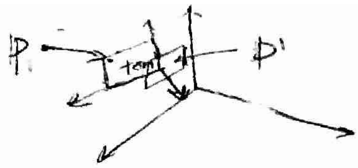

$P = [\, u, v, 1 \,]$

$P' = w(X) \cdot P = [\, u', v', w' \,]$

where $w(e) \cdot (p) = P$

$$\begin{cases} w(e) \cdot p = f(H(x) \cdot p) \\ H(x) = e^{A(x)}, \quad A(x) = \sum_{i=1}^{8} x_i \cdot G_i, \quad G_i \in Sl(3) \\ f([x_1, x_2, x_3]) = [\frac{x_1}{x_3}, \frac{x_2}{x_3}] \text{ (translate from 3DH} \to \text{2D)} \end{cases}$$

$Sl(3)$ is set of matrices, in which each G_i have $\mathrm{trace}(G_i) = 0$
→ So $\mathrm{trace}(A(x)) = 0$
$\underline{\det(e^{A(x)}) = 1}$

Region - Base tracking, basic Idea

$$\Delta s = \sum_{k=1}^{n} (I(P_k) - I(w(x_c) \cdot (P_k)))^2 = 0$$

Using Lie Algebra, suppose: (X_c)

$$w(\Delta X \circ X_c)(p) = w(\Delta x) \cdot (w(x)(p)))$$

$$\Rightarrow H(\Delta x) \cdot H(X) = H(\underline{\Delta x} \circ \underline{x})$$

(The reason of defining $H(x) = e^{A(x)}$)

We want

$$\Delta X = \underset{\Delta X}{\operatorname{argmin}} \sum_{k=1}^{n} (I(w(x_c) \cdot P_k) - I(w(\Delta x \circ x_c) \cdot P_k))^2$$

⇒ second-order Taylor equation: simplify this into,

$$\Delta S \approx -\frac{1}{2} (J_{esm}) \Delta X, \quad J_{esm} = \left(\frac{\partial I\, w(e)(p)}{\partial P} + \frac{\partial I(w(x_c) \cdot p)}{\partial p} \right) \frac{\partial w(x)}{\partial x}\Big|_e$$

$$\Delta x \approx -2\, J_{esm}^{+} \Delta s, \quad J_{esm}^{+} = (J_{esm}^{T} J_{esm})^{-1} J_{esm}^{T}$$

for k point

$$\Delta S_k \approx -\frac{1}{2} (J_{esmk}) \Delta X$$

$$J_{esmk} = \left(\frac{\partial I(w(e)(P_k))}{\partial P_k} + \frac{\partial I(w(x)\, P_k)}{\partial P_k} \right) \cdot \frac{\partial w(x) \cdot P_k}{\partial x}\Big|_e$$

1. Let: $\dfrac{\partial I(w(x), p_k)}{\partial p_k} = [I_x, I_y]$ template gradient,

$\dfrac{\partial I(w(x), p_k)}{\partial p_k} = [I_{wx}, I_{wy}]$ warp Image gradient (Current Image)

$\dfrac{\partial I(w(x), p_k)}{\partial p_k} + \dfrac{\partial I(w(x), p_k)}{\partial p_k} = [I_{sx}, I_{sy}]$

$$\left.\frac{\partial (w(x), p_k)}{\partial x}\right|_e = \left.\frac{\partial f(e^{A(x)} \cdot p_k)}{\partial x}\right|_e \quad // H(x) = e^{A(x)}$$

$$= \left.\frac{\partial f(\tilde{x})}{\partial \tilde{x}} \cdot p_k e^{A(x)} \cdot \frac{\partial A(x)}{\partial x}\right|_e \quad // \tilde{x} = e^{A(x)} \cdot p_k$$

$$= \left.\frac{\partial f(\tilde{x})}{\partial \tilde{x}} p_k \frac{\partial A(x)}{\partial x}\right|_e \quad // \text{since } H(e) \cdot p = p$$

Since

$$A(x) = \sum_{i=1}^{8} x_i G_i, \quad G_i \in Sl(3)$$

we can define:

$$A(x) = \begin{bmatrix} x_7 & x_1 & x_2 \\ x_3 & -x_7 - x_8 & x_4 \\ x_5 & x_6 & x_8 \end{bmatrix}$$

where $G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $G_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $G_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $G_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $G_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $G_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $G_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so

$$p_k \frac{\partial A(x)}{\partial x} = \frac{\partial (A(x) \cdot p_k)}{\partial x}$$

$$= \begin{bmatrix} v & 1 & 0 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & u & 1 & 0 & 0 & -v & -v \\ 0 & 0 & 0 & 0 & u & v & 0 & 1 \end{bmatrix}$$

Let $\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$

where $\tilde{x} = \left.\begin{bmatrix} H_1(x) \cdot p_k \\ H_2(x) \cdot p_k \\ H_3(x) \cdot p_k \end{bmatrix}\right|_e = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$

$$\frac{\partial f(\tilde{x})}{\partial \tilde{x}} = \frac{\partial \begin{bmatrix} \frac{\tilde{x}_1}{\tilde{x}_3} \\ \frac{\tilde{x}_2}{\tilde{x}_3} \end{bmatrix}}{\partial \tilde{x}} = \begin{bmatrix} \frac{1}{\tilde{x}_3} & 0 & -\frac{\tilde{x}_1}{\tilde{x}_3^2} \\ 0 & \frac{1}{\tilde{x}_3} & -\frac{\tilde{x}_2}{\tilde{x}_3^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -u \\ 0 & 1 & -v \end{bmatrix}$$

So,

$$\frac{\partial(w(x), p_k)}{\partial x}\Big|_e = \begin{bmatrix} 1 & 0 & -u \\ 0 & 1 & -v \end{bmatrix} \begin{bmatrix} v & 1 & 0 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & u & 1 & 0 & 0 & -v & -v \\ 0 & 0 & 0 & 0 & u & v & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} v & 1 & 0 & 0 & -u^2 & -uv & u & -u \\ 0 & 0 & u & 1 & -uv & -v^2 & -v & -2v \end{bmatrix}$$

Recall,

$$J_{esm_k} = \left(\frac{\partial I(w(x),p)}{\partial p_k} + \frac{\partial I(w(x), p_k)}{p_k}\right)\frac{\partial w(x), p_k}{\partial x}\Big|_e$$

$$= [I_{sx}, I_{sy}] \begin{bmatrix} v & 1 & 0 & 0 & -u^2 & -uv & u & u \\ 0 & 0 & u & 1 & -uv & -v^2 & -v & -2v \end{bmatrix}$$

$$= [vI_{sx}, I_{sx}, uI_{sy}, I_{sy}, -u^2 I_{sx} - uvI_{sy},$$
$$-uvI_{sx} - v^2 I_{sy}, uI_{sx} - vI_{sy}, uI_{sx} - 2vI_{sy}]$$

$$\Delta s_k = -\frac{1}{2} J_{esm_k} \Delta x.$$