习题 1.7 初等变换和初等矩阵

1. 证明 $E(i,j)=E(j(-1))E(i,j(1))E(j,i(-1))E(i,j(1))$.

证明：

$$\begin{aligned}
&E(j(-1))E(i,j(1))E(j,i(-1))E(i,j(1))\\
&\quad=(\varepsilon_1,\varepsilon_2,...,\varepsilon_i,...,-\varepsilon_j,...,\varepsilon_n)E(i,j(1))E(j,i(-1))E(i,j(1))\\
&\quad=(\varepsilon_1,\varepsilon_2,...,\varepsilon_i,...,\varepsilon_i-\varepsilon_j,...,\varepsilon_n)E(j,i(-1))E(i,j(1))\\
&\quad=(\varepsilon_1,\varepsilon_2,...\varepsilon_j,...,\varepsilon_i-\varepsilon_j,...,\varepsilon_n)E(i,j(1))\\
&\quad=(\varepsilon_1,\varepsilon_2,...,\varepsilon_j,...,\varepsilon_i,...,\varepsilon_n)\\
&\quad=E(i,j)
\end{aligned}.$$

□

2. 计算

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}^{2011}\begin{pmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}^{2012}.$$

解：因为 $\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}^2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, 所以

$$\text{原式}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}c_1&c_2&c_3\\b_1&b_2&b_3\\a_1&a_2&a_3\end{pmatrix}.$$

3. 设 A 为 3 阶方阵， $P^TAP=\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}$, 若 $P=(X_1,X_2,X_3)$, $Q=(X_1+X_2,X_2,X_3)$, 计算 Q^TAQ.

解：显然， $Q=PE(2,1(1))$, 则

$$\begin{aligned}
Q^TAQ&=(PE(2,1(1)))^TAPE(2,1(1))\\
&=E(1,2(1))P^TAPE(2,1(1))\\
&=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\\
&=\begin{pmatrix}2&1&0\\1&1&0\\0&0&2\end{pmatrix}.
\end{aligned}$$

4. 已知

$$A=\begin{pmatrix} 3 & 0 & -1 & 1 \\ -3 & 2 & -5 & 3 \\ 0 & 1 & -3 & 2 \end{pmatrix},$$

求可逆阵 P, Q, 使得

$$PAQ=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

解:

$$\begin{aligned} A &= \begin{pmatrix} 3 & 0 & -1 & 1 \\ -3 & 2 & -5 & 3 \\ 0 & 1 & -3 & 2 \end{pmatrix} \xrightarrow{E(2,1(1))} \begin{pmatrix} 3 & 0 & -1 & 1 \\ 0 & 2 & -6 & 4 \\ 0 & 1 & -3 & 2 \end{pmatrix} \\ &\xrightarrow{E(2,2(-\frac{1}{2}))} \begin{pmatrix} 3 & 0 & -1 & 1 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[E(1,3(\frac{1}{3}))E(1,4(-\frac{1}{3}))E(1(\frac{1}{3}))]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\xrightarrow[E(2,3(3))E(2,4(-2))E(2(\frac{1}{2}))]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

令

$$P=E(2,2(-\frac{1}{2}))E(2,1(1))=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix},$$

$$Q=E(1,3(\frac{1}{3}))E(1,4(-\frac{1}{3}))E(1(\frac{1}{3}))E(2,3(3))E(2,4(-2))E(2(\frac{1}{2}))=\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 3 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

则 P,Q(不唯一) 即为所求.

5. 设 $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $A^*BA=2BA-8E$, 计算 B.

解:因为 $\det A=-2$, 所以 A 可逆. 又因为 $A^*BA=2BA-8E$, 将该式两边同时左乘 A, 再同时右乘 A^{-1}, 得 $(\det A)B=2AB-8E$, 整理得:$(A+E)B=4E$, 所以 $B=4(A+E)^{-1}=\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ -1 & 0 & 2 \end{pmatrix}$.

6. 设矩阵 A 的伴随矩阵 $A^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 8 \end{pmatrix}$ 且 $ABA^{-1} = BA^{-1} + 3E$. 求 B.

解：直接计算得 $\det A^* = 8$, 但 $AA^* = \det \mathrm{A} E$, A 是 4 阶矩阵，两边同取行列式，可得 $\det A = 2$. 又 $ABA^{-1} = BA^{-1} + 3E$, 两边同时左乘 A^*, 再同时右乘 A, 有 $A^*ABA^{-1}A = A^*BA^{-1}A + 3A^*EA$, 整理得，$(2E - A^*)B = 6E$. 所以
$B = 6(2E - A^*)^{-1} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 6 & 0 & 6 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix}$.

7. 求解矩阵方程

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}.$$

解：通过行初等变换有

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & -1 \end{pmatrix},$$

$$X = \begin{pmatrix} -2 & 3 \\ 3 & -1 \end{pmatrix}.$$

8. 计算

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}^{-1}.$$

解：(法一) 行初等变换法. 对 (A, E) 做一系列的行初等变换

$$(A, E) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & -1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}
$$

$$
\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix},
$$

则

$$
A^{-1} = -\frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}.
$$

(法二) 直接计算得

$$
\det A = -3, A^* = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}.
$$

从而

$$
A^{-1} = \frac{A^*}{\det A} = -\frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}.
$$

9. 计算行列式

$$
\begin{vmatrix} a_1^2 & a_1a_2+1 & \cdots & a_1a_n+1 \\ a_2a_1+1 & a_2^2 & \cdots & a_2a_n+1 \\ \vdots & \vdots & & \vdots \\ a_na_1+1 & a_na_2+1 & \cdots & a_n^2 \end{vmatrix}.
$$

解：对任意 A_{m*n}, B_{n*m}, 下证 $\det(E_m - AB) = \det(E_n - BA)$.

事实上

$$
\begin{pmatrix} E_n & 0 \\ -A & E_m \end{pmatrix} \begin{pmatrix} E_n & B \\ A & E_m \end{pmatrix} = \begin{pmatrix} E_n & B \\ 0 & E_m - AB \end{pmatrix},
$$

两边取行列式得：

$$
\begin{vmatrix} E_n & B \\ A & E_m \end{vmatrix} = \det(E_m - AB),
$$

$$\begin{pmatrix} E_n & B \\ A & E_m \end{pmatrix} \begin{pmatrix} E_n & 0 \\ -A & E_m \end{pmatrix} = \begin{pmatrix} E_n - BA & B \\ 0 & E_m \end{pmatrix},$$

两边取行列式得：

$$\begin{vmatrix} E_n & B \\ A & E_m \end{vmatrix} = \det(E_n - BA),$$

因此 $\det(E_m - AB) = \det(E_n - BA)$.

现令

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

则

$$\begin{aligned} \text{原式} \quad &= \det(A^T A - E_n) = (-1)^n \det(E_n - A^T A) = (-1)^n \det(E_2 - AA^T) \\ &= (-1)^n \begin{vmatrix} 1 - \sum_{i=1}^n a_i^2 & \sum_{i=1}^n a_i \\ \sum_{i=1}^n a_i & 1 - n \end{vmatrix} \\ &= (-1)^n [(1-n)(1 - \sum_{i=1}^n a_i^2) - (\sum_{i=1}^n a_i)^2]. \end{aligned}$$

(万琴解答)