Distributed Kalman Consensus Filters for Distributed Parameter Systems with Unknown Boundary Conditions

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Abstract— **This work describes the distributed Kalman consensus filter for a class of large-scale distributed parameter systems monitored by a sensor network. This system boundary conditions are unknown or inaccessible in advance. Local Kalman information filters are developed on the reduced local observation models to estimate the state and the local unknown boundary condition, simultaneously. On the diffusion processing, we fuse the state estimations that are common among the local information filters using consensus averaging algorithms and algebraic graph theory. Stability and performance analysis is provided for this distributed Kalman consensus filtering algorithm. Finally, we consider an application of sensor networks to a heat conduction process. The performance of the proposed distributed algorithm is compared to the centralized Kalman filter.**

I. INTRODUCTION

Distributed parameter systems have been used extensively in a large cutting-edge technology-based social sectors, ranging from ocean and atmospheric sciences [1], [2], oil reservoir simulations [3], to hydrological and intelligent transportation systems [4], [5], [6]. Distributed parameter systems are based on sets of partial differential equations, able to accurately express their nonlinear distributed dynamical behavior. In recent years, more attention has been given to cyber-physical systems, built of physical plants, usually described with distributed parameter systems, sensor networks and execution units [1], [4], [7].

The estimation of distributed parameter systems is a challenging task. One troublesome problem is dealing with the system boundary conditions, which are often unknown or inaccessible in some applications. Assume that we could predict these from other channels, such as statistical inference in advance [8], [9]. Moreover, boundary conditions are treated as unknown model parameters and estimated as extended system state variables. In [10], which describes an adaptive extended Kalman filter method for freeway traffic system estimation, boundary conditions, i.e. the upstream and downstream traffic demands, are integrated into the traffic flow model using random-walk equations.

This work was supported by the grants from National Science Foundation, Grant CNS-1136141, Natural Science Foundation of China (No. 60904069, 61111130119), and the Doctoral Fund of Ministry of Education of China (No. 20091103120008)

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The distributed estimation [13], decomposes the large-scale system into low-dimensional subsystems and distributes the estimation algorithm, which is with a low order Kalman filter, at each subsystem. Consensus and adaptive consensus filters are designed in [14] for infinite dimensional distributed parameter systems based on Kalman consensus filters [15]-[21].

In this work, we present a distributed Kalman consensus filtering algorithm for distributed parameter systems with unknown boundary conditions based on simultaneous state and input estimation and consensus averaging algorithms.

The aims of this paper are: (1) to decompose the large-scale distributed parameter system into spatial subsystems with reduced observation models. Locations of the sensor group throughout the spatial domain significantly affect the outcome and quality of state estimation we model sensors placed in different locations with different output operators. (2) to develop the information filtering algorithm for simultaneous state and boundary condition estimation for local sensor groups. (3) to design consensus estimator for every sensor so that the overall state can be estimated via local exchange of messages among neighboring nodes. (4) to present a stability analysis distributed Kalman consensus filtering algorithm.

The subsequent sections are organized as follows. Some background on centralized and decentralized observation models is provided in Section II. Simultaneous state and input estimation is studied in Section III: Section III-A reviews the Gillijins-De Moor's recursive filtering algorithm while Section III-B studies the information filter for the state and the input. Distributed Kalman consensus filters for distributed parameter systems with unknown boundary conditions are introduced in Section IV-A. Stability analysis is provided in Section IV-B. Simulation results and performance comparisons for a 1D heat conduction process are presented in Section V. Conclusions and future work are discussed in Section VI.

II. PRELIMINARIES

A. Distributed Parameter Systems

Consider the well known distributed parameter systems represented by the convection-diffusion equation (1), commonly used to model physical processes.

$$\frac{\partial u}{\partial t}(t,\xi) - e\frac{\partial^2 u}{\partial \xi^2}(t,\xi) + C(t,\xi)\frac{\partial u}{\partial \xi}(t,\xi) = F(t,\xi), \quad (1)$$

$t \in [0,\infty), 0 < \xi < L$, L is the length of one spatial dimension, $F(t,\xi)$ is the given outside environment, e is the

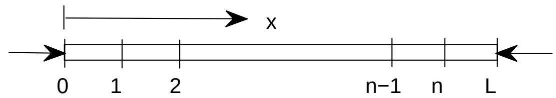

Fig. 1. One dimension heat conduction model with unknown boundary conditions

small diffusion coefficient, $C(t,\xi)$ is the given convection coefficient.

Assume initial conditions are given by:

$$u(0,\xi) = u_0(\xi), \quad 0 < \xi < L, \tag{2}$$

where $u_0(\xi)$ is a given function. The boundary conditions are imposed on both edges as:

$$u(t,0) = G_0(t), \quad u(t,L) = G_L(t), \quad t \in [0,\infty), \tag{3}$$

where $G_0(t)$, $G_L(t)$ are boundary input functions.

When the above initial and boundary conditions are compatible, equation (1) is well posed in the sense that it has a unique solution $u(t,\xi)$.

In general, the PDEs is solved numerically on a spatial uniformed mesh grid with the finite-difference discretization method. The central approximations of the first and the second order derivative are:

$$C(t,jh)\frac{\partial u}{\partial \xi}(t,jh) \approx C(t,jh)\frac{u_{j+1}-u_{j-1}}{2}, \tag{4}$$

$$\frac{\partial^2 u}{\partial \xi^2}(t,jh) \approx \frac{u_{j+1}-2u_j+u_{j-1}}{h^2}. \tag{5}$$

where u_j is the value of the random field $u(t,jh)$, at the jth location of the spatial grid, and h is the discretization interval.

We model the discretized boundary conditions $G_0(t)$, $G_L(t)$ by d_k, for those endpoint grids whose outer grids are missing. The spatio-temporal discretization of the PDEs (1) is formulated by the following difference equation

$$x_{k+1} = A_k x_k + B_k d_k + w_k, \quad k \geq 0, \tag{6}$$

where, k is the discrete time index, the state vector $x_k \in R^n$ is the collection of the sorted variables $\{u_j\}$, $d_k \in R^m$ is the discrete boundary condition, $w_k \in R^n$ is the model approximation error assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matric $Q_k = E[w_k w_k^T]$, $A_k \in R^{n\times n}$ and $B_k \in R^{n\times m}$ are the system state matrix and boundary input matrix, respectively.

Remarks 1: Under the spatio-temporal discretization, the unknown boundary functions $G_0(t)$, $G_L(t)$ in the distributed parameter systesm (1) regarded as nknown outside inputs to the stochastic discrete-time system (6).

Remarks 2: Here, for the PDEs system (1), the temporal discretization approximates the time derivative using the forward Euler method. For any given rightside function $F(t,\xi)$, which could be removed from the difference equation (6) with variable substitution.

B. Centralized Observation Model

Let the systems be monitored by a network of N sensors, in which the boundary conditions are not accessible directly. The local observations at sensor i at time k are

$$z_k^{(i)} = H_k^{(i)} x_k + v_k^{(i)}, \tag{7}$$

where $H_k^{(i)} \in R^{p_i\times n}$ is the local observation matrix, p_i is the number of simultaneous observations made by sensor i at time k, and $v_k^{(i)} \in R^{p_i}$ is the local observation noise assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matric $R_k^{(i)} = E[v_k^{(i)} v_k^{(i)T}]$.

The local measurement $z_k^{(i)}$ maybe temperature or sone other physical attribute at location i. The observation matrix $H_k^{(i)}$ is particular to the spatial position of the local sensor i in the discrete uniformed mesh grid.

We collect all observations in the sensor network to get the global observation model. Let p be the total number of observations at all the sensors. Let the global observation vector, $z_k \in R^p$, the global observation matrix, $H \in R^{p\times n}$, and the global observation noise vector $v_k \in R^p$ be

$$z_k = \begin{bmatrix} z_k^{(1)} \\ \vdots \\ z_k^{(N)} \end{bmatrix}, H_k = \begin{bmatrix} H_k^{(1)} \\ \vdots \\ H_k^{(N)} \end{bmatrix}, v_k = \begin{bmatrix} v_k^{(1)} \\ \vdots \\ v_k^{(N)} \end{bmatrix}. \tag{8}$$

Then the global observation model is given by

$$z_k = H_k x_k + v_k. \tag{9}$$

Since the observation noises at the different sensors are independent, we can combine the local observation noise covariance matrices at each sensor into the global observation noise covariance matrix, R_k, as

$$R_k = blockdiag[R_k^{(1)}, \cdots, R_k^{(N)}].$$

The centralized Kalman filter in this paper is used to distributed parameter systems (6)-(8) with unknown boundary condition, that is, no prior information about the unknown boundary input, d_k in the system model (6), is available.

Remarks 3: Since the unknown boundary conditions d_k be regarded as the outside input of the system (6). Therefor, state estimation under unknown inputs is directly converted to a simultaneous input and state estimation problem. Unbiased minimum variance linear state filters are described in [22], [23], [24].

We assume that the following sufficient conditions for the existence of an unbiased state estimator are satisfied.

Assumption 4 ([22]): $rank H_k B_{k-1} = rank B_{k-1} = m$, for all k.

C. Reduced Model at Decentralized Sensor

We model decentralized state estimation. Each sensor has a capability to generate its own estimate using locally available measurements. We define the local system models based on the different spatial positions of the local sensors i.

Case 1. The local sensor i is placed at the boundary grids of the spatio-temporal discretization model (6), which means $Rank(H_k^{(i)} B_{k-1}) = m_i \leq Rank(B_{k-1})$, $m_i \neq 0$.

We select submatrices $B_{k-1}^{(i)}$ containing the elements taken from the fullorder matrices B_{k-1} with indices belonging to $n \times I_i^d$, where the index set $I_i^d = \{j_1^i, \cdots, j_{m_i}^i\}$, $j_1^i, \cdots, j_{m_i}^i \in \{1, \cdots, m\}$, $j_1^i < \cdots < j_{m_i}^i$, $m_i \leq m$, such that

$$Rank(H_k^{(i)} B_{k-1}^{(i)}) = Rank(B_{k-1}^{(i)}) = Rank(d_k^{(i)}) = m_i,$$

where, $d_k^{(i)}$ is an m_i-dimensional vector composed of the components of d_k selected by I_i^d.

Then, the local system models available to the ith sensor can be defined as

$$x_{k+1} = A_k x_k + B_k^{(i)} d_k^{(i)} + w_k, \tag{10}$$

$i \in N$. In order to estimate all boundary inputs, we further assume that $\bigcup I_i^d = \{1, \cdots, m\}$.

Case 2. The local sensor i is placed at the inside grids of the spatio-temporal discretization model (6), which means $Rank(H_k^{(i)} B_{k-1}) = 0$, $m_i = 0$.

This situation could be considered as a special case of the former. We let $I_i^d = \{1, \cdots, m\}$ for those sensors, and define the local boundary matrices as $B_{k-1}^{(i)} = 0$, and local boundary conditions as $d_k^{(i)} = d_k$.

Example 5: Consider the one dimensional distributed parameter heat conduction model on a rod. Spatial discretization results in the grid shown in Fig. 1. A number of sensors are located on the specified grid points and each sensor measures the specified temperature. Unknown boundary conditions on the end of the rod are $d_k = [d_k^L \quad d_k^R]^T$, and the boundary system matrix is

$$B_{k-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}^T.$$

When the sensor location is at the grid point 1, the local observation matrix should be $H_k^{(1)} = [1 \quad 0 \quad \cdots \quad 0]$. Then, $B_{k-1}^{(i)}$ be obtained as a submatrix including the first column of the full-order boundary matrix B_{k-1}, and we can define $d_k^{(1)} = d_k^L$.

When the sensor location is at the grid point 2, or the other inside grids, the local observation matrix should be $H_k^{(2)} = [0 \quad 1 \quad \cdots \quad 0]$. In this case, $Rank(H_k^{(2)} B_{k-1}) = 0$, we define $B_{k-1}^{(2)} = 0$, and $d_k^{(2)} = d_k$.

In general, the local system model of the distributed parameter systems can be represented by

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k^{(i)} d_k^{(i)} + w_k, \\ z_k^{(i)} &= H_k^{(i)} x_k + v_k^{(i)}, \end{aligned} \tag{11}$$

$i \in \{1, \cdots, N\}$, and matrices A_k, $B_k^{(i)}$, $H_k^{(i)}$ have appreciably dimensions, respectively.

III. SIMULTANEOUS STATE AND BOUNDARY INPUT ESTIMATION

A. Unbiased Minimum-Variance State and Input Estimation

We use the Gillijins-De Moor's algorithm for simultaneous state and input estimation in this subsection. The main purpose is to design optimal state estimator for x_k in the sense of being unbiased and minimum variance for the system (6)-(8) without any information about the unknown input d_k. Readers could review [22], [23] for more information.

The recursive state and input filters are as follows:

$$\hat{x}_{k|k} = \bar{x}_{k|k} + K_k(z_k - H_k \bar{x}_{k|k}), \tag{12}$$

$$\bar{x}_{k|k} = \hat{x}_{k|k-1} + B_{k-1}\hat{d}_{k-1}, \tag{13}$$

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1}, \tag{14}$$

$$\hat{d}_{k-1} = M_k(z_k - H_k \hat{x}_{k|k-1}), \tag{15}$$

where $\hat{x}_{k|k}$ and $\hat{d}_{k-1}$ represent the state and input estimate, $\bar{x}_{k|k}$ and $\hat{x}_{k|k-1}$ represent the state prediction with or without the input information. $K_k \in R^{n \times p}$, $M_k \in R^{p \times m}$ are the gain matrices to be determined.

The error covariance matrices are defined as

$$P_{k|k}^+ = E[\eta_k \eta_k^T], \quad P_{k|k}^- = E[\bar{\eta}_k \bar{\eta}_k^T],$$

where $\eta_k = x_k - \hat{x}_k$, $\bar{\eta}_k = x_k - \bar{x}_k$. It is straightforward to see that matrices $P_{k|k}^+, P_{k|k}^-$ are symmetric positive definite.

By minimizing the error variance of system (6), the optimal gain matrices are obtained as

$$M_k = (D_k^T \tilde{R}_k^{-1} D_k)^{-1} D_k^T \tilde{R}_k^{-1}, \tag{16}$$

$$K_k = (P_{k|k}^- H_k^T + S_k)\breve{R}_k^{-1}, \tag{17}$$

where $D_k = H_k B_{k-1}$, $S_k = -B_{k-1} M_k R_k$, and

$$\tilde{R}_k = H_k(A_{k-1} P_{k-1|k-1}^+ A_{k-1}^T + Q_k) H_k^T + R_k, \tag{18}$$

$$\breve{R}_k = H_k P_{k|k}^- H_k^T + R_k + H_k S_k + S_k^T H_k^T, \tag{19}$$

The error covariance matrices update according to

$$P_{k|k}^+ = P_{k|k}^- - K_k(P_{k|k}^- H_k^T + S_k)^T, \tag{20}$$

$$P_{k|k}^- = \tilde{A}_{k-1} P_{k-1|k-1}^+ \tilde{A}_{k-1}^T + \tilde{Q}_{k-1}, \tag{21}$$

where, $\tilde{A}_{k-1} = (I_n - B_{k-1} M_k H_k) A_{k-1}$, and $\tilde{Q}_{k-1} = B_{k-1} M_k R_k M_k^T B_{k-1}^T$.

Remarks 6: For any k, $\breve{R}_k$ in (19) is singular and means the optimal gain K_k is not unique. However, we can assume that $\breve{R}_k$ is nonsingular and maintain the above results by choosing its full rank submatrix [22].

The recursive filter in (12)-(15) is the centralized Kalman filter, in which the measurements z_k are treated in one measurement matrix.

B. Information Filter

In this subsection,we extend the form in (12)-(15) to another form of the Kalman filter called the information filter. This is an implementation of the Kalman filter that propagates the inverse of the estimation error covariance matrix rather than the error covariance itself. The classical information filter frequently appears in distributed Kalman filtering.

Recall from equation (17) that the Kalman gain K_k can be written as

$$K_k = (P_{k|k}^- H_k^T + S_k)\breve{R}_k^{-1} = P_{k|k}^-[H_k + S_k^T (P_{k|k}^-)^{-1}]^T \breve{R}_k^{-1}$$

$$= P^-_{k|k}[H_k + S_k^T(P^-_{k|k})^{-1}]^T[H_kP^-_{k|k}H_k^T + R_k$$
$$+H_kS_k + S_k^TH_k^T]^{-1}$$
$$= P^-_{k|k}[H_k + S_k^T(P^-_{k|k})^{-1}]^T[(H_k + S_k^T(P^-_{k|k})^{-1})P^-_{k|k}$$
$$(H_k + S_k^T(P^-_{k|k})^{-1})^T + R_k - S_k^T(P^-_{k|k})^{-1}S_k]^{-1}.$$

Defining

$$\bar{H}_k = H_k + S_k^T(P^-_{k|k})^{-1}, \tag{22}$$

$$\bar{R}_k = R_k - S_k^T(P^-_{k|k})^{-1}S_k, \tag{23}$$

the equation of the Kalman gain matrix can be equivalently expressed as

$$K_k = P^-_{k|k}\bar{H}_k^T(\bar{H}_kP^-_{k|k}\bar{H}_k^T + \bar{R}_k)^{-1}, \tag{24}$$

and the error covariance matrix $P^+_{k|k}$ can be expressed as

$$P^+_{k|k} = P^-_{k|k} - K_k(P^-_{k|k}H_k^T + S_k)^T$$
$$= P^-_{k|k} - K_k[H_k + S_k^T(P^-_{k|k})^{-1}]P^-_{k|k}$$
$$= P^-_{k|k} - K_k\bar{H}_kP^-_{k|k}$$
$$= P^-_{k|k} - P^-_{k|k}\bar{H}_k^T(\bar{H}_kP^-_{k|k}\bar{H}_k^T + \bar{R}_k)^{-1}\bar{H}_kP^-_{k|k}. \tag{25}$$

The alternate Kalman filter formulations (24) and (25)are similar in form to the classical Kalman filter. Then the information filter for simultaneous state and input estimation can be directly obtained from these standard forms.

Defining $F_k = I - K_k\bar{H}_k$ (I denotes the identity matrix), we further have

$$P^+_{k|k} = (I - K_k\bar{H}_k)P^-_{k|k}(I - K_k\bar{H}_k^T) + K_k\bar{R}_kK_k^T$$
$$= F_kP^-_{k|k}F_k^T + K_k\bar{R}_kK_k^T, \tag{26}$$

and

$$K_k = P^+_{k|k}\bar{H}_k^T\bar{R}_k^{-1}. \tag{27}$$

Derivation details of equation (26) and (27) are omitted here for conciseness.

Using the matrix inversion lemma,

$$(A + BD^{-1}C)^{-1} = A^{-1} - A^{-1}B(D + CA^{-1}B)^{-1}CA^{-1},$$

equation (26) can be written as

$$P^+_{k|k} = [(P^-_{k|k})^{-1} + \bar{H}_k^T\bar{R}_k^{-1}\bar{H}_k]^{-1}. \tag{28}$$

Substituting the definition of $\bar{H}_k$ into the state time-update equation (12) gives

$$\hat{x}_{k|k} = \bar{x}_{k|k} + K_k(z_k - H_k\bar{x}_{k|k})$$
$$= \bar{x}_{k|k} + K_k[z_k + S_k^T(P^-_{k|k})^{-1}\bar{x}_{k|k} - \bar{H}_k\bar{x}_{k|k}]. \tag{29}$$

Defining the weighted sensor data y_k and the information matrix U_k as

$$y_k = \bar{H}_k^T\bar{R}_k^{-1}[z_k + S_k^T(P^-_{k|k})^{-1}\bar{x}_{k|k}], \tag{30}$$

$$U_k = \bar{H}_k^T\bar{R}_k^{-1}\bar{H}_k, \tag{31}$$

the state estimation equation and error covariance matrix can be restated as

$$\hat{x}_{k|k} = \bar{x}_{k|k} + P^+_{k|k}(y_k - U_k\bar{x}_{k|k}), \tag{32}$$

$$P^+_{k|k} = [(P^-_{k|k})^{-1} + U_k]^{-1}. \tag{33}$$

The information filter for state estimation of distributed paameter systems can be summarized as follows. The computations at each time step $k = 1, 2, \cdots$: are

$$\hat{x}_{k|k} = \bar{x}_{k|k} + P^+_{k|k}(y_k - U_k\bar{x}_{k|k}), \tag{34}$$

$$U_k = \bar{H}_k^T\bar{R}_k^{-1}\bar{H}_k, \tag{35}$$

$$P^+_{k|k} = [(P^-_{k|k})^{-1} + U_k]^{-1}, \tag{36}$$

$$P^-_{k|k} = \tilde{A}_{k-1}P^+_{k-1|k-1}\tilde{A}_{k-1}^T + \tilde{Q}_{k-1}, \tag{37}$$

$$\bar{x}_{k|k} = \hat{x}_{k|k-1} + B_{k-1}\hat{d}_{k-1}, \tag{38}$$

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1}, \tag{39}$$

with input estimation $\hat{d}_{k-1}$, and weighted sensor data y_k.

The information filter played a key role in the derivation of the distributed Kalman filtering. This form will be used in the next section to derive our distributed Kalman consensus filter algorithms.

Remarks 7: For the simultaneous state and input estimation problem [22], if the prior input estimate is unbiased, then in the state estimation step, the Kalman state gain matrix K_k and the error covariance matrix $P^+_{k|k}$ are the familiar updating forms of the standard Kalman filter.

Remarks 8: When $B_k = 0$, the classical information filter is obtained.

IV. DISTRIBUTED KALMAN CONSENSUS FILTER

A. Kalman Consensus Information Filter

In this section, consider a sensor network consisting of N sensor nodes monitoring a distributed parameter system. The nodes are connected each other in some network topology. The set of neighbors of node i including node i itself is denoted by the set N_i.

We propose consensus filters consist of local filters with a disagreement term

$$\hat{x}^{(i)}_{k|k} = \bar{x}^{(i)}_{k|k} + K^{(i)}_k(z^{(i)}_k - H^{(i)}_k\bar{x}^{(i)}_{k|k}) + C^{(i)}_k\sum_{j\in N_i}(\bar{x}^{(j)}_{k|k} - \bar{x}^{(i)}_{k|k}) \tag{40}$$

where $K^{(i)}_k$, $C^{(i)}_k$ are the Kalman and consensus gains of nodes i, respectively, and we choose

$$K^{(i)}_k = P^{-(i)}_{k|k}\bar{H}^{(i)T}_k(\bar{H}^{(i)}_kP^{-(i)}_{k|k}\bar{H}^{(i)T}_k + \bar{R}^{(i)}_k)^{-1}, \tag{41}$$

$$C^{(i)}_k = \gamma P^{-(i)}_{k|k} = \epsilon\frac{P^{-(i)}_{k|k}}{1 + \|P^{-(i)}_{k|k}\|}. \tag{42}$$

where $\|\cdot\|$ is the frobenius norm of a matrix and $\epsilon > 0$ is a relatively small constant. $\bar{H}^{(i)}_k$, $P^{-(i)}_{k|k}$, and $\bar{R}^{(i)}_k$ are calculated from the equations (22), (37), and (23) for the sensor i with the local observation matrix $H_k(i)$.

The distributed Kalman consensus filter for distributed parameter systems (6) can be expressed as

$$\hat{x}^{(i)}_{k|k} = \bar{x}^{(i)}_{k|k} + P^{+(i)}_{k|k}(y^{(i)}_k - U^{(i)}_k \bar{x}^{(i)}_{k|k}) + C^{(i)}_k \sum_{j\in N_i}(\bar{x}^{(j)}_{k|k} - \bar{x}^{(i)}_{k|k}), \tag{43}$$

$$P^{+(i)}_{k|k} = [(P^{-(i)}_{k|k})^{-1} + U^{(i)}_k]^{-1}, \tag{44}$$

$$P^{-(i)}_{k|k} = \tilde{A}^{(i)}_{k-1} P^{+(i)}_{k-1|k-1} \tilde{A}^{(i)T}_{k-1} + \tilde{Q}^{(i)}_{k-1}, \tag{45}$$

$$\bar{x}^{(i)}_{k|k} = \hat{x}^{(i)}_{k|k-1} + B^{(i)}_{k-1}\hat{d}^{(i)}_{k-1}, \tag{46}$$

$$\hat{x}^{(i)}_{k|k-1} = A_{k-1}\hat{x}^{(i)}_{k-1|k-1}, \tag{47}$$

with the weighted measurement $y^{(i)}_k = \bar{H}^{(i)T}_k \bar{R}^{(i)-1}_k [z^{(i)}_k + S^{(i)T}_k (P^{-(i)}_{k|k})^{-1}\bar{x}^{(i)}_{k|k}]$, and the information matrix $U^{(i)}_k = \bar{H}^{(i)T}_k \bar{R}^{(i)-1}_k \bar{H}^{(i)}_k$.

The local input estimate

$$\hat{d}^{(i)}_{k-1} = M^{(i)}_k (z^{(i)}_k - H^{(i)}_k \hat{x}^{(i)}_{k|k-1}), \tag{48}$$

and $\bar{H}^{(i)}_k$, $\bar{R}^{(i)}_k$, $S^{(i)}_k$, $\tilde{A}^{(i)}_{k-1}$, and $\tilde{Q}^{(i)}_{k-1}$ are calculated from the corresponding equations in section II, with global observation matrix H_k is replaced by local matrix $H^{(i)}_k$.

Remarks 9: From the definition of local boundary conditions $d^{(i)}_k$ in section IIC, if there exist sensors located at the boundary grids of the distributed parameter system, the assumption 5 in the Gillijins-De Moor's algorithm for the simultaneous state and input estimation (the centralized Kalman filter), is satisfied in this above distributed Kalman consensus filter algorithm.

B. Stability of Distributed Kalman Consensus Filter

Stability of the classical estimation are often formulated in terms of stabilizability and detectability conditions. However, these conditions are not sufficient to guarantee stability in the more complicated simultaneous state and input estimations.

In this subsection, we present the stability analysis for distributed parameter systems with unknown boundary conditions following the results in [18].

Lemma 10: For the information filter (34)-(39), $F_k = I - P^+_{k|k}U_k = P^+_{k|k}(P^-_{k|k})^{-1}$, and $P^+_{k+1|k+1} = F_k G_k F^T_k$ with $G_k = \tilde{A}_{k-1}P^+_{k-1|k-1}\tilde{A}^T_{k-1} + \tilde{Q}_{k-1} + P^-_{k-1|k-1}U_k P^-_{k-1|k-1}$. Moreover, if U_k is positive definite, then G_k is positive definite as well.

Lemma 11: Suppose information matrix U_k is positive definite. Then, the error dynamics of the discrete-time information filter (34)-(39) is globally asymptotically stable with a Lyapunov function $V(\eta_k) = \eta^T_k (P^+_{k|k})^{-1}\eta_k$.

Proofs of Lemma 10, 11 are deduced from the equation (27), (28), (31), and with $P^+_{k|k}(P^-_{k|k} + U_k) = I$.

Theorem 12: Consider the distributed filter (43)-(47) with consensus gain $C^{(i)}_k = \gamma F^{(i)}_k G^{(i)}_k$ and $F^{(i)}_k = I - P^{+(i)}_{k|k}U^{(i)}_k$,

$$G^{(i)}_k = \tilde{A}^{(i)}_{k-1}P^{+(i)}_{k-1|k-1}\tilde{A}^{(i)T}_{k-1} + \tilde{Q}^{(i)}_{k-1} + P^{-(i)}_{k-1|k-1}U^{(i)}_k P^{-(i)}_{k-1|k-1}, \tag{50}$$

Suppose the information matrix $U^{(i)}_k$ is positive definite for all i. Then, the error dynamics of the distributed Kalman consensus filter (43)-(47) is globally asymptotically stable for a sufficient small $\gamma > 0$. Furthermore, all estimators asymptotically reach a consensus on state estimates, i.e. $\hat{x}^{(1)}_{k|k} = \hat{x}^{(2)}_{k|k} = \cdots = \hat{x}^{(N)}_{k|k} = x$.

Proof: The error dynamics (without noise) of the consensus filter can be written as

$$\eta^{(i)}_{k+1} = F^{(i)}_k \tilde{A}^{(i)}_k \eta^{(i)}_k + C^{(i)}_k \tilde{A}^{(i)}_k \sum_{j\in N_i}(\eta^{(j)}_k - \eta^{(i)}_k), \tag{51}$$

Using the Lyapunov function in Lemma 11, and calculating the change $\delta V(\eta_k) = V(\eta_{k+1}) - V(\eta_k)$, we have

$$\begin{aligned}
\delta V(\eta_k) &= \sum_i \eta^{(i)T}_{k+1}(P^{+(i)}_{k+1|k+1})^{-1}\eta^{(i)}_{k+1} - \eta^{(i)T}_k (P^{+(i)}_{k|k})^{-1}\eta^{(i)}_k \\
&= \textstyle\sum_i (F^{(i)}_k \tilde{A}^{(i)}_k \eta^{(i)}_k + C^{(i)}_k \mu^{(i)}_k)^T (P^{+(i)}_{k+1|k+1})^{-1}(F^{(i)}_k \tilde{A}^{(i)}_k \eta^{(i)}_k \\
&\quad + C^{(i)}_k \mu^{(i)}_k) - \eta^{(i)T}_k (P^{+(i)}_{k|k})^{-1}\eta^{(i)}_k, \\
&= \textstyle\sum_i \eta^{(i)T}_k [\tilde{A}^{(i)T}_k F^{(i)T}_k (P^{+(i)}_{k+1|k+1})^{-1} F^{(i)}_k \tilde{A}^{(i)}_k - (P^{+(i)}_{k|k})^{-1}] \\
&\quad \cdot \eta^{(i)}_k + 2\sum_i \bar{\eta}^{(i)T}_{k+1} F^{(i)T}_k (P^{+(i)}_{k+1|k+1})^{-1} C^{(i)}_k \mu^{(i)}_k \\
&\quad + \sum_i \mu^{(i)T}_k C^{(i)T}_k (P^{+(i)}_{k+1|k+1})^{-1} C^{(i)}_k \mu^{(i)}_k,
\end{aligned} \tag{52}$$

with $\mu^{(i)}_k = \tilde{A}^{(i)}_k \sum_{j\in N_i}(\eta^{(j)}_k - \eta^{(i)}_k)$.

The first term in (52) is negative semidefinite from Lemma 11, and the second term can be designed negative semidefinite using the quadrtic property of graph Laplacians [20], [21], by setting the consensus gain $C^{(i)}_k = \gamma F^{(i)}_k G^{(i)}_k$.

The last term in the expression of δV is positive semidefinite, while we could choose γ sufficient small so that the sum of the first and third term is negative semidefinite. The following proof is analogous to the Theorem 2 in [18].

C. Unknown Boundary Conditions Estimation

Input estimate of the unknown boundary conditions $\hat{d}^{(i)}_{k-1}$ to a distributed parameter system is obtained from the the innovation by least squares estimation. This is an unbiased optimal estimate.

In the construction of our reduced models at the decentralized sensors, we require $\bigcup I^d_i = \{1, \cdots, m\}$, which means the global boundary condition is covered by the set of local boundary conditions.

Therefore, we can reconstruct the estimate of the boundary conditions by choosing the corresponding components from the local estimates, that is

$$\hat{d}_{k-1}(j) = \hat{d}^{(i)}_{k-1}(j), \quad j \in I^d_i, \tag{53}$$

$j \in \{1, 2, \cdots, m\}$, where $\hat{d}_{k-1}(j)$ is the jth component of the estimate vector $\hat{d}_{k-1}$.

Remark 13: Index j may belong to some different subsets I^d_i. The ones with the better estimates have smaller local sensor noise signals $v^{(i)}_k$.

TABLE I

MODEL AND SIMULATION PARAMETERS

Paramaters	Values	Units
ρ	8700	kgm^{-3}
κ	400	$Wm^{-1}K^{-1}$
C_p	385	$Jkg^{-1}K^{-1}$
λ	10	$Wm^{-1}K^{-1}$
T_e	0	K
T_b	298.15	K
L	4	m
n	20	$-$
T_0	50	K
x_0	80	K
Q_k	0.5	$-$
R_k	0.08	$-$

V. HEAT CONDUCTION PROCESS MODEL AND SIMULATIONS

Consider a rod with length L and cross-section radius r [9], [25]. The density, heat capacity, and thermal conductivity of the material are denoted by ρ, C_p, and κ respectively, see Fig.1. Using the energy balance equation, we get the following partial differential equation

$$\frac{\partial T}{\partial t} = \frac{1}{\rho C_p}[\kappa \frac{\partial^2 T}{\partial l^2} + \frac{\lambda P_e}{A_T}(T_e - T)], \quad (54)$$

where T is the temperature of the rod, T_e the temperature of the environment, λ the heat transfer coefficient of the surface of the rod, l the spatial coordinate of the length, $P_e = 2\pi r$ the perimeter of the rod, and $A_T = \pi r^2$ the area of the longitudinal section.

For the distributed parameter system (52), we assume the initial condition is $T(l, 0) = T_0$, and he boundary conditions $T(0, t)$, $T(L, t)$ are unknown inputs, respectively.

Employing the central approximation (5), the system (52) becomes the ordinary differential equation

$$\frac{dT_i}{dt} = C_x T_{i-1} - (2C_x + C_{Peh})T_i + C_x T_{i+1} + C_{Peh} T_e, \quad (55)$$

with i the node index corresponding to the grid point index increasing from the left to the right and

$$C_x = \frac{\kappa}{\rho C_p h^2}, \quad C_{Peh} = \frac{\lambda P_e}{\rho C_p A_T}.$$

Using the temporal Euler approximation (time increment assumed to be less than $1/2C_x$ for convergence [25]), the state equation (53) is discretized to get the discrete time linear equation

$$x_{k+1} = Ax_k + Bd_k + w_k, \quad (56)$$

with state vector $x = [T_1 \quad \cdots \quad T_n]^T$, unknown boundary conditions $d_k = [T_k^L \quad T_k^R]^T$, model approximation error w_k with the covariance matric Q_k.

In this simulation, the boundary conditions are taken as

$$T_k^L = T_k^R = \frac{T_b}{2} sin(k)\zeta_k, \quad (57)$$

where ζ_k is a normal Gaussian random variable. Suppose that no information about T_k^L, T_k^R are available for use.

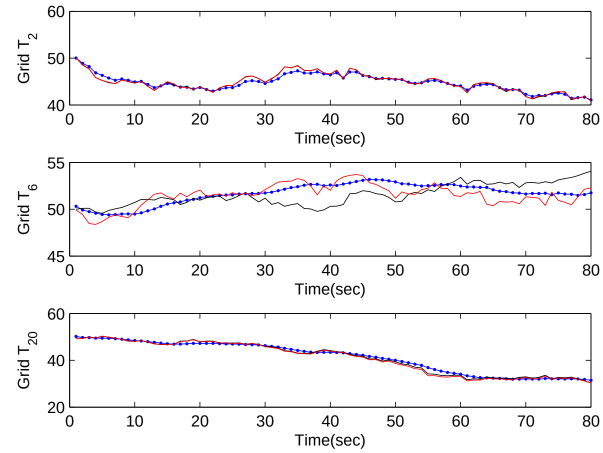

Fig. 2. Actual grid temperature values (red solid line), centralized Kalman filter (black dashed line), and distributed Kalman consensus filter (blue dot-solid line).

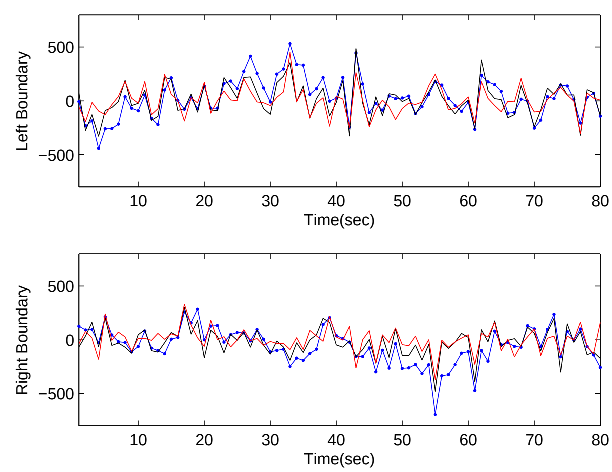

Fig. 3. Actual boundary conditions values (red solid line), centralized Kalman filter (black dashed line), and distributed Kalman consensus filter (blue dot-solid line).

The model and simulation parameters are shown in Table 1. Three local sensor groups are located separately at the left, central and right grids of the rod. We get the following subsystem observation matrices:

$$H_k^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}, \quad B_k^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T,$$

$$H_k^{(2)} = \begin{bmatrix} \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots \end{bmatrix}, \quad B_k^{(2)} = 0,$$

$$H_k^{(3)} = \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad B_k^{(3)} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T.$$

The simulation final time is 80 seconds and the consensus gains are $C^{(i)} = 0.2$. All sensor groups are full connected. We compare the estimation error of the distributed Kalman

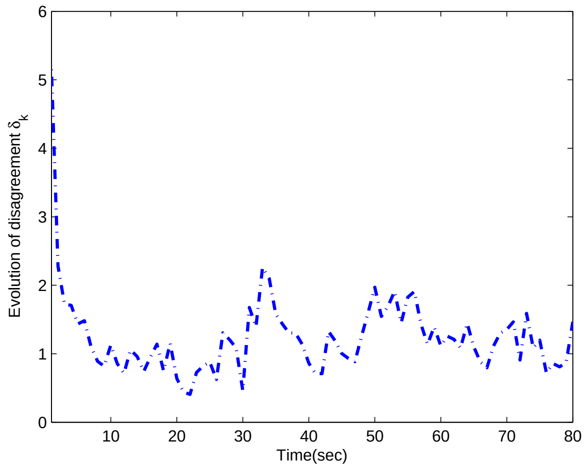

Fig. 4. Evolution of consensus disagreement δ_k.

consensus filter to the centralized one. The state estimation results for grid 2, 6, and 20 are shown in Fig.2, and left and right boundary condition estimation are shown in Fig.3.

Following Olfati-Saber [17] on the appropriate measure of the disagreement of the estimates, we use the following disagreement measure

$$\|\delta_k\|^2 = \sum_{i=1}^{N} (\delta_k^{(i)})^2, \tag{58}$$

where $\delta_k^{(i)} = \hat{x}_k^{(i)} - \mu_k$, and $\mu_k = \frac{1}{N}\sum_i \hat{x}_k^{(i)}$.

The evolution of the disagreement term is depicted in Fig.4, which shows that a relatively high disagreement at the beginning is bounded asymptotically.

VI. CONCLUSIONS

The concept of consensus filters are applied to distributed parameter systems with unknown boundary conditions. The local sensor groups have different reduced observation models according to their spatial locations, and we decompose the whole large-scale distributed parameter systems into different local subsystems. Local information filtering algorithms are developed to estimate the state and the boundary condition simultaneously and extended to the distributed Kalman consensus filter through a penalty term that enforces consensus. A 1-dimension heat conduction process model with two known boundary conditions is used to illustrated the spatially distributed consensus filter with three sensor groups.

Future work will consider applications of the distributed consensus filtering estimation to some concrete distributed parameter systems, such as the networks of traffic flows with changing traffic demands, and hydrological and atmospheric transport models.

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