2\ell-1)!!} \mathcal{J}_{A_\ell} , \tag{3.19}$$

where $\mathcal{M}_{A_\ell}$ and $\mathcal{J}_{A_\ell}$ are the Geroch-Hansen moments.

3.2 Gravitational wave generation by slow-motion sources: $\lambda\!\!\!^{-} \gg L \gtrsim M$

3.2.1 Metric in the weak-field near zone

Turn attention now from stationary systems to a system with slowly changing gravitational field:

$$\lambda\!\!\!^{-} \equiv \begin{pmatrix}\text{timescale}\\ \text{of changes}\end{pmatrix} \gg L \equiv \begin{pmatrix}\text{size of}\\ \text{system}\end{pmatrix} \gtrsim M \equiv \begin{pmatrix}\text{mass of}\\ \text{system}\end{pmatrix}. \tag{3.20}$$

Such a "slow-motion" system possesses a weak-field near zone (WFNZ)

$$(10M \text{ and } L) < r < \lambda\!\!\!^{-}/10 \tag{3.21}$$

(Fig. 2 and associated discussion). In that WFNZ and in de Donder gauge I have developed an algorithm for computing the general "gravitational field" $\bar{h}^{\alpha\beta}$ and the spacetime metric $g_{\alpha\beta}$; see §IX of RMP. That algorithm is based on a simultaneous "nonlinearity expansion" like that used above for stationary systems, and "slow-motion expansion" — i.e., expansion of the time evolution of the metric in powers of $r/\lambda\!\!\!^{-}$.

At lowest order in $r/\lambda\!\!\!^{-}$, $\bar{h}^{\alpha\beta}$ and $g_{\alpha\beta}$ are identical to the general stationary solution (eqs. 3.13, 3.16, 3.17); except that now the multipole moments of order $\ell \geq 2$ are slowly changing functions of time t rather than constants. [The $\ell = 0$ and $\ell = 1$ moments, M = (mass) and $\mathcal{S}_j$ = (angular momentum) are forced, by the field equations, to be constant at lowest order in $r/\lambda\!\!\!^{-}$; but they change due to radiation reaction at orders $(r/\lambda\!\!\!^{-})^5$ and higher.] The slow time changes of $\mathcal{I}_{A_\ell}(t)$ and $\mathcal{S}_{A_\ell}(t)$ produce, through the field equations (3.8') and gauge conditions (3.7) and through matching to outgoing waves at $r \gtrsim \lambda\!\!\!^{-}$, the "motional" corrections of order $r/\lambda\!\!\!^{-}$, $(r/\lambda\!\!\!^{-})^2, \ldots$ to $\bar{h}^{\alpha\beta}$ and $g_{\alpha\beta}$.

3.2.2 Metric in the induction zone and local wave zone

In the inner parts $r \ll \lambda\!\!\!^{-}$ of the WFNZ the motional corrections are very small but the nonlinear corrections may be large; and $\bar{h}^{\alpha\beta}$, $g_{\alpha\beta}$ are essentially those of a stationary system (eqs. 3.13, 3.16, 3.17) with slowly changing moments. In the outer parts $r \gg L \gtrsim M$ of the WFNZ the nonlinear corrections are very small but as r nears $\lambda\!\!\!^{-}$ the motional corrections become large. This allows us to ignore nonlinearities when extending the $\bar{h}^{\alpha\beta}$ of the outer part of the WFNZ into the induction zone and local wave zone. In other words, we can compute $\bar{h}^{\alpha\beta}$ in the induction zone and local wave zone by constructing the general outgoing-wave solution of the linearized, time-dependent, vacuum field equations and gauge conditions

$$\eta^{\mu\nu}\,\bar{h}^{\alpha\beta}{}_{,\mu\nu} = 0, \qquad \bar{h}^{\alpha\beta}{}_{,\beta} = 0; \tag{3.22}$$

and by matching that solution onto the $O([r/\lambda\!\!\!^{-}]^0)$ solution (3.13), (3.16) in the WFNZ. The result is

$$\bar{h}^{00} = \frac{4M}{r} + \sum_{\ell=2}^{\infty} (-1)^\ell \frac{4}{\ell!}\left[\frac{1}{r}\mathcal{I}_{A_\ell}(t-r)\right]_{,A_\ell} + \begin{pmatrix}\text{small nonlinear}\\ \text{terms}\end{pmatrix}, \tag{3.23a}$$

$$\begin{aligned}\bar{h}^{0j} = \frac{2\epsilon_{jpq}\mathcal{S}_p n_q}{r^2} &+ \sum_{\ell=2}^{\infty} (-1)^\ell \frac{4\ell}{(\ell+1)!}\left[\frac{1}{r}\epsilon_{jpq}\mathcal{S}_{pA_{\ell-1}}(t-r)\right]_{,qA_{\ell-1}} \\ &- \sum_{\ell=2}^{\infty} (-1)^\ell \frac{4}{\ell!}\left[\frac{1}{r}\dot{\mathcal{I}}_{jA_{\ell-1}}(t-r)\right]_{,A_{\ell-1}} + \begin{pmatrix}\text{small nonlinear}\\ \text{terms}\end{pmatrix},\end{aligned} \tag{3.23b}$$

$$\begin{aligned}\bar{h}^{jk} = \sum_{\ell=2}^{\infty} \Bigg\{ &(-1)^\ell \frac{4}{\ell!}\left[\frac{1}{r}\ddot{\mathcal{I}}_{jkA_{\ell-2}}(t-r)\right]_{,A_{\ell-2}} + (-1)^{\ell+1}\frac{8\ell}{(\ell+1)!} \times \\ &\times \left[\frac{1}{r}\epsilon_{pq(j}\dot{\mathcal{S}}_{k)pA_{\ell-2}}(t-r)\right]_{,qA_{\ell-2}}\Bigg\} + (\text{small nonlinear terms}).\end{aligned} \tag{3.23c}$$

Here dots denote $\partial/\partial t$, and as indicated the moments with $\ell \geq 2$ are to be regarded as functions of t-r. The dominant nonlinear corrections to this solution are discussed in §IX of RMP; see also equation (3.25) below. The metric can be computed from (3.23) via equations (3.6), which reduce to

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + \text{nonlinearities}, \qquad h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \tfrac{1}{2}\bar{h}\eta_{\alpha\beta}, \tag{3.24}$$

where indices on $\bar{h}^{\alpha\beta}$ are lowered (as usual) using $\eta_{\mu\nu}$.

The matching of the solution (3.23) onto that of the WFNZ can be done either by elementary techniques, which require care and thought, or by the sophisticated technique of "matched asymptotic expansions" (see the lecture by Kates in this volume; also those of Damour), which do the job with less danger of error.

* * * * *

Exercise 19. Show that (3.23) without nonlinear terms is an exact solution of the linearized field equations and gauge conditions (3.22). Then, at radii $r \ll \lambda\!\!\!^{-}$, expand (3.23) in powers of $r/\lambda\!\!\!^{-}$ and show that the leading terms, of

$O([r/\lambda\!\!\!^{-}]^0)$, is identical to the linear part (3.13) of the WFNZ field.

3.2.3 Gravitational-wave field in local wave zone

The gravitational-wave field h^{TT}_{jk} of the local wave zone can be computed from expression (3.23c) by letting the spatial derivatives all act on $\mathcal{I}$ and $\mathcal{S}$ (so as to keep only the $1/r$ part of the field); and by then taking the TT part. The result is

$$h^{TT}_{jk} = \left\{ \sum_{\ell=2}^{\infty} \frac{1}{r} \frac{4}{\ell!} \, {}^{(\ell)}\mathcal{I}_{jkA_{\ell-2}}(t-r)\, N_{A_{\ell-2}} + \sum_{\ell=2}^{\infty} \frac{1}{r} \frac{8\ell}{(\ell+1)!}\, \epsilon_{pq(j}\, {}^{(\ell)}\mathcal{S}_{k)pA_{\ell-2}}(t-r)\, n_q N_{A_{\ell-2}} \right\}^{TT} \underbrace{\left\{1 + O\!\left(\frac{M}{\lambda\!\!\!^{-}} \ln \frac{\lambda\!\!\!^{-}}{L}\right)\right\}}_{\text{effects of nonlinearities}} . \tag{3.25}$$

Here a prefix superscript (ℓ) means "take ℓ time derivatives"

$${}^{(\ell)}\mathcal{I} \equiv (\partial/\partial t)^{\ell}\mathcal{I} \;; \tag{3.26}$$

and I have indicated the magnitude of the cumulative effects of nonlinearities, integrated up from the inner part of the WFNZ into the local wave zone, which in effect cause the multipole moments of the radiation field to differ slightly from those one would measure in the inner part of the weak-field near zone; see §IX.H of RMP.

Note that the mass quadrupole part of the radiation field (3.25) has the familiar form first derived (in different notation) by Einstein (1918):

$$h^{TT}_{jk} = \frac{2}{r}\, \ddot{\mathcal{I}}^{TT}_{jk}(t-r). \tag{3.27}$$

For most slow-motion systems these mass quadrupole waves will dominate; but when quadrupole motions are suppressed by special symmetries (e.g., in torsional oscillations of neutron stars, §3.2.7 below), other moments may dominate. Note that, in the absence of suppression due to symmetries, the magnitudes of the various multipole components of the waves are

$$\left(h^{TT}_{jk}\right)_{\text{mass } \ell\text{-pole}} \sim \frac{M}{r}\left(\frac{L}{\lambda\!\!\!^{-}}\right)^{\ell}, \quad \left(h^{TT}_{jk}\right)_{\text{current } \ell\text{-pole}} \sim \frac{M}{r}\, v\left(\frac{L}{\lambda\!\!\!^{-}}\right)^{\ell} \tag{3.28}$$

(cf. eq. 3.4). Typically the internal velocity v will be of order $L/\lambda\!\!\!^{-}$, so

$$\begin{gathered}(\text{current quadrupole waves}) \sim (\text{mass octupole waves}),\\ (\text{current } \ell\text{-pole waves}) \sim (\text{mass } [\ell+1]\text{-pole waves}).\end{gathered} \tag{3.29}$$

This is the same pattern as one sees in electromagnetism, for which "electric" multipoles are the analogs of "mass" multipoles and "magnetic" multipoles are the analogs of "current" multipoles.

3.2.4 Slow-motion method of computing wave generation

Equations (3.17) and (3.25) are the foundation for the slow-motion method of computing gravitational-wave generation (RMP §XII): (i) Analyze the near-zone structure and evolution of any slow-motion ($\lambda\!\!\!^{-} \gg L \gtrsim M$) system in any convenient coordinate system and by any approximation scheme that gives, with reasonable fractional accuracy, the time evolution of the system's asymmetries. [One attractive approximation scheme is the "instantaneous gravity" method, in which one sets to zero all time derivatives of the metric (but not of the matter variables) when solving the near-zone Einstein field equations; see, e.g., Thorne (1983) and Schumaker and Thorne (1983).] (ii) From the near-zone analysis obtain an approximation to the system's external gravitational field which, at any moment, satisfies the time-independent, vacuum Einstein equations. (iii) Compute the dominant multipole moments of that quasistationary field (moments with largest values of ${}^{(\ell)}\mathcal{I}_{A_\ell}$ or ${}^{(\ell)}\mathcal{S}_{A_\ell}$) either by transforming to de Donder or ACMC coordinates and comparing with equations (3.17), or by the methods of Geroch (1970) and Hansen (1974) plus equation (3.19). (iv) Insert those moments into equation (3.25) to obtain the radiation field in the local wave zone.

3.2.5 Example: Rigidly rotating neutron star

As an example, consider the gravitational waves produced by a slowly rotating neutron star (pulsar) idealized as a non-axisymmetric, fully relativistic body which rotates rigidly. Full details are given in Gürsel and Thorne (1983); the main ideas will be sketched here.

The star can rotate rigidly (distance between every pair of neighboring "material particles" forever fixed) only to first order in the angular velocity Ω. At order $(\Omega L)^2$ there is a Lorentz contraction of distances; and as the star's angular velocity precesses that Lorentz contraction changes. Thus, the Gürsel-Thorne analysis, which assumes rigid rotation and works to first order in Ω, has fractional errors of order ΩL.

Gürsel and Thorne show, by a de Donder-gauge analysis of the star's interior [step (i) of "slow-motion wave-generation method"] that to first order in Ω the star's angular momentum $\mathcal{S}_j$ and its angular velocity Ω_j (which are both spatial vectors residing in the nearly flat, weak-field near zone) are proportional to each other

$$\mathcal{S}_j = I_{jk}\Omega_k; \tag{3.30a}$$

and that their ratio, the moment of inertia tensor, is symmetric and rotates with angular velocity Ω_j

$$I_{jk} = I_{kj}, \quad \dot{I}_{jk} = \epsilon_{jpq}\Omega_p I_{qk} + \epsilon_{kpq}\Omega_p I_{qj}. \tag{3.30b}$$

Of course, the angular momentum is conserved (aside from negligible radiation-reaction changes)

$$\dot{\mathcal{S}}_j = 0. \tag{3.30c}$$

Equations (3.30) are identical to the classical Euler equations which govern the precession of a rigidly rotating, nongravitating body (Goldstein 1980). Thus, any fully relativistic, slowly ($\Omega L \ll 1$) and rigidly rotating body undergoes a free precession which is identical to that of a nongravitating body with the same moment of inertia I_{jk} and the same angular momentum $\mathcal{S}_j$. The only influence of relativistic gravity will be through its influence on the values of the components of the moment of inertia tensor I_{jk}; cf. Hartle (1973). Gürsel and Thorne go on to show that the mass moments $\mathcal{I}_{A_\ell}$, which characterize g_{00} in the weak-field near zone, like I_{jk} rotate with angular velocity Ω_j:

$$\dot{\mathcal{I}}_{jk} = \epsilon_{jpq}\Omega_p\mathcal{I}_{qk} + \epsilon_{kpq}\Omega_p\mathcal{I}_{jq}, \quad \text{and similarly for } \mathcal{I}_{A_\ell}. \tag{3.31}$$

The mass quadrupole $\mathcal{I}_{jk}$ will be the dominant source of gravitational waves unless the star has very unexpected symmetries. The waves that it produces are described by the standard quadrupole-moment formula

$$h_{jk}^{TT} = (2/r)\ \ddot{\mathcal{I}}_{jk}^{TT}(t-r). \tag{3.32}$$

Because the precessional equations (3.30) are identical to those of Euler and the waves are given by the same standard quadrupole moment formula (3.32) as one often uses for weakly gravitating systems, one might expect the waves from a fully relativistic, rigidly and slowly rotating body to be the same as those from a weakly gravitating body with the same moment of inertia. However, I doubt that this is so, because I suspect that relativistic gravity destroys the classical relationship

$$\mathcal{I}_{jk} = I_{jk} - \frac{1}{3} I_{ii}\,\delta_{jk} \tag{3.33}$$

between the quadrupole moment and the moment of inertia.

Zimmermann and Szedenitz (1979) and Zimmermann (1980) have computed in detail the quadrupole waves from a rigidly and slowly rotating body under the assumption that the classical relationship (3.33) is preserved. They show that the spectrum of the waves is rather rich and contains much detailed information about the star's angular momentum vector and moment of inertia tensor. Future theoretical studies should probe the possible breakdown of the classical relation (3.33) and should quantify deviations from rigid-body rotation due to the finiteness of the shear modulus and bulk modulus of neutron-star matter.

3.2.6 Example: compact binary system

Consider a binary system with stars sufficiently compact that tidal distortions of each other can be ignored (this is frequently true), and with separation between stars that is large compared to gravitational radii. Then general relativistic "near-zone" analyses (e.g., Damour in this volume; or, for the case of two black holes, D'Eath 1975) show that the orbital motions are Keplerian, aside from post-Newtonian corrections of size (gravitational radii)/(separation of stars); and this is true no matter how strong the stars' internal gravity may be. Moreover, the quadrupole moment which one reads off the weak-field, near-zone metric in de Donder gauge (eq. 3.17; Damour in this volume) is the same and evolves the same as that which one would compute for the Kepler problem using Newtonian techniques; and thus the gravitational waves obtained by inserting that quadrupole moment into $h_{jk}^{TT} = (2/r)\ddot{\mathcal{I}}_{jk}^{TT}$ are the same as one would compute for a nearly Newtonian system with the same masses and semimajor axis. For details of those waves see Peters and Mathews (1963).

3.2.7 Example: torsional oscillations of a neutron star

"Glitches" observed in the timing of pulsars are thought to be due to starquakes, i.e., due to ruptures in the crystalline crust of the neutron star. Such ruptures may trigger torsional oscillations of the star with a restoring force, due to the crystal's shear modulus, which is sufficiently small that the oscillations are slow ($\lambda \gg L$). In such oscillations its mass-energy density T^{00} remains constant while the momentum density T^{0j} oscillates; and, as a result, the star's mass quadrupole moment $\mathcal{I}_{jk}$ is constant but its current quadrupole moment $\mathcal{S}_{jk}$ oscillates. The resulting gravitational waves are thus current quadrupole rather than mass quadrupole

$$h_{jk}^{TT} = \left[\frac{8}{3r}\,\epsilon_{pq(j}\ddot{\mathcal{S}}_{k)p}(t-r)\,n_q\right]^{TT}. \tag{3.34}$$

Schumaker and Thorne (1983) have analyzed such torsional oscillations in detail using perturbation theory and have derived, in the "instantaneous gravity approximation", an eigenequation that governs the oscillations and determines the current quadrupole moment $\mathcal{S}_{jk}(t)$ for insertion into the gravity-wave formula (3.34).

3.3 Multipole decomposition of arbitrary waves in the local wave zone

3.3.1 The radiation field

The gravitational waves from any source — slow-motion or fast, weak-gravity or strong — can be decomposed into multipole components in the local wave zone. The multipole moments can be computed as surface integrals of the radiation field (RMP eq. 4.11):

$$^{(\ell)}\mathcal{I}_{A_\ell} = \left[\frac{\ell(\ell-1)(2\ell+1)!!}{2(\ell+1)(\ell+2)} \frac{r}{4\pi} \int h^{TT}_{a_1 a_2} n_{a_3} \ldots n_{a_\ell} \, d\Omega\right]^{STF} , \tag{3.35a}$$

$$^{(\ell)}\mathcal{S}_{A_\ell} = \left[\frac{(\ell-1)(2\ell+1)!!}{4(\ell+2)} \frac{r}{4\pi} \int \epsilon_{a_1 jk} n_j h^{TT}_{ka_2} n_{a_3} \ldots n_{a_\ell} \, d\Omega\right]^{STF} ; \tag{3.35b}$$

and the field can be reconstructed as a sum over the multipole moments — the same sum as we encountered in the theory of slow-motion sources (eq. 3.25)

$$h^{TT}_{jk} = \sum_{\ell=2}^{\infty} \left\{ \frac{1}{r} \frac{4}{\ell!} \, {}^{(\ell)}\mathcal{I}_{jkA_{\ell-2}}(t-r) N_{A_{\ell-2}} + \frac{1}{r} \frac{8\ell}{(\ell+1)!} \epsilon_{pq(j} \, {}^{(\ell)}\mathcal{S}_{k)pA_{\ell-2}}(t-r) n_q N_{A_{\ell-2}} \right\}^{TT} . \tag{3.36}$$

For slow-motion sources the lowest few moments will dominate; but for fast-motion sources the radiation may be highly directional and many moments may contribute. See, e.g., Kovács and Thorne (1978) for the example of "gravitational bremsstrahlung radiation", in which the radiation from one particle flying past another at a speed $v \simeq 1$ is beamed forward into a cone of half angle $\sim \gamma^{-1} = (1-v^2)^{1/2} \ll 1$, and moments $\ell = 2, 3, 4, \ldots, \gamma$ all contribute significantly to the waves. In such cases multipole expansions are not very useful.

3.3.2 The energy, momentum, and angular momentum carried by the waves

In the local wave zone the gravitational waves, which have $\ell \geq 2$, coexist with the (nearly) time-independent $\ell = 0$ (mass) and $\ell = 1$ (angular momentum) parts of the source's gravitational field; in TT gauge and neglecting nonlinearities and induction terms the total spacetime metric is

$$g_{00} = \underbrace{-1 + 2M/r}_{\ell=0} , \quad g_{0j} = \underbrace{(-2/r^2)\epsilon_{jk\ell}\mathcal{S}_k n_\ell}_{\ell=1} , \quad g_{jk} = (1 + \underbrace{2M/r}_{\ell=0})\delta_{jk} + \underbrace{h^{TT}_{jk}}_{\ell\geq 2} . \tag{3.37}$$

This metric is written in coordinates that coincide with the asymptotic rest frame of the source; in this frame the source's linear momentum P_j vanishes; i.e., the 4-momentum (a 4-vector residing in the asymptotically flat region) is $\vec{P} = M\partial/\partial t$.

The gravitational waves carry 4-momentum and angular momentum away from the source, thereby causing changes in the asymptotic rest frame, in M, and in S_j. A detailed analysis of those changes is given in chapters 19 and 20 of MTW; here I shall sketch only one variant of the main ideas.

The foundation for the analysis is the quantity

$$H^{\mu\alpha\nu\beta} \equiv \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta}, \tag{3.38}$$

where $\mathfrak{g}^{\alpha\beta}$ is the metric density of equations (3.6) and Lorentz gauge is <u>not</u> being imposed. In the asymptotic rest frame of the source, where the metric is (3.37) plus nonlinearities and induction terms, the surface integral of $H^{\mu\alpha 0j}{}_{,\alpha}$ plucks out the $1/r$ $\ell = 0$ and $\ell = 1$ parts of $g_{\alpha\beta}$ (which have zero contribution from nonlinearities and induction terms); i.e., it gives the 4-momentum of the source:

$$P^{\mu} = \frac{1}{16\pi}\oint H^{\mu\alpha 0j}{}_{,\alpha}\, d^2S_j = \begin{cases} 0 \text{ for } P^j \\ M \text{ for } P^0, \end{cases} \tag{3.39}$$

where d^2S_j is the surface element computed as though spacetime were flat. The rate of change of the 4-momentum is computed, with the help of Einstein's vacuum field equations in the form $H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = -H^{\mu\alpha\beta\nu}{}_{,\alpha\beta} = 16\pi(-g)t^{\alpha\beta}_{LL}$ (MTW eq. 20.21):

$$\begin{aligned} \frac{dP^{\mu}}{dt} &= \frac{1}{16\pi}\oint H^{\mu\alpha 0j}{}_{,\alpha 0}\, d^2S_j = \frac{1}{16\pi}\oint \left[H^{\mu\alpha\beta j}{}_{,\alpha\beta} - H^{\mu\alpha ij}{}_{,\alpha i}\right] d^2S_j \\ &= -\oint(-g)t^{\mu j}_{LL}\, d^2S_j - \frac{1}{16\pi}\int \underbrace{H^{\mu\alpha ij}{}_{,\alpha ij}}_{0}\, dx^1dx^2dx^3. \end{aligned} \tag{3.40}$$

The volume integral vanishes by symmetry; and the conversion from surface integral to volume integral requires the topology of the space slices to be Euclidean — which it always can be for astrophysically realistic systems, including those with black holes (see Fig. 5). By averaging equation (3.40) over $\Delta t =$ (several $\lambda\!\!\!^{-}$) and noting that the average of the Landau-Lifshitz pseudotensor is equal to the Isaacson stress-energy tensor for the gravitational waves (MTW exercise 35.19), we obtain

$$\langle dM/dt\rangle = -\oint T^{0r}_{(W)}\, r^2 d\Omega, \quad \langle dP^j/dt\rangle = -\oint T^{jr}_{(W)}\, r^2 d\Omega. \tag{3.41}$$

When the waves are decomposed into multipoles (eq. 3.36) these integrals give (RMP eqs. 4.16' and 4.20'):

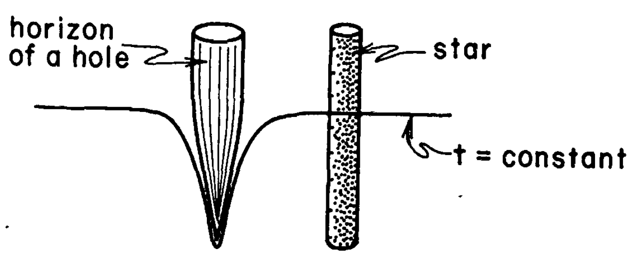

Fig. 5. Spacetime diagram showing how the slices of constant time can be chosen everywhere spacelike and have Euclidean topology even in the presence of a black hole.

$$\left\langle \frac{dM}{dt} \right\rangle = -\sum_{\ell=2}^{\infty} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\cdot\ell!(2\ell+1)!!} \left\langle {}^{(\ell+1)}\mathcal{I}_{A_\ell} \; {}^{(\ell+1)}\mathcal{I}_{A_\ell} \right\rangle \right.$$

$$\left. + \frac{4\ell(\ell+2)}{(\ell-1)\cdot(\ell+1)!(2\ell+1)!!} \left\langle {}^{(\ell+1)}\mathcal{S}_{A_\ell} \; {}^{(\ell+1)}\mathcal{S}_{A_\ell} \right\rangle \right\} , \tag{3.42}$$

$$\left\langle \frac{dP_j}{dt} \right\rangle = -\sum_{\ell=2}^{\infty} \left\{ \frac{2(\ell+2)(\ell+3)}{\ell(\ell+1)!(2\ell+3)!!} \left\langle {}^{(\ell+2)}\mathcal{I}_{jA_\ell} \; {}^{(\ell+1)}\mathcal{I}_{A_\ell} \right\rangle \right.$$

$$+ \frac{8(\ell+3)}{(\ell+1)!(2\ell+3)!!} \left\langle {}^{(\ell+2)}\mathcal{S}_{jA_\ell} \; {}^{(\ell+1)}\mathcal{S}_{A_\ell} \right\rangle \tag{3.43}$$

$$\left. + \frac{8(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!} \epsilon_{jpq} \left\langle {}^{(\ell+1)}\mathcal{I}_{pA_{\ell-1}} \; {}^{(\ell+1)}\mathcal{S}_{qA_{\ell-1}} \right\rangle \right\} .$$

Note that for typical slow-motion sources, with moments of order (3.4) and $v \sim L/\lambda\!\!\!^{-}$, the mass loss is predominantly due to the mass quadrupole moment beating against itself

$$\langle dM/dt \rangle = -(1/5) \langle \dddot{\mathcal{I}}_{jk} \dddot{\mathcal{I}}_{jk} \rangle \sim M^2L^4/\lambda\!\!\!^{-6} ; \tag{3.44}$$

and the momentum change is due to the mass quadrupole beating against the mass octupole <u>and</u> against the current quadrupole:

$$\left\langle \frac{dP_j}{dt} \right\rangle = -\frac{2}{63} \langle \dddot{\mathcal{I}}_{ab} \ddddot{\mathcal{I}}_{abj} \rangle - \frac{16}{45} \epsilon_{jpq} \langle \dddot{\mathcal{I}}_{pa} \dddot{\mathcal{S}}_{qa} \rangle \sim \frac{M^2L^5}{\lambda\!\!\!^{-7}} . \tag{3.45}$$

As the momentum of the source changes, its asymptotic rest frame changes. Since I have formulated my discussion of the external fields of slow-motion sources in mass-centered de Donder coordinates which coincide with the asymptotic rest frame, in applying my equations one must continually readjust the coordinates as time passes.

The intrinsic angular momentum of the source can be computed by a surface integral analogous to (3.39), which picks out the $1/r^2$ dipole part of the metric (3.37):

$$\mathcal{S}_j = \frac{1}{16\pi} \oint \epsilon_{jpq} \left(x^p \, H^{q\alpha 0i}{}_{,\alpha} + H^{pi0q} \right) d^2S_i . \tag{3.46}$$

By manipulations analogous to (3.40) one can show that

$$d\mathcal{S}_j/dt = -\oint \epsilon_{jpq} \, x^p (-g) \, t^{qr}_{LL} \, r^2 d\Omega . \tag{3.47}$$

By computing the Landau-Lifshitz pseudotensor (MTW eq. 20.22) for the local-wave-zone metric (3.37), inserting it into (3.47), and then averaging over Δt = (several $\lambda\!\!\!^{-}$), we obtain (RMP eq. 4.22)

$$\left\langle \frac{d\mathcal{S}_j}{dt} \right\rangle = \frac{1}{16\pi} \oint \epsilon_{jpq} \, x^p \left\langle -\left(h^{TT}_{qa} \dot{h}^{TT}_{ab}\right)_{,b} + \tfrac{1}{2} h^{TT}_{ab,q} \dot{h}^{TT}_{ab} \right\rangle r^2 d\Omega . \tag{3.48}$$

When the multipole expansion (3.36) is inserted this becomes (RMP eq. 4.23'):

$$\left\langle\frac{dS_j}{dt}\right\rangle = -\sum_{\ell=2}^{\infty}\left\{\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!}\,\epsilon_{jpq}\left\langle {}^{(\ell)}\mathcal{I}_{pA_{\ell-1}}\;{}^{(\ell+1)}\mathcal{I}_{qA_{\ell-1}}\right\rangle\right.$$
$$\left. + \frac{4\ell^2(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!}\,\epsilon_{jpq}\left\langle {}^{(\ell)}\mathcal{S}_{pA_{\ell-1}}\;{}^{(\ell+1)}\mathcal{S}_{qA_{\ell-1}}\right\rangle\right\} . \tag{3.49}$$

For typical slow-motion sources the dominant term is mass quadrupole beating against mass quadrupole:

$$\left\langle\frac{dS_j}{dt}\right\rangle = \frac{2}{5}\,\epsilon_{jpq}\left\langle \ddot{\mathcal{I}}_{pa}\,\dddot{\mathcal{I}}_{qa}\right\rangle \sim \frac{M^2L^4}{\lambda\!\!\!^{-5}} . \tag{3.50}$$

The above analysis encounters serious difficulties for a source which changes its asymptotic rest frame significantly in a few gravity-wave periods, i.e., for which $|(dP_j/dt)\lambda\!\!\!^-| \sim M$. Because the local wave zone, where one constructs the above surface integrals, must have a size $\Delta r \gg \lambda\!\!\!^-$, the momentum of such a source is not well defined there (it is changing too fast); and thus there is no clean prescription for constructing the source's asymptotic rest frame or for "mass centering" the coordinates in it. As a result, the instantaneous mass M and linear momentum P_j of the source (which depend on the choice of time t) are somewhat ill defined; and the instantaneous angular momentum S_j is even more ill defined because it is sensitive to the mass centering (factor of x^p in eqs. 3.47, 3.48). This difficulty is discussed, using the Bondi-Sachs formulation of gravitational waves at "future null infinity", in a lecture by Ashtekar in this volume.

Fortunately for theorists (unfortunately for experimenters) all realistic astrophysical sources are believed to radiate momentum only weakly

$$|dP_j/dt| \ll M/\lambda\!\!\!^- \tag{3.51}$$

(see the lectures by Eardley) and thus have asymptotic rest frames that are well enough defined for the above analysis to be well founded.

3.3.3 Order-of-magnitude formulas

For typical slow-motion sources the gravitational-wave amplitude at earth (eq. 3.27 propagated on out to earth through a nearly flat universe) will be

$$h_{jk}^{TT} \simeq \frac{2}{r}\,\ddot{\mathcal{I}}_{jk}^{TT} \sim \frac{M}{r}\left(\frac{L}{\lambda\!\!\!^-}\right)^2 \sim \frac{G}{c^4}\,\frac{(\text{internal kinetic energy of source})}{r}$$

$$\sim (\text{Newtonian potential at earth produced by internal kinetic energy of source})$$

$$\sim 10^{-17}\times\frac{(\text{internal kinetic energy})}{(\text{total mass-energy of Sun})}\times\frac{(\text{distance to galactic center})}{(\text{distance to source})} . \quad (3.$$

In using this formula one must include only the internal kinetic energy associatec with quadrupolar-type (nonspherical) motions. The total power carried by such sources is expressed most conveniently in terms of the "universal power unit"

$$\mathcal{L}_o = c^5/G = 1 = 3.63\times10^{59}\ \text{erg/sec} = 2.03\times10^5\ M_\odot c^2/\text{sec} \tag{3.53}$$

and the source's internal power flow $\mathcal{L}_{\rm int}$ = (internal kinetic energy)/$\lambdabar$:

$$\mathcal{L}_{GW} \simeq \frac{1}{5} \langle \dddot{\mathcal{I}}_{jk} \dddot{\mathcal{I}}_{jk} \rangle \sim \left(\frac{ML^2}{\lambdabar^3}\right)^2 \sim \left(\frac{\mathcal{L}_{\rm int}}{\mathcal{L}_o}\right)^2 \mathcal{L}_o \, . \tag{3.54}$$

Realistic astrophysical sources — even those with fast, large-amplitude motions and strong internal gravity — are not expected to deviate strongly from these order-of-magnitude formulas; see the lectures of Eardley. Moreover, all calculations to date suggest that no realistic source can radiate away a substantial fraction of its mass more quickly than the light travel time across its gravitational radius; i.e.,

$$\mathcal{L}_{GW} \lesssim M/M = 1 = \mathcal{L}_o \quad \text{for all sources} \tag{3.55a}$$

(a limit first suggested, so far as I know, by Dyson 1963); and, correspondingly, that the gravity-wave amplitude will always be smaller than

$$h^{TT}_{jk} \lesssim \lambdabar/r \qquad \text{for all sources.} \tag{3.55b}$$

3.4 Radiation reaction in slow-motion sources

There are three approaches to the theory of gravitational radiation reaction in slow-motion sources, each of which is sufficiently rigorous to make me happy; but none of which is sufficiently rigorous to make the most mathematically careful of my colleagues happy (see, e.g., Ehlers, Rosenblum, Goldberg, and Havas 1976). I shall discuss each of these approaches in turn.

3.4.1 Method of conservation laws

The method of conservation laws is based on equations (3.41) and (3.48), which express the rates of change of the source's mass M, momentum P_j, and angular momentum $\mathcal{S}_j$ in terms of integrals over its gravitational waves, and thence is based on expressions (3.42)-(3.45) and (3.49)-(3.50) for $\dot{M}$ and $\dot{P}_j$ in terms of multipole moments that are computable by "instantaneous-gravity", near-zone analyses. These formulas for $\dot{M}$, $\dot{P}_j$, and $\dot{\mathcal{S}}_j$ ("conservation laws") rely, ultimately, on the vacuum Einstein equations (e.g., through the third equality of equation 3.40).

It is crucial for radiation-reaction theory that the M, P_j, and $\mathcal{S}_j$ of these conservation laws are physically measurable (e.g., by Kepler's laws and the precession of gyroscopes) in the weak-field near zone or the local wave zone, and correspondingly that they are computable in terms of the physical near-zone properties of the gravitating system. For example, in the case of a compact binary system (e.g., the binary pulsar), near-zone analyses give

$$\begin{aligned} M &= m_1 + m_2 - \frac{m_1 m_2}{2a} + M_{pN} + M_{p^2N} \\ \mathcal{S} \equiv |\underset{\sim}{\mathcal{S}}| &= \left[\frac{m_1^2 m_2^2}{m_1 + m_2} a(1-e^2)\right]^{1/2} + \mathcal{S}_{pN} + \mathcal{S}_{p^2N} \, . \end{aligned} \tag{3.56}$$

Here m_1 and m_2 are the masses of each of the stars as measured by Kepler's laws and as manifest in the stars' external metrics; a and e are the semimajor axis and eccentricity of the orbits at Newtonian order; and M_{pN}, M_{p^2N}, $\mathcal{S}_{pN}$, $\mathcal{S}_{p^2N}$ are post-

Newtonian and post-post Newtonian contributions. Moreover, for such a binary the conservation laws (3.44) and (3.50) reduce to

$$\left\langle\frac{dM}{dt}\right\rangle = -\frac{32}{5}\frac{m_1^{\,2}m_2^{\,2}(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right),$$
$$\left\langle\frac{d\mathcal{S}}{dt}\right\rangle = -\frac{32}{5}\frac{m_1^{\,2}m_2^{\,2}(m_1+m_2)^{\frac{1}{2}}}{a^{7/2}(1-e^2)^2}\left(1+\frac{7}{8}e^2\right) \tag{3.57}$$

(Peters and Mathews 1963, Peters 1964).

Consider the evolution of such a binary over time scales $\Delta t \gg M_{pN}/\langle dM/dt\rangle \simeq$ (10^3 years for the binary pulsar). It is "<u>physically obvious</u>" (or one can show by a careful analysis such as that in Damour's lectures) that m_1 and m_2 are unaffected by the gravity-wave emission. Moreover, over these long time scales the changes of M_{pN}, M_{p^2N}, $\mathcal{S}_{pN}$, and $\mathcal{S}_{p^2N}$ are negligible compared to the much larger M and $\mathcal{S}$ carried off in the radiation. Thus, the changes of M and $\mathcal{S}$ must be fully accounted for by changes of a and e — changes that are fully determined by equations (3.56) and (3.57). And from those changes one can compute the change of orbital period $P = 2\pi[a^3/(m_1+m_2)]^{1/2}$, the most directly measurable quantity.

For evolution of the binary pulsar on time scales $P = 7.75$ hours $\ll \Delta t \ll$ 1000 years (corresponding to current measurements), the same argument gives the same result (which agrees with the measurements), if one makes a "<u>highly plausible</u>" <u>assumption</u>: that $M_{pN} + M_{p^2N}$ and $\mathcal{S}_{pN} + \mathcal{S}_{p^2N}$ are <u>not</u> sharply changing functions of the (nearly conserved) orbital parameters such as a and e, and thus cannot account for any significant piece of the changes in M and $\mathcal{S}$. Of course, one can only feel fully comfortable about this conclusion after detailed pN and p^2N calculations have verified this assumption; see the lectures of Damour.

3.4.2 Radiation reaction potential

For systems which, unlike the binary pulsar, have weak internal gravity as well as slow motions, one can compute radiation reaction using Newtonian gravity augmented by Burke's (1969) radiation-reaction potential:

$$\underset{\sim}{F}_{grav} = -m\underset{\sim}{\nabla}\Phi; \qquad \Phi = \Phi_{Newton} + \Phi_{react};$$
$$\Phi_{react} = \frac{1}{5}\,{}^{(5)}\mathcal{I}_{jk}\,x^j x^k . \tag{3.58}$$

Here $\underset{\sim}{F}_{grav}$ is the gravitational force that acts on a material element of mass m.

Physically Φ_{react} results from matching the near-zone gravitational field onto outgoing gravitational waves. If the near-zone field were matched onto standing waves Φ_{react} would be zero; if it were matched onto ingoing waves Φ_{react} would change sign. Mathematically one derives the equation of motion (3.58) by constructing the outgoing-wave solution of the Einstein equations in any convenient gauge, matching it onto the near-zone solution, identifying the largest terms in the near-zone metric which are sensitive to outgoing waves versus ingoing waves, and discarding all terms except these sensitive ones and the terms of Newton. By an appropriate change of gauge one then obtains equations of motion of the form (3.58).

I have a confession to make: The derivation along these lines given in §36.11 of MTW is flawed. As Walker and Will (1980) point out, when one works in de Donder gauge (as I did in writing §36.11), one obtains reaction terms of magnitude ${}^{(3)}\mathcal{I}_{jk}$ in the near-zone metric when one matches onto outgoing waves. Although these terms are "pure gauge", i.e., have no direct physical consequences, they produce nonlinear corrections in the final gauge change, corrections which I mistakenly ignored in MTW but which are cancelled by a nonlinear iteration of the Einstein equations that I also mistakenly ignored. The reason for my sloppiness in writing MTW is that I had previously derived the radiation-reaction potential (3.58) working not in de Donder gauge but in "Regge-Wheeler gauge" (Thorne 1969); and in that gauge ${}^{(3)}\mathcal{I}_{jk}$ terms never arise and a final gauge change and nonlinear iteration are not needed. Having been very careful in my Regge-Wheeler-type derivation, I was highly confident of the final result — and, buoyed by this confidence, I became careless when constructing the de Donder-gauge proof in MTW.

Historically, Peres (1960) gave the first correct analysis of gravitational radiation reaction by the technique of identifying the dominant terms sensitive to the outgoing-wave boundary condition. However, Peres did not write his answer in terms of a Newton-type radiation-reaction potential Φ_{react}; that was first done by Burke (1969).

3.4.3 pN, p^2N, and $p^{2.5}$N iteration of the field equations

The method (3.58) of the radiation reaction potential discards all post-Newtonian and post-post-Newtonian effects in the evolution of the system — even though radiation reaction is a somewhat smaller, post$^{2.5}$-Newtonian phenomenon over times of order $\lambda\!\!\!^-$. The justification is that radiation reaction, unlike pN and p^2N effects, produces secular changes of such quantities as the period of a binary system; and those secular changes can build up over long times $\Delta t \gg \lambda\!\!\!^-$, becoming ultimately much larger than post-Newtonian order. However, over shorter time-scales ($\Delta t \lesssim 1000$ years in the case of the binary pulsar) the cumulative effects do not exceed post-Newtonian order; and thus for full confidence in one's results one should augment the Newtonian forces and $p^{2.5}$N radiation-reaction forces by a full pN and p^2N analysis of the system.

Damour, in his lectures in this volume, describes the history of attempts at such full analyses and presents a beautiful, full analysis of his own for the special case of binary systems with compact (white-dwarf, neutron-star, and black-hole) members.

3.5 Gravitational-wave generation by fast-motion sources: $\lambda\!\!\!^- \lesssim L$

Elsewhere (Thorne 1977) I have reviewed methods for computing gravitational waves from fast-motion sources. Here I shall give only a brief classification of the various methods and a few recent references where each method is used.

There are three ways to classify methods for computing wave generation: by strength of the source's internal gravity ("weak" if a Newtonian analysis gives $|\Phi| \ll 1$ everywhere; "strong" if $|\Phi| \gtrsim 1$ somewhere); by speed of the source's internal motions ("slow" if $\lambda\!\!\!^- \gg L$; "fast" if $\lambda\!\!\!^- \lesssim L$); and by the fractional amount that the source deviates from a nonradiating spacetime ("large deviations" or "small deviations"). Here is a list of frequently used methods of computing wave generation, classified in these three ways:

1. Slow-Motion Method
   - Arbitrary gravity, slow speed, arbitrary deviations.
   - § 3.2 above.
2. Post-Newtonian and Post-Post-Newtonian Wave-Generation Methods
   - Moderate gravity, moderate speed, arbitrary deviations.
   - Epstein and Wagoner (1975); Wagoner and Will (1976); RMP §§V.D,E.

3. Post-Linear (≡ Post-Minkowski) Wave-Generation Method

- Weak gravity, arbitrary speed, arbitrary deviations.
- Kovács and Thorne (1978).

4. First-Order Perturbations of Nonradiating Solutions

- Arbitrary gravity, arbitrary speed, small deviations.
- Cunningham, Price, and Moncrief (1978); Schumaker and Thorne (1983); Detweiler (1980); lecture by Nakamura in this volume; eq. (2.29) above.

5. Numerical Relativity

- Arbitrary gravity, arbitrary speed, arbitrary deviations.
- Lectures by York, Piran, Nakamura, and Isaacson in this volume.

The strongest radiators of gravity waves will be those with strong gravity, fast motions, and large deviations (e.g., collisions between two black holes); and such systems can be analyzed quantitively by only one technique: numerical relativity.

The results of wave-generation calculations for various astrophysical sources are reviewed in the lectures of Eardley in this volume.

4 THE DETECTION OF GRAVITATIONAL WAVES

Turn attention now from methods of analyzing the generation of gravitational waves to methods of analyzing their detection. If the size of the detector is small compared to a reduced wavelength, $L \ll \lambdabar$, it can be analyzed in the "proper reference frame" of the detector's center of mass (§4.1). If $L \gtrsim \lambdabar$, the proper reference frame is not a useful concept; and the detector is usually analyzed, instead, using "post-linear" (≡ "post-Minkowski") techniques and some carefully chosen gauge (§4.2).

4.1 Detectors with size $L \ll \lambdabar$

4.1.1 The proper reference frame of an accelerated, rotating laboratory

Most gravity-wave detectors reside in earth-bound laboratories whose walls and floor accelerate relative to local inertial frames ("acceleration of gravity") and rotate relative to local gyroscopes ("Foucault pendulum effect"). In such laboratories the mathematical analog of a LIF is a "proper reference frame" (PRF). A PRF is constructed by choosing a fiducial world line which is usually attached to the detector's center of mass and thus accelerates, by next constructing spatial slices of simultaneity, $t = \text{const}$, which are orthogonal to the fiducial world line and are as flat as the spacetime curvature permits; and by then constructing in each slice of simultaneity a spatial coordinate grid which is as Cartesian as the spacetime curvature permits and is attached to the laboratory walls and thus rotates. The origin of the spatial grid is on the fiducial world line, and the time coordinate t of the slices of simultaneity is equal to proper time along the fiducial world line. Such a PRF is a mathematical realization of the type of coordinate system that a very careful experimental physicist who knows little relativity theory would likely set up in his earth-bound laboratory.

One version of such a PRF is the rotating, accelerating analog of a Fermi normal coordinate system (eq. 2.15); its spacetime metric is (Ni and Zimmermann 1978)

$$ds^2 = -dt^2\left[1 + \underbrace{2\underset{\sim}{a}\cdot\underset{\sim}{x} + (\underset{\sim}{a}\cdot\underset{\sim}{x})^2}_{\text{grav'l redshift}} - \underbrace{(\underset{\sim}{\Omega}\times\underset{\sim}{x})^2}_{\text{Lorentz time dilation}} + R_{0\ell 0m}x^\ell x^m\right]$$

$$+ 2dtdx^i\left[\underbrace{\epsilon_{ijk}\Omega^j x^k}_{\text{Sagnac effect}} - \frac{2}{3}R_{0\ell im}x^\ell x^m\right] + dx^i dx^j\left[\delta_{ij} - \frac{1}{3}R_{i\ell jm}x^\ell x^m\right]$$

$$+ O(r^3\,dx^\alpha dx^\beta). \qquad (4.1)$$

Here $\underset{\sim}{a}$ is the acceleration of the fiducial world line (minus the local "acceleration of gravity") and $\underset{\sim}{\Omega}$ is the angular velocity of the spatial grid, i.e., of the laboratory walls, relative to local gyroscopes. Other versions of a PRF have spatial grids and slices of simultaneity that are bent from those of (4.1) by amounts of the order of the bending enforced by the spacetime curvature: $x^{j'} = x^j + O(r^3/\Re^2)$, $t' = t + O(r^3/\Re^2)$ where $\Re \sim (R_{\alpha\beta\gamma\delta})^{-1/2}$; cf. eq. (2.16). In them g_{00} will be the same as (4.1), but g_{0i} and g_{jk} may be different by amounts of $O(r^2/\Re^2)$.

In the PRF (4.1) a test particle acted on by an external force acquires a coordinate acceleration (obtained from the geodesic equation with a force term added)

$$\frac{d^2x^i}{dt^2} = \underbrace{-a^i}_{\substack{\text{"acceleration}\\\text{of gravity"}}} \underbrace{-2(\underset{\sim}{\Omega}\times\underset{\sim}{v})^i}_{\substack{\text{Coriolis}\\\text{acceleration}}} \underbrace{-[\underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{x})]^i}_{\substack{\text{centrifugal}\\\text{acceleration}}} \underbrace{-(\dot{\underset{\sim}{\Omega}}\times x)^i}_{\substack{\text{effect of}\\\text{changing }\underset{\sim}{\Omega}}} \underbrace{+f^i/m}_{\substack{\text{external}\\\text{force}}}$$

$$+\left[\begin{matrix}\text{special relativistic corrections to these inertial effects,}\\ \text{equation (20) of Ni and Zimmermann (1978)}\end{matrix}\right]$$

$$\underbrace{-R_{0i0\ell}x^\ell}_{\text{geodesic deviation}} + \underbrace{2R_{ij0\ell}v^j x^\ell}_{\substack{\text{source of "spin-}\\\text{curvature coupling"}}} + \frac{2}{3}R_{ijk\ell}v^j v^k x^\ell \qquad (4.2)$$

$$+ 2v^i R_{0j0\ell}v^j x^\ell + \frac{2}{3}v^i R_{0jk\ell}v^j v^k x^\ell.$$

Here v^j is the coordinate velocity of the particle. The terms on the first line are far bigger than those on the last three lines; they are familiar from nonrelativistic mechanics in a uniform, Newtonian gravitational field.

4.1.2 Examples of detectors

Figure 6 shows three types of gravitational-wave detectors with $L \ll \lambdabar$ that are now under construction or look favorable for future construction.

"Weber-type resonant-bar detectors" have been under development for twenty years and are discussed in detail in the lectures of Blair, Braginsky, Hamilton, Michelson, and Pallotino. In such a detector the waves couple to and drive normal modes of oscillation of a mechanical system ("antenna"), usually a solid bar made

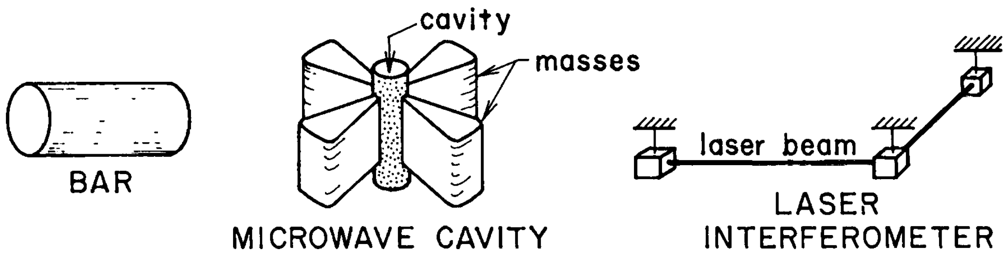

Fig. 6 Examples of gravitational-wave detectors with size $L \ll \lambda\!\!\!^{-}$.

of aluminum, and those oscillations are monitored by a transducer which is attached to the ends or sides of the bar.

In a "microwave-cavity detector" with $L \ll \lambda\!\!\!^{-}$ the gravitational waves drive oscillatory deformations of the walls of a microwave cavity; and those wall motions pump microwave quanta from one normal mode of the cavity into another. To enhance the wall deformations, big masses may be attached to the walls at strategic locations. (See, e.g., Braginsky et al. (1974), Caves (1979), Pegoraro and Radicati (1980), Grishchuk and Polnarev (1981), and references therein.) Although the design sensitivities of such detectors are comparable to those of bars, no serious efforts are now under way to develop them.

"Laser interferometer detectors" have been under development for about a decade and are discussed in detail in the lectures of Drever and Brillet. In such a detector three (or more) masses are suspended as pendula from overhead supports and swing back and forth in response to gravitational waves; and their relative motions are monitored by laser interferometry.

4.1.3 Method of analyzing detectors

For me the conceptually clearest way to analyze these three detectors, and any other with $L \ll \lambda\!\!\!^{-}$, is using the PRF of the detector's center of mass. The gravitational waves enter such an analysis entirely through the Riemann curvature terms of the metric (4.1), which have sizes

$$g_{\alpha\beta}^{(W)} \sim R_{\alpha\beta\gamma\delta}^{(W)} L^2 \sim h_{jk}^{TT} (L/\lambda\!\!\!^{-})^2 \ll h_{jk}^{TT} \quad \text{in PRF.} \tag{4.3}$$

By contrast, in TT gauge the waves would contribute $g_{\alpha\beta}^{(W)} \sim h_{jk}^{TT}$ to the metric. An important consequence is this: In the PRF analysis the direct coupling of the gravitational waves to the detector's electromagnetic field can be ignored; and this is true whether the EM field is in a transducer on the bar, or in a microwave cavity, or in a laser beam. The direct coupling produces terms in Maxwell's equations for the vector potential with size $\delta A/A \sim g_{\mu\nu}^{(W)} \sim h_{jk}^{TT} (L/\lambda\!\!\!^{-})^2$, which are smaller by $(L/\lambda\!\!\!^{-})^2$ than the "indirect" coupling effects

$$\begin{pmatrix}\text{gravity waves deform}\\ \text{or move masses}\end{pmatrix} \rightarrow \frac{\delta L}{L} \sim h_{jk}^{TT} \rightarrow \begin{pmatrix}\text{changes of boundaries}\\ \text{for Maxwell equations}\end{pmatrix} \rightarrow \frac{\delta A}{A} \gtrsim h_{jk}^{TT}. \tag{4.4}$$

By contrast, in TT gauge the direct coupling is not negligible, and one must consider the direct interaction of the gravitational waves with both the electromagnetic field and the mechanical parts of the detector.

For all three detectors in Figure 6 and all other promising ones, the velocities of the mechanical parts of the system relative to the center of mass are $|v^j| \ll 1$. Consequently, in the mechanical equation of motion (4.2) all Riemann curvature terms except $-R_{0i0\ell}x^\ell$ can be ignored. Because $R^{(W)}_{0i0\ell} = -\frac{1}{2}\ddot{h}^{TT}_{i\ell}$ (where a dot means $\partial/\partial t$), the equation of motion for each mass element in the detector becomes

$$\ddot{x}^i = \frac{1}{2}\ddot{h}^{TT}_{ij}\,x^j + \left(\begin{array}{c}\text{all acceleration terms associated with}\\ \text{non-gravitational-wave effects}\end{array}\right). \qquad (4.5)$$

In summary, if one knows how to analyze the detector in the absence of gravitational waves, one can take account of the waves by simply adding the driving acceleration $\frac{1}{2}\ddot{h}^{TT}_{jk}\,x^j$ to the equation of motion of the detector's mass elements and by ignoring direct coupling of the waves to the detector's electromagnetic field.

This conclusion is valid even if the detector is large compared to inhomogeneities in the Newtonian gravity of the earth or solar system — e.g., if the detector is a normal mode of the earth itself. Then one cannot use the proper reference frame (4.1), so far as Newtonian-gravity effects are concerned; but one can still use (4.1) so far as gravitational-wave effects are concerned ($g^{(W)}_{\alpha\beta} \sim \ddot{h}^{TT}_{jk}\,L^2$). In other words, one can graft the waves onto a Newtonian analysis by means of

$$g^{(W)}_{00} = -R^{(W)}_{0\ell 0m}\,x^\ell x^m\,,$$

$$|g^{(W)}_{0i}| \sim |g^{(W)}_{ij}| \sim R^{(W)}_{\alpha\beta\gamma\delta}\,L^2\left(\begin{array}{c}\text{details depend on specific variant of PRF}\\ \text{and are unimportant because } |v^j| \ll 1\end{array}\right), \qquad (4.6)$$

and by means of the resulting equation of motion (4.5).

The gravitational-wave driving acceleration $\frac{1}{2}\ddot{h}^{TT}_{jk}\,x^j$ can be described by a quadrupole-shaped "line-of-force" diagram (Fig. 7; MTW Box 37.2).

4.1.4 Resonant-bar detectors

Consider, as an example, the analysis of a Weber-type resonant-bar gravitational-wave detector (MTW Box 37.4). One begins by computing the normal-mode eigenfrequencies ω_n and eigenfunctions $\underset{\sim}{u}^{(n)}$ of the antenna ignoring the weak frictional and fluctuational coupling between modes and ignoring external forces (e.g., gravity waves). The resulting eigenfunctions, which are real (not complex) are normalized so that

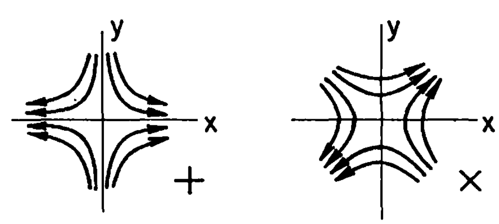

Fig. 7 "Lines of force" for the gravitational-wave acceleration (4.5) in the PRF of a detector. The two drawings correspond to waves with + polarization and with × polarization.

$$\int \rho \underset{\sim}{u}^{(n)} \cdot \underset{\sim}{u}^{(m)}\, d^3x = M\delta_{nm}; \qquad \rho = \text{density}, \quad M = \text{antenna mass}. \tag{4.7}$$

One then expands the vibrational displacement of the bar's material in terms of normal modes

$$\delta \underset{\sim}{x} = \sum_n X^{(n)} \underset{\sim}{u}^{(n)}(\underset{\sim}{x})\, e^{-i\omega_n t}; \qquad X^{(n)} \equiv \text{"complex amplitude of mode n"}. \tag{4.8}$$

Next one writes down the equation of motion for $\delta \underset{\sim}{x}$ in the presence of gravity waves [force per unit volume equal to $\frac{1}{2}\rho \ddot{h}^{TT}_{jk} x^j$], internal friction, Nyquist forces [i.e., weak fluctuational couplings between normal modes], and coupling to the transducer. When one resolves that full equation of motion into normal modes one obtains

$$\dot{X}^{(n)} = -(2/\tau_n)X^{(n)} + \frac{ie^{i\omega_n t}}{M\omega_n} \int \underset{\sim}{f} \cdot \underset{\sim}{u}^{(n)}\, d^3x\,, \tag{4.9}$$

where τ_n is the (very long) frictional damping time for mode n and $\underset{\sim}{f}$ is the force per unit volume in the antenna including gravity-wave force, Nyquist forces, and "back-action forces" of the transducer on the antenna. The gravity-wave force, when integrated over the normal mode, $\underset{\sim}{u}^{(n)}$, gives

$$\int \underset{\sim}{f}^{(W)} \cdot \underset{\sim}{u}^{(n)}\, d^3x = \frac{1}{4}\, \ddot{h}^{TT}_{jk}\, \mathcal{I}^{(n)}_{jk}, \tag{4.10a}$$

$$\begin{aligned} \mathcal{I}^{(n)}_{jk} &= \int \rho\left(u^j_{(n)} x^k + u^k_{(n)} x^j - \frac{2}{3}\delta_{jk}\, \underset{\sim}{u}_{(n)} \cdot \underset{\sim}{x}\right) d^3x \\ &= \left[X^{(n)} e^{-i\omega_n t}\right]^{-1} \times \begin{pmatrix}\text{contribution of mode n to antenna's} \\ \text{quadrupole moment}\end{pmatrix}. \end{aligned} \tag{4.10b}$$

Thus, the same quadrupole moment as would govern emission of gravitational waves by mode n also governs their reception; this is an aspect of the principle of detailed balance (MTW §37.7).

If the antenna is hit by a broad-band burst of gravitational waves with spectral energy flux $\mathcal{F}_\nu$ (ergs cm^{-2} Hz^{-1}) and polarization e_{jk} (normalized so $e_{jk}e_{jk} = 2$)

$$h^{TT}_{jk} = A(t)e_{jk}, \qquad \mathcal{F}_\nu = \frac{\omega^2}{4}\left|\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(t)\, e^{i\omega t}\, dt\right|^2, \qquad \omega = 2\pi\nu > 0, \tag{4.11}$$

then equations (4.9) and (4.10) give for the wave-induced change of complex amplitude $\delta X^{(n)}$

$$\begin{aligned} \frac{1}{2} M\omega_n{}^2 |\delta X^{(n)}|^2 &= \begin{pmatrix}\text{energy that would be absorbed by antenna mode n} \\ \text{if } X^{(n)} \text{ were zero when the wave burst hit}\end{pmatrix} \\ &= \mathcal{F}_\nu(\omega = \omega_n) \int \sigma_n d\nu. \end{aligned} \tag{4.12}$$

Here $\int \sigma_n d\nu$ is the antenna cross section integrated over frequency

$$\int \sigma_n d\nu = \frac{\pi}{4}\frac{\omega_n^2}{M}\left(\mathcal{I}^{(n)}_{jk}\, e_{jk}\right)^2$$
$$\sim 10^{-21}\ \text{cm}^2\,\text{Hz} \text{ for typical antennas with } M \sim 1 \text{ ton},\ \omega_n/2\pi \sim 1 \text{ kHz}. \tag{4.13}$$

For further details and discussion see MTW, chapter 37; also the lectures by Blair, Braginsky, Hamilton, Michelson, and Pallotino in this volume.

* * * * *

Exercise 20. Derive equation (4.9) by resolving the equation of motion for $\delta \underline{x}$ into normal modes. Show that the driving term due to gravity waves has the form (4.10) and show that $\mathcal{I}^{(n)}_{jk}$ has the claimed relationship to the quadrupole moment.

Exercise 21. Show that (eq. 4.11) correctly represents the spectral energy flux of a gravity wave in the sense that $\int \mathcal{F}_\nu d\nu$ is the energy per unit area that passes the detector. Show that the broad-band burst of gravity waves (4.11) produces the change (4.12) of the antenna's complex amplitude. Show that for typical antennas the frequency-integrated cross section is $\sim 10^{-21}$ cm^2 Hz as claimed.

Exercise 22. Show that for a homogeneous, spherical antenna whose quadrupolar oscillations are being driven by gravity waves, the quadrupole moment $\mathcal{I}_{jk}$ obeys the equation of motion (2.39a).

4.2 Detectors with size $L \gtrsim \lambda\!\!\!^{-}$

Examples of gravitational-wave detectors with $L \gtrsim \lambda\!\!\!^{-}$ include the Doppler tracking of spacecraft (lectures by Hellings in this volume), and microwave-cavity detectors in which the gravity waves pump microwave quanta from one normal mode to another via direct interaction with the electromagnetic field as well as via deformation of the walls or wave guides which confine the field (Braginský et al. (1974), Caves (1979), Pegoraro and Radicati (1980), Grishchuk and Polnarev (1981) and references therein).

Because the Riemann tensor of the waves varies significantly over the volume of such a detector, the "proper reference frame" is not a useful tool in analyzing it. Instead, analyses are based on the linearized approximation to general relativity (MTW chapter 18), with the metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ including both Newtonian gravitational potentials Φ and gravitational waves; and the waves are usually treated in TT gauge:

$$g_{00} = -1 - 2\Phi, \qquad g_{0j} = 0, \qquad g_{jk} = \delta_{jk}(1-2\Phi) + h^{TT}_{jk}.$$

The analysis is actually "post-linear" (or "post-Minkowski") in that the "equations of motion" for the matter (sun, planets, detector) are not taken to be $T^{\alpha\beta}{}_{,\beta} = 0$, but rather $T^{\alpha\beta}{}_{;\beta} = 0$ with connection coefficients linear in $h_{\mu\nu}$ and Φ. See, e.g., §2 of Thorne (1977) for a careful discussion of the differences between linear theory and post-linear theory.

For an example of such an analysis see Hellings' treatment, in this volume, of the interaction of gravitational waves with NASA's doppler tracking system. As in that analysis, so in most analyses of detectors with $L \gtrsim \lambda\!\!\!^{-}$, direct interaction of the gravitational waves with the electromagnetic field is as important as interaction with the mechanical parts of the detector.

A mathematical trick that sometimes simplifies calculations of the interaction of gravity waves with electromagnetic fields is to write the curved-spacetime

Maxwell equations in a form identical to those for flat spacetime in a moving, anisotropic dielectric medium (e.g., Volkov, Izmest'ev, and Skrotskii 1970). Caves (unpublished) has combined this technique with gauge changes that attach the spatial coordinates onto the cavity walls even when the walls are wiggling, and has thereby produced an elegant and powerful formalism for analyzing microwave cavity detectors with $L \gtrsim \lambda\!\!\!^{-}$.

5 CONCLUSION

Most of my own research on gravitational-wave theory is motivated by the need to prepare for the day when gravitational waves are detected and astronomers confront the task of extracting astrophysical information from them. The task of preparing for that day is nontrivial. The ideas described in these lectures are a foundation for those preparations, but much further theoretical research is needed — especially the computation of gravitational wave forms $h_{jk}^{TT}(t-r;\ \theta,\ \varphi)$ from fast-motion, strong-gravity sources such as black-hole collisions.

We theorists often pay great lip service to a second motivation for our research: to give guidance to experimenters who are designing and constructing gravitational-wave detectors; and experimenters often follow with avid interest and thumping hearts the fluctuations in theoretical predictions of the waves bathing the earth. However, we theorists are far more ignorant than most experimenters imagine. For those strong sources whose wave characteristics are fairly well known (e.g., collisions between black holes), the event rate is uncertain by many orders of magnitude; and for those whose event rates are fairly well known (e.g., supernovae), the wave strengths are uncertain by many orders. Our ignorance has a simple cause: The information carried by electromagnetic waves, which is the foundation of today's theories, is nearly "orthogonal" to the information carried by gravitational waves. As a corollary, when they are ultimately detected, gravitational waves will likely give us a revolutionary new view of the universe; but until they are detected, we theorists can offer little precious advice to our experimental colleagues, and our colleagues should turn a half-deaf ear to our most confident remarks about the characteristics of the waves for which they search.

REFERENCES

Aichelburg, P. C., Ecker, G., and Sexl, R. U. 1971, Nuovo Cimento B, 2, 63.

Bontz, R. J. and Haugan, M. P. 1981, Astrophys. Space Sci., 78, 204.

Bondi, H., van der Burg, M. G. J., and Metzner, A. W. K. 1962, Proc. Roy. Soc., A269, 21.

Braginsky, V. B., Caves, C. M., and Thorne, K. S. 1977, Phys. Rev. D, 15, 2047.

Braginsky, V. B., Grishchuk, L. P., Doroshkevich, A. G., Zel'dovich, Ya. B., Novikov, I. D., and Sazhin, M. V. 1974, Sov. Phys.—JETP, 38, 865.

Burke, W. L. 1969, unpublished Ph.D. thesis, Caltech; see also Phys. Rev. A, 2, 1501 (1970).

Caves, C. M. 1979, Phys. Lett., 80B, 323.

Caves, C. M. 1980, Ann. Phys. (USA), 125, 35.

Crowley, R. J. and Thorne, K. S. 1977, Astrophys. J., 215, 624.

Cunningham, C. T., Price, R. H., and Moncrief, V. 1978, Astrophys. J., 224, 643.

Cyranski, J. F. and Lubkin, E. 1974, Ann. Phys. (USA), 87, 205.

Davis, M., Ruffini, R., Press, W. H., and Price, R. H. 1971, Phys. Rev. Lett., 27, 1466.

D'Eath, P. 1975, Phys. Rev. D, 12, 2183.

Detweiler, S. 1980, Astrophys. J., 239, 292.

DeWitt, B. S. and Brehme, R. W. 1960, Ann. Phys. (USA), 9, 220.

Dicke, R. H. 1964, in Relativity, Groups, and Topology, ed. B. and C. DeWitt (Gordon & Breach, New York).

Dyson, F. 1963, in Interstellar Communication, ed. A. G. W. Cameron (Benjamin, New York).

ELLWW: See Eardley, Lee, Lightman, Wagoner, and Will (1973).

Eardley, D. M., Lee, D. L., and Lightman, A. P. 1973, Phys. Rev. D, 8, 3308.

Eardley, D. M., Lee, D. L., Lightman, A. P., Wagoner, R. V., and Will, C. M. 1973, Phys. Rev. Lett., 30, 884; cited in text as ELLWW.

Ehlers, J., Rosenblum, A., Goldberg, J. N., and Havas, P. 1976, Astrophys. J. (Letters), 208, L77.

Einstein, A. 1918, Preuss. Akad. Wiss. Berlin, Sitzber. 1918, 154.

Epstein, R. and Wagoner, R. V. 1975, Astrophys. J., 197, 717.

Geroch, R. 1970, J. Math. Phys., 11, 2580.

Geroch, R., Held, A., and Penrose, R. 1973, J. Math. Phys., 14, 874.

Goldstein, H. 1980, Classical Mechanics, second edition (Addison Wesley, Reading, Mass.), §5-6.

Grishchuk, L. P. and Polnarev, A. G. 1981, chapter 10 of General Relativity and Gravitation, Vol. 2, ed. A. Held (Plenum Press, N.Y.).

Gürsel, Y. 1982, J. Gen. Rel. Grav., submitted.

Gürsel, Y. and Thorne, K. S. 1983, Mon. Not. Roy. Astron. Soc., submitted.

Hansen, R. O. 1974, J. Math. Phys., 15, 46.

Hartle, J. B. 1973, Astrophys. Space Sci., 24, 385.

Hartle, J. B. and Thorne, K. S. 1983, paper in preparation.

Hobson, E. W. 1931, The Theory of Spherical and Ellipsoidal Harmonics (Cambridge U. Press, Cambridge), p. 119. Republished in 1955 by Chelsea Publishing Co., New York.

Isaacson, R. A. 1968, Phys. Rev., 166, 1263 and 1272.

Kelvin, Lord (Sir William Thomson) and Tate, P. G. 1879, Treatise on Natural Philosophy, Appendix B of Chapter 1 of Volume 1 (Cambridge U. Press, Cambridge).

Kovács, S. J. and Thorne, K. S. 1978, Astrophys. J., 224, 62.

MTW: see Misner, Thorne, and Wheeler (1973).

Mashhoon, B. 1973, Astrophys. J. (Letters), 181, L65.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1973, Gravitation (Freeman, San Francisco); cited in text as MTW.

Ni, W.-T. and Zimmermann, M. 1978, Phys. Rev. D, 17, 1473.

Pegoraro, F. and Radicati, L. A. 1980, J. Phys. A, 13, 2411.

Peres, A. 1960, Nuovo Cimento, 15, 351.

Peters, P. C. 1964, Phys. Rev., 136, B1224.

Peters, P. C. and Mathews, J. 1963, Phys. Rev., 131, 435.

Pirani, F. A. E. 1964, in Lectures on General Relativity, ed. A. Trautman, F. A. E. Pirani, and H. Bondi (Prentice Hall, Englewood Cliffs, N.J.).

Press, W. H. 1979, J. Gen. Rel. Grav., 11, 105.

Price, R. H. 1972, Phys. Rev. D, 5, 2419.

RMP: see Thorne (1980a).

Rosen, N. 1973, J. Gen. Rel. Grav., 4, 435; see also N. Rosen, Ann. Phys. (USA), 84, 455 (1974).

Sachs, R. K. 1962, Proc. Roy. Soc., A270, 103.

Schumaker, B. L. and Thorne, K. S. 1983, Mon. Not. Roy. Astron. Soc., in press.

Sonnabend, D. 1979, <u>To the Solar Foci</u>, JPL Publication 79-18 (Jet Propulsion Laboratory, Pasadena).

Szekeres, P. 1971, Ann. Phys., <u>64</u>, 599.

Thorne, K. S. 1969, Astrophys. J., <u>158</u>, 997.

Thorne, K. S. 1977, in <u>Topics in Theoretical and Experimental Gravitation Physics</u>, ed. V. De Sabbata and J. Weber (Plenum Press, London).

Thorne, K. S. 1980a, Rev. Mod. Phys., <u>52</u>, 299; cited in text as RMP.

Thorne, K. S. 1981, Mon. Not. Roy. Astron. Soc., <u>194</u>, 439.

Thorne, K. S. 1983, Astrophys. J., in preparation.

Thorne, K. S. and Campolattaro, A. 1967, Astrophys. J., <u>149</u>, 591.

Turner, M. 1977, Astrophys. J., <u>216</u>, 914.

Volkov, A. M., Izmest'ev, A. A., and Skrotskii, G. V. 1970, Zh. Eksp. Teor. Fiz., <u>59</u>, 1254 [Sov. Phys.—JETP, <u>32</u>, 636 (1971)].

Wagoner, R. V. and Will, C. M. 1976, Astrophys. J., <u>210</u>, 764.

Walker, M. and Will, C. M. 1980, Astrophys. J. (Letters), <u>242</u>, L129.

Will, C. M. 1982, <u>Theory and Experimentation in Gravitational Physics</u> (Cambridge University Press, New York).

Zimmermann, M. 1980, Phys. Rev. D, <u>21</u>, 891.

Zimmermann, M. and Szedenits, E. 1979, Phys. Rev. D, <u>20</u>, 351.