Advanced Quantum Mechanics: Fall 2017
Solution to Midterm Exam

NOTE: Problems start on page 2. Answer the questions in italic fonts.

Possibly useful facts:

- Pauli matrices: $\sigma_0 = \mathbb{1}_{2\times 2}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

  $\sigma_1\sigma_2 = \mathbb{i}\sigma_3 = -\sigma_2\sigma_1$, $\sigma_2\sigma_3 = \mathbb{i}\sigma_1 = -\sigma_3\sigma_2$, $\sigma_3\sigma_1 = \mathbb{i}\sigma_2 = -\sigma_1\sigma_3$, $\sigma_1^2 = \sigma_2^2 = \sigma_3^3 = \sigma_0$.

  So $\sigma_{1,2,3}$ mutually anti-commute, $\{\sigma_1, \sigma_2\} = \{\sigma_2, \sigma_3\} = \{\sigma_3, \sigma_1\} = 0$,

  and $[\sigma_1, \sigma_2] = 2\mathbb{i}\sigma_3$, $[\sigma_2, \sigma_3] = 2\mathbb{i}\sigma_1$, $[\sigma_3, \sigma_1] = 2\mathbb{i}\sigma_2$.

- Some Taylor expansions:
  $\exp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$, $\sin(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $\cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

- Baker-Hausdorff formula: $\exp(\hat{A}) \cdot \hat{B} \cdot \exp(-\hat{A}) = \hat{B} + \sum_{n=1}^{+\infty} \frac{1}{n!} \underbrace{[\hat{A}, [\hat{A}, \ldots [\hat{A},}_{n\text{-fold commutator}} \hat{B}] \ldots]]$.

- If $[\hat{A}, \hat{B}]$ is a c-number, then $\exp(\hat{A} + \hat{B}) = \exp(\hat{A}) \exp(\hat{B}) \exp(-\frac{1}{2}[\hat{A}, \hat{B}])$.

- 1D harmonic oscillator: $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$.
  Here $\hat{x}$ is position operator, $\hat{p}$ is momentum operator, $[\hat{x}, \hat{p}] = \mathbb{i}\hbar$, and in position representation $\hat{p} = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{b} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \mathbb{i}\frac{1}{m\omega}\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{b}, \hat{b}^\dagger] = 1$ and $\hat{H}_0 = \hbar\omega\,(\hat{b}^\dagger\hat{b} + \frac{1}{2})$. It has a unique ground state $|0\rangle$ with $\hat{b}|0\rangle = 0$, and excited states $|n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{b}^\dagger)^n|0\rangle$ with energy $E_n = (n + \frac{1}{2})\hbar\omega$. These states $|n\rangle$ can be viewed as occupation basis of a single boson mode.

- Creation & annihilation operators:
  $\hat{\psi}^\dagger$ "creates" a particle in single particle state $|\psi\rangle$;
  $\hat{\psi}$ "destroys" a particle in single particle state $|\psi\rangle$; $\hat{\psi}^\dagger$ is hermitian conjugate of $\hat{\psi}$.

  - Given complete orthonormal basis $|e_i\rangle$ of single particle states, one set of complete orthonormal basis for the Fock space is the *occupation basis* $|n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1!}}(\hat{e}_1^\dagger)^{n_1}\frac{1}{\sqrt{n_2!}}(\hat{e}_2^\dagger)^{n_2}\cdots|\text{vac}\rangle$. Here $|\text{vac}\rangle$ is the particle "vacuum". $\hat{e}_i^\dagger$ are creation operators for state $|e_i\rangle$. For bosons, $[\hat{e}_i, \hat{e}_j^\dagger] = \delta_{i,j}$; for fermions, $\{\hat{e}_i, \hat{e}_j^\dagger\} = \delta_{i,j}$.
  - $[\hat{e}_i^\dagger\hat{e}_j,\ \hat{e}_k^\dagger] = \delta_{j,k}\hat{e}_i^\dagger$, for both bosons and fermions.

Problem 1. (10 points)

A 2-dimensional Hilbert space basis $|1\rangle$ and $2\rangle$, with overlaps $\langle 1|1\rangle = \langle 2|2\rangle = 1$ and $\langle 1|2\rangle = -\frac{2}{3}$. A linear operator $\hat{A}$ is defined by $\hat{A}|1\rangle = 3|1\rangle + |2\rangle$ and $\hat{A}|2\rangle = |1\rangle + 3|2\rangle$.

(a) (5pts)*Is $\hat{A}$ a hermitian operator? Is $\hat{A}$ a unitary operator?*

(b) (5pts)*Solve the eigenvalues and normalized eigenstates of $\hat{A}$.*

Solution: This is similar to Homework #1 Problem 1(c).

(a) $\hat{A}$ is hermitian, but not unitary.

Method #1: use definitions,

$\hat{A}|i\rangle = \sum_j |j\rangle A_{ji}$, where A_{ji} is the 2×2 matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

Define $G_{ij} = \langle i|j\rangle$, which is the 2×2 matrix $\begin{pmatrix} 1 & -2/3 \\ -2/3 & 1 \end{pmatrix}$.

Consider two generic states, $|a\rangle = \sum_i a_i |i\rangle$ and $|b\rangle = \sum_i b_i |i\rangle$, and consider the following two inner products,

$(\hat{A}|a\rangle,\, |b\rangle) = (\sum_{j,i} |j\rangle\, A_{ji} a_i, \sum_k |k\rangle\, b_k) = \sum_{i,j,k} a_i^* A_{ji}^* G_{jk} b_k = \vec{a}^\dagger \cdot (A^\dagger \cdot G) \cdot \vec{b}$.

$(|a\rangle,\, \hat{A}|b\rangle) = (\sum_i |i\rangle a_i, \sum_{j,k} |j\rangle\, A_{jk} b_k) = \sum_{i,j,k} a_i^* G_{ij} A_{jk} b_k = \vec{a}^\dagger \cdot (G \cdot A) \cdot \vec{b}$.

Here $\vec{a}$ and $\vec{b}$ are column vectors with a_i and b_i as elements, respectively.

It is easy to check that $A = 3\sigma_0 + \sigma_1$ and $G = \sigma_0 - \frac{2}{3}\sigma_1$ satisfy $A^\dagger \cdot G = G \cdot A = \frac{7}{3}\sigma_0 - \sigma_1$, therefore for any states $|a\rangle$ and $|b\rangle$, $(\hat{A}|a\rangle,\, |b\rangle) = (|a\rangle,\, \hat{A}|b\rangle)$.

By the definition of hermitian conjugate of operators, this shows $\hat{A} = \hat{A}^\dagger$.

Equivalent condition #1: If $\langle a|\hat{A}|a\rangle$ is real for any state $|a\rangle$, then $\hat{A}$ is real. Here $\langle a|\hat{A}|a\rangle = \vec{a}^\dagger \cdot (G\cdot A) \cdot \vec{a} = \vec{a}^\dagger \cdot (\frac{7}{3}\sigma_0 - \sigma_1)\cdot \vec{a} = \frac{7}{3}(|a_1|^2 + |a_2|^2) - (a_1^* a_2 + a_2^* a_1)$, is always real.

Equivalent condition #2: The above condition is equivalent to $\langle i|\hat{A}|j\rangle = G \cdot A$ is a hermitian matrix. Here $G \cdot A = \frac{7}{3}\sigma_0 - \sigma_1$ is a hermitian matrix.

To check whether $\hat{A}$ is unitary, we need to check whether $(\hat{A}|i\rangle, \hat{A}|j\rangle)$ equals to $\langle i|j\rangle$ for any pair of basis states $|i\rangle$ and $|j\rangle$.

But $(\hat{A}|1\rangle, \hat{A}|1\rangle = (3|1\rangle + |2\rangle, 3|1\rangle + |2\rangle) = 9 \cdot 1 + 3 \cdot (-\frac{2}{3}) + 3 \cdot (-\frac{2}{3}) + 1 \cdot 1 = 6$ $\neq \langle 1|1\rangle$. Therefore $\hat{A}$ is not unitary.

Equivalent condition #1: If the inner product $(\hat{A}|a\rangle, \hat{A}|a\rangle) = (|a\rangle, |a\rangle)$ for any state $|a\rangle$, then $\hat{A}$ is unitary. This condition is $\vec{a}^\dagger \cdot (A^\dagger \cdot G \cdot A) \cdot \vec{a} = \vec{a}^\dagger \cdot G \cdot \vec{a}$ for any complex vector $\vec{a}$, or equivalently, $A^\dagger \cdot G \cdot A = G$. Here $A^\dagger \cdot G \cdot A = 6\sigma_0 - \frac{2}{3}\sigma_1 \neq G = \sigma_0 - \frac{2}{3}\sigma_1$.

Equivalent condition #2: If $\hat{A}^\dagger \hat{A} = \mathbb{1}$, then $\hat{A}$ is unitary.

Knowing that $\hat{A}$ is hermitian, this condition is $\hat{A}^2 = \hat{\mathbb{1}}$, since $\hat{A}^2|i\rangle = \sum_k \sum_j |k\rangle A_{kj} A_{ji}$, this condition is equivalent to $A \cdot A = \mathbb{1}$. But here $A \cdot A = 10\sigma_0 + 6\sigma_1 \neq \mathbb{1} = \sigma_0$.

Method #2: change to complete orthonormal basis.

The $\langle i|j\rangle$ values and definition of $\hat{A}$ are invariant if we exchange the two basis $|1\rangle$ and $|2\rangle$.

This suggests to use the symmetric and anti-symmetric combinations, $|1\rangle + |2\rangle$ and $|1\rangle - |2\rangle$, as new basis. They are indeed orthogonal to each other.

Normalize them: $(|1\rangle + |2\rangle, |1\rangle + |2\rangle) = 1 + (-\frac{2}{3}) + (-\frac{2}{3}) + 1 = \frac{2}{3}$,
$(|1\rangle - |2\rangle, |1\rangle - |2\rangle) = 1 - (-\frac{2}{3}) - (-\frac{2}{3}) + 1 = \frac{10}{3}$.

We get the complete orthonormal basis, $|+\rangle \equiv \sqrt{\frac{3}{2}}(|1\rangle + |2\rangle)$ and $|-\rangle \equiv \sqrt{\frac{3}{10}}(|1\rangle - |2\rangle)$.

The action of $\hat{A}$ on this basis is $\hat{A}|+\rangle = 4|+\rangle$ and $\hat{A}|-\rangle = 2|-\rangle$. Namely the matrix representation of $\hat{A}$ under this complete orthonormal basis is $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$. This is obviously a hermitian matrix, but is not a unitary matrix. Therefore $\hat{A}$ is hermitian, but not unitary.

(b) The method #2 of (a) already produces the answer for (b).

$\hat{A}$ has eigenvalue 4 with normalized eigenstate $\sqrt{\frac{3}{2}}(|1\rangle + |2\rangle)$;
and eigenvalue 2 with normalized eigenstate $\sqrt{\frac{3}{10}}(|1\rangle - |2\rangle)$.

The eigenvalues and un-normalized eigenvectors can also be obtained from the eigenvalues and eigenvectors of the A_{ij} matrix, $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, defined in method #1 of (a).

Problem 2. (50 points)

Consider the 1D harmonic oscillator $\hat{H}_0$ defined on page 1.

(a) (10pts) Let $\hat{H}' = \hat{H}_0 - f \cdot \hat{x}$, where f is a real constant. $\hat{H}'$ is related to $\hat{H}_0$ by $\hat{U} \cdot \hat{H}_0 \cdot \hat{U}^\dagger = \hat{H}' + c$. Here c is a real constant, $\hat{U} = \exp(-\mathbb{i}X\hat{p} - \mathbb{i}P\hat{x})$ is a unitary operator with real parameters X and P. *Solve X and P and c in terms of $f, m, \omega, \hbar$.*

(b) (10pts) Denote the normalized ground state of $\hat{H}'$ by $|0'\rangle$. *Evaluate $\langle 0'|\hat{x}|0'\rangle$ and $\langle 0'|\hat{p}|0'\rangle$.* [Hint: result of (a) may help.]

(c) (10pts) At $t = 0$, let the state $|\psi(t=0)\rangle = |0'\rangle$, evolve this state under $\hat{H}_0$, namely $|\psi(t)\rangle = \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)|\psi(t=0)\rangle$. *Evaluate $\langle\psi(t)|\hat{x}|\psi(t)\rangle$ and $\langle\psi(t)|\hat{p}|\psi(t)\rangle$.* [Hint: you can use either Schrödinger or Heisenberg picture.]

(d) (5pts) *Further evaluate $\langle\psi(t)|\hat{x}^2|\psi(t)\rangle$ and $\langle\psi(t)|\hat{p}^2|\psi(t)\rangle$ for $|\psi(t)\rangle$ defined in (c). Check that the uncertainty relation for $\hat{x}$ and $\hat{p}$ is always satisfied.* [Hint: it'll be most efficient to use the Schrödinger picture and knowledge about boson coherent states.]

(e) (10pts) Define two Hermitian operators: $\hat{O}_1 = m^2\omega^2\hat{x}^2 - \hat{p}^2$, $\hat{O}_2 = m\omega(\hat{x}\hat{p} + \hat{p}\hat{x})$. Their Heisenberg picture under $\hat{H}_0$ are $\hat{O}_{i,\mathrm{H}}(t) = \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t) \cdot \hat{O}_i \cdot \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0 \cdot t)$. *Write down the Heisenberg equations of motion, $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_{i,H}(t) = \dots$ for $i = 1, 2$.* The right-hand side of these equations should be expressed in terms of $\hat{O}_{j,\mathrm{H}}(t)$ with $j = 1, 2$.

(f) (5pts) *Solve the equations in (e). Namely solve $\hat{O}_{i,H}(t)$ in terms of $\hat{O}_{j,H}(t=0)$.*

Solution: This is similar to Homework #3 Problem 1(d)(e).

(a) By the Baker-Hausdorff formula,
$\hat{U}\hat{x}\hat{U}^\dagger = \hat{x} + [-\mathbb{i}X\hat{p} - \mathbb{i}P\hat{x}, \hat{x}] + \cdots = \hat{x} + (-\mathbb{i}X)(-\mathbb{i}\hbar) + 0 + \cdots = \hat{x} - X\hbar$, and
$\hat{U}\hat{p}\hat{U}^\dagger = \hat{p} + [-\mathbb{i}X\hat{p} - \mathbb{i}P\hat{x}, \hat{p}] + \cdots = \hat{p} + (-\mathbb{i}P)(\mathbb{i}\hbar) + 0 + \cdots = \hat{p} + P\hbar$.

Then $\hat{U} \cdot \hat{H}_0 \cdot \hat{U}^\dagger = \frac{1}{2m}(\hat{U}\hat{p}\hat{U}^\dagger)^2 + \frac{m\omega^2}{2}(\hat{U}\hat{x}\hat{U}^\dagger)^2 = \frac{1}{2m}(\hat{p} + P\hbar)^2 + \frac{m\omega^2}{2}(\hat{x} - X\hbar)^2$.
Compare this with $\hat{H}' = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2 - f \cdot \hat{x} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}(\hat{x} - \frac{f}{m\omega^2})^2 - \frac{f^2}{2m\omega^2}$, we get
$X = \frac{f}{m\omega^2\hbar}$, $P = 0$, $c = \frac{f^2}{2m\omega^2}$.

(b) According to (a), $\hat{U}\hat{H}_0\hat{U}^\dagger = \hat{H}' + c$, there is one-to-one correspondence between the eigenstates of $\hat{H}_0$ and $\hat{H}'$:

if $\hat{H}_0|n\rangle = E_n|n\rangle$, then $\hat{H}'\cdot\hat{U}|n\rangle = (\hat{U}\hat{H}_0\hat{U}^\dagger - c)\cdot\hat{U}|n\rangle = \hat{U}\hat{H}_0|n\rangle - c\hat{U}|n\rangle = (E_n - c)\cdot\hat{U}|n\rangle$; conversely, if $\hat{H}'|n'\rangle = E'_n|n'\rangle$, then $\hat{H}_0\cdot\hat{U}^\dagger|n'\rangle = (E'_n + c)\cdot\hat{U}^\dagger|n'\rangle$.

The ground state of $\hat{H}'$ is $\hat{U}|0\rangle$ where $|0\rangle$ is the ground state of $\hat{H}_0$.

$\langle 0'|\hat{x}|0'\rangle = \langle 0|\hat{U}^\dagger\hat{x}\hat{U}|0\rangle$, $\langle 0'|\hat{p}|0'\rangle = \langle 0|\hat{U}^\dagger\hat{p}\hat{U}|0\rangle$.

Similar to the calculations in (a), $\hat{U}^\dagger\hat{x}\hat{U} = \hat{x} + X\hbar = \hat{x} + \frac{f}{m\omega}$, $\hat{U}^\dagger\hat{p}\hat{U} = \hat{p} - P\hbar = \hat{p}$.

In the ground state $|0\rangle$ of $\hat{H}_0$, $\langle 0|\hat{x}|0\rangle = 0$ and $\langle 0|\hat{p}|0\rangle = 0$.

This can be seen from $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b} + \hat{b}^\dagger)$, and $\hat{p} = -\mathbb{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{b} - \hat{b}^\dagger)$, and $\langle 0|\hat{b}|0\rangle = \langle 0|\hat{b}^\dagger|0\rangle^* = 0$.

Therefore $\langle 0'|\hat{x}|0'\rangle = \langle 0|(\hat{x} + \frac{f}{m\omega^2})|0\rangle = \frac{f}{m\omega^2}$, $\langle 0'|\hat{p}|0'\rangle = \langle 0|\hat{p}|0\rangle = 0$.

(c) Method #1: Schrödinger picture.

$|0'\rangle = \exp(-\mathbb{i}\frac{f}{m\omega^2\hbar}\hat{p})|0\rangle = \exp[-\frac{f}{m\omega^2\hbar}\sqrt{\frac{\hbar m\omega}{2}}(\hat{b} - \hat{b}^\dagger)]|0\rangle$.

For notation simplicity, define $z = \frac{f}{m\omega^2\hbar}\sqrt{\frac{\hbar m\omega}{2}}$, then $|0'\rangle = \exp(-z^*\hat{b} + z\hat{b}^\dagger)|0\rangle$ $= e^{-|z|^2/2}\exp(z\hat{b}^\dagger)\exp(-z\hat{b})|0\rangle = e^{-|z|^2/2}\exp(z\hat{b}^\dagger)|0\rangle$ is a boson coherent state.

Denote boson coherent states $e^{-|z|^2/2}\exp(z\hat{b}^\dagger)|0\rangle$ by $|z\rangle$ hereafter.

$|\psi(t)\rangle = \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)|0'\rangle = \exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\cdot e^{-|z|^2/2}\exp(z\hat{b}^\dagger)\cdot\exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\cdot\exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)|0\rangle$ $= e^{-|z|^2/2}\exp\left[z\cdot\exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\cdot\hat{b}^\dagger\cdot\exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\right]\cdot e^{-\frac{\mathbb{i}}{\hbar}E_0\cdot t}|0\rangle$. Here E_0 is the ground state energy of $\hat{H}_0$.

From $\hat{H}_0 = \hbar\omega\cdot(\hat{b}^\dagger\hat{b} + \frac{1}{2})$, the commutator $[-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t, \hat{b}^\dagger] = -\mathbb{i}\omega t\cdot\hat{b}$, then by the Baker-Hausdorff formula, $\exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\cdot\hat{b}^\dagger\cdot\exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t) = \sum_{n=0}^{\infty}\frac{(-\mathbb{i}\omega t)^n}{n!}\hat{b}^\dagger = e^{-\mathbb{i}\omega t}\hat{b}^\dagger$.

$|\psi(t)\rangle = e^{-|z|^2/2}\cdot\exp(ze^{-\mathbb{i}\omega t}\hat{b}^\dagger)\cdot e^{-\frac{\mathbb{i}}{\hbar}E_0\cdot t}|0\rangle = e^{-\frac{\mathbb{i}}{\hbar}E_0\cdot t}|ze^{-\mathbb{i}\omega t}\rangle$, is still a boson coherent state.

Then $\langle\psi(t)|\hat{b}|\psi(t)\rangle = ze^{-\mathbb{i}\omega t}$, $\langle\psi(t)|\hat{b}^\dagger|\psi(t)\rangle = z^*e^{\mathbb{i}\omega t}$.

Finally

$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \langle\psi(t)|\sqrt{\frac{\hbar}{2m\omega}}(\hat{b} + \hat{b}^\dagger)|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}(ze^{-\mathbb{i}\omega t} + z^*e^{\mathbb{i}\omega t}) = \frac{f}{m\omega^2}\cos(\omega t)$,

$\langle\psi(t)|\hat{p}|\psi(t)\rangle = \langle\psi(t)| - \mathbb{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{b} - \hat{b}^\dagger)|\psi(t)\rangle = \sqrt{\frac{\hbar m\omega}{2}}(-\mathbb{i}ze^{-\mathbb{i}\omega t} + \mathbb{i}z^*e^{\mathbb{i}\omega t}) = -\frac{f}{\omega}\sin(\omega t)$,

Method #2: Heisenberg picture.

Define the Heisenberg picture operators $\hat{x}_H(t) = \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\cdot\hat{x}\cdot\exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)$, and $\hat{p}_H(t) = \exp(\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)\cdot\hat{p}\cdot\exp(-\frac{\mathbb{i}}{\hbar}\hat{H}_0\cdot t)$.

They satisfy $[\hat{x}_H(t), \hat{p}_H(t)] = \mathbb{i}\hbar$. And the Heisenberg picture of $\hat{H}_0$ is simply $\hat{H}_{0,H}(t) =$

$\frac{1}{2m}[\hat{p}_H(t)]^2 + \frac{m\omega^2}{2}[\hat{x}_H(t)]^2$.

The Heisenberg equations of motion for $\hat{x}_H$ and $\hat{p}_H$ are

$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H(t) = \frac{\mathrm{i}}{\hbar}[\hat{H}_{0,H}(t), \hat{x}_H(t)] = \frac{1}{m}\hat{p}_H(t)$, and $\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H(t) = \frac{\mathrm{i}}{\hbar}[\hat{H}_{0,H}(t), \hat{p}_H(t)] = -m\omega^2\hat{x}_H(t)$.

The solution to these equations is

$\hat{x}_H(t) = \hat{x}_H(t=0)\cos(\omega t) + \frac{1}{m\omega}\hat{p}_H(t=0)\sin(\omega t) = \hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)$,

$\hat{p}_H(t) = \hat{p}_H(t=0)\cos(\omega t) - m\omega\hat{x}_H(t=0)\sin(\omega t) = \hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)$.

Finally,

$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \langle\psi(t=0)\hat{x}_H(t)|\psi(t=0)\rangle = \langle 0'|[\hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)]|0'\rangle = \frac{f}{m\omega^2}\cos(\omega t)$,

and $\langle\psi(t)|\hat{p}|\psi(t)\rangle = \langle\psi(t=0)\hat{p}_H(t)|\psi(t=0)\rangle = \langle 0'|[\hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)]|0'\rangle$

$= -m\omega\frac{f}{m\omega^2}\sin(\omega t) = -\frac{f}{\omega}\sin(\omega t)$.

(d) According to the method #1 of (c), $|\psi(t)\rangle = e^{-\frac{\mathrm{i}}{\hbar}E_0 \cdot t}|ze^{-\mathrm{i}\omega t}\rangle$ is a boson coherent state, $\hat{b}|\psi(t)\rangle = ze^{-\mathrm{i}\omega t}|\psi(t)\rangle$ with $z = \frac{f}{m\omega^2\hbar}\sqrt{\frac{\hbar m\omega}{2}}$.

$\hat{x}^2 = \frac{\hbar}{2m\omega}(\hat{b}+\hat{b}^\dagger)^2 = \frac{\hbar}{2m\omega}[\hat{b}^2 + (\hat{b}^\dagger)^2 + 2\hat{b}^\dagger\hat{b} + 1]$.

$\hat{p}^2 = -\frac{\hbar m\omega}{2}(\hat{b}-\hat{b}^\dagger)^2 = \frac{\hbar m\omega}{2}[-\hat{b}^2 - (\hat{b}^\dagger)^2 + 2\hat{b}^\dagger\hat{b} + 1]$.

Finally

$\langle\psi(t)|\hat{x}^2|\psi(t)\rangle = \frac{\hbar}{2m\omega}[z^2e^{-2\mathrm{i}\omega t} + (z^*)^2e^{2\mathrm{i}\omega t} + 2|z|^2 + 1] = \frac{\hbar}{2m\omega}[(ze^{-\mathrm{i}\omega t} + z^*e^{\mathrm{i}\omega t})^2 + 1]$

$= [\frac{f}{m\omega^2}\cos(\omega t)]^2 + \frac{\hbar}{2m\omega}$, and

$\langle\psi(t)|\hat{p}^2|\psi(t)\rangle = \frac{\hbar m\omega}{2}[-z^2e^{-2\mathrm{i}\omega t} - (z^*)^2e^{2\mathrm{i}\omega t} + 2|z|^2 + 1] = \frac{\hbar m\omega}{2}[-(ze^{-\mathrm{i}\omega t} - z^*e^{\mathrm{i}\omega t})^2 + 1]$

$= [\frac{f}{\omega}\sin(\omega t)]^2 + \frac{\hbar m\omega}{2}$.

Combine these with the result of (c), the variance of $\hat{x}$ and $\hat{p}$ under state $|\psi(t)\rangle$ are $\langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}$ and $\langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar m\omega}{2}$, independent of time, and satisfy the uncertainty relation $(\langle x^2\rangle - \langle x\rangle^2)(\langle p^2\rangle - \langle p\rangle^2) \geq \frac{\hbar^2}{4}$.

(e). $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_{1,H}(t) = 2\omega\hat{O}_{2,H}(t)$, and $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_{2,H}(t) = -2\omega\hat{O}_{1,H}(t)$.

Method #1: use the Heisenberg equations of motion, $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_H(t) = \frac{\mathrm{i}}{\hbar}[\hat{H}_{0,H}(t), \hat{O}_H(t)]$, and compute the commutators using $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B},\hat{C}]\hat{D} + [\hat{A},\hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B},\hat{D}] + \hat{C}[\hat{A},\hat{D}]\hat{B}$ and $[\hat{x}_H(t), \hat{p}_H(t)] = \mathrm{i}\hbar$.

Method #2: use the Heisenberg equations of motion for $\hat{x}_H$ and $\hat{p}_H$ in method #2 of (c).

$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H = \frac{1}{m}\hat{p}_H$, and $\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H = -m\omega^2\hat{x}_H$.

For notation simplicity, the argument t for Heisenberg picture operators are omitted here.

$\frac{d}{dt}(m^2\omega^2\hat{x}_H^2 - \hat{p}_H^2) = m^2\omega^2(\frac{d}{dt}\hat{x}_H \cdot \hat{x}_H + \hat{x}_H \cdot \frac{d}{dt}\hat{x}_H) - (\frac{d}{dt}\hat{p}_H \cdot \hat{p}_H + \hat{p}_H \cdot \frac{d}{dt}\hat{p}_H)$
$= m^2\omega^2 \cdot \frac{1}{m}(\hat{p}_H\hat{x}_H + \hat{x}_H\hat{p}_H) - (-m\omega^2)(\hat{x}_H\hat{p}_H + \hat{p}_H\hat{x}_H) = 2\omega \cdot m\omega(\hat{x}_H\hat{p}_H + \hat{p}_H\hat{x}_H)$

$\frac{d}{dt}[(m\omega(\hat{x}_H\hat{p}_H + \hat{p}_H\hat{x}_H)] = m\omega(\frac{d}{dt}\hat{x}_H \cdot \hat{p}_H + \hat{x}_H \cdot \frac{d}{dt}\hat{p}_H + \frac{d}{dt}\hat{p}_H \cdot \hat{x}_H + \hat{p}_H \cdot \frac{d}{dt}\hat{x}_H)$
$= m\omega(\frac{1}{m}\hat{p}_H \cdot \hat{p}_H - m\omega^2\hat{x}_H \cdot \hat{x}_H - m\omega^2\hat{x}_H \cdot \hat{x}_H + \frac{1}{m}\hat{p}_H \cdot \hat{p}_H) = -2\omega \cdot (m^2\omega^2\hat{x}_H^2 - \hat{p}^2)$.

(f). The solution is
$\hat{O}_{1,H}(t) = \hat{O}_{1,H}(t=0)\cos(2\omega t) + \hat{O}_{2,H}(t=0)\sin(2\omega t)$,
$\hat{O}_{2,H}(t) = \hat{O}_{2,H}(t=0)\cos(2\omega t) - \hat{O}_{1,H}(t=0)\sin(2\omega t)$.

Method #1: write the equations in (e) as $\frac{d}{dt}\begin{pmatrix}\hat{O}_{1,H}\\ \hat{O}_{2,H}\end{pmatrix} = \begin{pmatrix}0 & 2\omega\\ -2\omega & 0\end{pmatrix} \cdot \begin{pmatrix}\hat{O}_{1,H}\\ \hat{O}_{2,H}\end{pmatrix}$.

The solution is $\begin{pmatrix}\hat{O}_{1,H}\\ \hat{O}_{2,H}\end{pmatrix} = \exp\left[\begin{pmatrix}0 & 2\omega\\ -2\omega & 0\end{pmatrix} \cdot t\right] \cdot \begin{pmatrix}\hat{O}_{1,H}(t=0)\\ \hat{O}_{2,H}(t=0)\end{pmatrix}$.

$\exp[\begin{pmatrix}0 & 2\omega t\\ -2\omega t & 0\end{pmatrix}] = \exp[\mathbb{i}\cdot(2\omega t)\cdot\sigma_2] = \cos(2\omega t)\sigma_0 + \mathbb{i}\sin(2\omega t)\sigma_2 = \begin{pmatrix}\cos(2\omega t) & \sin(2\omega t)\\ -\sin(2\omega t) & \cos(2\omega t)\end{pmatrix}$.
[Check Homework #1 Problem 4(b)]

One can also first diagonalize the 2×2 matrix $\begin{pmatrix}0 & 2\omega t\\ -2\omega t & 0\end{pmatrix}$. Or equivalently consider $\frac{d}{dt}(\hat{O}_{1,H} \pm \mathbb{i}\hat{O}_{2,H}) = \pm(2\omega\mathbb{i}) \cdot (\hat{O}_{1,H} \pm \mathbb{i}\hat{O}_{2,H})$, whose solution is $(\hat{O}_{1,H} \pm \mathbb{i}\hat{O}_{2,H}) = e^{\pm 2\omega t\mathbb{i}}[\hat{O}_{1,H}(t=0) \pm \mathbb{i}\hat{O}_{2,H}(t=0)]$.

Method #2: In fact these can be obtained without using the equations of motion in (e). Use $\hat{x}_H = \hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)$, and $\hat{p}_H = \hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)$. Then
$\hat{O}_{1,H} = m^2\omega^2\hat{x}_H^2 - \hat{p}_H^2 = m^2\omega^2[\hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)]^2 - [\hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)]^2$
$= (m^2\omega^2\hat{x}^2 - \hat{p}^2) \cdot [\cos(\omega t)^2 - \sin(\omega t)^2] + m\omega(\hat{x}\hat{p} + \hat{p}\hat{x}) \cdot 2\cos(\omega t)\sin(\omega t)$, and
$\hat{O}_{2,H} = m\omega(\hat{x}_H\hat{p}_H + \hat{p}_H\hat{x}_H) = m\omega \cdot \Big\{[\hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)] \cdot [\hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)]$
$+[\hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)] \cdot [\hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t)]\Big\}$
$= (m^2\omega^2\hat{x}^2 - \hat{p}^2) \cdot [-2\cos(\omega t)\sin(\omega t)] + m\omega(\hat{x}\hat{p} + \hat{p}\hat{x}) \cdot [\cos(\omega t)^2 - \sin(\omega t)^2]$.

Problem 3. (40 points)
The single fermion Hilbert space has complete orthonormal basis $|1\rangle$ and $|2\rangle$. Denote the

corresponding creation operators by $\hat{f}_1^\dagger$ and $\hat{f}_2^\dagger$. Denote the fermion vacuum state by $|\text{vac}\rangle$. Then $\hat{f}_i|\text{vac}\rangle = 0$, and $|i\rangle = \hat{f}_i^\dagger|\text{vac}\rangle$ for $i = 1, 2$ respectively, and $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{i,j}$.

(a) (5pts) *Write down a complete orthonormal basis for the entire Fock space.*

(b) (5pts) Define four "Majorana fermion" operators, $\hat{\eta}_1 \equiv (\hat{f}_1 + \hat{f}_1^\dagger)$, $\hat{\eta}_2 \equiv -\mathbb{i}(\hat{f}_1 - \hat{f}_1^\dagger)$, $\hat{\eta}_3 \equiv (\hat{f}_2 + \hat{f}_2^\dagger)$, $\hat{\eta}_4 \equiv -\mathbb{i}(\hat{f}_2 - \hat{f}_2^\dagger)$. They are obviously hermitian. *Check the anti-commutation relations* $\{\hat{\eta}_i, \hat{\eta}_j\} = 2\delta_{i,j} = \begin{cases} 0, & \text{if } i \neq j; \\ 2, & \text{if } i = j. \end{cases}$ [Hint: use the multi-linearity of anti-commutators. Only $i \leq j$ cases need to be checked.]

(c) (10pts) *Under the basis in (a), write down the matrix representations for each $\hat{\eta}_i$ operator.* [Hint: first compute the matrices for $\hat{f}_{1,2}$, then $\hat{f}_{1,2}^\dagger$ are just hermitan conjugates; for later convenience it may help to write these results as tensor products of Pauli matrices.]

(d) (5pts) Define hermitian operators $\hat{L}_x = \mathbb{i}\hat{\eta}_3\hat{\eta}_4$, $\hat{L}_y = \mathbb{i}\hat{\eta}_4\hat{\eta}_2$, and $\hat{L}_z = \mathbb{i}\hat{\eta}_2\hat{\eta}_3$. *Compute their commutators $[\hat{L}_a, \hat{L}_b]$ for $a, b = x, y, z$ and $a \neq b$, express the results in terms of $\hat{L}_{x,y,z}$. Check that $\hat{L}_x^2 = \hat{L}_y^2 = \hat{L}_z^2 = 1$.* [Hint: use the anti-commutation relations in (b). Only three combinations of a, b need to be considered.]

(e) (5pts) *Solve the eigenvalues and normalized eigenstates of $\hat{L}_x$ in the entire Fock space.*

(f) (5pts) *Solve the eigenvalues and normalized eigenstates of $\hat{L}_z$ in the entire Fock space.*

(g) (5pts) *Compute* $\exp(\mathbb{i}\theta\hat{L}_z) \cdot (c_1\hat{L}_x + c_2\hat{L}_y + c_3\hat{L}_z) \cdot \exp(-\mathbb{i}\theta\hat{L}_z)$, *where θ and $c_{1,2,3}$ are c-numbers. The result should be a finite degree polynomial of $\hat{L}_{x,y,z}$.* [Hint: either use the Baker-Hausdorff formula and the result of (d), or expand and compute $\exp(\mathbb{i}\theta\hat{L}_z)$ explicitly.]

Solution:

(a) Complete orthonormal basis can be chosen as the occupation basis,
$|\text{vac}\rangle \equiv |n_1 = 0, n_2 = 0\rangle$, $\hat{f}_1^\dagger|\text{vac}\rangle \equiv |n_1 = 1, n_2 = 0\rangle$, $\hat{f}_2^\dagger|\text{vac}\rangle \equiv |n_1 = 0, n_2 = 1\rangle$, $\hat{f}_1^\dagger\hat{f}_2^\dagger|\text{vac}\rangle \equiv |n_1 = 1, n_2 = 1\rangle$.

Other choices are possible, but this choice(sequence) will be convenient later.

(b) Use $\{\hat{f}_i, \hat{f}_j\} = 0$, $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{i,j}$.

$\{\eta_1, \eta_1\} = \{\hat{f}_1 + \hat{f}_1^\dagger, \hat{f}_1 + \hat{f}_1^\dagger\} = 0 + \{\hat{f}_1, \hat{f}_1^\dagger\} + \{\hat{f}_1^\dagger, \hat{f}_1\} + 0 = 2,$

$\{\eta_2, \eta_2\} = -\{\hat{f}_1 - \hat{f}_1^\dagger, \hat{f}_1 - \hat{f}_1^\dagger\} = -0 + \{\hat{f}_1, \hat{f}_1^\dagger\} + \{\hat{f}_1^\dagger, \hat{f}_1\} - 0 = 2,$

similarly, $\{\eta_3, \eta_3\} = \{\eta_4, \eta_4\} = \{\hat{f}_2, \hat{f}_2^\dagger\} + \{\hat{f}_2^\dagger, \hat{f}_2\} = 2.$

It is also easy to see that $\{\eta_1, \eta_3\} = \{\eta_1, \eta_4\} = \{\eta_2, \eta_3\} = \{\eta_2, \eta_4\} = 0$, because each term in the expansion involves creation/annihilation operators of different fermion modes.

$\{\eta_1, \eta_2\} = -\mathbb{i}\hat{f}_1 + \hat{f}_1^\dagger, \hat{f}_1 - \hat{f}_1^\dagger\} = -\mathbb{i}(0 - \{\hat{f}_1, \hat{f}_1^\dagger\} + \{\hat{f}_1^\dagger, \hat{f}_1\} - 0) = 0,$

similarly $\{\eta_3, \eta_4\} = -\mathbb{i}(0 - \{\hat{f}_2, \hat{f}_2^\dagger\} + \{\hat{f}_2^\dagger, \hat{f}_2\} - 0) = 0.$

(c) Denote the complete orthonormal basis in (a) as $|e_i\rangle$ with $i = 1, 2, 3, 4$, the matrix element $O_{ji} \equiv \langle e_j|\hat{O}|e_i\rangle$ of an operator $\hat{O}$ can be obtained by $\hat{O}|e_i\rangle = \sum_j |e_j\rangle \cdot O_{ji}$.

$\hat{f}_1$ is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, because $\hat{f}_1|0,0\rangle = 0$, $\hat{f}_1|1,0\rangle = |0,0\rangle$, $\hat{f}_1|0,1\rangle = 0$, $\hat{f}_1|1,1\rangle = |0,1\rangle$.

$\hat{f}_2$ is $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, because $\hat{f}_2|0,0\rangle = 0$, $\hat{f}_2|1,0\rangle = 0$, $\hat{f}_2|0,1\rangle = |0,0\rangle$, $\hat{f}_2|1,1\rangle = -|1,0\rangle$.

Be careful about the minus sign here.

$\hat{\eta}_1$ is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \sigma_0 \otimes \sigma_1$, $\hat{\eta}_2$ is $\begin{pmatrix} 0 & -\mathbb{i} & 0 & 0 \\ \mathbb{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{i} \\ 0 & 0 & \mathbb{i} & 0 \end{pmatrix} = \sigma_0 \otimes \sigma_2,$

$\hat{\eta}_3$ is $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3$, $\hat{\eta}_4$ is $\begin{pmatrix} 0 & 0 & -\mathbb{i} & 0 \\ 0 & 0 & 0 & \mathbb{i} \\ \mathbb{i} & 0 & 0 & 0 \\ 0 & -\mathbb{i} & 0 & 0 \end{pmatrix} = \sigma_2 \otimes \sigma_3.$

(d) Method #1: use the anti-commutation relations in (a).

Note that $\hat{\eta}_i^2 = 1$, and $\hat{\eta}_i\hat{\eta}_j = -\hat{\eta}_j\hat{\eta}_i$ if $i \neq j$.

$\hat{L}_x\hat{L}_y = -\hat{\eta}_3\hat{\eta}_4\hat{\eta}_4\hat{\eta}_2 = -\hat{\eta}_3\hat{\eta}_2 = \hat{\eta}_2\hat{\eta}_3 = -\mathbb{i}\hat{L}_z$.

$\hat{L}_y\hat{L}_x = -\hat{\eta}_4\hat{\eta}_2\hat{\eta}_3\hat{\eta}_4 = -\hat{\eta}_2\hat{\eta}_4\hat{\eta}_4\hat{\eta}_3 = -\hat{\eta}_2\hat{\eta}_3 = \mathbb{i}\hat{L}_z$.

Therefore $[\hat{L}_x, \hat{L}_y] = -2\mathbb{i}\hat{L}_z$. By cyclic permutation of subscripts $2, 3, 4$ of $\hat{\eta}$, we have $[\hat{L}_y, \hat{L}_z] = -2\mathbb{i}\hat{L}_x$, and $[\hat{L}_z, \hat{L}_x] = -2\mathbb{i}\hat{L}_y$.

$\hat{L}_x^2 = -\hat{\eta}_3\hat{\eta}_4\hat{\eta}_3\hat{\eta}_4 = \hat{\eta}_3\hat{\eta}_4\hat{\eta}_4\hat{\eta}_3 = \hat{\eta}_3\hat{\eta}_3 = 1$, and by cyclic permutation of subscripts $2, 3, 4$ of $\hat{\eta}$, we have $\hat{L}_y^2 = \hat{L}_z^2 = 1$.

Method #2: use the result of (c),

The matrix form of $\hat{L}_{x,y,z}$ under the basis of (a) are

$\hat{L}_x$: $\mathbb{i}(\sigma_1 \otimes \sigma_3) \cdot (\sigma_2 \otimes \sigma_3) = \mathbb{i}(\sigma_1\sigma_2 \otimes \sigma_3\sigma_3) = -\sigma_3 \otimes \sigma_0$;

$\hat{L}_y$: $\mathbb{i}(\sigma_2 \otimes \sigma_3) \cdot (\sigma_0 \otimes \sigma_2) = \mathbb{i}(\sigma_2\sigma_0 \otimes \sigma_3\sigma_2) = \sigma_2 \otimes \sigma_1$;

$\hat{L}_z$: $\mathbb{i}(\sigma_0 \otimes \sigma_2) \cdot (\sigma_1 \otimes \sigma_3) = \mathbb{i}(\sigma_0\sigma_1 \otimes \sigma_2\sigma_3) = -\sigma_1 \otimes \sigma_1$.

One can then use the multiplication rules for Pauli matrices to produce their commutation relations.

(e) From the result of (c), $\hat{L}_x$ is $(-\sigma_3 \otimes \sigma_0)$ under the basis of (a), which is already diagonal.

In fact $\hat{L}_x = 2\hat{f}_2^\dagger \hat{f}_2 - 1$, according to the definition of $\hat{\eta}_i$.

The occupation basis are normalized eigenstates of $\hat{L}_x$.

$\hat{L}_x$ has

eigenvalue $+1$ for $|n_1 = 0, n_2 = 1\rangle$ and $|n_1 = 1, n_2 = 1\rangle$;

and eigenvalue -1 for $|n_1 = 0, n_2 = 0\rangle$ and $|n_1 = 1, n_2 = 0\rangle$.

(f) From the result of (c), $\hat{L}_z$ is $(-\sigma_1 \otimes \sigma_1)$ under the basis of (a).

This is similar to the operator in Homework #1 Problem 5(a), with an additional minus sign.

σ_1 has eigenvalue $+1$ with normalized eigenvector $\frac{1}{\sqrt{2}}(1, 1)^T$; and eigenvalue -1 with normalized eigenvector $\frac{1}{\sqrt{2}}(1, -1)$.

Therefore $\hat{L}_z$ has

eigenvalue $+1$ for normalized eigenvectors $\frac{1}{\sqrt{2}}(1, 1)^T \otimes \frac{1}{\sqrt{2}}(1, -1)^T = \frac{1}{2}(1, -1, 1, -1)^T$ and $\frac{1}{\sqrt{2}}(1, -1)^T \otimes \frac{1}{\sqrt{2}}(1, 1)^T = \frac{1}{2}(1, 1, -1, -1)^T$; and

eigenvalue -1 for normalized eigenvectors $\frac{1}{\sqrt{2}}(1, 1)^T \otimes \frac{1}{\sqrt{2}}(1, 1)^T = \frac{1}{2}(1, 1, 1, 1)^T$ and

$\frac{1}{\sqrt{2}}(1,-1)^T \otimes \frac{1}{\sqrt{2}}(1,-1)^T = \frac{1}{2}(1,-1,-1,1)^T$.

In fact, the 4×4 matrix $(-\sigma_1 \otimes \sigma_1)$ is already block-diagonalized, with two copies of 2×2 diagonal block $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ for the subspaces spanned by basis $(|\mathrm{vac}\rangle, \hat{f}_1^\dagger \hat{f}_2^\dagger|\mathrm{vac}\rangle)$ and $(\hat{f}_1^\dagger|\mathrm{vac}\rangle, \hat{f}_2^\dagger|\mathrm{vac}\rangle)$ respectively. It has

eigenvalue -1 for $\frac{1}{\sqrt{2}}(|\mathrm{vac}\rangle + \hat{f}_1^\dagger \hat{f}_2^\dagger|\mathrm{vac}\rangle)$ and $\frac{1}{\sqrt{2}}(\hat{f}_1^\dagger|\mathrm{vac}\rangle + \hat{f}_2^\dagger|\mathrm{vac}\rangle)$; and

eigenvalue $+1$ for $\frac{1}{\sqrt{2}}(|\mathrm{vac}\rangle - \hat{f}_1^\dagger \hat{f}_2^\dagger|\mathrm{vac}\rangle)$ and $\frac{1}{\sqrt{2}}(\hat{f}_1^\dagger|\mathrm{vac}\rangle - \hat{f}_2^\dagger|\mathrm{vac}\rangle)$.

(g) Method #1: use Baker-Hausdorff formula.

This problem is the same as Homework #1 Problem 3, if you make the following replacement there, $\hat{A} \to \frac{1}{2\mathbb{i}}\hat{L}_z$, $\hat{B} \to \frac{1}{2\mathbb{i}}\hat{L}_x$, $\hat{C} \to \frac{1}{2\mathbb{i}}\hat{L}_x$, $\theta \to -2\theta$, $a \to 2\mathbb{i}c_3$, $b \to 2\mathbb{i}c_1$, $c \to 2\mathbb{i}c_2$.

The result of Homework #1 Problem 3 is $a\hat{A} + \hat{B}(b\cos\theta + c\sin\theta) + \hat{C}(c\cos\theta - b\sin\theta)$ (check the Solution to Homework #1). Therefore $\exp(\mathbb{i}\theta\hat{L}_z)\cdot(c_1\hat{L}_x + c_2\hat{L}_y + c_3\hat{L}_z)\cdot\exp(-\mathbb{i}\theta\hat{L}_z)$ $= c_3\hat{L}_z + \hat{L}_x[c_1\cos(2\theta) - c_2\sin(2\theta)] + \hat{L}_y[c_2\cos(2\theta) + c_1\sin(2\theta)]$

Method #2: use the result of (d).

Use $\hat{L}_z^2 = 1$ from (d),

$\exp(\mathbb{i}\theta\hat{L}_z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\theta^{2m} + \sum_{m=0}^{\infty} \mathbb{i}\frac{(-1)^m}{(2m+1)!}\theta^{2m+1}\hat{L}_z = \cos\theta + \mathbb{i}\sin\theta\hat{L}_z$.

Further use $\hat{L}_z\hat{L}_x = -\hat{L}_x\hat{L}_z = -\mathbb{i}\hat{L}_y$, and $\hat{L}_z\hat{L}_y = -\hat{L}_y\hat{L}_z = \mathbb{i}\hat{L}_x$, we have

$\exp(\mathbb{i}\theta\hat{L}_z)\cdot\hat{L}_x\cdot\exp(-\mathbb{i}\theta\hat{L}_z) = (\cos\theta + \mathbb{i}\sin\theta\hat{L}_z)\cdot\hat{L}_x\cdot(\cos\theta - \mathbb{i}\sin\theta\hat{L}_z)$
$= (\cos\theta\hat{L}_x + \sin\theta\hat{L}_y)\cdot(\cos\theta - \mathbb{i}\sin\theta\hat{L}_z) = (\cos^2\theta - \sin^2\theta)\hat{L}_x + 2\cos\theta\sin\theta\hat{L}_y$
$= \cos(2\theta)\hat{L}_x + \sin(2\theta)\hat{L}_y$, and
$\exp(\mathbb{i}\theta\hat{L}_z)\cdot\hat{L}_y\cdot\exp(-\mathbb{i}\theta\hat{L}_z) = (\cos\theta + \mathbb{i}\sin\theta\hat{L}_z)\cdot\hat{L}_y\cdot(\cos\theta - \mathbb{i}\sin\theta\hat{L}_z)$
$= (\cos\theta\hat{L}_y - \sin\theta\hat{L}_y)\cdot(\cos\theta - \mathbb{i}\sin\theta\hat{L}_z) = (\cos^2\theta - \sin^2\theta)\hat{L}_y - 2\cos\theta\sin\theta\hat{L}_x$
$= \cos(2\theta)\hat{L}_y - \sin(2\theta)\hat{L}_x$, and obviously
$\exp(\mathbb{i}\theta\hat{L}_z)\cdot\hat{L}_z\cdot\exp(-\mathbb{i}\theta\hat{L}_z) = \hat{L}_z$.

These lead to $\exp(\mathbb{i}\theta\hat{L}_z)\cdot(c_1\hat{L}_x + c_2\hat{L}_y + c_3\hat{L}_z)\cdot\exp(-\mathbb{i}\theta\hat{L}_z)$
$= c_1\cdot[\cos(2\theta)\hat{L}_x + \sin(2\theta)\hat{L}_y] + c_2\cdot[\cos(2\theta)\hat{L}_y - \sin(2\theta)\hat{L}_x] + c_3\hat{L}_z$
$= c_3\hat{L}_z + \hat{L}_x[c_1\cos(2\theta) - c_2\sin(2\theta)] + \hat{L}_y[c_2\cos(2\theta) + c_1\sin(2\theta)]$