Finite Element Exterior Calculus With Lower-order Terms

Finite Element Circus 2013 @ University of Delaware

Lizao (Larry) Li
joint work with Douglas Arnold

School of Mathematics, University of Minnesota, Twin Cities

Motivation: Maxwell equations

Part of a simple model in MHD: for $\operatorname{div} f = 0$,

$$\operatorname{curl}\operatorname{curl} B - \operatorname{curl}(v \times B) = f, \quad \operatorname{div} B = 0.$$

More suitable for discretization: for any f,

$$-\operatorname{grad}\operatorname{div} B + \operatorname{curl}\operatorname{curl} B - \operatorname{curl}(v \times B) = f.$$

Mixed method: $(\sigma, B) \in H(\operatorname{curl}) \times H(\operatorname{div})$,

$$\begin{aligned} (\sigma, \tau) - (B, \operatorname{curl}\tau) + (v \times B, \tau) &= 0, && \forall \tau, \\ (\operatorname{curl}\sigma, G) + (\operatorname{div} B, \operatorname{div} G) &= (f, G), && \forall G. \end{aligned}$$

More complicated lower-order terms arise as well.

Question: Stability and error estimates?

FEEC abstract framework [AFW2006, 2010]

FEEC includes Hodge Laplacian, linear elasticity, ...

Hilbert complex: $W^{k-1} \xrightarrow{d^{k-1}} W^k \xrightarrow{d^k} W^{k+1}$ with compactness.

Abstract Hodge Laplacian: $d^* du + dd^* u = f$.

Mixed method:

$$
\begin{aligned}
(\sigma, \tau) - (u, d\tau) &= 0, && \forall \tau, \\
(d\sigma, v) + (du, dv) &= (f, v), && \forall v.
\end{aligned}
$$

Main theorem (rough)

$V_h^k \underset{\pi_h}{\overset{\text{dense}}{\rightleftarrows}} W^k$, $\quad d^k$ is bounded on V_h^k and $d\pi_h = \pi_h d$

$\implies$ stability and errors like $\|\sigma - \sigma_h\|_W$ can be controlled.

Mathematical question

The Hodge Laplace equation is special: it gives the Hodge decomposition of the data.

$$
\begin{aligned}
\operatorname{grad} H^1 \oplus \operatorname{curl} H(\operatorname{curl}) &= L^2, \\
-\operatorname{grad}\operatorname{div} B + \operatorname{curl}\sigma &= f.
\end{aligned}
$$

Orthogonality is the key for W-norm error estimates. But

$$
-\operatorname{grad}\operatorname{div} B + \operatorname{curl}\sigma + B = f
$$

completely destroys this nice structure of the equation.

Question: How robust is the FEEC against perturbations?

FEEC with lower-order terms

We systematically treat all possible relatively bounded linear perturbations.

Main equation: perturbed abstract Hodge Laplacian

$$\mathsf{L}u = (d^* + l_4)(d + l_1)u + (d + l_3)(d^* + l_2)u + l_5 u = f$$

where $l_i \in \mathcal{L}(W, W)$ with some mild regularity conditions.

Example:

$$(-\operatorname{grad} + l_4)(\operatorname{div} + l_1 \cdot)u + (\operatorname{curl} + l_3 \times)(\operatorname{curl} + l_2 \times)u + l_5 u = f.$$

Mixed method:

$$\begin{aligned}
&(\sigma, \tau) - (u, d\tau) - (l_2 u, \tau) = 0, \\
&(d\sigma, v) + (du, dv) + (l_1 u, dv) + (l_3 \sigma, v) + (l_4 du, v) + (\ell_5 u, v) = (f, v).
\end{aligned}$$

Known results: scalar Laplacian (n-forms)

Complex: $L^2 \xrightarrow{\text{div}} L^2 \to 0$.

Equation: $(\text{div} + l_3 \cdot)(-\text{grad} + l_2)u + l_5 u = f, \quad u \in \mathring{H}^1$.

Mixed method: $\sigma = -\text{grad}\, u + l_2 u \in H(\text{div}), u \in L^2$.

Theorem [Douglas-Roberts 1982,1985]

$(\sigma_h, u_h) \in RT_r \times DG_r \implies$ stability and optimal L^2 rates.

Known results: scalar Laplacian (n-forms)

Complex: $L^2 \xrightarrow{\mathrm{div}} L^2 \to 0$.

Equation: $(\mathrm{div} + l_3 \cdot)(-\mathrm{grad} + l_2)u + l_5 u = f, \quad u \in \mathring{H}^1$.

Mixed method: $\sigma = -\mathrm{grad}\, u + l_2 u \in H(\mathrm{div}), u \in L^2$.

Theorem [Douglas-Roberts 1982,1985]

$(\sigma_h, u_h) \in RT_r \times DG_r \implies$ stability and optimal L^2 rates.

Theorem [Demlow 2002]

$(\sigma_h, u_h) \in BDM_r \times DG_r \implies$ stability and optimal L^2 rates, except when $l_2 \neq 0$, the L^2-rate for σ is dropped by 1.

Our new results

$$\mathrm{d}^*\mathrm{d}u + \mathrm{d}\underbrace{\mathrm{d}^*u}_{\sigma} = f$$

$$\Downarrow$$

$$(\mathrm{d}^* + l_4)(\mathrm{d} + l_1)u + (\mathrm{d} + l_3)\underbrace{(\mathrm{d}^* + l_2)u}_{\sigma} + l_5 u = f.$$

Theorem: Stability is robust against perturbation

Any mixed finite elements which are stable for the unperturbed problem remains stable for the perturbed problem, for sufficiently fine discretization.

Our new results

Theorem: W-norm estimates

$$\begin{aligned}\|\mathsf{e}_\sigma\| \lesssim \|\mathsf{E}_\sigma\| &+ [\eta + (\chi_2+\chi_3+\chi_4+\chi_5)\delta]\|\mathsf{dE}_\sigma\| \\ &+ [\chi_2+\chi_4+\chi_5+\chi_1\chi_3(\eta+\gamma_1)]\|\mathsf{E}_\mathsf{u}\| \\ &+ [(\chi_2+\chi_4+\chi_5)(\eta+\gamma_4)+\chi_4(\chi_2+\chi_3+\chi_4+\chi_5)\delta \\ &+ \chi_1\chi_3(\eta+\gamma_1)(\eta+\gamma_4)+\bar{\gamma}_4]\|\mathsf{dE}_\mathsf{u}\|,\end{aligned}$$

$$\begin{aligned}\|\mathsf{de}_\sigma\| &\lesssim \|\mathsf{dE}_\sigma\| + [\chi_4+\chi_5(\gamma_4+\eta)]\|\mathsf{dE}_\mathsf{u}\| \\ &+(\chi_1\chi_4+\chi_4+\chi_5)\|\mathsf{E}_\mathsf{u}\|+[(\chi_1\chi_4+\chi_4+\chi_5)(\eta+\gamma_3)+\chi_3]\|\mathsf{e}_\sigma\|,\end{aligned}$$

$$\begin{aligned}\|\mathsf{e}_\mathsf{u}\| \lesssim \|\mathsf{E}_\mathsf{u}\| &+ (\eta^2+\delta+\eta\gamma_4)\|\mathsf{dE}_\sigma\| + (\chi_4\delta+\gamma_4+\eta)\|\mathsf{dE}_\mathsf{u}\| \\ &+ [\eta+\gamma_3+\chi_3(\eta+\delta+\gamma_4)]\|\mathsf{e}_\sigma\|,\end{aligned}$$

$$\begin{aligned}\|\mathsf{de}_\mathsf{u}\| \lesssim \|\mathsf{dE}_\mathsf{u}\| &+ \eta\|\mathsf{dE}_\sigma\| + (\chi_1+\chi_4+\chi_5)\|\mathsf{E}_\mathsf{u}\| \\ &+ [(\chi_1+\chi_4+\chi_5)(\eta+\gamma_3)+\chi_3]\|\mathsf{e}_\sigma\|.\end{aligned}$$

Our new results

de Rham complex: $L^2\Lambda^{k-1} \xrightarrow{d} L^2\Lambda^k \xrightarrow{d} L^2\Lambda^{k+1}$.

Hodge Laplacian:
$Lu = (\delta + l_4)(d + l_1)u + (d + l_3)(\delta + l_2)u + l_5 u = f$

Four pairs of canonical elements: $P_{r+1}\Lambda^{k-1} \times P_r\Lambda^k$, $P^-_{r+1}\Lambda^{k-1} \times P_r\Lambda^k, P_r\Lambda^{k-1} \times P^-_r\Lambda^k, P^-_r\Lambda^{k-1} \times P^-_r\Lambda^k$.

L^2 error estimates theorem

For sufficiently regular problems, L^2-rates are all optimal, except when $P\Lambda^{k-1}$ is used for σ, and
(1) $l_4 \neq 0$, $\|d(\sigma - \sigma_h)\|$, $\|\sigma - \sigma_h\|$ rates are 1 order lower,
(2) $l_2, l_5 \neq 0$, $\|\sigma - \sigma_h\|$ rate is 1 order lower.

L^2-estimates in [AFW 06,10], [Douglas-Roberts 82,85], [Demlow 02] are special cases of this theorem*.

Numerical example: 2-forms in 3D.

On the unit cube,

$$-\operatorname{grad}\operatorname{div}\mathsf{u} + \operatorname{curl}\underbrace{(\operatorname{curl} + \mathsf{l}_2 \times)\mathsf{u}}_{=\sigma} = \mathsf{f},$$

with $\mathsf{u} \times \mathsf{n} = 0$, $\operatorname{div}\mathsf{u} = 0$ on the boundary.

Mixed method: $(\sigma, \mathsf{u}) \in H(\operatorname{curl}) \times H(\operatorname{div})$.

Finite element pairs: $(\sigma_h, u_h) \in \begin{pmatrix}\mathrm{Ned}_1 \\ \mathrm{Ned}_2\end{pmatrix} \times \begin{pmatrix}\mathrm{RT} \\ \mathrm{BDM}\end{pmatrix}$, four choices.

Numerical example: 2-forms in 3D.

$P_2\Lambda^1 \times P_1\Lambda^2 \cong \mathrm{Ned}_2(2) \times \mathrm{BDM}(1)$

Unperturbed problem:

h	u		div u		σ		curl σ	
8.66e-01	7.467e-03	1.57	5.044e-02	-0.13	4.504e-03	2.72	7.549e-02	1.69
4.33e-01	2.262e-03	1.72	2.916e-02	0.79	6.501e-04	2.79	2.071e-02	1.87
2.17e-01	6.005e-04	1.91	1.512e-02	0.95	8.657e-05	2.91	5.363e-03	1.95
1.08e-01	1.528e-04	1.97	7.631e-03	0.99	1.111e-05	2.96	1.361e-03	1.98

With a smooth l_2 term:

h	u		div u		σ		curl σ	
8.66e-01	7.667e-03	1.55	5.046e-02	-0.12	1.226e-02	2.07	1.059e-01	1.66
4.33e-01	2.362e-03	1.70	2.916e-02	0.79	2.888e-03	2.09	2.899e-02	1.87
2.17e-01	6.303e-04	1.91	1.512e-02	0.95	7.043e-04	2.04	7.527e-03	1.95
1.08e-01	1.605e-04	1.97	7.631e-03	0.99	1.733e-04	2.02	1.913e-03	1.98

Numerical example: 2-forms in 3D.

$\mathsf{P}_2^- \Lambda^1 \times \mathsf{P}_1 \Lambda^2 \cong$ $\mathsf{Ned}_1(2) \times$ $\mathsf{BDM}(1)$

Unperturbed problem:

h	u		div u		σ		curl σ	
8.66e-01	7.480e-03	1.57	5.044e-02	-0.13	1.633e-02	1.95	8.360e-02	1.85
4.33e-01	2.264e-03	1.72	2.916e-02	0.79	4.317e-03	1.92	2.260e-02	1.89
2.17e-01	6.006e-04	1.91	1.512e-02	0.95	1.113e-03	1.96	5.825e-03	1.96
1.08e-01	1.528e-04	1.98	7.631e-03	0.99	2.828e-04	1.98	1.475e-03	1.98

With a smooth l_2 term:

h	u		div u		σ		curl σ	
8.66e-01	7.727e-03	1.55	5.046e-02	-0.12	2.259e-02	1.79	1.158e-01	1.66
4.33e-01	2.367e-03	1.71	2.916e-02	0.79	6.132e-03	1.88	3.178e-02	1.87
2.17e-01	6.307e-04	1.91	1.512e-02	0.95	1.601e-03	1.94	8.188e-03	1.96
1.08e-01	1.606e-04	1.97	7.631e-03	0.99	4.088e-04	1.97	2.068e-03	1.99

Numerical examples

We tested numerically vector Laplacian in 2D and 3D with all possible combinations of lower-order terms. The results of the scalar Laplacian are well-known. All the numerical results agree with our theorem.

The L^2-estimates in the theorem are sharp.

Summary

A general theory for analyzing perturbations for mixed methods.

- The abstract FEEC framework was extended to include lower-order terms.
- The stability result in FEEC is robust.
- The W-norm estimates sometimes are not robust. The details were worked out and were nontrivial.
- For the de Rham complex, $\mathrm{P}^{-}\Lambda$ family is more robust than $\mathrm{P}\Lambda$ family.