Risk and Reward in Spatial Queuing Theory

Jiangchuan Huang and Raja Sengupta

***Abstract*— We introduce an m-Vehicle Spatial Queuing Problem (m-SQP) extending the m-vehicle Dynamic Traveling Repairman Problem (m-DTRP) introduced by Bertsimas and van Ryzin [7]. The m-DTRP is focused on the mean response time (E). We extend it to also consider the variance (V) as a measure of risk, and the distribution of response time where possible. This enables an $E-V$ analysis of system performance to suit individual agencies with different $E-V$ or reward-risk preferences. We decompose the policy design problem into allocation and sequencing following the literature, but permit a spatial equivalent of only the class of polling policies for allocation. One can restrict attention to this class without loss of optimality in the heavy and light load cases because it includes some of the policies proven to be optimal in the m-DTRP literature. To enable E-V analysis we start by proving that the entire class of spatial polling policies is Markovian, irreducible, and aperiodic at each station. We then formulate a property on the sequencing phase called Economy of Scale (EoS) and show it holds for several policies in the literature. We then prove that the Bertsimas stability condition for the m-DTRP is sufficient for the ergodicity and stability of any sequencing policy satisfying EoS as long as the polling discipline is unlimited. We then describe computational approximations to compute the stationary distributions when the system is stable for some common policies thereby enabling $E-V$ analysis.**

***Key Words* – $E-V$ analysis, queuing theory, DTRP, task allocation.**

I. Introduction

We extend the results of the m-vehicle Dynamic Traveling Repairman Problem (m-DTRP) introduced by Bertsimas and van Ryzin [7]. The performance measure of this problem is the mean response time [7], [6], [8], [24], [10], where the response time T is defined as the difference between task completion time and arrival time. We extend this problem to focus on the variance of T and its distribution when possible. We call this extension the m-Vehicle Spatial Queuing Problem (m-SQP).

The extension to m-SQP is to characterize policy performance by reward E, which is the inverse of the mean response time, and its risk as measured by the variance V of the the response time. Following Markowitz in Modern Portfolio Theory (MPT) [22] we call this E-V analysis. This $E-V$ m-SQP is motivated by best effort systems. On entering a McDonalds one may ask not just "What is my expected service time?" but also "How certain is this value?" Accordingly both E and V have been characterized for many policies in classical non-spatial queueing theory [35]. We write this paper to do the same for queueing systems with moving servers (vehicles). Some systems in the robostics [1], [15], [5], [28] and logistics [11], [18], [19] are like this. The research is thus focused on providing a customer with mean response time, together with a measure of predictability or risk. We show in Section IV that two policies at the same load level can be incomparable in the sense that one has high reward and high risk while the other has low reward but also low risk. Highly variable response time can be even more frustrating than large mean response times [13], [20].

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Wierman [35] went beyond the mean response time to the distribution of response times in scheduling computer systems. Wierman [34] proposed the predictability criterion $\frac{Var[T(x)]^P}{x} \leq \frac{\lambda E[X^2]}{(1-\rho)^3}$, where $Var[T(x)]^P$ is the variance of the system time of a task of size x under policy P. The literature on financial risk analysis also uses two moments. Markowitz [22] and Elton and Gruber [16] used variance or standard deviation of return as a measure of risk. Accordingly, we explore $E[T]-Var[T]$ analysis for m-vehicle spatial queues. The customers of such systems could be modeled by a utility function on the two measures just as an investor is modeled in MPT by a risk-reward preference [22], [16].

A. Problem Statement

Based on the considerations above, we formulate the m-Vehicle Spatial Queuing Problem (m-SQP) by extending the m-vehicle Dynamic Traveling Repairman Problem (m-DTRP) introduced by Bertsimas and van Ryzin [7] in E and V and the distribution of response time as follows: A convex region $\mathbf{A}$ of area A contains m vehicles (servers) that travel at constant velocity v between task locations. Tasks arrive according to a Poisson process with rate λ and have a location that is independent and identically distributed (i.i.d.) according to the pdf $f_X(x)$ within $\mathbf{A}$. Each task i requires an i.i.d. execution time B, with mean b, which is assumed to be finite. Define load $\rho = \lambda b$. The system time of task i, denoted T_i, is defined as the elapsed time between the arrival of task i and the time task i is completed. When the system is stable, T_i converges to some T in distribution. T is called the steady state system time. Similarly, we can define the steady state waiting time, W, service time, T_S, and travel time to serve a task T_D. Intuitively, $T_S = T_D + B$ and $T = W + T_S$. Our aim is to find some verifiable stability condition, derive the mean and variance of T for some reasonable scheduling policies, and seek the distribution of T where possible.

In classic queueing theory, understanding the distribution of system time T is known to be a difficult task. In particular, exact derivations of the response time distribution are only known in specialized settings such as the M/M/1 and for simple scheduling policies, such as FCFS [35].

We decompose the policy design problem into allocation and sequencing but permit a spatial equivalent of only the class of polling policies for allocation. We call this the A-S class. The PART-FCFS, Batch Traveling Salesman Problem (TSP), Stochastic Queue Median (SQM) [7], Divide-And-Conquer policies [24] fit this class. Like these policies we use sequencing policies such as FCFS, NN, or TSP. The SQM policy in [6], which attains the $E[T]$ minimum under light load ($\rho \to 0^+$) is the 1-Polling-FCFS policy in the A-S class and the Divide-and-Conquer TSP policy in [24], which attains the $E[T]$ minimum under heavy load ($\rho \to 1^-$) is the r-Polling-TSP in the A-S class. So the $E[T]$ optimality of the A-S class under heavy and light loads follows from these inclusions.

B. Main Results

To enable E-V analysis we start by proving that the entire class of spatial polling policies is Markovian, irreducible, and aperiodic at each station in *Theorems 1 and 2* based on *Assumptions A1 - A4* in Section II-A. We then formulate a property on the sequencing phase called *Economy of Scale* (EOS) in Definition 3, a property common in microeconomics [23], and show it holds for several policies in the literature in Section II-B.

Bertsimas and van Ryzin (1991) defined $\overline{d} \equiv \lim_{x\to\infty} E[d_i]$, and proved that if $\overline{s} + \overline{d} \leq \frac{1}{\lambda}$, then $T^* \geq \frac{E[\|X-x^*\|]}{1-\rho}$ when $\rho\overline{s} \to 0$, and $T^* \geq \gamma^2 \frac{\lambda A}{(1-\rho)^2} - \frac{1-2\rho}{2\lambda}$ when $\rho\overline{s} \to 1^-$, where $T^* = \min(E[T])$. They also checked policies such as FCFS, PART, "Batch" TSP, SFC, NN, SQM, UTSP and get $E[T]$ analytically for FCFS and by simulation for other policies. Xu (1994) analyzed the heavy-load (when $\rho \to 1^-$) behavior of $Var[T]$ is $O\left(\frac{1}{(1-\rho)^4}\right)$ under the PART_TSP policy. The literature does not discuss how to find $\overline{d}$ or establish its existence. Unlike $\overline{s}$, $\overline{d}$ is policy dependent. This complication does not arise in classic queues, meaning there are no policy independent stability conditions in spatial queues.

In section II-C we prove that the Bertsimas stability condition for the m-DTRP is sufficient for the ergodicity and stability of any sequencing policy satisfying EOS as long as the polling discipline is unlimited. We then describe computational approximations to compute the stationary distributions when the system is stable for some common policies thereby enabling $E - V$ analysis. We focus in particular on the one-partition case of the polling policies, the gated policies. If the service time distribution in each gate is in a certain exponential family, e.g. Gamma or Erlang, then the transition probability matrix of the Markov chain has a closed form and we find the stationary distribution numerically. The service time distribution in the gate depends upon the policy used to sequence the tasks in a gate. We then focus on sequencing tasks by solving traveling salesman problems, since this has been a focus in the DTRP literature [7], [6], [8], [24], [10]. Using simulation techniques and the Kolmogorov-Smirnov ($K-S$) test we show that the Gamma distribution is a good fit for the TSP sequencing policy when the number of tasks in the gate is greater than 50. Based on this goodness of fit, we characterize the service time distribution for the TSP sequencing policy in closed form and solve for the parameters of the distribution using least squares regression. Since TSP approximation algorithms are more likely to be used in practice than exact TSP solvers, we do the same for two polynomial time approximation algorithms, NN and the Daganzo's algorithm [12], [26]. From the stationary distribution of the number of tasks inside each gate, we are able to derive the distribution of T, together with $E[T]$ and $Var[T]$. $Var[T]$ can also be connected to more conservative measures of risk by tail inequalities like the Chebyshevs Inequality [35].

$$\Pr(|T - E[T]| \geq a) \leq \frac{Var[T]}{a^2} \tag{1}$$

II. Scheduling Policies: Markov Chain, Economy of Scale and Stability

This section and section III are on the 1-SQP. In section IV we generalize the results to m-SQP.

We propose a class of *Allocation-Sequencing* (A-S) scheduling policies that take advantage of both the arrival time and location of the tasks. The A-S policies are composed of *allocation policies* and *sequencing policies*. An allocation policy determines a finite set of tasks for the vehicle in a finite horizon. A sequencing policy determines the order in which the tasks are served.

A. Allocation: Spatial-Polling Policies and Markov Chain

We propose a class of *Spatial-Polling* policies for the Allocation policies in the A-S class.

Polling policies are well-established in the queueing theory literature. Overviews and surveys of polling systems can be found in [30], [32], [33]. Stability and ergodicity criteria for polling systems are well established and can be found in [3], [17], [21], [27], [9], [4].

For Spatial-Polling policies, we divide the region $\mathbf{A}$ into an r-partition $\{\mathbf{A}^k\}_{k=1}^r$ of $\mathbf{A}$, label the r partitions as $1, 2, \ldots, r$. We regard each partition as a station in classic polling systems. In this way we generalize the polling system from classic queue theory to the spatial case. We follow the same construction made by Altman [3] and the mathematical definition by Fricker [17] with some necessary chages for the spatial case.

The spatial polling system contains a single vehicle and r partitions with infinite capacities. Partition k has area $A^k, k = 1, 2, \ldots, r$, $A = \sum_{k=1}^r A^k$. The vehicle visits the partitions in cyclic order, $1, 2, \ldots, r, 1, 2, \ldots$, and serves the queue in each partition. Without loss of generality, we assume that the vehicle is initially at partition 1. Thus, the n-th queue that the vehicle visits is in partition $I(n) = (n-1) \pmod r + 1$, where $n \pmod r$ means the remainder of the division of n by r.

We denote by:

$G^k(N)$ the number of tasks that are served in partition k when the queue is of length N.

$T_S^k(N)$ the total service time of N tasks in partition k. The formal definition of $T_S^k(N)$ will be given in section II-B.

$S^k(N)$ the duration of the service in partition k when the queue is of length N.

$$S^k(N) = T_S^k\left(G^k(N)\right) \tag{2}$$

Function $G^k(.)$ characterizes the polling discipline and $T_S^k(.)$ characterizes the sequencing policy.

The *switch time* the vehicle takes from a random point in partition k to a random point in partition $k+1$ is denoted by $\Delta^k, k = 1, \ldots, r-1$. The value from partition r to partition 1 is denoted by Δ^r. $\Delta^k, k = 1, \ldots, r$ are bounded above by the diameter of the region $\mathbf{A}$ divided by the speed of the vehicle, v. The first moments of Δ^k is denoted by $\delta^k, k = 1, \ldots, r$. Let $\Delta = \sum_{k=1}^r \Delta^k$ be the total switch time in a cycle and denote by δ the first moment of Δ.

The tasks arrive at partition k with a Poisson process of parameter $\lambda^k = \int_{A^k} f(x)\,dx\lambda$. The task sizes are i.i.d. with common distribution B and mean b. Define $\rho^k = \lambda^k b$, and $\rho = \sum_{k=1}^r \rho^k$, $1 \leq k \leq r$. Let $N^k\left(t_1, t_2\right]$ denote the number of Poisson arrivals to partition k, $1 \leq k \leq r$, during a (random) time interval $(t_1, t_2]$. $N^k(t) \equiv N^k\left(0, t\right]$ is the number of arrivals in a time interval of length t.

The n-th value of the polling system is described by the random variables N_n^k, $1 \leq k \leq r$, $n \geq 1$, where N_n^k represents the number of tasks in partition k when the vehicle arrives at the n-th queue. Let $N_n = \left(N_n^1, \ldots, N_n^r\right)$, taking values in $\mathbf{N}^r$, where $\mathbf{N}$ is the set of nonnegative integers.

Denote by S_n, the *station time*, the time interval between the arrival times of the vehicle to the n-th queue and the $(n+1)$st queue.

$$S_n = S^{I(n)}(N_n^{I(n)}) + \Delta^{I(n)} \tag{3}$$

Denote by C_n, the *cycle time*, the time interval between two successive arrivals of the vehicle to the same partition. $C_n = S_n + \ldots + S_{n+r-1}$.

The arrival times, the service times, the switch times are mutually independent, and are independent of the past and present system states. We adopt the rigorous independence definitions from Fricker [17] with some changes for the spatial case.

Consider a queue service starting at stopping time τ at partition k while N tasks are waiting and N^- tasks have already been served for the whole system. Let $\mathbf{F}_\tau$ be any σ-field containing the history of the service process up to random time τ. $\mathbf{F}_\tau$ is independent of the process $N(\tau, \tau + .]$ of arrivals after τ and of the task sizes $\{B_{N^-+i}\}_{i>0}$ of the tasks that have not been served up to time τ. The following four assumptions hold for all $k = 1, \ldots, r$.

A1: (G^k, S^k) is conditionally independent of $\mathbf{F}_\tau$ given N, and has the disribution of $\left(G^k(N), S^k(N)\right)$ where the random functions $(G^k(.), T_S^k(.))$ are taken independent of N. i.e. The A-S policies do not depend on the past history of the service process such as the number of tasks being already served and the time spent serving them.

A2: (G^k, S^k) is independent of $\left((B_{N^-+G^k+i})_{i>0}, N\left(\tau + S^k, \tau + S^k + .\right]\right)$, i.e. The selection of a task for service is independent of the required execution time and of possible future arrivals.

A3: $G^k(0) = S^k(0) = 0$ and there exists $x > 0$ such that $G^k(x) > 0$. i.e. The vehicle leaves immediately a queue which is or becomes empty, but provides service with a positive probability once there are enough task(s) in the queue.

A4: $\left(G^k(x), S^k(x)\right)$ is *monotonic* and *contractive* in x.

N_n evolve according to the following evolution equations:

$$N_{n+1}^k = \begin{cases} N_n^k + N^k\left(S_n\right), & I(n) \neq k \\ N_n^k - G^k\left(N_n^k\right) + N^k\left(S_n\right), & I(n) = k \end{cases} \tag{4}$$

where S_n is given by (3) and (2)

Typical *service disciplines* G^k were classified by Vishnevskii [33] into mainly two categories: Deterministic and Random. We focus on the deterministic disciplines, where the maximum number of tasks treated by the vehicle in one visit to the queue is constant. Common diciplines include exhaustive, gated, l_k-limited, l_k-decrementing and limited sojourn time. The first two are unlimited disciplines. The last three limited cases are not efficient in the sense that they are not stable for some $\rho < 1$ even in classic queues [17]. So we focus on the *unlimited* disciplines.

Definition 1: An allocation policy is called an Unlimited-Polling policy if it is a polling policy and its $G^k(.)$ satisfies $G^k(x) \to \infty$, when $x \to \infty$.

Definition 2: An allocation policy is called a Gated-Polling policy if it is a Polling policy and its $G^k(.)$ satisfies $G^k(x) = x$.

The spatial polling system has a Markovian structure as specified by the following two theorems, which is almost identical to the theorems given in [17].

Theorem 1: The sequence $\{N_n\}_{n=0}^\infty$ is a Markov chain.

Proof: At the n-th polling instant τ, the server starts serving queue n (if not empty, otherwise he starts switching to queue $n+1$) according to policy $G^{I(n)}$ while the state of all queues is given by N_n. The arrival processes after n are Poisson and are independent of F_τ; the sevice times and the switch times involved after τ are also independent of F_τ. Because these quantities are mutually independent, it follows that given N_n, the evolution of the system after τ is independent of F_τ, which ensures the Markov property of the sequence . ∎

Remark: This Markov chain is in general not homogeneous because its transitions depend on n through $G^{I(n)}$ and $\Delta^{I(n)}$, and $I(n)$ is different for each n.

Theorem 2: $\{N_{nr+k}\}_{n=0}^\infty$ is a homogeneous, irreducible and aperiodic Markov chain with state space $\mathbf{N}^r$.

Proof: $\{N_{nr+k}\}_{n=0}^\infty$ is a subsequence of the Markov chain $\{N_n\}_{n=0}^\infty$ and is thus also a Markov chain which is homogeneous because $I(nr+k) = k$ and $G^{I(nr+k)} = G^k$ for $n = 0, 1, 2, \ldots$.

It is irreducible because all states communicate. Indeed, $N = (N^1, \ldots, N^r)$ can be reached in one step from the state

$(0,\ldots,0)$: this is realized when first no arrivals occur to all queues during the whole cycle but the last switch time Δ^{r-1}, and then the last switch time is positive and $(N^1,\ldots,N^r)$ arrivals occur during it, all this having a positive probability because the arrival processes are Poisson. On the other hand, $(0,\ldots,0)$ is reached in (possibly) many steps from any state $(N^1,\ldots,N^r)$ with positive probability too: this is realized when there are no arrivals until it happens. By the same arguments, the state $(0,\ldots,0)$ is aperiodic and so is the (irreducible) Markov chain. ■

B. Sequencing: Economy of Scale

The Spatial-Polling policies give a finite set of tasks for the vehicle in a finite time horizon. The set of tasks can be sequenced based on their arrival time or location. Common policies include FCFS, NN and TSP. For a set of N tasks $\{B_l, X_l\}_{l=1}^N$, each with size B_k and location X_k, denote by $T_D^P\left(\{X_l\}_{l=1}^N\right)$ and $T_S^P\left(\{B_l, X_l\}_{l=1}^N\right)$ the travel time and service time for the N tasks $\{B_l, X_l\}_{l=1}^N$ under sequencing policy P.

$$T_D^P\left(\{X_l\}_{l=1}^N\right) \equiv E_X\left[\frac{1}{v} D^P\left(X, \{X_l\}_{l=1}^N\right)\right] \qquad (5)$$

where v is the vehicle speed, and $D^P\left(X, \{X_l\}_{l=1}^N\right)$ is the distance travelled by the vehicle to serve the tasks $\{B_l, X_l\}_{l=1}^N$ starting from a random point X in region $\mathbf{A}$. Note

$$T_S^P\left(\{B_l, X_l\}_{l=1}^N\right) = T_D^P\left(\{X_l\}_{l=1}^N\right) + \sum_{l=1}^N B_l \qquad (6)$$

Define $T_D^P(N) \equiv E_{\{X_l\}}\left[T_D^P\left(\{X_l\}_{l=1}^N\right)\right]$,
and $T_S^P(N) \equiv E_{\{X_l\}}\left[E_{\{B_l\}}\left[T_S^P\left(\{B_l, X_l\}_{l=1}^N\right)\right]\right]$, then

$$T_S^P(N) = T_D^P(N) + Nb \qquad (7)$$

$T_S^k(N) \equiv T_S^P(N)$ and $T_D^k(N) \equiv T_D^P(N)$ when sequencing policy P is used in partition k.

Definition 3: A sequencing policy P is said to have economy of scale (EOS) if i) $T_D^P(n)$ is nondecreasing in n. ii) $T_D^P(n+1) - T_D^P(n)$ is nonincreasing in n.

Lemma 1: Under a sequencing policy P with economy of scale, $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d \geq 0$, thus $\frac{T_D^P(n)}{n}$ is bounded.

Proof: Stolz-Cesàro theorem: Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of real numbers. Assume that b_n is strictly incresing and unbounded and $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = l$.

$T_D^P(n)$ is nondecreasing implies $T_D^P(n+1) - T_D^P(n) \geq 0$, Then $T_D^P(n+1) - T_D^P(n)$ is nonincreasing and lower bounded. So $\lim_{n\to\infty}\left(T_D^P(n+1) - T_D^P(n)\right)$ exists and ≥ 0, say it is b_d. Let $a_n = T_D^P(n)$ and $b_n = n$ in Stolz-Cesàro theorem we have $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d \geq 0$. ■

Remark: b_d is a measure of how well the sequencing policy can take advantage of the task locations.

From [25] we know that for the TSP policy the average distance traveled to serve a task $\frac{L_i}{i}$ satisfies $\lim_{i\to\infty} \frac{L_i}{i} = \beta_{TSP}\frac{\sqrt{A}}{\sqrt{i}}$, where L_i is the expected tour length connecting i points in a square of area A. This implies that TSP policy has an economy of scale property. Furthermore it takes most avantage of the task locations, i.e. $b_d = 0$ for TSP. Our simulation shows that NN also has economy of scale. Spatial FCFS has (trivial) economy of scale in the sense that $T_D^P(i+1) - T_D^P(i)$ is a constant.

C. Stability Condition

Stability in spatial queues is more complicated in the sense that the stability of spatial queues is policy dependent, whereas in queueing theory we have the universal stability condition $\rho < 1$ for work conserving queues.

Lemma 2: (Foster's Criterion): Suppose a Markov chain is irreducible and let E_0 be a finite subset of the state space E. Then the chain is *positive recurrent* if for some $h: E \to \mathbf{R}$ and some $\epsilon > 0$ we have $\inf_x h(x) > -\infty$ and

i) $\sum_{k\in E} p_{jk} h(k) < \infty, j \in E_0$,

ii) $\sum_{k\in E} p_{jk} h(k) \leq h(j) - \epsilon, j \notin E_0$.

where p_{jk} is the transition probability of the chain.

Theorem 3: (Stability theorem): For any A-S policy with allocation policy satisfying *Definition 1* (Unlimited-Polling) and sequencing policy P satisfying *Definition 3* (EOS), when $\rho + \lambda b_d < 1$, for all $1 \leq k \leq r$ the Markov chains $\{N_{nr+k}\}_{n=0}^\infty$ are ergodic, thus $\{N_{nr+k}\}_{n=0}^\infty$ together with the sequence of station times $\{S_{nr+k}\}_{n=0}^\infty$ and the cycle times $\{C_{nr+k}\}_{n=0}^\infty$ have stationary distributions.

Remark: $\rho + \lambda b_d = \lambda(b + b_d) < 1$ is the stability condition for spatial queues while in classic queues we have $\rho = \lambda b < 1$. b_d is the "task size" due to traveling to execute the task.

Proof: Taking a conditional expectation in (4) and summing over k, we obtain:

$$\begin{aligned}
&E\left[\sum_{k=1}^r bN_{n+1}^k | N_n\right] = \sum_{k=1}^r bN_n^k - bG^{I(n)}\left(N_n^{I(n)}\right) \\
&+ E\left[\sum_{k=1}^r bN^k\left(\Delta^{I(n)}\right) | N_n\right] \\
&+ E\left[\sum_{k=1}^r bN^k\left(T_S^{I(n)}\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right) | N_n\right] \\
&= \sum_{k=1}^r bN_n^k - bG^{I(n)}\left(N_n^{I(n)}\right) \\
&+ E\left[\sum_{k=1}^r bN^k\left(\Delta^{I(n)}\right)\right] + \\
&\sum_{k=1}^r bE\left[N^k\left(\sum_{i=1}^{G^{I(n)}\left(N_n^{I(n)}\right)} b + T_D^{I(n)}\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right) | N_n\right] \\
&= \sum_{k=1}^r bN_n^k - bG^{I(n)}\left(N_n^{I(n)}\right) + \sum_{k=1}^r b\lambda_k E\left[\Delta^{I(n)}\right] \\
&+ \sum_{k=1}^r b\lambda_k\left(G^{I(n)}\left(N_n^{I(n)}\right)b + T_D^{I(n)}\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right) \\
&= \sum_{k=1}^r bN_n^k - bG^{I(n)}\left(N_n^{I(n)}\right) + \sum_{k=1}^r b\lambda_k \delta^{I(n)} \\
&+ \rho\left(G^{I(n)}\left(N_n^{I(n)}\right)b + T_D^P\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right) \\
&= \sum_{k=1}^r bN_n^k + \rho\delta^{I(n)} \\
&+ \left(\rho - 1 + \lambda\frac{T_D^P\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)}{G^{I(n)}\left(N_n^{I(n)}\right)}\right) bG^{I(n)}\left(N_n^{I(n)}\right).
\end{aligned}$$

Define $\gamma^k = \rho - 1 + \lambda\frac{T_D^P\left(G^{I(n+k)}\left(N_{n+k}^{I(n+k)}\right)\right)}{G^{I(n+k)}\left(N_{n+k}^{I(n+k)}\right)}, k = 0,\ldots,r-1$,

then $E\left[\sum_{k=1}^r bN_{n+1}^k | N_n\right]$

$= \sum_{k=1}^r bN_n^k + \rho\delta^{I(n)} + \gamma^0 bG^{I(n)}\left(N_n^{I(n)}\right)$.

Similarly, $E\left[\sum_{k=1}^r bN_{n+2}^k | N_n\right]$

$= E\left[E\left[\sum_{k=1}^r bN_{n+2}^k | N_{n+1}, N_n\right] | N_n\right]$

$= E\left[E\left[\sum_{k=1}^r bN_{n+2}^k | N_{n+1}\right] | N_n\right]$

$= E\left[\sum_{k=1}^{r} bN_{n+1}^{k} \,|N_n\right] + \rho\delta^{I(n+1)}$
$+ E\left[\gamma^1 bG^{I(n+1)}\left(N_{n+1}^{I(n+1)}\right)|N_n\right]$.

Since $N_{n+1}^{I(n+1)} = N_n^{I(n+1)} + N^{I(n+1)}\left(\Delta^{I(n)}\right)$
$+ N^{I(n+1)}\left(T_S^{I(n)}\left(G^{I(n)}\left(N_n^{I(n)}\right)\right)\right) \geq N_n^{I(n+1)}$,
and $G_k(.)$ is nondecreasing, then
$E\left[G^{I(n+1)}\left(N_{n+1}^{I(n+1)}\right)|N_n\right] \geq$
$E\left[G^{I(n+1)}\left(N_n^{I(n+1)}\right)|N_n\right] = G^{I(n+1)}\left(N_n^{I(n+1)}\right)$

$\rho + \lambda b_d < 1$ implies $\epsilon_1 = \frac{1-\rho-\lambda b_d}{\lambda} > 0$

Since $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d \geq 0$ by *Lemma 1*,
then $\exists M_1 > 0$, s.t. $n > M_1$ implies $\frac{T_D^P(n)}{n} - b_d < \epsilon_1$,
i.e. $\rho - 1 + \lambda\frac{T_D^P(n)}{n} < 0$.

Thus when $G^{I(n+1)}\left(N_n^{I(n+1)}\right) > M_1$,
$G^{I(n+1)}\left(N_{n+1}^{I(n+1)}\right) > M_1$, $\gamma^1 < 0$, so
$E\left[\gamma^1 bG^{I(n+1)}\left(N_{n+1}^{I(n+1)}\right)|N_n\right] \leq$
$\gamma^1 bG^{I(n+1)}\left(N_n^{I(n+1)}\right)$.

So when $G^{I(n+1)}\left(N_n^{I(n+1)}\right) > M_1$,
$E\left[\sum_{k=1}^{r} bN_{n+2}^{k} \,|N_n\right] \leq$
$E\left[\sum_{k=1}^{r} bN_{n+1}^{k} \,|N_n\right] + \rho\delta^{I(n+1)} + \gamma^1 bG^{I(n+1)}\left(N_n^{I(n+1)}\right)$
$= \sum_{k=1}^{r} bN_n^k + \rho\left(\delta^{I(n)} + \delta^{I(n+1)}\right)$
$+ \gamma^0 bG^{I(n+1)}\left(N_n^{I(n)}\right) + \gamma^1 bG^{I(n+1)}\left(N_n^{I(n+1)}\right)$.

Repeating the above calculation, we obtain

$E\left[\sum_{k=1}^{r} bN_{n+r}^{k} \,|N_n\right] \leq \sum_{k=1}^{r} bN_n^k + \rho\delta +$
$\sum_{k=0}^{r-1}\gamma^k bG^{I(n+k)}\left(N_n^{I(n+k)}\right)$,
when $G^{I(n+k)}\left(N_n^{I(n+k)}\right) > M_1$, $k = 1, \ldots, r-1$.

Since $\gamma^k < 0$, when $G^{I(n+k)}\left(N_n^{I(n+k)}\right) > M_1$, $k = 0, \ldots, r-1$,

then $\exists M > M_1$, s.t.
$G^{I(n+k)}\left(N_n^{I(n+k)}\right) > M$ implies
$-\epsilon = \rho\delta + \sum_{k=0}^{r-1}\gamma^k bG^{I(n+k)}\left(N_n^{I(n+k)}\right) < 0$.

Define
$E_0 = \left\{N_n \in \mathbf{N}^r \middle| G^{I(n+k)}\left(N_n^{I(n+k)}\right) \leq M, k = 1, \ldots, r\right\}$,
then E_0 is a finite subset of the state space $\mathbf{N}^r$.

Define $h(N) = \sum_{k=1}^{r} bN^k$, since $b \geq 0$ and $N \in \mathbf{N}^r$, then $\inf_N h(N) > -\infty$.

It then follows that

$E\left[h\left(N_{n+r}\right)|N_n\right] \leq h\left(N_n\right) - \epsilon$, when $N_n \notin E_0$,

$E\left[h\left(N_{n+r}\right)|N_n\right] \leq \sum_{k=1}^{r} bN_n^k + \rho\delta +$
$\sum_{k=0}^{r-1}\gamma^k bG^{I(n+k)}\left(N_n^{I(n+k)}\right)$, when $N_n \in E_0$.

Then $\{N_{lr+k}\}_{l=0}^{\infty}$ is positive recurrent by *Lemma 2* (Foster's Criterion), thus it is ergodic (irreducible, aperiodic and positive recurrent). ■

III. Gated Policies: Stationary Distribution

A. The Class of Gated Policies

Usually finding the stationary distribution of $\{N_{nr+k}\}_{n=0}^{\infty}$ is difficult, in this section we solve the stationary distribution of $\{N_{nr+k}\}_{n=0}^{\infty}$ for the special case when $r = 1$, i.e., the *Gated policy*, where $N_n, S_n, C_n \in \mathbf{N}$, $S_n = C_n$. In a system with gated policies [31], an arriving task that finds the server idle causes a gate to close. When this task service is completed, the gate opens and admits into a waiting room all the tasks that arrived during the service time and then closes. When all the tasks in the waiting room have been served, the gate opens and admits into the waiting room all the tasks that have arrived during the collective service times of the preceding group of tasks, after which it closes. The process continues in this manner. Inside each gate, different sequencing policies can be used, e.g. FCFS, NN, TSP, thus we have a class of gated policies.

Now N_n becomes the number of tasks served in the n-th gate. By *Theorem 3* it has a stationary distribution and converges to N when $\rho + \lambda b_d < 1$.

$\Pr\left(N(t) = j\right) = \frac{(\lambda t)^j}{j!}e^{-\lambda t}$, is the probability that j tasks arrive during time interval $(0, t]$ according to a Poisson process. The transition probability of $\{N_n\}_{n=1}^{\infty}$ is

$$p_{ij} = \begin{cases} 1, & i = 0, j = 1 \\ 0, & i = 0, j \neq 1 \\ \displaystyle\int_0^{\infty} \Pr\left(N(t) = j\right)\, dF_{T_S^P(i)}(t), & i > 0 \end{cases} \tag{8}$$

In general, $\int_0^{\infty} \Pr\left(N(t) = j\right)\, dF_{T_S^P(i)}(t)$ cannot be integrated analytically because the service distribution inside each gate, $F_{T_S^P(i)}(t)$, is usually complicated. For certain distributions in the exponential family, $F_{T_S^P(i)}(t) = h(x)g(\theta)\exp\left[\eta(\theta)T(t)\right]$, when $h(x)$ is a polynomial function and $T(x)$ is a monomial function $\int_0^{\infty} \Pr\left(N(t) = j\right)\, dF_{T_S^P(i)}(t)$ has a closed form, and the stationary distribution can be solved numerically for these distributions. Common examples are the Gamma and Erlang distributions.

For the Gamma distribution, we obtain

$$f_{T_S^P(i)}(t) = \frac{\beta^{\alpha}t^{\alpha-1}e^{-\beta t}}{\Gamma(\alpha)} \tag{9}$$

where α and β are functions of i.

Thus when $i > 0$, $p_{ij} = \int_0^{\infty} \frac{(\lambda t)^j}{j!}e^{-\lambda t}\frac{\beta^{\alpha}t^{\alpha-1}e^{-\beta t}}{\Gamma(\alpha)}\, dt$

$$= \frac{\lambda^j\beta^k}{j!\Gamma(\alpha)}\int_0^{\infty} t^{j+\alpha-1}e^{-(\lambda+\beta)}\, dt = \frac{\Gamma(j+\alpha)\lambda^j\beta^{\alpha}}{j!\Gamma(\alpha)\left(\lambda+\beta\right)^{j+\alpha}}$$
.

B. Gamma Approximation for NN and TSP

We show the Gamma distribution is a good approximation for the service time $T_S^P(i)$ when the sequencing policy P is NN, TSP or the Daganzo Algorithm (DA) for the TSP [12], [26].

Simulations summarized in Table I, II and III show the null hypothesis that the tour lengths of NN, TSP and DA are distributed according to the Gamma distribution is not rejected at the 5% level based on the Kolmogorov-Smirnov (K-S) test when the number of points is greater than 12 for the NN, 50 for the exact TSP by Mathematica and 27 for the DA. When there are less than 50 points in the region, The exact TSP tour can be solved easily.

TABLE I

NEAREST NEIGHBOR TOUR DISTRIBUTION

#spls	#pts	E	Var	α	β	h	p-value
10K	192	12.56	0.2632	600.20	47.77	0	0.6781
10K	147	11.00	0.2571	471.66	42.86	0	0.7975
10K	108	9.43	0.2508	354.98	37.65	0	0.9732
10K	75	7.84	0.2399	256.67	32.74	0	0.9218
10K	70	7.57	0.2403	239.02	31.56	0	0.7724
10K	65	7.29	0.2392	222.73	30.57	0	0.2679
10K	48	6.23	0.2260	172.03	27.61	0	0.8907
10K	27	4.59	0.2125	98.92	21.57	0	0.9877
10K	12	2.87	0.1737	47.07	16.40	0	0.5245

#spls is the sample size.
$h = 1(0)$: The null hypothesis that the tour length is distributed according to the Gamma distribution (α, β) is (not) rejected at the 5% level based on the K-S test.

TABLE II

TSP TOUR DISTRIBUTION

#spls	#pts	E	Var	α	β	h	p-value
10K	192	10.83	0.0626	1872.29	172.80	0	0.2633
10K	147	9.53	0.0644	1408.49	147.78	0	0.3189
10K	108	8.23	0.0658	1026.85	124.72	0	0.0946
10K	75	6.95	0.0703	684.46	98.52	0	0.0549
10K	70	6.74	0.0704	641.60	95.26	0	0.1253
10K	65	6.50	0.0713	591.32	90.90	0	0.2311
10K	48	5.59	0.0642	483.03	86.38	1	1.99E-04
10K	27	4.37	0.0901	209.68	47.94	1	3.64E-05
10K	12	3.09	0.1109	84.31	27.28	1	1.14E-05

C. Deriving T from N

When the system is stable, the duration of a gate, the station time, S_n converges to $S = S(N) + \Delta = T_S^P(N) + \Delta$. $F_S(t) = \Pr(S \leq t) = \sum_{i=0}^{\infty} \Pr\left(T_S^P(N) + \Delta \leq t\,|i\right) \Pr(N = i)$ $\approx \sum_{i=0}^{i_m} \Pr\left(T_S^P(i) + \Delta \leq t\right) \Pr(N = i)$ for some i_m large enough, since $\Pr(N = i) \to 0$ when $i \to \infty$.

T is composed of two random variables: waiting time outside a gate, W_O, and waiting time inside a gate, W_I, i.e. $T = W_O + W_I$.

W_O is the *excess* or *residual life* [29] (sec. 5-5) of S, defined as the time interval from an arbitrary point (an arrival point in a Poisson process) during S to the end of S. The pdf of W_O, $f_{W_O}(t) = \frac{1-F_S(t)}{E[S]}$, $t \geq 0$.

W_I depends on the order of tasks in the sequencing policy. It can be approximated by the *age* [29] (sec. 5-5) of S, defined as the time interval from the beginning of S to an arbitrary point during S. The *age* of S has the same distribution as the excess of S. Since W_O and W_I are independent for non-arrival-time-based sequencing, the pdf of T is given by the convolution of the pdfs of the two.

Fig. 1 shows the pdf of T under TSP with different load in a square region with size 1×1 with each task size uniformly distributed in $[0, 1]$. $E[T]$ and $Var[T]$ can be obtained from the pdf of T. Fig. 2 shows the $E-V-\rho$ profile for different policies in the same region and task size distribution setting. Fig. 3, 4, 5, 6, 7 and 8 show the cut of Fig. 2 with different ρ. Note that TSP, DA and NN almost overlap on each other in Fig. 3, 4, 5 and 6 because their $E-V$ are very close to each other compare to the scale of the plot. We observe that FCFS has higer $E[T]$ (lower reward) but lower $Var[T]$ (lower risk) than SJF, TSP, DA and NN when $\rho = 0.1$. SJF has lower $E-V$ than TSP, DA and NN when ρ is small. As ρ becomes greater, the $E-V$ of FCFS and SJF increase much faster than TSP, DA and NN. The system becomes unstable under FCFS and SJF when ρ approaches 0.5, while the $E-V$ of TSP, DA and NN are still low. When ρ approaches 1, the system become unstable for all policies, which agrees with *Theorem 3*. DA has lower $E-V$ than NN when $\rho = 0.675$ while NN has lower $E-V$ than DA when $\rho = 0.85$.

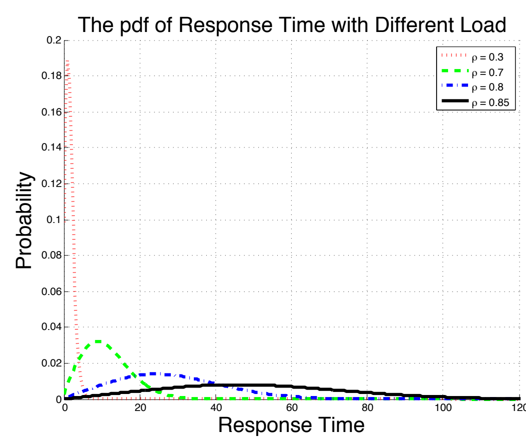

Fig. 1. The pdf of Response Time under TSP with Different Load.

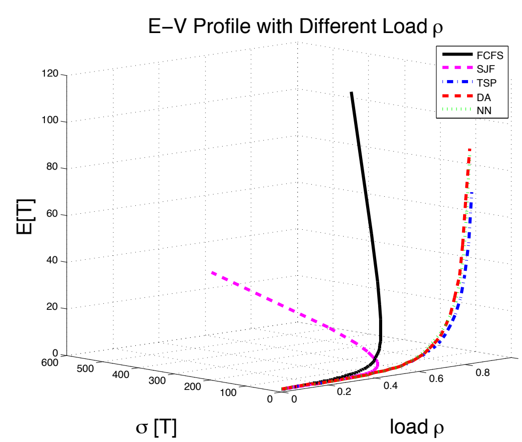

Fig. 2. $E-V-\rho$ Profile for Different Policies.

TABLE III

TOUR DISTRIBUTION FROM DAGANZO'S ALGORITHM

#spls	#pts	#s	E	Var	α	β	h	p-value
10K	192	8	12.92	0.1492	1119.78	86.65	0	0.8655
10K	147	7	11.32	0.1481	865.03	76.44	0	0.5011
10K	108	6	9.69	0.1532	613.04	63.25	0	0.9881
10K	75	5	8.06	0.1524	426.03	52.84	0	0.6412
10K	48	4	6.41	0.1616	254.62	39.70	0	0.7582
10K	27	3	4.73	0.1668	133.33	28.22	0	0.8647
10K	12	2	2.97	0.1731	50.46	16.97	1	0.0403

#s is the number of swaths in Daganzo's Algorithm.

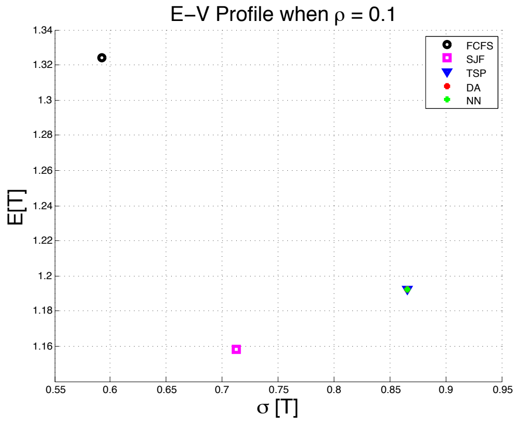

Fig. 3. $E-V$ Profile when $\rho = 0.1$ for Different Policies.

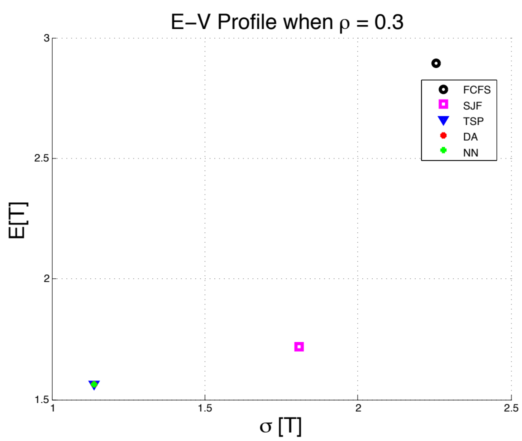

Fig. 4. $E-V$ Profile when $\rho = 0.3$ for Different Policies.

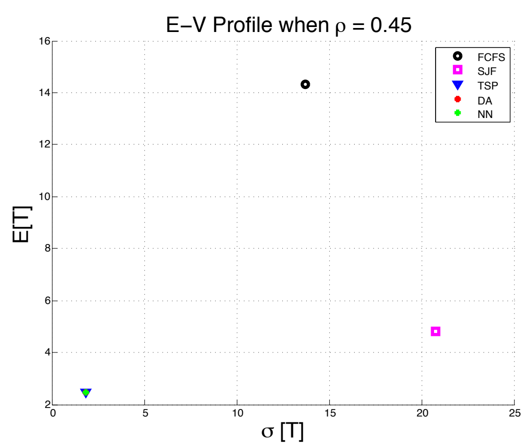

Fig. 5. $E-V$ Profile when $\rho = 0.45$ for Different Policies.

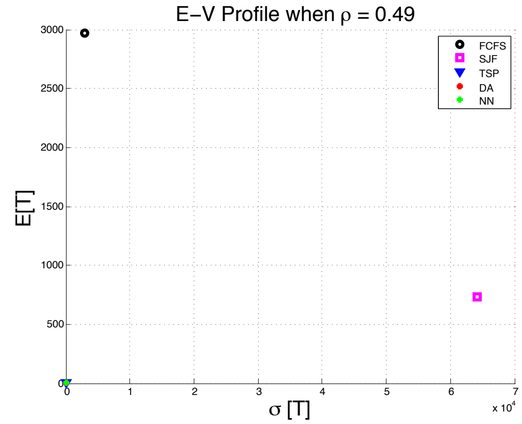

Fig. 6. $E-V$ Profile when $\rho = 0.49$ for Different Policies.

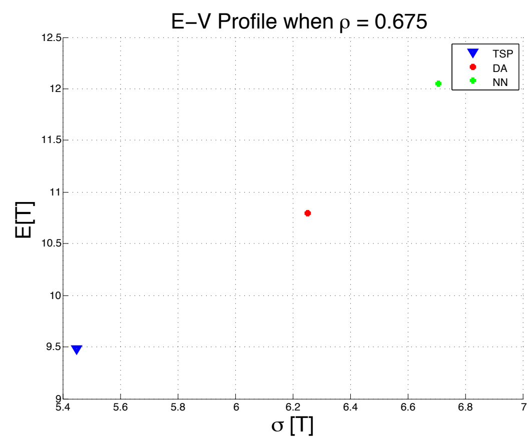

Fig. 7. $E-V$ Profile when $\rho = 0.675$ for Different Policies.

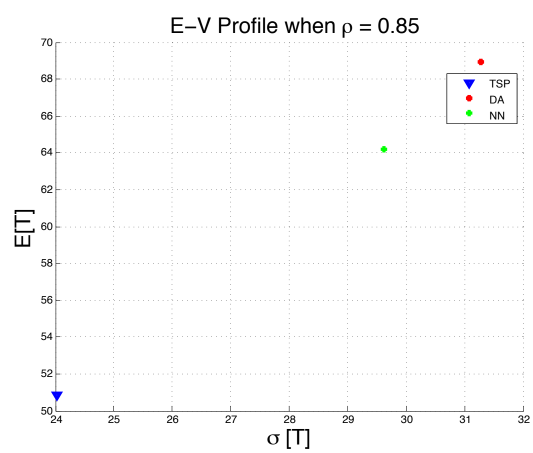

Fig. 8. $E-V$ Profile when $\rho = 0.85$ for Different Policies.

IV. From Single Vehicle to Multi-Vehicles

Frazzoli and Bullo [24] gave a way to break the m-DTRP into m 1-DTRP's: First produce an m-center Voronoi tessellation based on the Weber function or the continuous multi-median function [2], [14] that is *equitable* with respect to $f(.)$ and $f^{\frac{1}{2}}(.)$ and allocate a single vehicle to each Voronoi cell. An m-partition $\{A_k\}_{k=1}^{m}$ is *equitable* with respoect to $f(.)$ if $\int_{A_k} f(x)\,dx = \frac{1}{m}$ for all $k \in \{1, \ldots, m\}$. Then run the 1-DTRP policies in each cell independently. All theorems discussed in the 1-SQT can be extended to the m-SQT in this way.

Definition 4: A policy for an m-SQP is called an m-A-S policy if it allocates a single vehicle to each Voronoi cell of region $\mathbf{A}$ that is *equitable* with respect to $f(.)$ and $f^{\frac{1}{2}}(.)$, and runs the A-S policy independently in each Voronoi cell.

It is clear that there will be m independ Markov processes under an m-A-S policy each satisfying *Theorem 1 and 2*

Theorem 4: (Stability theorem for m-SQP): For any m-A-S policy satisfying *Definition 4*, when $\rho + \lambda b_d < m$, the m-SQP is stable.

Proof: (Sketch) Since the m-center Voronoi tessellation is equitable to $f(.)$, then $\lambda^k = \int_{A^k} f(x)\,dx\lambda = \frac{\lambda}{m}$. So each independent cell has a 1-SQP with load $\frac{\rho}{m}$. When $\frac{\rho}{m} + \frac{\lambda}{m} b_d < 1$, i.e. $\rho + \lambda b_d < m$, each independent cell is stable, thus the system is stable. ■

V. Conclusion

This paper introduces an m-Vehicle Spatial Queuing Problem (m-SQP) extending the m-vehicle Dynamic Traveling Repairman Problem (m-DTRP). We propose a new way to evaluate the performance of m-SQP in both reward (E) and risk (V), and also characterize the distribuiton of T where possible. In order to achieve that, we take a new approach by formulating a class of Allocation-Sequencing policies. We proposed the Spatial-Polling policies. We show the number of tasks in each station of the polling system is a Markov chain. Within the class of polling policies we define a sub-class of policies having economy of scale, and we give the sufficient condition stability based on economy of scale.

We solve the Markov chain of 1-SQP under the special case of A-S policies, the gated policies, for service distribution in certain exponential families and show that some important sequencing policies, including the NN, TSP and Daganzo's Algorithm, can be well-approximated with Gamma. We also generalise the analysis from the 1-SQP to the m-SQP.

Prediction based on the distribution of response time, and Resource and Administrative control base on the $E - V - \rho$ Profile for different policies for different customers with different utility preference can be done as future work.

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