Stability of Dynamic Traveling Repairman Problem under Polling-Sequencing Policies

Jiangchuan Huang and Raja Sengupta

***Abstract*— We establish a necessary and sufficient condition for stability in the Dynamic Traveling Repairman Problem (DTRP) [5] under the class of Polling-Sequencing (P-S) policies satisfying unlimited-polling and economy of scale. The P-S class includes some of the policies proven to be optimal for the expectation of system time under light and heavy loads in the DTRP literature. The number of tasks inside each polling partition is shown to be a Markov chain. Policies such as First Come First Serve, Traveling Salesman Policy, Nearest Neighbor and Daganzo's Algorithm are shown to have economy of scale.**

Key Words – **Dynamic Traveling Repairman Problem, Queuing Theory, Economy of Scale.**

I. INTRODUCTION

We establish a necessary and sufficient stability condition for the Dynamic Traveling Repairman Problem (DTRP) introduced by Bertsimas et al. [5] under the class of Polling-Sequencing (P-S) policies satisfying unlimited-polling and economy of scale. For the convenience of the reader, we restate the definition of the DTRP [5]: A convex region $\mathbf{A}$ of area A contains a vehicle (server) that travels at constant speed v. Tasks arrive according to a Poisson process with rate λ and have a location that is independent and identically distributed (i.i.d.) according to the probability density function (pdf) $f_X(x)$ within $\mathbf{A}$. Each task i has size B_i, B_i is i.i.d. according to $f_B(s)$. $E[B_i] = b$, which is assumed to be finite. Define load $\rho = \lambda b$. The system time of task i, denoted T_i, is defined as the elapsed time between the arrival of task i and the time task i is completed. When the system is stable, T_i converges to some T in distribution. T is called the steady state system time. This paper is focused on stability conditions for this problem and the existence of stationary distributions for T and the queue length N.

A. Literature and Summary of Results

In queueing theory, $\rho < 1$ is a necessary and sufficient condition for all *work conserving* M/G/1 queues [8] (sec. II.4.2), where work conserving means the server will not be idle when there are tasks waiting in the queue. However, there is no such policy-independent stability condition for the DTRP, which seems to be a "spatial version" of the M/G/1 queue. Bertsimas et al. [5] showed that $\rho < 1$ is not a sufficient stability condition for the work conserving policies of DTRP. For example, the DTRP can be unstable when $\rho < 1$ under the First Come First Serve (FCFS) policy.

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The known stability conditions for the DTRP are policy-dependent. Bertsimas et al. [6] argued $\rho + \lambda\frac{\bar{d}}{v} \leq 1$ as a necessary condition for stability of the DTRP under a policy, where $\bar{d} = \lim_{i\to\infty} E[d_i]$, and d_i denotes the distance traveled from task i to the next task served after i under the policy. The sufficiency of this condition for stability is established for some policies for 1-DTRP, by deriving an expression for the mean response time, $E[T]$, or an upper bound $\overline{E[T]}$, when $\rho \to 1^-$ [5], [20], [12]. The case of many capacitated vehicles with arbitrary demand distribution was investigated in [4], [6]. Table I summarizes known results and references. Since the sufficiency of this condition is policy dependent, stability is not proven for some reasonable policies such as Nearest Neighbor (NN) [5] or Shrotest Job First (SJF) [19] discussed in the literature. Equation (45) in [5] is an assumption. This paper makes progress towards finding stability conditions for the DTRP that are less policy dependent than those in this literature as described next.

TABLE I

STABILITY CONDITIONS FOR DIFFERENT POLICIES FOR THE 1-DTRP, NA = NOT APPLICABLE

Policies	$E[T]$	$\overline{E[T]}$	Stability conditions	Ref.
FCFS	Yes	NA	$\rho + 0.52\lambda\sqrt{A} < 1$	[5]
SQM	Yes	NA	$\rho + 0.766\lambda\sqrt{A} < 1$	[5]
PART-FCFS	Yes	NA	$\rho + \frac{0.52}{r}\lambda\sqrt{A} < 1$	[5]
n-TSP	No	Yes	$\rho + \lambda\beta_{TSP}\frac{\sqrt{A}}{n} < 1$	[5]
SFC	No	Yes	$\rho < 1$	[5]
NN	No	No	No	[5]
U-TSP	No	Yes	$\rho < 1$	[6]
B-TSP	No	Yes	$\rho < 1$	[6]
PART-TSP	No	Yes	$\rho < 1$	[20]
DC	No	Yes	$\rho < 1$	[12]
RH	No	Yes	$\rho < 1$	[12]

We establish $\rho + \lambda b_d < 1$ as a necessary and sufficient condition for the class of Polling-Sequencing (P-S) policies satisfying *unlimited-polling* (ULP, Definition 3) and *economy of scale* (EoS, Definition 2) in Theorem 6 in Section II-C. The Unlimited Polling property is adapted from the one in the literature to this spatial queueing problem. The Economy of Scale property is particular to the spatial aspect of the DTRP. This is the main result. Theorem 1 shows the queue length is Markovian. Theorem 2 shows it to be aperiodic and irreducible. These are separated from Theorem 6 because the ULP assumption is sufficient for Theorems 1 and 2, while the ergodicity proof (Theorem 6) also requires policies to satisfy Economy of Scale.

A P-S policy has two phases: the polling phase allocates the arriving tasks to each polling station. The vehicle visits each polling station in cyclic order. The sequencing phase is a vehical routing problem inside each polling station. ULP means in each polling station the number of tasks served goes to infinity as the number of waiting tasks goes to infinity. EoS is a property of the sequencing phase policy. EoS means the mean travel time to serve a task decreases as the number of tasks increases. The extra term b_d is defined as the limit of mean travel time as the number of tasks in a polling station goes to infinity. We prove the existence of b_d is a consequence of EoS (Theorem 5). b_d is policy-dependent, but it only depends on the sequencing phase of the P-S policy. Thus the value of b_d can be derived in the static setting, we do not need to analyse or simulate the dynamic queueing system of DTRP to get b_d. One only needs to analyze or simulate to obtain the statistics of sequencing N tasks, with N a random variable, and the task locations distributed in some fashion.

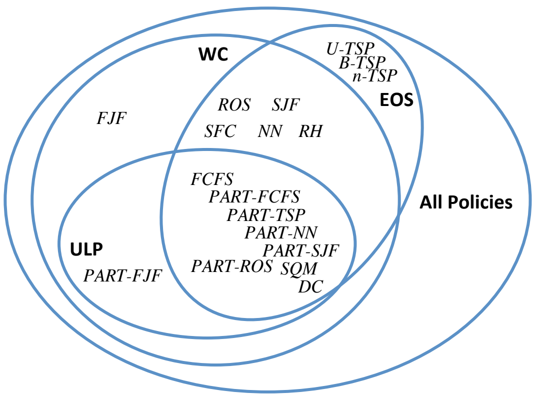

Fig. 1. Classification of Policies, WC = Work Conserving, EoS = Economy of Scale, ULP = Unlimited-Polling.

One can check that policies considered in [5], [20], [12] such as FCFS, Stochastic Queue Median (SQM), Partition-FCFS (PART-FCFS), Partition-Traveling Salesman Policy (PART-TSP) and Divide & Conquer (DC) satisfy unlimited-polling and EoS. Other policies considered in [5], [20], [12] such as Space Filling Curve (SFC), NN, Receding Horizon (RH), and policies important in queueing theory but not considered for the DTRP such as SJF, and Random Order of Service (ROS) satisfy EoS but not unlimited-polling. We define a polling version for these policies in Section II, and name them PART-NN and PART-SJF. These satisfy both unlimited-polling and EoS. This paragraph is summarized by Figure 1. Theorems 3 and 4 prove the inclusions in the figure.

II. Polling-Sequencing Policies

The class of Polling-Sequencing policies include a polling phase and a sequencing phase.

A. Spatial-Polling: Markov Chain

Polling policies are well-established in the queueing theory literature. Overviews and surveys of polling systems can be found in [16], [17], [18]. Stability and ergodicity criteria for polling systems are well established and can be found in [1], [10], [11], [14], [7], [2].

The polling phase of the P-S class is a *Spatial-Polling* policy, we divide the region $\mathbf{A}$ into an r-partition $\{\mathbf{A}^k\}_{k=1}^r$ of $\mathbf{A}$, label the r partitions as $1, 2, \ldots, r$. We regard each partition as a station in classic polling systems. In this way we generalize the polling system from classic queue theory to the spatial case. We follow the same construction made by Altman [1] and the mathematical definition by Fricker [10] with some necessary chages for the spatial case.

Partition k has area $A^k, k = 1, 2, \ldots, r$, $A = \sum_{k=1}^r A^k$. The vehicle visits the partitions in cyclic order, $1, 2, \ldots, r, 1, 2, \ldots$, and serves the queue in each partition. Without loss of generality, we assume that the vehicle is initially at partition 1. Thus, the n-th queue that the vehicle visits is in partition $I(n) = (n-1) \pmod r + 1$, where $n \pmod r$ means the remainder of the division of n by r.

We denote by:

$G^k(N)$ the number of tasks that are served in partition k when the queue is of length N.

$T_S^k(N)$ the total service time of N tasks in partition k. The formal definition of $T_S^k(N)$ will be given in section II-B.

$S^k(N)$ the duration of the service in partition k when the queue is of length N.

$$S^k(N) = T_S^k\left(G^k(N)\right) \tag{1}$$

Function $G^k(.)$ characterizes the polling policy and $T_S^k(.)$ characterizes the sequencing policy.

The *switch time* the vehicle takes from a random point in partition k to a random point in partition $k+1$ is denoted by $\Delta^k, k = 1, \ldots, r-1$. The value from partition r to partition 1 is denoted by Δ^r. $\Delta^k, k = 1, \ldots, r$ are bounded above by the diameter of the region $\mathbf{A}$ divided by the speed of the vehicle, v. The first moments of Δ^k is denoted by $\delta^k, k = 1, \ldots, r$. Let $\Delta = \sum_{k=1}^r \Delta^k$ be the total switch time in a cycle and denote by δ the first moment of Δ.

The tasks arrive at partition k with a Poisson process of parameter $\lambda^k = \int_{A^k} f_X(x)\, dx \lambda$. The task sizes are i.i.d. with common distribution B and mean b. Define $\rho^k = \lambda^k b$, and $\rho = \sum_{k=1}^r \rho^k$, $1 \le k \le r$. Let $N^k(t_1, t_2]$ denote the number of Poisson arrivals to partition k, $1 \le k \le r$, during a (random) time interval $(t_1, t_2]$. $N^k(t) \equiv N^k(0, t]$ is the number of Poisson arrivals in a time interval of length t.

The n-th value of the polling system is described by the random variables N_n^k, $1 \le k \le r$, $n \ge 1$, where N_n^k represents the number of tasks in partition k when the vehicle arrives at the n-th queue. Let $N_n = \left(N_n^1, \ldots, N_n^r\right)$, taking values in $\{\mathbb{N}\}^r$, where $\mathbb{N}$ is the set of nonnegative integers.

Denote by S_n, the *station time*, the time interval between the arrival times of the vehicle to the n-th queue and the $(n+1)$st queue.

$$S_n = S^{I(n)}(N_n^{I(n)}) + \Delta^{I(n)} \tag{2}$$

Denote by C_n, the *cycle time*, the time interval between two successive arrivals of the vehicle to the same partition. $C_n = S_n + \ldots + S_{n+r-1}$.

The arrival times, the service times, the switch times are mutually independent, and are independent of the past and present system states. We adopt the rigorous independence definitions from Fricker [10] with some changes for the spatial case.

Consider a queue service starting at stopping time τ at partition k while N tasks are waiting and N^- tasks have already been served for the whole system. Let $\mathbf{F}_\tau$ be any σ-field containing the history of the service process up to random time τ. $\mathbf{F}_\tau$ is independent of the process $N(\tau, \tau+.]$ of arrivals after τ and of the task sizes $\{B_{N^-+i}\}_{i>0}$ of the tasks that have not been served up to time τ. The following four assumptions hold for all $k = 1, \ldots, r$.

A1: (G^k, S^k) is conditionally independent of $\mathbf{F}_\tau$ given N, and has the disribution of $\left(G^k(N), S^k(N)\right)$ where the expressions of the random functions $(G^k(.), T_S^k(.))$ are taken independent of N. i.e. The A-S policies do not depend on the past history of the service process such as the number of tasks being already served and the time spent serving them.

A2: (G^k, S^k) is independent of $\left((B_{N^-+G^k+i})_{i>0}, N\left(\tau+S^k, \tau+S^k+.\right]\right)$, i.e. The selection of a task for service is independent of the required execution time and of possible future arrivals.

A3: $G^k(0) = S^k(0) = 0$ and there exists $N > 0$ such that $G^k(N) > 0$. i.e. The vehicle leaves immediately a queue which is or becomes empty, but provides service with a positive probability once there are "enough" task(s) in the queue.

A4: $\left(G^k(N), S^k(N)\right)$ is *monotonic* and *contractive* in N. A function $g(.)$ is contractive if for every $x \geq y$, $g(x) - g(y) \leq x - y$.

N_n evolve according to the following evolution equations:

$$N_{n+1}^k = \begin{cases} N_n^k + N^k\left(S_n\right), & \text{if } I(n) \neq k \\ N_n^k - G^k\left(N_n^k\right) + N^k\left(S_n\right), & \text{if } I(n) = k \end{cases} \tag{3}$$

where $I(n) = (n-1) \pmod r + 1$.

The spatial polling system has a Markovian structure as specified by the following two theorems, which is almost identical to the theorems given in [10].

Theorem 1: The sequence $\{N_n\}_{n=0}^\infty$ is a Markov chain.

Proof: At the n-th polling instant τ, the server starts serving queue n (if not empty, otherwise he starts switching to queue $n+1$) according to policy $G^{I(n)}$ while the state of all queues is given by N_n. The arrival processes after n are Poisson and are independent of F_τ; the sevice times and the switch times involved after τ are also independent of F_τ. Because these quantities are mutually independent, it follows that given N_n, the evolution of the system after τ is independent of F_τ, which ensures the Markov property of the sequence. ■

Remark: This Markov chain is in general not homogeneous because its transitions depend on n through $G^{I(n)}$ and $\Delta^{I(n)}$, and $I(n)$ is different for each n. One can check that theorem 1 also holds when the task arrival process is renewal. This guarantees the arrival processes after n are independent of F_τ.

Theorem 2: $\{N_{nr+k}\}_{n=0}^\infty$ is a homogeneous, irreducible and aperiodic Markov chain with state space $\{\mathbb{N}\}^r$, $k = 1, \ldots, r$, where r is the number of polling stations.

Proof: $\{N_{nr+k}\}_{n=0}^\infty$ is a subsequence of the Markov chain $\{N_n\}_{n=0}^\infty$ and is thus also a Markov chain which is homogeneous because $I(nr+k) = k$ and $G^{I(nr+k)} = G^k$ for $n = 0, 1, 2, \ldots$.

It is irreducible because all states communicate. Indeed, $N = (N^1, \ldots, N^r)$ can be reached in one step from the state $(0, \ldots, 0)$: this is realized when first no arrivals occur to all queues during the whole cycle but the last switch time Δ^{r-1}, and then the last switch time is positive and $(N^1, \ldots, N^r)$ arrivals occur during it, all this having a positive probability because the arrival processes are Poisson. On the other hand, $(0, \ldots, 0)$ is reached in (possibly) many steps from any state $(N^1, \ldots, N^r)$ with positive probability too: this is realized when there are no arrivals until it happens. By the same arguments, the state $(0, \ldots, 0)$ is aperiodic and so is the (irreducible) Markov chain. ■

B. Sequencing: Economy of Scale

Under a spatial-polling policy, the number and locations of tasks are determined in each polling station in each polling cycle, which is a static vehicle routing problem. The sequencing policies sequence the set of tasks in each polling station. For the notation of P-S policies, we use "PART-" to denote the polling (partitioning) phase, followed with the sequencing policies for the sequencing phase. For example, PART-TSP means first partition the region $\mathbf{A}$ into polling stations, and use TSP to sequence the tasks inside each polling station. Similarly, we can define PART-NN, PART-DA, PART-SJF, etc.

Definition 1: A policy for the 1-DTRP is called a P-S policy if it runs a spatial-polling policy in region $\mathbf{A}$, and sequences the set of tasks in each polling station using some sequencing policy.

A sequencing policy is called *non-location based* if it does not sequence according to the locations of the tasks. Examples include FCFS, SJF, ROS and Longest Job First (LJF). A sequencing policy is called *location based* if it sequences according to the locations of the tasks. Examples include NN, Furtherest Job First (FJF), TSP and the approximation algorithms for TSP such as DA.

For a set of N tasks $\{B_l, X_l\}_{l=1}^N$, each with size B_k and location X_k, denote by $T_D^P\left(\{X_l\}_{l=1}^N\right)$ and $T_S^P\left(\{B_l, X_l\}_{l=1}^N\right)$ the travel time and service time for the N tasks $\{B_l, X_l\}_{l=1}^N$ under sequencing policy P.

$$T_D^P\left(\{X_l\}_{l=1}^N\right) \equiv E_X\left[\frac{1}{v} D^P\left(X, \{X_l\}_{l=1}^N\right)\right] \tag{4}$$

where v is the vehicle speed, and $D^P\left(X, \{X_l\}_{l=1}^N\right)$ is the distance travelled by the vehicle to serve the tasks $\{B_l, X_l\}_{l=1}^N$ starting from a random point X in region $\mathbf{A}$. Note

$$T_S^P\left(\{B_l, X_l\}_{l=1}^N\right) = T_D^P\left(\{X_l\}_{l=1}^N\right) + \sum_{l=1}^N B_l \tag{5}$$

Define $T_D^P(N) \equiv E_{\{X_l\}}\left[T_D^P\left(\{X_l\}_{l=1}^N\right)\right]$,
and $T_S^P(N) \equiv E_{\{X_l\}}\left[E_{\{B_l\}}\left[T_S^P\left(\{B_l, X_l\}_{l=1}^N\right)\right]\right]$, then

$$T_S^P(N) = T_D^P(N) + Nb \tag{6}$$

$T_S^k(N) \equiv T_S^P(N)$ and $T_D^k(N) \equiv T_D^P(N)$ when sequencing policy P is used in partition k.

Definition 2: A sequencing policy P is said to have economy of scale (EoS) if $\frac{T_D^P(n)}{n}$ is nonincreasing in n.

Theorem 3: Non-location based policies satisfy economy of scale.

Proof: This is because that X_i is independent of the arrival process, and X_i is independent of X_{i-1}. Non-location based policies do not sequence based on the locations of tasks, so $T_D^P(n) = E\left[\frac{\sum_{i=1}^n D_i}{v}\right] = \frac{\sum_{i=1}^n E[D_i]}{v}$, where $D_i = \| X_i - X_{i-1} \|$ when $i > 1$, $D_1 = \| X_1 - X_v \|$, where X_i is the location of the i-th task and X_v is the initial position of the vehicle. X_i and X_v are i.i.d. with pdf $f_X(x)$. Thus $E[D_i]$ is a constant, say d. Then $\frac{T_D^P(n)}{n} = \frac{nd}{n} = d$, which is nonincreasing in n. So non-location based policies satisfy economy of scale. ■

Remark: Non-location based policies have trival economy of scale in the sense that $\frac{T_D^P(n)}{n}$ is a constant.

For location based policies, there are two categories. One category try to find a shorter path connecting the locations of the tasks, which we call *smart*. Examples include TSP, NN and the approximation algorithms of TSP such as Daganzo's Algorithm (DA) [9], [13]. The other category tries to find a longer path connecting the locations of the tasks, which we call *stupid*. Examples include Furthest Job First (FJF). This category does not has economy of scale, and is not practical. Theorem 4 below proves that the common policies in the smart category such as TSP, NN and DA have economy of scale. Other policies in this category can be checked by the similar analysis or through simulation. Theorem 4 also proves that FJF does not satisfy EoS. NN and TSP are well known. In DA, one cuts a swath of approximate width, w, covering the region $\mathbf{A}$. One possible patten is shown in the left of Figure 2 with a swath of width $\frac{\sqrt{A}}{6}$. The vehicle visits the task locations by moving along the swath without backtracking.

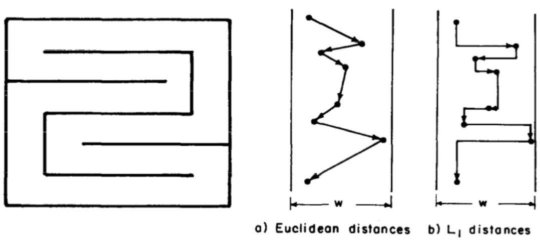

Fig. 2. Daganzo's Algorithm, cited from [9].

Theorem 4: NN, TSP and DA satisfy economy of scale, FJF does not satisfy economy of scale.

Proof: Under TSP, when there are n tasks, there are $n!$ *Hamiltonian paths* starting from the initial position of the vehicle. A Hamiltonian path is a path that visits each X_i exactly once. Each Hamiltonian path is the sum of n i.i.d. D_i's, $HP = \sum_{i=1}^n D_i$. The TSP tour is the Hamiltonian path with minimum lengths among the $n!$ Hamiltonian paths. Let L_n be the tour length, then $\frac{T_D^P(n)}{n} = \frac{E[L_n]}{nv}$. When there are $n+1$ tasks, there are $(n+1)!$ Hamiltonian paths, and L_{n+1} is the shortest of them. $P\left(\frac{L_{n+1}}{n+1} > l\right) \le P\left(\frac{L_n}{n} > l\right)$ because L_{n+1} is the minimum of $(n+1)!$ Hamiltonian paths and L_n is the minimum of $n!$ Hamiltonian paths. $\frac{E[L_{n+1}]}{n+1} = \int_0^\infty P\left(\frac{L_{n+1}}{n+1} > l\right) dl \le \int_0^\infty P\left(\frac{L_n}{n} > l\right) dl = \frac{E[L_n]}{n}$. Then $\frac{T_D^P(n+1)}{n+1} \le \frac{T_D^P(n)}{n}$. TSP satisfies economy of scale.

Under NN, when there are n tasks, Let L_n^{NN} be the length of the tour connecting the initial vehicle position and the locations of the n tasks, then $\frac{T_D^P(n)}{n} = \frac{E\left[L_n^{NN}\right]}{nv}$. L_n^{NN} is composed of n segments, $L_n^{NN} = \sum_{i=1}^n D_i^{NN}$, label i backwards such that D_i^{NN} is the distance from the $(n-i)$-th point to the $(n-i+1)$-th point when $i = 1, \ldots, n-1$, D_n is the distance from the initial position of the vehicle to the 1st point. So D_i^{NN} is the minimum of i D_j's, where each D_j is the distance between two random points in the region $\mathbf{A}$. Thus $P\left(D_{i+1}^{NN} > l\right) \le P\left(D_i^{NN} > l\right)$, this implies $E\left[D_{i+1}^{NN}\right] \le E\left[D_i^{NN}\right]$, thus $\frac{E\left[L_{n+1}^{NN}\right]}{n+1} = \frac{\sum_{i=1}^{n+1} E\left[D_i^{NN}\right]}{n+1} \le \frac{\sum_{i=1}^n E\left[D_i^{NN}\right] + \sum_{i=1}^n \frac{1}{n} E\left[D_i^{NN}\right]}{n+1} = \frac{\sum_{i=1}^n \frac{n+1}{n} E\left[D_i^{NN}\right]}{n+1} = \frac{E\left[L_n^{NN}\right]}{n}$. So $\frac{T_D^P(n+1)}{n+1} \le \frac{T_D^P(n)}{n}$. NN satisfies economy of scale.

In [9], the swath was approximated to be a infinitely long strip of width w neglecting corner effect as shown in the right two of Figure 2. The mean travel time per task when serving n tasks, $\frac{T_D^P(n)}{n} = \frac{nd_w}{n} = d_w$, where d_w is the expected distance between two consecutive locations. Let X denote the random distance between two consecutive points along the width of the strip, and Y the distance along the side of the strip, then $E[X] = \frac{w}{3}$, $E[Y] = \frac{A}{nw}$ according to [9]. $d_w = E_{X,Y}\left(\sqrt{X^2 + Y^2}\right)$ for the Euclidean metric. $d_w \approx \frac{w}{3} + \frac{A}{nw}\psi\left(\frac{nw^2}{A}\right)$, where $\psi(x) = \frac{2}{x^2}\left((1+x)log(1+x) - x\right)$. $w^* = \sqrt{\frac{2.95A}{n}}$ minimizes d_w. Plug in w^*, we see d_w is decreasing with n. Thus $\frac{T_D^P(n)}{n}$ is nonincreasing in n.

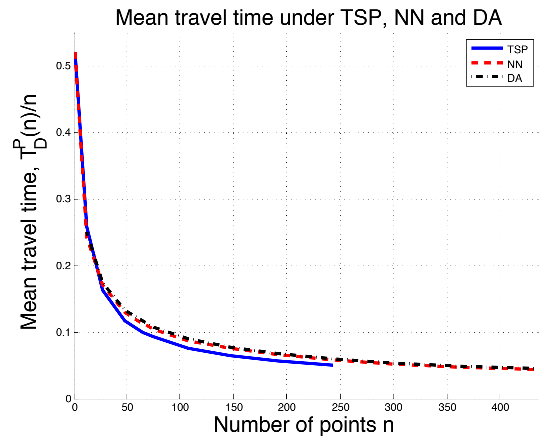

Fig. 3. Mean Travel Time under TSP, NN and DA.

We show that FJF does not satisfy EoS by a counterexample. Consider a square of size 1×1 with uniformly distributed task locations. $\frac{T_D^P(1)}{1} = E[D_1] = 0.52$, where $D_1 = \| X_1 - X_v \|$, where X_i is the location of the i-th task and X_v is the initial position of the vehicle. X_i and X_v are i.i.d. with pdf $f_X(x) = 1$. When there are two tasks, the vehicle will choose the task further away, thus $\frac{T_D^P(2)}{2} > 0.52 = \frac{T_D^P(1)}{1}$. Thus FJF does not satisfy EoS. ■

Theorem 4 is supported by the simulation results in Figure 3. The simulations are done in a square $\mathbf{A}$ of size 1×1. The task locations and the initial vehicle position are generated independently from a uniform distribution with pdf $f_X(x) = 1$. The length of the path connecting the vehicle and the tasks is calculated under TSP, NN and DA for different number of points n.

Theorem 5: Under a sequencing policy P with economy of scale, $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d \geq 0$, and $\exists M > 0$, s.t. $M \geq \frac{T_D^P(n)}{n} \geq b_d$ for all n.

Proof: $\frac{T_D^P(n)}{n} \geq 0$ and $\frac{T_D^P(n)}{n}$ is nonincreasing in n imply that $\lim_{n\to\infty} \frac{T_D^P(n)}{n}$ exists, say b_d.
Thus we have $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d$ and $M = \frac{T_D^P(1)}{1} = \frac{T_D^P(n)}{n} \geq b_d \geq 0$. ■

Remark: b_d is a measure of how well the sequencing policy can take advantage of the task locations. Let L_n denote the length of the tour connecting n points in a square of area A under TSP. From [15] we know that $\lim_{n\to\infty} \frac{L_n}{\sqrt{n}} = \beta_{TSP}\sqrt{A}$, where $\beta_{TSP} \approx 0.72$, thus $b_d = \lim_{n\to\infty} \frac{T_D^P(n)}{n} = \lim_{n\to\infty} \frac{E[L_n]}{vn} = \lim_{n\to\infty} \beta_{TSP}\frac{\sqrt{A}}{v\sqrt{n}} = 0$. This implies that TSP does best in taking avantage of the task locations.

C. Stability Condition

Stability in DTRP is more complicated in the sense that the stability of DTRP is policy dependent, whereas in queueing theory we have the policy-independent stability condition $\rho < 1$ for work conserving M/G/1 queues. Theorem 1 and 2 showed that $\{N_n\}_{n=0}^{\infty}$ is a Markov chain, and $\{N_{nr+k}\}_{n=0}^{\infty}$ is a homogeneous, irreducible and aperiodic Markov chain. We check the ergodicity of $\{N_{nr+k}\}_{n=0}^{\infty}$ and the stability of the DTRP under the P-S policies in this section.

Definition 3: A polling policy characterized by $G^k(.)$ is called an *unlimited-polling* policy if $G^k(N) \to \infty$, when $N \to \infty$, $k = 1, \ldots, r$.

Lemma 1: (LEMMA 3.1 in [1]) If for all $1 \leq k \leq r$ the Markov chains $\{N_{nr+k}\}_{n=0}^{\infty}$ are ergodic, then for all $1 \leq k \leq r$ $\{N_{nr+k}\}_{n=0}^{\infty}$ together with the sequence of station times $\{S_{nr+k}\}_{n=0}^{\infty}$ and the cycle times $\{C_{nr+k}\}_{n=0}^{\infty}$ converge weakly to finite random variables.

Definition 4: The DTRP under a P-S policy is said to be *stable* if all the r Markov chains $\{N_{nr+k}\}_{n=0}^{\infty}$ are *ergodic*.

Lemma 2: (Foster's Criterion [3], p.19): Suppose a Markov chain is irreducible and let E_0 be a finite subset of the state space E. Then the chain is *positive recurrent* if for some $h : E \to \mathbb{R}$ and some $\epsilon > 0$ we have $\inf_x h(x) > -\infty$ and

i) $\sum_{k\in E} p_{jk}h(k) < \infty, j \in E_0$,

ii) $\sum_{k\in E} p_{jk}h(k) \leq h(j) - \epsilon, j \notin E_0$.

where p_{jk} is the transition probability of the chain.

Theorem 6: (Stability theorem): For any P-S policy with polling policy satisfying *Definition 3* (Unlimited-Polling) and sequencing policy P satisfying $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d$, when $\rho + \lambda b_d < 1$, for all $1 \leq k \leq r$ the Markov chains $\{N_{nr+k}\}_{n=0}^{\infty}$ are ergodic. Moreover, if the sequencing policy P satisfying *Definition 2* (EoS), then $\rho + \lambda b_d < 1$ is necessary for the ergodicity of $\{N_{nr+k}\}_{n=0}^{\infty}$.

Proof: Sufficiency: taking a conditional expectation in (3), summing over k, and plugging in (2) and (1) we obtain:

$$E\left[\sum_{k=1}^r bN_{n+1}^k\,|N_n\right] = \sum_{k=1}^r bN_n^k - bG^{I(n)}\left(N_n^{I(n)}\right)$$
$$+ E\left[\sum_{k=1}^r bN^k\left(\Delta^{I(n)}\right)|N_n\right]$$
$$+ E\left[\sum_{k=1}^r bN^k\left(T_S^{I(n)}\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right)|N_n\right]$$
$$= \sum_{k=1}^r bN_n^k - bG^{I(n)}\left(N_n^{I(n)}\right)$$
$$+ E\left[\sum_{k=1}^r bN^k\left(\Delta^{I(n)}\right)\right] +$$
$$\sum_{k=1}^r bE\left[N^k\left(\sum_{i=1}^{G^{I(n)}\left(N_n^{I(n)}\right)} B_i + T_D^{I(n)}\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right)|N_n\right]$$
$$= \sum_{k=1}^r bN_n^k - bG^{I(n)}\left(N_n^{I(n)}\right) + \sum_{k=1}^r b\lambda_k E\left[\Delta^{I(n)}\right]$$
$$+ \sum_{k=1}^r b\lambda_k\left(G^{I(n)}\left(N_n^{I(n)}\right)b + T_D^{I(n)}\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right)$$
$$= \sum_{k=1}^r bN_n^k - bG^{I(n)}\left(N_n^{I(n)}\right) + \sum_{k=1}^r b\lambda_k\delta^{I(n)}$$
$$+ \rho\left(G^{I(n)}\left(N_n^{I(n)}\right)b + T_D^P\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right)$$
$$= \sum_{k=1}^r bN_n^k + \rho\delta^{I(n)}$$
$$+ \left(\rho - 1 + \lambda\frac{T_D^P\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)}{G^{I(n)}\left(N_n^{I(n)}\right)}\right)bG^{I(n)}\left(N_n^{I(n)}\right).$$

Define $\gamma^k = \rho - 1 + \lambda\frac{T_D^P\left(G^{I(n+k)}\left(N_{n+k}^{I(n+k)}\right)\right)}{G^{I(n+k)}\left(N_{n+k}^{I(n+k)}\right)}, k = 0, \ldots, r-1$,

then $E\left[\sum_{k=1}^r bN_{n+1}^k\,|N_n\right]$
$= \sum_{k=1}^r bN_n^k + \rho\delta^{I(n)} + \gamma^0 bG^{I(n)}\left(N_n^{I(n)}\right)$.

Similarly, $E\left[\sum_{k=1}^r bN_{n+2}^k\,|N_n\right]$

$$= E\left[E\left[\sum_{k=1}^r bN_{n+2}^k\,|N_{n+1}, N_n\right]|N_n\right]$$
$$= E\left[E\left[\sum_{k=1}^r bN_{n+2}^k\,|N_{n+1}\right]|N_n\right]$$
$$= E\left[\sum_{k=1}^r bN_{n+1}^k\,|N_n\right] + \rho\delta^{I(n+1)}$$
$$+ E\left[\gamma^1 bG^{I(n+1)}\left(N_{n+1}^{I(n+1)}\right)|N_n\right].$$

Since $N_{n+1}^{I(n+1)} = N_n^{I(n+1)} + N^{I(n+1)}(S_n) \geq N_n^{I(n+1)}$, and $G_k(.)$ is nondecreasing, then

$$E\left[G^{I(n+1)}\left(N_{n+1}^{I(n+1)}\right)|N_n\right] \geq$$
$$E\left[G^{I(n+1)}\left(N_n^{I(n+1)}\right)|N_n\right] = G^{I(n+1)}\left(N_n^{I(n+1)}\right)$$

$\rho + \lambda b_d < 1$ implies $\epsilon_1 = \frac{1-\rho-\lambda b_d}{\lambda} > 0$

Since $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d \geq 0$ by *Theorem 5*, then $\exists M_1 > 0$, s.t. $n > M_1$ implies $\frac{T_D^P(n)}{n} - b_d < \epsilon_1$, i.e. $\rho - 1 + \lambda\frac{T_D^P(n)}{n} < 0$.

Thus when $G^{I(n+1)}\left(N_n^{I(n+1)}\right) > M_1$, $G^{I(n+1)}\left(N_{n+1}^{I(n+1)}\right) > M_1$, $\gamma^1 < 0$, so
$E\left[\gamma^1 bG^{I(n+1)}\left(N_{n+1}^{I(n+1)}\right)|N_n\right] \leq \gamma^1 bG^{I(n+1)}\left(N_n^{I(n+1)}\right)$.

So when $G^{I(n+1)}\left(N_n^{I(n+1)}\right) > M_1$,
$E\left[\sum_{k=1}^r bN_{n+2}^k\,|N_n\right] \leq$

$E\left[\sum_{k=1}^{r} bN_{n+1}^{k} \left| N_n \right.\right] + \rho\delta^{I(n+1)} + \gamma^{1} bG^{I(n+1)}\left(N_n^{I(n+1)}\right)$
$= \sum_{k=1}^{r} bN_n^k + \rho\left(\delta^{I(n)} + \delta^{I(n+1)}\right)$
$+ \gamma^0 bG^{I(n+1)}\left(N_n^{I(n)}\right) + \gamma^1 bG^{I(n+1)}\left(N_n^{I(n+1)}\right).$

Repeating the above calculation, we obtain
$E\left[\sum_{k=1}^{r} bN_{n+r}^{k} \left| N_n \right.\right] \leq \sum_{k=1}^{r} bN_n^k + \rho\delta +$
$\sum_{k=0}^{r-1} \gamma^k bG^{I(n+k)}\left(N_n^{I(n+k)}\right),$
when $G^{I(n+k)}\left(N_n^{I(n+k)}\right) > M_1$, $k = 1, \ldots, r-1$.

Since $\gamma^k < 0$, when $G^{I(n+k)}\left(N_n^{I(n+k)}\right) > M_1$, $k = 0, \ldots, r-1$,
then $\exists M > M_1$, s.t.
$G^{I(n+k)}\left(N_n^{I(n+k)}\right) > M$ implies
$-\epsilon = \rho\delta + \sum_{k=0}^{r-1} \gamma^k bG^{I(n+k)}\left(N_n^{I(n+k)}\right) < 0.$

Define
$E_0 = \left\{N_n \in \mathbb{N}^r \left| G^{I(n+k)}\left(N_n^{I(n+k)}\right) \leq M, k = 1, \ldots, r \right.\right\}$,
then E_0 is a finite subset of the state space $\mathbb{N}^r$.

Define $h(N) = \sum_{k=1}^{r} bN^k$, since $b \geq 0$ and $N \in \mathbb{N}^r$, then $\inf_N h(N) > -\infty$.

It then follows that
$E\left[h\left(N_{n+r}\right) \left| N_n \right.\right] \leq h\left(N_n\right) - \epsilon$, when $N_n \notin E_0$,
$E\left[h\left(N_{n+r}\right) \left| N_n \right.\right] \leq \sum_{k=1}^{r} bN_n^k + \rho\delta +$
$\sum_{k=0}^{r-1} \gamma^k bG^{I(n+k)}\left(N_n^{I(n+k)}\right)$, when $N_n \in E_0$.

Then $\{N_{lr+k}\}_{l=0}^{\infty}$ is positive recurrent by *Lemma 2* (Foster's Criterion), thus it is ergodic (irreducible, aperiodic and positive recurrent).

Necessity when economy of scale applies: Bertsimas et al. gave the necessary condition for stability in [6] $\rho + \lambda\frac{\bar{d}}{v} \leq 1$, where $\bar{d} = \lim_{i\to\infty} E\left[d_i\right]$, where d_i denotes the distance traveled from task i to the next task served after i, i.e. $\bar{d}$ is the steady state expected value of d_i. Let N^k be the number of tasks served in partition k in steady state. Then $\frac{\bar{d}}{v} = \sum_{k=1}^{r} \sum_{n=1}^{\infty} \frac{T_D^P(n)+\Delta}{n} P\left(N^k = n, I(i) = k\right)$
$> \sum_{k=1}^{r} \sum_{n=1}^{\infty} \lim_{n\to\infty} \frac{T_D^P(n)}{n} P\left(N^k = n, I(i) = k\right)$
$= b_d \sum_{k=1}^{r} \sum_{n=1}^{\infty} P\left(N^k = n, I(i) = k\right) = b_d,$
where $P\left(N^k = n, I(i) = k\right)$ means the probability that task i is in partition k and there are n tasks served in partition k in steady state. So $\rho + \lambda b_d \leq \rho + \lambda\frac{\bar{d}}{v} < 1$ ■

Remark1: The stability condition $\rho + \lambda b_d < 1$ has an additional term λb_d compared to $\rho < 1$ in queueing theory, where b_d is a measure of the mean travel time per task when $n \to \infty$.

Remark2: Theorem 6 is for the case when all the r partitions produce the same b_d. This can be generalised to the case when the values of b_d are different, say b_d^k for the k-th partition, then b_d in Theorem 6 should be $b_d = max_k\left\{b_d^k\right\}$, $k = 1, \ldots, r$.

Remark3: By Lemma 1, ergodicity implies that the sequence of station times $\{S_{nr+k}\}_{n=0}^{\infty}$ and the cycle times $\{C_{nr+k}\}_{n=0}^{\infty}$ converge weakly to finite random variables. The i-th task arriving in partition k to be served in station time S_{nr+k} first spends time W_{Oi} to wait outside the previous cycle, $C_{(n-1)r+k}$, and spends time W_{Ii} inside the current station time S_{nr+k}. Both W_{Oi} and W_{Ii} are well defined based on $C_{(n-1)r+k}$ and S_{nr+k} under the P-S policy, and $W_{Oi} \leq C_{(n-1)r+k}$ and $W_{Ii} \leq S_{nr+k}$. So both W_{Oi} and W_{Ii} converge weakly to finite random variables. Thus the system time $T_i = W_{Oi} + W_{Ii}$ converges weakly to finite random variable.

III. Conclusion

We prove a necessary and sufficient condition for stability in the Dynamic Traveling Repairman Problem (DTRP) [5] under the class of Polling-Sequencing (P-S) policies satisfying unlimited-polling and economy of scale. The number of tasks inside each polling partition is shown to be a Markov chain. Non-location based policies and some common location based policies such as TSP, NN and DA are shown to have economy of scale. The P-S class includes some of the policies proven to be optimal for the expectation of system time under light and heavy loads in the DTRP literature.

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