Optimal Output Synchronization Control of a Class of Complex Dynamical Networks With Partially Unknown System Dynamics

Ning Wang and Xiao-Jian Li, *Member, IEEE*

***Abstract*—This paper is concerned with the optimal output synchronization control problem for a class of heterogeneous complex dynamical networks (CDNs) with partially unknown system dynamics. By defining a new quadratic performance index including the information of state couplings, an optimal controller consisting of a feedback control and a compensated feedforward control is then designed. A necessary and sufficient condition for the optimal control minimizing the quadratic performance index is established. Especially, an algebraic Riccati equation (ARE) associated with a discount factor is obtained, and a novel online iteration algorithm by iteratively solving the ARE is proposed to compute the optimal controller gain for the case that the CDNs are with partially unknown system dynamics. Moreover, it is proved that the output synchronization errors converge to zero asymptotically by applying the optimal control laws. Finally, two simulation examples are provided to verify the effectiveness of the proposed control method.**

***Index Terms*—Complex dynamical networks (CDNs), optimal control, output synchronization, policy iteration.**

I. Introduction

COMPLEX networks exist in nature and human society extensively. The study on complex dynamical networks (CDNs), which has attracted growing attention of scholars recently, is of important meaning for people in biology, physics, social science, and engineering [1]–[8]. For CDNs, every node's controller is designed based on information about itself and neighbor nodes to realize certain qualities. For instance, the controller can be designed to realize minimal energy consumption which has attracted many scholars to study the optimal control problem in [9]–[14]. Furthermore, the synchronization problem also plays an imperative role in all kinds of networked systems, and rich works have been developed in [15]–[20]. For CDNs, most of existing achievements about synchronization focus on state by utilizing different control policies, such as robust control, adaptive control, pinning control, optimal control, etc. It can be seen in [21]–[32]. Delellis *et al.* [30] was concerned with how to realize the synchronization of networked nonlinear oscillators and how to assure the asymptotic stability of synchronization errors by using a local adaptive approach. A class of CDNs with stochastic couplings and uncertain dynamics were introduced in [31], and the robust pinning synchronization problem was solved by an evolutionary algorithm. Besides, under the condition of actuator faults and unavailable states in [32], the synchronization errors were all bounded by using the adaptive control policy. These mentioned studies in [21]–[32], while, were limited to state synchronization. The output synchronization, compared with the state synchronization, is more general, which attracted some scholars to consider the output synchronization problems of the CDNs [33]–[36]. It is noted that the node dynamics of CDNs in [21]–[36] were required to be identical, and such a requirement may be restrict.

In order to relax the limitation, the output synchronization control problems, then, have been extended to the heterogeneous CDNs in [37]–[43]. For instance, the output quasi-passivity property was considered in [41] to realize the generalized output synchronization of the CDNs with heterogeneous dynamics. With the help of output quasi-passivity of each node, the output synchronization problems can be simplified. In [42], for the consensus problem of multiple Euler–Lagrange systems, a novel sampled-data communication strategy was proposed. For achieving asymptotic stability of output synchronization errors in [43], an output feedback controller was designed by solving an algebraic Riccati equation (ARE) with a discount factor.

Nevertheless, these results above only focused on the output synchronization problems of heterogeneous CDNs. How to realize output synchronization control by using minimal energy consumption has not been considered. And the optimal output synchronization control of the heterogeneous CDNs is a meaningful research topic. Specially, for the heterogeneous CDNs with unknown dynamics, how to resolve the optimal output synchronization control is a challenging task and still remains open. Hence, it is necessary to develop new methods to address these problems, which motivates the current research.

Manuscript received June 13, 2018; revised September 22, 2018; accepted November 17, 2018. Date of publication December 13, 2018; date of current version January 19, 2021. This work was supported in part by the National Natural Science Foundation of China under Grant 61873050, and in part by the State Key Laboratory of Synthetical Automation for Process Industries under Grant 2018ZCX14. This paper was recommended by Associate Editor J. Sarangapani. *(Corresponding author: Xiao-Jian Li.)*

N. Wang is with the College of Information Science and Engineering, Northeastern University, Shenyang 110819, China (e-mail: wangnsun@qq.com).

X.-J. Li is with the State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang 110819, China, also with the Key Laboratory of Vibration and Control of Aero-Propulsion System Ministry of Education, Northeastern University, Shenyang 110819, China, and also with the College of Information Science and Engineering, Northeastern University, Shenyang 110819, China (e-mail: lixiaojian@ise.neu.edu.cn).

Color versions of one or more of the figures in this paper are available online at https://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSMC.2018.2882827

In this paper, an optimal output synchronization control policy for the heterogeneous CDNs with partially unknown dynamics is developed. By defining a new quadratic performance index including the information of state couplings, an optimal controller consisting of a feedback control and a compensated feedforward control is then designed. Especially, a necessary and sufficient condition of the optimal control minimizing the performance index is established, and an ARE associated with a discount factor is obtained. Motivated by [44], an online iteration algorithm to iteratively solve the ARE by using the information of states, inputs, and couplings is proposed to compute the optimal controller gain for the case that the CDNs are with partially uncertain system dynamics. Besides, by using the Lyapunov function method, the output synchronization errors are proven to be convergent asymptotically. Finally, the simulation results confirm the effectiveness of the proposed algorithm.

This paper is organized as follows. In Section II, some necessary concepts about the graph theory and the problem of the CDNs are presented. An optimal output synchronization control protocol is proposed in Section III. In Section IV, an online iteration algorithm is put forward. In Section V, two examples of the CDNs are provided. The conclusions are finally drawn in Section VI.

Notation: In this paper, $\|.\|$ is used to represent the Euclidean norm for vectors or matrices. Define $\mathrm{vec}(A)$ as an mn-vector for $A \in \mathbb{R}^{m\times n}$, i.e., $\mathrm{vec}(A) = [\alpha_1^T, \alpha_2^T, \ldots, \alpha_n^T]$, where $\alpha_i \in \mathbb{R}^n$ is the ith column of A. $P = [p_{ij}]_{n\times n}$ denotes a matrix $P \in \mathbb{R}^{n\times n}$ with the ith row and jth column being p_{ij}. Besides, $A > 0$ denotes that A is a positive definite matrix.

II. Preliminaries and Problem Formulation

In this section, some necessary theoretical background about graph theory is provided, and the optimal output synchronization problem for a class of CDNs is also presented.

A. Graph Theory

A directed graph (digraph) $\mathcal{G} = VE$ contains a set $V = 1, 2, \ldots, N$ of vertices and a set $E = 1, 2, \ldots, M$ of arcs (i, j) leading from initial vertex i to terminal vertex j. Each arc $(i, j) \in E$ is associated with a real-valued weight with $l_{ij} > 0$, and only if there is no arc connection between the ith node and the jth node, $l_{ij} = 0$. Moreover, the Laplacian matrix of a digraph is defined as

$$L = \begin{bmatrix} \sum_{k\neq 1} l_{1k} & -l_{12} & \cdots & -l_{1N} \\ -l_{21} & \sum_{k\neq 2} l_{2k} & \cdots & -l_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -l_{N1} & -l_{N2} & \cdots & \sum_{k\neq N} l_{Nk} \end{bmatrix}.$$

More detailed concepts on the digraph can be found in [45].

B. Problem Statement

The leader dynamic is assumed to be as

$$\begin{aligned} \dot{z}_0 &= S z_0 \\ y_0 &= G z_0 \end{aligned} \tag{1}$$

where $z_0 \in R^q$ is the leader's state, and $S \in R^{q\times q}$ is the leader's dynamic matrix. Besides, $y_0 \in R^p$, with the output matrix $G \in R^{p\times q}$, is the reference trajectory to be followed by followers.

Assumption 1: S has no eigenvalues with positive real parts.

Consider a CDNs model with N linear nodes which can be described as

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + \sum_{j=1}^{N} l_{ij}\Gamma_i\big(x_j(t) - x_i(t)\big) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{2}$$

where $x_i \in R^n$ is the state of the ith node, $B_i \in R^{n\times n}$ is the constant invertible matrix, $u_i \in R^n$ is the controlled input, and $y_i \in R^p$ is the measurement output of ith node to follow y_0. l_{ij} are known coupling weights. The matrix $\Gamma_i \in R^{n\times n}$ is the inner connecting matrix of the ith node.

Assumption 2: (A_i, B_i) is controllable and (A_i, C_i) is observable.

Remark 1: Assumption 1 ensures that the reference trajectory is a bounded oscillation signal or converges to a constant. That is to say the state of leader is stable. Similar to [49], Assumption 2 is a fundamental requirement for controller design.

Now, the considered problem in this paper is presented as follows.

Problem: For the CDNs (1) and (2) with unknown dynamic matrices S and A_i, the objective is to design an optimal distributed controller $u_i(t)$ minimizing the quadratic performance index (6), such that all the outputs $y_i(t)$ of the followers (2) synchronize to the output $y_0(t)$ of the leader (1).

III. Optimal Output Synchronization Controller

An optimal output synchronization control policy is proposed in this section, and a necessary and sufficient condition for the optimal control (7) which minimizes the performance index (6) is established.

A. Optimal Control Protocol

Define the augmented system state for ith node as follows:

$$X_i(t) = \begin{bmatrix} x_i^T(t) & z_0^T(t) \end{bmatrix}^T \tag{3}$$

where x_i is defined by (2) and z_0 is given in (1). Putting (1) and (2) together results in the augmented dynamic as

$$\begin{aligned} \dot{X}_i &= \begin{bmatrix} A_i & 0 \\ 0 & S \end{bmatrix} X_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i + \begin{bmatrix} \sum_{j=1}^{N} l_{ij}\Gamma_i\big(x_j - x_i\big) \\ 0 \end{bmatrix} \\ &= T_i X_i + B_{1i}\left(u_i + M_i \sum_{j=1}^{N} l_{ij}\Gamma_i x_j\right) \end{aligned} \tag{4}$$

with $T_i = \begin{bmatrix} A_i - \sum_{j=1}^{N} l_{ij}\Gamma_i & 0 \\ 0 & S \end{bmatrix}$, $B_{1i} = \begin{bmatrix} B_i \\ 0 \end{bmatrix}$, and M_i satisfying $B_i M_i = I_n, I_n \in R^{n\times n}$.

Using (1) and (2) gives the output synchronization error for ith node

$$e_i = y_i - y_0 = C_{1i} X_i \tag{5}$$

with $C_{1i} = \begin{bmatrix} C_i & -G \end{bmatrix}$.

Consider the discounted performance function of system (2) as

$$J_i = \frac{1}{2}\int_t^{\infty} e^{-\gamma_i(\tau-t)}\big(X_i^T C_{1i}^T Q_i C_{1i} X_i + u_i^T R_i u_i + 2u_i^T R_i M_i \omega_i + \omega_i^T M_i^T R_i M_i \omega_i\big)d\tau \tag{6}$$

where $w_i = \sum_{j=1}^{N} l_{ij}\Gamma_i x_j$, Q_i and R_i are symmetric positive weight matrices, and γ_i is a discount factor.

Remark 2: According to [46], the discount factor γ_i ($\gamma_i > 0$) is significant to ensure that the performance index (6) is bounded for a given control protocol. Moreover, it is also used to achieve the output synchronization.

Theorem 1: For the augmented dynamics of CDNs (4) with the quadratic performance index (6), a control protocol is an optimal one if and only if it has the following form:

$$u_i = -M_i \sum_{j=1}^{N} l_{ij}\Gamma_i x_j - R_i^{-1} B_{1i}^T P_i X_i \tag{7}$$

where P_i is a symmetric positive matrix and satisfies the following ARE:

$$T_i^T P_i + P_i T_i - \gamma_i P_i + C_{1i}^T Q_i C_{1i} - P_i B_{1i} R_i^{-1} B_{1i}^T P_i = 0. \tag{8}$$

Proof: First, the necessary of Theorem 1 is proved as follows. Based on (6), define the Hamiltonian for (4) as

$$H_i = \frac{1}{2}e^{-\gamma_i(\tau-t)}\big(X_i^T C_{1i}^T Q_i C_{1i} X_i + u_i^T R_i u_i + 2u_i^T R_i M_i \omega_i + \omega_i^T M_i^T R_i M_i \omega_i\big) + \lambda_i^T(t)\left(T_i X_i + B_{1i}\left(u_i + M_i \sum_{j=1}^{N} l_{ij}\Gamma_i x_j\right)\right) \tag{9}$$

where $\lambda_i(t)$ is defined as an undetermined multiplier. The optimal control satisfies

$$\frac{\partial H_i}{\partial u_i} = e^{-\gamma_i(\tau-t)}(R_i u_i + R_i M_i w_i) + B_{1i}^T \lambda_i = 0 \tag{10}$$

which results in

$$u_i = -M_i \sum_{j=1}^{N} l_{ij}\Gamma_i x_j - e^{\gamma_i(\tau-t)} R_i^{-1} B_{1i}^T \lambda_i. \tag{11}$$

Since $[(\partial^2 H_i)/(\partial u_i^2)] = e^{-\gamma_i(\tau-t)} R_i > 0$, (11) is a minimum point of the Hamiltonian (9). Putting (11) into canonical equation, one can get

$$\frac{\partial H_i}{\partial \lambda_i} = \frac{\partial X_i}{\partial \tau} = T_i X_i - e^{\gamma_i(\tau-t)} B_{1i} R_i^{-1} B_{1i}^T \lambda_i \tag{12}$$

and

$$-\frac{\partial H_i}{\partial X_i} = \frac{\partial \lambda_i}{\partial \tau} = -e^{-\gamma_i(\tau-t)} C_{1i}^T Q_i C_{1i} - T_i^T \lambda_i. \tag{13}$$

Assume that X_i and λ_i satisfy the following linear equation:

$$\lambda_i(t) = e^{-\gamma_i(\tau-t)} P_i X_i \tag{14}$$

where P_i is obtained by solving the ARE (8). If (14) holds, (12) can be written as

$$\dot{X} = \Big(T_i - B_{1i} R_i^{-1} B_{1i}^T P_i\Big) X_i. \tag{15}$$

Differentiating (14) and using (15) yield (16)

$$\begin{aligned}\frac{\partial \lambda_i}{\partial \tau} &= -\gamma_i e^{-\gamma_i(\tau-t)} P_i X_i + e^{-\gamma_i(\tau-t)} P_i \dot{X}_i \\ &= e^{-\gamma_i(\tau-t)}\Big(P_i T_i - \gamma_i P_i - P_i B_{1i} R_i^{-1} B_{1i}^T P_i\Big) X_i.\end{aligned} \tag{16}$$

By substituting (14) into (13), the ARE with the discount factor γ_i is presented as

$$T_i^T P_i + P_i T_i - \gamma_i P_i + C_{1i}^T Q_i C_{1i} - P_i B_{1i} R_i^{-1} B_{1i}^T P_i = 0.$$

Using (11) and (14), it yields the control (7) as

$$u_i = -M_i \sum_{j=1}^{N} l_{ij}\Gamma_i x_j - R_i^{-1} B_{1i}^T P_i X_i.$$

Therefore, the necessity is established.

Next, the proof process of sufficiency for Theorem 1 will be given, which can be divided into two steps.

Step 1: It is necessary to prove that applying the control (7) to system (4) results in the following equation:

$$\lim_{t\to\infty} e^{-\gamma_i(t-t_0)} X_i^T(t) P_i X_i(t) = 0 \tag{17}$$

if the discount factor γ_i satisfies

$$\gamma_i \le 2\left\|\left(B_i R_i^{-1} B_i^T C_i^T Q_i C_i\right)^{\frac{1}{2}}\right\|. \tag{18}$$

According to [49], suppose that

$$P_i = \begin{bmatrix} P_{11}^i & P_{12}^i \\ P_{21}^i & P_{22}^i \end{bmatrix}. \tag{19}$$

The upper left-side of (8) can be written as

$$A_{1i}^T P_{11}^i + P_{11}^i A_{1i} - \gamma_i P_{11}^i + C_i^T Q_i C_i - P_{11}^i B_i R_i^{-1} B_i^T P_{11}^i = 0 \tag{20}$$

with $A_{1i} = A_i - \sum_{j=1}^{N} l_{ij}\Gamma_i$.

Consider the optimal control has the following form:

$$u_i = -M_i \sum_{j=1}^{N} l_{ij}\Gamma_i x_j - K_i X_i \tag{21}$$

where

$$\begin{aligned} K_i &= \begin{bmatrix} K_{1i} & K_{2i} \end{bmatrix} = R_i^{-1} B_{1i}^T P_i \\ K_{1i} &= R_i^{-1} B_i^T P_{11}^i. \end{aligned} \tag{22}$$

Substituting (22) into (4), the augmented dynamic can be rewritten as

$$\dot{X}_i = \begin{bmatrix} A_{ci} & -B_i K_{2i} \\ 0 & S \end{bmatrix} X_i = H_i X_i \tag{23}$$

with $A_{ci} = A_{1i} - B_i K_{1i}$ and $H_i = \begin{bmatrix} A_{ci} & -B_i K_{2i} \\ 0 & S \end{bmatrix}$. Furthermore, the ith node's dynamic can be presented as

$$\begin{aligned}\dot{x}_i &= A_{ci} x_i - B_i K_{2i} z_0 \\ &= \Big(A_{1i} - B_i R_i^{-1} B_i^T P_{11}^i\Big) x_i - B_i K_{2i} z_0.\end{aligned} \tag{24}$$

Based on (24), (20) becomes

$$\left(A_{1i}-B_iR_i^{-1}B_i^TP_{11}^i\right)^TP_{11}^i+P_{11}^i\left(A_{1i}-B_iR_i^{-1}B_i^TP_{11}^i\right) - \gamma_iP_{11}^i+C_i^TQ_iC_i+P_{11}^iB_iR_i^{-1}B_i^TP_{11}^i=0. \tag{25}$$

That is,

$$A_{ci}^TP_{11}^i+P_{11}^iA_{ci}-\gamma_iP_{11}^i+C_i^TQ_iC_i+P_{11}^iB_iR_i^{-1}B_i^TP_{11}^i=0. \tag{26}$$

According to [47], it can be proved that $A_{1i}-B_iK_{1i}$ is Hurwitz if the discount factor γ_i satisfies $\gamma_i \leq 2\|(B_iR_i^{-1}B_i^TC_i^TQ_iC_i)^{(1/2)}\|$. Similar to [49], suppose the Lyapunov function of system (4) as

$$V_i(t)={X_i}^TP_iX_i\geq 0. \tag{27}$$

Then, one can get the derivative of Lyapunov function (27) as

$$\begin{aligned}\dot{V}_i(t)&=\dot{X}_i^TP_iX_i+X_i^TP_i\dot{X}_i^T\\&=X_i^T\left(P_iH_i+H_i^TP_i\right)X_i.\end{aligned} \tag{28}$$

Since A_{ci} is Hurwitz and S has no eigenvalues with positive real parts, there exists a matrix $Q\geq 0$ that $\dot{V}_i(t)=-X_i^TQX_i\leq 0$. By using LaSalle's invariance principle, X_i will converge to $X_i^T(P_iH_i+H_i^TP_i)X_i=0$, the largest invariant subspace. That is to say, X_i converges to the largest invariant subspace $P_iX_i=0$ because all eigenvalues of H_i are on the left side of imaginary axis or the imaginary axis. So $\lim_{t\to\infty}e^{-\gamma_i(t-t_0)}X_i^T(t)P_iX_i(t)=0$.

Step 2: Consider the following equation:

$$\begin{aligned}&\lim_{t\to\infty}e^{-\gamma_i(t-t_0)}X_i^T(t)P_iX_i(t)-X_i^T(t_0)P_iX_i(t_0)\\&=\int_{t_0}^{\infty}\frac{d\left(e^{-\gamma_i(\tau-t_0)}X_i^TP_iX_i\right)}{d\tau}d\tau\\&=\int_{t_0}^{\infty}\Big(-\gamma_ie^{-\gamma_i(\tau-t_0)}X_i^TP_iX_i+e^{-\gamma_i(\tau-t_0)}\dot{X}_i^TP_iX_i\\&\qquad+e^{-\gamma_i(\tau-t_0)}X_i^TP_i\dot{X}_i\Big)d\tau\\&=\int_{t_0}^{\infty}e^{-\gamma_i(\tau-t_0)}\\&\quad\times\left(X_i^T\left(T_i^TP_i+P_iT_i-\gamma_iP_i\right)X_i+(u_i+M_i\omega_i)^TB_{1i}^TP_iX_i\right.\\&\qquad\left.+X_i^TP_iB_{1i}(u_i+M_i\omega_i)\right)d\tau.\end{aligned} \tag{29}$$

By using (8), (29) can be rewritten as

$$\begin{aligned}&\lim_{t\to\infty}e^{-\gamma_i(t-t_0)}X_i^T(t)P_iX_i(t)-X_i^T(t_0)P_iX_i(t_0)\\&=\int_{t_0}^{\infty}-e^{-\gamma_i(\tau-t_0)}\left(X_i^TC_{1i}^TQ_iC_{1i}X_i+{u_i}^TR_iu_i\right.\\&\qquad\left.+2u_i^TR_iM_i\omega_i+\omega_i^TM_i^TR_iM_i\omega_i\right)d\tau\\&\quad+\int_{t_0}^{\infty}e^{-\gamma_i(\tau-t_0)}\left(u_i-{u_i}^*\right)^TR_i\left(u_i-{u_i}^*\right)d\tau\end{aligned} \tag{30}$$

with ${u_i}^*=-M_i\sum_{j=1}^{N}l_{ij}\Gamma_ix_j-R_i^{-1}B_{1i}^TP_iX_i$. According to step 1, $\lim_{t\to\infty}e^{-\gamma_i(t-t_0)}X_i^T(t)P_iX_i(t)=0$, then the performance function becomes

$$J_i=\frac{1}{2}X_i^T(t_0)P_iX_i(t_0)+\frac{1}{2}\int_{t_0}^{\infty}e^{-\gamma_i(\tau-t_0)}\left(u_i-{u_i}^*\right)^TR_i\left(u_i-{u_i}^*\right)d\tau. \tag{31}$$

Since $X_i^T(t_0)P_iX_i(t_0)$ is a constant, $R_i>0$, and $e^{-\gamma_i(\tau-t_0)}>0$, the minimum value ${J_i}^*=(1/2)X_i^T(t_0)P_iX_i(t_0)$ of the performance function (31) can be obtained if and only if $u_i={u_i}^*$. This completes the proof. ■

B. Asymptotic Stability Analysis of Output Synchronization Errors

By applying the optimal control (7) in system (2), the output synchronization errors (5) will achieve asymptotic stability. Next, motivated by [49], the asymptotic stability of output synchronization errors is proved in Theorem 2.

Theorem 2: Applying the control protocol (7) to the CDNs (2), the output synchronization error $e_i=y_i-y_0$ will converge to zero asymptotically if the discount factor satisfies

$$\gamma_i\leq 2\left\|\left(B_iR_i^{-1}B_i^TC_i^TQ_iC_i\right)^{\frac{1}{2}}\right\|.$$

Proof: If both left- and right-hand sides of ARE (8) are, respectively, multiplied by X_i^T and X_i, one has

$$\begin{aligned}&X_i^TT_i^TP_iX_i+(P_iX_i)^TT_iX_i-\gamma_iX_i^TP_iX_i\\&\quad+(C_{1i}Xi)^TQ_iC_{1i}X_i-(P_iX_i)^TB_{1i}R^{-1}B_{1i}^TP_iX_i=0.\end{aligned} \tag{32}$$

Equation (32) indicates that if $P_iX_i=0$ then $C_{1i}X_i=0$, for the reason of $Q_i>0$, which leads to $e_i=y_i-y_0=0$. Step 1 has proven that X_i converges to the largest invariant subspace $P_iX_i=0$ if the discount factor satisfies $\gamma_i\leq 2\|(B_iR_i^{-1}B_i^TC_i^TQ_iC_i)^{(1/2)}\|$. Thus, the output synchronization error e_i is stable asymptotically. ■

Remark 3: The weight matrix Q_i and the discount factor γ_i are related to the convergence of the output synchronization error e_i. Specifically, the more larger Q_i is, the faster the output synchronization error e_i decreases to zero. Also, the smaller γ_i is, the faster the output synchronization error e_i decreases.

Remark 4: This paper is concerned with the optimal output synchronization control of the CDNs with heterogeneous dynamics. One of the difficulties is how to establish a necessary and sufficient condition of the optimal control minimizing a quadratic performance index for system (2). Another one is how to guarantee the convergence of the output synchronization errors. To overcome such difficulties, by defining the new quadratic performance index (6), the ARE (8) is obtained to solve the optimal controller gain K_i in (7). Besides, based on the ARE, the necessary and sufficient condition of the optimal control (7) is established in Theorem 1. Particularly, due to the existence of the discount factor in the ARE, the adaptive online iterative algorithm in [44] is no longer applicable in this paper. A novel online iteration algorithm by iteratively solving the ARE will be proposed in the next section to compute the optimal controller gain K_i.

IV. Optimal Control Design With Partially Unknown System Dynamics

When all system dynamics are accurately known, the feedback gain K_i is available by directly solving the ARE (8). In fact, (8) can be modified as

$$\left(T_i - \frac{1}{2}\gamma_i I\right)^T P_i + P_i\left(T_i - \frac{1}{2}\gamma_i I\right) + C_{1i}^T Q_i C_{1i} - P_i B_{1i} R_i^{-1} B_{1i}^T P_i = 0. \tag{33}$$

Based on [46], one can get that $(T_i - (1/2)\gamma_i I)$ is stable if γ_i satisfies (18). Thus, the form of (33) is the same as the classical ARE in [48]. That is, the optimal feedback gain K_i^* is determined by the unique solution P_i^* of the ARE (33) as

$$K_i^* = R_i^{-1} B_{1i}^T P_i^*. \tag{34}$$

However, when matrices A_i and S are unavailable, the feedback gain K_i cannot be obtained directly by solving (8). To overcome this difficulty, a new policy to iteratively solve the ARE (8) relying on states, outputs, and couplings is presented as follows.

First, suppose that the γ_i based on (18) and K_i^0 which stabilize system (4) have been known. According to the idea of an iterative algorithm, for each $k \in \mathbb{Z}_+$ which means the number of iteration, a symmetric positive matrix P_i^k can be solved from (36), then K_i^{k+1} is obtained by using

$$K_i^{k+1} = R_i^{-1} B_{1i}^T P_i^k. \tag{35}$$

Now, rewrite the augmented system as

$$\dot{X}_i = \bar{T}_i^k X_i + B_{1i}\left(K_i^k X_i + u_i + M_i w_i\right) \tag{36}$$

with $\bar{T}_i^k = T_i - B_{1i}K_i^k$. Based on (36), it is easy to obtain that

$$\left(\bar{T}_i^k\right)^T P_i^k + P_i^k \bar{T}_i^k + C_{1i}^T Q_i C_{1i} - \gamma_i P_i + {K_i^k}^T R_i K_i^k = 0. \tag{37}$$

Consider the following equation:

$$\begin{aligned}
&e^{-\gamma_i \delta t} X_i^T(t+\delta t) P_i^k X_i(t+\delta t) - X_i^T(t) P_i^k X_i(t)\\
&= \int_t^{t+\delta t} \frac{d\left(e^{-\gamma_i(\tau-t)} X_i^T P_i^k X_i\right)}{d\tau} d\tau\\
&= \int_t^{t+\delta t} e^{-\gamma_i(\tau-t)}\bigg(X_i^T\left(\bar{T}_i^k\right)^T P_i^k X_i + X_i^T P_i^k \bar{T}_i^K X_i - \gamma_i X_i^T P_i^k X_i\\
&\qquad + 2\left(u_i + K_i^k X_i + M_i w_i\right)^T B_{1i}^T P_i^k X_i\bigg) d\tau\\
&= \int_t^{t+\delta t} -e^{-\gamma_i(\tau-t)} X_i^T Q_i^k X_i d\tau\\
&\quad + 2\int_t^{t+\delta t} e^{-\gamma_i(\tau-t)}\left(u_i + K_i^k X_i + M_i w_i\right)^T R_i K_i^{k+1} X_i d\tau
\end{aligned} \tag{38}$$

where $Q_i^k = C_{1i}^T Q_i C_{1i} + {K_i^k}^T R_i K_i^k$.

Remark 5: The discount factor and the augmented state in (38) make it different to [14]. One can use $u_i = -M_i \sum_{j=1}^N l_{ij}\Gamma_i x_j - K_i X_i + e$, with e the exploration noise as the input signal for learning.

For a matrix $P = [p_{ij}]_{n\times n}$ and a vector $x = [x_1, x_2, \ldots, x_n]^T$, two operators are defined as

$$\hat{P} = \left[p_{11}, 2p_{12}, \ldots, p_{22}, 2p_{23}, \ldots, 2p_{n-1,n}, p_{nn}\right]^T$$
$$\hat{x} = \left[x_1^2, x_1x_2, \ldots, x_1x_n, x_2^2, x_2x_3, \ldots, x_{n-1}x_n, x_n^2\right]^T.$$

In addition, Kronecker product is introduced as

$$e^{-\gamma_i(\tau-t)} X_i^T Q_i^k X_i = e^{-\gamma_i(\tau-t)} X_i^T \otimes X_i^T \operatorname{vec}\left(Q_i^k\right)$$

and

$$\begin{aligned}
&e^{-\gamma_i(\tau-t)}\left(u_i + K_i^k X_i + M_i w_i\right)^T R_i K_i^{k+1} X_i\\
&= e^{-\gamma_i(\tau-t)}\left(X_i^T \otimes X_i^T\right)\left(I_{nq} \otimes \left(K_i^k\right)^T R_i\right)\operatorname{vec}\left(K_i^{k+1}\right)\\
&\quad + e^{-\gamma_i(\tau-t)}\left(X_i^T \otimes u_i^T\right)\left(I_{nq} \otimes R_i\right)\operatorname{vec}\left(K_i^{k+1}\right)\\
&\quad + e^{-\gamma_i(\tau-t)}\left(X_i^T \otimes w_i^T M_i^T\right)\left(I_{nq} \otimes R_i\right)\operatorname{vec}\left(K_i^{k+1}\right)
\end{aligned}$$

with $I_{nq} \in \mathbb{R}^{(n+q)^2}$. Furthermore, for $l \in \mathbb{Z}_+$, define matrices $\delta_{ixx} \in \mathbb{R}^{l\times(1/2)(n+q)(n+q+1)}$, $I_{ixx} \in \mathbb{R}^{l\times(n+q)^2}$, $I_{ixu} \in \mathbb{R}^{l\times n(n+q)}$, $I_{ixw} \in \mathbb{R}^{l\times n(n+q)}$, so that

$$\delta_{ixx} = \left[e^{-\gamma_i\Delta T}\hat{X}_i(t_1) - \hat{X}_i(t_0), e^{-\gamma_i\Delta T}\hat{X}_i(t_2) - \hat{X}_i(t_1)\right.$$
$$\left.\cdots, e^{-\gamma_i\Delta T}\hat{X}_i(t_l) - \hat{X}_i(t_{l-1})\right]^T \tag{39}$$

$$I_{ixx} = \left[e^{\gamma_i t_0}\int_{t_0}^{t_1} e^{-\gamma_i\tau} X_i \otimes X_i d\tau,\ e^{\gamma_i t_1}\int_{t_1}^{t_2} e^{-\gamma_i\tau} X_i \otimes X_i d\tau\right.$$
$$\left.\cdots, e^{\gamma_i t_{l-1}}\int_{t_{l-1}}^{t_l} e^{-\gamma_i\tau} X_i \otimes X_i d\tau\right]^T \tag{40}$$

$$I_{ixu} = \left[e^{\gamma_i t_0}\int_{t_0}^{t_1} e^{-\gamma_i\tau} X_i \otimes u_i d\tau,\ e^{\gamma_i t_1}\int_{t_1}^{t_2} e^{-\gamma_i\tau} X_i \otimes u_i d\tau\right.$$
$$\left.\cdots, e^{\gamma_i t_{l-1}}\int_{t_{l-1}}^{t_l} e^{-\gamma_i\tau} X_i \otimes u_i d\tau\right]^T \tag{41}$$

$$I_{ixw} = \left[e^{\gamma_i t_0}\int_{t_0}^{t_1} e^{-\gamma_i\tau}(X_i \otimes M_i w_i) d\tau\right.$$
$$e^{\gamma_i t_1}\int_{t_1}^{t_2} e^{-\gamma_i\tau}(X_i \otimes M_i w_i) d\tau$$
$$\left.\cdots, e^{\gamma_i t_{l-1}}\int_{t_{l-1}}^{t_l} e^{-\gamma_i\tau}(X_i \otimes M_i w_i) d\tau\right]^T \tag{42}$$

where $0 \le t_0 \le t_1 \le \cdots \le t_l$, $\Delta T = t_l - t_0 = t_2 - t_1 = \cdots = t_l - t_{l-1}$. For a known gain matrix K_i^k which is given in the $(k-1)$th step, (38) can be written as the following equation:

$$\Theta_i^k \begin{bmatrix} \hat{P}_i^k \\ \operatorname{vec}\left(K_i^{k+1}\right)\end{bmatrix} = \Xi_i^k \tag{43}$$

where $\Theta_i^k \in \mathbb{R}^{l\times[(1/2)(n+q)(n+q+1)+n(n+q)]}$ and $\Xi_i^k \in \mathbb{R}^l$ are presented as

$$\Theta_i^k = \left[\delta_{ixx}, -2I_{ixx}\left(I_{nq} \otimes \left(K_i^k\right)^T R_i\right) - 2I_{ixu}\left(I_{nq} \otimes R_i\right)\right.$$
$$\left.-\ 2I_{ixw}\left(I_{nq} \otimes R_i\right)\right]$$
$$\Xi_i^k = -I_{ixx}\operatorname{vec}\left(Q_i^k\right).$$

Algorithm 1 Adaptive Optimal Control Algorithm

1. Base on (18), choose γ_i for every node. Apply $u_i = -M_i w_i - K_i^0 X_i + e$ to the ith node over time $[t_0, t_l]$, where the feedback gain matrix K_i^0 stabilizes system (4) and e is the exploration noise. Change $[t_0, t_l]$ until the rank condition in (45) is satisfied. Let $k = 0$.
2. Solve P_i^k and K_i^{k+1} from (44).
3. Let $k = k+1$, and repeat Step 2 until $\|P_i^k - P_i^{k-1}\| \le \varepsilon$ for $k \ge 1$, where ε is a small positive constant as threshold.
4. $u_i = -M_i \sum_{j=1}^{N} l_{ij}\Gamma_i x_j - K_i^k X_i$ is regarded as the optimal control.

If Θ_i^k has full column rank, various $\hat{P}_i$ and K_i^{k+1} can be easily solved by

$$\begin{bmatrix} \hat{P}_i^k \\ \mathrm{vec}\left(K_i^{k+1}\right) \end{bmatrix} = \left(\Theta_i^{k^T}\Theta_i^k\right)^{-1}\left(\Theta_i^k\right)^T \Xi_i^k. \tag{44}$$

Similar to [44], the rank condition of Θ_i^k is estimated by Lemma 1.

Lemma 1: If there is an integer $l_0 > 0$, for all $l \ge l_0$

$$\mathrm{rank}([I_{ixx}, (I_{ixu} + I_{ixw})]) = \frac{(n+q)(n+q+1)}{2} + n(n+q). \tag{45}$$

Then, Θ_i^k has full column rank for all $k \in \mathbb{Z}_+$.

To compute the optimal control law only by using the information of system input and state, the ARE (37) with the controller gain (35) is rewritten as (43) by using (38) and the two operators $\hat{P}$ and $\hat{x}$. Furthermore, it is shown that (43) is equivalent to (44) if the condition in Lemma 1 holds. The controller gain is then derived by solving (44) directly. On the other hand, with the condition in Lemma 1 holding, the solution of (44) is unique. Therefore, the solution of (44) still satisfies the ARE (37). According to [44, Th. 1], given initial K_i^0 and the discount factor γ_i satisfying (18), and let the symmetric positive definite matrix P_i^k be the solution of (37) with the control gain K_i^{k+1} satisfying (35) ($k = 1, 2, \ldots, $). Then the sequence of P_i^k and K_i^{k+1} are convergent (that is, $\lim_{k\to\infty} K_i^k = K_i^*$, $\lim_{k\to\infty} P_i^{k-1} = P_i^*$), which ensures the constringency of the proposed Algorithm 1.

Remark 6: Note that, for the CDNs (4) in this paper, the couplings $\sum_{j=1}^{N} l_{ij}\Gamma_i(x_j - x_i)$ are involved in node dynamic equations and they need to be compensated to realize output synchronization. However, for the multiagent system in [49], the couplings were not contained in node dynamics and the compensation problem of couplings was not considered. So, the control method in [49] is infeasible in the CDNs (4) to achieve the output synchronization.

V. Simulation Example

In this section, two CDN examples are taken to demonstrate the effectiveness of above method.

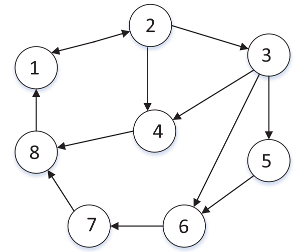

Fig. 1. Graph structure for the CDNs.

Example 1: Consider CDNs with eight nodes, which are coupled through the topology as

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Besides, the correlative graph is shown in Fig. 1. Assume the system dynamics of leader node described as the following equations:

$$\dot{z}_0(t) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} z_0, \;\; y_0 = \begin{bmatrix} 1 & 1 \end{bmatrix} z_0.$$

The system dynamics of eight nodes are shown as

$$A_1 = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \; B_1 = 0.8I, \; C_1 = \begin{bmatrix} 2 & 3 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B_2 = 0.8I, \; C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}, \; B_3 = 0.8I, \; C_3 = \begin{bmatrix} 1 & 1.1 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \; B_4 = 0.8I, \; C_4 = \begin{bmatrix} 2 & 3 \end{bmatrix}$$
$$A_5 = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}, \; B_5 = 0.8I, \; C_5 = \begin{bmatrix} 2 & 1.6 \end{bmatrix}$$
$$A_6 = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}, \; B_6 = 0.8I, \; C_6 = \begin{bmatrix} 1.5 & 1 \end{bmatrix}$$
$$A_7 = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \; B_7 = 0.8I, \; C_7 = \begin{bmatrix} 1.5 & 2 \end{bmatrix}$$
$$A_8 = \begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix}, \; B_8 = 0.8I, \; C_8 = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

and $\Gamma_i = I, i = 1, 2, \ldots, 8$.

First, the optimal solution can be obtained by solving the ARE (33) directly

$$P_1^* = \begin{bmatrix} 32.4104 & 45.3208 & -15.2470 & -16.8800 \\ 45.3208 & 65.0091 & -22.0925 & -23.0058 \\ -15.2470 & -22.0925 & 8.2908 & 7.6399 \\ -16.8800 & -23.0058 & 7.6399 & 9.4715 \end{bmatrix}$$

$$P_2^* = \begin{bmatrix} 21.3835 & 19.8212 & -17.9939 & -21.4520 \\ 19.8212 & 18.6616 & -17.5886 & -20.0444 \\ -17.9939 & -17.5886 & 19.4449 & 18.8365 \\ -21.4520 & -20.0444 & 18.8365 & 23.6882 \end{bmatrix}$$

$$P_3^* = \begin{bmatrix} 16.1576 & 19.2764 & -16.5889 & -17.6458 \\ 19.2764 & 24.8686 & -19.1828 & -23.3188 \\ -16.5889 & -19.1828 & 18.3241 & 17.5114 \\ -17.6458 & -23.3188 & 17.5114 & 22.7537 \end{bmatrix}$$

$$P_4^* = \begin{bmatrix} 32.0817 & 45.6075 & -15.1720 & -16.5985 \\ 45.6075 & 65.8332 & -22.1483 & -23.2244 \\ -15.1720 & -22.1483 & 8.1097 & 7.6549 \\ -16.5985 & -23.2244 & 7.6549 & 9.0941 \end{bmatrix}$$

$$P_5^* = \begin{bmatrix} 46.1527 & 32.3371 & -21.0417 & -23.8844 \\ 32.3371 & 24.8279 & -16.0130 & -15.9842 \\ -21.0417 & -16.0130 & 10.9230 & 10.4020 \\ -23.8844 & -15.9842 & 10.4020 & 12.9943 \end{bmatrix}$$

$$P_6^* = \begin{bmatrix} 33.0908 & 22.1604 & -21.3809 & -24.1307 \\ 22.1604 & 14.8417 & -14.2905 & -16.1919 \\ -21.3809 & -14.2905 & 16.4719 & 15.0638 \\ -24.1307 & -16.1919 & 15.0638 & 19.7949 \end{bmatrix}$$

$$P_7^* = \begin{bmatrix} 21.8950 & 29.6732 & -14.9965 & -14.1850 \\ 29.6732 & 47.5664 & -21.7248 & -24.9788 \\ -14.9965 & -21.7248 & 11.4751 & 10.6338 \\ -14.1850 & -24.9788 & 10.6338 & 14.4355 \end{bmatrix}$$

$$P_8^* = \begin{bmatrix} 34.5796 & 37.0600 & -17.9690 & -18.2956 \\ 37.0600 & 42.6071 & -19.5520 & -22.0467 \\ -17.9690 & -19.5520 & 10.6435 & 9.5125 \\ -18.2956 & -22.0467 & 9.5125 & 12.7480 \end{bmatrix} \tag{46}$$

and

$$K_1^* = \begin{bmatrix} 25.9283 & 36.2566 & -12.1976 & -13.5040 \\ 36.2566 & 52.0073 & -17.6740 & -18.4046 \end{bmatrix}$$

$$K_2^* = \begin{bmatrix} 17.1068 & 15.8569 & -14.3951 & -17.1616 \\ 15.8569 & 14.9293 & -14.0709 & -16.0355 \end{bmatrix}$$

$$K_3^* = \begin{bmatrix} 12.9261 & 15.4211 & -13.2711 & -14.1167 \\ 15.4211 & 19.8949 & -15.3462 & -18.6551 \end{bmatrix}$$

$$K_4^* = \begin{bmatrix} 25.6653 & 36.4860 & -12.1376 & -13.2788 \\ 36.4860 & 52.6666 & -17.7186 & -18.5795 \end{bmatrix}$$

$$K_5^* = \begin{bmatrix} 36.9222 & 25.8697 & -16.8334 & -19.1075 \\ 25.8697 & 19.8623 & -12.8104 & -12.7874 \end{bmatrix}$$

$$K_6^* = \begin{bmatrix} 26.4727 & 17.7283 & -17.1047 & -19.3046 \\ 17.7283 & 11.8734 & -11.4324 & -12.9536 \end{bmatrix}$$

$$K_7^* = \begin{bmatrix} 17.5160 & 23.7386 & -11.9972 & -11.3480 \\ 23.7386 & 38.0531 & -17.3798 & -19.9831 \end{bmatrix}$$

$$K_8^* = \begin{bmatrix} 27.6637 & 29.6480 & -14.3752 & -14.6364 \\ 29.6480 & 34.0857 & -15.6416 & -17.6374 \end{bmatrix}. \tag{47}$$

Then, the proposed Algorithm 1 is applied to the CDNs Example 1 to verify its effectiveness.

The initial state of leader is chosen as $z_0(0) = [5, 5]$, and suppose that eight followers have initial states $X_i(0) = [1, 1]$, with $i = 1, 2, \ldots, 8$, respectively. It is assumed that the weighting matrices Q_i, R_i and the discount factor γ_i are as

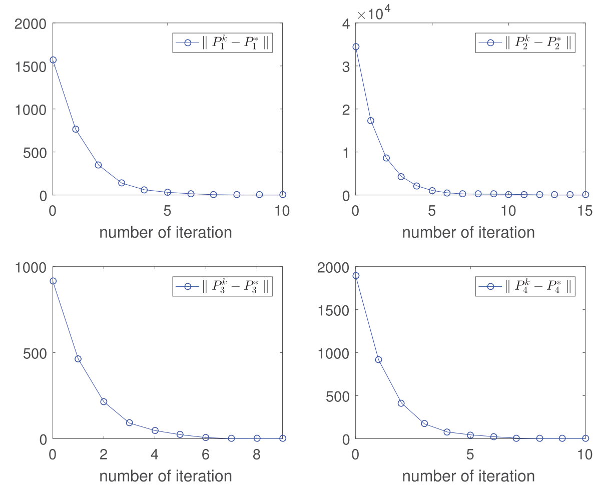

Fig. 2. Convergence of P_i^k to the optimal values P_i^* $(i = 1, \ldots, 4)$.

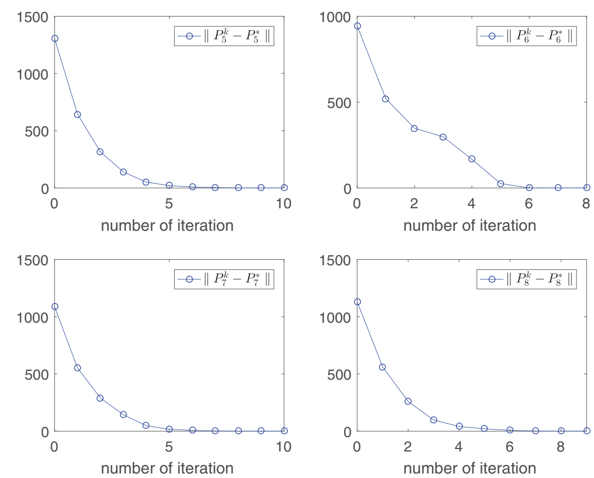

Fig. 3. Convergence of P_i^k to the optimal values P_i^* $(i = 5, \ldots, 8)$.

$Q_i = 500$, $R_i = I$, and $\gamma_i = 1$, with $i = 1, 2, \ldots, 8$. Since $\gamma_i = 1$ satisfies (18) which ensures the stability of system (4), the initial feedback gain can be chosen as $K_0 = 0$. The exploration noise is the sum of sinusoidal signals with different frequencies from $t = 0$ s to $t = 4$ s as

$$e = 5 \sum_{i=1}^{100} \sin(\omega_i t) \tag{48}$$

where ω_i, with $i = 1, \ldots, 50$, is randomly selected from $[-50, 50]$.

Choose internal $\Delta T = 0.01$ s, and collect the information of states, inputs, and couplings. The iteration policies of eight nodes are convergent after 10, 15, 9, 10, 10, 8, 10, and 9 iterations, respectively. It is shown in Figs. 2–5 that P_i and K_i matrices, respectively, converge to their optimal values when the stopping criterions $\|P_i^k - P_i^{k-1}\| \leq 0.05$ $(i = 1, 2, \ldots, 8)$ are satisfied. In fact, the simulation results indicate that the solutions P_i and K_i solved by solving Algorithm 1, are the same as P_i^* and K_i^* in (46) and (47), respectively $(i = 1, 2, \ldots, 8)$.

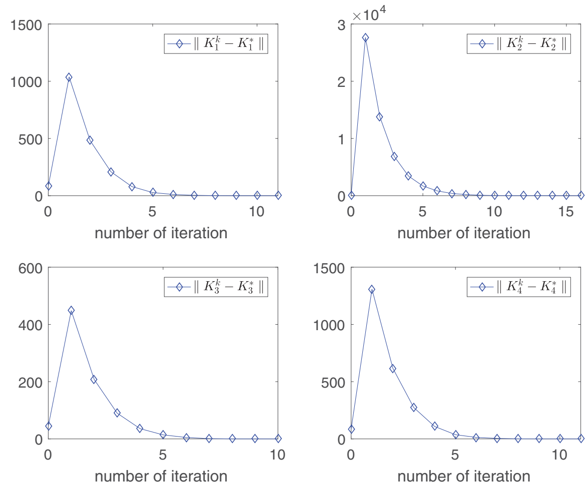

Fig. 4. Convergence of K_i^k to the optimal values K_i^* $(i = 1, \ldots, 4)$.

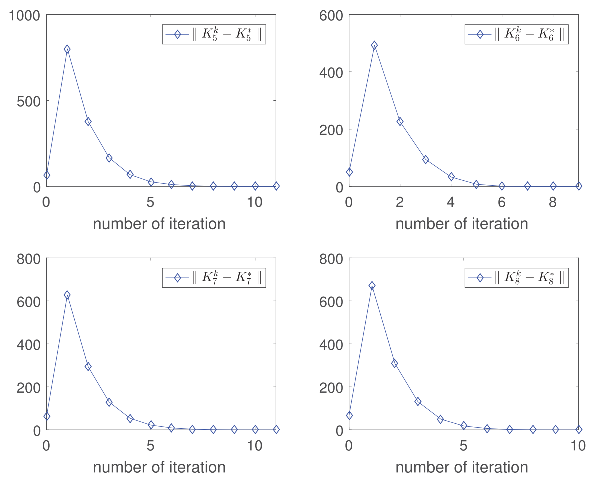

Fig. 5. Convergence of K_i^k to the optimal values K_i^* $(i = 5, \ldots, 8)$.

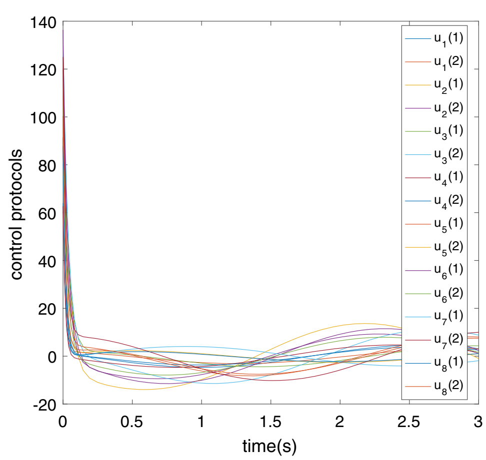

Fig. 6. Control protocols of nodes 1–8.

In addition, the inputs of nodes 1–8 are shown in Fig. 6. The output trajectories of leader and eight followers are given in Fig. 7. Also the output synchronization errors are plotted in Fig. 8.

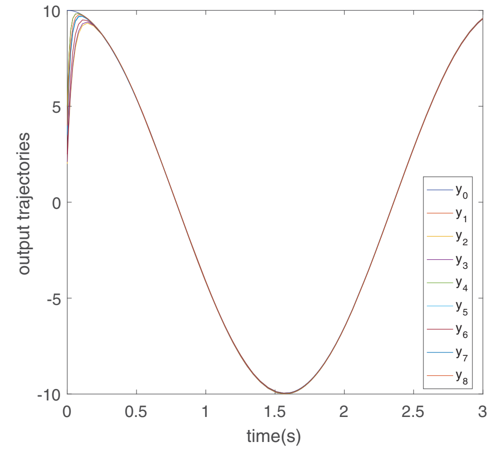

Fig. 7. Output trajectories of nodes 1–8.

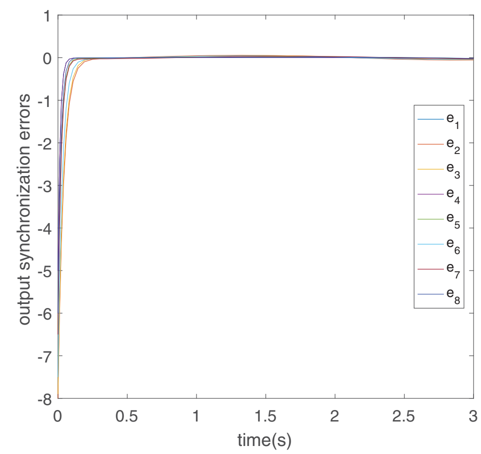

Fig. 8. Synchronization errors of nodes 1–8.

As shown in Figs. 6–8, the inputs and outputs are all bounded and the output synchronization errors converge to zero asymptotically, which confirm the effectiveness of the proposed optimal output synchronization control method.

Example 2: Consider a spring-connected multivehicle system in [50], which is composed of an isolated master vehicle, four slave vehicles, and springs considered as the interconnection between two vehicles. Assume that the system dynamics of master vehicle are described as

$$\begin{aligned} \dot{\mathbf{x}}_0 &= \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_0. \end{aligned} \tag{49}$$

The slave vehicle dynamics can be converted as

$$\begin{aligned} \dot{\mathbf{x}}_i &= A_i \mathbf{x}_i + \Sigma_{j=1}^N l_{ij} \Gamma_i \big(\mathbf{x}_j - \mathbf{x}_i\big) + u_i \\ \mathbf{y}_i &= C_i \mathbf{x}_i \end{aligned} \tag{50}$$

where $\mathbf{x}_i = [x_i, \dot{x}_i]^T$, and x_i is the position of ith vehicle from its equilibrium position $(i = 1, 2, 3, 4)$. Besides

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \Gamma_i = \begin{bmatrix} 0 & 0 \\ \frac{k}{m_i} & 0 \end{bmatrix}, C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

are all borrowed from [50] with m_i representing the weight of ith vehicle and k representing the stiffness coefficient of spring.

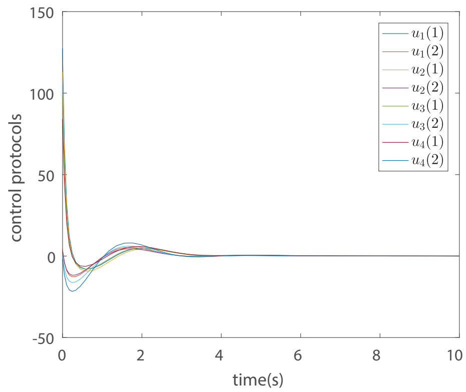

Fig. 9. Control protocols of vehicles 1–4.

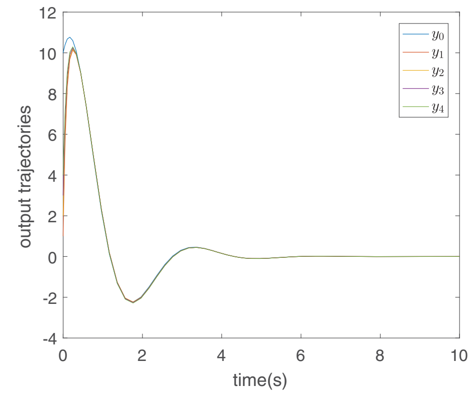

Fig. 10. Output trajectories of vehicles 1–4.

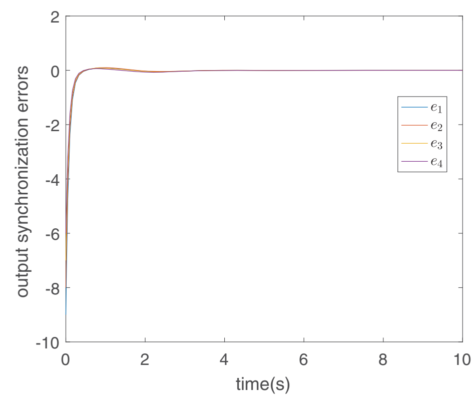

Fig. 11. Synchronization errors of vehicles 1–4.

Here, choose $m_1 = 1, m_2 = 2.5, m_3 = 1.5, m_4 = 0.5$, and $k = 1$. Choose the discount factor $\gamma_i = 1$ satisfying (18), and the optimal solution can be obtained by solving the ARE (8) directly

$$P_1^* = \begin{bmatrix} 13.5646 & 0.8539 & -12.5998 & -1.5718 \\ 0.8539 & 0.6084 & -0.4019 & -0.3962 \\ -12.5998 & -0.4019 & 12.2261 & 1.1012 \\ -1.5718 & -0.3962 & 1.1012 & 0.5923 \end{bmatrix}$$
$$P_2^* = \begin{bmatrix} 13.5758 & 0.8619 & -12.5974 & -1.5776 \\ 0.8619 & 0.6095 & -0.3797 & -0.3745 \\ -12.5974 & -0.3797 & 12.2523 & 1.1014 \\ -1.5776 & -0.3745 & 1.1014 & 0.5922 \end{bmatrix}$$
$$P_3^* = \begin{bmatrix} 13.5464 & 0.8406 & -12.6025 & -1.5608 \\ 0.8406 & 0.6066 & -0.4390 & -0.4323 \\ -12.6025 & -0.4390 & 12.1875 & 1.1053 \\ -1.5608 & -0.4323 & 1.1053 & 0.5994 \end{bmatrix}$$
$$P_4^* = \begin{bmatrix} 13.5118 & 0.8142 & -12.6027 & -1.5336 \\ 0.8142 & 0.6025 & -0.5126 & -0.5034 \\ -12.6027 & -0.5126 & 12.1286 & 1.1309 \\ -1.5336 & -0.5034 & 1.1309 & 0.6395 \end{bmatrix} \tag{51}$$

and

$$K_1^* = \begin{bmatrix} 13.5646 & 0.8539 & -12.5998 & -1.5718 \\ 0.8539 & 0.6084 & -0.4019 & -0.3962 \end{bmatrix}$$
$$K_2^* = \begin{bmatrix} 13.5758 & 0.8619 & -12.5974 & -1.5776 \\ 0.8619 & 0.6095 & -0.3797 & -0.3745 \end{bmatrix}$$
$$K_3^* = \begin{bmatrix} 13.5464 & 0.8406 & -12.6025 & -1.5608 \\ 0.8406 & 0.6066 & -0.4390 & -0.4323 \end{bmatrix}$$
$$K_4^* = \begin{bmatrix} 13.5118 & 0.8142 & -12.6027 & -1.5336 \\ 0.8142 & 0.6025 & -0.5126 & -0.5034 \end{bmatrix}. \tag{52}$$

Suppose that the initial state of leader is $\mathbf{x}_0(0) = [10, 10]$, and four followers have initial states $\mathbf{x}_1(0) = [1, 1]$, $\mathbf{x}_2(0) = [2, 2]$, $\mathbf{x}_3(0) = [3, 3]$, and $\mathbf{x}_4(0) = [4, 4]$. Then assume the weighting matrices $Q_i = 200$ and $R_i = I$. Choose the initial feedback gains $K_i^0 = 0$, the same exploration noise e as (48), and the internal $\Delta T = 0.01$ s. The matrices P_i and K_i all converge to their optimal values after ten iterations when the stopping criterions $\|P_i^k - P_i^{k-1}\| \leq 0.05$ $(i = 1, 2, \ldots, 8)$ are satisfied. Also, the simulation results indicate that the solutions P_i and K_i obtained by solving Algorithm 1, are the same as P_i^* and K_i^* in (51) and (52), respectively $(i = 1, 2, 3, 4)$.

In addition, the inputs of four vehicles are shown in Fig. 9. The output trajectories of leader and four vehicles are given in Fig. 10. Also the output synchronization errors are plotted in Fig. 11.

The results of Figs. 9–11 have shown that the output synchronization errors are asymptotically stable, which verify the effectiveness of the proposed control method again.

VI. Conclusion

In this paper, the optimal output synchronization control problem of the heterogeneous CDNs with partially unknown system dynamics has been proposed. An optimal controller consisting of a feedback control and a compensated feedforward control can minimize a new quadratic performance index including the information of state couplings. A necessary and sufficient condition for the optimal control minimizing the performance index has been established. Specially, when the CNDs are with partially unknown dynamics, an online iteration algorithm to iteratively compute an ARE with a discount factor by using the information of states, inputs, and couplings has been proposed to obtain the optimal controller gain. Besides, the output synchronization errors converge to zero asymptotically by applying the optimal control laws to the CDNs. Finally, two simulation examples have confirmed the effectiveness of the proposed optimal output synchronization control method. It should be pointed out that this paper is mainly concerned about the CDNs with linear state couplings. While, the method cannot be applied to the CDNs with nonlinear state couplings. Our future research will consider the optimal synchronization control of CDNs with nonlinear couplings and completely unknown dynamics.

REFERENCES

[1] X. F. Wang and G. R. Chen, "Complex networks: Small-world, scale-free and beyond," *IEEE Trans. Circuits Syst. Mag.*, vol. 3, no. 1, pp. 6–20, 1st Quart., 2003.

[2] P. Yi and Y. G. Hong, "Quantized subgradient algorithm and data-rate analysis for distributed optimization," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 4, pp. 380–392, Dec. 2014.

[3] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, "Complex networks: Structure and dynamics," *Phys. Rep.*, vol. 424, nos. 4–5, pp. 175–308, 2006.

[4] Z. Li, W. Ren, X. Liu, and M. Fu, "Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols," *IEEE Trans. Autom. Control*, vol. 58, no. 7, pp. 1786–1791, Jul. 2013.

[5] H. Ma *et al.*, "Neural-network-based distributed adaptive robust control for a class of nonlinear multiagent systems with time delays and external noises," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 46, no. 6, pp. 750–758, Jun. 2016.

[6] X. Li and G. Chen, "Synchronization and desynchronization of complex dynamical networks: An engineering viewpoint," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 11, pp. 1381–1390, Nov. 2003.

[7] J. Xiang and G. Chen, "On the V-stability of complex dynamical networks," *Automatica*, vol. 43, no. 6, pp. 1049–1057, 2007.

[8] X.-J. Li and G.-H. Yang, "Fuzzy approximation-based global pinning synchronization control of uncertain complex dynamical networks," *IEEE Trans. Cybern.*, vol. 47, no. 4, pp. 873–883, Apr. 2017.

[9] F. L. Lewis, H. Zhang, K. Hengster-Movric, and A. Das, *Cooperative Control of Multi-Agent Systems: Optimal and Adaptive Design Approaches*. London, U.K.: Springer, 2014.

[10] H. Zhang, C. Qin, and Y. Luo, "Neural-network-based constrained optimal control scheme for discrete-time switched nonlinear system using dual heuristic programming," *IEEE Trans. Autom. Sci. Eng.*, vol. 11, no. 3, pp. 839–849, Jul. 2014.

[11] X. Zhong, H. He, H. Zhang, and Z. Wang, "Optimal control for unknown discrete-time nonlinear Markov jump systems using adaptive dynamic programming," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 12, pp. 2141–2155, Dec. 2014.

[12] Z.-Y. Jiang, M.-G. Liang, Q. Li, and D. C. Guo, "Optimal dynamic bandwidth allocation for complex networks," *Physica A Stat. Mech. Appl.* vol. 392, no. 5, pp. 1256–1262, 2013.

[13] H. Zhang, L. Cui, X. Zhang, and Y. Luo, "Data-driven robust approximate optimal tracking control for unknown general nonlinear systems using adaptive dynamic programming method," *IEEE Trans. Neural Netw.*, vol. 22, no. 12, pp. 2226–2236, Dec. 2011.

[14] Y. W. Cao, G. H. Yang, and X. J. Li, "Optimal synchronization controller design for complex dynamical networks with unknown system dynamics," *J. Frankl. Inst.*.

[15] H. Li, Y. Wang, and L. Xie, "Output tracking control of Boolean control networks via state feedback: Constant reference signal case," *Automatica*, vol. 59, pp. 54–59, Sep. 2015.

[16] Y. Li, S. Tong, and T. Li, "Adaptive fuzzy output feedback dynamic surface control of interconnected nonlinear pure-feedback systems," *IEEE Trans. Cybern.*, vol. 45, no. 1, pp. 138–149, Jan. 2015.

[17] Z.-H. Pang, G.-P. Liu, D. Zhou, and M. Chen, "Output tracking control for networked systems: A model-based prediction approach," *IEEE Trans. Ind. Electron.*, vol. 61, no. 9, pp. 4867–4877, Sep. 2014.

[18] C.-X. Shi, G.-H. Yang, and X.-J. Li, "Data-based fault-tolerant consensus control for uncertain multiagent systems via weighted edge dynamics," *IEEE Trans. Syst., Man, Cybern., Syst.*, to be published, doi: 10.1109/TSMC.2017.2743261.

[19] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.

[20] Y.-X. Li, G.-H. Yang, and S. Tong, "Fuzzy adaptive distributed event-triggered consensus control of uncertain nonlinear multiagent systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, to be published, doi: 10.1109/TSMC.2018.2812216.

[21] T. H. Lee, Z.-G. Wu, and J. H. Park, "Synchronization of a complex dynamical network with coupling time-varying delays via sampled-data control," *Appl. Math. Comput.*, vol. 219, no. 3, pp. 1354–1366, 2012.

[22] H. Zhang, T. Feng, G.-H. Yang, and H. Liang, "Distributed cooperative optimal control for multiagent systems on directed graphs: An inverse optimal approach," *IEEE Trans. Cybern.*, vol. 45, no. 7, pp. 1315–1326, Jul. 2015.

[23] M. Xu, J.-L. Wang, Y.-L. Huang, P.-C. Wei, and S.-X. Wang, "Pinning synchronization of complex dynamical networks with and without time-varying delay," *Neurocomputing*, vol. 266, pp. 263–273, Nov. 2017.

[24] H. Dai, W. S. Chen, J. P. Jia, J. Y. Liu, and Z. Q. Zhang, "Exponential synchronization of complex dynamical networks with time-varying inner coupling via event-triggered communication," *Neurocomputing*, vol. 245, pp. 124–132, Jul. 2017.

[25] J.-L. Wang, H.-N. Wu, and T. Huang, "Passivity-based synchronization of a class of complex dynamical networks with time-varying delay," *Automatica*, vol. 56, pp. 105–112, Jun. 2015.

[26] W. Yu, G. Chen, and J. Lü, "On pinning synchronization of complex dynamical networks," *Automatica*, vol. 45, no. 2, pp. 429–435, 2009.

[27] J. Zhou, J.-A. Lu, and J. Lu, "Adaptive synchronization of an uncertain complex dynamical network," *IEEE Trans. Autom. Control*, vol. 51, no. 4, pp. 652–656, Apr. 2006.

[28] A. Das and F. L. Lewis, "Distributed adaptive control for synchronization of unknown nonlinear networked systems," *Automatica*, vol. 46, no. 12, pp. 2014–2021, 2010.

[29] J. Zhou, J.-A. Lu, and J. Lü, "Pinning adaptive synchronization of a general complex dynamical network," *Automatica*, vol. 44, no. 4, pp. 996–1003, 2008.

[30] P. Delellis, M. Dibernardo, and F. Garofalo, "Novel decentralized adaptive strategies for the synchronization of complex networks," *Automatica*, vol. 45, no. 5, pp. 1312–1318, 2009.

[31] Y. Tang, H. J. Gao, and J. Kurths, "Distributed robust synchronization of dynamical networks with stochastic coupling," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 5, pp. 1508–1519, May 2014.

[32] X.-J. Li, C.-X. Shi, and G.-H. Yang, "Observer-based adaptive output-feedback fault-tolerant control of a class of complex dynamical networks," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 48, no. 12, pp. 2407–2418, Dec. 2018, doi: 10.1109/TSMC.2017.2688581.

[33] J. Xiang, W. Wei, and Y. Li, "Synchronized output regulation of linear networked systems," *IEEE Trans. Autom. Control*, vol. 54, no. 6, pp. 1336–1341, Jun. 2009.

[34] J.-L. Wang, H.-N. Wu, T. Huang, S.-Y. Ren, and J. Wu, "Passivity and output synchronization of complex dynamical networks with fixed and adaptive coupling strength," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 2, pp. 364–376, Feb. 2018.

[35] T. Liu, D. J. Hill, and J. Zhao, "Output synchronization of dynamical networks with incrementally-dissipative nodes and switching topology," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 62, no. 9, pp. 2312–2323, Sep. 2015.

[36] T. H. Lee and J. H. Park, "Improved criteria for sampled-data synchronization of chaotic Lur'e systems using two new approaches," *Nonlin. Anal. Hybrid Syst.*, vol. 24, pp. 132–145, May 2017.

[37] J. Zhao, D. J. Hill, and T. Liu, "Synchronization of dynamical networks with nonidentical nodes: Criteria and control," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 3, pp. 584–594, Mar. 2011.

[38] Q. Song, J. Cao, and F. Liu, "Synchronization of complex dynamical networks with nonidentical nodes," *Phys. Lett. A*, vol. 374, no. 4, pp. 544–551, 2010.

[39] B. Liu and D. J. Hill, "Impulsive consensus for complex dynamical networks with nonidentical nodes and coupling time-delays," *SIAM J. Control Optim.*, vol. 49, no. 2, pp. 315–338, 2011.

[40] F. Wang, B. Chen, C. Lin, and X. Li, "Distributed adaptive neural control for stochastic nonlinear multiagent systems," *IEEE Trans. Cybern.*, vol. 47, no. 7, pp. 1795–1803, Jul. 2017.

[41] Y. Liu and J. Zhao, "Generalized output synchronization of dynamical networks using output quasi-passivity," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 59, no. 6, pp. 1290–1298, Jun. 2012.

[42] X. D. Zhao, X. L. Zheng, C. Ma, and R. Li, "Distributed consensus of multiple Euler–Lagrange systems networked by sampled-data information with transmission delays and data packet dropouts," *IEEE Trans. Autom. Sci. Eng.*, vol. 14, no. 3, pp. 1440–1450, Jul. 2017.

[43] X. Xiao, X.-J. Li, X.-Z. Jin, and Y. Cui, "Output synchronization control for a class of complex dynamical networks with non-identical dynamics," *Appl. Math. Comput.*, vol. 335, pp. 38–49, Oct. 2018.
[44] Y. Jiang and Z.-P. Jiang, "Computational adaptive optimal control for continuous-time linear systems with completely unknown dynamics," *Automatica*, vol. 48, no. 10, pp. 2699–2704, 2012.
[45] M. Y. Li and Z. Shuai, "Global-stability problem for coupled systems of differential equations on networks," *J. Differ. Equ.*, vol. 248, no. 1, pp. 1–20, 2010.
[46] H. Modares and F. L. Lewis, "Linear quadratic tracking control of partially-unknown continuous-time systems using reinforcement learning," *IEEE Trans. Autom. Control*, vol. 59, no. 11, pp. 3051–3056, Nov. 2014.
[47] H. Modares, F. L. Lewis, and Z.-P. Jiang, "H_∞ tracking control of completely unknown continuous-time systems via off-policy reinforcement learning," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 10, pp. 2550–2562, Oct. 2015.
[48] F. L. Lewis, D. Vrabie, and V. Syrmos, *Optimal Control*, 3rd ed. New York, NY, USA: Wiley, 2012.
[49] H. Modares, S. P. Nageshrao, G. A. D. Lopes, R. Babuška, and F. L. Lewis, "Optimal model-free output synchronization of heterogeneous systems using off-policy reinforcement learning," *Automatica*, vol. 71, pp. 334–341, Sep. 2016.
[50] X.-Z. Jin, G.-H. Yang, and W.-W. Che, "Adaptive synchronization of master–slave large-scale systems against bias actuators and network attenuations," *Int. J. Control Autom. Syst.*, vol. 10, no. 6, pp. 1102–1110, 2012.

Ning Wang received the B.S. degree in measuring and control instrument from Northeastern University at Qinhuangdao, Qinhuangdao, China, in 2017. She is currently pursuing the M.S. degree in control theory and engineering with Northeastern University, Shenyang, China.

Her current research interests include complex networks, adaptive dynamics program, and cyber-physical systems.

Xiao-Jian Li (M'16) received the B.S. and M.S. degrees in mathematics from Northeast Normal University, Changchun, China, in 2003 and 2006, respectively, and the Ph.D. degree in control theory and engineering from Northeastern University, Shenyang, China, in 2011.

He is currently a Professor with the College of Information Science and Engineering, Northeastern University. His current research interests include fault diagnosis, fault-tolerant control, fuzzy control, and complex networks.