PART **VIII**

GRAVITATIONAL WAVES

Wherein the reader voyages on stormy seas of curvature ripples, searching for the ripple-generating storm gods, and battles through an electromagnetic and thermal fog that allows only uncertain visibility upon those seas.

CHAPTER 35

PROPAGATION OF GRAVITATIONAL WAVES

Born: *"I should like to put to Herr Einstein a question, namely, how quickly the action of gravitation is propagated in your theory. That it happens with the speed of light does not elucidate it to me. There must be a very complicated connection between these ideas."*
Einstein: *"It is extremely simple to write down the equations for the case when the perturbations that one introduces in the field are infinitely small. Then the g's differ only infinitesimally from those that would be present without the perturbation. The perturbations then propagate with the same velocity as light."*
Born: *"But for great perturbations things are surely very complicated?"*
Einstein: *"Yes, it is a mathematically complicated problem. It is especially difficult to find exact solutions of the equations, as the equations are nonlinear."*

Excerpts from discussion after Einstein's Fall 1913 lecture in Vienna on "The present position of the problem of gravitation," already two years before he had the final field equations [EINSTEIN, 1913a]

§35.1. VIEWPOINTS

Study one idealization after another. Build a catalog of idealizations, of their properties, of techniques for analyzing them. This is the only way to come to grips with so complicated a subject as general relativity!

Spherical symmetry is the idealization that has dominated most of the last 12 chapters. Together with the idealization of matter as a perfect fluid, and of the universe as homogeneous, it has yielded insight into stars, into cosmology, into gravitational collapse.

Turn attention now to an idealization of an entirely different type, one independent of any symmetry considerations at all: the idealization of a "gravitational wave."

Gravitational waves compared to water waves on ocean:

Just as one identifies as "water waves" small ripples rolling across the ocean, so one gives the name "gravitational waves" to small ripples rolling across spacetime.

We are deeply indebted to Mr. James M. Nester, who found and corrected many errors in the equations of this chapter and of a dozen others throughout the book.

(1) approximate nature of a wave

Ripples of what? Ripples in the shape of the ocean's surface; ripples in the shape (i.e., curvature) of spacetime. Both types of waves are idealizations. One cannot, with infinite accuracy, delineate at any moment which drops of water are in the waves and which are in the underlying ocean: Similarly, one cannot delineate precisely which parts of the spacetime curvature are in the ripples and which are in the cosmological background. But one can almost do so; otherwise one would not speak of "waves"!

(2) local viewpoint vs. large-scale viewpoint

Look at the ocean from a rowboat. Waves dominate the seascape. Changes in angle and level of the surface occur every 30 feet or less. These changes propagate. They obey a simple wave equation

$$\left(\frac{1}{g^2}\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)(\text{height of surface}) = 0.$$

Now get more sophisticated. Notice from a spaceship the large-scale curvature of the ocean's surface—curvature because the Earth is round, curvature because the sun and the moon pull on the water. As waves propagate long distances, this curvature bends their fronts and changes slightly their simple wave equation. Also important over large distance are nonlinear interactions between waves, interaction with the wind, Coriolis forces due to the Earth's rotation, etc.

Spacetime is similar. Propagating through the universe, according to Einstein's theory, must be a complex pattern of small-scale ripples in the spacetime curvature, ripples produced by binary stars, by supernovae, by gravitational collapse, by explosions in galactic nuclei. Locally ("rowboat viewpoint") one can ignore the interaction of these ripples with the large-scale curvature of spacetime and their nonlinear interaction with each other. One can pretend the waves propagate in flat spacetime; and one can write down a simple wave equation for them. But globally one cannot. The large-scale curvature due to quiescent stars and galaxies will produce redshifts and will deform wave fronts; and the energy carried by the waves themselves will help to produce the large-scale curvature. This chapter treats both viewpoints, the local (§§35.2–6), and the global (§§35.7–15).

§35.2. REVIEW OF "LINEARIZED THEORY" IN VACUUM

Linearized theory of gravitational waves:

Idealize, for awhile, the gravitational waves of our universe as propagating through flat, empty spacetime (local viewpoint). Then they can be analyzed using the "linearized theory of gravity," which was introduced in Chapter 18.

Linearized theory, one recalls, is a weak-field approximation to general relativity. The equations of linearized theory are written and solved as though spacetime were flat (special-relativity viewpoint); but the connection to experiment is made through the curved-space formalism of general relativity.

More specifically, linearized theory describes gravity by a symmetric, second-rank tensor field $\bar{h}_{\mu\nu}$. In the standard ("Lorentz," or Hilbert) gauge, this field satisfies the "gauge" or "subsidiary" conditions (coordinate conditions)

(1) Lorentz gauge condition

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0. \tag{35.1a}$$

(Here, and throughout linearized theory, indices of $\bar{h}_{\mu\nu}$ are raised and lowered with the Minkowski metric $\eta_{\alpha\beta}$.) In this gauge the *propagation equations* for vacuum gravitational fields are the familiar wave equations

$$\Box \bar{h}_{\mu\nu} \equiv \bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0. \tag{35.1b}$$

(2) propagation equation

Spacetime is really curved in linearized theory, although equations (35.1) are written and solved as though it were not. The global inertial frames of equations (35.1) are only *almost* inertial. In them the metric components are actually

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O([h_{\mu\nu}]^2); \tag{35.2a*}$$

(3) metric

and the "metric perturbation" $h_{\mu\nu}$ is related to the "gravitational field" $\bar{h}_{\mu\nu}$ by

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu},$$
$$h \equiv h_{\alpha}{}^{\alpha} = -\bar{h} = -\bar{h}_{\alpha}{}^{\alpha}. \tag{35.2b}$$

The metric (35.2a) governs the motion of test particles, the propagation of light, etc., in the usual general-relativistic manner.

Recall the origin of the equations (35.1) that govern $\bar{h}_{\mu\nu}$. The subsidiary conditions $\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha} = 0$ were imposed by specializing the coordinate system; and the Einstein field equations in vacuum then reduced to $\Box\bar{h}_{\mu\nu} = 0$.

Actually, as was shown in Box 18.2, the coordinates of linearized theory are not fully fixed by the conditions $\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha} = 0$. There remains an ambiguity embodied in further "gauge changes" (infinitesimal coordinate transformations), ξ_{μ}, which satisfy a restrictive condition

(4) residual gauge freedom

$$\xi_{\mu,\alpha}{}^{\alpha} = 0 \tag{35.3a}$$

in order to preserve conditions (35.1a). Then

$$x^{\mu}{}_{\text{new}} = x^{\mu}{}_{\text{old}} + \xi^{\mu} \tag{35.3b}$$

is the coordinate transformation and

$$\bar{h}_{\mu\nu\,\text{new}} = \bar{h}_{\mu\nu\,\text{old}} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} \tag{35.3c}$$

is the gauge change. All this was derived and discussed in Chapter 18.

§35.3. PLANE-WAVE SOLUTIONS IN LINEARIZED THEORY

The simplest of all solutions to the linearized equations $\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = \bar{h}_{\mu}{}^{\alpha}{}_{,\alpha} = 0$ is the monochromatic, plane-wave solution,

Monochromatic, plane wave

$$\bar{h}_{\mu\nu} = \Re[A_{\mu\nu} \exp(ik_{\alpha}x^{\alpha})]. \tag{35.4a}$$

*A more nearly rigorous treatment defines $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$, and puts the small corrections $O([h_{\mu\nu}]^2)$ into the field equations:

$$\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha} = O([h_{\mu\nu}]^2{}_{,\alpha}), \qquad \bar{h}_{\mu\nu,\alpha}{}^{\alpha} = O([h_{\mu\nu}]^2{}_{,\alpha\beta}).$$

Here $\Re[\ldots.]$ means that one must take the real part of the expression in brackets; while $A_{\mu\nu}$ (*amplitude*) and k_μ (*wave vector*) are constants satisfying

$$k_\alpha k^\alpha = 0 \qquad (\boldsymbol{k} \text{ a null vector}), \tag{35.4b}$$

$$A_{\mu\alpha} k^\alpha = 0 \qquad (\boldsymbol{A} \text{ orthogonal to } \boldsymbol{k}) \tag{35.4c}$$

[consequences of $\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0$ and $\bar{h}_{\mu,\alpha}^{\ \alpha} = 0$, respectively; see (35.10) below for the true physics associated with this wave, the curvature tensor]. Clearly, this solution describes a wave with frequency

$$\omega \equiv k^0 = (k_x{}^2 + k_y{}^2 + k_z{}^2)^{1/2}, \tag{35.5}$$

which propagates with the speed of light in the direction $(1/k^0)(k_x, k_y, k_z)$.

Plane wave has two degrees of freedom in amplitude (two polarizations)

At first sight the amplitude $A_{\mu\nu}$ of this plane wave appears to have six independent components (ten, less the four orthogonality constraints $A_{\mu\alpha}k^\alpha = 0$). But this cannot be right! As Track-2 readers have learned in Chapter 21, the gravitational field in general relativity has two dynamic degrees of freedom, not six. Where has the analysis gone astray?

One went astray by forgetting the arbitrariness tied up in the gauge freedom (35.3). The plane-wave vector

$$\xi^\mu \equiv -iC^\mu \exp(ik_\alpha x^\alpha), \tag{35.6}$$

with four arbitrary constants C^μ, generates a gauge transformation that can change arbitrarily four of the six independent components of $A_{\mu\nu}$. One gets rid of this arbitrariness by choosing a specific gauge.

§35.4. THE TRANSVERSE TRACELESS (TT) GAUGE

Transverse-traceless (TT) gauge:

(1) for plane wave

Select a 4-velocity $\boldsymbol{u}$—not at just one event, but the same $\boldsymbol{u}$ throughout all of spacetime (special-relativistic viewpoint!). By a specific gauge transformation (exercise 35.1), impose the conditions

$$A_{\mu\nu}u^\nu = 0. \tag{35.7a}$$

These are only three constraints on $A_{\mu\nu}$, not four, because one of them—$k^\mu(A_{\mu\nu}u^\nu) = 0$—is already satisfied (35.4c). As a fourth constraint, use a gauge transformation (exercise 35.1) to set

$$A^\mu{}_\mu = 0. \tag{35.7b}$$

One now has eight constraints in all, $A_{\mu\alpha}u^\alpha = A_{\mu\alpha}k^\alpha = A_\alpha{}^\alpha = 0$, on the ten components of the amplitude; and the coordinate system (gauge) is now fixed rigidly. Thus, the two remaining free components of $A_{\mu\nu}$ represent the two degrees of freedom (two polarizations) in the plane gravitational wave.

It is useful to restate the eight constraints $A_{\mu\alpha}u^\alpha = A_{\mu\alpha}k^\alpha = A^\mu{}_\mu = 0$ in a Lorentz frame where $u^0 = 1$, $u^j = 0$, and in a form where k^α does not appear explicitly:

$$h_{\mu 0} = 0, \quad \text{i.e., only the spatial components } h_{jk} \text{ are nonzero;} \tag{35.8a}$$

$$h_{kj,j} = 0, \quad \text{i.e., the spatial components are divergence-free;} \tag{35.8b}$$

$$h_{kk} = 0, \quad \text{i.e., the spatial components are trace-free.} \tag{35.8c}$$

(Here and henceforth repeated spatial indices are to be summed, even if both are down; e.g., $h_{kk} \equiv \Sigma_{k=1}^{3} h_{kk}$.) Notice that, since $h = h_\mu{}^\mu = h_{kk} = 0$, *there is no distinction between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ in this gauge.*

Turn attention now away from plane waves to arbitrary gravitational waves in linearized theory. Any electromagnetic wave can be resolved into a superposition of plane waves, and so can any gravitational wave. For each plane wave in the superposition, introduce the special gauge (35.8). Note that the gauge conditions are all linear in $h_{\mu\nu}$. Therefore the arbitrary wave will also satisfy conditions (35.8). Thus arises the theorem: *Pick a specific global Lorentz frame of linearized theory (i.e., pick a specific 4-velocity $\boldsymbol{u}$). In that frame* (*where $u^\alpha = \delta^\alpha{}_0$*), *examine a specific gravitational wave of arbitrary form. One can always find a gauge in which $h_{\mu\nu}$ satisfies the constraints* (35.8). Moreover, in this gauge only the h_{jk} are nonzero. Therefore, one need only impose the six wave equations (2) for any wave

$$\Box h_{jk} = h_{jk,\alpha}{}^{,\alpha} = 0. \tag{35.9}$$

Any symmetric tensor satisfying constraints (35.8) [but not necessarily the wave equations (35.9)] is called a *transverse-traceless* (*TT*) *tensor*—transverse because it is purely spatial ($h_{0\mu} = 0$) and, if thought of as a wave, is transverse to its own direction of propagation ($h_{ij,j} = h_{ij}k_j = 0$); traceless because $h_{kk} = 0$. The most general purely spatial tensor S_{ij} can be decomposed [see Arnowitt, Deser, and Misner (1962) or Box 35.1] into a part S_{ij}^{TT}, which is "transverse and traceless"; a part $S_{ij}^{T} = \frac{1}{2}(\delta_{ij}f_{,kk} - f_{,ij})$, which is "transverse" ($S_{ij,j}^{T} = 0$) but is determined entirely by one function f giving the trace of S ($S_{kk}^{T} = \nabla^2 f$); and a part $S_{ij}^{L} = S_{i,j}^{L} + S_{j,i}^{L}$, which is "longitudinal" and is determined by the vector field S_i^L. In linearized theory h_{ij}^L is a purely gauge part of $h_{\mu\nu}$, whereas h_{ij}^T and h_{ij}^{TT} are gauge-invariant parts of $h_{\mu\nu}$. The special gauge in which $h_{\mu\nu}$ reduces to its transverse-traceless part is called the *TT* or transverse-traceless gauge. The conditions (35.8) defining this gauge can be summarized as

Decomposition of spatial tensors

$$h_{\mu\nu} = h_{\mu\nu}^{TT}. \tag{35.8d}$$

As exercise 35.2 illustrates, only pure waves (and not more general solutions of the linearized field equations with source, $\Box h_{\mu\nu} = -16\pi T_{\mu\nu}$) can be reduced to *TT* gauge.

Curvature tensor in TT gauge

In the *TT* gauge, the time-space components

$$R_{j0k0} = R_{0j0k} = -R_{j00k} = -R_{0jk0}$$

of the Riemann curvature tensor have an especially simple form [see equation (18.9) and exercise 18.4]:

$$R_{j0k0} = -\frac{1}{2} h^{TT}_{jk,00}. \tag{35.10}$$

Recall that the curvature tensor is gauge-invariant (exercise 18.1). It follows that $h_{\mu\nu}$ cannot be reduced to still fewer components than it has in the *TT* gauge.

Box 35.1 describes methods to calculate $h^{TT}_{\mu\nu}$ from a knowledge of $h_{\mu\nu}$ in some other gauge.

Box 35.1 METHODS TO CALCULATE "TRANSVERSE-TRACELESS PART" OF A WAVE

Problem: Let a gravitational wave $h_{\mu\nu}(t, x^j)$ in an arbitrary gauge of linearized theory be known. How can one calculate the metric perturbation $h^{TT}_{\mu\nu}(t, x^j)$ for this wave in the transverse-traceless gauge?

Solution 1 (valid only for waves; i.e., when $\Box\bar{h}_{\mu\nu} = 0$). Calculate the components R_{j0k0} of ***Riemann*** in the initial gauge; then integrate equation (35.10)

$$h^{TT}_{jk,00} = -2R_{j0k0} \tag{1}$$

to obtain h^{TT}_{jk}. When the wave is monochromatic, $h_{\mu\nu} = h_{\mu\nu}(x^l)e^{-i\omega t}$; then the solution of (1) has the simple form

$$h^{TT}_{jk} = 2\omega^{-2}R_{j0k0}. \tag{2}$$

Solution 2 (valid only for plane waves). "Project out" the *TT* components in an algebraic manner using the operator

$$P_{jk} = \delta_{jk} - n_j n_k. \tag{3}$$

Here

$$n_k = k_k/|\boldsymbol{k}|$$

is the unit vector in the direction of propagation. Verify that P_{jk} is a projection operator onto the transverse plane:

$$P_{jl}P_{lk} = P_{jk}, \qquad P_{jk}n_k = 0, \qquad P_{kk} = 2.$$

Then the transverse part of h_{jk} is $P_{jl}h_{lm}P_{mk}$ (or in matrix notation, PhP); and the *TT* part is this quantity diminished by its trace:

$$h^{TT}_{jk} = P_{jl}P_{mk}h_{lm} - \frac{1}{2}P_{jk}(P_{ml}h_{lm}) \tag{4}$$

(index notation),

$$h^{TT} = PhP - \frac{1}{2}P\,\mathrm{Tr}\,(Ph) \tag{4'}$$

(matrix notation).

The sequence of operations that gives h^{TT}_{ij} cuts two parts out of h_{ij}. The first part cut out is

$$h^{T}_{jk} = \frac{1}{2}P_{jk}(P_{lm}h_{lm}), \tag{5}$$

which is transverse but is built from its own trace,

$$h^T \equiv \mathrm{Tr}\,(PhP) = \mathrm{Tr}\,(Ph) = P_{lm}h_{ml}.$$

EXERCISES

Exercise 35.1. TRANSFORMATION OF PLANE WAVE TO TT GAUGE

Let a plane wave of the form (35.4) be given, in some arbitrary gauge of linearized theory. Exhibit explicitly the transformation that puts it into the TT gauge. [*Hint:* Work in a Lorentz frame where the 4-velocity u^μ of the TT gauge is $u^0 = 1$, $u^j = 0$. Solve for the four constants C^μ of the generating function (35.6) by demanding that $\bar{h}_{\mu\nu}$ satisfy the TT constraints (35.7).]

Exercise 35.2. LIMITATION ON EXISTENCE OF TT GAUGE

Although the metric perturbation $h_{\mu\nu}$ for any *gravitational wave* in linearized theory can be put into the TT form (35.8), nonradiative $h_{\mu\nu}$'s cannot. Consider, for example, the external field of a rotating, spherical star, which cannot be written as a superposition of plane waves:

The second part cut out of h_{ij} is the longitudinal part

$$h^L{}_{jk} = h_{jk} - P_{jl}P_{mk}h_{lm}$$
$$= n_l n_k h_{jl} + n_j n_l h_{lk} - n_j n_k(n_l n_m h_{lm}); \quad (6)$$

or

$$h^L = h - PhP \quad (6')$$

Solution 3 (general case). Fourier analyze any symmetric array $h_{ij} = \int h_{ij}(k, t) \exp(ik_m x^m)\, d^3k$, and apply the formulas (4) from solution 2 to each Fourier component individually. But note that in this case one can write the projection operator in the direction-independent form

$$P_{jk} = \delta_{jk} - \frac{1}{\nabla^2}\partial_j\,\partial_k \quad (7)$$

or

$$n_l n_m = \frac{1}{\nabla^2}\partial_l\,\partial_m \quad (8)$$

(provided the formulas are written with all h's standing on the right), since $\partial_l = ik_l$ under the Fourier integral. Of course the operation $1/\nabla^2$ can be evaluated by other methods, e.g., by Green's functions, as well as by Fourier analysis. [The quantity $\psi \equiv \nabla^{-2}f$ stands for the solution ψ of the Poisson equation $\nabla^2\psi = f$.] The advantage of this method is its power in certain analytic computations (see, e.g., below).

Gauge Transformations. The change in $h_{\mu\nu}$ due to a gauge transformation is

$$\delta h_{\mu\nu} = -(\partial_\nu\xi_\mu + \partial_\mu\xi_\nu). \quad (9)$$

The transverse part of this change is

$$P_{jl}P_{km}(\delta h_{lm}) = -P_{jl}P_{km}(\partial_l\xi_m + \partial_m\xi_l) = 0. \quad (10)$$

To verify this formula for a plane wave (solution 2), note that $\partial_l = i|k|n_l$ and $P_{jl}n_l = 0$. To verify the same result in general, use equation (7) to give the result

$$P_{jl}\,\partial_l = 0. \quad (11)$$

Thus both h^{TT}_{ij} of equation (4), and h^T_{ij} of equation (5) are gauge-invariant:

$$\delta h^{TT}_{ij} = \delta h^T_{ij} = 0. \quad (12)$$

In empty space ($T_{\mu\nu} = 0$), both h^T_{ij} and another gauge-invariant quantity $\bar{h}_{0k}$ (discussed in exercise 35.4) vanish, by virtue of the field equations.

$$h_{00} = \frac{2M}{r}, \qquad h_{jk} = \frac{2M}{r}\delta_{jk}, \qquad h_{0k} = -2\epsilon_{klm}\frac{S^l x^m}{r^3},$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$

[see equation (19.5)]. Here M is the star's mass and S is its angular momentum. Show that this *cannot* be put into TT gauge. [Hint: Calculate R_{j0k0} and from it, by means of (35.10), infer h_{jk}^{TT}. Then calculate R_{0xyz} in both the original gauge and the new gauge, and discover that they disagree—not only by virtue of the mass term, but also by virtue of the angular-momentum term.]

Exercise 35.3. A CYLINDRICAL GRAVITATIONAL WAVE

To restore one's faith, which may have been shaken by exercise 35.2, one can consider the radiative solution whose only nonvanishing component $h_{\mu\nu}$ is

$$\bar{h}_{zz} = 4A\cos(\omega t)J_0(\omega\sqrt{x^2 + y^2}),$$

where J_0 is the Bessel function. This solution represents a superposition of ingoing and outgoing cylindrical gravitational waves. For this gravitational field calculate R_{j0k0}, and from it infer h_{jk}^{TT}. Then calculate several other components of $R_{\alpha\beta\gamma\delta}$ (e.g., R_{zyzy}) in the original gauge and in TT gauge, and verify that the answers are the same.

Exercise 35.4. NON-TT PARTS OF METRIC PERTURBATION [Track 2]

From Box 35.1 establish the formula $h^T = \nabla^{-2}(h_{kk,ll} - h_{kl,kl})$; then verify the gauge invariance of h^T directly, by showing that $h_{kk,ll} - h_{kl,kl}$ is gauge-invariant. Use $\delta h_{ij} = \xi_{i,j} + \xi_{j,i}$. Show similarly that the quantities $\tilde{h}_{0k}$ defined by

$$\tilde{h}_{0k} = \bar{h}_{0k} - \nabla^{-2}(\bar{h}^{\mu}_{0,\mu k} + \bar{h}_{kl,l0})$$

are gauge-invariant. Show from the gauge-invariant linearized field equations (18.5) that

$$\nabla^2 h^T = -16\pi T^{00},$$

$$\nabla^2 \tilde{h}_{0k} = -16\pi T_{0k},$$

so h^T and $\tilde{h}_{0k}$ must vanish for waves in empty space.

§35.5. GEODESIC DEVIATION IN A LINEARIZED GRAVITATIONAL WAVE

Action of a gravitational wave on separation of two test particles

The oscillating curvature tensor of a gravitational wave produces oscillations in the separation between two neighboring test particles, A and B. Examine the oscillations from the viewpoint of A. Use a coordinate system ("*proper reference frame of A*"), with spatial origin $x^{\hat{j}} = 0$, attached to A's world line (comoving coordinates); with coordinate time equal to A's proper time ($x^{\hat{0}} = \tau$ on world line $x^{\hat{j}} = 0$); and with orthonormal spatial axes attached to gyroscopes carried by A ("nonrotating frame"). This coordinate system, appropriately specialized, is a local Lorentz frame not just at one event $\mathcal{P}_0$ on A's geodesic world line, but all along A's world line:

$$ds^2 = -dx^{\hat{0}2} + \delta_{\hat{j}\hat{k}}\,dx^{\hat{j}}\,dx^{\hat{k}} + O(|x^{\hat{j}}|^2)\,dx^{\hat{\alpha}}\,dx^{\hat{\beta}}. \tag{35.11}$$

[Proof: such a "proper reference frame" was set up for accelerated particles in Track 2's §13.6. The line element (13.71) derived there, when specialized to particle A ($a_{\hat{j}} = 0$ because A falls freely; $\omega^{\hat{l}} = 0$ because the spatial axes are attached to gyroscopes) reduces to the above form, as in equation (13.73).]

As the gravitational wave passes, it produces an oscillating curvature tensor, which wiggles the separation vector $\boldsymbol{n}$ reaching from particle A to particle B:

$$D^2 n^{\hat{j}}/d\tau^2 = -R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} n^{\hat{k}} = -R_{\hat{j}\hat{0}\hat{k}\hat{0}} n^{\hat{k}}. \tag{35.12}$$

The components of the separation vector are nothing but the coordinates of particle B, since particle A is at the origin of its own proper reference frame; thus,

$$n^{\hat{j}} = x_B{}^{\hat{j}} - x_A{}^{\hat{j}} = x_B{}^{\hat{j}}.$$

Moreover, at $x^{\hat{j}} = 0$ [where the calculation (35.12) is being performed], the $\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}$ vanish for all $x^{\hat{0}}$; so $d\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}/d\tau$ also vanish. This eliminates all Christoffel-symbol corrections in $D^2 n^{\hat{j}}/D\tau^2$. Hence, equation (35.12) reduces to

$$d^2 x_B{}^{\hat{j}}/d\tau^2 = -R_{\hat{j}\hat{0}\hat{k}\hat{0}} x_B{}^{\hat{k}}. \tag{35.13}$$

There is a TT coordinate system that, to first order in the metric perturbation h^{TT}_{jk}, moves with particle A and with its proper reference frame. To first order in h^{TT}_{jk}, the TT coordinate time t is the same as proper time τ, and $R^{TT}_{j0k0} = R_{\hat{j}\hat{0}\hat{k}\hat{0}}$. Hence, equation (35.13) can be rewritten

$$d^2 x_B{}^{\hat{j}}/dt^2 = -R^{TT}_{j0k0} x_B{}^{\hat{k}} = \frac{1}{2}(\partial^2 h^{TT}_{jk}/\partial t^2) x_B{}^{\hat{k}}. \tag{35.14}$$

Suppose, for concreteness, that the particles are at rest relative to each other before the wave arrives ($x_B{}^{\hat{j}} = x_{B(0)}{}^{\hat{j}}$ when $h^{TT}_{jk} = 0$). Then the equation of motion (35.14) can be integrated to yield

$$x_B{}^{\hat{j}}(\tau) = x_{B(0)}{}^{\hat{k}} \left[\delta_{jk} + \frac{1}{2} h^{TT}_{jk} \right]_{\text{at position of } A} \tag{35.15}$$

This equation describes the wave-induced oscillations of B's location, as measured in the proper reference frame of A.

Turn to the special case of a plane wave. Suppose the test-particle separation lies in the direction of propagation of the wave. Then the wave cannot affect the separation; there is no oscillation:

Transverse character of relative accelerations

$$h^{TT}_{jk} x_{B(0)}{}^{\hat{k}} \propto h^{TT}_{jk} k_k = 0.$$

Only separations in the transverse direction oscillate; *the wave is transverse not only in its mathematical description (h^{TT}_{jk}), but also in its physical effects (geodesic deviation)!*

EXERCISE

Exercise 35.5. ALTERNATIVE CALCULATION OF RELATIVE OSCILLATIONS

Introduce a TT coordinate system in which, at time $t = 0$, the two particles are both at rest. Use the geodesic equation to *show* that subsequently they both always remain at rest in the TT coordinates, despite the action of the wave. This means that the contravariant components of the separation vector are always constant in the TT coordinate frame:

$$n^j = x_B{}^j - x_A{}^j = \text{const.}$$

Call this constant $x_{B(0)}{}^j$. *Transform* these components to the comoving orthonormal frame; the answer should be equation (35.15).

§35.6. POLARIZATION OF A PLANE WAVE

Polarization of gravitational waves:

(1) States of linear polarization, "+" and "×"

Geodesic deviation in the transverse direction provides a means for studying and characterizing the polarizations of plane waves.

Consider a plane, monochromatic wave propagating in the z direction. In the TT gauge the constraints $h^{TT}_{0\mu} = 0$, $h^{TT}_{ij,j} \equiv ik_j h^{TT}_{ij} = 0$, and $h^{TT}_{kk} = 0$ reveal that the only nonvanishing components of $h^{TT}_{\mu\nu}$ are

$$\begin{aligned} h^{TT}_{xx} &= -h^{TT}_{yy} = \Re\{A_+ e^{-i\omega(t-z)}\}, \\ h^{TT}_{xy} &= h^{TT}_{yx} = \Re\{A_\times e^{-i\omega(t-z)}\}. \end{aligned} \tag{35.16}$$

The amplitudes A_+ and $A_\times$ represent two independent modes of polarization.

As for electromagnetic plane waves (Figure 35.1), so also for gravitational plane waves (Figure 35.2), one can resolve a given wave into two linearly polarized components, or, alternatively, into two circularly polarized components.

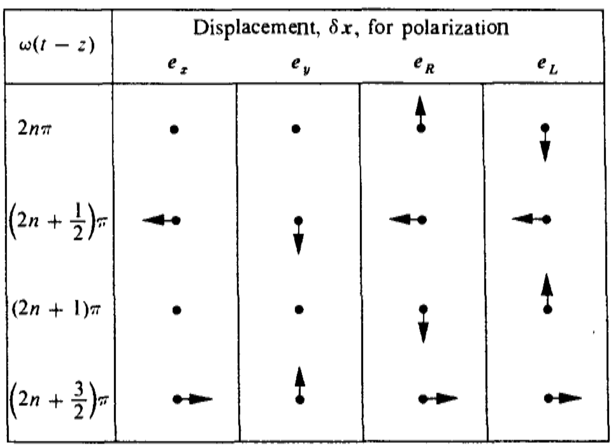


Figure 35.1.
Plane Electromagnetic Waves.
Polarization vector: e_P
Vector Potential

$$A = \Re\,[A_0 e^{-i\omega(t-z)} e_P]$$

Acceleration of a test charge:

$$a = (q/m)E = (q/m)(-\partial A/\partial t) = \Re[i\omega(q/m)A_0 e^{-i\omega(t-z)} e_P]$$

Displacement of charge relative to inertial frame:

$$\delta x = \Re\left[\frac{q/m}{i\omega} A_0 e^{-i\omega(t-z)} e_P\right]$$

For *linearly polarized waves,* the unit polarization vectors of electromagnetic theory are $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$. A test charge hit by a plane wave with polarization vector $\boldsymbol{e}_x$ oscillates in the x-direction relative to an inertial frame; and similarly for $\boldsymbol{e}_y$. By analogy, the *unit linear-polarization tensors* for gravitational waves are

$$\boldsymbol{e}_+ \equiv \boldsymbol{e}_x \otimes \boldsymbol{e}_x - \boldsymbol{e}_y \otimes \boldsymbol{e}_y, \tag{35.17a}$$

$$\boldsymbol{e}_\times \equiv \boldsymbol{e}_x \otimes \boldsymbol{e}_y + \boldsymbol{e}_y \otimes \boldsymbol{e}_x. \tag{35.17b}$$

The plane wave (35.16), when $A_\times = 0$, has polarization $\boldsymbol{e}_+$ and can be rewritten

$$h_{jk} = \Re\{A_+ e^{-i\omega(t-z)} e_{+jk}\}. \tag{35.18}$$

Its effect in altering the geodesic separation between two test particles depends on the direction of their separation. To see the effect in all directions at once, consider a circular ring of test particles in the transverse (x, y) plane, surrounding a central particle (Figure 35.2). As the plane wave (35.18) (polarization $\boldsymbol{e}_+$) passes, it deforms what was a ring as measured in the proper reference frame of the central particle into an ellipse with axes in the x and y directions that pulsate in and out:

etc. By contrast (Figure 35.2), a wave of polarization $\boldsymbol{e}_\times$ deforms the ring at a 45-degree angle to the x and y directions: etc.

(2) States of circular polarization

For circularly polarized waves, the unit polarization vectors of electromagnetic theory are

$$e_R = \frac{1}{\sqrt{2}}(e_x + ie_y), \qquad e_L = \frac{1}{\sqrt{2}}(e_x - ie_y) \tag{35.19}$$

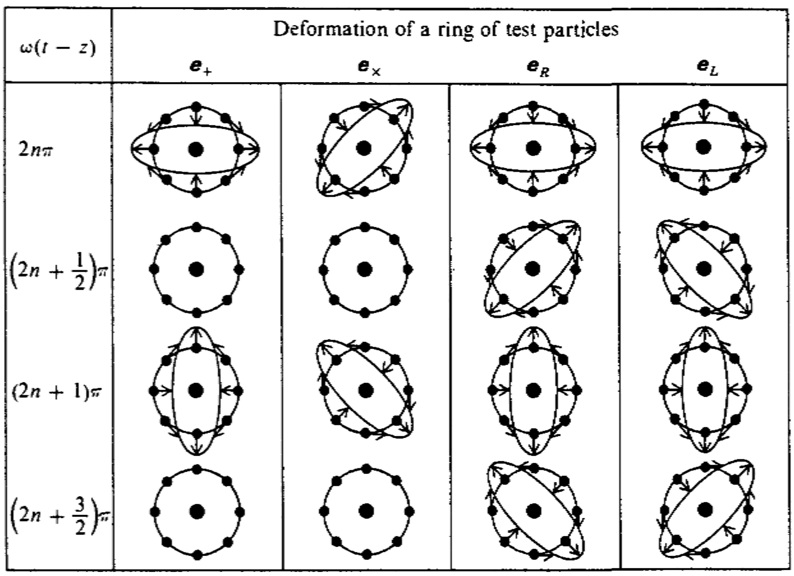


Figure 35.2.
Plane Gravitational Waves. Polarization tensor: $\boldsymbol{e}_P$

Metric perturbation:

$$h_{jk} = \Re\,[A_0 e^{-i\omega(t-z)} e_{Pjk}]$$

Tidal acceleration between two test particles:

$$\frac{D^2 n_j}{D\tau^2} = -R_{j0k0} n_k = \frac{1}{2}\frac{\partial^2 h_{jk}}{\partial t^2} n_k$$
$$= \Re\left[-\frac{1}{2}\omega^2 A_0 e^{-i\omega(t-z)} e_{Pjk} n_k\right]$$

Separation between two test particles:

$$n_j = n_j^{(0)} + \Re\left[\frac{1}{2} A_0 e^{-i\omega(t-z)} e_{Pjk} n_k^{(0)}\right]$$

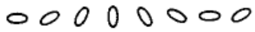
Position of test particle B in proper reference frame of test particle A. (In drawing, A is the central particle and B is any peripheral particle):

$$x_B{}^j = x_{B(0)}{}^j + \Re\left[\frac{1}{2} A_0 e^{-i\omega(t-z)} e_{Pjk} x_{B(0)}{}^k\right]$$

Similarly, the *unit circular polarization tensors* of gravitation theory are

$$\boldsymbol{e}_R = \frac{1}{\sqrt{2}}(\boldsymbol{e}_+ + i\boldsymbol{e}_\times), \qquad \boldsymbol{e}_L = \frac{1}{\sqrt{2}}(\boldsymbol{e}_+ - i\boldsymbol{e}_\times). \tag{35.20}$$

A test charge hit by an electromagnetic wave of polarization e_R moves around and around in a circle in the righthanded direction (counterclockwise for a wave propagating toward the reader); for e_L it circles in the lefthanded (clockwise) direction (see Figure 35.1). Similarly (Figure 35.2), a gravitational wave of polarization $\boldsymbol{e}_R$ rotates the deformation of a test-particle ring in the righthanded direction,



while a wave of $\boldsymbol{e}_L$ rotates it in the lefthanded direction. The individual test particles in the ring rotate in small circles relative to the central particle. However, just as the drops in an ocean wave do not move along with the wave, so the particles on the ring do not move *around* the central particle with the rotating ellipse.

Spin-2 character of gravitational field and its relation to symmetries of waves

Notice from Figure 35.2 that, at any moment of time, a gravitational wave is invariant under a rotation of 180° about its direction of propagation. The analogous angle for electromagnetic waves (Figure 35.1) is 360°, and for neutrino waves it is 720°. This behavior is intimately related to the spin of the zero-mass particles associated with the quantum-mechanical versions of these waves: gravitons have spin 2, photons spin 1, and neutrinos spin 1/2. The classical radiation field of a spin-S particle is always invariant under a rotation of 360°/S about its propagation direction.

A radiation field of any spin S has precisely two orthogonal states of linear polarization. They are inclined to each other at an angle of 90°/S; thus, for a neutrino field, with $S = \frac{1}{2}$, the two states are distinguished as $|\uparrow\rangle$ and $|\downarrow\rangle$ (spin up and spin down; 180° angle). For an electromagnetic wave $S = 1$ and two orthogonal states of polarization are e_x and e_y (90° angle). For a gravitational wave $S = 2$, and two orthogonal states are $\boldsymbol{e}_+$ and $\boldsymbol{e}_\times$ (45° angle).

EXERCISES

Exercise 35.6. ROTATIONAL TRANSFORMATIONS FOR POLARIZATION STATES

Consider two Lorentz coordinate systems, one rotated by an angle θ about the z direction relative to the other:

$$t' = t, \qquad x' = x\cos\theta + y\sin\theta, \qquad y' = y\cos\theta - x\sin\theta, \qquad z' = z. \tag{35.21}$$

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be quantum-mechanical states of a neutrino with spin-up and spin-down relative to the x direction; and similarly for $|\uparrow'\rangle$ and $|\downarrow'\rangle$. Let $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_{x'}$, $\boldsymbol{e}_{y'}$ be the unit polarization vectors in the two coordinate systems for an electromagnetic wave traveling in the z-direction; and similarly $\boldsymbol{e}_+$, $\boldsymbol{e}_\times$, $\boldsymbol{e}_{+'}$, $\boldsymbol{e}_{\times'}$ for a gravitational wave in linearized theory. Derive the following transformation laws:

$$\begin{aligned}
|\uparrow'\rangle &= |\uparrow\rangle \cos\tfrac{1}{2}\theta + |\downarrow\rangle \sin\tfrac{1}{2}\theta; & |\downarrow'\rangle &= -|\uparrow\rangle \sin\tfrac{1}{2}\theta + |\downarrow\rangle \cos\tfrac{1}{2}\theta;\\
\boldsymbol{e}_{x'} &= \boldsymbol{e}_x \cos\theta + \boldsymbol{e}_y \sin\theta; & \boldsymbol{e}_{y'} &= -\boldsymbol{e}_x \sin\theta + \boldsymbol{e}_y \cos\theta;\\
\boldsymbol{e}_{+'} &= \boldsymbol{e}_+ \cos 2\theta + \boldsymbol{e}_\times \sin 2\theta; & \boldsymbol{e}_{\times'} &= -\boldsymbol{e}_+ \sin 2\theta + \boldsymbol{e}_\times \cos 2\theta.
\end{aligned} \tag{35.22}$$

What is the generalization to the linear-polarization basis states for a radiation field of arbitrary spin S?

Exercise 35.7. ELLIPTICAL POLARIZATION

Discuss elliptically polarized gravitational waves in a manner analogous to the discussion of linearly and circularly polarized waves in Figure 35.2.

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§35.7. THE STRESS-ENERGY CARRIED BY A GRAVITATIONAL WAVE

Exercise 18.5 showed that in principle one can build detectors which extract energy from gravitational waves. Hence, it is clear that the waves must carry energy.

Unfortunately, to derive and justify an expression for their energy requires a somewhat more sophisticated viewpoint than linearized theory. Such a viewpoint will be developed later in this chapter (§§35.13 and 35.15). But for the benefit of Track-1 readers, the key result is stated here.

Approximate localization of energy in a gravitational wave

In accordance with the discussions in §§19.4 and 20.4, the stress-energy carried by gravitational waves cannot be localized inside a wavelength. One cannot say whether the energy is carried by the crest of a wave, by its trough, or by its "walls." However, one *can* say that a certain amount of stress-energy is contained in a given "macroscopic" region (region of several wavelengths' size), and one can thus talk about a tensor for an *effective* smeared-out stress-energy of gravitational waves, $T^{(\mathrm{GW})}_{\mu\nu}$. In a (nearly) inertial frame of linearized theory, $T^{(\mathrm{GW})}_{\mu\nu}$ is given by

Effective stress-energy tensor for gravitational waves:

(1) expressed in terms of metric perturbations

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\langle h^{TT}_{jk,\mu} h^{TT}_{jk,\nu}\rangle, \tag{35.23}$$

where $\langle\ \rangle$ denotes an average over several wavelengths and h^{TT}_{jk} means the (gauge-invariant) transverse-traceless part of $h_{\mu\nu}$, which is simply h_{jk} in the TT gauge. Another formula for $T^{(\mathrm{GW})}_{\mu\nu}$, valid in any arbitrary gauge, with $\bar{h} \neq 0$, $\bar{h}_\mu{}^\alpha{}_{,\alpha} \neq 0$, and $\bar{h}_{0\mu} \neq 0$ is

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\left\langle \bar{h}_{\alpha\beta,\mu}\bar{h}^{\alpha\beta}{}_{,\nu} - \frac{1}{2}\,\bar{h}_{,\mu}\bar{h}_{,\nu} - \bar{h}^{\alpha\beta}{}_{,\beta}\bar{h}_{\alpha\mu,\nu} - \bar{h}^{\alpha\beta}{}_{,\beta}\bar{h}_{\alpha\nu,\mu}\right\rangle \tag{35.23$'$}$$

This stress-energy tensor, like any other, is divergence-free in vacuum

(2) subject to conservation law

$$T^{(\mathrm{GW})}{}_\mu{}^\nu{}_{,\nu} = 0; \tag{35.24}$$

and it contributes to the large-scale background curvature (which linearized theory ignores) just as any other stress-energy does:

(3) role as source of background curvature

$$G^{(\mathrm{B})}_{\mu\nu} = 8\pi(T^{(\mathrm{GW})}_{\mu\nu} + T^{(\mathrm{matter})}_{\mu\nu} + T^{(\mathrm{other\ fields})}_{\mu\nu}). \tag{35.25}$$

In writing here the term $T^{(\mathrm{GW})}_{\mu\nu}$ for the effective smeared-out energy density of the gravitational wave, one is foregoing any further insertion of the gravitational wave into the Einstein equation. Otherwise one might end up counting twice over the contribution of the same wave to the background curvature of space, even though expressed in very different formalisms.

(4) for a plane, monochromatic wave

According to equation (35.23), the stress-energy tensor for the plane wave,

$$h_{\mu\nu} = \mathfrak{R}\{(A_+ e_{+\mu\nu} + A_\times e_{\times\mu\nu})e^{-i\omega(t-z)}\}, \tag{35.26}$$

is

$$T^{(\mathrm{GW})}_{tt} = T^{(\mathrm{GW})}_{zz} = -T^{(\mathrm{GW})}_{tz} = \frac{1}{32\pi}\omega^2(|A_+|^2 + |A_\times|^2). \tag{35.27}$$

Notice that the background radius of curvature $\mathcal{R}$ (ignored by linearized theory), and the mean reduced wavelength $\lambdabar$ (= wavelength/2π) and amplitude $\mathcal{A}$ of the gravitational waves satisfy

$$\begin{aligned}\mathcal{R}^{-2} &\sim \text{typical magnitude of components of } R^{(\mathrm{B})}_{\alpha\beta\gamma\delta}\\ &\sim T^{(\mathrm{GW})}_{\mu\nu} \sim \mathcal{A}^2/\lambdabar^2 \text{ if } T^{(\mathrm{GW})}_{\mu\nu} \text{ is chief source of background curvature,}\\ &\gg T^{(\mathrm{GW})}_{\mu\nu} \sim \mathcal{A}^2/\lambdabar^2 \text{ if } T^{(\mathrm{GW})}_{\mu\nu} \text{ is not chief source.}\end{aligned}$$

Consequently, the dimensionless numbers $\mathcal{A}$ and $\lambdabar/\mathcal{R}$ are related by

$$\mathcal{A} \lesssim \lambdabar/\mathcal{R}. \tag{35.28}$$

Conditions for validity of gravitational-wave formalism

Thus, *the whole concept of a small-scale ripple propagating in a background of large-scale curvature breaks down, and the whole formalism of this chapter becomes meaningless, if the dimensionless amplitude of the wave approaches unity. One must always have $\mathcal{A} \ll 1$ as well as $\lambdabar \ll \mathcal{R}$ if the concept of a gravitational wave is to make any sense!*

§35.8. GRAVITATIONAL WAVES IN THE FULL THEORY OF GENERAL RELATIVITY

Nonlinear effects in gravitational waves:

Curving up of the background spacetime by the energy of the waves is but one of many new effects that enter, when one passes from linearized theory to the full, nonlinear general theory of relativity.

(1) radiation damping

In linearized theory one can consider a localized source of gravitational waves (e.g., a vibrating bar) in steady oscillation, radiating a strictly periodic wave. But the exact theory insists that the energy of the source decrease secularly, to counterbalance the energy carried off by the radiation (energy conservation; gravitational radiation damping; see §§36.8 and 36.11). This makes an exactly periodic wave impossible, though a very nearly periodic one can certainly be emitted [Papapetrou (1958); Arnowitt, Deser, and Misner as reported by Misner (1964b)].

(2) refraction

In the real universe there are spacetime curvatures due not only to the energy of gravitational waves, but also, and more importantly, to the material content of the universe (planets, stars, gas, galaxies). As a gravitational wave propagates through these curvatures, its wave fronts change shape ("refraction"), its wavelength changes (gravitational redshift), and it backscatters off the curvatures to some extent. If the wave is a pulse, the backscatter will cause its shape and polarization to keep changing and will produce "tails" that spread out behind the moving pulse, traveling slower than light [see exercise 32.10; also Riesz (1949), DeWitt and Brehme (1960), DeWitt and DeWitt (1964a), Kundt and Newman (1968), Couch *et. al.* (1968)]. However, so long as $\mathcal{A} \ll 1$ and $\lambda\!\!\!^{-}/\mathcal{R} \ll 1$, these effects will be extremely small locally. They can only build up over distances of the order of $\mathcal{R}$—and sometimes not even then. Thus, locally, linearized theory will remain highly accurate.

(3) redshift

(4) backscatter

(5) tails

Even in an idealized universe containing nothing but gravitational waves, backscatter and tails are produced by the interaction of the waves with the background curvature that they themselves produce.

If the reduced wavelength $\lambda\!\!\!^{-} = \lambda/2\pi$ and the mass-energy m of a pulse of waves satisfy $\lambda\!\!\!^{-} \ll m$, it is possible (in principle) to focus the pulse into a region of size $r < m$, whereupon a part of the energy of the pulse will undergo gravitational collapse to a singularity, leaving behind a black hole [see Ruffini and Wheeler (1970), and pp. 7–24 of Christodoulou (1971)]. Short of a certain critical strength, no part of the pulse undergoes such a collapse. But it does undergo a time delay before reexpanding. This time delay is definable and measurable in the asymptotically flat space, far from the domain where the energy a little earlier underwent temporary focusing into dimensions of order $\lambda\!\!\!^{-}$.

(6) self-gravitational attraction

All these effects can be analyzed in general relativity theory using approximation schemes which, in first order, are similar to or identical to linearized theory. Later in this chapter (§§35.13–35.15), one such approximation scheme will be developed. But first it is helpful to study an exact solution that exhibits some of these effects.

1

2

§35.9. AN EXACT PLANE-WAVE SOLUTION

The rest of this chapter is Track 2. No earlier Track-2 material is needed as preparation for it, but Chapter 20 (conservation laws) and §22.5 (geometric optics) will be found to be helpful. It is not needed as preparation for any later chapter.

Any exact gravitational-wave solution that can be given in closed mathematical form must be highly idealized; otherwise it could not begin to cope with the complexities outlined above. Consequently, mathematically exact solutions are useful for pedagogical purposes only. However, pedogogy should not be condemned: it is needed not only by students, but also by veteran workers in the field of relativity, who even today are only beginning to develop intuition into the nonlinear regime of geometrodynamics!

From the extensive literature on exact solutions, we have chosen, as a compromise between realism and complexity, the following plane wave [Bondi *et. al.* (1959), Ehlers and Kundt (1962)]:

Exact plane-wave solution of vacuum field equation:

(1) form of metric

$$\begin{aligned} ds^2 &= L^2(e^{2\beta}\,dx^2 + e^{-2\beta}\,dy^2) + dz^2 - dt^2 \\ &= L^2(e^{2\beta}\,dx^2 + e^{-2\beta}\,dy^2) - du\,dv. \end{aligned} \tag{35.29a}$$

Here

$$u = t - z, \qquad v = t + z, \qquad L = L(u), \qquad \beta = \beta(u). \tag{35.29b}$$

The forms that the functions $L(u)$ ("background factor") and $\beta(u)$ ("wave factor") can take are determined by the vacuum field equations. In the null coordinate system u, v, x, y, the only component of the Ricci tensor that does not vanish identically is (see Box 14.4, allowing for the difference in coordinates, $2v_{\text{there}} = v_{\text{here}}$)

$$R_{uu} = -2L^{-1}[L'' + (\beta')^2 L]. \tag{35.30}$$

where the prime denotes d/du. Thus, Einstein's equations in vacuum read

(2) generation of "background factor" L by "wave factor" β

$$L'' + (\beta')^2 L = 0. \tag{35.31}$$

("effect of wave factor on background factor")

(3) linearized limit

The linearized version of this equation is $L'' = 0$, since $(\beta')^2$ is a second-order quantity. Therefore the solution corresponding to linearized theory is

$$L = 1, \qquad \beta(u) \text{ arbitrary but small.}$$

The corresponding metric is

$$ds^2 = (1 + 2\beta)\, dx^2 + (1 - 2\beta)\, dy^2 + dz^2 - dt^2, \qquad \beta = \beta(t - z). \tag{35.32}$$

Notice that this is a plane wave of polarization $\boldsymbol{e}_+$ propagating in the z-direction. (See exercise 35.10 at end of §35.12 for the extension to a wave possessing both polarizations, $\boldsymbol{e}_+$ and $\boldsymbol{e}_\times$.)

(4) special case: a plane-wave pulse

Return attention to the exact plane wave, and focus on the case where the "wave factor" $\beta(u)$ is a pulse of duration $2T$, and $|\beta'| \ll 1/T$ throughout the pulse. Then the exact solution (Figure 35.3) is: (1) for $u < -T$ (flat spacetime; pulse has not yet arrived),

$$\beta = 0, \qquad L = 1; \tag{35.33a}$$

(2) for $-T < u < +T$ (interior of pulse),

$$\beta = \beta(u) \text{ is arbitrary, except that } |\beta'| \ll 1/T,$$

$$L(u) = 1 - \int_{-T}^{u} \left\{ \int_{-T}^{\bar{u}} [\beta'(\bar{\bar{u}})]^2\, d\bar{\bar{u}} \right\} d\bar{u} + O([\beta' T]^4); \tag{35.33b}$$

(3) for $u > T$ (after the pulse has passed),

$$\beta = 0, \qquad L = 1 - \frac{u}{a}, \qquad a \equiv \frac{1}{\int_{-T}^{T} (\beta')^2\, du} + \frac{O([\beta' T]^2)}{\int_{-T}^{T} (\beta')^2\, du}. \tag{35.33c}$$

Before discussing the physical interpretation of this exact solution, one must come to grips with the singularity in the metric coefficients at $u = a \gg T$. (There $L = 0$, so $g_{xx} = g_{yy} = 0$.) Is this a physical singularity like the region $r = 0$ of the Schwarzschild geometry, or is it merely a coordinate singularity as $r = 2M$ is in Schwarzschild coordinates (Chapters 31, 32, and 33)? The only nonzero components of the Riemann tensor for the metric (35.29) are (see Box 14.4)

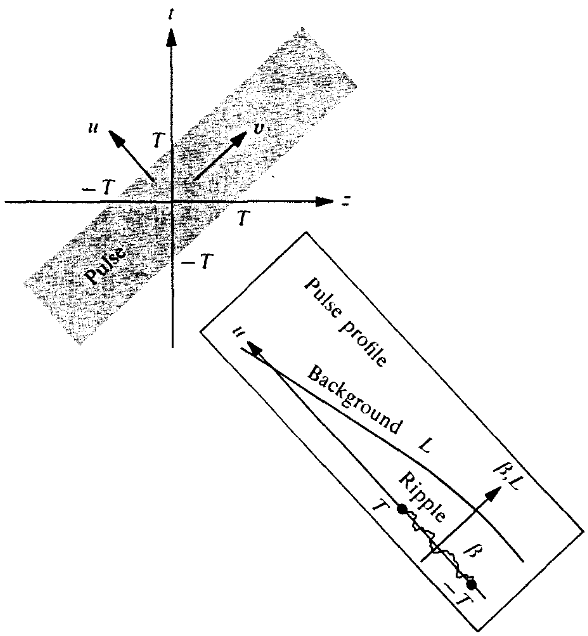

Figure 35.3.
Spacetime diagram and pulse profile for an exact plane-wave solution to Einstein's vacuum field equations. The metric has the form

$$ds^2 = L^2(e^{2\beta}\,dx^2 + e^{-2\beta}\,dy^2) + dz^2 - dt^2.$$

The "wave factor" $\beta(u) \equiv \beta(t - z)$ (short-scale ripples) and the "background factor" $L(u) \equiv L(t - z)$ (large-scale bending of the background geometry by the effective mass-energy of the "ripply" gravitational wave) are shown in the drawing and are given analytically by equations (35.33).

$$R^x{}_{uxu} = \frac{1}{2}R_{uu} - \beta'' - 2(L'/L)\beta', \tag{35.34}$$

$$R^y{}_{uyu} = \frac{1}{2}R_{uu} + \beta'' + 2(L'/L)\beta'.$$

Moreover, these components both vanish in any extended region where $\beta = 0$. Thus, *spacetime is completely flat in regions where the "wave factor" vanishes—which is everywhere outside the pulse!* In particular, spacetime is flat near $u = a$, so the singularity there must be a coordinate singularity, not a physical singularity. To eliminate this singularity, one can perform the coordinate transformation

(5) spacetime is flat outside the pulse

$$x = \frac{X}{1 - U/a}, \qquad y = \frac{Y}{1 - U/a}, \qquad u = U, \qquad v = V + \frac{X^2 + Y^2}{a - U} \tag{35.35}$$

throughout the region to the future of the pulse ($u > T$), where

$$ds^2 = (1 - u/a)^2(dx^2 + dy^2) - du\,dv. \tag{35.36a}$$

In the new X, Y, U, V coordinates the metric has the explicitly flat form

$$ds^2 = dX^2 + dY^2 - dU\,dV \qquad \text{for } U = u > T. \tag{35.36b}$$

EXERCISES

Exercise 35.8. GLOBALLY WELL-BEHAVED COORDINATES FOR PLANE WAVE [based on Ehlers and Kundt (1962)]

Find a coordinate transformation similar to (35.35), which puts the exact plane-wave solution (35.29a), (35.31), into the form

$$ds^2 = dX^2 + dY^2 - dU\,dV + (X^2 - Y^2)F\,dU^2, \tag{35.37}$$
$$F = F(U) \text{ completely arbitrary.}$$

This coordinate system has the advantage of no coordinate singularities anywhere; while the original coordinate system has the advantages of an easy transition to linearized theory, and easy interpretation of the action of the wave on test particles.

Exercise 35.9. GEODESIC COMPLETENESS FOR PLANE-WAVE MANIFOLD [based on Ehlers and Kundt (1962)]

Prove that the coordinate system (X, Y, U, V) of exercise 35.8 completely covers its spacetime manifold. More specifically, show that every geodesic can be extended in both directions for an arbitrarily large affine-parameter length without leaving the X, Y, U, V coordinate system. (This property is called *geodesic completeness.*) [*Hint:* Choose an arbitrary event and an arbitrary tangent vector $d/d\lambda$ there. They determine an arbitrary geodesic. Perform a coordinate transformation that leaves the form of the metric unchanged and puts $d/d\lambda$ either in the $(\tilde{U}, \tilde{V})$ = constant 2-surface, or in the $\tilde{X}, \tilde{Y})$ = constant 2-surface. Verify that the two coordinate systems cover the same region of spacetime. Then analyze completeness of $d/d\lambda$'s geodesic in $(\tilde{X}, \tilde{Y}, \tilde{U}, \tilde{V})$ coordinates.]

§35.10. PHYSICAL PROPERTIES OF THE EXACT PLANE WAVE

Spacetime is completely flat both before the arrival of the plane-wave pulse ($u < -T$) and after it has passed ($u > T$). This is the message of the last paragraph.

Flatness outside gravitational-wave pulses is unusual

Complete flatness outside the pulse is very atypical for gravitational waves in the full, nonlinear general theory of relativity. It occurs in this example only because the wave fronts (surfaces of constant u and v, i.e., constant z and t) are perfectly flat 2-surfaces. If the wave fronts were bent (e.g., spherical), the energy carried by the pulse would produce spacetime curvature outside it.

Action of exact gravitational-wave pulse on test particles:

To see nonlinear effects in action, turn from the flat geometry outside the pulse to the action of the pulse on freely falling test particles. Consider a family of particles that are all at rest in the original t, x, y, z coordinate system (world lines: $[x, y, z]$ = constant) before the pulse arrives. Then even while the pulse is passing, and after it has gone, the particles remain at rest in the coordinate system. (This is true for any metric, such as (35.29a), with $g_{0\mu} = -\delta^0{}_\mu$, for then $\Gamma^\mu{}_{00} = 0$, so $x^\mu = \delta^\mu{}_0\tau +$ const. satisfies the geodesic equation.)

Two particles whose separation is in the direction of propagation of the pulse (z-direction) have not only constant coordinate separation, $\Delta x = \Delta y = 0$ and $\Delta z \neq 0$; they also have constant proper separation, $\Delta s = g_{zz}{}^{1/2}\,\Delta z = \Delta z$. Hence, the exact plane wave is completely transverse, like a plane wave of linearized theory.

(1) transverse character of relative accelerations

Neighboring particles transverse to the propagation direction, ($\Delta x \neq 0$, $\Delta y \neq 0$, $\Delta z = 0$) have a proper separation that wiggles as the pulse passes:

$$\begin{aligned}\Delta s &= L(t-z)[e^{2\beta(t-z)}(\Delta x)^2 + e^{-2\beta(t-z)}(\Delta y)^2]^{1/2}\\ &\approx L[(1+2\beta)(\Delta x)^2 + (1-2\beta)(\Delta y)^2]^{1/2}.\end{aligned} \tag{35.38}$$

Superimposed on the usual linearized-theory type of wiggling, due to the "wave factor" β, is a very small net acceleration of the particles toward each other, due to the "background factor" L [note the form of $L(u)$ in Figure 35.3]. This is an acceleration of almost Newtonian type, caused by the gravitational attraction of the energy that the gravitational wave carries between the two particles. The total effect of all the energy that passes is to convert the particles from an initial state of relative rest, to a final state of relative motion with speed

(2) gravitational attraction due to energy in pulse

$$v_{\text{final}} = d\,\Delta s/dt = d(L\,\Delta s_i)/dt = -\Delta s_i/a, \tag{35.39}$$

where

$$\Delta s_i = [(\Delta x)^2 + (\Delta y)^2]^{1/2} = (\text{initial particle separation}).$$

[Recall: $L_{\text{initial}} = 1$, $L_{\text{final}} = 1 - u/a = 1 - (t-z)/a$; equation (35.33).]

Precisely the same effect can be produced by a pulse of electromagnetic waves (§35.11).

§35.11. COMPARISON OF AN EXACT ELECTROMAGNETIC PLANE WAVE WITH THE GRAVITATIONAL PLANE WAVE

Consider the metric

$$ds^2 = L^2(u)(dx^2 + dy^2) - du\,dv, \qquad \begin{Bmatrix} u = t - z \\ v = t + z \end{Bmatrix}, \tag{35.40}$$

which is always flat if it satisfies the vacuum Einstein equations ($R_{\mu\nu} = 0$ or $L'' = 0$), and therefore cannot represent a gravitational wave. In this metric the electromagnetic potential

An electromagnetic plane-wave pulse

$$\boldsymbol{A} = A_\mu\,\boldsymbol{d}x^\mu = \mathcal{A}(u)\,\boldsymbol{d}x \tag{35.41}$$

satisfies Maxwell's equations for arbitrary $\mathcal{A}(u)$. It represents an electromagnetic plane wave analogous to the gravitational plane wave of the last few sections. The only nonzero field components of this wave are

$$F_{ux} = \mathcal{A}', \text{ i.e., } F_{tx} = -F_{zx} = \mathcal{A}'; \tag{35.42}$$

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so the electric vector oscillates back and forth in the x-direction, the magnetic vector oscillates in the y-direction, and the wave propagates in the z-direction. The stress-energy tensor in x, y, u, v, coordinates has only

$$T_{uu} = (4\pi L^2)^{-1}(\mathcal{A}')^2 \tag{35.43}$$

nonzero.

The Maxwell equations are already satisfied by the potential (35.41) in the background metric (35.40), as the reader can verify. In order to make that metric itself equally acceptable, one need only impose the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$. They read [see equation (35.30) with $\beta = 0$]

$$L'' + (4\pi T_{uu})L = 0. \tag{35.44}$$

Electromagnetic plane wave and gravitational plane wave produce same gravitational attractions

This has exactly the form of the equation $L'' + (\beta')^2 L = 0$ for the gravitational plane wave. Consequently, the relative motions of uncharged test particles produced by the "background factor" $L(u)$ are the same whether $L(u) \neq 1$ is produced by the stress-energy of an electromagnetic wave, or by a corresponding gravitational wave with

$$[(\beta')^2/4\pi]_{\text{grav wave}} = [T_{uu}]_{\text{em wave}} = (\mathcal{A}')^2/4\pi L^2. \tag{35.45}$$

The analogy can be made even closer. Decrease the wavelength of the waves, while holding $(\beta')^2/4\pi$ and $(\mathcal{A}')^2/4\pi L^2$ fixed:

$$\langle(\beta')^2/4\pi\rangle = \langle(\mathcal{A}')^2/4\pi L^2\rangle = \text{const}; \qquad \lambdabar \longrightarrow 0.$$

In the limit of very small wavelength (i.e., from a viewpoint whose characteristic length is $\gg \lambdabar$), the two solutions are completely indistinguishable. Their metrics are identical ($\lambdabar \longrightarrow 0$ and $\langle(\beta')^2\rangle = \text{const.}$ imply $\beta \longrightarrow 0$), and their jigglings of test particles are too small to be seen. Only their curving up of spacetime ($L \neq 1$) and the associated gravitational pull of their energy are detectable.

§35.12. A NEW VIEWPOINT ON THE EXACT PLANE WAVE

Exact gravitational plane waves reexamined in the language of "short-wave approximation":

(1) ripples vs. background

The above comparison suggests a viewpoint that was sketched briefly in the introduction to this chapter and in §35.8. Think of the exact gravitational plane-wave solution [Figure 35.3; equations (35.29) and (35.33)] as ripples in the spacetime curvature, described by $\beta(u)$, propagating on a very slightly curved background spacetime, characterized by $L(u)$. The most striking difference between the background and the ripples is not in the magnitude of their spacetime curvatures, but in their characteristic lengths. The ripples have characteristic length

$$\lambdabar \equiv (\text{typical reduced wavelength, } \lambda/2\pi, \text{ of waves}); \tag{35.46}$$

the background has characteristic length ("radius of curvature of background geometry")

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$$\mathcal{R} \sim |L/L''|^{1/2}{}_{\text{inside wave}} \sim 1/|\beta'|. \tag{35.47}$$

Recall that $\lambda\!\!\!^{-}$ is somewhat smaller than the pulse length, $2T$. Recall also that $|\beta' T| \ll 1$. Conclude that the characteristic lengths of the "wave factor" and the "background factor" differ greatly:

$$\lambda\!\!\!^{-} \ll \mathcal{R}. \tag{35.48}$$

This difference in scales enables one to separate out the background from the ripples.

The ripples are very much smaller in scale ($\lambda\!\!\!^{-} \ll \mathcal{R}$) than the background. Nevertheless the local curvature in a ripple is correspondingly larger than the background curvature [equations (35.30), (35.34)]; thus,

$$\begin{aligned}(R^x{}_{uxu})_{\text{background}} &= (R^y{}_{uyu})_{\text{background}} = -L''/L \sim 1/\mathcal{R}^2,\\ (R^x{}_{uxu})_{\text{waves}} &= -(R^y{}_{uyu})_{\text{waves}} = -\beta'' \sim |\beta'|/\lambda\!\!\!^{-} \sim 1/(\lambda\!\!\!^{-}\mathcal{R})\\ &\sim (\mathcal{R}/\lambda\!\!\!^{-})(R^x{}_{uxu})_{\text{background}}.\end{aligned} \tag{35.49}$$

One is reminded of the mottled surface of an orange!

The metric for the background of the gravitational plane wave is the same as for the electromagnetic one [equation (35.40)]:

$$ds^2 = g^{(B)}_{\mu\nu}\, dx^\mu\, dx^\nu = L^2(dx^2 + dy^2) - du\, dv. \tag{35.50}$$

By comparison with equation (35.29a), one sees that the metric for the full spacetime (background plus ripple) is

$$ds^2 = (g^{(B)}_{\mu\nu} + h_{\mu\nu})\, dx^\mu\, dx^\nu, \tag{35.51}$$

$$h_{xx} = -h_{yy} = 2\beta, \text{ all other } h_{\mu\nu} = 0. \tag{35.52}$$

(Recall, in the region where $\beta \neq 0$, L is very nearly 1.) One can think of the ripples as a transverse, traceless, symmetric tensor field $h_{\mu\nu}$ analogous to the electromagnetic field A_μ, propagating in the background geometry. Just as the electromagnetic field produces the background curvature via

(2) propagation of ripples in background

$$G_{uu} = -2L''/L = 8\pi T_{uu},$$

so the gravitational-wave ripples $h_{\mu\nu}$ produce the background curvature via equation (35.31), which one can rewrite as

$$G^{(B)}_{uu} = -2L''/L = 8\pi T^{(\text{EFF})}_{uu}. \tag{35.53}$$

Here

$$T^{(\text{EFF})}_{uu} \equiv \frac{1}{4\pi}(\beta')^2 = \frac{1}{32\pi} h_{jk,u} h_{jk,u} \tag{35.54}$$

(3) effective stress-energy tensor for ripples

is the "effective stress-energy tensor" for the gravitational waves. Notice that it agrees, except for averaging, with the expression (35.23) that was written down without justification in §35.7.

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EXERCISE

Exercise 35.10. PLANE WAVE WITH TWO POLARIZATIONS PRESENT

The exact plane-wave solution (35.29) has polarization $\boldsymbol{e}_{+}$. Construct a similar solution, containing two arbitrary amplitudes, $\beta(u)$ and $\gamma(u)$, for polarizations $\boldsymbol{e}_{+}$ and $\boldsymbol{e}_{\times}$. Extend the discussions of §§35.9–35.12 to this solution.

§35.13. THE SHORTWAVE APPROXIMATION

The remainder of this chapter extends the above viewpoint in a rigorous manner to very general gravitational-wave solutions. This extension is called the "shortwave formalism"; it was largely devised by Isaacson (1968a,b), though it was built on foundations laid by Wheeler (1964a) and by Brill and Hartle (1964). Versions that are even more rigorous have been given in the W.K.B. or geometric-optics limit by Choquet-Bruhat (1969), and by MacCallum and Taub (1973).

Foundations for shortwave formalism:

(1) $\mathcal{R}$, $\bar{\lambda}$, and $\mathcal{A}$ defined

(2) demand that $\mathcal{A} \ll 1$ and $\bar{\lambda}/\mathcal{R} \ll 1$

Consider gravitational waves propagating through a *vacuum* background spacetime. As in §35.7, let $\mathcal{R}$ be the typical radius of curvature of the background; let $\bar{\lambda}$ and $\mathcal{A}$ be the typical reduced wavelength ($\lambda/2\pi$) and amplitude of the waves; and demand both $\mathcal{A} \ll 1$ and $\bar{\lambda}/\mathcal{R} \ll 1$. The background curvature might be due entirely to the waves, or partly to waves and partly to nearby matter and nongravitational fields.

(3) split of metric into background plus perturbation; "steady coordinates"

The analysis uses a coordinate system closely "tuned" to spacetime in the sense that the metric coefficients can be split into "background" coefficients plus perturbations

$$g_{\mu\nu} = g^{(\mathrm{B})}_{\mu\nu} + h_{\mu\nu} \tag{35.55}$$

with these properties: (1) the amplitude of the perturbation is $\mathcal{A}$

$$h_{\mu\nu} \lesssim (\text{typical value of } g^{(\mathrm{B})}_{\mu\nu}) \cdot \mathcal{A}; \tag{35.56a}$$

(2) the scale on which $g^{(\mathrm{B})}_{\mu\nu}$ varies is $\gtrsim \mathcal{R}$

$$g^{(\mathrm{B})}_{\mu\nu,\alpha} \lesssim (\text{typical value of } g^{(\mathrm{B})}_{\mu\nu})/\mathcal{R}; \tag{35.56b}$$

(3) the scale on which $h_{\mu\nu}$ varies is $\sim \bar{\lambda}$

$$h_{\mu\nu,\alpha} \sim (\text{typical value of } h_{\mu\nu})/\bar{\lambda}. \tag{35.56c}$$

Such coordinates are called "*steady*."

(4) Split of Ricci curvature tensor

A rather long computation (exercise 35.11) shows that the Ricci tensor for an expanded metric of the form (35.55) is

$$\begin{array}{ccccccccc} R_{\mu\nu} = & R^{(\mathrm{B})}_{\mu\nu} & + & R^{(1)}_{\mu\nu}(h) & + & R^{(2)}_{\mu\nu}(h) & + & \text{error.} \\ & ? & & \mathcal{A}/\bar{\lambda}^2 & & \mathcal{A}^2/\bar{\lambda}^2 & & \mathcal{A}^3/\bar{\lambda}^2 \end{array} \tag{35.57}$$

Here a marker ($\mathcal{A}/\bar{\lambda}^2$, etc.) has been placed under each term to show its typical order of magnitude; $R^{(B)}_{\mu\nu}$ is the Ricci tensor for the background metric $g^{(B)}_{\mu\nu}$; and $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are expressions defined by

$$R^{(1)}_{\mu\nu}(h) \equiv \frac{1}{2}\left(-h_{|\mu\nu} - h_{\mu\nu|\alpha}{}^{\alpha} + h_{\alpha\mu|\nu}{}^{\alpha} + h_{\alpha\nu|\mu}{}^{\alpha}\right), \tag{35.58a}$$

$$\begin{aligned} R^{(2)}_{\mu\nu}(h) \equiv \frac{1}{2}\Big[\frac{1}{2}h_{\alpha\beta|\mu}h^{\alpha\beta}{}_{|\nu} &+ h^{\alpha\beta}\left(h_{\alpha\beta|\mu\nu} + h_{\mu\nu|\alpha\beta} - h_{\alpha\mu|\nu\beta} - h_{\alpha\nu|\mu\beta}\right) \\ &+ h_{\nu}{}^{\alpha|\beta}\left(h_{\alpha\mu|\beta} - h_{\beta\mu|\alpha}\right) - \left(h^{\alpha\beta}{}_{|\beta} - \frac{1}{2}h^{|\alpha}\right)\left(h_{\alpha\mu|\nu} + h_{\alpha\nu|\mu} - h_{\mu\nu|\alpha}\right)\Big]. \end{aligned} \tag{35.58b}$$

In these expressions and everywhere below, indices are raised and lowered with $g^{(B)}_{\mu\nu}$, and an upright line denotes a covariant derivative with respect to $g^{(B)}_{\mu\nu}$ (whereas in Chapter 21 it denoted covariant derivative with respect to 3-geometry).

Split of vacuum field equations into "wave part" ($\propto \mathcal{A}$) plus "coarse-grain part" ($\propto \mathcal{A}^2$ and smooth on scale $\bar{\lambda}$) plus "fluctuational corrections" ($\propto \mathcal{A}^2$ and ripply on scale $\bar{\lambda}$)

At the heart of the shortwave formalism is its method for solving the vacuum field equations $R_{\mu\nu} = 0$. One begins by selecting out of expression (35.57) the part linear in the amplitude of the wave $\mathcal{A}$, and setting it equal to zero. The action of the waves to curve up the background is a nonlinear phenomenon (linearized theory shows no sign of it); so $R^{(B)}_{\mu\nu}$ cannot be linear in $\mathcal{A}$. Hence, in expression (35.57), $R^{(1)}_{\mu\nu}(h)$ is the only linear term, and it must vanish by itself

$$R^{(1)}_{\mu\nu}(h) = 0. \tag{35.59a}$$

[Of course $h_{\mu\nu}$ may contain nonlinear correction terms—call them $j_{\mu\nu}$—of order $\mathcal{A}^2$, which must not be constrained by this linear equation. They will be determined by (35.59c), below.]

One next splits the remainder of $R_{\mu\nu}$ into a part that is free of ripples—i.e., that varies only on scales far larger than $\bar{\lambda}$ ("coarse-grain viewpoint"), and a second part that contains the fluctuations. This split can be accomplished by averaging over several wavelengths (see exercise 35.14 for a precise treatment of the averaging process, also see Choquet-Bruhat (1969) for a class of solutions where such averaging is not required):

$$\begin{array}{ccccccc} R^{(B)}_{\mu\nu} & + & \langle R^{(2)}_{\mu\nu}(h)\rangle & + & \text{error} = 0 & \begin{bmatrix}\text{smooth}\\ \text{part}\end{bmatrix} \\ ? & & \mathcal{A}^2/\bar{\lambda}^2 & & \mathcal{A}^3/\bar{\lambda}^2 & \end{array} \tag{35.59b}$$

$$\begin{array}{ccccccccc} R^{(1)}_{\mu\nu}(j) & + & R^{(2)}_{\mu\nu}(h) & - & \langle R^{(2)}_{\mu\nu}(h)\rangle & + & \text{error} = 0 & \begin{bmatrix}\text{fluctuating}\\ \text{part}\end{bmatrix} \\ \mathcal{A}^2/\bar{\lambda}^2 & & \mathcal{A}^2/\bar{\lambda}^2 & & \mathcal{A}^2/\bar{\lambda}^2 & & \mathcal{A}^3/\bar{\lambda}^2 & \end{array} \tag{35.59c}$$

$\begin{bmatrix}\text{nonlinear cor-}\\ \text{rection to } h\end{bmatrix}$

That's all there is to it!—except for reducing the equations to manageable form, and a fuller interpretation of the physics.

Physical interpretation of the three parts of field equations:

Begin with the interpretation.

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(1) propagation of waves

Equation (35.59a) is an equation for the propagation of the gravitational waves $h_{\mu\nu}$.

(2) production of background curvature by energy of waves; $T^{(GW)}_{\mu\nu}$ defined

Equation (35.59b) shows how the stress-energy in the waves creates the background curvature. It can be rewritten in the more suggestive form

$$G^{(B)}_{\mu\nu} \equiv R^{(B)}_{\mu\nu} - \frac{1}{2} R^{(B)} g^{(B)}_{\mu\nu} = 8\pi T^{(GW)}_{\mu\nu} \text{ in vacuum,} \tag{35.60}$$

where

$$T^{(GW)}_{\mu\nu} \equiv -\frac{1}{8\pi} \left\{ \langle R^{(2)}_{\mu\nu}(h) \rangle - \frac{1}{2} g^{(B)}_{\mu\nu} \langle R^{(2)}(h) \rangle \right\} \tag{35.61}$$

is the stress-energy tensor for the gravitational waves. Now one sees the origin of the statement in §35.7, that the stress-energy of gravitational waves is well-defined only in a smeared-out sense.

(3) nonlinear self-interaction of waves

Finally, equation (35.59c) shows how the gravitational waves h generate nonlinear corrections j to themselves (wave-wave scattering, harmonics of the fundamental frequency, etc.). These higher-order effects will not be investigated in this chapter.

EXERCISE

Exercise 35.11. CONNECTION COEFFICIENTS AND CURVATURE TENSORS FOR A PERTURBED METRIC

In a specific coordinate frame of an arbitrary spacetime, write the metric coefficients in covariant representation in the form

$$g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu}. \tag{35.62a}$$

(At the end of the calculation, one can split $h_{\mu\nu}$ into two parts, $h_{\mu\nu} \longrightarrow h_{\mu\nu} + j_{\mu\nu}$; and out of this split obtain the formulas used in the text.) Assume that the typical components of $h_{\mu\nu}$ are much less than those of $g^{(B)}_{\mu\nu}$; so one can expand Christoffel symbols and curvature tensors in $h_{\mu\nu}$. Raise and lower indices of $h_{\mu\nu}$ with $g^{(B)}_{\mu\nu}$; and denote by a "$|$" covariant derivatives relative to $g^{(B)}_{\mu\nu}$ and by a "$;$" covariant derivatives relative to $g_{\mu\nu}$.

(a) Here $g_{\mu\nu}$ and $g^{(B)}_{\mu\nu}$ can be thought of as two different metrics coexisting in the spacetime manifold. Show that the difference between the corresponding covariant derivatives, $\boldsymbol{\nabla} - \boldsymbol{\nabla}^{(B)} \equiv \boldsymbol{S}$—indeed, the difference between any two covariant derivatives!—is a tensor with components

$$S^{\mu}{}_{\beta\gamma} = \Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{(B)\mu}{}_{\beta\gamma} \tag{35.62b}$$

[Hint: See part B of Box 10.3.]

(b) Show that

$$g^{\mu\nu} = g^{(B)\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_{\alpha}{}^{\nu} - h^{\mu\alpha}h_{\alpha}{}^{\beta}h_{\beta}{}^{\nu} + \cdots, \tag{35.62c}$$

and also that

$$g^{\mu\nu} = g^{(B)\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_{\alpha}{}^{\nu} - h^{\mu\alpha}h_{\alpha}{}^{\beta}h_{\beta\gamma}g^{\gamma\nu}. \tag{35.62c'}$$

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(c) By calculating in a local Lorentz frame of $g^{(B)}_{\mu\nu}$ and then transforming back to the original frame, show that

$$S^{\mu}{}_{\beta\gamma} = \frac{1}{2} g^{\mu\alpha}(h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha}), \qquad (35.62d)$$

$$R^{\alpha}{}_{\beta\gamma\delta} - R^{(B)\alpha}{}_{\beta\gamma\delta} = S^{\alpha}{}_{\beta\delta|\gamma} - S^{\alpha}{}_{\beta\gamma|\delta} + S^{\alpha}{}_{\mu\gamma} S^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\mu\delta} S^{\mu}{}_{\beta\gamma}, \qquad (35.62e)$$

$$R_{\beta\delta} - R^{(B)}{}_{\beta\delta} = S^{\alpha}{}_{\beta\delta|\alpha} - S^{\alpha}{}_{\beta\alpha|\delta} + S^{\alpha}{}_{\mu\alpha} S^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\mu\delta} S^{\mu}{}_{\beta\alpha}. \qquad (35.62f)$$

(d) Show that expression (35.62f) reduces to

$$R_{\beta\delta} = R^{(B)}{}_{\beta\delta} + R^{(1)}_{\beta\delta}(h) + R^{(2)}_{\beta\delta}(h) + \cdots \qquad (35.62g)$$

where $R^{(1)}$ and $R^{(2)}$ are defined by equations (35.58).

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§35.14. EFFECT OF BACKGROUND CURVATURE ON WAVE PROPAGATION

Focus attention on the propagation equation $R^{(1)}_{\mu\nu}(h) = 0$. As in linearized theory, so also here, the propagation is studied more simply in terms of

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h g^{(B)}_{\mu\nu}, \qquad (35.63)$$

$\bar{h}_{\mu\nu}$ defined

than in terms of $h_{\mu\nu}$. Rewritten in terms of $\bar{h}_{\mu\nu}$, $R^{(1)}_{\mu\nu}(h) = 0$ says

$$\bar{h}_{\mu\nu|\alpha}{}^{\alpha} + g^{(B)}_{\mu\nu} \bar{h}^{\alpha\beta}{}_{|\beta\alpha} - 2\bar{h}_{\alpha(\mu|}{}^{\alpha}{}_{\nu)} + 2R^{(B)}_{\alpha\mu\beta\nu} \bar{h}^{\alpha\beta} - 2R^{(B)}_{\alpha(\mu} \bar{h}_{\nu)}{}^{\alpha} = 0. \qquad (35.64)$$

Propagation equation for waves on curved background

[To obtain this, invert equation (35.63) obtaining $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} g^{(B)}_{\mu\nu} \bar{h}$; insert this into (35.58a) and equate to zero; then commute covariant derivatives using the identity (16.6b); finally contract to obtain an expression for $\bar{h}_{|\alpha}{}^{\alpha}$ and substitute that back in.]

The propagation equation (35.64) can be simplified by a special choice of gauge. An infinitesimal coordinate transformation

Specialization to "Lorentz gauge"

$$x^{\mu}{}_{\text{new}}(\mathscr{P}) = x^{\mu}{}_{\text{old}}(\mathscr{P}) + \xi^{\mu}(\mathscr{P}) \qquad (35.65a)$$

induces a first-order change in the functional forms of the metric coefficients given by

$$h_{\mu\nu\,\text{new}}(x^{\alpha}{}_{\text{new}}) = h_{\mu\nu\,\text{old}}(x^{\alpha}{}_{\text{new}}) - 2\xi_{(\mu|\nu)} \qquad (35.65b)$$

[analog of the gauge transformation of linearized theory, equation (35.3c); see exercise 35.12]. By an appropriate choice of the four functions ξ^{μ}, one can enforce the four "Lorentz gauge conditions"

$$\bar{h}_{\mu}{}^{\alpha}{}_{|\alpha} = 0 \qquad (35.66)$$

in the new coordinate system (exercise 35.13). This choice of gauge is analogous to that of linearized theory. It makes the second and third terms in the propagation equation vanish. (For additional gauge conditions of the "TT" type. see exercise 35.13.)

Coupling of waves to Ricci tensor can be ignored

The last term of the propagation equation, $-2R^{(B)}_{\alpha(\mu}\bar{h}_{\nu)}{}^{\alpha}$, vanishes to within the precision of the analysis, for this reason: attention has been confined to vacuum; so the only source of a nonvanishing Ricci tensor is the stress-energy carried by the gravitational waves themselves [equation (35.60)]; hence $R^{(B)}_{\alpha\beta} \sim \mathcal{A}^2/\lambda\!\!\!^{-2}$ and

$$R^{(B)}{}_{\alpha(\mu}\bar{h}_{\nu)}{}^{\alpha} \sim \mathcal{A}^3/\lambda\!\!\!^{-2}. \tag{35.67}$$

This is of the same order as $R^{(3)}_{\mu\nu}(h)$, the third-order correction to the Ricci tensor, which is far below the precision of the analysis. For consistency in the analysis it will therefore be neglected.

Summary of this section thus far: by choosing a gauge where $\bar{h}_{\mu}{}^{\alpha}{}_{|\alpha} = 0$, and by discarding terms of higher order than the precision of the analysis, one obtains the vacuum propagation equation

Propagation equation in Lorentz gauge and its realm of validity

$$\bar{h}_{\mu\nu|\alpha}{}^{\alpha} + 2R^{(B)}{}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} = 0, \tag{35.68}$$

subject to the Lorentz gauge condition

$$\bar{h}_{\mu\alpha}{}^{|\alpha} = 0.$$

Equation (35.68) is accurate to first order in the amplitude [corrections $\propto \mathcal{A}^2$ are embodied in equation (35.59c)]; and its accuracy is independent of the ratio $\lambda\!\!\!^-/\mathcal{R}$, as one sees from equations (35.59). Thus, *it can be applied whenever the waves are weak, even if the wavelength is large!*

Lists of effects absent from and contained in propagation equation

All nonlinear interactions of the wave with itself are neglected in this first-order propagation equation. Absent is the mechanism for waves to scatter off each other and off the background curvature that they themselves produce. Also absent are any hints of a change in shape of pulse due to self-interaction as a pulse of waves propagates. There are no signs of the gravitational collapse that one knows must occur when a mass-energy m of gravitational waves gets compressed into a region of size $\lesssim m$. To see all these effects, one must turn to corrections of second order in $\mathcal{A}$ and higher [e.g., equations (35.59c) and (35.60)].

Actually contained in the propagation equation are all effects due to the linear action of the background curvature on the propagating wave. These effects are explored, for short wavelengths ($\lambda\!\!\!^-/\mathcal{R} \ll 1$) and for nearly flat wave fronts, in exercises 35.15–35.17 at the end of the chapter. The effects considered include a gravitational redshift of gravitational radiation and gravitational deflection of the direction of propagation of gravitational radiation, identical to those for light; and also a rotation of the polarization tensor. When the wavelength is not small ($\lambda\!\!\!^-/\mathcal{R}$ not $\ll 1$), the propagation equation includes a back-scatter of the gravitational waves off the background curvature and a resultant pattern of wave "tails" analogous to that explored in exercise 32.10 [see, e.g., Couch *et al.* (1968), Price (1971a), Bardeen and Press (1972), Unt and Keres (1972)].

EXERCISES

Exercise 35.12. GAUGE TRANSFORMATIONS IN A CURVED BACKGROUND

(a) Show that the infinitesimal coordinate transformation (35.65a) induces the change (35.65b) in the functional form of the metric perturbation.

(b) Discuss the relationship between this gauge transformation and the concept of a Killing vector (§25.2).

Exercise 35.13. TRANSVERSE-TRACELESS GAUGE FOR GRAVITATIONAL WAVES PROPAGATING IN A CURVED BACKGROUND

(a) Show that, in vacuum in a curved background spacetime, the gauge condition $\bar{h}_{\mu}{}^{\alpha}{}_{|\alpha} = 0$ is preserved by transformations whose generator satisfies the wave equation $\xi_{\mu}{}^{|\alpha}{}_{\alpha} = 0$.

(b) Locally (over distances much smaller than $\mathcal{R}$) linearized theory is applicable; so there exists such a transformation which makes [see equations (35.7b) and (35.8a)]

$$\bar{h} = 0 + \text{error}, \qquad \bar{h}_{\mu\alpha}u^{\alpha} = 0 + \text{error}. \tag{35.69}$$

Here u^{α} is a vector field that is as nearly covariantly constant as possible ($u^{\alpha}{}_{|\beta} = 0$); i.e., it is a constant vector in the inertial coordinates of linearized theory; and the errors are small over distances much less than $\mathcal{R}$. Show that $\bar{h} = 0$ *can be imposed globally along with* $\bar{h}_{\mu\alpha}{}^{|\alpha} = 0$; i.e., show that, if it is imposed on an initial hypersurface, the propagation equation (35.68) preserves it.

(c) Show that in general, the background curvature prevents any vector field from being covariantly constant ($u^{\hat{\alpha}}{}_{|\beta} \sim u^{\hat{\alpha}}/\mathcal{R}$ at best); and from this *show that* $\bar{h}_{\mu\alpha}u^{\alpha} = 0$ *cannot be imposed globally along with* $\bar{h}_{\mu}{}^{\alpha}{}_{|\alpha} = 0$.

§35.15. STRESS-ENERGY TENSOR FOR GRAVITATIONAL WAVES

Turn now to an evaluation of the effective stress-energy tensor $T^{(\mathrm{GW})}_{\mu\nu}$ of equation (35.61). The evaluation requires averaging various quantities over several wavelengths. Useful rules for manipulating quantities inside the averaging brackets $\langle\ \ \rangle$ are the following (see exercise 35.14 for justification).

The averaging process involved in "coarse-grain" viewpoint

(1) Covariant derivatives commute; e.g., $\langle h\ h_{\mu\nu|\alpha\beta}\rangle = \langle h\ h_{\mu\nu|\beta\alpha}\rangle$. The fractional errors made by freely commuting are $\sim(\lambda\!\!\!^{-}/\mathcal{R})^2$, well below the inaccuracy of the computation.

(2) Gradients average out to zero; e.g., $\langle(h_{|\alpha}h_{\mu\nu})_{|\beta}\rangle = 0$. Fractional errors made here are $\lesssim \lambda\!\!\!^{-}/\mathcal{R}$.

(3) As a corollary, one can freely integrate by parts, flipping derivatives from one h to the other; e.g., $\langle h\ h_{\mu\nu|\alpha\beta}\rangle = \langle -h_{|\beta}h_{\mu\nu|\alpha}\rangle$.

Evaluation of effective stress-energy tensor for gravitational waves, $T^{(\mathrm{GW})}_{\mu\nu}$

A straightforward but long calculation using these rules, using equation (35.58b) for $R^{(2)}_{\mu\nu}(h)$, using definition (35.63) of $\bar{h}_{\mu\nu}$, using the propagation equation (35.64), and using the definition (35.61) of $T^{(\mathrm{GW})}_{\mu\nu}$, yields $\langle R^{(2)}(h)\rangle = 0$, and

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\langle \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}{}_{|\nu} - \frac{1}{2}\bar{h}_{|\mu}\bar{h}_{|\nu} - 2\bar{h}^{\alpha\beta}{}_{|\beta}\bar{h}_{\alpha(\mu|\nu)}\rangle. \tag{35.70}$$

This is the result quoted in equation (35.23′), except that there one used an inertial

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coordinate system of linearized theory, where covariant derivatives and ordinary derivatives are the same. In a gauge where $\bar{h}_\mu{}^\alpha{}_{|\alpha} = 0$, the last term vanishes. When, in addition, $\bar{h}_{\mu\nu}$ is traceless (see exercise 35.13), the second term vanishes; and there remains only

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\langle \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}{}_{|\nu}\rangle \qquad \text{if } \bar{h}_\mu{}^\alpha{}_{|\alpha} = \bar{h} = 0. \tag{35.70$'$}$$

Accuracy of expression for $T^{(\mathrm{GW})}_{\mu\nu}$

These expressions for the effective stress-energy of a gravitational wave have fractional errors of order $\mathcal{a}$, due to the neglect of second-order corrections to $h_{\mu\nu}$; they also have fractional errors of order $\lambda\!\!\!^{-}/\mathcal{R}$, due to the averaging process, which makes no sense when $\lambda\!\!\!^{-}$ approaches $\mathcal{R}$ in magnitude. Since $\mathcal{a} \lesssim \lambda\!\!\!^{-}/\mathcal{R}$ (35.28), the dominant errors in $T^{(\mathrm{GW})}_{\mu\nu}$ are $\sim\lambda\!\!\!^{-}/\mathcal{R}$.

Properties of $T^{(\mathrm{GW})}_{\mu\nu}$

To this accuracy, the stress-energy tensor for gravitational waves is on an equal footing with any other stress-energy tensor. It plays the same role in producing background curvature; and it enters into conservation laws in the same way. For example, one can show, either by direct calculation or from the identity $G^{(\mathrm{B})\mu\nu}{}_{|\nu} = 0$, that

$$T^{(\mathrm{GW})\mu\nu}{}_{|\nu} = 0 + \text{error}, \tag{35.71}$$

where the error $\sim(\lambda\!\!\!^{-}/\mathcal{R})(T^{(\mathrm{GW})\mu\nu}/\mathcal{R})$ is negligible in the shortwave approximation.

Some of the properties of $T^{(\mathrm{GW})}_{\mu\nu}$ have already been explored in §35.7. Further properties are explored in exercises 35.18 and 35.19.

EXERCISES

Exercise 35.14. BRILL-HARTLE AVERAGE

Isaacson (1968b) introduces the following averaging scheme, which he names "Brill-Hartle averaging."

(a) In the small region, of size several times $\lambda\!\!\!^{-}$, where the averaging occurs, there will be a unique geodesic of $g^{(\mathrm{B})}_{\mu\nu}$ connecting any two points $\mathcal{P}'$ and $\mathcal{P}$; so given a tensor $\boldsymbol{E}(\mathcal{P}')$ at $\mathcal{P}'$, one can parallel transport it along this geodesic to $\mathcal{P}$, getting there a tensor $\boldsymbol{E}(\mathcal{P}')_{\to\mathcal{P}}$.

(b) Let $f(\mathcal{P}', \mathcal{P})$ be a weighting function that falls smoothly to zero when $\mathcal{P}'$ and $\mathcal{P}$ are separated by many wavelengths, and such that

$$\int f(\mathcal{P}', \mathcal{P})\sqrt{-g^{(\mathrm{B})}(\mathcal{P}')}\, d^4x' = 1. \tag{35.72}$$

(c) Then the average of the tensor field $\boldsymbol{E}(\mathcal{P}')$ over several wavelengths about the point $\mathcal{P}$ is

$$\langle \boldsymbol{E}\rangle_{\mathcal{P}} \equiv \int \boldsymbol{E}(\mathcal{P}')_{\to\mathcal{P}} f(\mathcal{P}', \mathcal{P})\sqrt{-g^{(\mathrm{B})}(\mathcal{P}')}\, d^4x'. \tag{35.73}$$

(i) Show that there exists an entity $g^{(\mathrm{B})\alpha'}_{\mu}(\mathcal{P}, \mathcal{P}')$, whose primed index transforms as a tensor at $\mathcal{P}'$ and whose unprimed index transforms as a tensor at $\mathcal{P}$, such that (for $\boldsymbol{E}$ second rank)

$$E_{\alpha\beta}(\mathcal{P}')_{\to\mathcal{P}} = g^{(\mathrm{B})\mu'}_{\alpha} g^{(\mathrm{B})\nu'}_{\beta} E_{\mu'\nu'}(\mathcal{P}'). \tag{35.74}$$

This entity is called the "bivector of geodesic parallel displacement"; see DeWitt and Brehme (1960) or Synge (1960a).

(ii) Rewriting expression (35.73) in coördinate language as

$$\langle E_{\alpha\beta}(x)\rangle = \int g_{\alpha}^{(\mathrm{B})\mu'}(x,x')g_{\beta}^{(\mathrm{B})\nu'}(x,x')E_{\mu'\nu'}(x')f(x,x')\sqrt{-g^{(\mathrm{B})}(x')}\,d^4x', \qquad (35.73')$$

derive the three averaging rules cited at the beginning of §35.15. [For solution, see Appendix of Isaacson (1968b).]

Exercise 35.15. GEOMETRIC OPTICS

Develop geometric optics for gravitational waves of small amplitude propagating in a curved background. Pattern the analysis after geometric optics for electromagnetic waves (§22.5). In particular, let $\bar{h}_{\mu\nu}$ have an amplitude that varies slowly (on a scale $\mathcal{L} \lesssim \mathcal{R}$) and a phase θ that varies rapidly ($\theta_{,\hat{\alpha}} \sim 1/\lambda\!\!\!^{-}$). Expand the amplitude in powers of $\lambda\!\!\!^{-}/\mathcal{L}$, so that

$$\bar{h}_{\mu\nu} = \Re\{A_{\mu\nu} + \epsilon B_{\mu\nu} + \epsilon^2 C_{\mu\nu} + \cdots\}e^{i\theta/\epsilon}. \qquad (35.75)$$

Here ϵ is a formal expansion parameter, actually equal to unity, which reminds one that the terms attached to ϵ^n are proportional to $(\lambda\!\!\!^{-}/\mathcal{R})^n$. Define the following quantities (with $A^*_{\mu\nu}$ denoting the complex conjugate of $A_{\mu\nu}$):

$$\text{“wave vector”: } k_\alpha \equiv \theta_{,\alpha} \qquad (35.76a)$$

$$\text{“scalar amplitude”: } \mathcal{A} \equiv \left(\frac{1}{2}A^*_{\mu\nu}A^{\mu\nu}\right)^{1/2} \qquad (35.76b)$$

$$\text{“polarization”: } e_{\mu\nu} \equiv A_{\mu\nu}/\mathcal{A}. \qquad (35.76c)$$

By inserting expression (35.75) into the gauge condition (35.66) and the propagation equation (35.68), derive the fundamental equations of geometrical optics as follows.

(a) The rays (curves perpendicular to surfaces of constant phase) are null geodesics; i.e.

$$k_\alpha k^\alpha = 0, \qquad (35.77a)$$

$$k_{\alpha|\beta}k^\beta = 0. \qquad (35.77b)$$

(b) The polarization is orthogonal to the rays and is parallel transported along them;

$$e_{\mu\alpha}k^\alpha = 0, \qquad (35.77c)$$

$$e_{\mu\nu|\alpha}k^\alpha = 0. \qquad (35.77d)$$

(c) The scalar amplitude decreases as the rays diverge away from each other in accordance with

$$\mathcal{A}_{,\alpha}k^\alpha = -\frac{1}{2}k^\alpha{}_{|\alpha}\mathcal{A}. \qquad (35.77e)$$

i.e.,

$$(\mathcal{A}^2k^\alpha)_{|\alpha} = 0 \text{ (“conservation of gravitons”).} \qquad (35.77f)$$

(d) The correction $B_{\mu\nu}$ to the amplitude obeys

$$B_{\mu\alpha}k^\alpha = iA_{\mu\alpha}{}^{|\alpha}, \qquad (35.77g)$$

$$B_{\mu\nu|\alpha}k^\alpha = -\frac{1}{2}k^\alpha{}_{|\alpha}B_{\mu\nu} + \frac{1}{2}iA_{\mu\nu|\alpha}{}^{\alpha} + iR^{(\mathrm{B})}_{\alpha\mu\beta\nu}A^{\alpha\beta}. \qquad (35.77h)$$

In accordance with exercise 35.13, specialize the gauge so that $\bar{h} = 0$, i.e.,

$$e_\alpha{}^\alpha = 0. \qquad (35.77i)$$

2

Then show that the stress-energy tensor (35.70′) for the waves is

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\mathcal{A}^2 k_\mu k_\nu. \tag{35.77j}$$

This has the same form as the stress-energy tensor for a beam of particles with zero rest mass (see §5.4). Show explicitly that $T^{(\mathrm{GW})\mu\nu}{}_{|\nu} = 0$.

Exercise 35.16. GRAVITONS

Show that geometric optics, as developed in the preceding exercise, is equivalent to the following: "A graviton is postulated to be a particle of zero rest mass and 4-momentum $\boldsymbol{p}$, which moves along a null geodesic ($\boldsymbol{\nabla}_{\boldsymbol{p}}\boldsymbol{p} = 0$). It parallel transports with itself ($\boldsymbol{\nabla}_{\boldsymbol{p}}\boldsymbol{e} = 0$) a transverse ($\boldsymbol{e}\cdot\boldsymbol{p} = 0$) traceless ($e_\alpha{}^\alpha = 0$) polarization tensor $\boldsymbol{e}$. Geometric optics is the theory of a stream of such gravitons moving through spacetime." Exhibit the relationship between the quantities in this version of geometric optics and the quantities in the preceding version (e.g., $\boldsymbol{p} = \hbar\boldsymbol{k}$, where $\hbar$ is Planck's reduced constant $h/2\pi$).

Exercise 35.17. GRAVITATIONAL DEFLECTION OF GRAVITATIONAL WAVES

Show that gravitational waves of short wavelength passing through the solar system experience the same redshift and gravitational deflection as does light. (One should be able to infer this directly from exercise 35.15.)

Exercise 35.18. GAUGE INVARIANCE OF $T^{(\mathrm{GW})}_{\mu\nu}$

Show that the stress-energy tensor $T^{(\mathrm{GW})}_{\mu\nu}$ of equation (35.70) is invariant under gauge transformations of the form (35.65).

Exercise 35.19. $T^{(\mathrm{GW})}_{\mu\nu}$ EXPRESSED AS THE AVERAGE OF A STRESS-ENERGY PSEUDOTENSOR

Calculate the average over several wavelengths of the Landau-Lifshitz stress-energy pseudotensor [equation (20.22)] for gravitational waves with $\lambda\!\!\!^{-}/\mathcal{R} \ll 1$. The result should be equal to $T^{(\mathrm{GW})}_{\mu\nu}$. [Hint: Work in a gauge where $\bar{h}_\mu{}^\alpha{}_{|\alpha} = \bar{h} = 0$, to simplify the calculation.]

Exercise 35.20. SHORTWAVE APPROXIMATION FROM A VARIATIONAL VIEWPOINT

Readers who have studied the variational approach to gravitation theory in Chapter 21 may find attractive the following derivation of the basic equations of the shortwave approximation. It was devised, independently, by Sándor Kovács and Bernard Schutz, and by Bryce DeWitt (unpublished, 1971). MacCallum and Taub (1973) give a "non-Palatini" version.

(a) Define

$$g_{\mu\nu} \equiv g^{(\mathrm{B})}_{\mu\nu} + h_{\mu\nu}, \qquad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}g^{(\mathrm{B})}_{\mu\nu}h, \tag{35.78a}$$

$$W^\mu{}_{\beta\gamma} \equiv \frac{1}{2}g^{\mu\alpha}_{(\mathrm{B})}(h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha}). \tag{35.78b}$$

Raise and lower indices on $h_{\mu\nu}$ and $W^\mu{}_{\beta\gamma}$ with the background metric. Using the results of exercise 35.11, derive the following expression for the Lagrangian of the gravitational field:

$$\mathcal{L} \equiv \frac{1}{16\pi}(-g)^{1/2}R = \mathcal{L}' + \begin{pmatrix}\text{perfect divergence}\\ \text{of form } \partial\mathcal{Q}^\alpha/\partial x^\alpha\end{pmatrix} + \begin{pmatrix}\text{corrections of order}\\ \mathcal{A}^3/\lambda\!\!\!^{-2},\ R^{(\mathrm{B})}_{\mu\nu}\mathcal{A}\text{, and smaller}\end{pmatrix}, \tag{35.78c}$$

where

$$\mathcal{L}' \equiv \frac{1}{16\pi}(-g^{(B)})^{1/2}[R^{(B)} - \bar{h}^{\mu\nu}(W^{\alpha}{}_{\mu\nu|\alpha} - W^{\alpha}{}_{\mu\alpha|\nu}) + g^{(B)\mu\nu}(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu} - W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha})]. \tag{35.78d}$$

[*Hint:* recall that

$$(-g^{(B)})^{1/2}B^{\alpha}{}_{|\alpha} = \partial[(-g^{(B)})^{1/2}B^{\alpha}]/\partial x^{\alpha}$$

for any B^{α}.] Drop the corrections of order $\mathcal{a}^3/\lambda\!\!\!^{-2}$ from $\mathcal{L}$; and, knowing in advance that the field equations will demand $R^{(B)}_{\mu\nu} \sim \mathcal{a}^2/\lambda\!\!\!^{-2}$, drop also the corrections of order $R^{(B)}_{\mu\nu}\mathcal{a}$. Knowing that a perfect divergence contributes nothing in an extremization calculation, drop the divergence term from $\mathcal{L}$. Then $\mathcal{L}'$ is the only remaining part of $\mathcal{L}$.

(b) Extremize $I \equiv \int \mathcal{L}'\, d^4x$ by the Palatini method (§21.2); i.e., abandon (temporarily) definition (35.78b) of $W^{\mu}{}_{\beta\gamma}$, and extremize I with respect to independent variations of $W^{\mu}{}_{\beta\gamma} = W^{\mu}{}_{\gamma\beta}$, $\bar{h}^{\mu\nu} = \bar{h}^{\nu\mu}$, and $g^{\mu\nu}_{(B)} = g^{\nu\mu}_{(B)}$. Show that extremization with respect to $W^{\mu}{}_{\beta\gamma}$ leads back to equation (35.78b) for $W^{\mu}{}_{\beta\gamma}$ in terms of $h_{\mu\nu}$. Show that extremization with respect to $\bar{h}^{\mu\nu}$, when combined with equations (35.78a,b), leads to the propagation equation for gravitational waves (35.64). Show that extremization with respect to $g^{(B)\mu\nu}$, when combined with equations (35.78a,b) and with the propagation equation (35.64), and when averaged over several wavelengths, leads to

$$G^{(B)}_{\mu\nu} = 8\pi T^{(GW)}_{\mu\nu},$$

where $T^{(GW)}_{\mu\nu}$ is given by equation (35.70). [*Warning:* The amount of algebra in this exercise is enormous, unless one chooses to impose the gauge conditions $\bar{h} = \bar{h}_{\alpha}{}^{\beta}{}_{|\beta} = 0$ from the outset.]

CHAPTER 36

GENERATION OF GRAVITATIONAL WAVES

Matter is represented by curvature, but not every curvature does represent matter; there may be curvature "in vacuo."

G. LEMAITRE in Schilpp (1949), p. 440

§36.1. THE QUADRUPOLE NATURE OF GRAVITATIONAL WAVES

Masses in an isolated, nearly Newtonian system move about each other. How much gravitational radiation do they emit?

Generation of gravitational waves analyzed by electromagnetic analog

For an order-of-magnitude estimate, one can apply the familiar radiation formulas of electromagnetic theory, with the replacement $e^2 \longrightarrow -m^2$, which converts the static coulomb law into Newton's law of attraction. This procedure treats gravity as though it were a spin-one (vector) field, rather than a spin-two (tensor) field; consequently, it introduces moderate errors in numerical factors and changes angular distributions. But it gives an adequate estimate of the total power radiated.

In electromagnetic theory, electric-dipole radiation dominates, with a power output or "luminosity," L, given (see §4.4 and Figure 4.6) by

$$L_{\text{electric dipole}} = (2/3)e^2\boldsymbol{a}^2$$

for a single particle with acceleration $\boldsymbol{a}$ and dipole moment changing as $\ddot{\boldsymbol{d}} = e\ddot{\boldsymbol{x}} = e\boldsymbol{a}$;

$$L_{\text{electric dipole}} = (2/3)\,\ddot{\boldsymbol{d}}^2$$

for a general system with dipole moment $\boldsymbol{d}$. [Geometric units: luminosity in cm of mass-energy per cm of light travel time; charge in cm, $e = (G^{1/2}/c^2)e_{\text{conv}} = (2.87 \times 10^{-25}\,\text{cm/esu}) \times (4.8 \times 10^{-10}\,\text{esu}) = 1.38 \times 10^{-34}\,\text{cm}$, acceleration in cm of distance per cm of time per cm of time. For conventional units, with e in esu or $(\text{g cm}^3/\text{sec}^2)^{1/2}$, insert a factor c^{-3} on the right and get L in erg/sec]. The gravitational analog of the electric dipole moment is the mass dipole moment

Why gravitational waves cannot be dipolar

$$\boldsymbol{d} = \sum_{\text{particles } A} m_A \boldsymbol{x}_A.$$

Its first time-rate of change is the total momentum of the system,

$$\dot{\boldsymbol{d}} = \sum_{\text{particles } A} m_A \dot{\boldsymbol{x}}_A = \boldsymbol{p}.$$

The second time-rate of change of the mass dipole moment has to vanish because of the law of conservation of momentum, $\ddot{\boldsymbol{d}} = \dot{\boldsymbol{p}} = 0$. Therefore *there can be no mass dipole radiation in gravitation physics*.

The next strongest types of electromagnetic radiation are magnetic-dipole and electric-quadrupole. Magnetic-dipole radiation is generated by the second time-derivative of the magnetic moment, $\ddot{\boldsymbol{\mu}}$. Here again the gravitational analog is a constant of the motion, the angular momentum,

$$\boldsymbol{\mu} = \sum_A (\text{position of } A) \times (\text{current due to } A) = \sum_A \boldsymbol{r}_A \times (m\boldsymbol{v}_A) = \boldsymbol{J};$$

so it cannot radiate. *Thus, there can be no gravitational dipole radiation of any sort.*

When one turns to quadrupole radiation, one finally gets a nonzero result (see Figure 36.1). The power output predicted by electromagnetic theory,

$$L_{\text{electric quadrupole}} = \frac{1}{20}\dddot{Q}^2 \equiv \frac{1}{20}\dddot{Q}_{jk}\dddot{Q}_{jk},$$

$$Q_{jk} \equiv \sum_A e_A\left(x_{Aj}x_{Ak} - \frac{1}{3}\delta_{jk}r_A{}^2\right)$$

($Q_{jk\,\text{here}} = Q_{jk\,\text{in much other literature}}$), has as its gravitational counterpart

$$L_{\text{mass quadrupole}} = \frac{1}{5}\langle \dddot{I\!\!\!-}^2\rangle \equiv \frac{1}{5}\langle \dddot{I\!\!\!-}_{jk}\dddot{I\!\!\!-}_{jk}\rangle, \tag{36.1}$$

$$I\!\!\!-_{jk} \equiv \sum_A m_A\left(x_{Aj}x_{Ak} - \frac{1}{3}\delta_{jk}r_A{}^2\right) = \int \rho\left(x_j x_k - \frac{1}{3}\delta_{jk}r^2\right)d^3x. \tag{36.2}$$

Gravitational-wave power output expressed in terms of "reduced quadrupole moment" of source

Formula (36.1) contains the correct factor of 1/5, which comes from tensor calculations (see §36.10), instead of the incorrect factor 1/20 suggested by the electromagnetic analog; and the righthand side of (36.1) has been averaged ("⟨ ⟩") over several characteristic periods of the source to accord with one's inability to localize the energy of gravitational radiation inside a wavelength.

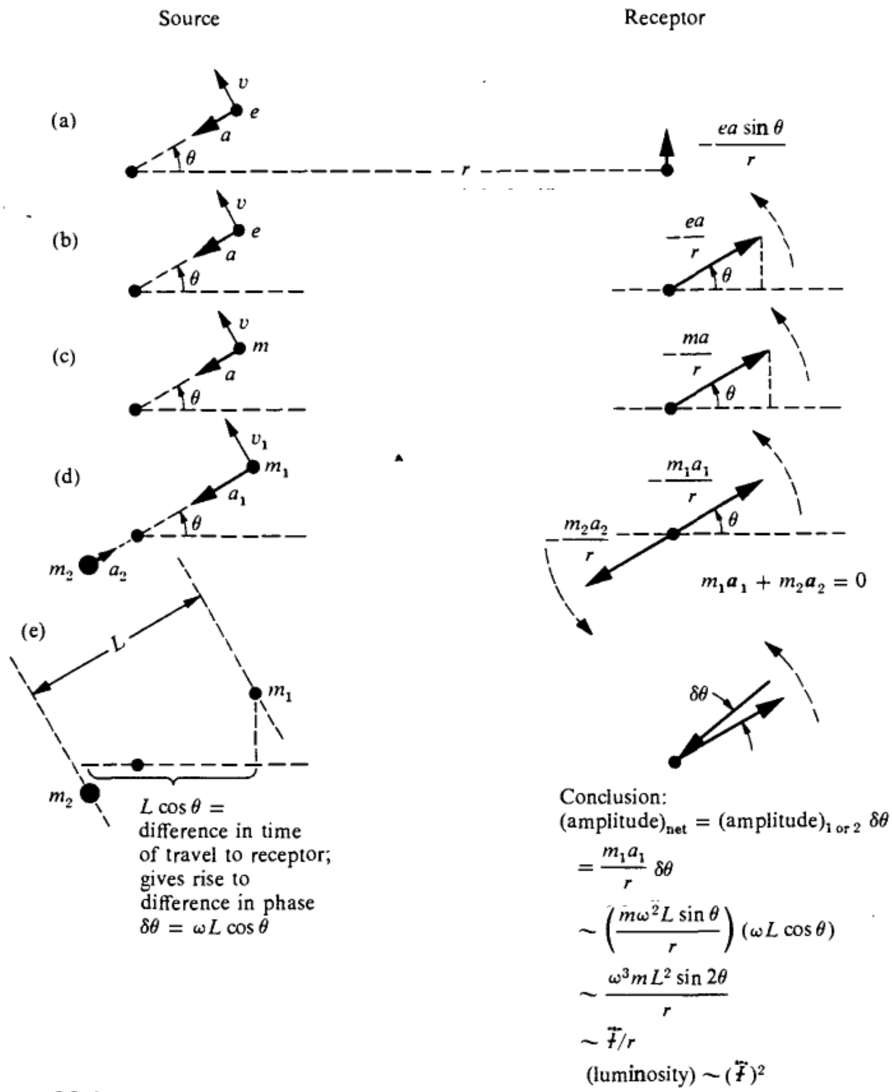

Figure 36.1.

Why gravitational radiation is ordinarily weak. In brief, contributions to the amplitude of the outgoing wave from the mass dipole moments of the separate masses cancel, $(m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2)/r = 0$ (principle that action equals reaction).

(a) Radiation from an accelerated charge (see §4.4 and Figure 4.6).

(b) Representation of the field at the great distance r in terms of the typical rotating-vector diagram of electrical engineering; however, here, for ease of visualization, the *vertical* projection of the rotating vector gives the observed field (usual dipole-radiation field produced by a charge in circular orbit).

(c) Corresponding rotating-vector diagram for gravitational radiation, based on the simplified model of the gravitation field as a spin-one or vector field (to be contrasted with its true tensor character; hence details of angular distribution and total radiation as given by this simple diagram are not correct; but order of magnitude of luminosity is correct).

(d) The two masses m_1 and m_2 that hold each other in orbit give equal and opposite contributions to the amplitude of the outgoing wave because of the principle that action equals reaction. (In electromagnetic radiation from a hydrogen atom, the corresponding radiation amplitudes do not cancel: $e_{\text{elec}}\boldsymbol{a}_{\text{elec}} + e_{\text{prot}}\boldsymbol{a}_{\text{prot}} \sim e_{\text{elec}}\boldsymbol{a}_{\text{elec}} \neq 0$).

(e) In a better approximation, one has to allow for the difference in time of arrival at the receptor of the effects from the two masses. The two vectors that formerly opposed each other exactly are now drawn inclined, at the phase angle $\delta\theta$. The amplitude of the resulting field goes as $\dddot{I\!\!\!-}$, where $I\!\!\!-$ is the reduced quadrupole moment; and the luminosity is proportional to $\dddot{I\!\!\!-}^2$.

Notation: There is no ambiguity about the definition of the *"second moment of the mass distribution"* as it appears throughout the physics and mathematics literature

$$I_{jk} \equiv \int \rho x_j x_k \, d^3x.$$

Nor is there any ambiguity about how one constructs the moment of inertia tensor $\mathcal{I}_{jk}$ from this second moment of the mass distribution:

$$\mathcal{I}_{jk} = \delta_{jk} \text{ trace } (I_{ab}) - I_{jk} = \int \rho(r^2 \, \delta_{jk} - x_j x_k) \, d^3x.$$

The moments that characterize a source radiating quadrupole gravitational radiation are here taken, equally unambiguously, to be the *"trace-free part of the second moment of the mass distribution"*:

$$\text{I}\!\!\!-_{jk} = I_{jk} - \frac{1}{3} \delta_{jk} \text{ trace } (I_{ab}) = I_{jk} - \frac{1}{3} \delta_{jk} I = \int \rho(x_j x_k - \frac{1}{3} \delta_{jk} r^2) \, d^3x. \quad (36.3)$$

This notation is adopted because it simplifies formulas, it simplifies calculations, it meshes well with much of the literature of gravitational-wave theory [e.g. Peters (1964), Peres and Rosen (1964)], and it is easy to remember. Another name for the quantities $\text{I}\!\!\!-_{jk}$ is *reduced quadrupole moment.* This terminology makes clear the distinction between the quantities used here and the three-times-larger quantities that are called quadrupole moments in the standard text of Landau and Lifshitz (1962) and in the literature on nuclear quadrupole moments, and the 3/2-times-larger quantities used in the theory of spherical harmonics:

$$Q_{zz} \begin{pmatrix} \text{Landau and Lifshitz; also} \\ \text{nuclear quadrupole moments} \end{pmatrix} = \int \rho(3z^2 - r^2) \, d^3x,$$

$$Q_{zz} \text{ (theory of spherical harmonics)} = \int \rho\left(\frac{3}{2} z^2 - \frac{1}{2} r^2\right) d^3x,$$

$$\text{I}\!\!\!-_{zz} \begin{pmatrix} \text{reduced quadrupole moment;} \\ \text{unambiguous measure of} \\ \text{source strength adopted here} \end{pmatrix} = \int \rho\left(z^2 - \frac{1}{3} r^2\right) d^3x.$$

Thus the $\text{I}\!\!\!-_{jk}$ notation has the merit of circumventing the existing ambiguity in the literature.

Why gravitational waves are ordinarily quadrupolar

That electromagnetic radiation is predominantly dipolar (spherical-harmonic index $l = 1$), and gravitational radiation is quadrupolar ($l = 2$) are consequences of a general theorem. Consider a classical radiation field, whose associated quantum mechanical particles have integer spin S, and zero rest mass. Resolve that radiation field into spherical harmonics—i.e., into multipole moments. All components with $l < S$ will vanish; in general those with $l \geq S$ will not; and this is independent of the nature of the source! [See, e.g., Couch and Newman (1972).] Since the lowest nonvanishing multipoles generally dominate for a slowly moving source (speeds $\ll c$), electromagnetic radiation ($S = 1$) is ordinarily dipolar ($l = S = 1$), while gravitational radiation ($S = 2$) is ordinarily quadrupolar ($l = S = 2$). Closely connected with this theorem is the "topological fixed-point theorem" [e.g., Lifshitz (1949)], which distinguishes between scalar, vector, and tensor fields. For a scalar disturbance, such as a pressure wave, there is no difficulty in having a spherically symmetric source. Thus, over a sphere of a great radius r, there is no difficulty in having a pressure field that everywhere, at any one time, takes on the same value p. In contrast, there is no way to lay down on the surface of a 2-sphere a continuous vector field, the magnitude of which is non-zero and everywhere the same ("no way to comb smooth the hair on the surface of a billiard ball"). Likewise, there is no way to lay down on the surface of a 2-sphere a continuous non-zero transverse-traceless 2×2 matrix field that differs from one point to another at most by a rotation. Topology thus excludes the possibility of any spherically symmetric source of gravitational radiation whatsoever.

§36.2. POWER RADIATED IN TERMS OF INTERNAL POWER FLOW

Expression (36.1) for the power output can be rewritten in a form that is easier to use in order-of-magnitude estimates. Notice that the reduced quadrupole moment is

$$\dddot{I\!\!\!-}_{jk} \sim \frac{\begin{pmatrix}\text{mass of that part of}\\ \text{system which moves}\end{pmatrix} \times \begin{pmatrix}\text{size of}\\ \text{system}\end{pmatrix}^2}{\begin{pmatrix}\text{time for masses to move from}\\ \text{one side of system to other}\end{pmatrix}^3} = \frac{MR^2}{T^3}$$

$$\sim \frac{M(R/T)^2}{T} \sim \frac{\begin{pmatrix}\text{nonspherical part}\\ \text{of kinetic energy}\end{pmatrix}}{T};$$

$$\dddot{I\!\!\!-}_{jk} \sim L_{\text{internal}} \equiv \begin{pmatrix}\text{power flowing from one}\\ \text{side of system to other}\end{pmatrix}: \tag{36.4}$$

Gravitational-wave power output in terms of internal power flow of source

Consequently, equation (36.1) says that *the power output in gravitational waves ("luminosity") is roughly the square of the internal power flow*

$$L_{\text{GW}} \sim (L_{\text{internal}})^2. \tag{36.5}$$

If this equation seems crazy (who but a fool would equate a power to the square of a power?), recall that in geometrized units power is dimensionless. The conversion factor to conventional units is

$$L_o \equiv c^5/G = 3.63 \times 10^{59}\ \text{erg/sec} = 2.03 \times 10^5 M_\odot c^2/\text{sec}. \tag{36.6}$$

One may freely insert this factor of $L_o = 1$ wherever one wishes in order to feel more comfortable with the appearance of the equations. For example, one can rewrite equation (36.5) in the form

$$L_{\text{GW}}/L_{\text{internal}} \sim L_{\text{internal}}/L_o. \tag{36.7}$$

In applying the equation $L_{GW} \sim (L_{\text{internal}})^2$, one must be careful to ignore those internal power flows that cannot radiate at all, i.e., those that do not accompany a time-changing quadrupole moment. For example, in a star the internal power flows associated with spherical pulsation and axially symmetric rotation must be ignored.

Conservation of energy guarantees that radiation reaction forces will pull down the internal energy of the system at the same rate as gravitational waves carry energy away (see Box 19.1). The characteristic time-scale for radiation reaction to change the system markedly is

Characteristic time-scale for radiation-reaction effects

$$\begin{aligned}\tau_{\text{react}} &\sim [1/(\text{rate at which energy is lost})] \times [\text{energy in motions that radiate}] \\ &\sim [1/L_{GW}] \times [(L_{\text{internal}}) \times (\text{characteristic period } T \text{ of internal motions})] \\ &\sim (L_{\text{internal}}/L_{GW})T \sim (L_0/L_{\text{internal}})T. \qquad (36.8)\end{aligned}$$

Consequently, *radiation reaction is important in one characteristic period only if the system achieves the enormous internal power flow*

$$L_{\text{internal}} \gtrsim L_o = 3.63 \times 10^{59}\ \text{ergs/sec} = 1!$$

§36.3. LABORATORY GENERATORS OF GRAVITATIONAL WAVES

As a laboratory generator of gravitational waves, consider a massive steel beam of radius $r = 1$ meter, length $l = 20$ meters, density $\rho = 7.8$ g/cm^3, mass $M = 4.9 \times 10^8$ g (490 tons), and tensile strength $\mathrm{t} = 40{,}000$ pounds per square inch or 3×10^9 dyne/cm^2. Let the beam rotate about its middle (so it rotates end over end), with an angular velocity ω limited by the balance between centrifugal force and tensile strength

Power output from a rotating steel beam

$$\omega = (8\mathrm{t}/\rho l^2)^{1/2} = 28\ \text{radians/sec}.$$

The internal power flow is

$$\begin{aligned}L_{\text{internal}} &= \left(\frac{1}{2} I\omega^2\right)\omega = \frac{1}{24} Ml^2\omega^3 \\ &\approx 2 \times 10^{18}\ \text{erg/sec} \approx 10^{-41} L_o.\end{aligned}$$

The order of magnitude of the power radiated is

$$L_{GW} \sim (10^{-41})^2 L_o \sim 10^{-23}\ \text{erg/sec}. \qquad (36.9)$$

(An exact calculation using equation (36.1) gives 2.2×10^{-22} erg/sec; see Exercise 36.1.) Evidently the construction of a laboratory generator of gravitational radiation is an unattractive enterprise in the absence of new engineering or a new idea or both.

To rely on an astrophysical source and to build a laboratory or solar-system detector is a more natural policy to consider. Detection will be discussed in the next chapter. Here attention focuses on astrophysical sources.

EXERCISE

Exercise 36.1. GRAVITATIONAL WAVES FROM ROTATING BEAM

A long steel beam of length l and mass M rotates end over end with angular velocity ω. Show that the power it radiates as gravitational waves is

$$L_{\mathrm{GW}} = \frac{2}{45} M^2 l^4 \omega^6. \tag{36.10}$$

Use this formula to verify that the rod described in the text radiates 2.2×10^{-22} ergs/sec.

§36.4. ASTROPHYSICAL SOURCES OF GRAVITATIONAL WAVES: GENERAL DISCUSSION

Consider a highly dynamic astrophysical system (a star pulsating and rotating wildly, or a collapsing star, or an exploding star, or a chaotic system of many stars). If its mass is M and its size is R, then according to the virial theorem (exercise 39.6) its kinetic energy is $\sim M^2/R$. The characteristic time-scale for mass to move from one side of the system to the other, T, is

$$T \sim \frac{R}{(\text{mean velocity})} \sim \frac{R}{(M/R)^{1/2}} = \left(\frac{R^3}{M}\right)^{1/2} \tag{36.11a}$$

($\sim$ time of free fall; $\sim$ time to turn one radian in Kepler orbit; see Chapter 25). Consequently, the internal power flow is

$$L_{\text{internal}} \sim \frac{(\text{kinetic energy})}{T} \sim \left(\frac{M^2}{R}\right)\left(\frac{M}{R^3}\right)^{1/2} \sim \left(\frac{M}{R}\right)^{5/2}. \tag{36.11b}$$

The gravitational-wave output or "luminosity" is the square of this quantity, or

Power output from violent astrophysical sources, in terms of mass and radius

$$L_{\mathrm{GW}} \sim (M/R)^5 L_o. \tag{36.11c}$$

(If the system is rather symmetric, or if only a small portion of its mass is in motion, then its quadrupole moment does not change much, and the estimate of L_{GW} must be reduced accordingly. The wave amplitude goes down in proportion to the fraction of the mass in motion, and the power is reduced in proportion to the square of that fraction.)

Clearly, *the maximum power output occurs when the system is near its gravitational radius;* and because nothing, not even gravitational waves, can escape from inside the gravitational radius, *the maximum value of the output is $\sim L_o = 3.63 \times 10^{59}$ ergs/second, regardless of the nature of the system!*

Upper limit on power output

Actually, the above derivation of this limit and of equation (36.11c) uses approximations to general relativity that break down near the gravitational radius. [Velocities small compared to light are required in deriving the standard formula (36.1) for

L_{GW} (see §36.7); nearly Newtonian fields are required for the virial theorem arguments of (36.11a), as well as for the L_{GW} formula.] Nevertheless, in rough order of magnitude, equation (36.11c) is valid to quite near the Schwarzschild radius, say, $R \sim 3M$; and inside that point gravity is so strong that no system can resist collapse for an effective length of time much longer than $T \sim M$.

Radiation reaction in astrophysical sources

The time required for radiation-reaction forces to affect a system substantially [equation (36.8)] is of the order

$$\tau_{\text{react}} \sim (L_o/L_{\text{internal}})T \sim (R/M)^{5/2}T, \tag{36.11d}$$

where T is the characteristic time (36.11a) of rotation or free fall. (Note how one inserts and removes the factor $L_o = 1$ at will!) Consequently, *the effect of radiation reaction, as integrated over one period, is unimportant except when the system is near its gravitational radius.*

When a system such as a pulsating star is settling down into an equilibrium state, the radiation reaction will damp its internal motions. On the other hand, when the system, like a binary star system, is far from any state of equilibrium, then loss of energy (and angular momentum) to radiation under certain circumstances may speed up the angular velocity or speed up the internal motions and augment the radiation.

§36.5. GRAVITATIONAL COLLAPSE, BLACK HOLES, SUPERNOVAE, AND PULSARS AS SOURCES

Gravitational waves from:
(1) stellar collapse and formation of a black hole

Since $L_{GW} \sim (M/R)^5 L_o$, the most intense gravitational waves reaching Earth must come from a dynamic, deformed system near its gravitational radius (L_{GW} drops by a factor 100,000 with each increase by 10 of R!). The scenario of Figure 24.3 gives an impression of some of the dynamic processes that not only may happen but probably must happen. The sequence of events sketched out there includes pulses of gravitational radiation interspersed with intervals of continuous radiation of gradually increasing frequency. Pulse number one comes at the time of the original collapse of a star with white-dwarf core to a pancake-shaped neutron star. The details of what then goes on will differ enormously depending on the original mass and angular momentum of this "pancake." In the illustration, this pancake fragments into a constellation of corevolving neutron stars, which then one by one undergo "pursuit and plunge."

Whether in this kind of scenario or otherwise, perhaps the most favorable source of gravitational radiation is a star (the original very temporary "pancake" or one of the fragments therefrom) collapsing through its gravitational radius in a highly nonspherical manner. Such a star should terminate life with a last blast of gravitational waves, which carry off a sizeable fraction of its rest mass. Thus an order-of-magnitude estimate gives

$$\begin{aligned}(\text{energy radiated}) = \int L_{GW}\, dt &\sim L_o \cdot \begin{pmatrix}\text{time during which}\\ \text{peak luminosity occurs}\end{pmatrix} \\ &\sim L_o M = M.\end{aligned} \tag{36.12}$$

(Whether the energy radiated is $0.9M$, or $0.1M$, or $0.01M$ is not known for certain today; but it must lie in this range of orders of magnitude.) The radiation should be weak at low frequencies; it should rise to a peak at a frequency a little smaller than $1/M$; and it should cut off sharply for circular frequencies above $\omega \sim 1/M$.

(2) the fall of debris into a black hole

Matter ("debris"; see Figure 24.3) falling into a black hole can also be a significant source of gravitational waves. The infalling matter will radiate only weakly when it is far from the gravitational radius; but as it falls through the gravitational radius (between $r \sim 4M$ and $r = 2M$), it should emit a strong burst. If m is the mass of an infalling lump of matter and M is the total mass of the black hole, then the total energy in the final burst is

$$E_{\text{radiated}} \sim m^2/M, \tag{36.13}$$

and it comes off in a time $\sim M$ with a power output of $L_{\text{GW}} \sim (m/M)^2 L_o$. (See exercise 36.2.) Actually, this is an extremely rough estimate of the energy output. In the limit where the infalling lump is small in both size and mass [(size of lump) $\ll$ (gravitational radius of black hole); $m \ll M$; "delta-function lump"], one can perform an exact calculation of the spectrum and energy radiated by treating the lump and the waves as small perturbations on the Schwarzschild geometry of the black hole. The foundations for such a treatment were given by Zerilli (1970b). Zerilli's formula was corrected and applied to the case of head-on impact by Davis, Ruffini, Press, and Price (1971). They predict the spectrum of Figure 36.2 and the total energy output

$$E_{\text{radiated}} = 0.0104 m^2/M \tag{36.14}$$

for $m \ll M$ and (size of lump) $\ll M$.

(3) collisions of black holes

A collision between black holes should also produce a strong burst of gravitational waves—through such collisions are probably very rare!

(4) supernova explosions

Not quite so rare, but still not common, are supernova explosions (about one per galaxy per 100 years). According to current theory as verified by pulsar observations, a supernova is triggered by the collapse of the core of a highly evolved star (see §24.3). The collapse itself and the subsequent wild gyrations of the collapsed core (neutron star) should produce a short, powerful burst of gravitational waves. The characteristics of the burst, as estimated with formulas (36.11), and assuming large departures from sphericity, are

$$\begin{aligned}
&(\text{energy radiated}) \sim (\text{neutron-star binding energy}) \\
&\qquad\qquad \sim M^2/R \sim 0.1M \sim 10^{53}\ \text{ergs}, \\
&(\text{mean frequency}) \sim 1/T \sim (M/R^3)^{1/2} \sim 0.03M^{-1} \sim 3000\ \text{Hz}, \qquad (36.15) \\
&(\text{power output}) \sim (M/R)^5 L_o \sim 10^{-5} L_o \sim 3 \times 10^{54}\ \text{ergs/sec}, \\
&\begin{pmatrix}\text{time for gravitational} \\ \text{radiation to damp the} \\ \text{motion if turbulence,} \\ \text{heat conduction, and other} \\ \text{effects do not damp it} \\ \text{sooner}\end{pmatrix} = \tau \sim M(M/R)^{-4} \sim 0.1\ \text{sec} \sim 300\ \text{periods}.
\end{aligned}$$

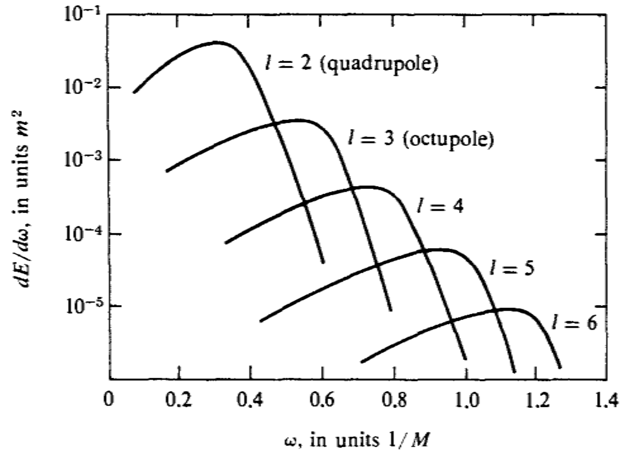

Figure 36.2.
Spectrum of the gravitational waves emitted by a "delta-function" lump of matter of mass m, falling head-on into a nonrotating (Schwarzschild) black hole of mass $M \gg m$. The total energy radiated is distributed among multipoles according to the empirical law

$$(\text{energy in } l\text{-pole waves}) \approx (0.44\ m^2/M)e^{-2l},$$

and the total spectrum peaks at angular frequency

$$\omega_{\max} = 0.32/M.$$

These results were calculated by treating the infalling lump and the gravitational waves as small perturbations on the Schwarzschild geometry of the black hole. The relevant perturbation-theory equations were derived by Zerilli (1970), and were solved numerically to give these results by Davis, Ruffini, Press, and Price (1971).

In the last stages of the stellar pulsations, when the amplitude $\xi = \delta r$ has dropped to $\delta r/r \ll 1$, one can calculate the pulsation frequencies and damping times exactly by treating the fluid motions and gravitational waves as small perturbations of an equilibrium stellar model. The results of such a calculation, which are in good agreement with the above rough estimates, are shown in Box 36.1.

Long after the pulsations of the neutron star have been damped out by gravitational radiation reaction and by other forces, the star will continue to rotate; and as it rotates, carrying along with its rotation an off-axis-pointing magnetic moment, it will beam out the radio waves, light, and x-rays that astronomers identify as "pulsar radiation." In this pulsar phase, gravitational radiation is important only if the star is somewhat deformed from axial symmetry (axial symmetry $\Longrightarrow$ constant quadrupole moment $\Longrightarrow$ no gravitational waves). According to estimates in exercise 36.3, a deformation that contains only 0.001 of the star's mass could radiate 10^{38} ergs per second for the youngest known pulsar (Crab nebula); and the accompanying radiation reaction could be a significant source of the pulsar's slowdown. However, it is not at all clear today (1973)—indeed, it seems unlikely—that the neutron star could support even so small a deformation. (5) young pulsars

(continued on page 986)

Box 36.1 GRAVITATIONAL WAVES FROM PULSATING NEUTRON STARS

The table given here, taken from Thorne (1969a), shows various characteristics of the quadrupole oscillations of several typical neutron-star models. Note that the gravitational waves emitted by the most massive models (1) have frequencies $\nu = 1/T_n \sim 3{,}000$ Hz, (2) last for a time of $\sim\frac{1}{3}$ second, (3) damp out the stellar vibrations after only $\sim$1,000 oscillations, and (4) carry off a total energy of $\sim(10^{54}\text{ ergs}) \times (\delta R/R)^2$, where $\delta R/R$ is the initial fractional amplitude of vibration of the star's surface.

These results are *not* based on the nearly Newtonian slow-motion formalism of this chapter [equation (36.1), §§36.7 and 36.8], because that formalism is invalid here: the reduced wavelength of the radiation, $\lambda\!\!\!^{-} \sim 15$ km for waves from the most massive star, is not large compared to the star's gravitational radius, $2M \sim 6$ km; and the star's internal gravitational field is not weak (M/R as large as 0.29). Consequently, these results were derived using an alternative technique, which is valid for rapid motions and strong internal fields, but which assumes small perturbations away from the equilibrium stellar model. See Thorne (1969a) and papers cited therein for details.

QUADRUPOLE PULSATIONS OF NEUTRON STARS

Equation of state	ρ_c ($g\ cm^{-3}$)	$M/M_\odot$	$2M/R$	n	T_n (msec)	τ_n (sec)	τ_n/T_n	$\frac{E_H}{\langle(\delta R/R)^2\rangle}$ (ergs)	Power $\frac{}{\langle(\delta R/R)^2\rangle}$ (ergs sec^{-1})	$\frac{(\delta R/R)}{(\delta r/r)_c}$	$\delta\theta_s/\delta\theta_c$
H-W	3×10^{14}	0.405	0.0574	0	1.197	13.	11000	7.8×10^{50}	1.2×10^{50}	+ 7.4	+ 3.1
H-W	6×10^{15}	0.682	0.240	0	0.3109	0.19	610	2.8×10^{52}	2.9×10^{53}	+ 5.2	+ 3.7
				1	0.1713	0.28	1600	3.6×10^{51}	2.6×10^{52}	− 14.	− 3.3
				2	0.1179	1.3	11000	2.6×10^{50}	3.9×10^{50}	+ 55.	+ 5.9
				3	0.0938	24.	250000	8.9×10^{48}	$7. \times 10^{47}$	−350.	−24.
V_γ	5.15×10^{14}	0.677	0.159	0	0.6991	1.7	2400	5.7×10^{52}	$7. \times 10^{52}$	+ 1.4	+ 1.3
				1	0.2358	11.	47000	6.0×10^{50}	1.1×10^{50}	− 38.	− 4.7
V_γ	3×10^{15}	1.954	0.580	0	0.3777	0.22	600	1.7×10^{54}	1.6×10^{55}	+ 1.9	+ 3.1
				1	0.1556	1.6	10000	1.5×10^{54}	1.9×10^{54}	− 2.1	− 0.66
				2	0.1026	2.6	25000	5.2×10^{53}	4.0×10^{53}	+ 2.9	+ 0.40

The columns in the table have the following meanings.

Equation of state: the equation of state $p(\rho)$ used in constructing the equilibrium stellar model and in calculating the adiabatic index from $\gamma = [(\rho + p)/p]\, dp/d\rho$; H-W is the Harrison-Wheeler equation of state in the tabular form given by Hartle and Thorne (1968), Table 1; V_γ is the Levinger-Simmons-Tsuruta-Cameron V_γ equation of state in the tabular form given by Hartle and Thorne (1968), Table 2.

ρ_c: central density of total mass-energy for the equilibrium stellar model.

$M/M_\odot$: total mass-energy of the equilibrium model (i.e., the mass that governs distant Keplerian orbits), in units of the sun's mass.

$2M/R = 2GM/Rc^2$: ratio of the gravitational radius of the equilibrium model to its actual radius (radii are defined by $4\pi R^2 =$ surface area).

n: the "order" of the pulsational normal mode under study (for all models given here, n is also the number of nodes in the radial relative eigenfunction, $\delta r/r$.). Note: $n = 0$ is the fundamental (quadrupole) mode.

$T_n = 2\pi/\omega_n$: the pulsation period of the quasinormal mode measured in milliseconds.

τ_n: the damping time for the amplitude of the normal mode measured in seconds.

$\tau_n/T_n = \omega_n\tau_n/2\pi$: the number of pulsation periods required for the amplitude to drop by a factor of $1/e$.

$E_{\mathfrak{M}}/\langle(\delta R/R)^2\rangle$: energy of pulsation of the star, divided by the square of the relative amplitude of radial motion of the star's surface averaged over its surface.

Power$/\langle(\delta R/R)^2\rangle$: the power radiated as gravitational waves, divided by the averaged square of the relative amplitude at the star's surface.

$(\delta R/R)(\delta r/r)_c^{-1}$: relative amplitude of radial motion at the star's surface divided by relative amplitude at the star's center.

$\delta\theta_s/\delta\theta_c = \delta\phi_s/\delta\phi_c$: amplitude of the angular displacement of the star's fluid at its surface divided by the same amplitude at its center.

Of the sources discussed in this section, most are "impulsive" rather than continuous (star collapsing through gravitational radius; debris falling into a black hole; collision between black holes; supernova explosion). They give rise to bursts of gravitational waves. An order-of-magnitude method of analyzing such bursts is spelled out in Box 36.2.

It is difficult and risky to pass from the above description of processes that should generate gravitational waves to an estimate of the characteristics of the waves that actually bathe the earth. For such an estimate, made in 1972 and subject to extensive revision as one's understanding of the universe improves, see Press and Thorne (1972).

EXERCISES

Exercise 36.2. GRAVITATIONAL WAVES FROM MATTER FALLING INTO A BLACK HOLE

A lump of matter with mass m falls into a black hole of mass M. Show that a burst of gravitational waves is emitted with duration $\sim M$ and power $L_{GW} \sim (m/M)^2 L_0$, so that the total energy radiated is given in crude order of magnitude by equation (36.13).

Exercise 36.3. GRAVITATIONAL WAVES FROM A VIBRATING NEUTRON STAR

Idealize a neutron star as a sphere of incompressible fluid of mass M and radius R, with structure governed by Newton's laws of gravity. Let the star pulsate in its fundamental quadrupole mode. Using Newtonian theory, calculate: the angular frequency of pulsation, ω; the energy of pulsation E_{puls}; the quantity $\frac{1}{5}\langle \dddot{I}^{\,2}\rangle$, which, according to equation (36.1), is the power radiated in gravitational waves, L_{GW}; and the e-folding time, $\tau = E_{\text{puls}}/L_{GW}$, for radiating away the energy of the pulsations. Compare the answers with equations (36.15)—which are based on a much cruder approximation—and with the results in Box 36.1, which are based on much better approximations. [For solution, see Table 13 of Wheeler (1966).]

Exercise 36.4. PULSAR SLOWDOWN

The pulsar NPO532 in the Crab Nebula has a period of 0.033 seconds and is slowing down at the rate $dP/dt = 1.35 \times 10^{-5}$ sec/yr. Assuming the pulsar is a typical neutron star, calculate the rate at which it is losing rotational energy. If this energy loss is due primarily to gravitational radiation reaction, what is the magnitude of the star's nonaxial deformation? [For solution, see Ferrari and Ruffini (1969); for a rigorous strong-field analysis, see Ipser (1970).]

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§36.6. BINARY STARS AS SOURCES

Binary stars as sources of gravitational waves:

The most numerous sources of weak gravitational waves are binary star systems. Moreover, roughly half of all stars are in binary or multiple systems [see, for example, the compilation of Allen (1962)]. According to Kepler's laws, two stars of masses m_1 and m_2 that circle each other have angular frequency ω and separation a coupled to each other by the formula

$$\omega^2 a^3 = m_1 + m_2 \equiv M.$$

Box 36.2 ANALYSIS OF BURSTS OF RADIATION FROM IMPULSE EVENTS*

	Electromagnetism	*Gravitation*
Typical moment relevant for radiation	$d_z(t)$	$\text{\it ł}_{zz}(t)$
Its Fourier transform	$(2\pi)^{-1/2}\int d_z \exp[i\omega t]\,dt$	$(2\pi)^{-1/2}\int \text{\it ł}_{zz} \exp[i\omega t]\,dt$
Name for this quantity	$d_z(\omega)$	$\text{\it ł}_{zz}(\omega)$
Time decomposition of total radiative energy loss ΔE	$c^{-3}\int \ddot{d}^2(t)\,dt$	$Gc^{-5}\int \dddot{\text{\it ł}}^2(t)\,dt$
Decomposition of ΔE according to circular frequency	$c^{-3}\int \ddot{d}^2(\omega)\,d\omega$	$Gc^{-5}\int \dddot{\text{\it ł}}^2(\omega)\,d\omega$
Integrand nearly constant with respect to ω from $\omega = 0$ up to a critical value of ω, beyond which radiation falls off very fast	$\omega_{\text{crit}} \sim 1/\Delta t$	$\omega_{\text{crit}} \sim 1/\Delta t$
$-d\,\Delta E/d\omega$ for $\omega < \omega_{\text{crit}}$	$\sim c^{-3}\ddot{d}^2(0)$	$\sim Gc^{-5}\dddot{\text{\it ł}}^2(0)$
Zero frequency moment that enters this formula	$\sim(e_1\,\Delta v_{z1} + e_2\,\Delta v_{z2})$	Δ(«Kinetic Energy»)$_{zz}$
Rewrite of $-d\,\Delta E/d\omega$	$\sim(e\,\Delta v)^2/c^3$	$\sim G[\Delta(\text{«K.E.»})_{zz}]^2/c^5$
Total energy of pulse	$\sim$This/Δt	$\sim$This/Δt

*Box adapted from pp. 113 and 114 of Wheeler (1962).

As sample applications of this analysis, Wheeler (1962) cites the following:

Parameter	*One atomic-nucleus fission of 180 MeV*	*Fission bomb yield 17 kilotons at 10% efficiency*	*Meteorite striking earth at escape velocity*	*Explosion of star when 10^{-4} of mass is released*
Mass	4×10^{-22} g	10^4 g	10^9 g	2×10^{33} g
Velocity	1.2×10^9 cm/s	4×10^8 cm/s	11×10^5 cm/s	4×10^8 cm/s
Energy	2.9×10^{-4} erg	7×10^{20} erg	6×10^{20} erg	1.8×10^{50} erg
Fraction assumed relevant to radiative moment	1	0.1	1	0.1
Time integral of this moment = «K.E.»$_{zz}$	2.9×10^{-4} erg	7×10^{19} erg	6×10^{20} erg	1.8×10^{49} erg
«K.E.»$_{zz}/c^2$	3.2×10^{-25} g	0.08 g	0.67 g	2×10^{28} g
$\dfrac{dE}{d\omega} \sim \dfrac{G}{c}\left(\dfrac{\text{«K.E.»}_{zz}}{c^2}\right)^2$	$2.3 \times 10^{-67}\dfrac{\text{erg}}{\text{rad/s}}$	$1.4 \times 10^{-20}\dfrac{\text{erg}}{\text{rad/s}}$	$1.0 \times 10^{-18}\dfrac{\text{erg}}{\text{rad/s}}$	$9 \times 10^{38}\dfrac{\text{erg}}{\text{rad/s}}$
Δt	10^{-21} s	10^{-8} s	10^{-3} s	10^4 s
$\Delta\omega \sim 1/\Delta t$	10^{21} rad/s	10^8 rad/s	10^3 rad/s	10^{-4} rad/s
$\Delta E_{\text{radiated}}$	10^{-46} erg	10^{-12} erg	10^{-15} erg	10^{35} erg
Assumed distance to detector	10^3 cm	10^3 cm	10^9 cm	10^{23} cm
$\Delta E/4\pi r^2$	10^{-53} erg/cm^2	10^{-19} erg/cm^2	10^{-34} erg/cm^2	10^{-12} erg/cm^2

The reader might find it informative to extend this table to the bursts of waves emitted by (1) debris falling into a black hole, (2) collisions between two black holes, and (3) a supernova explosion in which a star of two solar masses collapses to nuclear densities, ejecting half its mass in the process.

In this motion the kinetic energy is

$$\text{(kinetic energy)} = -\frac{1}{2}\text{(potential energy)} = \frac{1}{2}\frac{m_1 m_2}{a}.$$

The power that they radiate as gravitational waves can be estimated roughly as the square of the circulating power, $L \sim \omega \times$ (kinetic energy); thus,

$$L_{\mathrm{GW}} \sim \frac{\mu^2 M^3}{4a^5} L_o,$$

where $\mu = m_1 m_2/M$ is the familiar reduced mass, and $M = m_1 + m_2$ is the total mass of this binary system.

An *exact* calculation based on equation (36.1) gives a result larger than this by a factor ~ 30: for a binary system of semimajor axis a and eccentricity ε, the power output averaged over an orbital period is

(1) power output

$$L_{\mathrm{GW}} = \frac{32}{5}\frac{\mu^2 M^3}{a^5} f(\varepsilon) L_o, \qquad (36.16a)$$

where $f(\varepsilon)$ is the dimensionless "correction function,"

$$f(\varepsilon) = \left[1 + \frac{73}{24}\varepsilon^2 + \frac{37}{96}\varepsilon^4\right][1 - \varepsilon^2]^{-7/2}. \qquad (36.16b)$$

[See exercise 36.6 at end of §36.8; also Peters and Mathews (1963).]

(2) effects of radiation reaction

As the binary system loses energy by gravitational radiation, the stars spiral in toward each other (decrease of energy; tightening of gravitational binding). For circular orbits the energy, $E = -\frac{1}{2}m_1 m_2/a = -\frac{1}{2}\mu M/a$, decreases as

$$\begin{aligned} dE/dt &= 1/2(\mu M/a^2)(da/dt) \\ &= -L_{\mathrm{GW}} = -\frac{32}{5}\frac{\mu^2 M^3}{a^5}. \end{aligned}$$

Consequently, the evolution of the orbital radius is given by the formula

$$a = a_o(1 - t/\tau_o)^{1/4}, \qquad (36.17a)$$

where $a_o = a_{\mathrm{today}}$ and

$$\tau_o = \frac{1}{4}\left(\frac{-E}{L_{\mathrm{GW}}}\right)_{\mathrm{today}} = \frac{5}{256}\frac{a_o^4}{\mu M^2}. \qquad (36.17b)$$

Thus, unless nongravitational forces intervene, the two stars will spiral together in a time τ_o (*spiral time*). For an elliptical orbit, the eccentricity also evolves. Radiation is emitted primarily at periastron. Therefore the braking forces of radiation reaction act there with greatest force. This effect deprives the stars of some of the kinetic energy of the excursions in their separation ("radial kinetic energy"). In consequence, the orbit becomes more nearly circular. [See Peters and Mathews (1963) for detailed calculations.]

(3) particular binaries observed by astronomers

The calculated power output, flux at Earth, and damping times are shown in Box 36.3 for several known binary stars and several interesting hypothetical cases. Notice that in the most favorable known cases the period is a few hours; the damping time is the age of the universe (could the absence of better cases be due to radiation reaction's having destroyed them?); the output of power in the form of gravitational waves is $\sim 10^{30}$ to 10^{32} ergs/sec (approaching the light output of the sun, 3.9×10^{33} ergs/sec); and the calculated flux at the Earth is $\sim 10^{-10}$ to 10^{-12} ergs/sec (too small to detect in 1973, but perhaps not too small several decades hence; see Chapter 37).

The hypothetical cases in Box 36.3 illustrate the general relations for astrophysical systems that were derived in §36.4—namely, that only as the system approaches its gravitational radius can L_{GW} approach L_o, and only then can damping remove nearly the whole energy in a single period.

§36.7. FORMULAS FOR RADIATION FROM NEARLY NEWTONIAN SLOW-MOTION SOURCES

Turn now from illustrative astrophysical sources to rigorous formulas valid for a wide variety of sources. One such formula has already been written down,

$$L_{GW} = \frac{1}{5}\langle \dddot{I\!\!-}_{jk}\dddot{I\!\!-}_{jk}\rangle, \tag{36.1}$$

but it has not yet been derived, nor has its realm of validity been discussed.

The "nearly Newtonian, slow-motion approximation" for analyzing sources of gravitational waves

This formula for the power output is actually valid for any "nearly Newtonian, slow-motion source"—more particularly, for any source in which

$$\text{(size of source)/(reduced wavelength of waves)} \ll 1, \tag{36.18a}$$

$$|\text{Newtonian potential}| \ll \text{(size of source)/(reduced wavelength)}, \tag{36.18b}$$

$$\frac{|\text{typical stresses}|}{\text{(mass density)}} \ll \frac{\text{(size of source)}}{\text{(reduced wavelength)}}. \tag{36.18c}$$

It is not valid, except perhaps approximately, for fast-motion or strong-field sources. Moreover, there is no formalism available today which can handle effectively and *in general* the fast-motion case or the strong-field case.

The rest of this chapter is devoted to a detailed analysis of gravitational waves from nearly Newtonian, slow-motion sources. But the analysis (Track 2; §§36.9–36.11) will be preceded by a Track-1 summary in this section and the next.

For any source of size R and mean internal velocity v, the characteristic reduced wavelength ($\lambda\!\!\!^{-} = \lambda/2\pi$) of the radiation emitted is $\lambda\!\!\!^{-} \sim$ (amplitude of motions)$/v \lesssim R/v$. Consequently the demand (36.18a) that $R/\lambda\!\!\!^{-}$ be $\ll 1$ [i.e., that the source be confined to a small region deep inside the near (nonradiation) zone] enforces the slow-motion constraint

$$v \ll 1.$$

Box 36.3 GRAVITATIONAL RADIATION FROM SEVERAL BINARY STAR SYSTEMS[a]

Type of system	Name	Period	$\frac{m_1}{M_\odot}$	$\frac{m_2}{M_\odot}$	Distance from earth (pc)	Spiral time[b]	L_{GW} (ergs/sec)	Flux at earth (erg/sec cm^2)
Solar System (Sun + Jupiter)	Solar System	11.86 yr.	1.0	9.56×10^{-4}	Earth is in near zone	2.5×10^{23} yr	5.2×10^{10}	—
Typical resolved binaries from compilation of Van de Kamp (1958)	η Cas	480 yr.	0.94	0.58	5.9	9.5×10^{24}	5.6×10^{10}	1.4×10^{-29}
	ξ Boo	149.95 yr.	0.85	0.75	6.7	3.8×10^{23}	3.6×10^{12}	6.7×10^{-28}
	Sirius	49.94 yr.	2.28	0.98	2.6	7.2×10^{21}	1.1×10^{15}	1.3×10^{-24}
	Fu 46	13.12 yr.	0.31	0.25	6.5	3.2×10^{21}	3.6×10^{14}	7.1×10^{-26}
Typical eclipsing binaries from compilation of Gaposhkin (1958)	β Lyr	12.925 day	19.48	9.74	330	7.0×10^{11}	0.057×10^{30}	0.0004×10^{-11}
	UWCMa	4.395 day	40.0	31.0	1470	8.2×10^{9}	$49. \times 10^{30}$	0.019×10^{-11}
	β Per	2.867 day	4.70	0.94	30	3.2×10^{11}	0.014×10^{30}	0.013×10^{-11}
	WUMa	0.33 day	0.76	0.57	110	6.2×10^{9}	0.47×10^{30}	0.032×10^{-11}
Favorable cases from compilation of Braginsky (1965)	UV Leo	0.6 day	1.36	1.25	68	1.0×10^{10}	0.63×10^{30}	0.012×10^{-11}
	V Pup	1.45 day	16.6	9.8	390	2.3×10^{9}	$65. \times 10^{30}$	0.36×10^{-11}
	i Boo	0.268 day	1.35	0.68	12	2.0×10^{9}	3.2×10^{30}	$18. \times 10^{-11}$
	YY Eri	0.321 day	0.76	0.50	42	6.6×10^{9}	0.42×10^{30}	0.20×10^{-11}
	SW Lac	0.321 day	0.97	0.83	75	3.5×10^{9}	1.5×10^{30}	0.21×10^{-11}
	WZ Sge	81 min	0.6	0.03	100	1.1×10^{9} yr	0.5×10^{30}	0.04×10^{-11}
Hypothetical binaries (neutron stars or black holes)	10^4 km	12.2 sec	1.0	1.0	1000	3.2 yr	3.25×10^{41}	2.7×10^{-3}
	10^3 km	0.39 sec	1.0	1.0	1000	2.8 hr	3.25×10^{46}	2.7×10^{2}
	10^2 km	12.2 msec	1.0	1.0	1000	1.0 sec	3.25×10^{51}	2.7×10^{7}
	10 km	0.39 msec	1.0	1.0	1000	0.10 msec	3.25×10^{56}	2.7×10^{12}

[a] Based on tables by Braginsky (1965) and by Ruffini and Wheeler (1971b).
[b] The spiral time, τ_0, as given by equation (36.17b) is the time for the two stars to spiral into each other if no nongravitational forces intervene.

These related conditions, $v \ll 1$ and $R \ll \lambda\!\!\!^{-}$, are satisfied by all presently conceived laboratory generators of gravitational waves. No one has seen how to bring a macroscopic mass up to a speed $v \sim 1$. These conditions are also satisfied by every gravitationally bound, nearly Newtonian system. Thus, for such a system of mass M, the condition for gravitational binding, $\frac{1}{2}Mv^2 \lesssim M^2/R$ guarantees that $v \lesssim (M/R)^{1/2} \ll 1$.

The conditions $M/R \ll R/\lambda\!\!\!^{-}$ and $|T^{jk}|/T^{00} \ll R/\lambda\!\!\!^{-}$ are satisfied by all nearly Newtonian sources of conceivable interest. Typical sources (e.g. binary stars) have

$$\frac{M}{R} \sim \frac{|T^{jk}|}{T^{00}} \sim \left(\frac{R}{\lambda\!\!\!^{-}}\right)^2 \ll \frac{R}{\lambda\!\!\!^{-}}$$

(virial theorem). In those rare cases where (M/R or $|T^{jk}|/T^{00}) \gtrsim R/\lambda\!\!\!^{-}$ (e.g., a marginally stable, slowly vibrating star), the motion is so very slow that the radiation will be too weak to be interesting.

For any nearly Newtonian slow-motion system, there is a spacetime region deep inside the near zone ($r \ll \lambda\!\!\!^{-}$), but outside the boundary of the source ($r > R$), in which vacuum Newtonian gravitation theory is nearly valid. An observer in this Newtonian region can measure the Newtonian potential Φ and can expand it in powers of $1/r$:

Definitions of mass, dipole moment, and reduced quadrupole moment for a slow-motion source

$$\Phi = -\left(\frac{M}{r} + \frac{d_j n^j}{r^2} + \frac{3 I\!\!\!-_{jk} n^j n^k}{2r^3} + \cdots\right), \quad \text{where } n^j = x^j/r. \tag{36.19a}$$

He can then give names to the coefficients in this expansion:

$$\begin{aligned} M &\equiv \text{“total mass-energy”} = \text{“active gravitational mass”};\\ d_j &\equiv \text{“dipole moment” [if he chooses the origin of coordinates carefully, he can make } d_j = 0];\\ I\!\!\!-_{jk} &\equiv \text{“reduced quadrupole moment” [because the system is nearly Newtonian, } I\!\!\!-_{jk} \text{ is given by expression (36.3)].}\end{aligned} \tag{36.19b}$$

As this Newtonian potential reaches out into the radiation zone, the static portions of it ($-M/r - d_j n^j/r^2$) maintain their Newtonian form, unchanged. But the dynamic part ($-\frac{3}{2} I\!\!\!-_{jk} n^j n^k/r^3$) ceases to be describable in Newtonian terms. As retardation effects become noticeable (at increasing r values), it gradually changes over into outgoing gravitational waves, which must be described in the full general theory of relativity, or in linearized theory, or in the "shortwave" approximation of §35.13.

If one chooses to use linearized theory in the radiation zone, and if one imposes the transverse-traceless gauge there ($h^{TT}_{0\mu} = 0$, $h^{TT}_{jj} = 0$, $h^{TT}_{jk,k} = 0$), then the gravitational waves take the form [derived later as equation (36.47)]

Properties of gravitational waves in terms of reduced quadrupole moment:

$$h^{TT}_{jk} = \frac{2}{r}\ddot{I\!\!\!-}^{TT}_{jk}(t-r) + \text{corrections of order}\left[\frac{1}{r^2}\dot{I\!\!\!-}^{TT}_{jk}(t-r)\right]. \tag{36.20}$$

(1) the wave field h^{TT}_{jk}

Here $\ddot{\mathcal{I}}_{jk}^{TT}$ is the second time-derivative of the transverse-traceless part of the quadrupole moment (transverse to the radial direction; see §35.4); thus,

$$\ddot{\mathcal{I}}_{jk}^{TT} = P_{ja}\ddot{\mathcal{I}}_{ab}P_{bk} - \frac{1}{2}P_{jk}P_{ab}\ddot{\mathcal{I}}_{ab},$$

$$P_{ab} = (\delta_{ab} - n_a n_b) \qquad \text{(projection operator)}, \tag{36.21}$$

$$n_a = x^a/r \qquad \text{(unit radial vector)}.$$

The effective stress-energy tensor for these outgoing waves (§35.7) has the same form as for a swarm of zero-mass particles traveling radially outward with the speed of light; at large distances its components of lowest nonvanishing order are

(2) effective stress-energy tensor

$$T_{00}^{(\mathrm{GW})} = -T_{0r}^{(\mathrm{GW})} = T_{rr}^{(\mathrm{GW})} = \frac{1}{32\pi}\langle h_{jk,0}^{TT}h_{jk,0}^{TT}\rangle = \frac{1}{8\pi r^2}\langle \dddot{\mathcal{I}}_{jk}^{TT}\dddot{\mathcal{I}}_{jk}^{TT}\rangle$$

$$= \frac{1}{8\pi r^2}\left\langle \dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}_{jk} - 2n_i\dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}_{jk}n_k + \frac{1}{2}(n_j\dddot{\mathcal{I}}_{jk}n_k)^2\right\rangle, \tag{36.22}$$

where $\langle\ \ \rangle$ denotes an average over several wavelengths. (Recall that one cannot localize the energy more closely than a wavelength!) The total power crossing a sphere of radius r at time t is

(3) total power radiated

$$L_{\mathrm{GW}}(t,r) = \int T^{(\mathrm{GW})0r}r^2\,d\Omega = \frac{1}{5}\langle \dddot{\mathcal{I}}_{jk}(t-r)\dddot{\mathcal{I}}_{jk}(t-r)\rangle. \tag{36.23}$$

(See exercise 36.9.) This is the formula with which this chapter began: equation (36.1).

The wave fronts are not precisely spherical. For example, for a binary star system the wave fronts in the equatorial plane must be spirals. This means that there is a tiny nonradial component of the momentum flux, which decreases in strength as $1/r^3$. Associated with this nonradial momentum is an angular momentum density (angular momentum relative to the system's center, $r = 0$), which drops off as $1/r^2$ [Peters (1964), as corrected by DeWitt (1971), p. 286]:

(4) density of angular momentum

$$\mathcal{J}^i = \frac{1}{8\pi r^2}\epsilon^{ijk}\langle -6n_j\ddot{\mathcal{I}}_{km}\dddot{\mathcal{I}}_{mp}n_p + 9n_j\ddot{\mathcal{I}}_{km}n_m n_p\dddot{\mathcal{I}}_{pq}n_q\rangle. \tag{36.24}$$

The integral of this quantity over a sphere is the total angular momentum being transported outward per unit time,

(5) total angular momentum radiated

$$-dJ_j/dt = \int \mathcal{J}_j r^2\,d\Omega = \frac{2}{5}\epsilon^{jkl}\langle \ddot{\mathcal{I}}_{ka}\dddot{\mathcal{I}}_{al}\rangle. \tag{36.25}$$

(See exercise 36.9.)

§36.8. RADIATION REACTION IN SLOW-MOTION SOURCES*

The conservation laws discussed in Box 19.1 and derived in §20.5 guarantee that the source must lose energy and angular momentum at the same rate as the gravitational waves carry them off. The agent that produces these losses is a tiny component of the spacetime curvature inside the source, which reverses sign if one changes from a (realistic) outgoing-wave boundary condition at infinity to the opposite (unrealistic) ingoing-wave condition. These "radiation-reaction" pieces of the curvature can be described in Newtonian language when the source obeys the nearly Newtonian, slow-motion conditions (36.18).

Outgoing-wave boundary condition gives rise to a Newtonian-type radiation-reaction potential

The dynamical part of the Newtonian potential, in its "standard form"

$$\Phi = -\frac{3}{2}\,ɫ_{jk}(t)n_j n_k/r^3 + O(1/r^4);\ \text{equation (36.18)},$$

has no retardation in it. (Newtonian theory demands action at a distance!) Consequently, there is no way whatsoever for the standard potential to decide, at large radii, whether to join onto outgoing waves or onto ingoing waves. Being undecided, it takes the middle track of joining onto standing waves (half outgoing, plus half ingoing). But this is not what one wants. It turns out (see §36.11) that the join can be made to purely outgoing waves if and only if Φ is augmented by a tiny "radiation-reaction" potential

Form and magnitude of the radiation-reaction potential

$$\Phi = \Phi_{\text{standard Newtonian theory}} + \Phi^{(\text{react})}, \tag{36.26a}$$

$$\Phi^{(\text{react})} = \frac{1}{5}\frac{d^5 ɫ_{jk}}{dt^5}x^j x^k. \tag{36.26b}$$

If, instead, one sets $\Phi = \Phi_{\text{standard}} - \Phi^{(\text{react})}$, the potential will join onto purely ingoing waves.

In order of magnitude, the radiation-reaction potential is

$$\Phi^{(\text{react})} \sim \frac{1}{ƛ^5}(MR^2)r^2 \sim \frac{MR^2}{r^3}\left(\frac{r}{ƛ}\right)^5. \tag{36.27}$$

Consequently, near the source it is tiny compared to the standard Newtonian potential [a factor $(R/ƛ)^5 \sim v^5$ smaller!]. However, at the inner boundary of the radiation zone ($r \sim ƛ$), it is of the same order of magnitude as the dynamic, quadrupole part of the standard potential.

Effects of the potential:

The radiation-reaction part of the Newtonian potential plays the same role as a producer of accelerations that any other part of the Newtonian potential does. Any particle in the Newtonian region experiences a gravitational acceleration given by

(1) radiation-reaction accelerations

$$a_j = -\Phi_{,j} = -\Phi_{\text{standard},j} - \Phi^{(\text{react})}_{,j}. \tag{36.28}$$

*The ideas and formalism described in this section were devised by Burke (1970), Thorne (1969b), and Chandrasekhar and Esposito (1970). Among the forerunners of these ideas were the papers of Peters (1964), and Peres and Rosen (1964).

Inside the source, this acceleration leads to energy and angular momentum losses given by

$$dE/dt = \int \rho a_j v_j \, d^3x \tag{36.29a}$$

and

$$dJ_j/dt = \int \varepsilon_{jkl} x_k \rho a_l \, d^3x. \tag{36.29b}$$

(Here ρ is the density, v_j is the velocity, and a_j as above is the acceleration of the matter in the source.) Standard Newtonian theory conserves the energy and angular momentum. Therefore only the reaction part of the potential can produce losses:

(2) loss of energy and angular momentum

$$\begin{aligned} dE/dt &= -\int \rho \Phi^{(\text{react})}_{,j} v_j \, d^3x, \\ dJ_j/dt &= -\int \varepsilon_{jkl} x_k \rho \Phi^{(\text{react})}_{,l} \, d^3x. \end{aligned} \tag{36.30}$$

A straightforward calculation (exercise 36.5) using expression (36.26b) for the reaction potential yields, for the time-averaged losses,

$$\begin{aligned} dE/dt &= -\frac{1}{5}\langle \dddot{I\!\!\!-}_{jk} \dddot{I\!\!\!-}_{jk} \rangle, \\ dJ_j/dt &= -\frac{2}{5}\varepsilon_{jkl}\langle \ddot{I\!\!\!-}_{ka} \dddot{I\!\!\!-}_{al} \rangle. \end{aligned} \tag{36.31}$$

Notice that these results agree with the energy and angular momentum carried off by the radiation as given by equations (36.1) and (36.25). The agreement is an absolute imperative. The laws of conservation of total energy and angular momentum demand it.

A slow-motion electromagnetic system emitting electric dipole radiation has a radiation-reaction potential

Radiation-reaction potential for electromagnetic waves

$$A_j^{(\text{react})} = 0, \qquad A_0^{(\text{react})} = -\Phi^{(\text{react})} = \frac{2}{3}\dddot{d}_j x^j, \tag{36.32}$$

which is completely analogous to $\Phi^{(\text{react})}$ of gravitation theory [see, e.g., Burke (1971)]. However, attention does not usually focus on this potential and the reaction forces it produces. Instead, it focuses on the reaction force in a special case: that of an isolated charge being accelerated by nonelectromagnetic forces. For such a charge, the reaction force is

$$\boldsymbol{F}^{(\text{React})} = \frac{2}{3} e^2 \dddot{\boldsymbol{x}}. \tag{36.33}$$

No such formula is relevant to gravitation theory, because there is no such thing as a gravitationally isolated, radiating particle (i.e., one accelerated by forces that have no coupling to gravity).

EXERCISES

Exercise 36.5. ENERGY AND ANGULAR MOMENTUM LOSSES DUE TO RADIATION REACTION

Derive equations (36.31) for the rate at which gravitational radiation damping saps energy and angular momentum from a slow-motion source. Base the derivation on equations (36.26b) and (36.30).

Exercise 36.6. GRAVITATIONAL WAVES FROM BINARY STAR SYSTEMS

Apply the full formalism of §§36.7 and 36.8 to a binary star system with circular orbits. Calculate the angular distribution of the gravitational waves; the total power radiated; the total angular momentum radiated; the radiation-reaction forces; and the loss of energy and angular momentum due to radiation reaction. Compare the answers with the results quoted in §36.6. [For further details of the solution, see Peters and Mathews (1963).]

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§36.9. FOUNDATIONS FOR DERIVATION OF RADIATION FORMULAS

The rest of this chapter is Track 2. Chapter 20 (conservation laws) is needed as preparation for it. It will be helpful in Chapter 39 (post-Newtonian formalism), but is not needed as preparation for any other chapters.

Turn now from the formulas for radiation from a nearly Newtonian system in slow motion to a derivation of these formulas. Initially (this section) work in the full general theory of relativity without any approximations—not even that of slow motion. Impose only the constraint that the source be isolated, and that spacetime become asymptotically flat far away from it.

Derivation of formula for the gravitational-wave field produced by a slow-motion source:

Use a coordinate system that becomes asymptotically Lorentz as rapidly as spacetime curvature permits, when one moves radially outward from the source toward infinity. Everywhere in this coordinate system, even inside the source, which may be relativistic, define

$$h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}. \tag{36.34}$$

The $h_{\mu\nu}$ are clearly not the components of a tensor. Neither is $\eta_{\mu\nu}$ the true metric tensor. Nevertheless, one is free to raise and lower indices on $h_{\mu\nu}$ with $\eta_{\mu\nu}$ and to *define*

(1) definition of $\bar{h}_{\mu\nu}$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h; \qquad h = h_\alpha{}^\alpha = h_{\alpha\beta}\eta^{\alpha\beta}. \tag{36.35}$$

Moreover, one can always specialize the coordinates so that the four conditions

$$\bar{h}_\mu{}^\alpha{}_{,\alpha} = 0 \tag{36.36}$$

are exactly satisfied everywhere, including the interior of the source.

With these definitions and conventions, $\bar{h}_{\mu\nu}$ becomes the gravitational field of linearized theory far from the source, and also inside the source if gravity is weak there. But if the interior gravity is strong ($|\bar{h}_{\mu\nu}|$ not $\ll 1$), $\bar{h}_{\mu\nu}$ in the interior has no connection whatsoever to linearized theory.

2

The exact Einstein field equations can be written in terms of $\bar{h}^{\mu\nu}$ as [cf. §20.3; in particular, combine equations (20.14), (20.18), and (20.3); and impose the coordinate condition (36.36)]

(2) field equations in terms of $\bar{h}_{\mu\nu}$

$$\bar{h}^{\mu\nu}{}_{,\alpha\beta}\eta^{\alpha\beta} = -16\pi(T^{\mu\nu} + t^{\mu\nu}), \tag{36.37}$$

where $T^{\mu\nu}$ are the components of stress-energy tensor, and $t^{\mu\nu}$ are quantities (components of the "stress-energy pseudotensor for the gravitational field") that are of quadratic order and higher in $\bar{h}^{\mu\nu}$. Recall the "philosophy of controlled ignorance" expounded in §19.3. One is so ignorant that nowhere does one ever write down an explicit expression for $t^{\mu\nu}$ in terms of $\bar{h}^{\alpha\beta}$; and this ignorance is so controlled that one will never need such an expression in the calculations to follow! More specifically, the strength of the outgoing wave is proportional to the integral of a complicated expression over the interior of a system where "gravitational stresses" may be comparable to material stresses, $|t^{jk}| \sim |T^{jk}|$. No matter. All that will count for the radiation is the quadrupole part of the field. Moreover, that quadrupole moment is empirically definable by purely Newtonian measurements in the Newtonian region (1) well inside the wave zone, but (2) well outside the surface of the source. One does not have to know the inner workings of a star to define its mass (influence on Kepler orbits outside) nor does one have to know those inner workings to define its quadrupole moment *as sensed externally.*

(3) philosophy of controlled ignorance

Einstein's equations (36.37), augmented by an outgoing-wave boundary condition, are equivalent to the integral equations

(4) integral formulation of field equations

$$\bar{h}^{\mu\nu}(t, x^j) = 4\int_{\text{all space}} \frac{[T^{\mu\nu} + t^{\mu\nu}]_{\text{ret}}}{|x - x'|}\, d^3x', \tag{36.38}$$

where

$$|x - x'| \equiv \left[\sum_j (x^j - x^{j'})^2\right]^{1/2}, \qquad d^3x' \equiv dx^{1'}\, dx^{2'}\, dx^{3'},$$

and the subscript "ret" means the quantity is to be evaluated at the retarded spacetime point

$$(t' = t - |x - x'|, x^{j'}).$$

These are integral equations because the unknowns, $\bar{h}^{\mu\nu}$, appear both outside and inside the integral (inside they are contained in $t^{\mu\nu}$). Notice that in passing from the wave equations (36.37) to the integral equations (36.38), one has cavalierly behaved as though $\bar{h}^{\mu\nu}$ were fields in flat spacetime. This is certainly not true; but the mathematical manipulations are valid nevertheless!—and the integral equations (36.38) are valid for any field point (t, x^j), even inside the source.

§36.10. EVALUATION OF THE RADIATION FIELD IN THE SLOW-MOTION APPROXIMATION

(5) specialization to slow motion

Thus far the analysis has been exact. Now it is necessary to introduce the slow-motion assumption of §36.7: $R \ll \lambdabar$.

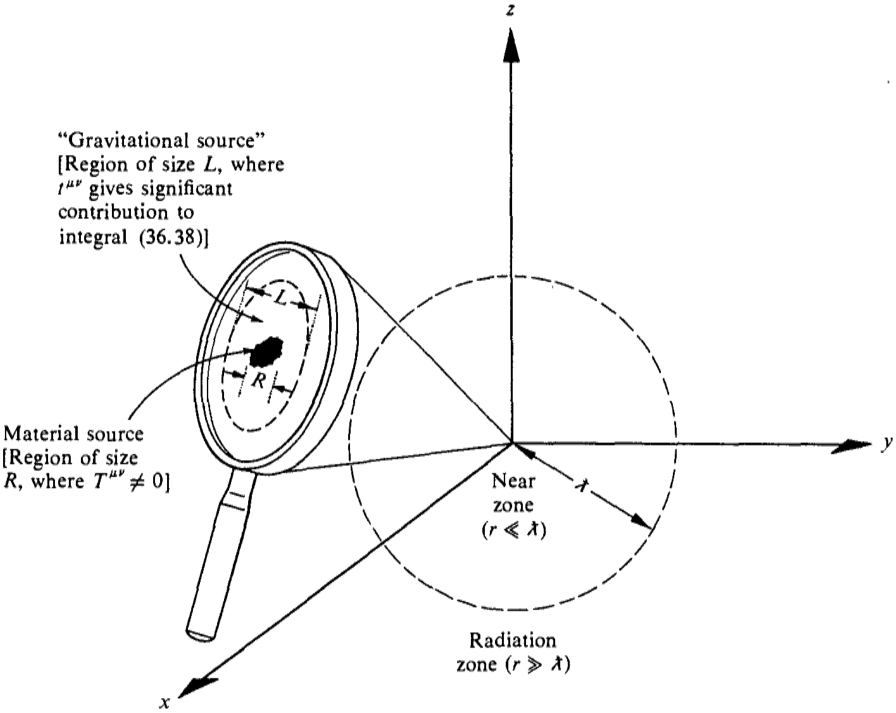


Figure 36.3.
A slow-motion source radiating gravitational waves. The origin of spatial coordinates is located inside the source. The size of the source, R, is very small compared to a reduced wavelength, $R \ll \lambda\!\!\!^{-}$. Significant contributions to the retarded integral (36.38) for $\bar{h}^{\mu\nu}$ come only from a region of size $L \sim R \ll \lambda\!\!\!^{-}$ surrounding the source, because outside the source—but in the near zone ($R \ll r \ll \lambda\!\!\!^{-}$)—the "stress-energy pseudotensor" $t^{\mu\nu}$ dies out as $1/r^4$ (see exercise 36.7).

In the radiation zone, $t^{\mu\nu}$ ceases to die out as $1/r^4$, and begins to die out as $1/r^2$; it is trying to describe (but cannot, really, without appropriate averaging) the stress-energy carried by the gravitational waves. If the source has been emitting waves long enough, contributions from the radiation zone to the retarded integral (36.38) may be nonnegligible:

$$\underbrace{[t^{\mu\nu}]_{\text{ret}} \sim \frac{1}{r'^2}}_{[\text{for } r > \lambda\!\!\!^{-}]} \Longrightarrow \int [t^{\mu\nu}]_{\text{ret}}\, d^3x' \sim \underbrace{\int \frac{1}{r'^2}\, r'^2\, d\Omega'\, dr'}_{\left[\substack{\text{may have significant contri-}\\ \text{butions from large } r'}\right]}.$$

Such contributions are ignored in the text, in calculations of the radiated waves, because they have nothing whatsoever to do with the emission process itself. Rather, they are part of the propagation process treated in the last chapter. They include the background curvature produced by the stress-energy of the waves, scattering of waves off the background curvature, wave-wave scattering, etc.; and they are totally negligible in the neighborhood of the source itself ($r \lesssim 1{,}000\ \lambda\!\!\!^{-}$, for example) because a slow-motion source radiates so very weakly.

Place the origin of spatial coordinates inside the source, as shown in Figure 36.3. For slow-motion systems, the only significant contributions to the retarded integrals (36.38) come from deep inside the near zone (from a region of size $L \sim R \ll \lambda\!\!\!^{-}$; see Figure 36.3). Confine attention to "field points" (points of observation) x^j far outside this "source region,"

$$|x| \equiv r \gg L \gtrsim |x'|, \tag{36.39a}$$

and expand the retarded integral (36.38) in powers of x'/r—in just the same manner as was done in §19.1. (Such an expansion is justified by and requires the slow-motion assumption, $\lambda\!\!\!^{-}/R \sim \lambda\!\!\!^{-}/L \ll 1$.) The result is

$$\begin{aligned}\bar{h}^{\mu\nu}(t, x) = \frac{4}{r}\int [T^{\mu\nu}(x', t - r) + t^{\mu\nu}(x', t - r)]\, d^3x' \\ + O\left\{\frac{x^j}{r^2\lambda\!\!\!^{-}}\int x^{j'}[T^{\mu\nu}(x', t - r) + t^{\mu\nu}(x', t - r)]\, d^3x'\right\}.\end{aligned} \tag{36.40}$$

(6) calculation of $\bar{h}^{jk}$ in radiation zone

Of the ten components of $\bar{h}^{\mu\nu}$, only the six spatial ones, $\bar{h}^{jk}$, are of interest, since only they are needed in projecting out the transverse-traceless radiation field $\bar{h}^{TT}_{jk}$. The spatial components are expressed by equations (36.40) in terms of integrals over the "stress distribution" $T^{jk} + t^{jk}$. It will be convenient, in making comparisons with Newtonian theory, to reexpress $\bar{h}^{jk}$ in terms of integrals over the "energy distribution" $T^{00} + t^{00}$. One can make the conversion with the help of the exact equations of motion $T^{\mu\nu}{}_{;\nu} = 0$, which have the special form

$$(T^{\mu\nu} + t^{\mu\nu})_{,\nu} = 0 \tag{36.41}$$

in the coordinate system being used [see equations (36.36) and (36.37); also the discussion in §20.3]. Applying these relations twice in succession, one obtains the identity

$$\begin{aligned}(T^{00} + t^{00})_{,00} &= -(T^{0l} + t^{0l})_{,l0} = -(T^{l0} + t^{l0})_{,0l} \\ &= +(T^{lm} + t^{lm})_{,ml}.\end{aligned}$$

From this and the elementary chain rule for differentiation, it follows that

$$\begin{aligned}[(T^{00} + t^{00})x^jx^k]_{,00} &= (T^{lm} + t^{lm})_{,ml}x^jx^k \\ &= [(T^{lm} + t^{lm})x^jx^k]_{,ml} - 2[(T^{lj} + t^{lj})x^k + (T^{lk} + t^{lk})x^j]_{,l} \\ &\quad + 2(T^{jk} + t^{jk}),\end{aligned}$$

whence

$$\int (T^{jk} + t^{jk})\, d^3x = \frac{1}{2}(d^2I_{jk}/dt^2), \tag{36.42a}$$

where

$$I_{jk}(t) \equiv \int [T^{00}(t, x) + t^{00}(t, x)]x^jx^k\, d^3x. \tag{36.42b}$$

(7) specialization to nearly Newtonian sources

Now introduce the nearly Newtonian assumption. It guarantees that gravitation contributes only a small fraction of the total energy:

$$t^{00} \sim (\Phi_{,j})^2 \sim M^2/R^4 \sim (M/R)T^{00} \ll T^{00};$$

hence

$$I_{jk}(t) = \int T^{00}(t, x)x^jx^k\, d^3x. \tag{36.42b$'$}$$

The quantity I_{jk} thus represents the second moment of the mass distribution.

By combining equations (36.42) and (36.40), and by noting that inside the source $|t^{jk}| \sim |\Phi_{,j}\Phi_{,k}| \sim T^{00}|\Phi|$, one obtains

$$\begin{aligned}\bar{h}^{jk}(t, x) &= \frac{2}{r}\frac{d^2 I_{jk}(t-r)}{dt^2} + O\left[\frac{1}{r}\left(\frac{|T^{jk}|}{T^{00}} + |\Phi|\right)\frac{R}{\lambda\!\!\!^{-}} M\right] \\ &= \frac{2}{r}\frac{d^2 I_{jk}(t-r)}{dt^2}\left\{1 + \underbrace{O\left[\frac{|T^{jk}|}{T^{00}} + \frac{M}{R}\right]\frac{\lambda\!\!\!^{-}}{R}}_{\text{[negligible by assumptions (36.18)]}}\right\}.\end{aligned} \tag{36.43}$$

Actually, what one wants are h^{TT}_{jk}, not $\bar{h}^{jk}$. They can be obtained by first lowering indices, using $\eta_{lm} = \delta_{lm}$, and then projecting out the TT part using the projection operator for radially traveling waves:

(8) conversion, by projection, to h^{TT}_{jk}

$$P_{lm} = \delta_{lm} - n_l n_m; \qquad n_l = x^l/r \tag{36.44}$$

(see Box 35.1). (Because $\bar{h}_{jk}$ and h_{jk} differ only in the trace, they have the same TT parts). The result is

$$h^{TT}_{jk}(t, x) = \frac{2}{r}\frac{d^2 I^{TT}_{jk}(t-r)}{dt^2}, \tag{36.45a}$$

where

$$I^{TT}_{jk} = P_{jl} I_{lm} P_{mk} - \frac{1}{2} P_{jk}(P_{lm} I_{ml}). \tag{36.45b}$$

This is not the best form in which to write the answer, because an external observer cannot measure directly the second moment of the mass distribution, I_{jk}. Fortunately, one can replace I_{jk} by the reduced quadrupole moment,

(9) reexpression of h^{TT}_{jk} in terms of reduced quadrupole moment

$$I\!\!\!-_{jk} \equiv I_{jk} - \frac{1}{3}\delta_{jk} I = \int (T^{00} + t^{00})\left(x^j x^k - \frac{1}{3}\delta_{jk} r^2\right) d^3x, \tag{36.46}$$

and write

$$h^{TT}_{jk}(t, x) = \frac{2}{r}\frac{d^2 I\!\!\!-^{TT}_{jk}(t-r)}{dt^2}. \tag{36.47}$$

This is allowed because the TT parts of I_{jk} and $I\!\!\!-_{jk}$ are identical (exercise 36.8).

The reduced quadrupole moment $I\!\!\!-_{jk}$ has a well-defined, elementary physical significance for an observer confined to the exterior of the source. In the near zone ($r \ll \lambda\!\!\!^{-}$), but outside the source so that vacuum Newtonian theory is very nearly valid, the Newtonian potential is

$$\begin{aligned}\Phi &= -\frac{1}{2}h_{00} = -\frac{1}{2}h^{00} = -\frac{1}{2}\left(\bar{h}^{00} + \frac{1}{2}\bar{h}\right) = -\frac{1}{4}(\bar{h}^{00} + \bar{h}^{jj}) \\ &= -\int_{\text{all space}} \frac{[T^{00} + t^{00} + T^{jj} + t^{jj}]_{\text{ret}}}{|x - x'|}\, d^3x'\end{aligned}$$

[see equation (36.38)]. Any nearly Newtonian, slow-motion source satisfies

$$|t^{00} + T^{jj} + t^{jj}| \ll T^{00}$$

[recall: $t^{\alpha\beta} \sim (\Phi_{,j})^2 \sim T^{00}|\Phi|$]. Hence, one can write

$$\Phi(x, t) = -\int \frac{[T^{00}(x', t)]}{|x - x'|} d^3x'. \tag{36.48}$$

Expanding $|x - x'|^{-1}$ in powers of $1/r$, one obtains

$$\Phi = -\left(\frac{M}{r} + \frac{d_j x^j}{r^3} + \frac{3 \text{⨍}_{jk} x^j x^k}{2r^5} + \cdots\right) \text{ for } \begin{pmatrix} r \ll \lambda\!\!\!^{-}, \text{ but } r \text{ neverthe-} \\ \text{less large enough that} \\ \text{vacuum Newtonian} \\ \text{theory is valid} \end{pmatrix}, \tag{36.49a}$$

where

$$M = (\text{total mass-energy of source}) = \int T^{00}\, d^3x,$$

$$d_j \equiv (\text{dipole moment of source}) = \int T^{00} x^j\, dx^3, \tag{36.49b}$$

$$\text{⨍}_{jk} \equiv (\text{reduced quadrupole moment of source}) = \text{expression (36.46)}.$$

Thus, *the quantities ⨍_{jk}, whose second time-derivatives determine the radiation field by equation (36.47), are precisely the components of the star's reduced quadrupole moment, as measured by an observer who explores its Newtonian potential Φ deep inside the near zone* ($r \ll \lambda\!\!\!^{-}$) ("empirical quadrupole moment.").

The final answer (36.47) for the radiation field in terms of ⨍_{jk}^{TT} was quoted in the summary of results given in §36.7. Also quoted there were expressions for the effective stress-energy tensor of the radiation and for the energy and angular momentum radiated [equations (36.22) to (36.25)]. Those expressions can be derived using the formalism of the shortwave approximation. (See exercise 36.9.)

EXERCISES

Exercise 36.7. MAGNITUDE OF $t^{\mu\nu}$

Consider a slow-motion source of gravitational waves. Show that far from the source, but in the near zone ($R \ll r \ll \lambda\!\!\!^{-}$) the components of the "stress-energy pseudotensor" $t^{\mu\nu}$ die out as $1/r^4$, but in the radiation zone ($r \gg \lambda\!\!\!^{-}$) they die out only as $1/r^2$.

Exercise 36.8. PROOF THAT THE TRANVERSE TRACELESS PARTS OF ⨍_{jk} AND I_{jk} ARE IDENTICAL

Prove by direct computation that the TT parts of I_{jk} (36.42b) and ⨍_{jk} (36.46) are identical, no matter where the observer is who does the TT projection (i.e., no matter what the unit vector $\boldsymbol{n}$ in the projection operator may be).

Exercise 36.9. ENERGY AND ANGULAR MOMENTUM RADIATED

(a) For the gravitational waves in asymptotically flat spacetime described by equation (36.47), calculate the smeared-out stress-energy tensor $T_{\mu\nu}^{(\mathrm{GW})}$ of equation (35.23). [*Answer:* equation (36.22).]

(b) Perform the integrals of equations (36.23) and (36.25) to obtain the total power and angular momentum radiated. [*Hint:* Derive and use the following averages over a sphere

$$\frac{1}{4\pi}\int n_i\, d\Omega = 0, \qquad \frac{1}{4\pi}\int n_i n_j\, d\Omega = \frac{1}{3}\delta_{ij}, \qquad \frac{1}{4\pi}\int n_i n_j n_k\, d\Omega = 0,$$

$$\frac{1}{4\pi}\int n_i n_j n_k n_l\, d\Omega = \frac{1}{15}(\delta_{ij}\,\delta_{kl} + \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}).$$

Here $\boldsymbol{n} \equiv \boldsymbol{x}/|\boldsymbol{x}|$ is the unit radial vector.]

§36.11. DERIVATION OF THE RADIATION-REACTION POTENTIAL

Derivation of formula for the radiation-reaction potential:

Turn, finally, to a derivation of the radiation-reaction results quoted in §36.8. The analysis starts with the solution (36.43) for the spatial part of the radiation field in the original (i.e., not *TT*) gauge:

$$\bar{h}^{jk}(t, x) = \frac{2}{r}\ddot{I}_{jk}(t - r). \tag{36.50}$$

Although this solution was originally derived by discarding all terms that die out faster than $1/r$, it is in fact an exact solution to the vacuum field equations $\bar{h}^{jk}{}_{,\alpha}{}^{\alpha} = 0$ of linearized theory. This means that it is valid in the intermediate and near zones ($r \lesssim \lambda\!\!\!^{-}$, but $r > R$) as well as in the radiation zone.

Were one to replace the outgoing-wave condition by an ingoing-wave condition at infinity, the exact solution (36.50) for $\bar{h}^{jk}$ would get replaced by

$$\bar{h}^{jk}(t, x) = \frac{2}{r}\ddot{I}_{jk}(t + r).$$

Thus, in order to delineate the effects of the outgoing-wave boundary condition, one can write the exact solution in the form

(1) formula for $\bar{h}_{jk}$ anywhere outside source, with either outgoing or ingoing waves

$$\bar{h}_{jk}(t, x) = \bar{h}^{jk}(t, x) = \frac{2}{r}\ddot{I}_{jk}(t - \epsilon r), \qquad \epsilon = \pm 1, \tag{36.51}$$

and then focus attention on the effects of the sign of ϵ.

In the near zone ($r \ll \lambda\!\!\!^{-}$), but outside the nearly Newtonian source, this solution for $\bar{h}_{jk}$, as expanded in powers of r, becomes

(2) $\bar{h}_{jk}$ specialized to near zone

$$\bar{h}_{jk} = 2\left[\frac{I^{(2)}_{jk}}{r} - \epsilon I^{(3)}_{jk} + \frac{I^{(4)}_{jk} r}{2!} - \epsilon\frac{I^{(5)}_{jk} r^2}{3!} + \cdots\right], \tag{36.52a}$$

where

$$I^{(n)}_{jk} \equiv d^n I_{jk}(t)/dt^n.$$

The corresponding forms of $\bar{h}_{0j}$ and $\bar{h}_{00}$ can be generated from this by the gauge conditions $\bar{h}_\alpha{}^\beta{}_{,\beta} = 0$; i.e., by $\bar{h}_{j0,0} = \bar{h}_{jk,k}$ and $\bar{h}_{00,0} = \bar{h}_{0j,j}$. The results are:

(3) $\bar{h}_{00}$ and $\bar{h}_{0j}$ in near zone calculated by gauge conditions

$$\bar{h}_{0j} = 2\left[-\frac{I^{(1)}_{jk}x^k}{r^3} + \frac{I^{(3)}_{jk}x^k}{2!r} - \epsilon\frac{2I^{(4)}_{jk}x^k}{3!} + \frac{3I^{(5)}_{jk}x^k r}{4!} - \epsilon\frac{4I^{(6)}_{jk}x^k r^2}{5!}\right]$$
$$+ \text{(static terms not associated with radiation);} \tag{36.52b}$$

$$\bar{h}_{00} = 2\left[\frac{(3x^jx^k - r^2\,\delta^{jk})}{r^5} I_{jk} - \frac{(x^jx^k - r^2\,\delta^{jk})}{2!r^3} I^{(2)}_{jk} - \epsilon\frac{2}{3!} I^{(3)}_{jj}\right.$$
$$\left. + \frac{3(x^jx^k + r^2\,\delta^{jk})}{4!r} I^{(4)}_{jk} - \epsilon\frac{4(2x^jx^k + r^2\,\delta^{jk})}{5!} I^{(5)}_{jk} + \cdots\right] \tag{36.52c}$$
$$+ \text{(static and time-linear terms not associated with radiation).}$$

The leading term in these expressions rises as $1/r^3$ when one approaches the source:

$$\bar{h}_{00} \approx \frac{2(3x^jx^k - r^2\,\delta^{jk})}{r^5} I_{jk} = \frac{6\text{I\!\!-}_{jk}n^jn^k}{r^3}.$$

It is precisely the leading term in the dynamic, quadrupole part of the Newtonian potential, $\Phi = -\frac{1}{2}h_{00} = -\frac{1}{4}\bar{h}_{00}$. All other terms without ϵ's in front of them are corrections to the Newtonian potential. They produce effects like the perihelion shift of Mercury that in no way deplete the energy and angular momentum of the system.

(4) plucking out the radiation-reaction potentials from $\bar{h}_{\alpha\beta}$

The terms with ϵ's are associated with radiation reaction. Pluck the leading ones out and call them "reaction potentials":

$$\begin{aligned}
\bar{h}^{(\text{react})}_{jk} &= -2I^{(3)}_{jk} - \frac{1}{3} I^{(5)}_{jk}r^2, \\
\bar{h}^{(\text{react})}_{0j} &= -\frac{2}{3} I^{(4)}_{jk}x^k - \frac{1}{15} I^{(6)}_{jk}x^kr^2, \\
\bar{h}^{(\text{react})}_{00} &= -\frac{2}{3} I^{(3)}_{jj} - \frac{1}{15}(2x^jx^k + r^2\,\delta^{jk})I^{(5)}_{jk}.
\end{aligned} \tag{36.53}$$

The corresponding metric perturbations $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\bar{h}\eta_{\alpha\beta}$ are

$$\begin{aligned}
h^{(\text{react})}_{jk} &= -2I^{(3)}_{jk} + \frac{2}{3} I^{(3)}_{ll}\,\delta_{jk} + 0(I^{(5)}_{jk}r^2), \\
h^{(\text{react})}_{0j} &= -\frac{2}{3} I^{(4)}_{jk}x^k + 0(I^{(6)}_{jk}r^3), \\
h^{(\text{react})}_{00} &= -\frac{4}{3} I^{(3)}_{ll} - \frac{1}{15}(x^jx^k + 3r^2\,\delta_{jk})I^{(5)}_{jk}.
\end{aligned} \tag{36.54}$$

These reaction potentials in the near zone are understood most clearly by a change of gauge that brings them into Newtonian form. Set

$$x^{\mu}{}_{\text{new}} = x^{\mu}{}_{\text{old}} + \xi^{\mu}(x), \qquad h_{\mu\nu_{\text{new}}} = h_{\mu\nu_{\text{old}}} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

(5) conversion of radiation-reaction potentials to Newtonian gauge

with

$$\xi_j = -I^{(3)}_{jk}x^k + \frac{1}{3}I^{(3)}_{ll}x^j,$$

$$\xi_0 = -\frac{2}{3}I^{(2)}_{ll} + \frac{1}{6}I^{(4)}_{jk}x^jx^k - \frac{1}{6}I^{(4)}_{ll}r^2. \tag{36.55}$$

Then in the new gauge

$$h^{(\text{react})}_{jk} = O(I^{(5)}_{jk}r^2), \qquad h^{(\text{react})}_{0j} = O(I^{(6)}_{jk}r^3),$$

$$h^{(\text{react})}_{00} = -\frac{2}{5}\not{I}^{(5)}_{jk}x^jx^k. \tag{36.56}$$

This gauge is ideally suited to a Newtonian interpretation, since in it the geodesic equation for slowly moving particles has the form

$$d^2x^j/dt^2 = -\Phi^{(\text{react})}_{,j} + \begin{pmatrix}\text{terms not sensitive to}\\ \text{outgoing-wave condition}\end{pmatrix}, \tag{36.57}$$

with

$$\Phi^{(\text{react})} = -\frac{1}{2}h^{(\text{react})}_{00} = \frac{1}{5}\not{I}^{(5)}_{jk}x^jx^k. \tag{36.58}$$

Thus, the leading radiation-reaction effects (with fractional errors $\sim [\lambda\!\!\!^{-}/r]^2$) can be described in the near zone of a nearly Newtonian source by appending the term $\frac{1}{5}\not{I}^{(5)}_{jk}x^jx^k$ to the Newtonian potential. The resulting formalism and a qualitative version of the above derivation were presented in §36.8.

CHAPTER 37

DETECTION OF GRAVITATIONAL WAVES

I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind: it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of science, whatever the matter may be.

WILLIAM THOMSON, LORD KELVIN [(1889), p. 73]

§37.1. COORDINATE SYSTEMS AND IMPINGING WAVES

Linearized description of gravitational waves propagating past Earth

The detector is even easier to analyze than the generator or the transmission when one deals with gravitational waves within the framework of general relativity. Man's potential detectors all lie in the solar system, where gravity is so weak and spacetime so nearly flat that a plane gravitational wave coming in remains for all practical purposes a plane gravitational wave. (Angle of deflection of wave front passing limb of sun is only 1.″75.) Moreover, the nearest source of significant waves is so far away that, for all practical purposes, one can consider the waves as plane-fronted when they reach the Earth. Consequently, as they propagate in the z-direction past a detector, they can be described to high accuracy by the following transverse-traceless linearized expressions

$$\text{Metric perturbation: } h^{TT}_{xx} = -h^{TT}_{yy} = A_+(t - z),\ h^{TT}_{xy} = h^{TT}_{yx} = A_\times(t - z), \tag{37.1a}$$

$$\begin{aligned} \text{Riemann tensor: } R_{x0x0} &= -R_{y0y0} = -\frac{1}{2}\ddot{A}_+(t - z), \\ R_{x0y0} &= R_{y0x0} = -\frac{1}{2}\ddot{A}_\times(t - z). \end{aligned} \tag{37.1b}$$

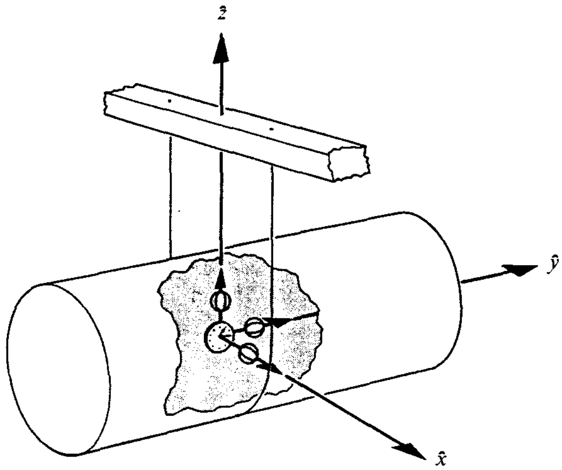


Figure 37.1.
The proper reference frame of a vibrating-bar detector. The bar hangs by a wire from a cross beam, which is supported by vertical posts (not shown) that are embedded in the Earth. Consequently, the bar experiences a 4-acceleration given, at the moment when this diagram is drawn, by $\boldsymbol{a} = g(\partial/\partial\hat{z})$, where g is the "local acceleration of gravity" ($g \sim 980$ cm/sec^2). Later, the spatial axes will have rotated relative to the bar ("Foucoult-pendulum effect" produced by Earth's rotation), so the components of $\boldsymbol{a}$ but not its magnitude will have changed.

The proper reference frame relies on an imaginary clock and three imaginary gyroscopes located at the bar's center of mass (and shown above in a cut-away view). Coordinate time is equal to proper time as measured by the clock, and the directions of the spatial axes $\partial/\partial x^{\hat{j}}$ are attached to the gyroscopes. The forces that prevent the gyroscopes from falling in the Earth's field must be applied at the centers of mass of the individual gyroscopes (no torque!).

$$\text{Stress-energy: } T_{00}^{(\mathrm{GW})} = T_{zz}^{(\mathrm{GW})} = -T_{0z}^{(\mathrm{GW})} = \frac{1}{16\pi}\langle \dot{A}_+^2 + \dot{A}_\times^2 \rangle_{\text{time avg.}} \qquad (37.1\text{c})$$

(See exercise 37.1.)

To analyze most easily the response of the detector to these impinging waves, use not the TT coordinate system $\{x^\alpha\}$ (which is specially "tuned" to the waves), but rather use coordinates $\{x^{\hat{\alpha}}\}$ specially "tuned" to the experimenter and his detector. The detector might be a vibrating bar, or the vibrating Earth, or a loop of tubing filled with fluid (see Figures 37.1 and 37.2). But whatever it is, it will have a center of mass. Attach the spatial origin, $x^{\hat{j}} = 0$, to this center of mass; and attach *orthonormal* spatial axes, $\partial/\partial x^{\hat{j}}$, to (possibly imaginary) gyroscopes located at this spatial origin (Figure 37.1). If the detector is accelerating (i.e., not falling freely), make the gyroscopes accelerate with it by applying the necessary forces at their centers of mass (no torque!). Use, as time coordinate, the proper time $x^{\hat{0}} = \tau$ measured by a clock at the spatial origin. Finally, extend these locally defined coordinates $x^{\hat{\alpha}}$ throughout all spacetime in the "straightest" manner possible. (See

Proper reference frame of a detector

Track 2's §13.6 for full details.) The metric in this "*proper reference frame of the detector*" will have the following form

$$ds^2 = -(1 + 2a_{\hat{j}}x^{\hat{j}})(dx^{\hat{0}})^2 + \delta_{\hat{j}\hat{k}}\, dx^{\hat{j}}\, dx^{\hat{k}} + O(|x^{\hat{j}}|^2)\, dx^{\hat{\alpha}}\, dx^{\hat{\beta}}. \tag{37.2}$$

[equation (13.71) with $\omega^{\hat{l}} = 0$]. Here $a_{\hat{j}}$ are the spatial components of the detector's 4-acceleration. (Since $\boldsymbol{a}$ must be orthogonal to the detector's 4-velocity, $a_{\hat{0}}$ vanishes.) Notice that, except for the acceleration term in $g_{\hat{0}\hat{0}}$ ("gravitational redshift term"; see §38.5 and exercise 6.6), this reference frame is locally Lorentz.

EXERCISES

Exercise 37.1. GENERAL PLANE WAVE IN TT GAUGE

Show that the most general linearized plane wave can be described in the transverse-traceless gauge of linearized theory by expressions (37.1). [*Hint:* Express the plane wave as a superposition (Fourier integral) of monochromatic plane waves, and describe each monochromatic plane wave by expressions (35.16). Use equations (35.10) and (35.23) to calculate $R_{\alpha\beta\gamma\delta}$ and $T^{(\mathrm{GW})}_{\mu\nu}$.]

Exercise 37.2. TEST-PARTICLE MOTION IN PROPER REFERENCE FRAME

Show that a slowly moving test particle, falling freely through the proper reference frame of equation (37.2), obeys the equation of motion (geodesic equation)

$$d^2x^{\hat{j}}/d\hat{t}^2 = -a_{\hat{j}} + O(|x^{\hat{k}}|).$$

Thus, one can interpret $-a_{\hat{j}}$ as the "local acceleration of gravity" (see caption of Figure 37.1).

§37.2. ACCELERATIONS IN MECHANICAL DETECTORS

Equations of motion for a mechanical detector

The proper reference frame of equation (37.2) is the closest thing that exists to the reference frame a Newtonian physicist would use in analyzing the detector. In fact, it is so nearly Newtonian that (according to the analysis of Box 37.1) *the equations of motion for a mechanical detector, when written in this proper reference frame, take their standard Newtonian form and can be viewed and dealt with in a fully Newtonian manner, with one exception: the gravitational waves produce a driving force of non-Newtonian origin, given by the familiar expression for geodesic deviation*

$$\begin{aligned}\begin{pmatrix}\text{force per unit mass (i.e., acceleration)}\\ \text{of a particle at } x^{\hat{j}} \text{ relative to detector's}\\ \text{center of mass at } x^{\hat{j}} = 0\end{pmatrix} &= \left(\frac{d^2x^{\hat{j}}}{d\hat{t}^2}\right)_{\text{due to waves}} \\ &= -(R_{\hat{j}\hat{0}\hat{k}\hat{0}})_{\text{due to waves}}\, x^{\hat{k}}.\end{aligned} \tag{37.3}$$

To use this equation, and to calculate detector cross sections later, one must know the components of the curvature tensor $R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}}$, and of the waves' stress-energy tensor, $T^{(\mathrm{GW})}_{\hat{\mu}\hat{\nu}}$, in the detector's proper reference frame. One cannot calculate $R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}}$ directly

Box 37.1 DERIVATION OF EQUATIONS OF MOTION FOR A MECHANICAL DETECTOR

Consider a "mass element" in a mechanical detector (e.g., a cube of aluminum one millimeter on each edge if the detector is the bar of Figure 37.1; or an element of fluid with volume 1 mm^3 if the detector is the tube filled with fluid shown in part h of Figure 37.2). This mass element gets pushed and pulled by adjacent matter and electromagnetic fields, as the medium of the detector vibrates or flows or does whatever it is supposed to do. Let

$$\boldsymbol{f} \equiv \begin{pmatrix} \text{4-force per unit mass exerted on mass-element} \\ \text{by adjacent matter and by electromagnetic fields} \end{pmatrix}. \tag{1}$$

This 4-force per unit mass gives the mass element a 4-acceleration $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = \boldsymbol{f}$; or, in terms of components in the detector's proper reference frame, $f^{\hat{j}} = Du^{\hat{j}}/d\tau$. Assume that the mass element has a very small velocity ($v \ll 1$) in the detector's proper reference frame (i.e., relative to the detector's center of mass). Then, ignoring terms of $O(v^2)$, $O(|x^{\hat{j}}|^2)$, and $O(|x^{\hat{j}}|v)$, one has [see equation (37.2)]

$$d\hat{t}/d\tau = u^{\hat{0}} = 1 - a_{\hat{j}}x^{\hat{j}} \equiv 1 - \boldsymbol{a}\cdot\boldsymbol{x}, \tag{2}$$

and

$$f^{\hat{j}} = d^2x^{\hat{j}}/d\tau^2 + \Gamma^{\hat{j}}{}_{\hat{\alpha}\hat{\beta}}u^{\hat{\alpha}}u^{\hat{\beta}} = (d^2x^{\hat{j}}/d\hat{t}^2 + \Gamma^{\hat{j}}{}_{\hat{0}\hat{0}})(1 - 2\boldsymbol{a}\cdot\boldsymbol{x}). \tag{3}$$

Exercise 37.3 calculates $\Gamma^{\hat{j}}{}_{\hat{0}\hat{0}}$ to precision of $O(|x^{\hat{j}}|)$. Inserting its result and rearranging terms, one finds that

$$d^2x^{\hat{j}}/d\hat{t}^2 = (1 + 2\boldsymbol{a}\cdot\boldsymbol{x})f^{\hat{j}} - a^{\hat{j}}(1 + \boldsymbol{a}\cdot\boldsymbol{x}) - R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}}x^{\hat{k}} \tag{4}$$

("*equation of motion for mass element*").

Examine this equation, first from the viewpoint of an Einsteinian physicist, and then from the viewpoint of a Newtonian physicist.

The Einsteinian physicist recognizes $d^2x^{\hat{j}}/d\hat{t}^2$ as the "coordinate acceleration" of the mass element—but he keeps in mind that, to precision of $O(|x^{\hat{j}}|^2)$, coordinate lengths and proper lengths are the same [see equation (37.2)]. The coordinate acceleration $d^2x^{\hat{j}}/d\hat{t}^2$ has three causes: (1) *the externally applied force,*

$$\begin{aligned}(1 + 2\boldsymbol{a}\cdot\boldsymbol{x})f^{\hat{j}} &= (d^2x^{\hat{j}}/d\hat{t}^2)_{\text{external force}} \\ &= (1 + 2\boldsymbol{a}\cdot\boldsymbol{x})(d^2x^{\hat{j}}/d\tau^2)_{\text{external force}}\end{aligned} \tag{5a}$$

(the origin of the $\boldsymbol{a}\cdot\boldsymbol{x}$ correction is simply the conversion between coordinate time

Box 37.1 *(continued)*

and proper time); (2) the "inertial force" due to the acceleration of the reference frame,

$$-a^{\hat{j}}(1 + \boldsymbol{a} \cdot \boldsymbol{x}) = (d^2x^{\hat{j}}/d\hat{t}^2)_{\text{inertial force}} \tag{5b}$$

(see exercise 37.4 for explanation of the $\boldsymbol{a} \cdot \boldsymbol{x}$ correction); and (3) a "*Riemann curvature force*," which will include Riemann curvature due to local, Newtonian gravitational fields (fields of Earth, sun, moon, etc.), plus Riemann curvature due to the impinging gravitational waves,

$$-(R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}})_{\text{waves}}x^{\hat{k}} - (R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}})_{\text{Newton fields}}x^{\hat{k}} = (d^2x^{\hat{j}}/d\hat{t}^2)_{\text{curvature}} \tag{5c}$$

(linear superposition because all gravitational fields in the solar system are so weak). This "Riemann curvature force" is not, of course, "felt" by the mass element; it does not produce any 4-acceleration. Rather, like the inertial force, it originates in the choice of reference frame: The spatial coordinates $x^{\hat{j}}$ measure proper distance and direction away from the detector's center of mass; and Riemann curvature tries to change this proper distance and direction ("relative acceleration;" "geodesic deviation").

A Newtonian physicist views the equation of motion (4) in a rather different manner. Having been told that the spatial coordinates $x^{\hat{j}}$ measure proper distance and direction away from the detector's center of mass, he thinks of them as the standard Euclidean spatial coordinates of Newtonian theory. He then rewrites equation (4) in the form

$$d^2x^{\hat{j}}/d\hat{t}^2 = F^{\hat{j}} - (R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}})_{\text{waves}}x^{\hat{k}}, \tag{6}$$

where

$$\begin{aligned} F^{\hat{j}} &\equiv \begin{pmatrix}\text{Newtonian force per unit mass}\\ \text{acting on mass element}\end{pmatrix} \\ &= (1 + 2\boldsymbol{a} \cdot \boldsymbol{x})f^{\hat{j}} - a^{\hat{j}}(1 + \boldsymbol{a} \cdot \boldsymbol{x}) - (R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}})_{\text{Newton fields}}x^{\hat{k}}. \end{aligned} \tag{7}$$

The Newtonian physicist is free to express $F^{\hat{j}}$ in a form more familiar than this. He can ignore the subtleties of the $\boldsymbol{a} \cdot \boldsymbol{x}$ "redshift effects" because (1) they are small

$$|a^{\hat{j}}(\boldsymbol{a} \cdot \boldsymbol{x})| \sim |f^{\hat{j}}(\boldsymbol{a} \cdot \boldsymbol{x})| \lesssim |(R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}})_{\text{waves}}x^{\hat{k}}|; \tag{8}$$

and (2) they are steady in time, and therefore—by contrast with the equally small wave-induced forces—they cannot excite resonant motions of the detector. Also, he can separate the "inertial acceleration," $-a^{\hat{j}}$, into a contribution from the local acceleration of gravity at the detector's center of mass, $-(\partial\Phi/\partial x^{\hat{j}})_{x^{\hat{j}}=0}$, plus a contribution $-a^{\hat{j}}_{\text{absolute}}$ due to acceleration of the detector relative to the "absolute space" of Newtonian theory. Finally, he can rewrite the Riemann curvature due to Newtonian gravity in the familiar form $R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} = \partial^2\Phi/\partial x^{\hat{j}}\,\partial x^{\hat{k}}$. The net result is

$$
\begin{aligned}
F^{\hat{j}} = \; & && \begin{bmatrix}\text{total Newtonian force per unit}\\ \text{mass acting on mass element}\end{bmatrix}\\
& + f^{\hat{j}} && \begin{bmatrix}\text{Newtonian force per unit mass exerted by}\\ \text{adjacent matter and by electromagnetic fields}\end{bmatrix}\\
& - a^{\hat{j}}{}_{\text{absolute}} && \begin{bmatrix}\text{inertial force per unit mass due to acceleration}\\ \text{of detector relative to Newtonian absolute space}\end{bmatrix}\\
& - \left(\frac{\partial\Phi}{\partial x^{\hat{j}}}\right)_{\text{at mass element}} && \begin{bmatrix}= -(\partial\Phi/\partial x^{\hat{j}})_{x^{\hat{i}}=0} - (\partial^2\Phi/\partial x^{\hat{j}}\,\partial x^{\hat{k}})x^{\hat{k}}\\ = \text{Newtonian gravitational acceleration}\end{bmatrix}.
\end{aligned}
\tag{9}
$$

Conclusion: The equation of motion for a mass element of a mechanical detector, when written in the detector's proper reference frame, has the standard Newtonian form (6), with standard Newtonian driving forces (9), plus a driving force due to the gravitational waves given by

$$(d^2x^{\hat{j}}/d\hat{t}^2)_{\text{due to waves}} = -(R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}})_{\text{waves}}x^{\hat{k}}. \tag{10}$$

from the metric coefficients $g_{\hat{\alpha}\hat{\beta}}$ of expression (37.2); to do so one would need the unknown corrections of $O(|x^{\hat{j}}|^2)$. However, one can easily obtain $R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}}$ and $T^{(\mathrm{GW})}_{\hat{\mu}\hat{\nu}}$ from the corresponding components in the TT coordinate frame [equations (37.1)] by applying the transformation matrix $\|\partial x^{\alpha}/\partial x^{\hat{\mu}}\|$. To make the transformation trivial, orient the TT coordinate frame so that, to a precision of $O(|h_{\mu\nu}|) \ll 1$, it coincides with the detector's proper reference frame near the detector's center of mass at the moment of interest, $t = \hat{t} = 0$. Then the transformation matrix will be

$$\partial x^{\alpha}/\partial x^{\hat{\mu}} = \underbrace{\delta^{\alpha}_{\mu} + O(h_{\mu\nu})}_{\begin{bmatrix}\text{corrections due to}\\ \text{ripples in spacetime}\\ \text{caused by waves}\end{bmatrix}} + \underbrace{O(a_{\hat{j}}\,x^{\hat{j}})}_{\begin{bmatrix}\text{redshift}\\ \text{corrections}\end{bmatrix}} + \underbrace{O(|\boldsymbol{a}|\,\hat{t})}_{\begin{bmatrix}\text{corrections due to relative}\\ \text{velocity of frames resulting}\\ \text{from detector's acceleration}\end{bmatrix}}. \tag{37.4}$$

The acceleration the detector experiences is typically

$$|\boldsymbol{a}| = \text{one ``Earth gravity''} = 980\ \text{cm/sec}^2 \sim 1/(\text{light-year}).$$

Description of waves in frame of detector

Therefore to enormous precision $||\partial x^{\alpha}/\partial x^{\hat{\mu}}|| = ||\delta^{\alpha}_{\mu}||$, and components of tensors are the same in the two reference frames:

$$R_{\hat{x}\hat{0}\hat{x}\hat{0}} = -R_{\hat{y}\hat{0}\hat{y}\hat{0}} = -\frac{1}{2}\ddot{A}_{+}, \qquad R_{\hat{x}\hat{0}\hat{y}\hat{0}} = R_{\hat{y}\hat{0}\hat{x}\hat{0}} = -\frac{1}{2}\ddot{A}_{\times},$$

$$T^{(\mathrm{GW})}_{\hat{0}\hat{0}} = T^{(\mathrm{GW})}_{\hat{z}\hat{z}} = -T^{(\mathrm{GW})}_{\hat{0}\hat{z}} = \frac{1}{16\pi}\langle \dot{A}_{+}^{2} + \dot{A}_{\times}^{2}\rangle_{\text{time avg.}} \tag{37.5}$$

[see equation (37.1)].

Combining equations (37.3) and (37.5), one obtains for the wave-induced accelerations relative to the center of mass of the detector

Explicit form of accelerations due to waves

$$\left(\frac{d^2\hat{x}}{d\hat{t}^2}\right)_{\text{due to waves}} = -R_{\hat{x}\hat{0}\hat{x}\hat{0}}\hat{x} - R_{\hat{x}\hat{0}\hat{y}\hat{0}}\hat{y} = \frac{1}{2}(\ddot{A}_{+}\hat{x} + \ddot{A}_{\times}\hat{y}),$$

$$\left(\frac{d^2\hat{y}}{d\hat{t}^2}\right)_{\text{due to waves}} = -R_{\hat{y}\hat{0}\hat{y}\hat{0}}\hat{y} - R_{\hat{y}\hat{0}\hat{x}\hat{0}}\hat{x} = \frac{1}{2}(-\ddot{A}_{+}\hat{y} + \ddot{A}_{\times}\hat{x}), \tag{37.6}$$

$$\left(\frac{d^2\hat{z}}{d\hat{t}^2}\right)_{\text{due to waves}} = 0.$$

This analysis is valid only for "small" detectors ($L \ll \lambda\!\!\!^{-}$)

These expressions, like the equation of geodesic deviation, are valid only over regions small compared to one wavelength. Second derivatives of the metric (i.e., the components of the Riemann tensor) give a poor measure of geodesic deviation and of wave-induced forces over regions of size $L \gtrsim \lambda\!\!\!^{-}$. Thus, to analyze large detectors ($L \gtrsim \lambda\!\!\!^{-}$), one must abandon the "local mathematics" of the curvature tensor and replace it by "global mathematics"—e.g., an analysis in the *TT* coordinate frame using the metric components $h_{\mu\nu}$. For an example, see exercise 37.6.

All detectors of high sensitivity that have been designed up until now (1973) are small compared to a wavelength, and therefore can be analyzed using the techniques of Newtonian physics and the driving forces of equations (37.6).

It is useful to develop physical intuition for the driving forces, $-R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}}x^{\hat{k}}$, produced by waves of various polarizations. Figure 35.2 is one aid to such intuition; Box 37.2 is another. [The reader may find it interesting to examine, compare, and reconcile them!]

EXERCISES

Exercise 37.3. CONNECTION COEFFICIENTS IN PROPER REFERENCE FRAME

(a) Calculate $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}}$ for the metric (37.2), ignoring corrections of $O(|x^{\hat{j}}|)$. [Answer: Equations (13.69) with $\omega^{\hat{i}} = 0$.]

(b) Calculate $R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}}$ using the standard formula (8.44), and leaving spatial derivatives of the connection coefficients unevaluated because of the unknown corrections of $O(|x^{\hat{j}}|)$ in $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}}$. [Answer: $R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} = \Gamma^{\hat{j}}{}_{\hat{0}\hat{0},\hat{k}} - a^{\hat{j}}a^{\hat{k}}$.]

(c) Use the answer to part (b) to evaluate the $O(|x^{\hat{j}}|)$ corrections to $\Gamma^{\hat{j}}{}_{\hat{0}\hat{0}}$. [Answer:

$$\Gamma^{\hat{j}}{}_{\hat{0}\hat{0}} = a^{\hat{j}}(1 + a_{\hat{k}}x^{\hat{k}}) + R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}}x^{\hat{k}} + O(|x^{\hat{k}}|^2).] \tag{37.7}$$

Box 37.2 LINES OF FORCE FOR GRAVITATIONAL-WAVE ACCELERATIONS

A. Basic Idea

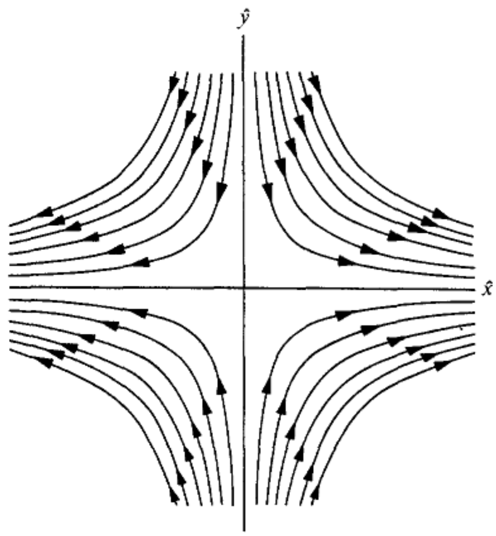
Consider a plane wave propagating in the $\hat{z}$ direction. Discuss it entirely in the proper reference frame of a detector. The relative accelerations due to the wave are entirely transverse. Relative to the center of mass of the detector (origin of spatial coordinates) they are

$$
\begin{aligned}
d^2\hat{x}/d\hat{t}^2 &= \frac{1}{2}(\ddot{A}_+\hat{x} + \ddot{A}_\times\hat{y}),\\
d^2\hat{y}/d\hat{t}^2 &= \frac{1}{2}(-\ddot{A}_+\hat{y} + \ddot{A}_\times\hat{x}), \qquad (1)\\
d^2\hat{z}/d\hat{t}^2 &= 0.
\end{aligned}
$$

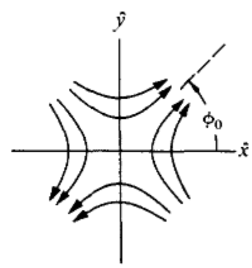
Notice that these accelerations are divergence-free. Consequently they can be represented by "lines of force," analogous to those of a vacuum electric field. At a value of $\hat{t} - \hat{z}$ where $\ddot{A}_\times = 0$ (so polarization is entirely $\boldsymbol{e}_+$), the lines of force are the hyperbolas shown here [sketch (a)]. The direction of the acceleration at any point is the direction of the arrow there; the magnitude of the acceleration is the density of force lines. Since acceleration is proportional to distance from center of mass, the force lines get twice as close together when one moves twice as far away from the origin in a given direction. When $\ddot{A}_+$ is positive, the arrows on the force lines are as shown in (a); when it is negative, they are reversed. As $|\ddot{A}_+|$ increases, the force lines move in toward the origin so their density goes up; as $|\ddot{A}_+|$ decreases, they move out toward infinity so their density goes down.

For polarization $\boldsymbol{e}_\times$ the force lines are rotated by 45° from the above diagram. For intermediate polarization (values of $\hat{t} - \hat{z}$ where $\ddot{A}_+$ and $\ddot{A}_\times$ are both nonzero), the diagram is rotated by an intermediate angle [sketch (b)]

$$\phi_0 = \frac{1}{2}\arctan(\ddot{A}_\times/\ddot{A}_+). \qquad (2)$$



(a) Force lines for $\ddot{A}_\times = 0$, $\ddot{A}_+ > 0$



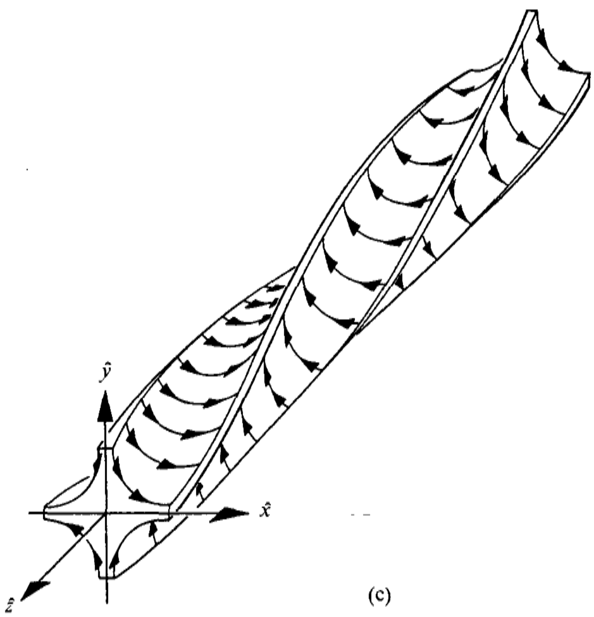
(b)

Box 37.2 ***(continued)***

B. Three-Dimensional Diagram

At each value of $\hat{t} - \hat{z}$, the wave-produced accelerations have a specific polarization [orientation angle ϕ_0 of sketch (b)] and a specific amplitude (density of lines of force). Draw the lines of force in a three-dimensional $(\hat{x}, \hat{y}, \hat{z})$ diagram for fixed $\hat{t}$. Then as time passes the over-all diagram will remain unchanged in form, but will propagate with the speed of light in the $\hat{z}$ direction.

Sketch (c) shows such a diagram for righthand circularly polarized waves of unchanging amplitude. *Note:* The authors are not aware of diagrams such as these [(a), (b), (c) above] and their use in analyzing detector response prior to William H. Press (1970).



(c)

Exercise 37.4. WHY THE $a \cdot x$?

Explain the origin of the $a \cdot x$ correction in equation (5b) of Box 37.1. [*Hint:* Take the viewpoint of an observer at rest at the spatial origin who watches two freely falling particles respond to the inertial force. At time $\hat{t} = 0$, put one particle at the origin and the other at $x^{\hat{j}}$. As time passes, the separation of the particles in their common Lorentz frame remains fixed; so there develops a Lorentz contraction from the viewpoint of the observer at $x^{\hat{j}} = 0$.]

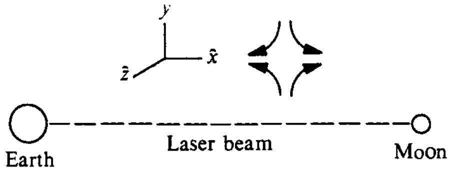
Exercise 37.5. ORIENTATION OF POLARIZATION DIAGRAM

Derive equation (2) of Box 37.2.

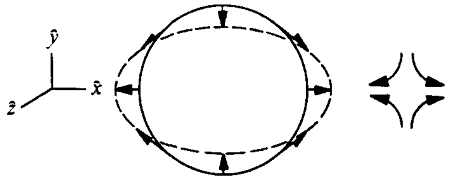
§37.3. TYPES OF MECHANICAL DETECTORS

Eight types of mechanical detectors:

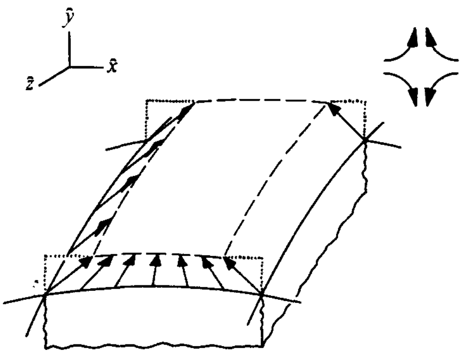
Figure 37.2 shows eight different types of mechanical detectors for gravitational waves. (By "mechanical detector" is meant a detector that relies on the relative



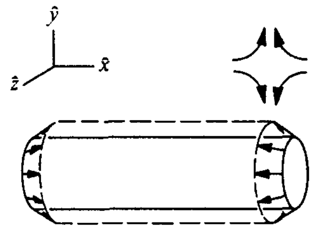
(a) Oscillations in Earth-moon separation (see exercise 37.7)



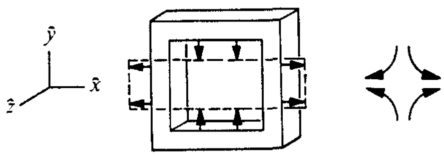
(b) Normal-mode vibrations of earth and moon [see Weber (1968)]



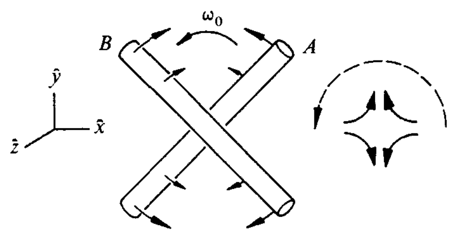
(c) Oscillations in Earth's crust [see Dyson (1969)]



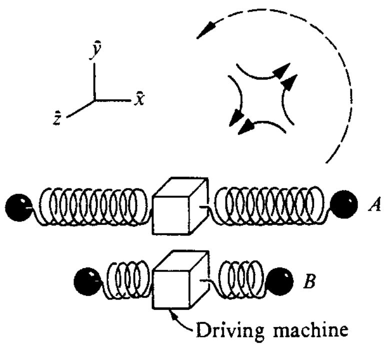
(d) Normal-mode vibrations of an elastic bar [see Weber (1969) and references cited therein]



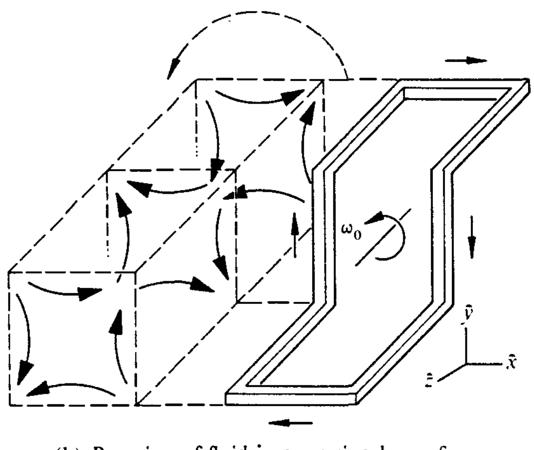
(e) Normal-mode vibrations of an elastic square, or hoop, or tuning fork [see Douglass (1971)]



(f) Angular accelerations of rotating bars ["Heterodyne detector": see Braginsky, Zel'dovich, and Rudenko (1969)]



(g) Angular accelerations of driven oscillators [Sakharov (1969)]



(h) Pumping of fluid in a rotating loop of pipe [Press (1970)]. The pipe rotates with the same angular velocity as the waves; so the position of the pipe in the righthand polarized lines of force remains forever fixed

Figure 37.2.
Various types of gravitational-wave detectors.

motions of matter. Nonmechanical detectors are described in §37.9, near end of this chapter.) These eight detectors, and others, can be analyzed easily using the force-line diagrams of Box 37.2. A *qualitative* discussion of each of the eight detectors is given below. (A full quantitative analysis for each one would entail experimental technicalities for which general relativity is irrelevant, and which are beyond the scope of this book. However, some quantitative details are spelled out in §§37.5–37.8.)

1. The Relative Motions of Two Freely Falling Bodies

(1) freely falling bodies

As a gravitational wave passes two freely falling bodies, their proper separation oscillates (Figure 37.3). This produces corresponding oscillations in the redshift and round-trip travel times for electromagnetic signals propagating back and forth between the two bodies. Either effect, oscillating redshift or oscillating travel time, could be used in principle to detect the passage of the waves. Examples of such detectors are the Earth-Moon separation, as monitored by laser ranging [Fig. 37.2(a)]; Earth-spacecraft separations as monitored by radio ranging; and the separation between two test masses in an Earth-orbiting laboratory, as monitored by redshift measurements or by laser interferometry. Several features of such detectors are explored in exercises 37.6 and 37.7. As shown in exercise 37.7, such detectors have so low a sensitivity that they are of little experimental interest.

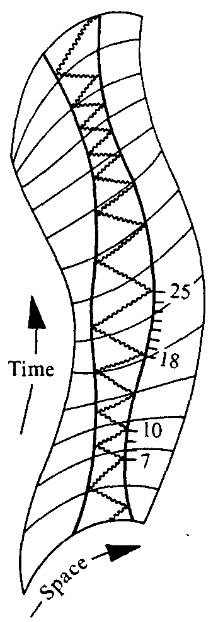


Figure 37.3.
Time of round-trip travel between two geodesics responds to oscillations in the curvature of spacetime (diagram is schematic only; symbolic of a laser pulse sent from the Earth to a corner reflector on the Moon and back at a time when a very powerful, long-wavelength gravitational wave passes by; the wave would have to be powerful because a direct measure of distance to better than 10 cm is difficult, and such precision produces a much less sensitive indicator of waves than the vibrations in length [10^{-14} cm or less] of a Weber bar; see exercise 37.7). The geodesics are curved toward each other in regions where the relevant component of the Riemann curvature tensor, call it $R_{\hat{x}\hat{0}\hat{x}\hat{0}}$, has one sign, and curved away from each other in regions where it has the opposite sign. The diagram allows one to see at a glance the answer to an often expressed puzzlement: Is not any change in round-trip travel time mere trumpery flummery? The metric perturbation, $\delta h_{\mu\nu}$, of the wave changes the scale of distances slightly but also correspondingly changes the scale of time. Therefore does not every possibility of any really meaningful and measurable effect cancel out? Answer: (1) The widened separation between the geodesics is not a local effect but a cumulative one. It does not arise from the local value of $\delta h_{\mu\nu}$ directly or even from the local value of the curvature. It arises from an accumulation of the bending process over an entire half-period of the gravitational wave. (2) The change in separation of the geodesics is a true change in proper distance, and shows up in a true change in proper time (see "ticks" on the world line of one of the particles). See exercise 37.6. Note: When one investigates the separation between the geodesics, not over a single period, as here, but over a large number of periods, he finds a cumulative, systematic, net slow bending of the rapidly wiggling geodesics toward each other. This small, attractive acceleration is evidence in gravitation physics of the effective mass-energy carried by the gravitational waves (see Chapter 35).

2. Normal-Mode Vibrations of the Earth and Moon

(2) Earth and Moon

A gravitational wave sweeping over the Earth will excite its quadrupole modes of vibration, since the driving forces in the wave have quadrupole spatial distributions [see Fig. 37.2(b)]. The fundamental quadrupole mode of the Earth has a period of 54 minutes, while that of the moon has a period of 15 minutes. Thus, the Earth and Moon should selectively pick out the 54-minute and 15-minute components of any passing wave train. Section 37.7 will analyze quantitatively the interaction between the wave and solid-body vibrations. By comparing that analysis with seismometer studies of the Earth's vibrations, Weber (1967) put the first observational limit ever on the cosmic flux of gravitational waves:

$$I_\nu \equiv \frac{d\text{ flux}}{d\text{ frequency}} < 3 \times 10^7 \text{ erg cm}^{-2}\text{ sec}^{-1}\text{ Hz}^{-1} \text{ at } \nu = 3.1 \times 10^{-4}\text{ Hz}. \quad (37.8)$$

3. Oscillations in the Earth's Crust

(3) Earth's crust

If the neutron star in a pulsar is slightly deformed from axial symmetry, its rotation will produce gravitational waves. The period of the waves is half the period of the pulsar (rotation of star through 180° produces one period of waves)—i.e., it should range from 0.017 sec for NP0532 (Crab Pulsar) to 1.87 sec. for NP0527. Such a wave train cannot excite the 54-minute quadrupole vibration or any of the other normal, low-frequency modes of vibration of the Earth. The kind of vibrations it *can* excite allow themselves in principle to be described in the language of normal modes. However, they are clearly and more conveniently envisaged as vibrations of localized regions of the Earth; or, more particularly, vibrations of the Earth's crust.

Dyson (1969) has analyzed the response of an elastic solid, such as the Earth, to an incident, off-resonance gravitational wave. He shows that the response depends on irregularities in the elastic modulus for shear waves, and that it is strongest at a free surface [Figure 37.2(c)]. For the fraction of gravitational-wave energy crossing a flat surface that is converted into energy of elastic motion of the solid, he finds the expression

$$(\text{fraction}) = (8\pi G\rho/\omega^2)(s/c)^3 \times \sin^2\theta|\cos\theta|^{-1}[1 + \cos^2\theta + (s/v)\sin^2\theta]. \quad (37.9)$$

Here s and v are the velocities of shear waves and compressive waves, respectively, and θ is the angle between the direction of propagation of the waves and the normal to the surface. Considering a flux of 2×10^{-5} erg/cm^2 sec (an optimistic but conceivable value for waves from a pulsar) incident horizontally ($\theta = \pi/2$; "divergent" factor $|\cos\theta|^{-1}$ cancels out in calculation!), and taking s to be 4.5×10^5 cm/sec and ω to be 6 rad/sec, he calculates that the 1-Hz horizontal displacement produced in the surface has an amplitude of $\xi_0 \sim 2 \times 10^{-17}$ cm, too small by a factor of the order of 10^5 to be detected against background seismic noise. He points to the possibilities of improvements, especially via resonance (elastic waves reflected back and forth between two surfaces; Antarctic ice sheet).

4. Normal-Mode Vibrations of an Elastic Bar

(4) elastic bar

As of 1972, the most often-discussed type of detector is the aluminum bar invented by Joseph Weber (1960, 1961) [see Figures 37.1 and 37.2(d)]. Weber's bars are cylindrical in shape, with length 153 cm, diameter 66 cm, and weight 1.4×10^6 g. Each bar is suspended by a wire in vacuum and is mechanically decoupled from its surroundings. Around its middle are attached piezoelectric strain transducers, which couple into electronic circuits that are sensitive to the bar's fundamental end-to-end mode of oscillation (frequency $\nu = 1{,}660$ Hz). When a gravitational wave hits the bar broadside, as shown in Figure 37.2(d), the relative accelerations carried by the wave will excite the fundamental mode of the bar. As of 1972, Weber observes sudden, simultaneous excitations in two such bars, one at the University of Maryland, near Washington, D.C.; the other at Argonne National Laboratory, near Chicago [see Weber (1969, 1970a,b)]. No one has yet come forward with a workable explanation for Weber's coincidences other than gravitational waves from outer space. However, the history of physics is rich with instances where supposedly new effects had to be attributed in the end to long familiar phenomena. Therefore it would seem difficult to rate the observed events as "battle-tested." To achieve that confidence rating would seem to require confirmation with different equipment, or under different circumstances, or both; experiments to provide that confirmation are now (1972) underway. If one makes this tentative assessment, one can be excused for expressing at the same time the greatest admiration for the experimental ingenuity, energy, and magnificent persistence that Joseph Weber has shown in his more than decade-long search for the most elusive radiation on the books of physics.

Mechanical detectors of the above four types represent systems on which measurements have been made; so practical difficulties and realizable noise levels can be estimated properly. In the continuing search for improved methods, more elaborate detectors are being studied, and in 1972 one can list a number of interesting proposals, as below. For these it is hard to know how much development would be required in order to achieve the desired performance.

5. Normal-Mode Vibrations of Elastic Bodies of Other Shapes

(5) elastic bodies of other shapes

The "bar" of a Weber detector need not be cylindrical in shape. For a discussion of a detector with the shape of a hollow square, a hoop, or a tuning fork, see Douglass (1971); such a detector might allow its fundamental frequency to be adjusted for the most favorable response, with given mass, or given maximum dimension, or both. Sections 37.4 and 37.7 and exercises 37.9 to 37.12 analyze in detail the operation of a "vibrating-bar" detector of arbitrary shape. See also Douglass and Tyson (1971).

6. Angular Accelerations of Rotating Bars

(6) rotating bars ("heterodyne detector")

All the potential detectors described thus far respond in the most obvious of manners to the tidal accelerations of a gravitational wave: relative distances oscillate in and out. But the tidal accelerations contain, in addition to a length-changing component, also a tangential, rotation-producing component. In picture (a) of Box 37.2, the length-changing component dominates near the $\hat{x}$ and $\hat{y}$ axes, whereas the rotation-producing component dominates half-way between the axes. Vladimir B. Braginsky was the first to propose a detector that responds to the rotation-producing accelerations [see Braginsky, Zel'dovich, and Rudenko (1969); Braginsky and Nazarenko (1971)]. It consists of two metal rods, oriented perpendicular to each other, and rotating freely with angular velocity ω_0 in their common plane [see Fig. 37.2(f)]. (The rotation is relative to the gyroscopes of the proper reference frame of the detector; equivalently, it is relative to the Lorentz frame local to the detector.) Let monochromatic gravitational waves of angular frequency $\omega = 2\omega_0$ (change of phase per unit of time equals twice the angular velocity at which the pattern of lines of force turns) impinge broadside on the rotating rods. The righthand circularly polarized component of the waves will then rotate with the rods; so their orientation in its lines-of-force diagram will remain forever fixed. With the orientation of Fig. 37.2(f), rod A will undergo angular acceleration, while rod B will decelerate. The experimenter can search for the constant relative angular acceleration of the two rods (constant so long as the angle between them does not depart significantly from 90°). Better yet, the experimenter can (all too easily) adjust the rotation rate ω_0 so it does not quite match the waves' frequency ω. Then for $\frac{1}{2}\omega_0/|\omega - 2\omega_0|$ rotations, rod A will accelerate and B will decelerate; then will follow $\frac{1}{2}\omega_0/|\omega - 2\omega_0|$ rotations in which A decelerates and B accelerates, and so on (frequency beating). The experimenter can search for oscillations in the relative orientation of the rods. One need not worry about the lefthand polarized waves marring the experiment. Since they do not rotate with the rods, their angular accelerations average out over one cycle.

Such a device is called a "heterodyne detector" by Braginsky. He envisages that such detectors might be placed in free-fall orbits about the Earth late in the 1970's. Heterodyne detectors would work most efficiently for long monochromatic wave trains such as those from pulsars; but even for short bursts of waves they may be more sensitive than vibrating bars [see Braginsky and Nazarenko (1971)].

7. Angular Accelerations of Driven Oscillators

(7) rotation of driven oscillators

Andrei D. Sakharov (1969) has proposed a different type of detector for the angular accelerations of a gravitational wave. Instead of two rotating bars, it consists of two identical, driven oscillators, initially parallel and nonrotating, but oscillating out of phase with each other. Each oscillator experiences angular accelerations in one direction at one phase of a passing wave, and in the opposite direction at the next phase, but the torques do not cancel out. When the oscillator is maximally distended, it experiences a greater torque (acceleration $\propto$ length; torque $\propto$ length^2) than when it is maximally contracted. Consequently, if the driven oscillations have the same angular frequency as a passing, monochromatic wave, and if the phases are as shown in Figure 37.2(g), then oscillator A will receive an angular acceleration in the righthand direction, while B receives an angular acceleration in the lefthand direction.

8. Pumping of Fluid in a Rotating Loop of Pipe

(8) fluid in pipe

A third type of detector that responds to angular accelerations has been described by William Press (1970). This detector would presumably be far less sensitive than others, and therefore not worth constructing; but it is intriguing in its novel design; and it illustrates features of gravitational waves ignored by other detectors. Press's detector consists of a loop of rotating pipe, containing a superfluid. The shape of the pipe and its constant rotation rate are chosen so that the gravitational waves will pump the fluid around inside the pipe. One conceivable pipe design (a bad one to build in practice, but an easy one to analyze) is shown in Fig. 37.2(h). Note that use is made of the variation in tidal acceleration along the direction of propagation of the wave as well as perpendicular to that direction. To analyze the response of the fluid to a righthand circularly polarized wave, one can mentally place the rotating pipe in the three-dimensional line-of-force diagram of Box 37.2(c).

EXERCISES

Exercise 37.6. RELATIVE MOTION OF FREELY FALLING BODIES AS A DETECTOR OF GRAVITATIONAL WAVES [see Figures 37.2(a) and 37.3.]

Consider two test bodies initially at rest with respect to each other in flat, empty spacetime. (The case where other, gravitating bodies are nearby can be treated without too much more difficulty; but this exercise concerns only the simplest example!) A plane, nearly monochromatic gravitational wave, with angular frequency ω and polarization $\boldsymbol{e}_+$, impinges on the bodies, coming from the $-z$ direction. As shown in exercise 35.5, the bodies remain forever at rest in those TT coordinates that constituted the bodies' global inertial frame before the wave arrived. Calculate, for arbitrary separations $(\Delta x, \Delta y, \Delta z)$ of the test bodies, the redshift and the round-trip travel time of photons going back and forth between them. Compare the answer, for large Δx, Δy, Δz, with the answer one would have obtained by using (without justification!) the equation of geodesic deviation. Physically, why does the correct answer *oscillate* with increasing separation? Discuss the feasibility and the potential sensitivity of such a detector using current technology.

Exercise 37.7. EARTH-MOON SEPARATION AS A GRAVITATIONAL-WAVE DETECTOR

In the early 1970's one can monitor the Earth-moon separation using laser ranging to a precision of 10 cm, with successive observations separated by at least one round-trip travel time. Suppose that no oscillations in round-trip travel time are observed except those (of rather long periods) to be expected from the Earth-moon-sun-planets gravitational interaction. What limits can one then place on the energy flux of gravitational waves that pass the Earth? The mathematical formula for the answer should yield numerically

$$\text{Flux} \lesssim 10^{18}\ \text{erg/cm}^2\ \text{sec for } 0.3\ \text{cycle/sec} \lesssim \nu \lesssim 1\ \text{cycle/day}, \tag{37.10a}$$

corresponding to a limit on the mass density in gravitational waves of

$$\text{Density} \lesssim 10^{-13}\ \text{g/cm}^3. \tag{37.10b}$$

Why is this an uninteresting limit?

---

§37.4. VIBRATING, MECHANICAL DETECTORS: INTRODUCTORY REMARKS

The rest of this chapter is Track 2. No earlier track-2 material is needed as preparation for it, nor is it needed for any later chapter.

The remainder of this chapter (except for §37.9) gives a detailed analysis of vibrating, mechanical detectors (Earth; Weber bar; "bars" with complex shapes; and so on).

The details of the analysis and its applications depend in a crucial way on the values of two dimensionless numbers: (1) the ratio τ_{GW}/τ_0, where

$$\tau_{GW} \equiv \begin{pmatrix}\text{characteristic time scale for changes in}\\ \text{gravitational-wave amplitude and spectrum}\end{pmatrix}, \qquad (37.11a)$$

$$\tau_0 \equiv \begin{pmatrix}\text{e-folding time for detector vibrations (in}\\ \text{normal mode of interest) to die out as}\\ \text{a result of internal damping}\end{pmatrix}; \qquad (37.11b)$$

and (2) the ratio $\bar{E}_{\text{vibration}}/kT$, where

$$\bar{E}_{\text{vibration}} \equiv \begin{pmatrix}\text{mean value of detector's vibration energy (in}\\ \text{normal mode of interest) while waves are}\\ \text{passing and driving detector}\end{pmatrix}, \qquad (37.12a)$$

$$\begin{aligned} kT &\equiv \begin{pmatrix}\text{Boltzman's}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{detector's}\\ \text{temperature}\end{pmatrix} \\ &= \begin{pmatrix}\text{Mean energy in normal mode}\\ \text{of interest when gravitational}\\ \text{waves are not exciting it}\end{pmatrix}. \end{aligned} \qquad (37.12b)$$

Definitions: "steady flux," "hammer-blow waves," "wave-dominated detector," "noisy detector"

When $\tau_{GW} \gg \tau_0$, the detector views the radiation as having a "*steady flux,*" and it responds with steady-state vibrations; when $\tau_{GW} \ll \tau_0$ (short burst of waves), the waves deal a "*hammer blow*" to the detector. When $\bar{E}_{\text{vibration}} \gg kT$, the driving force of the waves dominates over the detector's random, internal, Brownian-noise forces ("*wave-dominated detector*"); when $\bar{E}_{\text{vibration}} \lesssim kT$, the driving force of the waves must compete with the detector's random, internal, Brownian-noise forces ("*noisy detector*").

Sections 37.5 to 37.7 deal with wave-dominated detectors ($\bar{E}_{\text{vibration}} \gg kT$). The key results of those sections are summarized in Box 37.3, which appears here as a quick preview (though it may not be fully understandable in advance). Section 37.8 treats noisy detectors.

Design of detectors requires much more than the concept of cross section

Warning: Throughout the rest of this chapter prime attention focuses on the concept of cross section. This is fine for a first introduction to the theory of detectors. But cross section is not the entire story, especially when one wishes to study the detailed wave-form of the radiation. And sometimes (e.g., for the detector of Figure 37.2a), it is *none* of the story. A first-rate experimenter designing a new detector will not deal primarily in cross sections any more than a radio engineer will in designing a new radio telescope. Attention will also focus heavily on the bandwidth

(*continued on page 1022*)

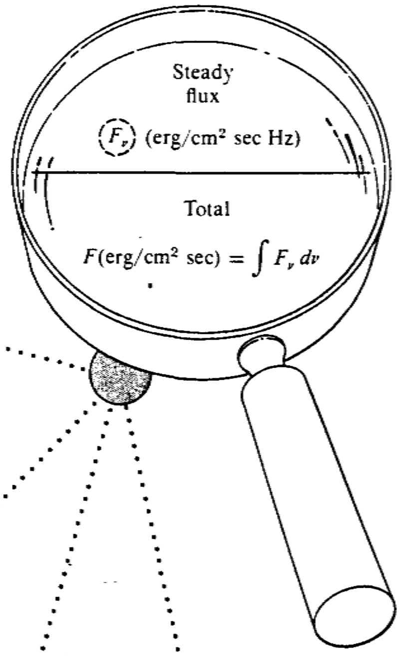
Box 37.3 WAYS TO USE CROSS SECTION FOR WAVE-DOMINATED DETECTORS

A. To Calculate *Rate* at which Detector Extracts Energy from a *Steady* Flux of Radiation

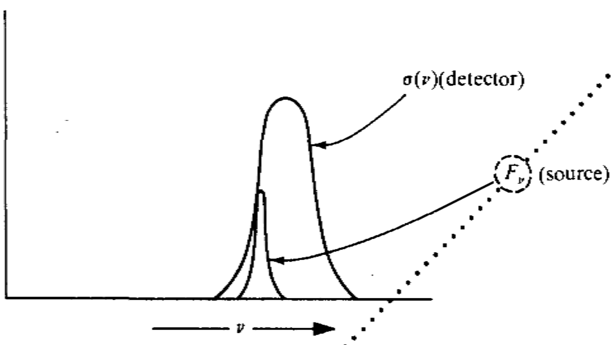
$(\tau_{GW} \gg \tau_0)$

1. Frequency distribution of radiation arbitrary:

$$\begin{pmatrix}\text{steady rate at which detector extracts}\\ \text{energy from gravitational waves}\end{pmatrix} = \int \underbrace{F_\nu(\nu)}_{\text{erg/cm}^2\text{ sec Hz}} \cdot \underbrace{\sigma(\nu)}_{\text{cm}^2} \underbrace{d\nu}_{\text{Hz}}$$



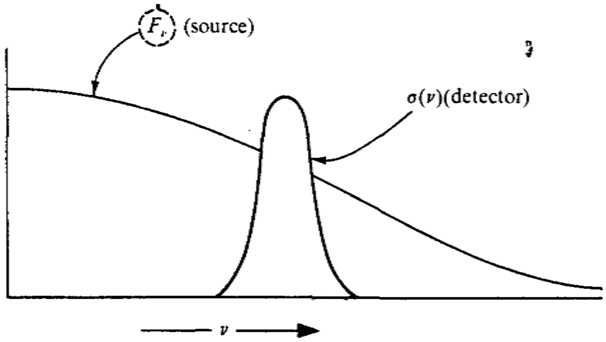
2. Frequency spread of radiation small compared to line width of detector:



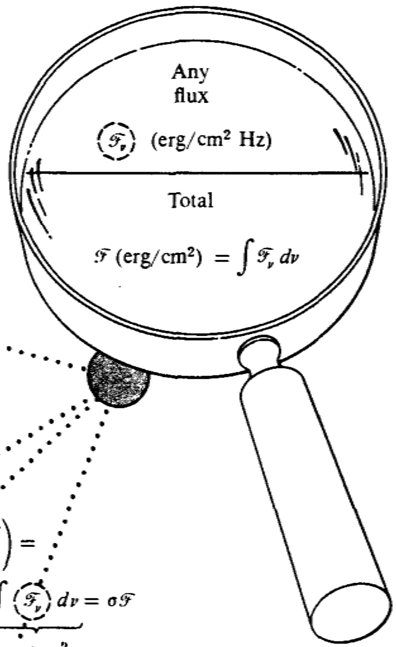
$$\begin{pmatrix}\text{Steady rate at which}\\ \text{detector extracts energy}\\ \text{from gravitational waves}\end{pmatrix} = \sigma(\nu_{\text{source}}) \int F_\nu\, d\nu = \sigma F$$

3. Frequency spread of radiation large compared to line width of detector:

$$\begin{pmatrix}\text{steady rate at which detector extracts}\\ \text{energy from gravitational waves}\end{pmatrix} = \underbrace{F_\nu(\nu_{\text{detector}})}_{\text{erg/cm}^2\text{ sec Hz}} \underbrace{\int \sigma(\nu)\, d\nu}_{\text{"resonance integral", cm}^2\text{ Hz}}$$



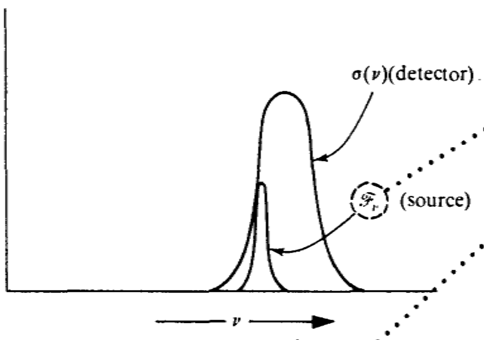
B. To Calculate *Total* Energy Deposited in Detector by *any* Pessing Wave train



1. If frequency distribution of radiation is arbitrary:

$$\begin{pmatrix}\text{total energy}\\ \text{deposited}\end{pmatrix} = \int \underbrace{\mathcal{F}_\nu(\nu)}_{\text{erg/cm}^2\,\text{Hz}} \cdot \underbrace{\sigma(\nu)}_{\text{cm}^2} \underbrace{d\nu}_{\text{Hz}}$$

2. If frequency spread of radiation is small compared to line width of detector ("monochromatic waves"):



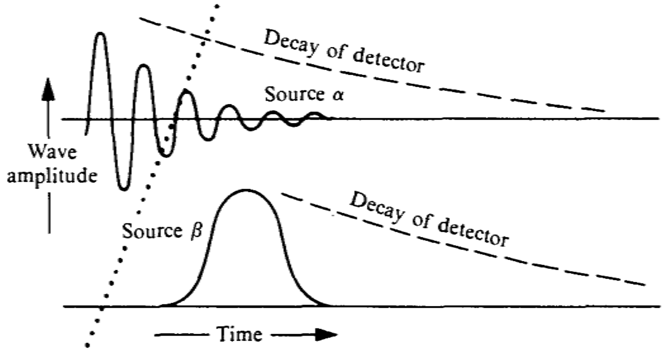
$$\begin{pmatrix}\text{total energy}\\ \text{deposited}\end{pmatrix} = \underbrace{\sigma(\nu_{\text{source}})}_{\text{cm}^2} \underbrace{\int \mathcal{F}_\nu \, d\nu}_{\text{erg/cm}^2} = \sigma\mathcal{F}$$

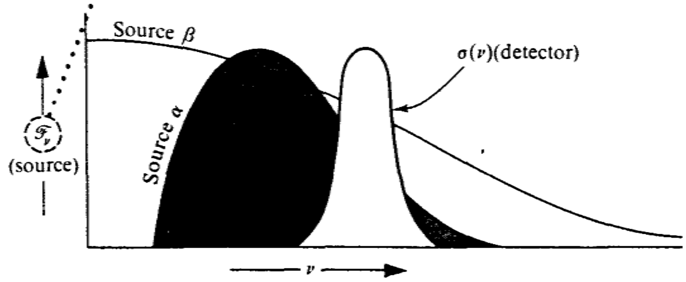
3. If frequency spread of radiation is large compared to line width of detector (as it *must* be for hammer-blow radiation, where

$\Delta\nu_{\text{source}} \gtrsim 1/4\pi\tau_{\text{GW}} \gg 1/4\pi\tau_0 = \Delta\nu_{\text{detector}}$):

(total energy deposited) =

$$\underbrace{\mathcal{F}_\nu(\nu_{\text{detector}})}_{\text{erg/cm}^2\,\text{Hz}} \underbrace{\int \sigma(\nu)\, d\nu}_{\substack{\text{cm}^2\ \text{Hz,}\\ \text{"resonance}\\ \text{integral"}}}$$





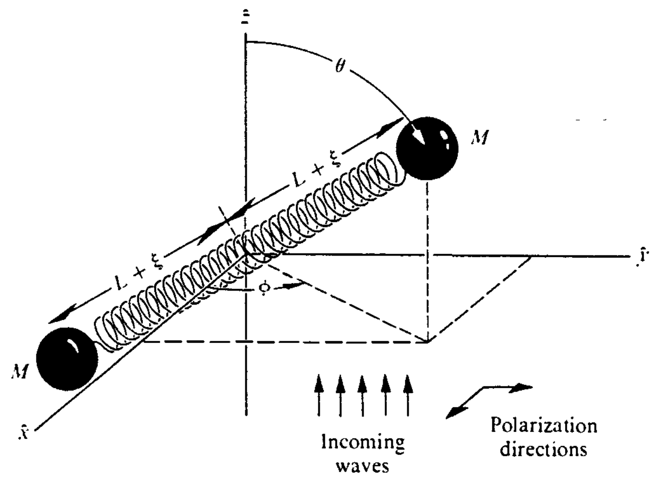


Figure 37.4.
An idealized detector (vibrator) responding to linearly polarized gravitational waves.

of the antenna, and on other, more detailed characteristics of its response, on coupling of the antenna to the displacement sensor, on response characteristics of the sensor, on antenna noise, on sensor noise, and so on. For an overview of these issues, and for discussions of detectors for which the concept of cross section is useless, see, e.g., Press and Thorne (1972).

§37.5. IDEALIZED WAVE-DOMINATED DETECTOR, EXCITED BY STEADY FLUX OF MONOCHROMATIC WAVES

Idealized detector: oscillator driven by a steady flux of monochromatic waves:

(1) derivation of equation of motion

Begin with the case of a *wave-dominated detector* ($\bar{E}_{\text{vibration}} \gg kT$) being driven by a steady flux of radiation ($\tau_{\text{GW}} \gg \tau_0$). Deal at first, not with a solid bar of arbitrary shape, but rather with the idealized detector of Figure 37.4: an oscillator made of two masses M on the ends of a spring of equilibrium length $2L$. Let the detector have a natural frequency of vibration ω_0 and a damping time $\tau_0 \gg 1/\omega_0$, so that its equation of motion (in the detector's proper reference frame) is

$$\ddot{\xi} + \dot{\xi}/\tau_0 + \omega_0^2\xi = \text{driving acceleration.} \tag{37.13}$$

Let gravitational waves of polarization $\boldsymbol{e}_+$ and angular frequency ω impinge on the detector from the $-\hat{z}$ direction; and let the polar angles of the detector relative to the wave-determined $\hat{x}, \hat{y}, \hat{z}$-axes be θ and ϕ.

The incoming waves are described by equations (37.1) with the amplitude

$$A_\times = 0, \qquad A_+ = \mathscr{A}_+ e^{-i\omega(t-z)}. \tag{37.14}$$

(Here and throughout one must take the real part of all complex expressions.)

Assume that the detector is much smaller than a wavelength, so that one can set $z \approx \hat{z} = 0$ throughout it. Then the tidal accelerations produced by the wave

$$\left(\frac{d^2\hat{x}}{d\hat{t}^2}\right)_{\text{due to wave}} = -R_{\hat{x}\hat{0}\hat{j}\hat{0}}x^{\hat{j}} = -\frac{1}{2}\omega^2 \mathcal{A}_+ e^{-i\omega\hat{t}}\hat{x},$$

$$\left(\frac{d^2\hat{y}}{d\hat{t}^2}\right)_{\text{due to wave}} = -R_{\hat{y}\hat{0}\hat{j}\hat{0}}x^{\hat{j}} = +\frac{1}{2}\omega^2 \mathcal{A}_+ e^{-i\omega\hat{t}}\hat{y},$$

have as their component along the oscillator

$$\frac{d^2\xi}{dt^2} = \frac{\hat{x}}{L}\frac{d^2\hat{x}}{dt^2} + \frac{\hat{y}}{L}\frac{d^2\hat{y}}{dt^2} + \frac{\hat{z}}{L}\frac{d^2\hat{z}}{dt^2} = -\frac{1}{2}\omega^2\mathcal{A}_+ L e^{-i\omega\hat{t}}\frac{\hat{x}^2 - \hat{y}^2}{L^2}$$
$$= -\frac{1}{2}\omega^2\mathcal{A}_+ L e^{-i\omega\hat{t}} \sin^2\theta \cos 2\phi.$$

Consequently, the equation of motion for the oscillator is

$$\ddot{\xi} + \dot{\xi}/\tau_0 + \omega_0{}^2\xi = -\frac{1}{2}\omega^2\mathcal{A}_+ L e^{-i\omega\hat{t}} \sin^2\theta \cos 2\phi. \tag{37.15}$$

The driving force varies as $\cos 2\phi$ because of the "spin-2" nature of gravitational waves: a rotation through 180° in the transverse plane leaves the waves unchanged; a rotation through 90° reverses the phase. The $\sin^2\theta$ term results from the transverse nature of the waves [one factor of $\sin\theta$ to account for projection onto the detector's direction], plus their tidal-force nature [another factor of $\sin\theta$ to account for (relative force) $\propto$ (distance *in transverse plane*)].

The straightforward steady-state solution of the equation of motion (37.15) is

$$\xi = \frac{\frac{1}{2}\omega^2\mathcal{A}_+ L \sin^2\theta \cos 2\phi}{\omega^2 - \omega_0{}^2 + i\omega/\tau_0} e^{-i\omega\hat{t}}. \tag{37.16}$$

(2) oscillator amplitude as function of frequency and orientation

When the incoming waves are near resonance with the detector, $|\omega \pm \omega_0| \lesssim 1/\tau_0$, the oscillator is excited to large amplitude. Otherwise the excitation is small. Focus attention henceforth on near-resonance excitations; then equation (37.16) can be simplified (*note:* ω_0 is positive, but ω may be negative or positive):

$$\xi = \frac{\frac{1}{4}\omega_0\mathcal{A}_+ L \sin^2\theta \cos 2\phi}{|\omega| - \omega_0 + \frac{1}{2}\text{sgn}(\omega) i/\tau_0} e^{-i\omega\hat{t}}. \tag{37.16'}$$

One measure of the detector's usefulness is its cross section for absorbing gravitational-wave energy. The steady-state vibrational energy in a detector with the above amplitude and with 2 masses M is

$$E_{\text{vibration}} = 2 \cdot \frac{1}{2} \cdot M \cdot (\dot{\xi}^2)_{\max} = \frac{\frac{1}{16} ML^2\omega_0{}^4\mathcal{A}_+{}^2 \sin^4\theta \cos^2 2\phi}{(|\omega| - \omega_0)^2 + (1/2\tau_0)^2}. \tag{37.17}$$

This energy is being dissipated internally at a rate $E_{\rm vibration}/\tau_0$. If one ignores reradiation of energy as gravitational waves (a negligible process!), one can equate the dissipation rate to the rate at which the detector absorbs energy from the incoming waves—which in turn equals the "cross section" σ times the incoming flux:

$$E_{\rm vibration}/\tau_0 = -dE_{\rm waves}/dt = \sigma T^{0z(\rm GW)} = \frac{1}{32\pi}\sigma\omega^2 \mathcal{a}_+{}^2.$$

(3) cross sections for polarized radiation

Consequently, *near resonance* ($|\omega \pm \omega_0| \ll \omega_0$), *the cross section* for interception of gravitational-wave energy is

$$\sigma = \frac{2\pi ML^2(\omega_0{}^2/\tau_0)\sin^4\theta\cos^2 2\phi}{(|\omega| - \omega_0)^2 + (1/2\tau_0)^2}, \qquad \text{for polarized radiation.} \tag{37.18}$$

This expression applies to monochromatic radiation. However, experience with many other kinds of waves has taught that one often has to deal with a broad continuum of frequencies, with the "bandwidth" of the incident radiation far greater than the width of the detector resonance (see Box 37.3). Under these conditions, the relevant quantity is not the cross section itself, but the "resonance integral" of the cross section,

$$\int_{\rm resonance} \sigma\, d\nu = \int \sigma(d\omega/2\pi) = 2\pi ML^2\omega_0{}^2\sin^4\theta\cos^2 2\phi, \quad \text{for polarized radiation.} \tag{37.19}$$

Before examining the magnitude of this cross section, scrutinize its directionality (the "antenna-beam pattern"). The factor of $\sin^4\theta\cos^2 2\phi$ refers to linearly polarized, $\boldsymbol{e}_+$ radiation (see Figure 37.4). For the orthogonal mode of polarization, $\boldsymbol{e}_\times$, $\cos^2 2\phi$ is to be replaced by $\sin^2 2\phi$; and for unpolarized (incoherent mixture) radiation or circularly polarized radiation, the cross section is the average of these two expressions; thus

(4) cross sections for unpolarized radiation

$$\sigma = \frac{\pi ML^2(\omega_0{}^2/\tau_0)\sin^4\theta}{(|\omega| - \omega_0)^2 + (1/2\tau_0)^2} \qquad \text{for unpolarized radiation.} \tag{37.20}$$

Notice that this unpolarized cross section is peaked, with half-width 33°, about the equatorial plane of the detector. Averaged over all possible directions of incoming waves, the cross section is

$$\begin{aligned}\langle\sigma\rangle_{\rm all\ directions} &= \frac{1}{2}\int_0^\pi \sigma\sin\theta\, d\theta = \frac{8}{15}\sigma_{\max} \\ &= \frac{(8\pi/15)ML^2(\omega_0{}^2/\tau_0)}{(|\omega| - \omega_0)^2 + (1/2\tau_0)^2} \ \text{for unpolarized radiation.}\end{aligned} \tag{37.21}$$

One can rewrite the above cross sections in several suggestive forms. For example, on resonance, the cross section (37.21) reads

$$\langle\sigma\rangle_{\rm all\ directions} = \frac{4\pi^2}{15}\frac{4M}{2\pi/\omega_0}(\omega_0\tau_0)(2L)^2.$$

Recall that $\omega_0\tau_0$ defines the "Q" of a detector, $1/Q \equiv$ (fraction of *energy* dissipated per radian of oscillation). Note that $2\pi/\omega_0$ is the wavelength λ_0 of resonant radiation. Finally, denote by $r_g = 4M$ the gravitational radius of the detector. In terms of these three familiar quantities, find for the cross section the formula

Magnitude of cross sections for any resonant detector

$$\frac{\langle\sigma\rangle_{\text{all directions}}}{(2L)^2} = \frac{(\text{cross section for absorbing waves on resonance})}{(\text{"geometric" cross section of detector})}$$
$$= (4\pi^2/15)(r_g/\lambda_0)Q \quad \text{for unpolarized radiation on resonance.} \tag{37.22}$$

This relation holds in order of magnitude for any resonant detector. It shows starkly that gravitational-wave astronomy must be a difficult enterprise. How large could *you* make the factor r_g/λ_0, given a reasonable budget? Weber's 1970 detectors have $2L_{\text{effective}} \approx 1$ meter, $r_g \approx (0.74 \times 10^{-28}\ \text{cm/g}) \times (10^6\ \text{g}) \approx 10^{-22}$ cm, $\nu_0 = \omega_0/2\pi = 1{,}660$ Hz, $\lambda_0 \approx 200$ km, $r_g/\lambda_0 \approx \frac{1}{2} \times 10^{-29}$, $\tau_0 \approx 20$ sec, $Q \approx 2 \times 10^5$; so that

$$\sigma_{\text{Weber}} \approx 3 \times 10^{-20}\ \text{cm}^2 \text{ on resonance.} \tag{37.23}$$

Flux required to excite a Weber-type detector

What flux of gravitational-wave energy would have to be incident to excite a cold detector ($\sim 0°$ K) into roughly steady-state vibrations with a vibration energy of (Boltzmann's constant) $\times$ (room temperature) $\sim 4 \times 10^{-14}$ erg? The vibrator, if cooled enough to be wave-dominated, dissipates its energy at the rate $E_{\text{vibration}}/\tau_0 \sim 2 \times 10^{-15}$ erg/sec. The incident flux has to make up this loss, at the rate

$$T_{00}^{(\text{GW})}\sigma \sim 2 \times 10^{-15}\ \text{erg/sec}, \tag{37.24a}$$

implying an incident flux of the order of $2 \times 10^{-15}/3 \times 10^{-20} \sim 10^5$ erg/cm^2 sec. Moreover, this flux has to be concentrated in the narrow range of resonance

$$\nu \approx \nu_0 \pm 1/4\pi\tau_0 = (1660 \pm 0.004)\ \text{Hz}. \tag{37.24b}$$

By anybody's standards, this is a very high flux of gravitational radiation for such a small bandwidth ($\sim 10^7$ erg/cm^2 sec Hz, as compared to the flux of blackbody gravitational radiation, $8\pi\nu^2 kT/c^2 = 3 \times 10^{-27}$ erg/cm^2 sec Hz, that would correspond to Planck equilibrium at the same temperature; the large factor of difference is a direct reflection of the difference in rate of damping of the oscillator by friction and by gravitational radiation).

A large Q is not necessarily optimal

Equation (37.22) makes it seem that an optimal detector must have a large Q. This is not necessarily so. Recall that the bandwidth, $\Delta\omega \approx \omega_0/Q$, over which the cross section is large, decreases with increasing Q. When an incoming steady flux of waves of bandwidth $\Delta\omega \gg \omega_0/Q \equiv 1/\tau_0$ and of specific flux

$$F_\nu(\text{erg/cm}^2\ \text{sec Hz})$$

drives the detector, it deposits energy at the rate

2

$$\begin{pmatrix}\text{rate of deposit}\\ \text{of energy}\end{pmatrix} = \frac{dE}{dt} = \int_{\text{resonance}} F_\nu \sigma\, d\nu \underset{\uparrow}{=} F_\nu(\nu_0) \int_{\text{resonance}} \sigma\, d\nu.$$
$$\begin{bmatrix}\text{for radiation with}\\ \text{bandwidth } \Delta\nu \gg 1/\tau_0\end{bmatrix}$$

Consequently, the relevant measure of detector effectiveness will be the integral of the cross section over the resonance, $\int \sigma\, d\nu$ (37.19). (See next section.) This frequency-integrated cross section is independent of the detector's Q, so one must use more sophisticated reasoning (e.g., signal-to-noise theory) in deciding whether a large Q is desirable. (See §37.8).

§37.6. IDEALIZED, WAVE-DOMINATED DETECTOR, EXCITED BY ARBITRARY FLUX OF RADIATION

Response of idealized detector to an arbitrary, non-monochromatic flux:

(1) derivation

Let plane-polarized waves of polarization $\boldsymbol{e}_+$ but *arbitrary* spectrum [equation (37.1) with $A_\times = 0$] impinge on the idealized detector of Figure 37.4. Then the equation of motion for the detector is the same as for monochromatic waves [equation (37.15)], but with $-\omega^2 \mathcal{A}_+ e^{-i\omega t}$ replaced by $\ddot{A}_+$:

$$\ddot{\xi} + \dot{\xi}/\tau_0 + \omega_0{}^2 \xi = \frac{1}{2} \ddot{A}_+ L \sin^2\theta \cos 2\phi. \tag{37.26}$$

[By now one is fully accustomed to the fact that all analyses of detectors (when the detector is much smaller than the wavelength of the waves) are performed in the proper reference frame, with coordinates $\hat{t}, \hat{x}, \hat{y}, \hat{z}$. *Henceforth, for ease of eyesight, abandon the "hats" on these "proper coordinates," and denote them as merely t, x, y, z.*]

Fourier-analyze the waves and the detector displacement,

$$A_+(t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \tilde{A}_+(\omega) e^{-i\omega t}, \tag{37.27a}$$

$$\xi(t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \tilde{\xi}(\omega) e^{-i\omega t}; \tag{37.27b}$$

and conclude from equation (37.26) that

$$\tilde{\xi} = \frac{\frac{1}{2}\omega^2 \tilde{A}_+ L \sin^2\theta \cos 2\phi}{\omega^2 - \omega_0{}^2 + i\omega/\tau_0}.$$

This Fourier amplitude is negligible unless $|\omega \pm \omega_0| \ll \omega_0$; consequently, without loss of accuracy, one can rewrite it as

$$\tilde{\xi} = \frac{\frac{1}{4}\,\omega_0 \tilde{A}_+ L \sin^2\theta \cos 2\phi}{|\omega| - \omega_0 + \frac{1}{2}\,\mathrm{sgn}(\omega) i/\tau_0}. \tag{37.28}$$

[Compare with the steady-state amplitude (37.16′).]

Ask how much total energy is deposited in the detector by the gravitational waves. Do *not* seek an answer by examining the amplitude of the vibrations, $\xi(t)$, directly; since that amplitude is governed by *both* internal damping and the driving force of the waves, it does not reflect directly the energy deposited. To get the total energy deposited, integrate over time the force acting on each mass multiplied by its velocity:

$$\begin{pmatrix}\text{total energy}\\ \text{deposited}\end{pmatrix} = \int_{-\infty}^{+\infty} \underbrace{2}_{\text{[2 masses]}} \underbrace{\left(\frac{1}{2} M \ddot{A}_+ L \sin^2\theta \cos 2\phi\right)}_{\text{[force on each mass]}} \underbrace{\dot{\xi}}_{\text{[velocity of each mass]}}\, dt.$$

Use Parseval's theorem (one of the most powerful tools of mathematical physics!) to replace the time integral by a frequency integral

$$\begin{pmatrix}\text{total energy}\\ \text{deposited}\end{pmatrix} = \Re \int_{-\infty}^{+\infty} (ML \sin^2\theta \cos 2\phi)(-\omega^2 \tilde{A}_+^*)(-i\omega\tilde{\xi})\, d\omega.$$

Then use equation (37.28) to rewrite this entirely in terms of the wave amplitude

$$\begin{pmatrix}\text{total energy}\\ \text{deposited}\end{pmatrix} = \int_{-\infty}^{+\infty} \left[\frac{2\pi(\omega_0{}^2/\tau_0) ML^2 \sin^4\theta \cos^2 2\phi}{(|\omega| - \omega_0)^2 + (1/2\tau_0)^2}\right]\left[\frac{\omega^2 |\tilde{A}_+|^2}{16\pi}\right] d\omega. \tag{37.29}$$

The first term in this expression is precisely the cross section for monochromatic waves, derived in the last section (37.18). The second term has an equally simple interpretation: the total energy that the gravitational waves carry past a unit surface area of detector is

$$\begin{aligned}\mathcal{F}(\text{ergs/cm}^2) &= \int T_{00}^{(\mathrm{GW})}\, dt = \int \frac{1}{16\pi} \dot{A}_+{}^2\, dt \\ &= \int \frac{\omega^2 |\tilde{A}_+|^2}{16\pi}\, d\omega = \int \frac{\omega^2 |\tilde{A}_+|^2}{8}\, d\nu\end{aligned} \tag{37.30}$$

(Parseval's theorem again!). Consequently, the energy per unit frequency interval, per unit area carried by the waves is

$$\mathcal{F}_\nu(\text{ergs/cm}^2\ \text{Hz}) = \frac{1}{8}\,\omega^2 |\tilde{A}_+|^2 \tag{37.31}$$

[for $-\infty < \nu < +\infty$; double this for $0 < \nu < +\infty$, a convention we use for the rest of this chapter]. This is 2π times the second term in (37.29).

Combining equations (37.18), (37.29), and (37.31), then, one finds

$$\begin{pmatrix}\text{total energy}\\ \text{deposited}\end{pmatrix} = \int \sigma(\nu) \mathcal{F}_\nu(\nu)\, d\nu. \tag{37.32}$$

(2) answer—

$$\begin{pmatrix}\text{energy}\\ \text{deposited}\end{pmatrix} = \int \sigma \mathcal{F}_\nu\, d\nu$$

2

This is the total energy deposited, regardless of the spectrum of the waves, and regardless of whether they come in a steady flux for a long time, or in a short burst, or in any other form. It is perfectly general—so long as the detector is wave-dominated ($E_{\text{vibration}} \gg kT$) while the waves are driving it.

How one can measure energy deposited

How can an experimenter measure the total energy deposited? He cannot measure it directly, in general, but he can measure a quantity equal to it: the total energy that goes into internal damping, i.e., into "friction." Energy is removed by "friction" at a rate $E_{\text{vibration}}/\tau_0$, when the vibration energy is much greater than kT (during period of wave-dominance). Therefore, the experimenter can measure

$$\begin{pmatrix}\text{total energy}\\ \text{deposited}\end{pmatrix} = \frac{1}{\tau_0}\underbrace{\int}_{\text{integrate over the period that } E_{\text{vibration}} \gg kT} E_{\text{vibration}}\, dt, \qquad \text{in general.} \tag{37.33}$$

In the special case of "hammer-blow waves" ($\tau_{\text{GW}} =$ duration of waves $\ll \tau_0$), the vibration energy is driven "instantaneously" from $\sim kT$ to a peak value, $E^{\text{peak}}_{\text{vibration}} \gg kT$, and then decays exponentially back to $\sim kT$; thus

$$\begin{pmatrix}\text{total energy}\\ \text{deposited}\end{pmatrix} = \frac{1}{\tau_0}\int_0^\infty E^{\text{peak}}_{\text{vibration}} e^{-t/\tau_0}\, dt = E^{\text{peak}}_{\text{vibration}} \tag{37.34}$$

for hammer-blow waves.

When the waves are steady for a long period of time ($\tau_{\text{GW}} \gg \tau_0$), with specific flux

$$F_\nu = \mathcal{F}_\nu/\tau_{\text{GW}} \qquad (\text{ergs/cm}^2\ \text{sec Hz}),$$

then the energy will be deposited at a constant rate

$$(dE/dt) = (\text{total energy deposited})/\tau_{\text{GW}};$$

and equation (37.32) can be rewritten

$$\begin{pmatrix}\text{rate of deposit}\\ \text{of energy}\end{pmatrix} = \int \sigma(\nu) F_\nu\, d\nu, \qquad \text{for steady waves } (\tau_{\text{GW}} \gg \tau_0). \tag{37.35}$$

Equations (37.32) and (37.35) are the key equations for application of the concept of cross section to realistic situations. They are applicable not only to polarized radiation, but also to unpolarized radiation and to radiation coming in from all directions, if one merely makes sure to use the appropriate cross section [equation (37.20) or (37.21) instead of (37.18)]. For examples of their application, see Box 37.3.

§37.7. GENERAL WAVE-DOMINATED DETECTOR, EXCITED BY ARBITRARY FLUX OF RADIATION

The cross sections of the idealized spring-plus-mass detector can be put into a form more elegant than equations (37.18) to (37.21)—a form that makes contact with many branches of physics, and is valid for *any* vibrating resonant detector whatsoever.

Introduce the "Einstein A-coefficients," which describe the rate at which a unit amount of detector energy is lost to internal damping and to reradiation of gravitational waves:

Cross sections reexpressed in terms of "Einstein A-coefficients"

$$A_{\rm diss} \equiv \left(\frac{\text{rate at which energy is dissipated internally}}{\text{energy in oscillations of detector}}\right) = \frac{1}{\tau_0}, \tag{37.36a}$$

$$A_{\rm GW} \equiv \left(\frac{\text{rate at which energy is reradiated}}{\text{energy in oscillations}}\right). \tag{37.36b}$$

For the idealized detector of Figure 37.4, the standard formula (36.1) for the emission of gravitational waves yields

$$(\text{power reradiated}) = \frac{32}{15}\omega^6 M^2 L^2 \langle \xi^2 \rangle_{\rm time\,avg.} \tag{37.37}$$

(see exercise 37.8). Consequently

$$A_{\rm GW} = \frac{16}{15} M L^2 \omega^4. \tag{37.38}$$

One can use these relations to rewrite the detector cross sections in terms of $A_{\rm diss}$, $A_{\rm GW}$, and the reduced wavelength

$$\lambdabar \equiv 1/\omega \tag{37.39}$$

of the radiation. For example, the cross section (37.21)—now with $\omega \geq 0$—is

$$\langle \sigma \rangle_{\rm all\,directions} = \frac{1}{2}\pi \lambdabar^2 \frac{A_{\rm GW} A_{\rm diss}}{(\omega - \omega_0)^2 + (A_{\rm diss}/2)^2} \quad \text{for unpolarized radiation} \tag{37.40}$$

(*recall the assumption* $|\omega - \omega_0| \ll \omega_0$ *in all cross-section formulas*) and the corresponding integral over the resonance is

$$\int \langle \sigma \rangle_{\rm all\,directions}\, d\nu = \frac{1}{2}\pi \lambdabar_0^2 A_{\rm GW} \quad \text{for polarized radiation.} \tag{37.41}$$

Generality of the A-coefficient formalism

These expressions for the cross section are comprehensive in their application. They apply to any vibrating, resonant, gravitational-wave detector whatsoever, as one sees from the "detailed balance" calculation of exercise 37.9, and from the dynamic calculations of exercise 37.10. They also apply, with obvious changes in statistical factors and notation, to compound-nucleus reactions in nuclear physics ("Breit-Wigner formula"; see Blatt and Weisskopf, pp. 392–94, 408–10, 555–59), to the absorption of photons by atoms and molecules, to reception of electromagnetic waves by a television antenna, etc. Equation (37.41) says in effect, "Calculate the rate at which the oscillator is damped by emission of gravitational radiation; multiply that rate by the geometric factor familiar in all work with antennas, $\frac{1}{2}\pi\lambdabar_0^2$, and immediately obtain the resonance integral of the cross section. The result is expressed in geometric

2

units (cm). To get the resonance integral in conventional units, multiply by the conversion factor $c = 3 \times 10^{10}$ cm Hz.

The 'dynamic analysis' of the idealized masses-on-spring detector, as developed in the last section, is readily extended to a vibrating detector of arbitrary shape (Earth; Weber's bar; an automobile fender; and so on). The extension is carried out in exercise 37.10 and its main results are summarized in Box 37.4.

Scattering of radiation by detector

Part of the energy that goes into a detector is reradiated as scattered gravitational radiation. For any detector of laboratory dimensions with laboratory damping coefficients, this fraction is fantastically small. However, in principle one can envisage a larger system and conditions where the reradiation is not at all negligible. In such an instance one is dealing with scattering. No attempt is made here to analyze such scattering processes. For a simple order-of-magnitude treatment, one can use the same type of Breit-Wigner scattering formula that one employs to calculate the scattering of neutrons at a nuclear resonance or photons at an optical resonance. A still more detailed account will analyze the correlation between the polarization of the scattered radiation and the polarization of the incident radiation. The kind of formalism useful here for gravitational radiation with its tensor character will be very much like that now used to treat polarization of radiation with a spin-1 character. Here notice especially the "Madison Convention" [Barschall and Haeberli (1971)] developed by the collaborative efforts of many workers after experience during many years with a variety of conflicting notations. Considering the way in which the best notation that is available today for spin-1 radiation was evolved, one can only feel that it is too early to canonize any one notation for describing the scattering parameters for an object that is scattering gravitational radiation.

EXERCISES

Exercise 37.8. POWER RERADIATED

The idealized gravitational wave detector of Figure 37.4 vibrates with angular frequency ω. Show that the power it radiates as gravitational waves is given by equation (37.37).

Exercise 37.9. CROSS SECTIONS CALCULATED BY DETAILED BALANCE

Use the principle of detailed balance to derive the cross sections (37.41) for a vibrating, resonant detector of any size, shape, or mass (e.g., for the vibrating Earth, or Weber's vibrating cylinder, or the idealized detector of Figure 37.4). [*Hints:* Let the detector be in thermal equilibrium with a bath of blackbody gravitational waves. Then it must be losing energy by reradiation as rapidly as it is absorbing it from the waves. (Internal damping can be ignored because, in true thermal equilibrium, energy loss by internal damping will match energy gain from random internal Brownian forces.) In detail, the balance of energy in and out reads [with I_ν = "specific intensity," equation (22.48)]

$$[4\pi I_\nu(\nu = \nu_0)]_{\text{blackbody}} \times \int \langle\sigma\rangle_{\text{all directions}}\, d\nu$$

$$= A_{\text{GW}} \times (\text{Energy in normal mode of detector}).$$

Solve for $\int\langle\sigma\rangle\, d\nu$, using the familiar form of the Planck spectrum and the fact that gravitational waves have two independent states of polarization.] *Note:* Because detailed balance

Box 37.4 VIBRATING, RESONANT DETECTOR OF ARBITRARY SHAPE

A. Physical Characteristics of Detector

1. Detector is a solid object (Earth, Weber bar, automobile fender, . . .) with density distribution $\rho(x)$ and total mass $M = \int \rho\, d^3x$.
2. Detector has normal modes of vibration. The nth normal mode is characterized by:

$$\omega_n = \text{angular frequency;}$$

$$\tau_n = \begin{pmatrix}\text{e-folding time for vibration energy}\\ \text{to decay as result of internal damping}\end{pmatrix} \gg 1/\omega_n; \tag{1}$$

$$\boldsymbol{u}_n(x) = \text{eigenfunction (defined here to be dimensionless and real).}$$

The eigenfunctions $\boldsymbol{u}_n$ are orthonormalized, so that

$$\int \rho \boldsymbol{u}_n \cdot \boldsymbol{u}_m\, d^3x = M\delta_{nm}. \tag{2}$$

3. During a normal-mode vibration with $E_{\text{vibration}} \gg kT$, a mass element originally at $\hat{x}$ receives the displacement

$$\delta \boldsymbol{x} = \boldsymbol{\xi} = \boldsymbol{u}_n(x)\underbrace{\mathscr{B}_n}_{\text{[constant amplitude]}} e^{-i\omega_n t - t/\tau_n}, \tag{3a}$$

the density at fixed x changes by

$$\delta\rho = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_n)\mathscr{B}_n e^{-i\omega_n t - t/\tau_n}, \tag{3b}$$

and the moment of inertia tensor oscillates

$$\delta I_{jk} = I_{(n)jk}\mathscr{B}_n e^{-i\omega_n t - t/\tau_n}. \tag{3c}$$

Here $I_{(n)jk}$ is the "*moment of inertia factor for the nth normal mode*":

$$\begin{aligned} I_{(n)jk} &\equiv \int -(\rho u_n{}^l)_{,l} x^j x^k\, d^3x \\ &= \int \rho(u_n{}^j x^k + u_n{}^k x^j)\, d^3x \end{aligned} \tag{4}$$

[dimensions: mass $\times$ length, multiply by $\mathscr{B}_n$ (length) to get I_{jk}].

The corresponding "*reduced quadrupole factor for the nth normal mode*" is

$$\text{I}\!\!\!-_{(n)jk} \equiv I_{(n)jk} - \frac{1}{3} I_{(n)ll}\, \delta_{jk}. \tag{5}$$

Box 37.4 (*continued*)

B. Cross Sections for Detector (exercise 37.10)

1. For *polarized radiation* with propagation direction $\boldsymbol{n}$ and polarization tensor $\boldsymbol{e}$:

$$h_{jk} = A(t - \boldsymbol{n} \cdot \boldsymbol{x})e_{jk},$$

$$e_{jk}n_k = 0, \qquad e_{jj} = 0, \qquad e_{jk}e_{jk} = 2; \tag{6}$$

$$\sigma_n(\nu) = \sigma_n(\omega/2\pi) = \frac{\pi}{4}\frac{|\text{\nolinebreak}I\!\!\!-_{(n)jk}e_{jk}|^2}{M}\frac{\omega_n{}^2/\tau_n}{(|\omega| - \omega_n)^2 + (1/2\tau_n)^2}, \tag{7a}$$

$$\int_{\text{resonance}} \sigma_n \, d\nu = \frac{\pi}{4}\frac{|I\!\!\!-_{(n)jk}e_{jk}|^2}{M}\omega_n{}^2. \tag{7b}$$

2. For *unpolarized radiation* (random mixture of polarizations) with propagation direction $\boldsymbol{n}$, cross sections are

$$\sigma_n(\nu) = \sigma_n(\omega/2\pi) = \frac{\pi}{4}\frac{(I\!\!\!-^{TT}_{(n)jk})^2}{M}\frac{\omega_n{}^2/\tau_n}{(|\omega| - \omega_n)^2 + (1/2\tau_n)^2}, \tag{8a}$$

$$\int_{\text{resonance}} \sigma_n \, d\nu = \frac{\pi}{4}\frac{(I\!\!\!-^{TT}_{(n)jk})^2}{M}\omega_n{}^2. \tag{8b}$$

Here $I\!\!\!-^{TT}_{(n)jk}$ is the transverse-traceless part of $I\!\!\!-_{(n)jk}$ (transverse and traceless relative to the propagation direction $\boldsymbol{n}$):

$$I\!\!\!-^{TT}_{(n)} = PI\!\!\!-_{(n)}P - \frac{1}{2}P\ \text{trace}\ (PI\!\!\!-_{(n)}), \qquad P_{jk} \equiv \delta_{jk} - n_j n_k. \tag{9}$$

(See Box 35.1)

3. Cross sections for *unpolarized radiation, averaged over all directions,* are

$$\langle\sigma_n(\nu)\rangle_{\text{all directions}} = \frac{1}{2}\pi\lambda\!\!\!^-{}^2\frac{A_{\text{GW}}A_{\text{diss}}}{(|\omega| - \omega_n)^2 + (A_{\text{diss}}/2)^2}, \tag{10a}$$

$$\int_{\text{resonance}} \langle\sigma_n\rangle_{\text{all directions}} \, d\nu = \frac{1}{2}\pi\lambda\!\!\!^-{}^2 A_{\text{GW}}, \tag{10b}$$

where the Einstein A coefficients are

$$A_{\text{diss}} = 1/\tau_n, \tag{11}$$

$$A_{\text{GW}} = \frac{1}{5}\frac{(I\!\!\!-_{(n)jk})^2}{M}\omega_n{}^4. \tag{12}$$

C. Spectrum Radiated by an Aperiodic Source (exercise 37.11)

It is instructive to compare these formulas with expressions for the radiation emitted by an aperiodic source.

1. Fourier-analyze the reduced quadrupole factor of the source

$$\mathcal{I}_{jk}(t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \tilde{\mathcal{I}}_{jk}(\omega) e^{-i\omega t}\, d\omega.$$

2. Then the total energy per unit frequency ($\nu \geq 0$) radiated over all time, into a unit solid angle about the direction $\boldsymbol{n}$, and with polarization tensor $\boldsymbol{e}$, is

$$\frac{dE}{d\nu\, d\Omega} = \frac{1}{8} \sum_{\omega = \pm 2\pi\nu} |\tilde{\mathcal{I}}_{jk} e_{jk}|^2 \omega^6 \tag{13a}$$

[compare with equations (7)]. Summed over polarizations, this is

$$\frac{dE}{d\nu\, d\Omega} = \frac{1}{2} \sum_{j,k} |\tilde{\mathcal{I}}_{jk}{}^{\mathrm{TT}}|^2 \omega^6 \tag{13b}$$

[compare with equations (8)]. Here $\nu \geq 0$.

3. The total energy radiated per unit frequency, integrated over all directions, still with $\nu \geq 0$, is

$$dE/d\nu = \frac{4\pi}{5} \sum_{j,k} |\tilde{\mathcal{I}}_{jk}|^2 \omega^6 \tag{14}$$

[compare with equations (10)–(12)].

can be applied to any kind of resonant system in interaction with any kind of thermal bath of radiation or particles, equations (37.40) and (37.41), with appropriate changes of statistical factors, have wide generality.

Exercise 37.10. NORMAL-MODE ANALYSIS OF VIBRATING, RESONANT DETECTORS

Derive all the results for vibrating, resonant detectors quoted in Box 37.4. Pattern the derivation after the treatment of the idealized detector in §37.6. [*Guidelines:* (a) Let the detector be driven by the polarized waves of equation (6), Box 37.4; and let it be wave-dominated ($E_{\text{vibration}} \gg kT$). Show that the displacements $\delta x = \xi(x, t)$ of its mass elements are described by

$$\xi = \sum_n B_n(t) \boldsymbol{u}_n(x), \tag{37.42a}$$

where the time-dependent amplitude for the nth mode satisfies the driven-oscillator equation

$$\ddot{B}_n + \dot{B}_n/\tau_n + \omega_n{}^2 B_n = R_n(t), \tag{37.42b}$$

and where the curvature-induced driving term is

$$\begin{aligned} R_n(t) &= -R_{\hat{j}\hat{0}\hat{k}\hat{0}} \int (\rho/M) u_n{}^j x^k \, d^3x \\ &= \frac{1}{4} \ddot{A}(\bar{I}_{(n)jk} e_{jk})/M. \end{aligned} \tag{37.42c}$$

(See Box 37.4 for notation.)

(b) Fourier-analyze the amplitudes of the detector and waves,

$$B_n(t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \tilde{B}_n(\omega) e^{-i\omega t} \, d\omega, \tag{37.42d}$$

$$A(t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \tilde{A}(\omega) e^{-i\omega t} \, d\omega, \tag{37.42e}$$

and solve the equation of motion (37.42b,c) to obtain, in the neighborhood of resonance,

$$\tilde{B}_n = \frac{\frac{1}{8}\omega_n \tilde{A}(\bar{I}_{(n)jk} e_{jk})/M}{|\omega| - \omega_n + \frac{1}{2} i/\tau_n} \qquad \text{for } |\omega \pm \omega_n| \ll \omega_n. \tag{37.42f}$$

(c) Calculate the total energy deposited in the detector by integrating

$$\begin{pmatrix}\text{energy}\\ \text{deposited}\end{pmatrix} = \int (\textbf{Force per unit volume}) \cdot (\textbf{Velocity}) \, d^3x \, dt.$$

Thereby obtain

$$\begin{pmatrix}\text{energy deposited in}\\ n\text{th normal mode}\end{pmatrix} = \frac{1}{4} (I_{(n)jk} e_{jk}) \int \ddot{A} \dot{B}_n \, dt.$$

(d) Apply Parseval's theorem and combine with expression (37.42f) to obtain

$$\begin{pmatrix}\text{energy deposited in}\\ n\text{th normal mode}\end{pmatrix} = \int \sigma_n(\nu) \mathcal{F}_\nu(\nu) \, d\nu, \tag{37.43}$$

where σ_n is given by equation (7a) of Box 37.4, and (for $-\infty < \omega < +\infty$)

$$\mathcal{F}_\nu(\nu) = \mathcal{F}_\nu(\omega/2\pi) = \frac{1}{8} \omega^2 |\tilde{A}|^2. \tag{37.44}$$

(e) Show that $\mathcal{F}_\nu(\nu)$ is the total energy per unit area per unit frequency carried by the waves past the detector.

(f) Obtain all the remaining cross sections quoted in Box 37.4 by appropriate manipulations of this cross section. Use the mathematical tools for projecting out and integrating "transverse-traceless parts," which were developed in Box 35.1 and exercise 36.9.

Exercise 37.11. SPECTRUM OF ENERGY RADIATED BY A SOURCE

Derive the results quoted in the last section of Box 37.4.

Exercise 37.12. PATTERNS OF EMISSION AND ABSORPTION

The elementary dumbbell oscillator of Figure 37.4, initially unexcited, has a cross section for absorption of unpolarized gravitational radiation proportional to $\sin^4\theta$, and when excited radiates with an intensity also proportional to $\sin^4\theta$ (Chapter 36). The patterns of emission and absorption are identical. Any other dumbbell oscillator gives the same pattern, apart from a possible difference of orientation. Consider a nonrotating oscillator of general shape undergoing free vibrations in a single nondegenerate (and therefore nonrotatory) mode, or excited from outside by *unpolarized* radiation.

(a) Show that its pattern of emission is identical with its pattern of absorption. [*Hint:* Make the comparisons suggested in the last few parts of Box 37.4.]

(b) Show that this emission pattern ($\equiv$ absorption pattern), apart from three Euler angles that describe the orientation of this pattern in space, and apart from a fourth parameter that determines total intensity, is uniquely fixed by a single ("fifth") parameter.

(c) Construct diagrams for the pattern of intensity for the two extreme values of this parameter and for a natural choice of parameter intermediate between these two extremes.

(d) Define the parameter in question in terms of a certain dimensionless combination of the principal moments of the reduced quadrupole tensor.

Exercise 37.13. MULTIMODE DETECTOR

Consider a cylindrical bar of length very long compared to its diameter. Designate the fundamental mode of end-to-end vibration of the bar as "$n = 1$," and call the mode with $n - 1$ nodes in its eigenfunction the "nth mode." Show that the cross section for the interception of unpolarized gravitational waves at the nth resonance, integrated over that resonance, and averaged over direction, is given by the formula [Ruffini and Wheeler (1971b)]

$$\int_{\substack{n\text{th}\\ \text{resonance;}\\ \text{random}}} \sigma(\nu)d\nu = \frac{32}{15\pi}\frac{v^2}{c^2}\frac{M}{n^2} \text{ for } n \text{ odd (zero for even } n), \tag{37.45}$$

where v is the speed of sound in the bar expressed in the same units as the speed of light, c; and M is the mass of the bar (geometric units; multiply the righthand side by the factor $G/c = 2.22 \times 10^{-18}\,\text{cm}^2\,\text{Hz/g}$ when employing conventional units). Show that this expression gives $\int\sigma\, d\nu = 1.0 \times 10^{-21}\,\text{cm}^2\,\text{Hz}$ for the lowest mode of Weber's bar. Multimode detectors are (1973) under construction by William Fairbank and William Hamilton, and by David Douglass and John A. Tyson.

Exercise 37.14. CROSS SECTION OF IDEALIZED MODEL OF EARTH FOR ABSORPTION OF GRAVITATIONAL RADIATION

The observed period of quadrupole vibration of the earth is 54 minutes [see, e.g., Bolt (1964) or Press (1965) for survey and bibliography]. To analyze that mode of vibration, with all due allowance for elasticity and the variation of density in the earth, is a major enterprise. Therefore, for a first estimate of the cross section of the earth for the absorption of quadrupole radiation, treat it as a globe of fluid of uniform density held in the shape of a sphere by gravitational forces alone (zero rigidity). Let the surface be displaced from $r = a$ to

$$r = a + a\alpha P_2(\cos\theta), \tag{37.46a}$$

where θ is polar angle measured from the North Pole and α is the fractional elongation of the principal axis. The motion of lowest energy compatible with this change of shape is described by the velocity field

$$\xi^x = -\frac{1}{2}\alpha x, \qquad \xi^y = -\frac{1}{2}\alpha y, \qquad \xi^z = \alpha z \tag{37.46b}$$

(zero divergence, zero curl).

(a) Show that the sum of the kinetic energy and the gravitational potential energy is

$$E \approx -(3/5)(M^2/a)(1 - \alpha^2/5) + (3/20)Ma^2\dot{\alpha}^2. \tag{37.46c}$$

(b) Show that the angular frequency of the free quadrupole vibration is

$$\omega = (16\pi/15)^{1/2}\rho^{1/2}. \tag{37.46d}$$

(c) Show that the reduced quadrupole moments are

$$\not{I}_{xx} = \not{I}_{yy} = -Ma^2\alpha/5, \qquad \not{I}_{zz} = 2Ma^2\alpha/5. \tag{37.46e}$$

(d) Show that the rate of emission of vibrational energy, averaged over a period, is

$$-\langle dE/dt\rangle = (3/125)M^2a^4\omega^6\alpha_{\text{peak}}^{\ 2}. \tag{37.46f}$$

(e) Show that the exponential rate of decay of energy by reason of gravitational wave damping, or "gravitational radiation line broadening," is

$$A_{\text{GW}} = (4/25)Ma^2\omega^4. \tag{37.46g}$$

(f) Show that the resonance integral of the absorption cross section for radiation incident from random directions with random polarization is

$$\int_{\text{resonance}} \langle\sigma(\nu)\rangle\, d\nu = (\pi/2)\bar{\lambda}^2 A_{\text{GW}} = (2\pi/25)Ma^2/\bar{\lambda}^2. \tag{37.46h}$$

(g) Evaluate this resonance integral. Note: This model of a globe of fluid of uniform density would imply for the earth, with average density 5.517 g/cm³, a quadrupole vibration period of 94 min, as compared to the observed 54 min; and a moment of inertia $(2/5)Ma^2$ as compared to the observed $0.33Ma^2$. Ruffini and Wheeler (1971b) have estimated correction factors for both effects and give for the final resonance integral ~5 cm² Hz.

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§37.8. NOISY DETECTORS

When the bandwidth of the incoming waves is large compared to the resonance width of the detector, the waves deposit a total energy in the detector given by

$$\underset{[\text{ergs}]}{(\text{total energy deposited})} = \int \sigma\mathcal{F}_\nu\, d\nu = \underset{[\text{erg cm}^{-2}\text{Hz}^{-1}]}{\mathcal{F}_\nu(\nu = \nu_0)} \underset{[\text{cm}^2\text{Hz}]}{\int \sigma\, d\nu}.$$

At least, this is so if the detector is wave-dominated (i.e., if $E_{\text{vibration}} \gg kT$ while waves act; i.e., if initial amplitude of oscillation, produced by Brownian forces, is too small to interfere constructively or destructively with the amplitude due to waves).

Extraction of small signal from large noise—general remarks

Unfortunately, all experiments today (1973) are faced with noisy detectors. Nobody has yet found waves so strong, or constructed a detector so sensitive, that the detector is wave-dominated. Consequently, a key experimental task today is to pick a small signal out of large noise. Many techniques for doing this have been developed and used in a variety of fields of physics, as well as in astronomy, psychology and engineering [see, e.g., Davenport and Root (1958), Blackman and Tukey (1959), and

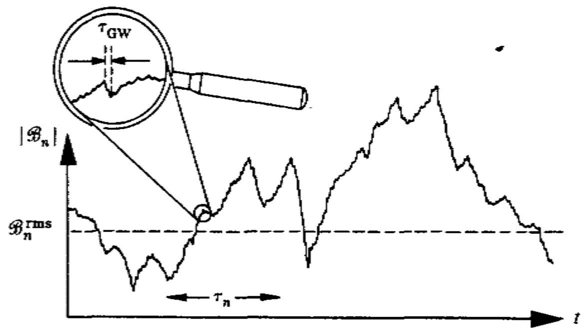

Figure 37.5.
Detection of hammer-blow gravitational waves with a noisy detector. Detection of even a weak pulse is possible if the time of the pulse is short enough. The amplitude $\mathcal{B}_n$ of the detector's vibrations changes by an amount $\sim\mathcal{B}_n^{\rm rms}(\Delta t/\tau_n)^{1/2}$ during a time interval Δt, due to thermal fluctuations (random-walk, Brownian-noise forces). Depicted in the inset is a change in amplitude produced by a burst of waves of duration $\tau_{\rm GW}$ arriving out of phase with the detector's thermal motions (energy extracted by waves!). The waves are detectable because

$$\Delta|\mathcal{B}_n|_{\text{due to waves}} \gg \mathcal{B}_n^{\rm rms}(\tau_{\rm GW}/\tau_n)^{1/2},$$

even though $\Delta|\mathcal{B}_n| \ll \mathcal{B}_n^{\rm rms}$.

references given there]. The key point is always to find some feature of the signal that is statistically more prominent than the same feature of the noise, plus a correlation to show that it arises from the expected signal source and not from elsewhere ("protection from systematic error"). Thus to detect steady gravitational waves from a pulsar, one might seek to define very precisely two numbers $\langle N^2\rangle$ and $\langle (N+S)^2\rangle = \langle N^2\rangle + \langle S^2\rangle$, where N and S are the noise and signal amplitudes respectively. A long series of observations (with the pulsar out of the antenna beam) gives one value of $\langle N^2\rangle$. Another equally long series of observations, interspersed with the first series, will be expected in zeroth approximation to give the same value of $\langle N^2\rangle$. In the next approximation one recognizes and calculates the influence of normal statistical fluctuations. In an illustrative example, theory, confirmed by statistical tests of other parameters drawn from the same data, guarantees that the fluctuations are less than $10^{-5}\langle N^2\rangle$ with 95 per cent confidence (only 5 per cent chance of exceeding $10^{-5}\langle N^2\rangle$; this limit is set by time and money, not by absolute limitations of physics). Let the second series of observations be carried out only at times when the pulsar is in the antenna beam. Let it give

$$\{\langle N^2\rangle + \langle S^2\rangle\}_{\text{2d series}} = (1 + 7.3\times 10^{-5})\{\langle N^2\rangle\}_{\text{1st series}}.$$

Then in first approximation one can say that $\langle S^2\rangle$ lies with 95 per cent confidence in the limits $(7.3 \pm 1.0)\times 10^{-5}\langle N^2\rangle$.

Rate-of-change of detector amplitude as a tool for extracting burst signals from thermal noise

Many conceivable sources of gravitational radiation produce bursts rather than a steady signal strength (Figure 37.5). Thus one is led to ask in what features "hammer-blow radiation" ($\tau_{\rm GW} \ll \tau_0$) differs from noise. The "Brownian motion" noise in the detector may be thought of as arising from large numbers of small (molecular) energy exchanges with a heat bath. The calculations below estimate the typical rate of change of amplitude that a series of such molecular "knocks" can produce in a detector, and compare it with the rapid amplitude change produced by a "hammer-blow" pulse of radiation. The calculations show that sudden thermally induced changes, even of very small amplitude, are rare. Thus sudden changes are a suitable feature for the observations to focus on. The actual detection of pulses requires a more extended analysis, however, which goes beyond the estimates made below. Such an analysis would calculate the probabilities that rare events (sudden changes in amplitude) occur by chance (i.e., due to thermal fluctuations) in specified periods of time, the still smaller probabilities that they occur in coincidence between two or more detectors, and the correlations with postulated sources.

Consider a realistic detector of the type described in Box 37.4. But examine it at a time when it is *not* radiation-dominated. Then its motions are being driven by internal Brownian forces (thermal fluctuations), and perhaps also by an occasional burst of gravitational waves. Focus attention on a particular normal mode (mode "n"), and describe that mode's contribution to the vibration of the detector by the vector field

$$\delta x = \xi = \mathscr{B}_n(t)e^{-i\omega_n t}\boldsymbol{u}_n(x). \tag{37.47}$$

Description of thermal noise in resonant detector

Since $\boldsymbol{u}_n$ is dimensionless with mean value unity ($\int \rho \boldsymbol{u}_n{}^2\, d^3x = M$), the complex number $\mathscr{B}_n(t)$ is the mass-weighted average of the amplitudes of motion of the detector's mass elements. This amplitude changes slowly with time (rate $\ll \omega_n$) as a result of driving by Brownian forces; but averaged over time it has a magnitude corresponding to a vibration energy of kT:

$$\langle E_{\text{vibration}}\rangle = 2\left\langle \frac{1}{2}\int \rho \dot{\xi}^2\, d^3x\right\rangle = \frac{1}{2}M\omega_n{}^2\langle|\mathscr{B}_n|^2\rangle = kT; \tag{37.48}$$

i.e.,

$$\mathscr{B}_n^{\text{rms}} \equiv \langle|\mathscr{B}_n|^2\rangle^{1/2} = (2kT/M\omega_n{}^2)^{1/2}. \tag{37.49}$$

Example: for Weber's detector ($M \sim 10^3$ kg, $\omega_0 \sim 10^4$/sec), the fundamental mode at room temperature has

$$\mathscr{B}_0^{\text{rms}} = \left(\frac{2 \times 1.38 \times 10^{-16} \times 300\ \text{erg}}{10^6\ \text{g} \times 10^8\ \text{sec}^{-2}}\right)^{1/2} = 3 \times 10^{-14}\ \text{cm}. \tag{37.50}$$

One's hope for detecting weak hammer-blow radiation lies not in an examination of the detector's vibration amplitude (or energy), but in an examination of its rate of change (Figure 37.5). The time-scale for large Brownian fluctuations in amplitude ($|\varDelta\mathscr{B}_n| \sim \mathscr{B}_n^{\text{rms}}$), when the detector is noisy, is the same as the time scale τ_n for internal forces to damp the detector, when it is driven to $E_{\text{vibration}} \gg kT$. Thus, *the amplitude $\mathscr{B}_n$ does a "random walk" under the influence of Brownian forces, with the mean time for "large walks" ($|\varDelta\mathscr{B}_n| \sim \mathscr{B}_n^{\text{rms}}$) being $\varDelta t \approx \tau_n$*. The change in $\mathscr{B}_n$ over shorter times $\varDelta t$ is smaller by the "$1/\sqrt{N}$ factor," which always enters into random-walk processes:

$$\sqrt{N} = \left(\frac{\text{number of vibration cycles in time } \tau_n}{\text{number of vibration cycles in time } \Delta t}\right)^{1/2} = \left(\frac{\tau_n}{\Delta t}\right)^{1/2}; \qquad (37.51)$$

$$\langle|\Delta\mathcal{B}_n^{(\text{thermal})}|\rangle \approx \mathcal{B}_0^{\text{rms}}\left(\frac{\Delta t}{\tau_n}\right)^{1/2} = \left(\frac{2kT}{M\omega_n{}^2}\right)^{1/2}\left(\frac{\Delta t}{\tau_n}\right)^{1/2} \begin{matrix}\text{during}\\ \text{time } \Delta t\end{matrix}. \qquad (37.52)^*$$

Now suppose that "hammer-blow" radiation (burst of duration $\Delta t = \tau_{\text{GW}} \ll \tau_n$) strikes the detector, producing a change $\Delta\mathcal{B}_n^{(\text{GW})}$ in the detector's amplitude. This change in amplitude, because it comes so quickly, (1) superposes linearly on any change in amplitude produced in the same time interval by the action of Brownian-motion forces; and (2) is therefore independent in value of the presence or absence of Brownian-motion forces, i.e., independent of all thermal agitation. Therefore $\Delta\mathcal{B}_n^{(\text{GW})}$ (a quantity with *both* magnitude and phase!) is identical to what it would have been if the detector were at zero temperature:

Effect of a burst of waves on a noisy, resonant detector

$$\underbrace{\frac{1}{2}M\omega_n^2|\Delta B_n^{(\text{GW})}|^2}_{\substack{\text{energy that would}\\ \text{be deposited if}\\ \text{detector were at}\\ \text{zero temperature}}} = \int \sigma_n(\nu)\mathcal{F}_\nu(\nu)\,d\nu \underset{\substack{\uparrow\\ \text{For hammer-blow radiation, bandwidth of}\\ \text{radiation is always} \gg \text{bandwidth of detector;}\\ \text{see Box 37.4}}}{=} \mathcal{F}_\nu(\omega_n/2\pi)\int \sigma_n(\nu)\,d\nu;$$

i.e.,

$$|\Delta\mathcal{B}_n^{(\text{GW})}| = \left(\frac{2\mathcal{F}_\nu(\omega_n/2\pi)\int\sigma_n\,d\nu}{M\omega_n{}^2}\right)^{1/2}. \qquad (37.53)$$

This wave-induced change in amplitude will be distinguishable from thermal changes only if it is significantly bigger than the thermal changes (37.52) expected during the same length of time τ_{GW}:

Criteria for detectability of burst

$$\left.\begin{aligned} &|\Delta\mathcal{B}_n^{(\text{GW})}| \gg \langle|\Delta\mathcal{B}_n^{(\text{thermal})}|\rangle \text{ during time } \tau_{\text{GW}}\\ &\text{equivalently: } F_\nu(\omega_n/2\pi) \gg \left(\frac{kT}{\int\sigma_n\,d\nu}\right)\left(\frac{\tau_{\text{GW}}}{\tau_n}\right)\end{aligned}\right\}\begin{matrix}\text{criteria for}\\ \text{detectability}\end{matrix} \qquad (37.54)$$

Of course, if one is equipped only to measure the magnitude of the detector's amplitude or energy, and not its phase, these criteria for detectability are not quite sufficient. The wave-induced change in squared amplitude (proportional to change in energy) will depend on the relative phases of the initial amplitude and amplitude change

*For a fuller derivation and discussion of this formula, see, e.g., Braginsky (1970). Two key points covered there are: (1) a statistical version of the formula, which describes the probability that in time Δt the amplitude will change by a given amount, from a given initial value; and (2) quantum-mechanical corrections, which come into play in the limit as $\tau_n \longrightarrow \infty$, but which are unimportant for detectors of the early 1970's.

$$\begin{aligned}
\Delta|\mathscr{B}_n|^2 &= |\mathscr{B}_n^{(\text{initial})} + \Delta\mathscr{B}_n^{(\text{GW})}|^2 - |\mathscr{B}_n^{(\text{initial})}|^2 \\
&\approx 2|\mathscr{B}_n^{(\text{initial})}||\Delta\mathscr{B}_n^{(\text{GW})}| && \text{if in phase} \\
&\approx 0 && \text{if phase difference is } \pm\pi/2 \\
&\approx -2|\mathscr{B}_n^{(\text{initial})}||\Delta\mathscr{B}_n^{(\text{GW})}| && \text{if phase difference is } \pi.
\end{aligned} \tag{37.55}$$

Thus, only a burst that arrives in phase with the initial motion of the detector or with reversed phase will be measurable. But for such a burst, the criteria (37.54) are sufficient.

Ways to improve sensitivity of detector

Equations (37.54) make it clear that *there are three ways to improve the sensitivity of vibratory detectors to hammer-blow radiation: (1) increase the detector's integrated cross-section* [*which can be done only by increasing the rate* A_{GW} *at which it reradiates gravitational waves; see equations (10b) and (11b) of* Box 37.4]; *(2) cool the detector; (3) increase the detector's damping time.*

Box 37.5 applies the above detectability criteria to some detectors that seem feasible in the 1970's, and to some bursts of waves predicted by theory. The conclusions of that comparison give one hope!

To be complete, the above discussion should have analyzed not only noise in the detector, but also the noise in the sensor which one uses to measure the amplitude of the detector's displacements. However, the theory of displacement sensors is beyond the scope of this book. For a brief discussion and for references, see Press and Thorne (1972).

§37.9. NON-MECHANICAL DETECTORS

Non-mechanical detectors

When gravitational waves flow through matter, they excite it into motion. Such excitations are the basis for all detectors described thus far. But gravitational waves interact not only with matter; they also interact with electromagnetic fields; and those interactions can also be exploited in detectors. One of the most promising detectors that may be built in the future, one designed by Braginsky and Menskii (1971), relies on a resonant interaction between gravitational waves and electromagnetic waves. It is described in Box 37.6.

§37.10. LOOKING TOWARD THE FUTURE

The future of gravitational-wave astronomy

As this book is being written, it is not at all clear whether the experimental results of Joseph Weber constitute a genuine detection of gravitational waves. (See §37.4, part 4.) But whether they do or not, gravitational-wave astronomy has begun, and seems to have a bright future. The technology of 1973 appears sufficient for the construction of detectors that will register waves from a star that collapses to form a black hole anywhere in our galaxy (Box 37.5); and detectors of the late 1970's and early 1980's may well register waves from pulsars and from supernovae in other galaxies. The technical difficulties to be surmounted in constructing such detectors are enormous. But physicists are ingenious; and with the impetus provided by Joseph Weber's pioneering work, and with the support of a broad lay public sincerely interested in pioneering in science, all obstacles will surely be overcome.

Box 37.5 DETECTABILITY OF HAMMER-BLOW WAVES FROM ASTROPHYSICAL SOURCES: TWO EXAMPLES
(The following calculations are accurate only to within an order of magnitude or so)

A. Waves from a Star of Ten Solar Masses Collapsing to Form a Black Hole; 1972 Detector with 1975 (?) Sensor

1. Predicted characteristics of radiation:

$$(\text{intensity at Earth}) = \mathcal{F}_\nu \sim \frac{M_\odot}{4\pi(\text{distance})^2\nu}$$
$$\sim (2 \times 10^5 \text{ ergs/cm}^2 \text{ Hz}) [(\text{distance to center of galaxy})/(\text{distance})]^2,$$
$$(\text{frequency of waves}) = \nu \sim 10^3 \text{ Hz},$$
$$(\text{bandwidth of waves}) = \Delta\nu \sim 10^3 \text{ Hz},$$
$$(\text{duration of burst}) = \tau_{\text{GW}} \sim 10^{-3} \text{ sec to } 10^{-1} \text{ sec}.$$

2. Detector properties: A Weber bar, vibrating in its fundamental mode, with

$$M = 10^6 \text{ g}, \qquad \int \sigma \, d\nu = 10^{-21} \text{ cm}^2 \text{ Hz (exercise 37.13)},$$
$$\nu_0 = \omega_0/2\pi = 1{,}660 \text{ Hz}, \quad T = 3 \text{ K (liquid Helium temperature)},$$
$$\tau_0 = 20 \text{ seconds},$$
$$\mathcal{B}_0^{\text{rms}} = \left(\frac{2 \times 1.37 \times 10^{-16} \times 3 \text{ erg}}{10^6 \text{ g} \times 10^8 \text{ sec}^{-2}}\right)^{1/2} = 3 \times 10^{-15} \text{ cm},$$
$$|\Delta\mathcal{B}_0^{(\text{thermal})}| = (3 \times 10^{-15} \text{ cm})(10^{-3}/20)^{1/2} = 2 \times 10^{-17} \text{ cm}, \quad \text{during } \Delta t = 10^{-3} \text{ sec},$$
$$|\Delta\mathcal{B}_0^{(\text{thermal})}| = 2 \times 10^{-16} \text{ cm, during } \Delta t = 0.1 \text{ sec}.$$

3. Effect of waves [equation (37.53)]:

$$|\Delta\mathcal{B}_0^{(\text{GW})}| = \left(\frac{2 \times 2 \times 10^5 \times 10^{-21} \text{ ergs}}{10^6 \times 10^8 \text{ sec}^{-2}}\right)^{1/2} \left(\frac{\text{center of Galaxy}}{\text{distance}}\right)$$
$$= 2 \times 10^{-15} \text{ cm} \left(\frac{\text{distance to center of Galaxy}}{\text{distance}}\right).$$

4. Conclusion: Gravitational waves from a massive star collapsing to form a black hole anywhere in our galaxy are readily detectable, *if* one can construct a "*sensor*" to measure changes in vibration amplitudes of magnitude $\lesssim 10^{-15}$ cm on time scales <0.1 seconds. This does appear to be feasible with 1972 technology; see Press and Thorne (1972).

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B. Waves from a Supernova Explosion in the Virgo Cluster of Galaxies; a Detector that might be constructable by late 1970's or early 1980's

1. Predicted characteristics of radiation:

$$\text{(intensity at Earth)} = F_\nu \sim \frac{0.03 M_\odot}{4\pi(11 \text{ megaparsecs})^2 \nu}$$
$$\sim 4 \times 10^{-3} \text{ ergs/cm}^2 \text{ Hz},$$

$$\text{(frequency of waves)} = \nu \sim 10^3 \text{ Hz},$$

$$\text{(bandwidth of waves)} \sim \nu \sim 10^3 \text{ Hz},$$

$$\text{(duration of burst)} = \tau_{GW} \sim 0.3 \text{ sec, or } \tau_{GW} \sim 2 \times 10^{-3} \text{ sec.*}$$

2. Detector: A Weber-type bar made not of metal, but of a 1,000-kg monocrystal of quartz, cooled to a temperature of 3×10^{-3} K. (For such a monocrystal, it is thought that the damping time would increase in inverse proportion to temperature, $\tau_0 \propto 1/T$.) Estimated properties of such a detector:

$$M \sim 10^6 \text{ g}, \qquad \int \sigma \, d\nu = 10^{-21} \text{ cm}^2 \text{ Hz (same as for Weber bar)},$$

$$\nu_0 = \omega_0/2\pi \sim 1{,}500 \text{ Hz}, \qquad T = 3 \times 10^{-3} \text{ K};$$

$$\tau_0 \sim 10^6 \text{ sec},$$

$$\mathcal{B}_0^{\text{rms}} = \left(\frac{2 \times 1.37 \times 10^{-16} \times 3 \times 10^{-3} \text{ erg}}{10^6 \text{ g} \times 10^8 \text{ sec}^{-2}}\right)^{1/2} = 1 \times 10^{-16} \text{ cm},$$

$$|\Delta\mathcal{B}_0^{\text{(thermal)}}| = (1 \times 10^{-16} \text{ cm}) \left(\frac{0.3 \text{ or } 2 \times 10^{-3}}{10^6}\right)^{1/2} = \begin{cases} 6 \times 10^{-20} \text{ cm}, \\ \text{or} \\ 5 \times 10^{-21} \text{ cm}. \end{cases}$$

3. Effect of waves [equation (37.53)]:

$$|\Delta\mathcal{B}_0^{\text{(GW)}}| = \left(\frac{2 \times 4 \times 10^{-3} \times 10^{-21} \text{ ergs}}{10^6 \times 10^8 \text{ sec}^{-2}}\right)^{1/2} = 3 \times 10^{-19} \text{ cm}.$$

4. Conclusion: Gravitational waves are detectable from a supernova in the Virgo cluster, if one can construct a sensor to measure changes in vibration amplitudes of magnitude $\lesssim 10^{-19}$ cm on time scales of $\lesssim 0.1$ seconds; and if one can construct a detector with the above characteristics.

*For the duration of waves from a supernova explosion, two time scales appear to be relevant: (1) the time required for the final stages of the collapse of the white-dwarf core to a neutron star or a neutron-star pancake, $\tau \sim$ (dimensions of neutron star)/(speed of sound in nuclear matter) $\sim 2 \times 10^{-3}$ sec ("pulse of gravitational radiation"); and (2) the time required for a vibrating neutron star to lose its energy of vibration by gravitational radiation ("damped train of waves"), $\tau \sim 0.3$ sec.

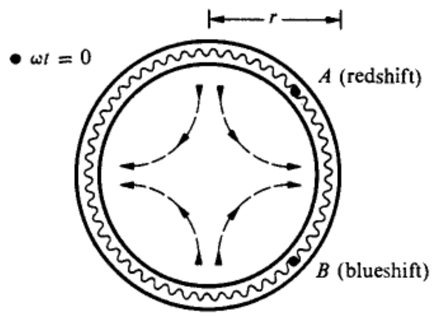
Box 37.6 A NONMECHANICAL DETECTOR OF GRAVITATIONAL WAVES [Braginsky and Menskii (1971)]

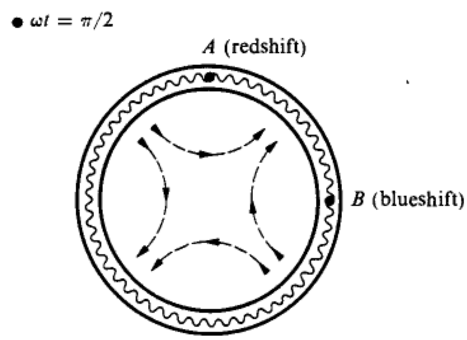
The Idea in Brief
(see diagram at right)

A toroidal waveguide contains a monochromatic train of electromagnetic waves, traveling around and around it. Gravitational waves propagate perpendicular to the plane of the torus. If the circuit time for the EM waves is twice the period of the gravitational waves, then one circularly polarized component of the gravitational waves will stay always in phase with the traveling EM waves. Result: a resonance develops. In one region of the EM wave train, gravitational tidal forces always "push" the waves forward (*blue shift!*) in another region the tidal forces "push" backward (*red shift!*). An EM frequency difference builds up linearly with time; a phase difference builds up quadratically.

ω = (angular frequency of gravitational waves) = (rate of change of phase of waves with time) = (two times angular velocity with which pattern of "lines of force" rotates)

r = (radius of torus), is adjusted so the speed of propagation of EM waves in waveguide is $v = \frac{1}{2}\omega r$.





[EM waves propagate counterclockwise; gravitational line-of-force diagram rotates counterclockwise; they stay in phase.]

Outline of Quantitative Analysis

1. Let waveguide fall freely in an Earth orbit. Orient axes of waveguide's proper reference frame ($\equiv$ local Lorentz frame) so (1) waveguide lies in $\hat{x}, \hat{y}$-plane, and (2) gravitational waves propagate in $\hat{z}$ direction.

2. Let gravitational waves have amplitudes

$$A_+ - iA_\times = \mathscr{A} e^{-i\omega(t-z)} \tag{1}$$

[Recall: $\hat{t} \approx t$, $\hat{z} \approx z$; i.e., proper frame and TT coordinates almost agree.] Then in plane of waveguide ($z = 0$),

$$\begin{aligned} R_{\hat{x}\hat{0}\hat{x}\hat{0}} &= -R_{\hat{y}\hat{0}\hat{y}\hat{0}} = \frac{1}{2}\omega^2 \mathscr{A} \cos(\omega t) \\ R_{\hat{x}\hat{0}\hat{y}\hat{0}} &= R_{\hat{y}\hat{0}\hat{x}\hat{0}} = \frac{1}{2}\omega^2 \mathscr{A} \sin(\omega t) \end{aligned} \tag{2}$$

3. Consider two neighboring parts of the EM wave, one at $\phi = \alpha + \frac{1}{2}\omega t$; the other at $\phi = \alpha + \delta\alpha + \frac{1}{2}\omega t$. Treat them as photons. Each moves along a null geodesic, except for

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Box 37.6 (*continued*)

the deflective guidance of the wave guide. Thus, their wave vectors $\boldsymbol{k}$ satisfy

$$\nabla_{k}\boldsymbol{k} = \begin{pmatrix}\text{deflective “acceleration”} \\ \text{of waveguide}\end{pmatrix}; \qquad (3)$$

and the difference $\delta\boldsymbol{k} = \nabla_{n}\boldsymbol{k}$ between the wave vectors of the two parts of the wave (difference measured via parallel transport) satisfies the equation

$$\begin{aligned}\nabla_{k}\delta\boldsymbol{k} &= \nabla_{k}\nabla_{n}\boldsymbol{k} = [\nabla_{k}, \nabla_{n}]\boldsymbol{k} + \nabla_{n}\nabla_{k}\boldsymbol{k} \qquad (4)\\ &= \boldsymbol{Riemann}\,(\ldots, \boldsymbol{k}, \boldsymbol{k}, \boldsymbol{n}) + \nabla_{n}\underbrace{\nabla_{k}\boldsymbol{k}}\end{aligned}$$

[deflective acceleration of wave guide]

The waveguide influences the direction of propagation of the waves, but not their frequency. Thus only ***Riemann*** enters into the 0 component of the above equation:

$$k^{\hat\alpha}\delta k^{\hat 0}{}_{,\hat\alpha} = R^{\hat 0}{}_{\hat\alpha\hat\beta\hat\gamma}k^{\hat\alpha}k^{\hat\beta}n^{\hat\gamma}. \qquad (5)$$

4. Let $k^{\hat 0} = \omega_e$ be the angular frequency of the electromagnetic wave. The direction of the space component $\boldsymbol{k}$ of the propagation 4-vector is along the purely spatial vector $\boldsymbol{n}$; so

$$k^{\hat 0} = \omega_e, \qquad \boldsymbol{k} = (v\omega_e/r\delta\alpha)\boldsymbol{n}, \qquad n^{\hat 0} = 0. \qquad (6)$$

Use these relations to rewrite equation (5) as

$$(d\,\delta\omega_e/d\hat t)_{\text{moving with photons}} = (v\omega_e/r\delta\alpha)R^{\hat 0}{}_{\hat i\hat 0\hat j}n^{\hat i}n^{\hat j}. \qquad (7)$$

5. Combine the expression for $\boldsymbol{n}$ in the spacetime diagram with equations (2) and (7), and with the world line $\phi = \alpha + \frac{1}{2}\omega t$ for the photons, to obtain

$$(d\,\delta\omega_e/d\hat t)_{\text{moving with photons}} = -\frac{1}{2}v\omega_e\omega^2\mathscr{A}r(\cos 2\alpha)\,\delta\alpha. \qquad (8)$$

6. Integrate over time and over α to obtain

$$\omega_e = \omega_{eo}\left[1 - \frac{1}{4}\mathscr{A}v(\sin 2\alpha)(\omega r)(\omega\hat t)\right]. \qquad (9)$$