Finite Element Relativity Simulation Using Regge Calculus

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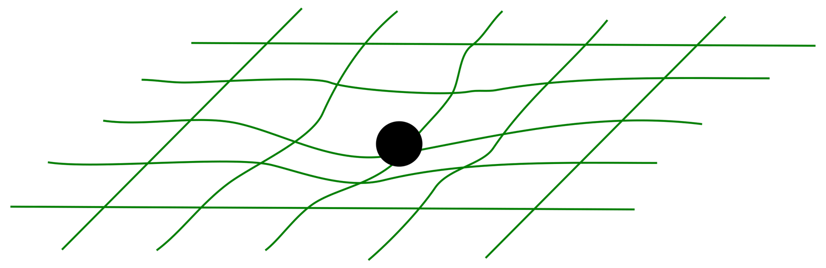
Summary

I propose a new finite element method based on Regge Calculus for solving the Einstein field equation in general relativity.

Background

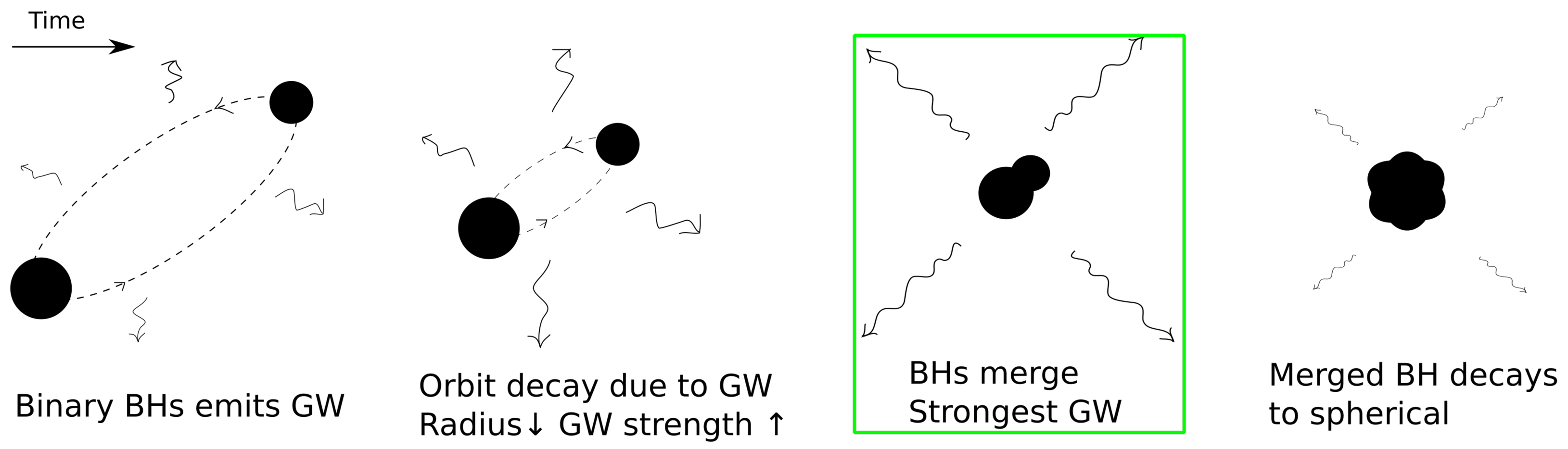
General Relativity: "Spacetime tells matter how to move; matter tells spacetime how to curve".



Einstein field equation makes this quantitative:

"curvature of space-time" $\propto$ "density of energy-matter".

Predictions: the existence of black holes (BH) and gravitational waves (GW). One of the most interesting phenomenon is the merger of BHs:



GWs are very weak and only the ones emitted during the merger is detectable by experiments like LIGO and GEO 600. The equations modeling the merger are analytically intractable. One of the main motivations for numerical relativity is to simulate this on a computer.

Mathematically, after splitting space and time, the vacuum Einstein equation becomes a *constrained evolution equation*:

$$\ddot{\gamma} + R(\gamma) = 0, \qquad C(\gamma, \dot{\gamma}) = 0,$$

where the unknown γ, called the *metric*, is a symmetric 3×3-matrix-valued function of time and space, characterizing the geometry of space. Both R and C are some known nonlinear differential operators related to the curvature.

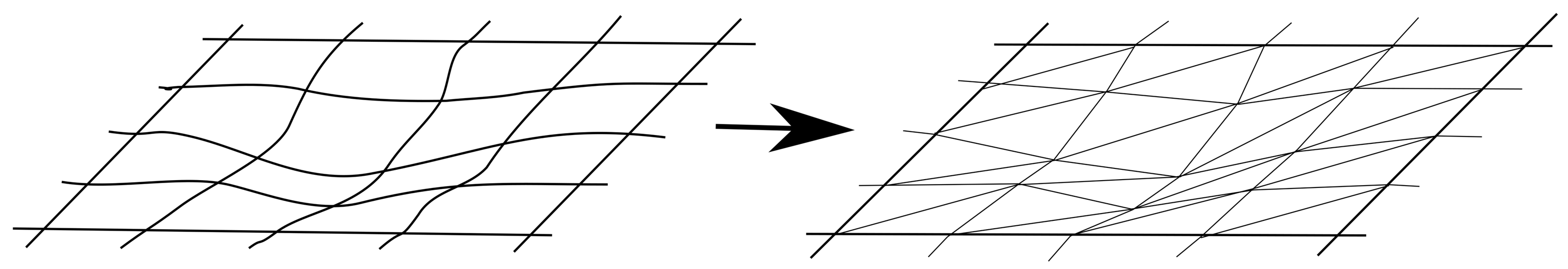
The major challenges in relativity simulation are:

- Gauge freedom. How the equations are formulated matters.
- Large number of degrees of freedoms: symmetric matrices in 4D.
- Long-time simulation is necessary. Reasonably good initial conditions can be obtained only when the binaries are far apart.
- Preservation of the constraints is important for physical problems.

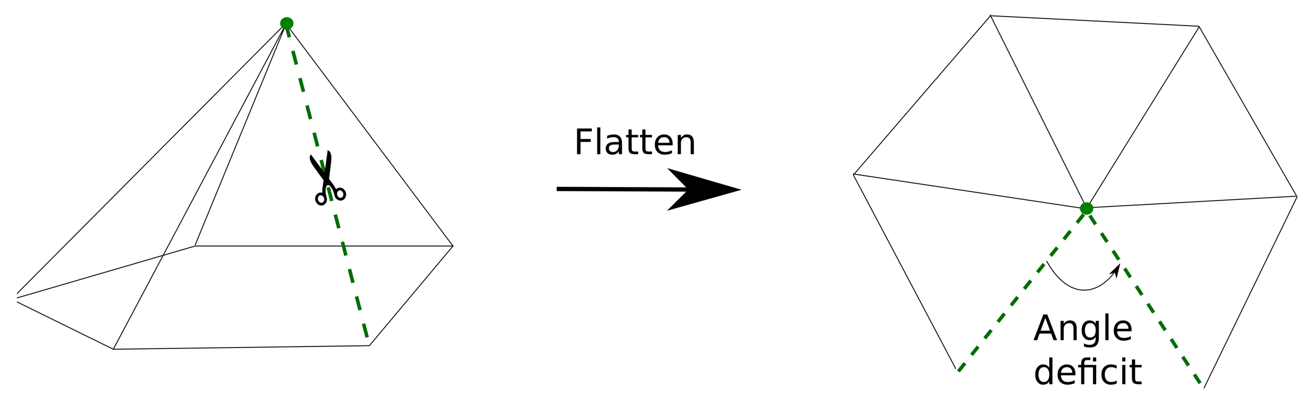
There has been tremendous progress in numerical relativity. Most current works relate the Einstein equation to a component-wise wave equation where standard finite difference and spectral method apply. These methods are restrictive and not flexible, requiring highly symmetric domains and so on. The rigorous analysis of the convergence is largely lacking. Moreover, it is well-known that similar approaches suffer subtle but severe problems when applied to problems in electromagnetism and solid mechanics.

Regge Calculus

Regge (1961) proposed Regge Calculus (RC) as a geometric approach to discretize general relativity. The space-time is obtained by gluing together a collection of Euclidean (or Minkowskian) simplices (triangles in 2D, tetrahedra in 3D). On a computer, the data consists of the connectivity of the edges and the length of the edges.



The curvature is quantified through *angle deficits*. They are defined on vertices in 2D, edges in 3D, and triangles in 4D. For example, in 2D:



Discrete Einstein equation reads:

"Quantity derived from angle deficits" $\propto$ "density of energy-matter".

Sorkin (1975) and many subsequent works by physicists made this a complete 4D spacetime method for solving the Einstein equation.

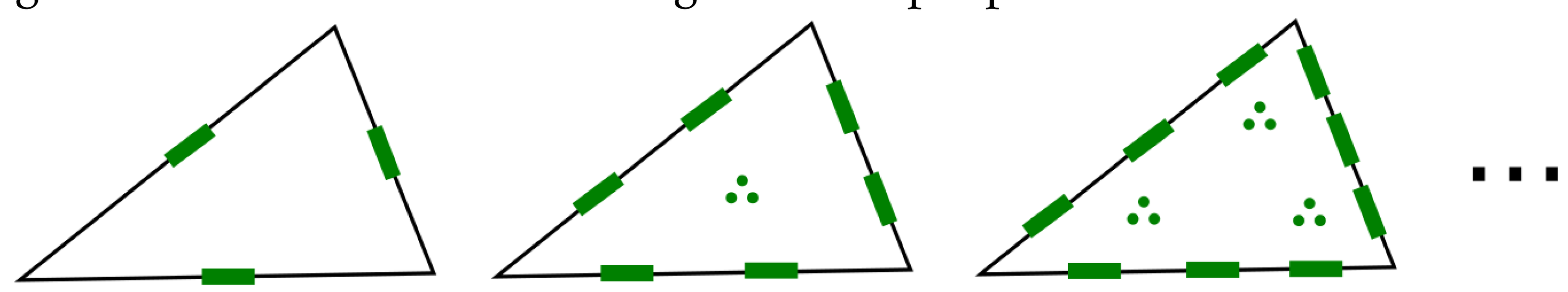
Despite a relative large literature, the success of RC is limited. I identified the following issues:

- The evolution equation does not control constraint violating modes.
- A similar method applied to the wave equation is subtly unstable.
- It works only on highly structured triangulations.

The first two make long-time simulation impossible while the third prevents local refinements and makes this method impractical. Thus I found this approach not suitable for numerical relativity simulations.

Finite Element Approach

Christiansen (2011) related RC to finite element. Gluing together Euclidean simplices is equivalent to having a piecewise constant metric on a triangulation satisfying certain weak continuity condition. I generalized this construction to piecewise polynomials of all degrees in all dimensions. The result is a discontinuous metric with well-defined geodesics and other favorable geometric properties.

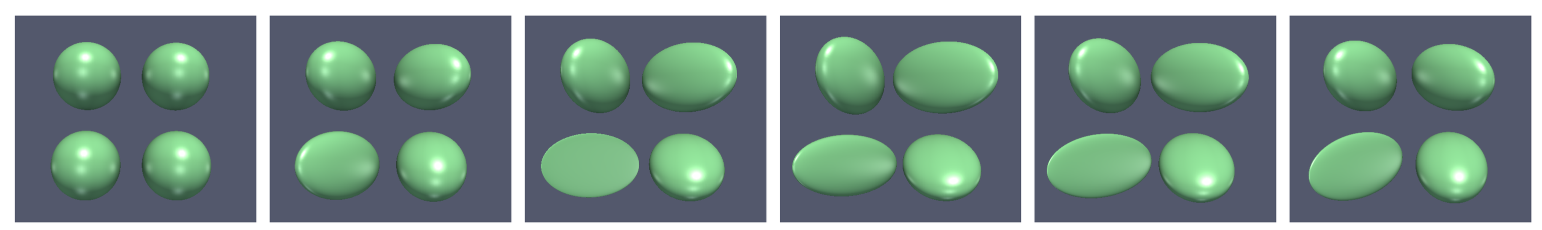


The curvature is a nonlinear second-order differential operator of the metric but our discrete metric is not even continuous. The discrete curvatures are distributions. This is mathematically justified by Cheeger-Müller-Schrader (1984) and Christiansen (2011). I partially generalized these results to my higher degree polynomial metrics.

Further, I proposed a new method for solving the Einstein equation where only the metric for spatial slices of the spacetime is discretized using RC. The discrete constrained evolution equation reads:

$$\ddot{\gamma}_h + R_h(\gamma_h) = 0, \qquad C_h(\gamma_h, \dot{\gamma}_h) = 0,$$

which holds in some weak sense. Using the ideas from the successful Finite Element Exterior Calculus framework by Arnold-Winther-Falk (2006), I further introduce some auxiliary variables and modified the equations to a first-order in time and second-order in space unconstrained evolution system. In the linearized case, I showed that the modified continuous system is stable and the solution satisfies the Einstein equation when the auxiliary variable vanishes. Moreover, I implemented this method for a linearized model problem and the numerical results are favorable. Below shows a gravitational wave obtained from my program visualized through the deformation of four spheres:



Currently ongoing work

- Efficient implementation using FEniCS.
- Convergence proof for the linearized model problem.
- Rigorous justifications in the nonlinear case.
- Computation with full scale physical problems.