STABILITY AND SYSTEM TIME DISTRIBUTION OF DYNAMIC TRAVELING REPAIRMAN PROBLEM*

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Abstract. A good model for the dynamic vehicle routing problem is the Dynamic Traveling Repairman Problem (DTRP) [4]. The DTRP literature has focused on optimizing the expected value of system time, defined as the elapsed time between the arrival and the completion of each task. We focus on the stability and distribution of system time, including its variance. This paper establishes a partially policy independent necessary and sufficient condition for stability in the DTRP. The policy class includes some of the policies proven to be optimal for system time expectation under light and heavy loads in the literature. We propose a new policy named PART-n-TSP and compute a good approximation for its system time distribution. PART-n-TSP has lower system time variance than PART-TSP [31] and Nearest Neighbor [4] when the load is neither too small or too large. We prove that PART-n-TSP is also optimal for system time expectation under light and heavy loads.

Key words. Dynamic Traveling Repairman Problem, Polling Systems, Economy of Scale

1. Introduction.

Dynamic vehicle routing problems arise when one needs to serve tasks that arrive in time and space. The objective is to schedule the tasks in an economic way to achieve good service level according to some performance metric, e.g., average waiting time, throughput, delivery probability of the tasks, or the total distance traveled by the moving servers (vehicles). There are many practical application where the dynamic vehicle routing problems arise. For example, (i) Google Street View, Google is running numerous data-collection vehicles with mounted cameras, lasers, a GPS and several computers to collect street views while minimizing the distance travelled [2]. (ii) Real-time traffic reporting, a radio station uses a helicopter to overfly accident scenes and other areas of high traffic volume for real-time traffic information. (iii) Unmanned aerial vehicle (UAV)-based sensing, a fleet of UAVs equipped with greenhouse gas sensors collect airborne measurements of greenhouse gases at several sites in California and Nevada, executing every task at its location before its deadline [18]. (iv) Enabling mobility in wireless sensor networks (WSN), mobile elements (vehicles) capable of short-range communications collect data from nearby sensor nodes as they approach on a schedule [23].

The Dynamic Traveling Repairman Problem (DTRP) [4] is a good way to model the dynamic vehicle routing problem: A convex region $\mathbf{A}$ of area A contains a vehicle (server) that travels at constant speed v. Tasks arrive according to a Poisson process with rate λ. Each task i is located at $X_i \in \mathbf{A}$, and has size B_i. X_i is independent and identically distributed (i.i.d.) with probability density function (pdf) $f_X(x)$, $x \in \mathbf{A}$. B_i is i.i.d. with pdf $f_B(s)$, $s \in [0, \infty)$. $E[B_i] = b$, which is assumed to be finite. Define load $\rho = \lambda b$. The system time of task i, denoted T_i, is defined as the elapsed time between the arrival of task i and the time task i is completed. It is a measure of system performance. An earlier formulation similar to the DTRP can be found in [10].

The DTRP resembles an M/G/1 queue in the time dimension but looks like a vehicle routing problem in the space dimension. As we know in queueing theory,

*This work was partly supported by NSF-CNS-1136141.

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$\rho = \lambda b < 1$ is a necessary and sufficient condition for all work conserving M/G/1 queues [9, sec. II.4.2]. However, there is no such policy-independent stability condition for the DTRP, which seems to be a "spatial version" of the M/G/1 queue. The known stability conditions for the DTRP are policy-dependent [4, 31, 20].

This paper makes progress towards finding stability conditions for the DTRP that are less policy dependent than those in the literature. We establish $\rho + \lambda b_d < 1$ as a necessary and sufficient stability condition for the class of Polling-Sequencing (P-S) policies satisfying *unlimited-polling* and *economy of scale* in Theorem 2.12. This stability condition is identical to the necessary condition for stability given in [5]. We show that important policies for the DTRP in the literature fall in the P-S class and satisfy the two properties. The extra term b_d is the limit of mean travel time as the number of tasks in a polling station goes to infinity. We prove that the existence of b_d is a consequence of economy of scale. b_d is policy-dependent, but it only depends on the sequencing phase of the P-S policy. Since the value of b_d can be derived in the static setting, we do not need to analyze or simulate the dynamic queueing system of DTRP to get b_d. We only need to analyze or simulate to obtain the statistics of sequencing N tasks, where N is a random variable, and the task locations are distributed in some fashion.

Our second contribution to the DTRP is the distribution of system time T, defined as the elapsed time between the arrival and the completion of each task. Knowing the distribution of the system time T, together with its expectation $E[T]$ and variance $Var[T]$ or standard deviation $\sigma[T]$, enables the expectation-variance analysis of the system under uncertainties [12, 22]. On entering a McDonalds, one may ask not just "What is my expected service time?" but also "How certain is this value?" We show in Tables 3.1 and 3.2 in Section 3 that two policies at the same load level can be incomparable in the sense that one has low expectation of system time but high variance while the other has high expectation of system time but low variance. In practice, highly variable system time can be even more frustrating than large mean system times [13, 17]. The literature discusses the distribution of system time T, together with its expectation and variance, for the FCFS policy and its variations such as the SQM and partitioning-FCFS [4]. This is in sharp contrast to queueing theory where the distribution of the system time or its moments are known for a wide variety of policies. See for example [26, 30]. To illustrate our point, the expected system time of the FCFS, SQM and partitioning-FCFS policy is not as good as NN [4] and PART-TSP [31] or DC [20] at most load levels.

In Section 3, we propose a policy in the P-S class called the PART-n-TSP policy. We give a good approximation for the distribution of the system time that is easy to compute. We do this by utilizing approximation results for the distribution of system time T, together with $E[T]$ and $Var[T]$ known for polling systems [8, 15]. Figure 3.5 shows that the cumulative distribution function (cdf) of the system time as computed by our method is very close to the cdf of the system time as obtained by Monte-Carlo simulation. We show that FCFS, partitioning-FCFS and n-TSP [4] are special cases of PART-n-TSP, meaning PART-n-TSP can be optimized to have better performance than the three. We also compare PART-n-TSP with PART-TSP [31] and Nearest Neighbor [4] on $E[T]$ and $\sigma[T]$ in Tables 3.1 and 3.2, since the latter two are considered near optimal in the literature. The $E[T]$ and $\sigma[T]$ under PART-n-TSP are obtained by our approximation. The $E[T]$ and $\sigma[T]$ under PART-TSP and NN are obtained by simulation. The results show that NN achieves lower $E[T]$ than both PART-n-TSP and PART-TSP for all loads $\rho \in \{0.1 \dots 0.9\}$ simulated by us. PART-

n-TSP achieves lower $E[T]$ than PART-TSP when ρ is not too small or too large, e.g. when $\rho \in \{0.3, \ldots, 0.7\}$. Also, PART-$n$-TSP achieves lower $\sigma[T]$ than PART-TSP and NN when ρ is not too small or too large, e.g. when $\rho \in \{0.3, \ldots, 0.7\}$. In real systems it may be desirable for ρ to be neither too small nor too large, since small ρ results in low server utilization, and large ρ in large system times. If so, PART-n-TSP would be good in practice as it achieves lower $\sigma[T]$ than PART-TSP and NN, and lower $E[T]$ than PART-TSP. We also prove that PART-n-TSP is $E[T]$ optimal under light load ($\rho \to 0^+$) and asymptotically optimal under heavy load ($\rho \to 1^-$). See Theorem 3.2.

2. Polling-Sequencing Policies. The class of Polling-Sequencing (P-S) policies include a polling phase and a sequencing phase. For the notation of P-S policies, we use “PART-” to denote the polling (partitioning) phase, followed with the sequencing policies for the sequencing phase. For example, PART-TSP means first partition the region $\mathbf{A}$ into polling stations, and use TSP to sequence the tasks inside each polling station. Similarly, we can define PART-NN, PART-SJF, etc.

2.1. Spatial-Polling: Markov Chain. Polling policies are well-established in the queueing theory literature. Overviews and surveys of polling systems can be found in [25, 27, 29]. Stability and ergodicity criteria for polling systems are well established and can be found in [1, 16, 21, 7].

The polling phase of the P-S class is a *spatial-polling* policy, we divide the region $\mathbf{A}$ into an r-partition $\{\mathbf{A}^k\}_{k=1}^r$, each of area A^k. We label the r partitions as $1, 2, \ldots, r$. We regard each partition as a station in classic polling systems. In this way we generalize the polling system [1, 16] in classic queueing theory to the spatial case.

The vehicle visits the partitions in cyclic order, $1, 2, \ldots, r, 1, 2, \ldots$, and serves the tasks in each partition. Without loss of generality, we assume that the vehicle is initially at partition 1. Thus, the l-th visit of the vehicle is partition $I(l) = (l-1) \pmod r + 1$, where $l \pmod r$ means the remainder of the division of l by r. The set of tasks waiting in partition $I(l)$ on the arrival of the l-th visit of the vehicle is called the l-th queue (the queue observed at l-th visit).

We denote by:

$G^k(N)$ the number of tasks that are served in partition k when the queue observed is of length N.

$T_S^k(n)$ the total service time of n tasks in partition k. The formal definition of function $T_S^k(.)$ will be given in section 2.2.

$S^k(N)$ the duration of the service in partition k when the queue observed is of length N.

$$S^k(N) = T_S^k\left(G^k(N)\right) \tag{2.1}$$

Function $G^k(.)$ characterizes the polling policy and $T_S^k(.)$ characterizes the sequencing policy.

The *switch time* the vehicle takes from a random point in partition k to a random point in partition $k+1$ is denoted by $\Delta^k, k = 1, \ldots, r-1$. The value from partition r to partition 1 is denoted by Δ^r. $\Delta^k, k = 1, \ldots, r$, are bounded above by the diameter of the region $\mathbf{A}$ divided by the speed of the vehicle v. The first moment of Δ^k is denoted by $\delta^k, k = 1, \ldots, r$. Let $\Delta = \sum_{k=1}^r \Delta^k$ be the total switch time in a cycle and denote by δ the first moment of Δ.

The tasks arrive at partition k with a Poisson process of parameter $\lambda^k = \int_{A^k} f_X(x)\, dx \lambda$. The task sizes are i.i.d. with common distribution B and mean b. Define $\rho^k = \lambda^k b$, and $\rho = \sum_{k=1}^r \rho^k$, $1 \le k \le r$. Let $N^k\left(t_1, t_2\right]$ denote the number of Poisson arrivals to partition k, $1 \le k \le r$, during a (random) time interval $(t_1, t_2]$. $N^k(t) \equiv N^k(0,t]$ is the number of Poisson arrivals in a time interval of length t.

The l-th value of the polling system is described by the random variables N_l^k, $1 \le k \le r$, $l \ge 1$, where N_l^k represents the number of tasks in partition k at the l-th visit of the vehicle. Let $N_l = \left(N_l^1, \dots, N_l^r\right)$, taking values in $\{\mathbb{N}\}^r$, where $\mathbb{N}$ is the set of nonnegative integers.

Denote by S_l, the *station time*, the time interval between the arrival times of the l-th visit and the $(l+1)$st visit of the vehicle.

$$S_l = S^{I(l)}(N_l^{I(l)}) + \Delta^{I(l)} \tag{2.2}$$

Denote by C_l, the *cycle time*, the time interval between two successive arrivals of the vehicle to the same partition. $C_l = S_l + \ldots + S_{l+r-1}$.

The arrival times, the service times, the switch times are mutually independent, and are independent of the past and present system states. We adopt the rigorous independence definitions from Fricker [16] with some changes for the spatial case.

Consider a queue service starting at stopping time τ at partition k while N tasks are waiting and N^- tasks have already been served for the whole system. Let $\mathcal{F}_\tau$ be any σ-field containing the history of the service process up to random time τ. $\mathcal{F}_\tau$ is independent of the process $N(\tau, \tau + .]$ of arrivals after τ and of the task sizes $\{B_{N^-+i}\}_{i>0}$ of the tasks that have not been served up to time τ. The following four assumptions hold for all $k = 1, \dots, r$.

A1: (G^k, S^k) is conditionally independent of $\mathcal{F}_\tau$ given N, and has the distribution of $\left(G^k(N), S^k(N)\right)$ where the expressions of the random functions $(G^k(.), T_S^k(.))$ are taken independent of N. i.e. The A-S policies do not depend on the past history of the service process such as the number of tasks being already served and the time spent serving them.

A2: (G^k, S^k) is independent of $\left((B_{N^-+G^k+i})_{i>0}, N\left(\tau + S^k, \tau + S^k + .\right]\right)$, i.e. The selection of a task for service is independent of the required execution time and of possible future arrivals.

A3: $G^k(0) = 0$, $S^k(0) = 0$ and there exists $N > 0$ such that $G^k(N) > 0$. i.e. The vehicle leaves immediately a queue which is or becomes empty, but provides service with a positive probability once there are "enough" task(s) in the queue.

A4: $\left(G^k(N), S^k(N)\right)$ is *monotonic* and *contractive* in N. A function $g(.)$ is contractive if for every $x \ge y$, $g(x) - g(y) \le x - y$.

N_l evolve according to the following evolution equations:

$$N_{l+1}^k = \begin{cases} N_l^k + N^k(S_l), & \text{if } I(l) \ne k \\ N_l^k - G^k\left(N_l^k\right) + N^k(S_l), & \text{if } I(l) = k \end{cases} \tag{2.3}$$

where $I(l) = (l-1) \pmod r + 1$.

The spatial polling system has a Markovian structure as specified by the following two theorems, which is almost identical to the theorems given in [16].

THEOREM 2.1. *The sequence $\{N_l\}_{l=0}^{\infty}$ is a Markov chain.*

Proof. At the l-th polling instant τ, the server starts serving queue l (if not empty, otherwise he starts switching to queue $l+1$) according to policy $G^{I(l)}$ while the state of all queues is given by N_l. The arrival processes after l are Poisson and are independent of $\mathcal{F}_\tau$; the service times and the switch times involved after τ are also independent of $\mathcal{F}_\tau$. Because these quantities are mutually independent, it follows that given N_l, the evolution of the system after τ is independent of $\mathcal{F}_\tau$, which ensures the Markov property of the sequence. □

REMARK 2.1.1. *This Markov chain is in general not homogeneous because its transitions depend on l through $G^{I(l)}$ and $\Delta^{I(l)}$, and $I(l)$ is different for each l. One can check that theorem 2.1 also holds when the task arrival process is renewal. This guarantees the arrival processes after l are independent of $\mathcal{F}_\tau$.*

THEOREM 2.2. *$\{N_{lr+k}\}_{l=0}^{\infty}$ is a homogeneous, irreducible and aperiodic Markov chain with state space $\{\mathbb{N}\}^r$, $k = 1, \ldots, r$, where r is the number of polling stations.*

Proof. $\{N_{lr+k}\}_{l=0}^{\infty}$ is a subsequence of the Markov chain $\{N_l\}_{l=0}^{\infty}$ and is thus also a Markov chain which is homogeneous because $I(lr + k) = k$ and $G^{I(lr+k)} = G^k$ for $l = 0, 1, 2, \ldots$.

It is irreducible because all states communicate. Indeed, $\left(N^1, \ldots, N^r\right)$ can be reached in one step from the state $(0, \ldots, 0)$: this is realized when first no arrivals occur to all queues during the whole cycle but the last switch time Δ^{r-1}, and then the last switch time is positive and $\left(N^1, \ldots, N^r\right)$ arrivals occur during it, all this having a positive probability because the arrival processes are Poisson. On the other hand, $(0, \ldots, 0)$ is reached in (possibly) many steps from any state $\left(N^1, \ldots, N^r\right)$ with positive probability too: this is realized when there are no arrivals until it happens. By the same arguments, the state $(0, \ldots, 0)$ is aperiodic and so is the (irreducible) Markov chain. ∎

2.2. Sequencing: Economy of Scale. Under a spatial-polling policy, the number and locations of tasks are determined in each polling station in each polling cycle, which is a static vehicle routing problem. The sequencing policies sequence the set of tasks in each polling station.

DEFINITION 2.3. *A policy for the 1-DTRP is called a Polling-Sequencing (P-S) policy if it runs a spatial-polling policy in region $\mathbf{A}$, and sequences the set of tasks in each polling station using some sequencing policy.*

For a set of n tasks $\{B_i, X_i\}_{i=1}^n$, each with size B_i and location X_i, denote by $T_D^P\left(\{X_i\}_{i=1}^n\right)$ and $T_S^P\left(\{B_i, X_i\}_{i=1}^n\right)$ the travel time and service time for the n tasks $\{B_i, X_i\}_{i=1}^n$ under sequencing policy P.

$$T_D^P\left(\{X_i\}_{i=1}^n\right) \equiv E_X\left[\frac{1}{v} D^P\left(X, \{X_i\}_{i=1}^n\right)\right] \tag{2.4}$$

where v is the vehicle speed, and $D^P\left(X, \{X_i\}_{i=1}^n\right)$ is the distance travelled by the vehicle to serve the tasks $\{B_i, X_i\}_{i=1}^n$ starting from a random point X in region $\mathbf{A}$ under sequencing policy P.

$$T_S^P\left(\{B_i, X_i\}_{i=1}^n\right) = T_D^P\left(\{X_i\}_{i=1}^n\right) + \sum_{i=1}^n B_i \tag{2.5}$$

Define $T_D^P(n) \equiv E_{\{X_i\}}\left[T_D^P\left(\{X_i\}_{i=1}^n\right)\right]$,
and $T_S^P(n) \equiv E_{\{X_i\}}\left[E_{\{B_i\}}\left[T_S^P\left(\{B_i, X_i\}_{i=1}^n\right)\right]\right]$, then

$$T_S^P(n) = T_D^P(n) + nb \tag{2.6}$$

$T_S^k(n) \equiv T_S^P(n)$ and $T_D^k(n) \equiv T_D^P(n)$ when sequencing policy P is used in partition k.

DEFINITION 2.4. *A sequencing policy P is said to have economy of scale (EoS) if $\frac{T_D^P(n)}{n}$ is nonincreasing in n.*

DEFINITION 2.5. *A scheduling policy is called non-location based if the distance between two consecutively executed tasks is i.i.d.. A scheduling policy is called location based if the distance between two consecutively executed tasks is dependent.*

Non-location based policies include FCFS, SJF, ROS and longest job first (LJF). Theorem 2.6(i) shows that non-location based policies satisfy economy of scale.

Location based policies include NN, furthest job first (FJF), TSP and the approximation algorithms for TSP such as Daganzo's algorithm (DA) [11]. For location based policies, there are two categories. One category try to find a shorter path connecting the locations of the tasks, which we call *smart*. Examples include TSP, NN and the approximation algorithms of TSP such as Daganzo's algorithm (DA) [11]. The other category tries to find a longer path connecting the locations of the tasks, which we call *foolish*. Examples include furthest job first (FJF). This category does not has EoS, and is not practical. Theorem 2.6(ii) below proves that the common policies in the smart category such as TSP, NN and DA have EoS. Other policies in this category can be checked by the similar analysis or through simulation. Theorem 2.6(ii) also proves that FJF does not satisfy EoS. NN and TSP are well known. In DA, one cuts a swath of approximate width, w, covering the region $\mathbf{A}$. One possible pattern is shown in the left of Figure 2.1 with a swath of width $\frac{\sqrt{A}}{6}$. The vehicle visits the task locations by moving along the swath without backtracking.

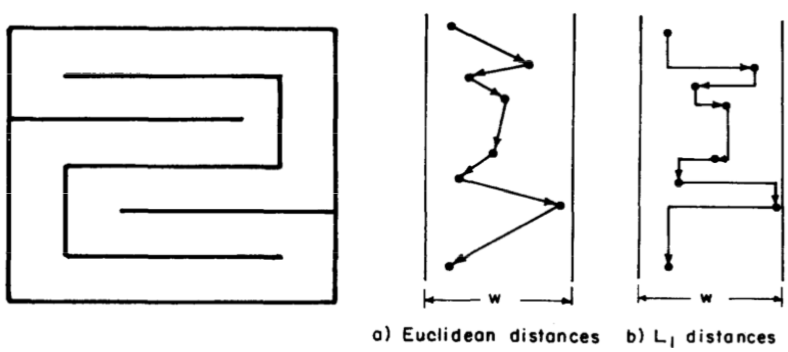

FIGURE 2.1. *Daganzo's Algorithm, cited from [11].*

THEOREM 2.6. *(i) Non-location based policies satisfy economy of scale. (ii) Nearest neighbor, traveling salesman policy and Daganzo's algorithm in the location based policy class satisfy economy of scale, furthest job first in the location based policy class does not satisfy economy of scale.*

Proof. (i) This is because that X_i is independent of the arrival process, and X_i is independent of X_{i-1}. Non-location based policies do not sequence based on the locations of tasks, so $T_D^P(n) = E\left[\frac{\sum_{i=1}^n D_i}{v}\right] = \frac{\sum_{i=1}^n E[D_i]}{v}$, where $D_i = \| X_i - X_{i-1} \|$ when $i > 1$, $D_1 = \| X_1 - X_v \|$, where X_i is the location of the i-th task and X_v is the initial position of the vehicle. X_i and X_v are i.i.d. with pdf $f_X(x)$. Thus $E[D_i]$ is a constant, say d. Then $\frac{T_D^P(n)}{n} = \frac{nd}{n} = d$, which is nonincreasing in n. So non-location based policies satisfy EoS.

(ii) Under TSP, when there are n tasks, there are $n!$ *Hamiltonian paths* starting from the initial position of the vehicle. A Hamiltonian path is a path that visits each X_i exactly once. The length of each Hamiltonian path is the sum of n i.i.d. D_i's, $HP = \sum_{i=1}^n D_i$. The TSP tour is the Hamiltonian path with minimum lengths among the $n!$ Hamiltonian paths. Let L_n be the tour length, then $\frac{T_D^P(n)}{n} = \frac{E[L_n]}{nv}$. When there are $n+1$ tasks, there are $(n+1)!$ Hamiltonian paths, and L_{n+1} is the shortest of them. $P\left(\frac{L_{n+1}}{n+1} > y\right) \leq P\left(\frac{L_n}{n} > y\right)$ because L_{n+1} is the minimum of $(n+1)!$

Hamiltonian paths and L_n is the minimum of $n!$ Hamiltonian paths. $\frac{E[L_{n+1}]}{n+1} = \int_0^\infty P\left(\frac{L_{n+1}}{n+1} > y\right) dy \le \int_0^\infty P\left(\frac{L_n}{n} > y\right) dy = \frac{E[L_n]}{n}$. Then $\frac{T_D^P(n+1)}{n+1} \le \frac{T_D^P(n)}{n}$. TSP satisfies EoS.

Under NN, when there are n tasks, Let L_n^{NN} be the length of the tour connecting the initial vehicle position and the locations of the n tasks, then $\frac{T_D^P(n)}{n} = \frac{E\left[L_n^{NN}\right]}{nv}$. L_n^{NN} is composed of n segments, $L_n^{NN} = \sum_{i=1}^n D_i^{NN}$, label i backwards such that D_i^{NN} is the distance from the $(n-i)$-th point to the $(n-i+1)$-th point when $i = 1, \dots, n-1$, and D_n is the distance from the initial position of the vehicle to the 1st point. So D_i^{NN} is the minimum of i D_j's, where each D_j is the distance between two random points in the region $\mathbf{A}$. Thus $P\left(D_{i+1}^{NN} > y\right) \le P\left(D_i^{NN} > y\right)$, this implies $E\left[D_{i+1}^{NN}\right] \le E\left[D_i^{NN}\right]$, thus $\frac{E\left[L_{n+1}^{NN}\right]}{n+1} = \frac{\sum_{i=1}^{n+1} E\left[D_i^{NN}\right]}{n+1} \le \frac{\sum_{i=1}^{n} E\left[D_i^{NN}\right] + \sum_{i=1}^{n} \frac{1}{n} E\left[D_i^{NN}\right]}{n+1} = \frac{\sum_{i=1}^{n} \frac{n+1}{n} E\left[D_i^{NN}\right]}{n+1} = \frac{E\left[L_n^{NN}\right]}{n}$. So $\frac{T_D^P(n+1)}{n+1} \le \frac{T_D^P(n)}{n}$. NN satisfies EoS.

In [11], the swath was approximated to be a infinitely long strip of width w neglecting corner effect as shown in the right two of Figure 2.1. The mean travel time per task when serving n tasks, $\frac{T_D^P(n)}{n} = \frac{nd_w}{n} = d_w$, where d_w is the expected distance between two consecutive locations. Let X denote the random distance between two consecutive points along the width of the strip, and Y the distance along the side of the strip, then $E[X] = \frac{w}{3}$, $E[Y] = \frac{A}{nw}$ according to [11]. $d_w = E_{X,Y}\left(\sqrt{X^2+Y^2}\right)$ for the Euclidean metric. $d_w \approx \frac{w}{3} + \frac{A}{nw}\psi\left(\frac{nw^2}{A}\right)$, where $\psi(x) = \frac{2}{x^2}\left((1+x)log(1+x) - x\right)$. $w^* = \sqrt{\frac{2.95A}{n}}$ minimizes d_w. Substituting w^*, we see d_w is decreasing with n. Thus $\frac{T_D^P(n)}{n}$ is nonincreasing in n.

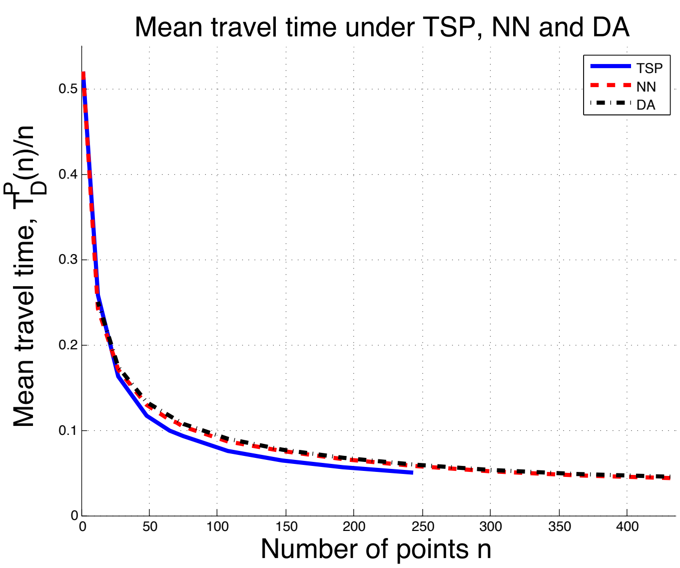

FIGURE 2.2. *Mean travel time under TSP, NN and DA.*

We show that FJF does not satisfy EoS by a counterexample. Consider a square of size 1×1 with uniformly distributed task locations. $\frac{T_D^P(1)}{1} = E[D_1] = 0.52$, where $D_1 = \| X_1 - X_v \|$, where X_i is the location of the i-th task and X_v is the initial position of the vehicle. X_i and X_v are i.i.d. with pdf $f_X(x) = 1$. When there are two tasks, the vehicle will choose the task further away, thus $\frac{T_D^P(2)}{2} > 0.52 = \frac{T_D^P(1)}{1}$. Thus FJF does not satisfy EoS. □

REMARK 2.6.1. *Non-location based policies have trivial EoS in the sense that $\frac{T_D^P(n)}{n}$ is a constant.*

Theorem 2.6(ii) is supported by the simulation results in Figure 2.2. The simulations are done in a square $\mathbf{A}$ of size 1×1. The task locations and the initial vehicle position are generated independently from a uniform distribution with pdf $f_X(x) = 1$. The length of the path connecting the vehicle and the tasks is calculated under TSP, NN and DA for different number of points n.

THEOREM 2.7. *Under a sequencing policy P with economy of scale, $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d \geq 0$, and $\exists M > 0$, s.t. $M \geq \frac{T_D^P(n)}{n} \geq b_d$ for all n.*

Proof. $\frac{T_D^P(n)}{n} \geq 0$ and $\frac{T_D^P(n)}{n}$ is nonincreasing in n imply that $\lim_{n\to\infty} \frac{T_D^P(n)}{n}$ exists, say b_d.
Thus we have $\lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d$ and $M = \frac{T_D^P(1)}{1} \geq \frac{T_D^P(n)}{n} \geq b_d \geq 0$. □

REMARK 2.7.1. *b_d is a measure of how well the sequencing policy can take advantage of the task locations. Let L_n denote the length of the tour connecting n points in a square of area A under TSP. From [24] we know that $\lim_{n\to\infty} \frac{L_n}{\sqrt{n}} = \beta_{TSP}\sqrt{A}$, where $\beta_{TSP} \approx 0.72$, thus $b_d = \lim_{n\to\infty} \frac{T_D^P(n)}{n} = \lim_{n\to\infty} \frac{E[L_n]}{vn} = \lim_{n\to\infty} \beta_{TSP}\frac{\sqrt{A}}{v\sqrt{n}} = 0$. This implies that TSP does best in taking advantage of the task locations.*

2.3. Stability Condition. Stability of DTRP is more complicated than in queueing theory because the stability of DTRP is policy dependent, whereas in queueing theory we have the policy-independent stability condition $\rho < 1$ for work conserving M/G/1 queues. Theorem 2.1 and 2.2 showed that $\{N_l\}_{l=0}^{\infty}$ is a Markov chain, and $\{N_{lr+k}\}_{l=0}^{\infty}$ is a homogeneous, irreducible and aperiodic Markov chain. We check the ergodicity of $\{N_{lr+k}\}_{l=0}^{\infty}$ and the stability of the DTRP under the P-S policies in this section.

DEFINITION 2.8. *A polling policy characterized by $G^k(.)$ is called an unlimited-polling policy if $G^k(N) \to \infty$, when $N \to \infty$, $k = 1, \ldots, r$.*

One can check that the common polling policies such as the exhaustive and gated policies in [25] are unlimited-polling policies.

LEMMA 2.9. *(LEMMA 3.1 in [1]) If for all $1 \leq k \leq r$ the Markov chains $\{N_{lr+k}\}_{l=0}^{\infty}$ are ergodic, then for all $1 \leq k \leq r$ $\{N_{lr+k}\}_{l=0}^{\infty}$ together with the sequence of station times $\{S_{lr+k}\}_{l=0}^{\infty}$ and the cycle times $\{C_{lr+k}\}_{l=0}^{\infty}$ converge weakly to finite random variables.*

DEFINITION 2.10. *The DTRP under a P-S policy is said to be stable if all the r Markov chains $\{N_{lr+k}\}_{l=0}^{\infty}$ are ergodic.*

LEMMA 2.11. *(Foster's Criterion [3, p.19]): Suppose a Markov chain is irreducible and let E_0 be a finite subset of the state space E. Then the chain is positive recurrent if for some $h : E \to \mathbb{R}$ and some $\epsilon > 0$ we have $\inf_x h(x) > -\infty$ and*

i) $\sum_{k\in E} p_{jk}h(k) < \infty, j \in E_0$,

ii) $\sum_{k\in E} p_{jk}h(k) \leq h(j) - \epsilon, j \notin E_0$.

where p_{jk} is the transition probability of the chain.

THEOREM 2.12. *(Stability theorem): For any P-S policy with polling policy satisfying Definition 2.8 (unlimited-polling) and sequencing policy P satisfying $\lim_{n\to\infty} \frac{T_D^k(n)}{n} = b_d^k$ in each partition $\mathbf{A}^k$, assuming the partitions $\{\mathbf{A}^k\}_{k=1}^{r}$ are divided such that $b_d^k = \lim_{n\to\infty} \frac{T_D^P(n)}{n} = b_d$, $\forall 1 \leq k \leq r$, then when $\rho + \lambda b_d < 1$, the Markov chains*

$\{N_{lr+k}\}_{l=0}^{\infty}$ *are ergodic,* $\forall 1 \le k \le r$. *Moreover, if the sequencing policy* P *satisfies Definition 2.4 (EoS), then* $\rho + \lambda b_d < 1$ *is necessary for the ergodicity of* $\{N_{lr+k}\}_{l=0}^{\infty}$.

Proof. Sufficiency: taking a conditional expectation in (2.3), summing over k, and substituting (2.2) and (2.1) we obtain:

$$E\left[\sum_{k=1}^r bN_{l+1}^k \,|N_l\right] = \sum_{k=1}^r bN_l^k - bG^{I(l)}\left(N_l^{I(l)}\right) + E\left[\sum_{k=1}^r bN^k\left(\Delta^{I(l)}\right)|N_l\right]$$
$$+ E\left[\sum_{k=1}^r bN^k\left(T_S^{I(l)}\left(G^{I(l)}\left(N_l^{I(l)}\right)\right)\right)|N_l\right]$$
$$= \sum_{k=1}^r bN_l^k - bG^{I(l)}\left(N_l^{I(l)}\right) + E\left[\sum_{k=1}^r bN^k\left(\Delta^{I(l)}\right)\right]$$
$$+ \sum_{k=1}^r bE[N^k(\sum_{i=1}^{G^{I(l)}(N_l^{I(l)})} B_i + T_D^{I(l)}(G^{I(l)}(N_l^{I(l)})))\,|N_l]$$
$$= \sum_{k=1}^r bN_l^k - bG^{I(l)}\left(N_l^{I(l)}\right) + \sum_{k=1}^r b\lambda_k E\left[\Delta^{I(l)}\right]$$
$$+ \sum_{k=1}^r b\lambda_k\left(G^{I(l)}\left(N_l^{I(l)}\right)b + T_D^{I(l)}\left(G^{I(l)}\left(N_l^{I(l)}\right)\right)\right)$$
$$= \sum_{k=1}^r bN_l^k - bG^{I(l)}\left(N_l^{I(l)}\right) + \sum_{k=1}^r b\lambda_k\delta^{I(l)}$$
$$+ \rho\left(G^{I(l)}\left(N_l^{I(l)}\right)b + T_D^P\left(G^{I(l)}\left(N_l^{I(l)}\right)\right)\right)$$
$$= \sum_{k=1}^r bN_l^k + \rho\delta^{I(l)} + \left(\rho - 1 + \lambda\frac{T_D^P\left(G^{I(l)}\left(N_l^{I(l)}\right)\right)}{G^{I(l)}\left(N_l^{I(l)}\right)}\right)bG^{I(l)}\left(N_l^{I(l)}\right).$$

Define $\gamma^k = \rho - 1 + \lambda\frac{T_D^P\left(G^{I(l+k)}\left(N_{l+k}^{I(l+k)}\right)\right)}{G^{I(l+k)}\left(N_{l+k}^{I(l+k)}\right)}, k = 0, \ldots, r-1,$

then $E\left[\sum_{k=1}^r bN_{l+1}^k\,|N_l\right] = \sum_{k=1}^r bN_l^k + \rho\delta^{I(l)} + \gamma^0 bG^{I(l)}\left(N_l^{I(l)}\right)$.

Similarly, $E\left[\sum_{k=1}^r bN_{l+2}^k\,|N_l\right] = E\left[E\left[\sum_{k=1}^r bN_{l+2}^k\,|N_{l+1}, N_l\right]|N_l\right]$
$= E\left[E\left[\sum_{k=1}^r bN_{l+2}^k\,|N_{l+1}\right]|N_l\right]$
$= E\left[\sum_{k=1}^r bN_{l+1}^k\,|N_l\right] + \rho\delta^{I(l+1)} + E\left[\gamma^1 bG^{I(l+1)}\left(N_{l+1}^{I(l+1)}\right)|N_l\right]$.

Since $N_{l+1}^{I(l+1)} = N_l^{I(l+1)} + N^{I(l+1)}\left(S_l\right) \ge N_l^{I(l+1)}$,
and $G_k(.)$ is nondecreasing, then
$E\left[G^{I(l+1)}\left(N_{l+1}^{I(l+1)}\right)|N_l\right] \ge E\left[G^{I(l+1)}\left(N_l^{I(l+1)}\right)|N_l\right] = G^{I(l+1)}\left(N_l^{I(l+1)}\right)$.

$\rho + \lambda b_d < 1$ implies $\epsilon_1 = \frac{1-\rho-\lambda b_d}{\lambda} > 0$.

Since $\lim_{n\to\infty}\frac{T_D^P(n)}{n} = b_d \ge 0$,
then $\exists M_1 > 0$, s.t. $n > M_1$ implies $\frac{T_D^P(n)}{n} - b_d < \epsilon_1$, i.e. $\rho - 1 + \lambda\frac{T_D^P(n)}{n} < 0$.

Thus when $G^{I(l+1)}\left(N_l^{I(l+1)}\right) > M_1$, $G^{I(l+1)}\left(N_{l+1}^{I(l+1)}\right) > M_1$, $\gamma^1 < 0$.
This implies $E\left[\gamma^1 bG^{I(l+1)}\left(N_{l+1}^{I(l+1)}\right)|N_l\right] \le \gamma^1 bG^{I(l+1)}\left(N_l^{I(l+1)}\right)$.

So when $G^{I(l+1)}\left(N_l^{I(l+1)}\right) > M_1$,
$E\left[\sum_{k=1}^r bN_{l+2}^k\,|N_l\right] \le E\left[\sum_{k=1}^r bN_{l+1}^k\,|N_l\right] + \rho\delta^{I(l+1)} + \gamma^1 bG^{I(l+1)}\left(N_l^{I(l+1)}\right)$
$= \sum_{k=1}^r bN_l^k + \rho\left(\delta^{I(l)} + \delta^{I(l+1)}\right) + \gamma^0 bG^{I(l+1)}\left(N_l^{I(l)}\right) + \gamma^1 bG^{I(l+1)}\left(N_l^{I(l+1)}\right)$.

Repeating the above calculation, we obtain
$$E\left[\sum_{k=1}^r bN_{l+r}^k\,|N_l\right] \le \sum_{k=1}^r bN_l^k + \rho\delta + \sum_{k=0}^{r-1}\gamma^k bG^{I(l+k)}\left(N_l^{I(l+k)}\right),$$
when $G^{I(l+k)}\left(N_l^{I(l+k)}\right) > M_1$, $k = 1, \ldots, r-1$.

Since $\gamma^k < 0$, when $G^{I(l+k)}\left(N_l^{I(l+k)}\right) > M_1$, $k = 0, \ldots, r-1$,
then $\exists M > M_1$, s.t.
$G^{I(l+k)}\left(N_l^{I(l+k)}\right) > M$ implies $-\epsilon = \rho\delta + \sum_{k=0}^{r-1}\gamma^k bG^{I(l+k)}\left(N_l^{I(l+k)}\right) < 0$.

Define $E_0 = \{N_l \in \mathbb{N}^r \left| G^{I(l+k)} \left(N_l^{I(l+k)} \right) \leq M, k = 1, \ldots, r \right\}$, then E_0 is a finite subset of the state space $\mathbb{N}^r$.

Define $h(N) = \sum_{k=1}^{r} bN^k$, since $b \geq 0$ and $N \in \mathbb{N}^r$, then $\inf_N h(N) > -\infty$.

It then follows that

$E\left[h\left(N_{l+r}\right)|N_l\right] \leq h\left(N_l\right) - \epsilon$, when $N_l \notin E_0$,

$E\left[h\left(N_{l+r}\right)|N_l\right] \leq \sum_{k=1}^{r} bN_l^k + \rho\delta + \sum_{k=0}^{r-1} \gamma^k b G^{I(l+k)} \left(N_l^{I(l+k)}\right)$, when $N_l \in E_0$.

Then $\{N_{lr+k}\}_{l=0}^{\infty}$ is positive recurrent by *Lemma 2.11* (Foster's Criterion), thus it is ergodic (irreducible, aperiodic and positive recurrent).

Necessity when economy of scale applies: Bertsimas et al. gave the necessary condition for stability in [5] $\rho + \lambda\frac{\bar{d}}{v} \leq 1$, where $\bar{d} = \lim_{i\to\infty} E\left[D_i\right]$, where D_i denotes the distance traveled from task i to the next task served after i, i.e. $\bar{d}$ is the steady state expected value of D_i. Let N^k be the number of tasks served in partition k in steady state. $P\left(N^k = n, X_i \in \mathbf{A}^k\right)$ denotes the probability that there are n tasks served in partition k in steady state and task i is one of them. Then

$\frac{\bar{d}}{v} = \sum_{k=1}^{r}\sum_{n=1}^{\infty} \frac{T_D^P(n)+\Delta}{n} P\left(N^k = n, X_i \in \mathbf{A}^k\right)$

$> \sum_{k=1}^{r}\sum_{n=1}^{\infty} \lim_{n\to\infty} \frac{T_D^P(n)}{n} P\left(N^k = n, X_i \in \mathbf{A}^k\right)$

$= b_d \sum_{k=1}^{r}\sum_{n=1}^{\infty} P\left(N^k = n, X_i \in \mathbf{A}^k\right) = b_d$.

So $\rho + \lambda b_d \leq \rho + \lambda\frac{\bar{d}}{v} < 1$. ∎

REMARK 2.12.1. *The stability condition $\rho + \lambda b_d < 1$ has an additional term λb_d compared to $\rho < 1$ in queueing theory, where b_d is the mean travel time per task when $n \to \infty$.*

REMARK 2.12.2. *By Lemma 2.9, ergodicity implies that the sequence of station times $\{S_{lr+k}\}_{l=0}^{\infty}$ and the cycle times $\{C_{lr+k}\}_{l=0}^{\infty}$ converge weakly to finite random variables. The i-th task arriving in partition k to be served in station time S_{lr+k} first spends time W_{Oi} to wait outside the previous cycle, $C_{(l-1)r+k}$, and spends time W_{Ii} inside the current station time S_{lr+k}. W_{Oi} and W_{Ii} are well defined based on $C_{(l-1)r+k}$ and S_{lr+k} under the P-S policy, and $W_{Oi} \leq C_{(l-1)r+k}$ and $W_{Ii} \leq S_{lr+k}$. So W_{Oi} and W_{Ii} converge weakly to finite random variables. Thus the system time $T_i = W_{Oi} + W_{Ii}$ converges weakly to finite random variable.*

3. System Time Distribution.

In the last section, we give a necessary and sufficient condition for the stability of the Dynamic Traveling Repairman Problem (DTRP) for the class of Polling-Sequencing (P-S) policies satisfying unlimited-polling and economy of scale. When the DTRP is stable, the distribution of the steady state system time T exists.

3.1. PART-n-Traveling Salesman Policy.

Bertsimas et al. [4] introduced the traveling salesman policy (TSP). It is based on collecting tasks into sets of size n that are then served in a TSP path. To be precise, we call this the n-TSP. This policy is a one-partition policy in the P-S class. We generalize it to multiple partitions as follows and call it the PART-n-TSP.

DEFINITION 3.1. *A Polling-Sequencing policy in Definition 2.3 is call the PART-n-TSP policy if the sequencing phase is an n-TSP policy that collects tasks into sets of cardinality n, and then serve them using an optimal traveling salesman path.*

The polling phase involving generating an r-partition $\left\{\mathbf{A}^k\right\}_{k=1}^{r}$ of $\mathbf{A}$ that is simultaneously equitable with respect to $f(x)$. In particular, when the region $\mathbf{A}$ is a square region $\mathbf{A}$ with size $a \times a$, and the tasks are uniformly distributed in $\mathbf{A}$ with pdf $f(x) = \frac{1}{A}$, $\mathbf{A}$ is divided into $r = m^2$ square partitions, each has size $\frac{a}{m} \times \frac{a}{m}$, where $m > 1$ is a given integer that parameterizes the policy. The vehicle visits the partitions in a cyclic order. Partitions are numbered so that for any $k = 1, \ldots, r-1$, partition $k+1$ is adjacent to partition k, and partition r is adjacent to partition 1 when m is even, or to the diagonal of partition 1 when m is odd, as illustrated in Figure 3.1 for the case $m = 4$ and $m = 5$. The vehicle cycles through the partitions in the order $1, \ldots, r, 1, \ldots, r, \ldots$. After the vehicle finishes the tasks polled in the current partition under a sequencing policy, the vehicle moves to an adjacent partition and serves it under the same sequencing policy.

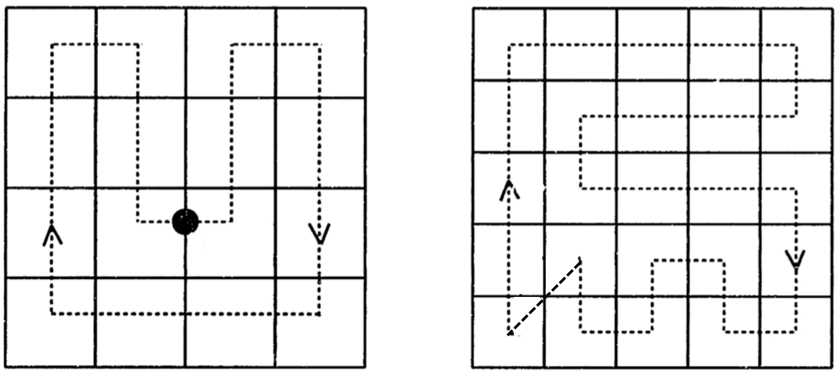

FIGURE 3.1. *Order for serving partitions under the polling policy, cited and revised from [31].*

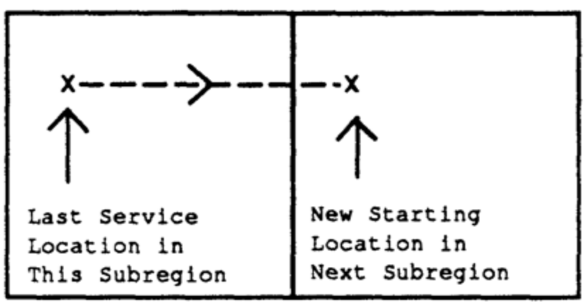

FIGURE 3.2. *Vehicle moving to adjacent partition, cited from [4].*

To move from one partition (polling station) to the next, the vehicle uses the projection rule shown in Figure 3.2 as introduced in [4]. Its last location in a given partition is simply "projected" onto the next partition to determine the server's new starting location. The vehicle then travels in a straight line between these two locations. This makes the distance traveled between partitions a constant, each starting location uniformly distributed, and independent of the locations of tasks in the new partition. In practice, one might use a more intelligent rule such as moving directly to the first task in the next partition. When m is even, the vehicle always travels to an adjacent partition. Thus the switch time between two consecutive partitions is always $\Delta^k = \frac{a}{m}$, $k = 1, \ldots, r$. Thus $\Delta = \sum_{k=1}^{r} \Delta^k = ma$. When m is odd, the vehicle always travels to an adjacent partition except the last one. Thus $\Delta^k = \frac{a}{m}$ when $k = 1, \ldots, r-1$, and $\Delta^r = \frac{\sqrt{2}a}{m}$ as shown in Figure 3.1. Thus $\Delta = \sum_{k=1}^{r} \Delta^k = \frac{(m^2-1+\sqrt{2})a}{m}$. Then we have

$$\Delta = \begin{cases} ma, & \text{if } m \text{ is even.} \\ \frac{(m^2-1+\sqrt{2})a}{m}, & \text{if } m \text{ is odd.} \end{cases} \tag{3.1}$$

In the sequencing phase, we use the exhaustive n-TSP policy to sequence the tasks in each partition. This policy is adopted from the TSP policy in [4]. We repeat it for the convenience of the reader. Let $\mathcal{N}_l^k$ denote the l-th set of n tasks to arrive in partition k. Each set $\mathcal{N}_l^k$ has cardinality n. For example, $\mathcal{N}_1^k$ is the set of tasks $1, \ldots, n$ in partition k, and $\mathcal{N}_2^k$ is the set of tasks $n+1, \ldots 2n$ in partition k, and so on. To serve a set, we form a TSP path of the n tasks in the set starting at the initial position of the vehicle and ending at the location of the last task in the TSP path. A TSP path is the Hamiltonian path with the minimum length among all the Hamiltonian paths. A Hamiltonian path is a path that visits each task location exactly once starting at the initial position of the vehicle.

The vehicle starts at some location in partition 1. If all tasks in set $\mathcal{N}_1^1$ have arrived, we form a TSP path over these tasks. Tasks are then served by following the TSP path. If all $\mathcal{N}_2^1$ tasks have arrived when the TSP path of $\mathcal{N}_1^1$ is completed, they are also served using a TSP path. Otherwise, the vehicle moves to partition 2, and so on. The sets N_l^k are served in an FCFS order in each partition k.

We know $\rho + \lambda b_d < 1$ is the stability condition by Theorem 2.12, and the sequencing phase TSP has $b_d = 0$ by Remark 2.7.1. Thus PART-n-TSP is stable if and only if $\rho < 1$.

3.2. Calculation of System Time Distribution. The system time T of a task has three components:

- W_O, the time a task waits for its set to form (wait for the last task in the set to arrive).
- W_P, waiting time of the set in the polling system.
- W_I, the time it takes to complete service of the task once the task's set enters service.

Thus,

$$T = W_O + W_P + W_I \tag{3.2}$$

where W_O, W_P and W_I are independent. The distribution of T can be obtained from the distributions of W_O, W_P and W_I through convolution.

3.2.1. Distribution of W_O. We first obtain the distribution of W_O, together with its expectation and variance. Pick a random task. Let W_{Ol} be the waiting time of a task outside a set if it is the $(n-l)$-th task arrived in the set, $l = 0, \ldots, n-1$. Since we have equitable partitions, then the task arrival process inside each partition is Poisson with arrival rate $\frac{\lambda}{r}$. Thus $W_{O0} = 0$ and W_{Ol} is Erlang distributed with parameters $\left(l, \frac{\lambda}{r}\right)$, $l = 1, \ldots, n-1$. Thus the cdf of W_{Ol}, $F_{W_{Ol}}\left(t; l, \frac{\lambda}{r}\right) = 1 - \sum_{j=0}^{l-1} \frac{1}{j!} e^{-\frac{\lambda}{r}t} \left(\frac{\lambda}{r}t\right)^j$, $E\left[W_{Ol}\right] = \frac{lr}{\lambda}$, and $E\left[W_{Ol}^2\right] = \frac{(l^2+l)r^2}{\lambda^2}$.

Since it is equally probable that a task is the $(n-l)$-th arrived task in the set, then

$P\left(W_O \le t\right) = \sum_{l=0}^{n-1} P\left(W_{Ol} \le t\right) \frac{1}{n}$
$= \frac{1}{n}\left(1 + \sum_{l=1}^{n-1}\left(1 - \sum_{j=0}^{l-1} \frac{1}{j!} e^{-\frac{\lambda}{r}t}\left(\frac{\lambda}{r}t\right)^j\right)\right)$.

$E\left[W_O\right] = \sum_{l=0}^{n-1} E\left[W_{Ol}\right] \frac{1}{n} = \frac{1}{n}\sum_{l=0}^{n-1} \frac{lr}{\lambda} = \frac{(n-1)r}{2\lambda}$.

$E\left[W_O^2\right] = \sum_{l=0}^{n-1} E\left[W_{Ol}^2\right]\frac{1}{n} = \frac{1}{n}\sum_{l=0}^{n-1}\frac{\left(l^2+l\right)r^2}{\lambda^2} = \frac{\left(n^2-1\right)r^2}{3\lambda^2}$.
$Var\left[W_O\right] = E\left[W_O^2\right] - E\left[W_O\right]^2 = \frac{\left(n^2+6n-7\right)r^2}{12\lambda^2}$.
To sum up,

$$P\left(W_O \le t\right) = \frac{1}{n}\left(1 + \sum_{l=1}^{n-1}\left(1 - \sum_{j=0}^{l-1}\frac{1}{j!}e^{-\frac{\lambda}{r}t}\left(\frac{\lambda}{r}t\right)^j\right)\right) \tag{3.3}$$

$$E\left[W_O\right] = \frac{(n-1)r}{2\lambda} \tag{3.4}$$

$$Var\left[W_O\right] = \frac{\left(n^2+6n-7\right)r^2}{12\lambda^2} \tag{3.5}$$

3.2.2. Distribution of W_I. Before discussing W_P, we compute the distribution of W_I, together with $E\left[W_I\right]$ and $Var\left[W_I\right]$ as follows. Let W_{Inj} be the waiting time of a task inside a set if it is the j-th task to be served in the set of n tasks, $j = 1, \ldots, n$. Then

$$W_{Inj} = \frac{D_{nj}}{v} + \sum_{i=1}^{j} B_i \tag{3.6}$$

where D_{nj} is the travel distance from the initial vehicle position to the location of the j-th task through a TSP path in a partition. B_i is i.i.d.. D_{nj} is independent of B_i. Thus $\sum_{j=1}^{k} B_i$ is the convolution of j B_i's, and W_{Inj} is the convolution of $\frac{D_{nj}}{v}$ and $\sum_{i=1}^{k} B_i$.

Let L_{nj} be the D_{nj} value when there is only one partition, i.e., $r = 1$. When $r > 1$, we assume that

$$D_{nj} =_d \frac{cL_{nj}}{\sqrt{r}} \tag{3.7}$$

where $=_d$ means identically distributed with, and c is a positive constant. In particular, when region $\mathbf{A}$ and all the partitions $\mathbf{A}^k$ are squares, and $r = m^2$ with m an integer, $c = 1$.

We obtain the empirical distribution of L_{nj}, together with $E\left[L_{nj}\right]$ and $Var\left[L_{nj}\right]$ for different n and $j = 1, \ldots, n$ through simulation on the TSP path. The ant colony optimization algorithm [14] is used to heuristically search for the TSP path. Both the number of ants and the number of iterations are set to 1000. The distribution of D_{nj} is calculated from L_{nj} by scaling in (3.7). We write L_n for L_{nn} and D_n for D_{nn}. Figure 3.3 shows the values of $E\left[L_n\right]$ and $Var\left[L_n\right]$ for different n when the region is a square of size 1×1.

Figure 3.4 shows the pdf of L_{nj} for a set of $n = 5$ tasks. The tasks are uniformly distributed on a square of 1×1.

Since it is equally probable that the task is the k-th served task in the TSP path, $k = 1, \ldots, n$, then

$$P\left(W_I \le t\right) = \frac{1}{n}\sum_{j=1}^{n} P\left(W_{Inj} \le t\right) \tag{3.8}$$

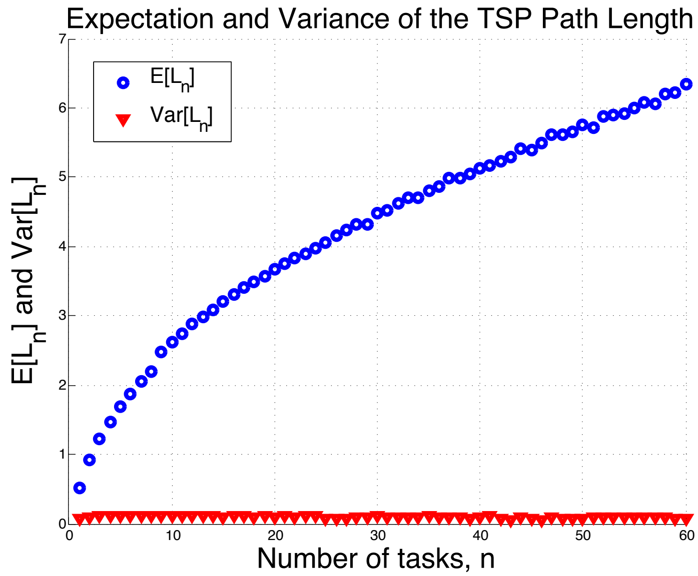

FIGURE 3.3. *Expectation and variance of the length of the TSP path for n tasks.*

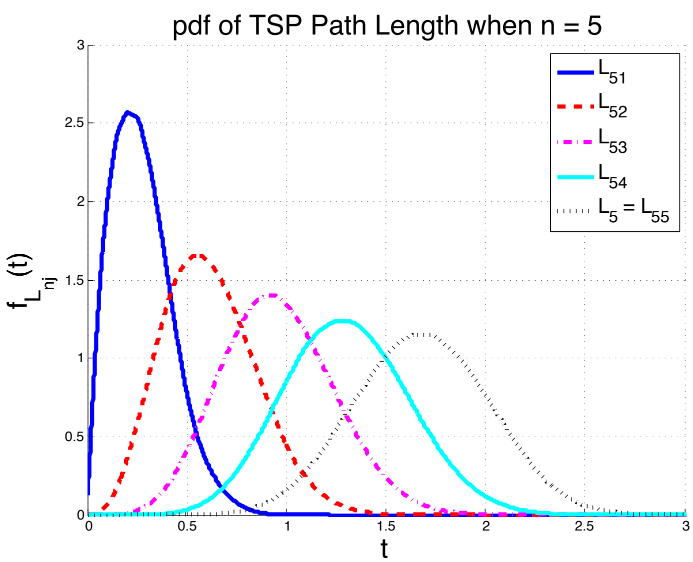

FIGURE 3.4. *The pdf of W_{Inj} and W_I.*

Then the pdf of W_I, $f_{W_I}(t) = \frac{1}{n}\sum_{j=1}^{n} f_{W_{Inj}}(t)$.
$E\left[W_I\right] = \frac{1}{n}\sum_{j=1}^{n} E\left[W_{Inj}\right]$.
$E\left[W_I^2\right] = \frac{1}{n}\sum_{j=1}^{n} E\left[W_{Inj}^2\right]$.
$Var\left[W_I\right] = E\left[W_I^2\right] - E\left[W_I\right]^2$.

The distribution of W_I can be calculated from (3.6) and (3.8). W_I does not have a closed form, but can be arbitrarily accurate through simulation. Observe that in order to obtain the distribution of D_{nj} and W_I for partitions with different sizes parameterized by r, we do not need to rerun the simulation for each partition with different size. We only need to run it once for L_{nj} and store the data. D_{nj} and W_I are obtained by scaling and convolution.

3.2.3. Distribution of W_P. The analysis of W_P uses the results from [15] by establishing the PART-n-TSP to be equivalent to a classic polling system over jobs that are the sets $\mathcal{N}_l^k$.

Since the task arrival process is Poisson with arrival rate λ, the distribution of the interarrival time of sets, A, is Erlang of order n and arrival rate λ, i.e., $A \sim Erlang(n, \lambda)$. Let A_k be the interarrival time of sets that fall in partition $\mathbf{A}^k$. Then $A_k \sim Erlang(n, \frac{\lambda}{r})$. Thus

$$E\left[A_k\right] = \frac{nr}{\lambda}, Var\left[A_k\right] = \frac{nr^2}{\lambda^2} \tag{3.9}$$

The arrival rate of a set is

$$\lambda^s = \frac{\lambda}{n} \tag{3.10}$$

The arrival rate of a set in partition $\mathbf{A}^k$, $k = 1, \ldots, r$, is

$$\lambda_k^s = \frac{\lambda^s}{r} = \frac{\lambda}{nr} \tag{3.11}$$

The size of a set, or the time needed to travel to and execute all the tasks in the set, is W_{Inn} as given in (3.6). We write W_n for W_{Inn}. The size of each set W_n is i.i.d.. Thus, if we treat each set as a job with size W_n, and each partition as a polling station, then the system is a classic polling system on r polling stations with renewal (Erlang) arrival of rate λ^s, job size W_n, and switch time Δ^k. The load is

$$\rho^s = \lambda^s E\left[W_n\right] \tag{3.12}$$

The load in partition $\mathbf{A}^k$, $k = 1, \ldots, r$, is

$$\rho_k^s = \frac{\rho^s}{r} \tag{3.13}$$

W_P is the waiting time of each set (job) in this classic polling system. Exhaustive or gated PART-n-TSP correspond to exhaustive or gated FCFS on sets, respectively.

Dorsman et al. [15] provide closed form approximations for the distribution of the steady state waiting time of a job, W_P, for polling systems under a renewal arrival process with gated or exhaustive policies when the sequencing policy is FCFS. They claim that for exhaustive-FCFS policies,

$$P\left(W_P \leq t\right) \approx P\left(UI \leq (1 - \rho^s)t\right) \tag{3.14}$$

where U is uniformly distributed on $[0, 1]$, and I is Gamma distributed with parameters

$$\alpha = \frac{2E[\Delta]\delta}{\sigma^2} + 1, \beta = \frac{2E[\Delta]\delta + \sigma^2}{2\sigma^2(1 - \rho^s)E\left[W_{Boon}\right]}. \tag{3.15}$$

where $\Delta = \sum_{k=1}^{r} \Delta^k$ is the total switch time in a cycle. When the region $\mathbf{A}$ and partitions $\mathbf{A}^k$ are squares as shown in Figure 3.1, Δ is given in (3.1). ρ^s is given in (3.12).

To explain δ, σ^2 and $E[W_{Boon}]$, we denote by $\hat{y}$ the value of each variable y that is a function of ρ^s evaluated at $\rho^s = 1$. $\delta = \sum_{j=1}^{r} \sum_{k=j+1}^{r} \hat{\rho^s}_j \hat{\rho^s}_k$, where $\hat{\rho^s}_k$ is given in (3.13) evaluated at $\rho^s = 1$. Since we have equitable partitions, then

$$\hat{\rho^s}_k = \frac{1}{r} \tag{3.16}$$

for all $k = 1, \ldots, r$. Thus

$$\delta = \frac{r(r-1)}{2r^2} = \frac{r-1}{2r}. \tag{3.17}$$

Again by [15]
$\sigma^2 = \sum_{k=1}^{r} \hat{\lambda^s}_k \left(Var[W_n] + \hat{\rho^s}_k^2 Var\left[\hat{A}_k\right]\right)$.

Since $\hat{\lambda} = n\hat{\lambda^s}$ by (3.11), then $Var\left[\hat{A}_k\right] = \frac{nr^2}{\hat{\lambda}^2} = \frac{r^2}{n\hat{\lambda^s}^2}$ by (3.9). Also, $\hat{\lambda^s}_k = \frac{\hat{\lambda^s}}{r}$ by (3.11), then substituting (3.16) we have $\sigma^2 = \hat{\lambda^s}\left(Var[W_n] + \frac{1}{r^2}\frac{r^2}{n\hat{\lambda^s}^2}\right)$. Thus,

$$\sigma^2 = \hat{\lambda^s}\left(Var[W_n] + \frac{1}{n\hat{\lambda^s}^2}\right), \tag{3.18}$$

where by (3.12)

$$\hat{\lambda^s} = \frac{1}{E[W_n]}. \tag{3.19}$$

From (3.6) we know

$$E[W_n] = \frac{E[D_n]}{v} + nb \tag{3.20}$$

$$Var[W_n] = \frac{Var[D_n]}{v^2} + n\sigma_B^2 \tag{3.21}$$

where $E[D_n]$ and $Var[D_n]$ are obtained from $E[L_n]$ and $Var[L_n]$ by (3.7), and $E[L_n]$ and $Var[L_n]$ are obtained from simulation as shown in Figure 3.3. Thus σ^2 is known substituting (3.19), (3.20) and (3.21).

Finally by Boon et al. [6], for equitable partitions

$$E[W_{Boon}] = \frac{K_0 + K_1\rho^s + K_2(\rho^s)^2}{1-\rho^s} \tag{3.22}$$

where $K_0 = E[\Delta^+]$. Δ^+ is called the residual of the random variable Δ with $E[\Delta^+] = \frac{E[\Delta^2]}{2E[\Delta]}$. In our case, Δ is deterministic. Thus,

$$K_0 = E\left[\Delta^+\right] = \frac{\Delta}{2} \tag{3.23}$$

with Δ given in (3.1). $K_1 = \hat{\rho^s}_k \left(\left(c^2_{\hat{A}_k}\right)^4 \mathbb{1}\{c^2_{\hat{A}_k} \le 1\} + 2\frac{c^2_{\hat{A}_k}}{c^2_{\hat{A}_k}+1}\mathbb{1}\{c^2_{\hat{A}_k} > 1\} - 1\right) E[W_n^+]$
$+ E[W_n^+] + \hat{\rho^s}_k (E[\Delta^+] - E[\Delta])$, where

$$c^2_{\hat{A}_k} = \frac{Var\left[\hat{A}_k\right]}{E\left[\hat{A}_k\right]^2}, \tag{3.24}$$

and $\mathbb{1}\{\Omega\}$ is the indicator function defined as $\mathbb{1}\{\Omega\} = 1$ if Ω is true, and $= 0$ otherwise. By (3.9) we have

$$c^2_{\hat{A}_k} = \frac{1}{n}. \tag{3.25}$$

Substituting (3.1), (3.16), (3.23) and (3.25) we have

$$K_1 = E\left[W_n^+\right]\left(\frac{1}{rn^4} - \frac{1}{r} + 1\right) - \frac{\Delta}{2r}, \tag{3.26}$$

where, by definition of a residual,

$$E\left[W_n^+\right] = \frac{E\left[W_n^2\right]}{2E\left[W_n\right]} = \frac{Var\left[W_n\right] + E\left[W_n\right]^2}{2E\left[W_n\right]} \tag{3.27}$$

with $E\left[W_n\right]$ and $Var\left[W_n\right]$ given in (3.20) and (3.21). Finally, by [6]

$$K_2 = \frac{1 - \hat{\rho^s}_k}{2}\left(\frac{\sigma^2}{2\delta} + E[\Delta]\right) - K_0 - K_1 \tag{3.28}$$

is known by substituting (3.1), (3.16), (3.17), (3.18), (3.23) and (3.26). Thus, we can calculate the closed form approximation for the cdf of W_P by (3.14).

After obtaining the distributions of W_O, W_P and W_I, we are able to calculate the distribution of $T = W_O + W_P + W_I$ through convolution. In the three components of T, W_O is accurate and known in closed form in (3.3). W_P has a closed form approximation in (3.14). W_I does not have a closed form, but can be arbitrarily accurate through simulation of TSP paths. It is also easy to obtain W_I because we only need to run the simulation once for the empirical distribution of L_{nj} and store the data, then calculate the distribution of D_{nj} and W_I by scaling in (3.7) and convolution in (3.8) for partitions with different sizes parameterized by r.

Figure 3.5 shows the pdf of T obtained from the convolution of its three components, and the empirical pdf of T obtained through simulation under the exhaustive PART-n-TSP when the region $\mathbf{A}$ is a square of size 1×1 with $m = 2$ and $n = 5$, and $m = 2$ and $n = 10$, separately. The tasks are uniformly distributed in the square with size $B_i \sim Unif[0, 1]$. The tasks arrive according to a Poisson process with rate $\lambda = 1$. Thus load $\rho = \lambda b = 0.5$. The simulated empirical pdf of T is regarded as the "true" value. We can see that the approximated values are very close to the true (simulated) values.

3.3. Comparison of PART-n-TSP, PART-TSP and Nearest Neighbor. Bertsimas et al. [4] compared the $E[T]$ of SQM, FCFS, PART-FCFS, SFC, NN and n-TSP through simulation, and concluded that NN achieves lower $E[T]$ than other policies simulated. PART-TSP [31] or DC [20] were proven to be $E[T]$ optimal under the light and heavy loads. Here the focus is on $Var[T]$ or $\sigma[T]$.

Since the approximation for the cdf of T for PART-n-TSP is both good and easy to compute, we can optimize the two parameters n and r to minimize $E[T]$ or other performance metrics when the region $\mathbf{A}$ and partitions $\mathbf{A}^k$ are squares. Table 3.1 gives the r^* and n^* in the range $r \in \left\{1^2, 2^2, \ldots, 10^2\right\}$ and $n = \{1, \ldots, 60\}$ that minimize $E[T]$ under exhaustive PART-n-TSP and the corresponding $E[T]$ and $\sigma[T]$ values for different ρ values. The region is a square of size 1×1 and $B_i \sim Unif[0, 0.5]$. Noting that FCFS is PART-n-TSP when $r = 1$ and $n = 1$, PART-FCFS is PART-n-TSP

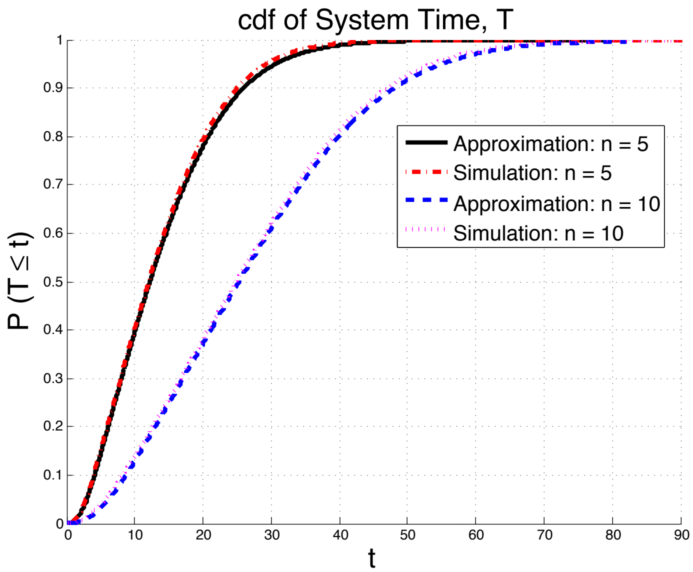

FIGURE 3.5. *Approximated and simulated values of the cdf of the system time.*

when $n = 1$, and n-TSP is PART-n-TSP when $r = 1$. Thus by optimizing on r and n, PART-n-TSP has better performance than FCFS, PART-FCFS and n-TSP.

We compare PART-n-TSP with PART-TSP [31] and Nearest Neighbor [4] since they are considered near optimal in the literature. We simulate PART-TSP and Nearest Neighbor under the same setting. The number of partitions for PART-TSP is set to be the optimal number of partitions for PART-n-TSP. The average number of tasks served inside each gate, denoted by $E[n]$, is also shown in the table. We generate $N = 100,000$ tasks are for each load ρ value. Only the 25,000th to the 75,000th tasks are used to calculate $E[T]$ and $\sigma[T]$ to make sure that the steady state data are used. We have checked this by randomly sampling time segments in this range. Figure 3.6 shows the truncation of 1000 data points for PART-TSP when $\rho = 0.9$, $B_i \sim [0, 0.5]$, and $r = 25$. Each time segments of about 50 data is the number of tasks inside each gate, indeed this is true as shown in Table 3.1 $E[n] = 49.7$ for this case. The system time is decreasing in general in each segment because tasks arrive earlier in a gate wait more than those arrive later. Table 3.2 gives the case when $B_i \sim Unif[0, 1]$.

The $E[T]$ and $\sigma[T]$ of PART-TSP and NN are compared to those of PART-n-TSP, The percentage following the $E[T]$ and $\sigma[T]$ of PART-TSP and NN are the ratio of these values over those of PART-n-TSP. The minimum $E[T]$ and $\sigma[T]$ at each load level of the three policies are bolded. From Tables 3.1 and 3.2, we can see that NN achieves lower $E[T]$ than PART-n-TSP and PART-TSP for all $\rho \in \{0.1 \dots 0.9\}$ in both $B_i \sim [0, 0.5]$ and $B_i \sim [0, 1]$. PART-n-TSP achieves lower $E[T]$ than PART-TSP when ρ is not too small or too large, e.g. when $\rho \in \{0.3, \dots, 0.7\}$ for $B_i \sim [0, 0.5]$, and when $\rho \in \{0.4, \dots, 0.8\}$ for $B_i \sim [0, 1]$. PART-n-TSP has higher $E[T]$ than PART-TSP when ρ is low because it is better to have the average number of tasks in a set to be between 1 and 2 as done by PART-TSP, but PART-n-TSP can only set it to be either 1 ro 2, resulting in higher $E[T]$. PART-n-TSP has higher $E[T]$ than PART-TSP when ρ is high because $r^* > 1$ when ρ is high. Then there is a switching time between partitions. By setting n to be a fixed number under PART-n-TSP, the vehicle might

TABLE 3.1
Comparison of PART-n-TSP, PART-TSP and Nearest Neighbor on $E[T]$ and $\sigma[T]$: $B_i \sim Unif[0, 0.5]$

ρ		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
r^* for PART-n-TSP		1	1	1	1	1	1	1	4	25
n^* for PART-n-TSP		1	2	2	4	12	24	57	59	57
$E[n]$ for PART-TSP		1.05	1.25	1.75	3.46	9.08	23.3	61.2	63.9	49.7
$E[T]$	PART-n-TSP	1.04	1.71	1.70	2.80	5.90	9.99	20.3	81.4	321
	PART-TSP	0.95 (91%)	1.26 (74%)	1.87 (110%)	3.30 (118%)	5.97 (101%)	10.9 (109%)	24.4 (120%)	51.7 (64%)	181 (56%)
	NN	**0.94** (90%)	**1.21** (71%)	**1.66** (98%)	**2.46** (88%)	**3.81** (65%)	**6.37** (64%)	**12.7** (63%)	**32.6** (40%)	**154** (48%)
$\sigma[T]$	PART-n-TSP	0.69	1.19	**0.94**	**1.39**	**2.74**	**4.36**	**8.56**	31.7	140
	PART-TSP	0.48 (70%)	0.77 (65%)	1.22 (130%)	2.11 (152%)	3.27 (119%)	5.35 (123%)	13.3 (155%)	**27.1** (85%)	**101** (72%)
	NN	**0.47** (68%)	**0.76** (64%)	1.26 (134%)	2.14 (154%)	3.57 (130%)	6.18 (142%)	12.5 (146%)	31.1 (98%)	147 (105%)

TABLE 3.2
Comparison of PART-n-TSP, PART-TSP and Nearest Neighbor on $E[T]$ and $\sigma[T]$: $B_i \sim Unif[0,1]$

ρ		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
r^* for PART-n-TSP		1	1	1	1	1	1	1	1	9
n^* for PART-n-TSP		1	1	1	2	3	6	15	42	41
$E[n]$ for PART-TSP		1.02	1.1	1.25	1.59	2.37	4.57	13.4	40	31.1
$E[T]$	PART-n-TSP	1.20	1.51	2.15	2.23	2.94	4.96	10.8	26.3	192
	PART-TSP	**1.16** (97%)	1.37 (91%)	1.71 (80%)	2.35 (105%)	3.63 (123%)	6.24 (126%)	12.9 (119%)	27.9 (106%)	93.8 (49%)
	NN	**1.16** (97%)	**1.36** (90%)	**1.66** (77%)	**2.16** (97%)	**2.93** (100%)	**4.50** (91%)	**8.10** (75%)	**18.0** (68%)	**78.7** (41%)
$\sigma[T]$	PART-n-TSP	0.69	0.91	1.59	**1.30**	**1.59**	**2.47**	**4.85**	**11.2**	80.7
	PART-TSP	**0.54** (78%)	**0.75** (82%)	**1.07** (67%)	1.64 (126%)	2.58 (162%)	4.22 (171%)	7.63 (157%)	14.6 (130%)	**56.2** (70%)
	NN	**0.54** (78%)	0.76 (84%)	1.10 (69%)	1.71 (132%)	2.64 (166%)	4.42 (179%)	8.24 (170%)	18.3 (163%)	75.9 (94%)

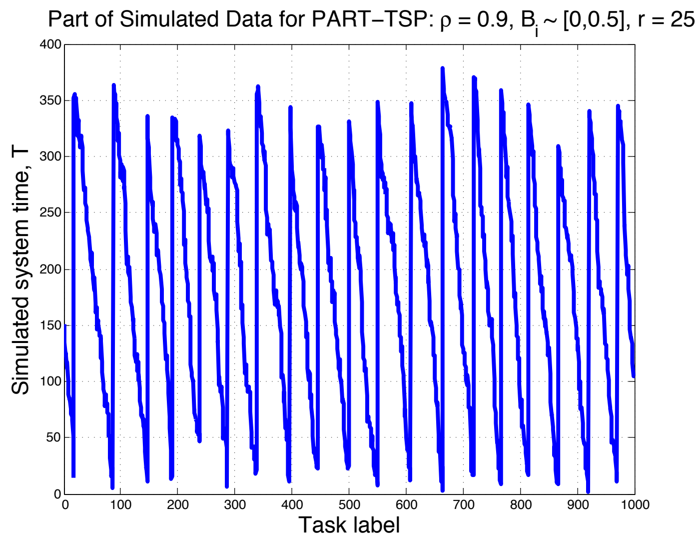

FIGURE 3.6. *Part of simulated data for PART-TSP: $\rho = 0.9$, $B_i \sim [0, 0.5]$, and $r = 25$.*

arrive at a partition, and find the number of tasks to be less than n. Then the vehicle would switch to the next partition without serving any task, resulting in a switching cost but no tasks served.

PART-n-TSP behaves like a "standardized" version of PART-TSP. While the fixed n reduces flexibility, it increases certainty. Thus $Var[T]$ or $\sigma[T]$ should be lower. Indeed, as shown in Tables 3.1 and 3.2, PART-n-TSP achieves lower $\sigma[T]$ than PART-TSP and NN when ρ is not too small or too large, e.g. when $\rho \in \{0.3, \ldots, 0.7\}$ for $B_i \sim [0, 0.5]$, and when $\rho \in \{0.4, \ldots, 0.8\}$ for $B_i \sim [0, 1]$. The performance of PART-n-TSP on $\sigma[T]$ when ρ is too small or too large is not as good for the same reasons affecting $E[T]$ as explained in the previous paragraph.

3.4. Optimality of PART-n-TSP under light and heavy loads. The PART-n-TSP can be modified to yield asymptotically optimal $E[T]$ under light load ($\rho \to 0^+$). First the PART-n-TSP becomes FCFS policy when setting $r = 1$ and $n = 1$. Then under FCFS policy, let the vehicle return to the median of region $\mathbf{A}$ when it becomes idle. Under light load this is the stochastic queue median (SQM) policy [4], where the vehicle travels directly to the task location from the median, executes the task, and then returns to the median after completion. SQM is proven to be $E[T]$ optimal under light load [4], proving the optimality of PART-n-TSP under light load.

Under heavy load ($\rho \to 1^-$), the following lower bound holds [5].

$$E[T] \geq \frac{\beta_{TSP,2}^2 \lambda \left(\int_A f_X^{\frac{1}{2}}(x)\, dx \right)^2}{2v^2(1-\rho)^2} \tag{3.29}$$

The following theorem shows that PART-n-TSP achieves the heavy-load lower bound (3.29) when $r \to \infty$. Thus PART-n-TSP is asymptotically optimal in $E[T]$ under heavy load.

THEOREM 3.2. *Under PART-n-TSP as per Definition 3.1, when $\rho \to 1^-$ and*

$n \to \infty$, the system time for the 1-DTRP satisfies

$$E[T] \leq \left(1+\frac{1}{r}\right)\frac{\beta_{TSP,2}^2 \lambda \left(\int_A f_X^{\frac{1}{2}}(x)\,dx\right)^2}{2v^2(1-\rho)^2} \tag{3.30}$$

where r is the number of partitions.

Proof. $E[T] = E[W_O] + E[W_P] + E[W_I]$ by (3.2).

And by (3.4)

$$E[W_O] = \frac{(n-1)r}{2\lambda} < \frac{nr}{2\lambda} \tag{3.31}$$

By (3.6) and (3.8), and conditioning on the position that a given task takes within its set, and noting that the travel time around the TSP path is no more than the length of the path itself, the expected wait for completion once a task's set enters service

$$E[W_I] \leq \frac{1}{v}E[D_n] + \frac{1}{n}\sum_{j=1}^{n} jb = \frac{1}{v}E[D_n] + \frac{n+1}{2}b \tag{3.32}$$

Given that a demand falls in partition $\mathbf{A^k}$, the conditional density for its location (whose support is $\mathbf{A^k}$) is $\frac{f_X(x)}{\int_{A^k} f_X(x)\,dx}$. From [24] we know that, almost surely, $\lim_{n\to\infty}\frac{D_n}{\sqrt{n}} = \beta_{TSP,2}\int_{A^k}\sqrt{\frac{f_X(x)}{\int_{A^k} f_X(x)\,dx}}\,dx$, where $\beta_{TSP,2}$ is a constant. Let $C = \frac{1}{v}\beta_{TSP,2}\int_{A^k}\sqrt{\frac{f_X(x)}{\int_{A^k} f_X(x)\,dx}}\,dx$, thus C is a constant. So $\lim_{n\to\infty}\frac{1}{v}E[D_n] = C\sqrt{n}$.

The load of a set $\rho^s = \lambda^s E[W_n] = \frac{\lambda}{n}\left(\frac{E[D_n]}{v} + nb\right) = \lambda b + \frac{\lambda}{v}\frac{E[D_n]}{n} = \rho + \lambda\frac{E[D_n]}{nv}$ by (3.11), (3.12) and (3.20). Thus $\lim_{n\to\infty}\rho^s = \rho + \lambda\lim_{n\to\infty}\frac{E[D_n]}{nv} = \rho + \lambda\lim_{n\to\infty}\frac{C\sqrt{n}}{n} = \rho$. So $\rho \to 1^-$ implies $\rho^s \to 1^-$ when $n \to \infty$.

As for $E[W_P]$, from [28] we know that the mean waiting time in a polling system with renewal arrivals as $\rho^s \to 1^-$ is

$$E[W_P] = \frac{\omega}{1-\rho^s} + o\left((1-\rho^s)^{-1}\right), \tag{3.33}$$

where $\omega = \frac{1-\hat{\rho^s}_k}{2}\left(\frac{\sigma^2}{\sum_{k=1}^r \hat{\rho^s}_k\left(1-\hat{\rho^s}_k\right)} + E[\Delta]\right)$ under the exhaustive policy, and $\omega = \frac{1+\hat{\rho^s}_k}{2}\left(\frac{\sigma^2}{\sum_{k=1}^r \hat{\rho^s}_k\left(1+\hat{\rho^s}_k\right)} + E[\Delta]\right)$ under the gated policy. Substituting (3.16) we have $\omega = \frac{\sigma^2}{2} + \frac{r-1}{2r}E[\Delta]$ under the exhaustive policy, and $\omega = \frac{\sigma^2}{2} + \frac{r+1}{2r}E[\Delta]$ under the gated policy. $\Delta = \sum_{k=1}^r \Delta^k$ is the total switch time of a polling cycle. Δ does not depend on ρ, ρ^s or n, and is given in (3.1) when the region $\mathbf{A}$ and partitions $\mathbf{A}^k$ are squares.

Thus $E[W_P] = \frac{1}{2(1-\rho^s)}\left(\sigma^2 + \frac{r\mp1}{r}E[\Delta]\right) + o\left((1-\rho^s)^{-1}\right)$ when $\rho^s \to 1^-$. Let $C' = \frac{r\mp1}{r}E[\Delta]$. So $E[W_P] = \frac{1}{2(1-\rho^s)}\left(\sigma^2 + C'\right)$ when $\rho^s \to 1^-$. Since r is a finite natural number, and Δ_k is upper bounded by the diameter of region $\mathbf{A}$, then $E[\Delta]$ is a positive finite number. Thus, C' is a positive finite number.

By (3.18) $\sigma^2 = \hat{\lambda^s}\left(Var[W_n] + \frac{1}{n\hat{\lambda^s}^2}\right)$, where $\hat{\lambda^s} = \frac{\hat{\lambda}}{n}$ by (3.11). Also, $\rho^s = \hat{\lambda^s}E[W_n]$ as $\rho^s \to 1^-$ by (3.12).

So $E\left[W_P\right] = \frac{\hat{\lambda^s}\left(Var[W_n]+\frac{1}{n\hat{\lambda^s}^2}\right)+C'}{2\left(1-\hat{\lambda^s}E[W_n]\right)}$
$= \frac{\frac{\hat{\lambda}}{n}\left(\frac{1}{v^2}Var[D_n]+n\sigma_B^2+\frac{n}{\hat{\lambda}^2}\right)+C'}{2\left(1-\frac{\hat{\lambda}}{n}\left(\frac{1}{v}E[D_n]+nb\right)\right)}$
$= \frac{\hat{\lambda}\left(\frac{1}{\hat{\lambda}^2}+\frac{Var[D_n]}{nv^2}+\sigma_B^2\right)+C'}{2\left(1-\hat{\lambda}b-\hat{\lambda}\frac{1}{v}\frac{E[D_n]}{n}\right)}$, as $\rho^s \to 1^-$, where we substituted $\hat{\lambda^s} = \frac{\hat{\lambda}}{n}$ and (3.20) and (3.21) in the first equality.

Since $\hat{\lambda}$ is the value of λ when $\rho^s = 1$, and $\rho \to 1^-$ implies $\rho^s \to 1^-$, then we can write

$$E\left[W_P\right] = \frac{\lambda\left(\frac{1}{\lambda^2}+\frac{Var[D_n]}{nv^2}+\sigma_B^2\right)+C'}{2\left(1-\rho-\lambda\frac{1}{v}\frac{E[D_n]}{n}\right)} \tag{3.34}$$

when $\rho \to 1^-$, where we substituted $\rho = \lambda b$.

From [19, p.189] we know $\lim_{n\to\infty} Var\left[D_n\right] = O(1)$, and therefore, $\lim_{n\to\infty}\frac{Var[D_n]}{n} = 0$.

Thus when $\rho \to 1^-$ and $n \to \infty$,
$E[T] = E\left[W_O\right] + E\left[W_P\right] + E\left[W_I\right]$
$\le \frac{(n-1)r}{2\lambda} + \frac{1}{v}E\left[D_n\right] + \frac{n+1}{2}b + \frac{\lambda\left(\frac{1}{\lambda^2}+\frac{Var[D_n]}{nv^2}+\sigma_B^2\right)+C'}{2\left(1-\rho-\lambda\frac{1}{v}\frac{E[D_n]}{n}\right)}$
$\le \frac{nr}{2\lambda} + C\sqrt{n} + \frac{n}{2}b + \frac{\lambda\left(\frac{1}{\lambda^2}+\sigma_B^2\right)+C'}{2\left(1-\rho-\lambda\frac{C}{\sqrt{n}}\right)}$.

Substituting $b = \frac{\rho}{\lambda}$ we have

$$E[T] \le \frac{\lambda\left(\frac{1}{\lambda^2}+\sigma_B^2\right)+C'}{2\left(1-\rho-\lambda\frac{C}{\sqrt{n}}\right)} + \frac{n(r+\rho)}{2\lambda} + C\sqrt{n} \tag{3.35}$$

We want to minimize (3.35) with respect to n to get the least upper bound. Noting that (3.35) is convex with respect to n, so there is indeed a minimum. First, however, consider a change of variable $y = \frac{\lambda C}{(1-\rho)\sqrt{n}}$. With this change,

$$E[T] \le \frac{\lambda\left(\frac{1}{\lambda^2}+\sigma_B^2\right)+C'}{2(1-\rho)(1-y)} + \frac{\lambda C^2(r+\rho)}{2(1-\rho)^2y^2} + \frac{\lambda C^2}{(1-\rho)y} \tag{3.36}$$

For $\rho \to 1^-$, one can verify that the optimum y approaches 1. Linearizing the last two terms above about $y = 1$ we have $\frac{\lambda C^2(r+\rho)}{2(1-\rho)^2y^2} = \frac{\lambda C^2(r+\rho)}{2(1-\rho)^2}(3-2y)$, and $\frac{\lambda C^2}{(1-\rho)y} = \frac{\lambda C^2}{1-\rho}(2-y)$.

Thus, $g(y) \equiv \frac{\lambda\left(\frac{1}{\lambda^2}+\sigma_B^2\right)+C'}{2(1-\rho)(1-y)} + \frac{\lambda C^2(r+\rho)}{2(1-\rho)^2y^2} + \frac{\lambda C^2}{(1-\rho)y}$
$\approx \frac{C_1}{1-y} + C_2(3-2y) + C_3(2-y)$
$= \frac{C_1}{1-y} + \left(2C_2+C_3\right)(1-y) + C_2 + C_3$, where $C_1 = \frac{\lambda\left(\frac{1}{\lambda^2}+\sigma_B^2\right)+C'}{2(1-\rho)}$, $C_2 = \frac{\lambda C^2(r+\rho)}{2(1-\rho)^2}$, and $C_3 = \frac{\lambda C^2}{1-\rho}$.

The approximation for $g(y)$ is minimized when $\frac{C_1}{1-y} = \left(2C_2+C_3\right)(1-y)$. Substituting C_1, C_2 and C_3 we have an approximate optimum value

$$y^* = 1 - \frac{1}{C}\sqrt{\frac{\left(\frac{1}{\lambda^2}+\sigma_B^2+\frac{C'}{\lambda}\right)(1-\rho)}{2(1+r)}} \tag{3.37}$$

Substituting (3.37) into (3.36) and noting that for $\rho \to 1^-$ the approximate y^* approaches 1 we have

$$E[T] \leq \frac{\lambda C^2(r+1)}{2(1-\rho)^2} + \frac{\lambda C\sqrt{2(r+1)\left(\frac{1}{\lambda^2}+\sigma_B^2+\frac{C'}{\lambda}\right)}}{2(1-\rho)^{\frac{3}{2}}} + \frac{\lambda C^2}{1-\rho} \tag{3.38}$$

when $\rho \to 1^-$.

Thus $E[T] \leq \frac{\lambda C^2(r+1)}{2(1-\rho)^2} + o\left((1-\rho)^{-2}\right)$ when $\rho \to 1^-$. We have

$$E[T] \leq \frac{\lambda C^2(r+1)}{2(1-\rho)^2} \tag{3.39}$$

when $\rho \to 1^-$.

$C = \frac{1}{v}\beta_{TSP,2}\int_{A^k}\sqrt{\frac{f_X(x)}{\int_{A^k} f_X(x)\,dx}}\,dx$
$= \frac{1}{v}\beta_{TSP,2}\sqrt{r}\int_{A^k}\sqrt{f_X(x)}\,dx$
$= \beta_{TSP,2}\frac{1}{v\sqrt{r}}\int_A\sqrt{f_X(x)}\,dx.$

Substituting C in (3.39) we have

$$E[T] \leq \left(1+\frac{1}{r}\right)\frac{\beta_{TSP,2}^2\lambda\left(\int_A f_X^{\frac{1}{2}}(x)\,dx\right)^2}{2v^2(1-\rho)^2}.$$

□

The PART-n-TSP is optimal under light load. Moreover, when $r \to \infty$, the PART-n-TSP policy achieves the heavy-load lower bound (3.29). Therefore the PART-n-TSP is both optimal under light load and arbitrarily close to optimality under heavy load, and stabilizes the system for every load $\rho \in [0, 1)$. Notice that with $r = 10$ the PART-n-TSP is already guaranteed to be within 10% of the optimal performance under heavy load.

4. Summary. We give a good approximation for the distribution of the system time that is easy to compute under the PART-n-TSP policy by utilizing the approximation results of the distribution of system time T, together with $E[T]$ and $Var[T]$ in the polling systems [8, 15]. We compare PART-n-TSP with PART-TSP [31] and Nearest Neighbor [4] on $E[T]$ and $\sigma[T]$ in Tables 3.1 and 3.2, since the latter two are considered near optimal in the literature. The results show that in practice PART-n-TSP achieves lower $\sigma[T]$ than PART-TSP and NN and lower $E[T]$ than PART-TSP when the load ρ is not too small or too large. We also prove that PART-n-TSP is $E[T]$ optimal under light load ($\rho \to 0^+$) and asymptotically optimal under heavy load ($\rho \to 1^-$) in Theorem 3.2.

5. Conclusion. We prove a necessary and sufficient condition for stability in the Dynamic Traveling Repairman Problem (DTRP) [4] under the class of Polling-Sequencing (P-S) policies satisfying unlimited-polling and economy of scale. The number of tasks inside each polling partition is shown to be a Markov chain. Non-location based policies and some common location based policies such as TSP, NN and DA are shown to have economy of scale. The P-S class includes some of the policies proven to be optimal for the expectation of system time under light and heavy loads in the DTRP literature. We give a close form approximation of the distribution of the steady state system time, together with its expectation and variance for PART-n-TSP policy in the P-S class, which is proved to be optimal for the expectation of system time under light and heavy loads.

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