Final Exam (Quantum Statistical Physics; 2018-2019 Fall Sem.; @PKU)

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I. NOTATION, CONVENTION, AND USEFUL EQUATIONS

k_F: Fermi wavelength. We set the speed of light to be 1, e: eletron charge (positive; $-e$ is the electron charge). m: electron mass. $\hbar$: Plank constant. ϵ_0: permittivity of vacuum. A step function $\theta(x)$ is defined as

$$\theta(x) = \begin{cases} 1 & (x>0) \\ 0 & (x<0) \end{cases} \tag{1}$$

A sign function sgn$x$ is defined as

$$\mathrm{sgn}\, x = \begin{cases} 1 & (x>0) \\ -1 & (x<0) \end{cases} \tag{2}$$

In Secs.V and VI, we use $\alpha, \beta, \delta, \gamma$ for the spin idex ($=\uparrow,\downarrow$), while use $\mu, \nu, \cdots$ for the spin component index ($=x,y,z$). 2 by 2 Pauli matrices are denoted by $[\sigma_\mu]_{\alpha\beta}$ ($\mu = x,y,z$);

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3}$$

A Fourier transform of the Coulomb interaction is often used;

$$\int d^3\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{|\boldsymbol{x}|} e^{-\lambda|\boldsymbol{x}|} = \frac{4\pi}{k^2+\lambda^2} \tag{4}$$

with $\lambda \geq 0$. with $k \equiv |\boldsymbol{k}|$. A Fourier transform of a convolution of two functions is a product of respective Fourier transforms;

$$\int d^3\boldsymbol{x} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} (f*g)(\boldsymbol{x}) = f(\boldsymbol{k})g(\boldsymbol{k}), \tag{5}$$

where

$$\begin{aligned} (f*g)(\boldsymbol{x}) &\equiv \int d^3\boldsymbol{y} f(\boldsymbol{y}) g(\boldsymbol{x}-\boldsymbol{y}), \\ f(\boldsymbol{k}) &\equiv \int d^3\boldsymbol{x} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}). \end{aligned}$$

II. PROBLEM.1

Consider an equation of motion for the longitudinal acoustic phonon in continous media;

$$\rho_m \partial_t^2 \boldsymbol{d} - B\boldsymbol{\nabla}^2 \boldsymbol{d} \equiv \rho_m \partial_t^2 \boldsymbol{d} - B\Big(\sum_{j=1,2,3} \nabla_j^2\Big)\boldsymbol{d} = 0 \tag{6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{d} = 0 \tag{7}$$

where $\boldsymbol{d}(\boldsymbol{x}, t)$ is the displacement field of lattice points at x. B is the bulk modulus and ρ_m is the mass of the lattice points per volume;

$$\rho_m \equiv \frac{M}{V} \tag{8}$$

with M being total mass of the lattice (In a simple-substance materials, M is simply mass of ion times total number of ions).

Show that the following Lagrangian yields the first equation of motion.

$$L_0 \equiv \frac{1}{2}\int dx^3 \int dt\Big[\rho_m \partial_t d_i \partial_t d_i - B\partial_j d_i \partial_j d_i\Big] \tag{9}$$

Show that, under the rotation-free condition, this Lagrangian is equivalent to the following Lagrangian,

$$L_0 = \frac{1}{2}\int dx^3 \int dt\Big[\rho_m \partial_t d_i \partial_t d_i - B\partial_j d_j \partial_i d_i\Big] \equiv \int dt \tilde{L}_0 \tag{10}$$

Based on the canonical procedure, the Hamiltonian is given by

$$H_0 = \int d^3x \tilde{\Pi}_j \partial_t \tilde{d}_j - \tilde{L}_0 = \frac{1}{2}\int d^3x \tilde{\Pi}_j \tilde{\Pi}_j + \frac{1}{2}\int d^3x \frac{B}{\rho_m}\left(\nabla\cdot \boldsymbol{d}\right)^2. \tag{11}$$

where $\tilde{\boldsymbol{\Pi}}(\boldsymbol{x})$ is the canonical conjugate momentum to the displacement field $\tilde{\boldsymbol{d}}(\boldsymbol{x})$;

$$\Pi_j = \frac{\delta L_0}{\delta \partial_t d_j} = \rho_m \partial_t d_j \tag{12}$$

or equivalently,

$$\tilde{\Pi}_j \equiv \frac{1}{\sqrt{\rho_m}}\Pi_j, \quad \tilde{d}_j \equiv \sqrt{\rho_m} d_j.$$

with $\tilde{\Pi}_j = \partial_t \tilde{d}_j$. Imposing the commutation relation of the momentum and displacement field,

$$[\tilde{d}_j(\boldsymbol{x}), \tilde{\Pi}_m(\boldsymbol{x}')] = i\hbar\delta_{jm}\delta^3(\boldsymbol{x}-\boldsymbol{x}'),$$
$$[\tilde{d}_j(\boldsymbol{x}), \tilde{d}_m(\boldsymbol{x}')] = [\tilde{\Pi}_j(\boldsymbol{x}), \tilde{\Pi}_m(\boldsymbol{x}')] = 0,$$

show that this Hamiltonian is second-quantized into the following quadratic form of a boson field,

$$H_0 \equiv \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\hbar\omega_{\boldsymbol{k}} a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + \text{constant} \tag{13}$$

with $[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')$. **Express** the boson field thus intrudced in terms of the displacement field and the conjugate momentum field. **Explain** why a phonon-contribution to the low-temperature specific heat vanishes as T^3.

III. PROBLEM.2 (ELECTRON GAS UNDER MAGNETIC FIELD)

Consider a two-dimensional free electron gas under magnetic field B (applied perpendicular to the 2-d plane);

$$\mathcal{H} = \frac{1}{2m}(\hat{\Pi}_x^2 + \hat{\Pi}_y^2), \tag{14}$$

with $\hat{\Pi}_\mu = \hat{p}_\mu + eA_\mu$ ($\mu = x, y$) and $[\hat{\Pi}_x, \hat{\Pi}_y] = i\hbar eB \equiv -i\frac{\hbar^2}{l^2}$ ($l \equiv \sqrt{\frac{\hbar}{eB}}$: magnetic length). **Show** that this Hamiltonian reduces to a harmonic oscillator's Hamilotnian;

$$H = \hbar\omega_c(\hat{a}^\dagger\hat{a} + \frac{1}{2}). \tag{15}$$

with $\hbar\omega_c \equiv \frac{\hbar eB}{m}$ and $[\hat{a}, \hat{a}^\dagger] = 1$. Consider that the 2DEG under the magnetic field B is further subjected under a uniform electric field E along the x-direction;

$$H = \frac{1}{2m}\left(\hat{p}_x^2 + (\hat{p}_y - eB\hat{x})^2\right) - eE\hat{x} \tag{16}$$

where we use the Landau gauge, respecting the translational symmetry along the y-direction. **Obtain** the eigenstates and eigenvalues.

Suppose a fermion-fermion interaction potential comprises of spin-independent and spin-dependent parts;

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \tag{17}$$

$$\mathcal{H}_0 = \sum_{\alpha=\uparrow,\downarrow} \int dx^3 \psi^\dagger_\alpha(\boldsymbol{x})\Big(-\frac{\hbar^2\boldsymbol{\nabla}^2}{2m}\Big)\psi_\alpha(\boldsymbol{x}), \tag{18}$$

$$\mathcal{H}_1 = \frac{1}{2}\int d^3\boldsymbol{x} d^3\boldsymbol{x}' V_{\alpha\beta,\gamma\delta}(\boldsymbol{x}-\boldsymbol{x}')\psi^\dagger_\alpha(\boldsymbol{x})\psi^\dagger_\gamma(\boldsymbol{x}')\psi_\delta(\boldsymbol{x}')\psi_\beta(\boldsymbol{x}), \tag{19}$$

$$V_{\alpha\beta,\gamma\delta}(\boldsymbol{x}-\boldsymbol{x}') = V_0(\boldsymbol{x}-\boldsymbol{x}')\,\delta_{\alpha\beta}\delta_{\gamma\delta} + V_1(\boldsymbol{x}-\boldsymbol{x}')\,[\sigma_\mu]_{\alpha\beta}[\sigma_\mu]_{\gamma\delta}. \tag{20}$$

The summation over the repeated spin indices (α,β, γ,δ=$\uparrow$,$\downarrow$) were and will be omitted henceforth unless dictated otherwise. The same convention for the repeated spin component indices $\mu, \nu = x, y, z$. Suppose that the system is subjected under an impulsive magnetic field;

$$\mathcal{H}_{\text{ext}} = \int d^3\boldsymbol{x} s_\nu(\boldsymbol{x}) h^{\text{ex}}_\nu(\boldsymbol{x}, t) \tag{21}$$

with

$$2s_\nu(\boldsymbol{x}) \equiv \psi^\dagger_\alpha(\boldsymbol{x})[\sigma_\nu]_{\alpha\beta}\psi_\beta(\boldsymbol{x}), \;\; h^{\text{ex}}_\nu(\boldsymbol{x}, t) = e^{i\boldsymbol{q}\boldsymbol{x}} h_\nu \delta(t). \tag{22}$$

<u>**Show**</u> that, within the linear order in the external field, the induced spin density is given by the retarded spin density correlation function;

$$\langle\Psi(t)|\hat{s}_\mu(\boldsymbol{x})|\Psi(t)\rangle \equiv \langle s_\mu(\boldsymbol{x}, t)\rangle = \frac{1}{\hbar}\int_{-\infty}^{+\infty} dt' D^R_{\mu\nu}(\boldsymbol{x}, t; \boldsymbol{x}', t') h^{\text{ex}}_\nu(\boldsymbol{x}', t'), \tag{23}$$

with

$$iD^R_{\mu\nu}(\boldsymbol{x}, t; \boldsymbol{x}', t') = \theta(t-t')\frac{\langle\Psi_0|\big[\bar{s}_{H,\mu}(\boldsymbol{x}, t), \bar{s}_{H,\nu}(\boldsymbol{x}', t')\big]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}, \tag{24}$$

and

$$\bar{s}_{H,\mu}(\boldsymbol{x}, t) \equiv e^{i\frac{\mathcal{H}t}{\hbar}} s_\mu(\boldsymbol{x}) e^{-i\frac{\mathcal{H}t}{\hbar}} - \langle\Psi_0|s_\mu(\boldsymbol{x})|\Psi_0\rangle. \tag{25}$$

Here $|\Psi_0\rangle$ denotes the ground state wavefunction of $\mathcal{H}$, while $|\Psi(t)\rangle$ evolves in time according to $\mathcal{H} + \mathcal{H}_{\text{ext}}$ starting from $|\Psi_0\rangle$;

$$i\partial_t|\Psi(t)\rangle = \big(\mathcal{H} + \mathcal{H}_{\text{ext}}\big)|\Psi(t)\rangle, \tag{26}$$

with $|\Psi(t = -\infty)\rangle \equiv |\Psi_0\rangle$.

Using the Lehmann representation, <u>**prove**</u> the following relation (Eqs. (28,29)) between the Fourier series of the retarded function and the time-ordered correlation function;

$$iD_{\mu\nu}(\boldsymbol{x}, t; \boldsymbol{x}', t') = \frac{\langle\Psi_0|T\{\bar{s}_{H,\mu}(\boldsymbol{x}, t)\bar{s}_{H,\nu}(\boldsymbol{x}', t')\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}. \tag{27}$$

The relation is given by

$$\text{Re}D_{\mu\nu}(\boldsymbol{k}, \omega) = \text{Re}D^R_{\mu\nu}(\boldsymbol{k}, \omega), \tag{28}$$

$$\text{Im}D_{\mu\nu}(\boldsymbol{k}, \omega) = \text{sign}(\omega)\text{Im}D^R_{\mu\nu}(\boldsymbol{k}, \omega). \tag{29}$$

The Fourier series are given by

$$D^{(R)}_{\mu\nu}(\boldsymbol{k}, \omega) = \int d^3\boldsymbol{x}\int dt\, e^{-i\boldsymbol{k}\boldsymbol{x}+i\omega t} D^{(R)}_{\mu\nu}(\boldsymbol{x}, t; \boldsymbol{0}, 0). \tag{30}$$

Using the Hint given in Sec.VA, **show** that, within the ring approximation, the Fourier series of the time-ordered density correlation function is given by

$$\frac{1}{\hbar}D_{\mu\nu}(\boldsymbol{k},\omega) = \frac{\delta_{\mu\nu}}{4}\frac{\frac{1}{\hbar}D^{\star}_{(r)}(\boldsymbol{k},\omega)}{1-\frac{1}{\hbar}D^{\star}_{(r)}(\boldsymbol{k},\omega)V_1(\boldsymbol{k})}. \tag{31}$$

$V_1(\boldsymbol{k})$ is the Fourier series of spin-dependent interaction potential;

$$V_1(\boldsymbol{k}) = \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{k}\boldsymbol{x}}\, V_1(\boldsymbol{x}).$$

The bare polarization part $\frac{1}{\hbar}D^{\star}_{(r)}(\boldsymbol{k},\omega)$ is given by the non-interacting single-particle Green's function as

$$\frac{1}{\hbar}D^{\star}_{(r)}(\boldsymbol{k},\omega) = -\frac{2i}{\hbar}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\int\frac{d\omega'}{2\pi}G^0(\boldsymbol{k}+\boldsymbol{k}',\omega+\omega')G^0(\boldsymbol{k}',\omega'), \tag{32}$$

$$G^0(\boldsymbol{k},\omega) = \frac{\theta(|\boldsymbol{k}|-k_F)}{\omega-\omega_{\boldsymbol{k}}+i\eta}+\frac{\theta(k_F-|\boldsymbol{k}|)}{\omega-\omega_{\boldsymbol{k}}-i\eta}, \tag{33}$$

with $\hbar\omega_{\boldsymbol{k}} = \frac{\hbar^2\boldsymbol{k}^2}{2m}$ and infinitesimally small positive η.

Without loss of generality, we can assume that $V_{0,1}(\boldsymbol{k})$ are real-valued ($V_{0,1}(\boldsymbol{x}) = V_{0,1}(-\boldsymbol{x})$). Thus, the Fourier series of the retarded function is obtained from eqs. (28,29) as;

$$\frac{1}{\hbar}D^{R}_{\mu\nu}(\boldsymbol{k},\omega) = \frac{\delta_{\mu\nu}}{4}\frac{\frac{1}{\hbar}D^{\star,R}_{(r)}(\boldsymbol{k},\omega)}{1-\frac{1}{\hbar}D^{\star,R}_{(r)}(\boldsymbol{k},\omega)V_1(\boldsymbol{k})}, \tag{34}$$

with

$$\mathrm{Re}D^{\star,R}_{(r)}(\boldsymbol{k},\omega) = \mathrm{Re}D^{\star}_{(r)}(\boldsymbol{k},\omega), \tag{35}$$

$$\mathrm{Im}D^{\star,R}_{(r)}(\boldsymbol{k},\omega) = \mathrm{sign}(\omega)\mathrm{Im}D^{\star}_{(r)}(\boldsymbol{k},\omega). \tag{36}$$

Using eqs. (23,30,34), **show** that, within the ring approximation, the spin density induced by the impulsive magnetic field (eqs. (21,22)) is given by

$$\langle s_\mu(\boldsymbol{x},t)\rangle \propto h_\mu e^{i\boldsymbol{q}\boldsymbol{x}-i\Omega_{\boldsymbol{q}}t-\gamma_{\boldsymbol{q}}t} \tag{37}$$

where $\Omega_{\boldsymbol{q}}$ and $\gamma_{\boldsymbol{q}}$ are obtained as a pole of eq. (34);

$$1 = V_1(\boldsymbol{q})\frac{1}{\hbar}D^{\star,R}_{(r)}(\boldsymbol{q},\Omega_{\boldsymbol{q}}-i\gamma_{\boldsymbol{q}}). \tag{38}$$

In the longe wavelength limit ($|\boldsymbol{q}|\to 0$), we can assume that $\Omega_{\boldsymbol{q}}$ thus determined is porportional to $|\boldsymbol{q}|$;

$$\Omega_{\boldsymbol{q}} = c_s|\boldsymbol{q}|. \tag{39}$$

Show that, in this limit, $\gamma_{\boldsymbol{q}} = 0$ when $\frac{mc_s}{\hbar k_F} > 1$. Assuming further that $\frac{mc_s}{\hbar k_F} > 1$ and $V_1(\boldsymbol{q}=0) > 0$, **derive** an expression for c_s in the following two limiting cases;

$$V_1(\boldsymbol{q}=0) \ll \frac{\pi^2\hbar^2}{mk_F}\quad\text{(weak coupling)} \tag{40}$$

$$V_1(\boldsymbol{q}=0) \gg \frac{\pi^2\hbar^2}{mk_F}\quad\text{(strong coupling)} \tag{41}$$

V. PROBLEM.4 (UNDER THE ZEEMAN FIELD)

Consider that the system is under a static and uniform Zeeman field along the z-direction (the fermion-fermion interaction part is same as in Problem.3; Eqs. (19,20));

$$\mathcal{H}_0 = \sum_{\alpha=\uparrow,\downarrow}\int dx^3\psi^\dagger_\alpha(\boldsymbol{x})\Big(-\frac{\hbar^2\boldsymbol{\nabla}^2}{2m}+H^0_z[\sigma_z]_{\alpha\alpha}\Big)\psi_\alpha(\boldsymbol{x}), \tag{42}$$

$iD_{\alpha\beta}(\cdots)$ with $\alpha, \beta = \uparrow, \downarrow$ are defined as follows

$$iD_{\alpha\beta}(\boldsymbol{x}, t; \boldsymbol{x}', t') = \frac{\langle\Psi_0|T\{\tilde{\rho}_{H,\alpha}(\boldsymbol{x}, t)\tilde{\rho}_{H,\beta}(\boldsymbol{x}', t')\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{44}$$

and

$$\tilde{\rho}_{H,\alpha}(\boldsymbol{x}, t) \equiv e^{i\frac{\mathcal{H}t}{\hbar}}\psi^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x})e^{-i\frac{\mathcal{H}t}{\hbar}} - \frac{\langle\Psi_0|\psi^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{45}$$

for $\alpha = \uparrow, \downarrow$ respectively (the summation over α is not assumed above). Using the hint in Sec.VA, **show** that the Dyson equation for these four generally takes the following form in the momentum space;

$$\begin{aligned}\frac{1}{\hbar}\begin{pmatrix} D_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix} &= \frac{1}{\hbar}\begin{pmatrix} D^\star_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D^\star_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix} \\ &+ \frac{1}{\hbar^2}\begin{pmatrix} D^\star_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D^\star_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix}\begin{pmatrix} V_0+V_1 & V_0-V_1 \\ V_0-V_1 & V_0+V_1\end{pmatrix}\begin{pmatrix} D_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix}.\end{aligned} \tag{46}$$

where $V_0 \pm V_1 \equiv V_0(\boldsymbol{k}) \pm V_1(\boldsymbol{k})$ and $D^\star_{\alpha\beta}(\boldsymbol{k},\omega)$ $(\alpha, \beta = \uparrow, \downarrow)$ denotes the proper part of the density correlation function.

Within the ring diagram approximation, one may approxiate the proper part by its lowest order in the interactions (i.e. the zero-th order);

$$\frac{1}{\hbar}\begin{pmatrix} D^\star_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D^\star_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix} \simeq \begin{pmatrix} \Pi_{(0),\uparrow}(\boldsymbol{k},\omega) & 0 \\ 0 & \Pi_{(0),\downarrow}(\boldsymbol{k},\omega)\end{pmatrix}. \tag{47}$$

where neglected terms are on the first order of V_0 and V_1.

$$\Pi_{(0),\alpha}(\boldsymbol{k},\omega) = -\frac{i}{\hbar}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\int\frac{d\omega'}{2\pi}G^0(\boldsymbol{k}+\boldsymbol{k}', \omega+\omega', \alpha)G^0(\boldsymbol{k}', \omega', \alpha), \tag{48}$$

$$G^0(\boldsymbol{k}, \omega, \alpha) = \frac{\theta(|\boldsymbol{k}| - k_{F,\alpha})}{\omega - \omega_{\boldsymbol{k}} - (\sigma_z)_{\alpha\alpha}H^0_z + i\eta} + \frac{\theta(k_{F,\alpha} - |\boldsymbol{k}|)}{\omega - \omega_{\boldsymbol{k}} - (\sigma_z)_{\alpha\alpha}H^0_z - i\eta}, \tag{49}$$

with $\alpha = \uparrow, \downarrow$ respectively. $k_{F,\alpha}$ denotes the Fermi momentum for α-spin in the presence of the static Zeeman field;

$$\frac{\hbar^2 k^2_{F,\uparrow}}{2m} + H^0_z = \frac{\hbar^2 k^2_{F,\downarrow}}{2m} - H^0_z \equiv \mu. \tag{50}$$

Obtain $\Pi_{0,\alpha}(\boldsymbol{k},\omega)$ for $\alpha = \uparrow, \downarrow$. By sovling eq. (46) with eq. (47), **show** that $D_{zz}(\boldsymbol{k},\omega)$ in eq. (43) is given by,

$$4D_{zz}(\boldsymbol{k},\omega) = \frac{\hbar\left(\Pi_{(0),\uparrow} + \Pi_{(0),\downarrow} - 4\Pi_{(0),\uparrow}\Pi_{(0),\downarrow}V_0\right)}{1 - (\Pi_{(0),\uparrow} + \Pi_{(0),\downarrow})(V_0+V_1) + 4\Pi_{(0),\uparrow}\Pi_{(0),\downarrow}V_0V_1} \tag{51}$$

Noting that $V_0(\boldsymbol{k})$ and $V_1(\boldsymbol{k})$ are real-valued, one obtain the corresponding retarded density correlation function as

$$4D^R_{zz}(\boldsymbol{k},\omega) = \frac{\hbar\left(\Pi^R_{(0),\uparrow} + \Pi^R_{(0),\downarrow} - 4\Pi^R_{(0),\uparrow}\Pi^R_{(0),\downarrow}V_0\right)}{1 - (\Pi^R_{(0),\uparrow} + \Pi^R_{(0),\downarrow})(V_0+V_1) + 4\Pi^R_{(0),\uparrow}\Pi^R_{(0),\downarrow}V_0V_1} \tag{52}$$

with

$$\begin{aligned}\mathrm{Re}\Pi^R_{(0),\alpha}(\boldsymbol{k},\omega) &= \mathrm{Re}\Pi_{(0),\alpha}(\boldsymbol{k},\omega), \\ \mathrm{Im}\Pi^R_{(0),\alpha}(\boldsymbol{k},\omega) &= \mathrm{sgn}\omega\,\mathrm{Im}\Pi_{(0),\alpha}(\boldsymbol{k},\omega).\end{aligned}$$

In order to derive eqs. (31,32,33) or eq. (46), let us begin with

$$G^{(c)}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) \equiv G_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) - G_{\alpha\delta}(x,x+)G_{\beta\gamma}(x',x'+). \tag{53}$$

where the time-ordered single-particle Green's function and two-particle Green's function are defined as usual;

$$iG_{\alpha\beta}(x,x') \equiv \frac{\langle\Psi_0|T\{\psi_{H,\alpha}(x)\psi^{\dagger}_{H,\beta}(x')\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}. \tag{54}$$

$$i^2 G_{\alpha\beta,\delta\gamma}(x_1,x_2,;x'_1,x'_2) \equiv \frac{\langle\Psi_0|T\{\psi_{H,\alpha}(x_1)\psi_{H,\beta}(x_2)\psi^{\dagger}_{H,\gamma}(x'_2)\psi^{\dagger}_{H,\delta}(x'_1)\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{55}$$

$|\Psi_0\rangle$ is the ground state wavefunction of $\mathcal{H}$. $x \equiv (\boldsymbol{x},t)$, $x' \equiv (\boldsymbol{x}',t')$, $x+ \equiv (\boldsymbol{x},t+)$ and $x'+ \equiv (\boldsymbol{x}',t'+)$. The time-ordered density correlation functions, such as eq. (27) or eq. (44), are given by the 'connected' two-point Green's function;

$$iD_{\mu\nu}(x,x') = -\frac{1}{4}[\sigma_\mu]_{\delta\alpha}[\sigma_\nu]_{\gamma\delta}G^{(c)}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+), \tag{56}$$

$$iD_{\alpha\beta}(x,x') = -G^{(c)}_{\alpha\beta,\alpha\beta}(x,x';x+,x'+), \tag{57}$$

Note that, in the first line, the summation over the spin indices $(\alpha,\beta,\gamma,\delta)$ are assumed, while, in the second line, the summation over the spin indices are NOT assumed; $\alpha,\beta = \uparrow,\downarrow$. To derive eq. (31) or eq. (46), one can begin with Fourier transform of the following Dyson equation for the 'connected' two-point Green's function;

$$\begin{aligned} G^{(c)}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) = G^{(c),\star}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) + \frac{i}{\hbar}\int d^4x_1 \int d^4x_2 \\ G^{(c),\star}_{\alpha\gamma_1,\delta\gamma_2}(x,x_1;x+,x_1+)\,V_{\gamma_2\gamma_1,\delta_2\delta_1}(\boldsymbol{x}_1-\boldsymbol{x}_2)\delta(t_1-t_2)\,G^{(c)}_{\delta_1\beta,\delta_2\gamma}(x_2,x';x_2+,x'+). \end{aligned} \tag{58}$$

where $G^{(c),\star}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+)$ denotes the so-called 'proper part' of the connected two-point Green's function. Note also that $D^{\star}_{\alpha\beta}(x,x')$, whose Fourier transform is used in eq. (46), is defined by this proper part;

$$iD^{\star}_{\alpha\beta}(x,x') \equiv -G^{(c),\star}_{\alpha\beta,\alpha\beta}(x,x';x+,x'+) \tag{59}$$

In the ring-diagram approximation, we replace the proper part as its bare contribution (lowest order in the interaction potential);

$$G^{(c),\star}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) = -G^0_{\alpha\gamma}(x,x'+)G^0_{\beta\delta}(x',x+) + \mathcal{O}(V). \tag{60}$$

where $G^0_{\alpha\beta}(x,x')$ denotes the bare (non-interacting) single-particle Green's function;

$$iG^0_{\alpha\beta}(x,x') = \frac{\langle\Phi_0|T\{\psi_{I,\alpha}(x)\psi^{\dagger}_{I,\beta}(x')\}|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle} = \delta_{\alpha\beta}\frac{\langle\Phi_0|T\{\psi_I(x)\psi^{\dagger}_I(x')\}|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle} \tag{61}$$

$|\Phi_0\rangle$ is the ground state wavefunction of the non-interacting Hamiltonian $\mathcal{H}_0$. The fourier transform of eq. (61) is calculated to be eq. (33). The minus sign in the r.h.s. of eq. (60) comes from an exchange of the fermion field.

B. Hint for problems 4 and 5

With $H^0_z = 0$, the bare polarization part is given by

$$\Pi_0(\boldsymbol{k},\omega) \equiv \frac{1}{\hbar}D^{\star}_{(r)}(\boldsymbol{k},\omega) = -\frac{2i}{\hbar}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\int\frac{d\omega'}{2\pi}G^0(\boldsymbol{k}+\boldsymbol{k}',\omega+\omega')G^0(\boldsymbol{k}',\omega'), \tag{62}$$

$$G^0(\boldsymbol{k},\omega) = \frac{\theta(|\boldsymbol{k}|-k_F)}{\omega-\omega_{\boldsymbol{k}}+i\eta} + \frac{\theta(k_F-|\boldsymbol{k}|)}{\omega-\omega_{\boldsymbol{k}}-i\eta}. \tag{63}$$

In a proper normalization, $k \equiv k_F q$ and $\omega \equiv \frac{\hbar k_F^2}{m}\nu$, its real part and imaginary part are calculated as follows,

$$\mathrm{Re}\Pi_0(k_F q, \frac{\hbar k_F^2}{m}\nu) = \frac{2mk_F}{\hbar^2}\frac{1}{4\pi^2} \times \Big\{ -1 + \frac{1}{2q}\Big(1 - \big(\frac{\nu}{q} - \frac{q}{2}\big)^2\Big)\ln\Big[\frac{|\nu - \frac{q^2}{2} + q|}{|\nu - \frac{q^2}{2} - q|}\Big] - \frac{1}{2q}\Big(1 - \big(\frac{\nu}{q} + \frac{q}{2}\big)^2\Big)\ln\Big[\frac{|\nu + \frac{q^2}{2} + q|}{|\nu + \frac{q^2}{2} - q|}\Big]\Big\} \quad (64)$$

and

$$\mathrm{Im}\Pi_0(k_F q, \frac{\hbar k_F^2}{m}\nu) = \begin{cases} -\frac{mk_F}{\hbar^2}\frac{1}{4\pi q}\Big[1 - \big(\frac{\nu}{q} - \frac{q}{2}\big)^2\Big] & (q > 2 \ \& \ \frac{q^2}{2} - q \le \nu \le \frac{q^2}{2} + q) \text{ or } (q < 2 \ \& \ -\frac{q^2}{2} + q \le \nu \le \frac{q^2}{2} + q) \\ -\frac{mk_F}{\hbar^2}\frac{\nu}{2\pi q} & (q < 2 \ \& \ 0 \le \nu \le -\frac{q^2}{2} + q) \\ 0 & (\text{otherwise}) \end{cases} \quad (65)$$

VI. PROBLEM 5 (RKKY INTERACTION IN A NON-INTERACTING ELECTRON GAS)

Suppose that a magnetic impurity spin is introduced into a non-interacting electron gas;

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \quad (66)$$

$$\mathcal{H}_0 = \sum_{\alpha=\uparrow,\downarrow}\int dx^3 \psi^\dagger_\alpha(\boldsymbol{x})\Big(-\frac{\hbar^2\boldsymbol{\nabla}^2}{2m}\Big)\psi_\alpha(\boldsymbol{x}), \quad (67)$$

$$\mathcal{H}_1 = J\sum_{\mu=x,y,z} s_\mu(\boldsymbol{x}_{\rm i})S_\mu(\boldsymbol{x}_{\rm i}), \quad (68)$$

$$2s_\mu(\boldsymbol{x}) \equiv \sum_{\alpha,\beta=\uparrow,\downarrow}\psi^\dagger_\alpha(\boldsymbol{x})[\sigma_\mu]_{\alpha,\beta}\psi_\beta(\boldsymbol{x}) \quad (69)$$

where $\boldsymbol{x}_{\rm i}$ stands for the position of the magnetic impurity, and σ_μ denote the 2 by 2 Pauli matrices. We assumed that the electron spin density and the impurity spin interact with each other via a short-range potential (a delta function; 'point-contact' interaction). Regarding the impurity spin as an external perturbation, one can expect that the impurity spin induces spin polariation in an electron gas. Using the linear response theory, <u>show</u> that the induced spin density is given as follows (use also eq. (64)),

$$\langle s_\mu(\boldsymbol{x})\rangle = JV(\boldsymbol{x} - \boldsymbol{x}_{\rm i})S_\mu(\boldsymbol{x}_{\rm i}). \quad (70)$$

$$V(\boldsymbol{y}) \equiv -\frac{mk_F}{8\pi^2\hbar^2}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{y}}\Big\{1 + \frac{1-x^2}{2x}\ln\frac{|1+x|}{|1-x|}\Big\}_{|x = \frac{|\boldsymbol{k}|}{2k_F}}. \quad (71)$$

Consider that two magnetic impurity spins are introduced at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively. The non-interacting electron gas is spin polarized by the impurity spin at $\boldsymbol{x}_1$,

$$\langle \boldsymbol{s}(\boldsymbol{x})\rangle = JV(\boldsymbol{x} - \boldsymbol{x}_1)\boldsymbol{S}(\boldsymbol{x}_1). \quad (72)$$

Such a spin polarization interacts with the other impurity spin at $\boldsymbol{x}_2$ via the short-range potential. Thus, the effective interaction between these two impurity spins can be effectively given by;

$$\mathcal{H}_{\rm RKKY} = J^2V(\boldsymbol{x}_1 - \boldsymbol{x}_2)\boldsymbol{S}(\boldsymbol{x}_1)\cdot\boldsymbol{S}(\boldsymbol{x}_2). \quad (73)$$

<u>Show</u> that

$$V(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \frac{mk_F}{\hbar^2}\Big(\frac{k_F}{\pi}\Big)^3\Big\{\frac{\cos(2k_F|\boldsymbol{x}_1 - \boldsymbol{x}_2|)}{(2k_F|\boldsymbol{x}_1 - \boldsymbol{x}_2|)^3} - \frac{\sin(2k_F|\boldsymbol{x}_1 - \boldsymbol{x}_2|)}{(2k_F|\boldsymbol{x}_1 - \boldsymbol{x}_2|)^4}\Big\} \quad (74)$$