Advanced Quantum Mechanics: Fall 2018
Midterm Exam: Brief Solutions

NOTE: Problems start on page 2. Answer the questions in italic fonts.

Possibly useful facts:

- Pauli matrices: $\sigma_0 = \mathbb{1}_{2\times 2}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

  $\sigma_1\sigma_2 = \mathbb{i}\sigma_3 = -\sigma_2\sigma_1$, $\sigma_2\sigma_3 = \mathbb{i}\sigma_1 = -\sigma_3\sigma_2$, $\sigma_3\sigma_1 = \mathbb{i}\sigma_2 = -\sigma_1\sigma_3$, $\sigma_1^2 = \sigma_2^2 = \sigma_3^3 = \sigma_0$.

  So $\sigma_{1,2,3}$ mutually anti-commute, $\{\sigma_1, \sigma_2\} = \{\sigma_2, \sigma_3\} = \{\sigma_3, \sigma_1\} = 0$,

  and $[\sigma_1, \sigma_2] = 2\mathbb{i}\sigma_3$, $[\sigma_2, \sigma_3] = 2\mathbb{i}\sigma_1$, $[\sigma_3, \sigma_1] = 2\mathbb{i}\sigma_2$.

- $\exp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$, $\sin(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $\cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

- Baker-Hausdorff formula: $\exp(\hat{A}) \cdot \hat{B} \cdot \exp(-\hat{A}) = \hat{B} + \sum_{n=1}^{+\infty} \frac{1}{n!} \underbrace{[\hat{A}, [\hat{A}, \dots [\hat{A},}_{n\text{-fold commutator}} \hat{B}] \dots]]$.

- If $[\hat{A}, \hat{B}]$ is a c-number, then $\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{B})\exp(\hat{A})\exp([\hat{A}, \hat{B}])$.

- $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}] + \hat{C}[\hat{A}, \hat{D}]\hat{B}$
  $= \hat{A}\{\hat{B}, \hat{C}\}\hat{D} - \{\hat{A}, \hat{C}\}\hat{B}\hat{D} + \hat{C}\hat{A}\{\hat{B}, \hat{D}\} - \hat{C}\{\hat{A}, \hat{D}\}\hat{B}$

- 1D harmonic oscillator: $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$.

  Here $\hat{x}$ is position operator, $\hat{p}$ is momentum operator, $[\hat{x}, \hat{p}] = \mathbb{i}\hbar$, and in position representation $\hat{p} = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \mathbb{i}\frac{1}{m\omega}\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{H}_0 = \hbar\omega\,(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. It has a unique ground state $|\psi_0\rangle$ with $\hat{a}|\psi_0\rangle = 0$, and excited states $|\psi_n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|\psi_0\rangle$ with energy $E_n = (n + \frac{1}{2})\hbar\omega$.

  Ground state wavefunction is $\psi_0(x) \equiv \langle x|\psi_0\rangle = (\frac{m\omega}{\hbar\pi})^{1/4}\exp(-\frac{x^2}{2\hbar/m\omega})$.

- Creation & annihilation operators:

  $\hat{\psi}^\dagger$ "creates" a particle in single particle state $|\psi\rangle$;

  $\hat{\psi}$ "destroys" a particle in single particle state $|\psi\rangle$; $\hat{\psi}^\dagger$ is hermitian conjugate of $\hat{\psi}$.

  - Given complete orthonormal basis $|e_i\rangle$ of single particle states, one set of complete orthonormal basis for the Fock space is the *occupation basis* $|n_1, n_2, \dots\rangle = \frac{1}{\sqrt{n_1!}}(\hat{e}_1^\dagger)^{n_1}\frac{1}{\sqrt{n_2!}}(\hat{e}_2^\dagger)^{n_2}\cdots|\mathrm{vac}\rangle$. Here $|\mathrm{vac}\rangle$ is the particle "vacuum". $\hat{e}_i^\dagger$ are creation operators for state $|e_i\rangle$. For bosons, $[\hat{e}_i, \hat{e}_j^\dagger] = \delta_{i,j}$; for fermions, $\{\hat{e}_i, \hat{e}_j^\dagger\} = \delta_{i,j}$.
  - $[\hat{e}_i^\dagger\hat{e}_j,\, \hat{e}_k^\dagger] = \delta_{j,k}\hat{e}_i^\dagger$, for both bosons and fermions.

Problem 1. (10pts) A Hilbert space has two basis $|1\rangle$ and $|2\rangle$, with $\langle 1|1\rangle = \langle 2|2\rangle = 1$ and $\langle 1|2\rangle = \frac{1}{2}$. An operator $\hat{A}$ is defined by $\hat{A}|1\rangle = |2\rangle$, $\hat{A}|2\rangle = -|1\rangle$.

(a) (5pts) *Is $\hat{A}$ a hermitian operator? Is $\hat{A}$ a unitary operator?*

(b) (5pts) *Solve the eigenvalues and normalized eigenstates of $\hat{A}$.*

Solution: this is similar to Homework #1 Problem 1.

(a) $\hat{A}$ is NOT hermitian, NOT unitary.

Method #1: quick test,

$(\hat{A}|1\rangle, |2\rangle) = (|2\rangle, |2\rangle) = 1 \neq (|1\rangle, \hat{A}|2\rangle) = (|1\rangle, -|1\rangle) = -1$, therefore $\hat{A}$ is not hermitian.

$(\hat{A}|1\rangle, \hat{A}|2\rangle) = (|2\rangle, -|1\rangle) = -\frac{1}{2} \neq (|1\rangle, |2\rangle) = \frac{1}{2}$, therefore $\hat{A}$ is not unitary.

Method #2: complete test,

$\hat{A}|i\rangle = \sum_j |j\rangle \cdot A_{j,i}$ with 2×2 matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Define $\hat{G}_{i,j} \equiv \langle i|j\rangle = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$.

$\hat{A}$ is hermitian if $(\hat{A}|i\rangle, |j\rangle) = (|i\rangle, \hat{A}|j\rangle)$ for all i, j, this condition is $(\sum_k |k\rangle \cdot A_{ki}, |j\rangle) = \sum_k A^*_{ki} G_{kj} = (|i\rangle, \sum_k |k\rangle \cdot A_{kj}) = \sum_k G_{ik} A_{kj}$, or $A^\dagger \cdot G = G \cdot A$. But here $A^\dagger \cdot G = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \neq G \cdot A = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & -\frac{1}{2} \end{pmatrix}$. Therefore $\hat{A}$ is not hermitian.

$\hat{A}$ is unitary if $(\hat{A}|i\rangle, \hat{A}|j\rangle) = (|i\rangle, |j\rangle)$ for all i, j, this condition is $A^\dagger \cdot G \cdot A = G$. But here $A^\dagger \cdot G \cdot A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \neq G$. Therefore $\hat{A}$ is not unitary.

(b) Suppose an eigenstate of $\hat{A}$ with eigenvalue λ is $c_1|1\rangle + c_2|2\rangle$, then $\hat{A}(\sum_i |i\rangle \cdot c_i) = \sum_j |j\rangle \cdot \sum_i A_{j,i} c_i = \lambda \cdot \sum_j |j\rangle \cdot c_j$. Here the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is defined in (a). Namely, $A \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. The normalization of this state is determined by $(c_1|1\rangle + c_2|2\rangle, c_1|1\rangle + c_2|2\rangle) = (c_1^*, c_2^*) \cdot G \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = |c_1|^2 + |c_2|^2 + \mathrm{Re}(c_1^* c_2)$. Here $G = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$ is defined in (a).

eigenvalue $+\mathbb{i}$, normalized eigenstate $\frac{1}{\sqrt{2}}(|1\rangle - \mathbb{i}|2\rangle)$.

eigenvalue $-\mathbb{i}$, normalized eigenstate $\frac{1}{\sqrt{2}}(|1\rangle + \mathbb{i}|2\rangle)$.

Problem 2. (50pts) Consider a 2D harmonic oscillator, $\hat{H}_0 = (\frac{\hat{p}_x^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}) + (\frac{\hat{p}_y^2}{2m} + \frac{m\omega^2\hat{y}^2}{2})$. Here m, ω are positive constants. $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = \mathbb{i}\hbar$, and other commutators between them vanish. Define ladder operators $\hat{a}_x = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{\mathbb{i}\hat{p}_x}{m\omega})$ and $\hat{a}_y = \sqrt{\frac{m\omega}{2\hbar}}(\hat{y} + \frac{\mathbb{i}\hat{p}_y}{m\omega})$. Then $[\hat{a}_x, \hat{a}_x^\dagger] = [\hat{a}_y, \hat{a}_y^\dagger] = 1$, and other commutators between them vanish. Then $\hat{H}_0 = \hbar\omega\cdot(\hat{a}_x^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_y + 1)$.

(a) (10pts) Denote the unique ground state of $\hat{H}_0$ by $|\varphi_0\rangle$. Note that $\hat{a}_x\varphi_0 = \hat{a}_y\varphi_0 = 0$. *Write down its wavefunction $\varphi_0(x, y)$. Write down all the eigenvalues and normalized eigenstates of $\hat{H}_0$ in terms of $|\varphi_0\rangle$ and ladder operators.* [Hint: make analogy to bosons]

(b) (10pts) Define the Heisenberg picture operator $\hat{O}_H(t) \equiv e^{\mathbb{i}\hat{H}_0 t/\hbar}\hat{O}_S e^{-\mathbb{i}\hat{H}_0 t/\hbar}$ for the Schrödinger picture operator $\hat{O}_S$. *Write down the Heisenberg equations of motion, $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_H(t) = \dots$, for $\hat{x}_H(t)$, $\hat{y}_H(t)$, $\hat{p}_{x,H}(t)$, $\hat{p}_{y,H}(t)$. The right-hand-side of these equations should be explicitly in terms of these four operators. Write down the solution to these equations of motion, namely these operators at time t in terms of their $t = 0$ values.*

(c) (10pts) Let $t = 0$ state be a coherent state, $|\psi(t = 0)\rangle = A\cdot\exp(z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger)|\varphi_0\rangle$. Here $z_{1,2}$ are two complex numbers. *Solve the constant A so that $\langle\psi(t = 0)|\psi(t = 0)\rangle = 1$.* Evolve this state under $\hat{H}_0$, $|\psi(t)\rangle = e^{-\mathbb{i}\hat{H}_0 t/\hbar}|\psi(t = 0)\rangle$. *Evaluate the expectation values $\langle\psi(t)|\hat{x}|\psi(t)\rangle$, $\langle\psi(t)|\hat{y}|\psi(t)\rangle$, $\langle\psi(t)|\hat{p}_x|\psi(t)\rangle$, $\langle\psi(t)|\hat{p}_y|\psi(t)\rangle$.*

(d) (5pts) *Show that $\hat{L} \equiv \frac{1}{\hbar}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)$ is a conserved quantity.* Namely its expectation value does not change over time. [Hint: consider its Heisenberg equations of motion]

(e) (10pts) *Solve all the eigenvalues and normalized eigenstates of $\hat{L}$ defined in (d).* [Hint: rewrite $\hat{L}$ by ladder operators, then do some basis change for ladder operators]

(f) (5pts) *Compute $e^{\mathbb{i}\theta\hat{L}}\cdot\hat{x}\cdot e^{-\mathbb{i}\theta\hat{L}}$ and $e^{\mathbb{i}\theta\hat{L}}\cdot\hat{y}\cdot e^{-\mathbb{i}\theta\hat{L}}$.* Here θ is a real number. The results should be finite-degree polynomials of $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$.

Solution: this is similar to Homework #3 Problem 2.

(a) $\varphi_0(x,y) = \psi_0(x)\cdot\psi_0(y) = (\frac{m\omega}{\hbar\pi})^{1/2}\exp(-\frac{x^2+y^2}{2\hbar/m\omega})$.

Here ψ_0 is the normalized ground state wavefunction for 1D harmonic oscillator. Then it is easy to see that φ_0 is normalized, $\int\int \mathrm{d}x\mathrm{d}y\,|\varphi_0(x,y)|^2 = 1$, and $\hat{a}_x\phi_0(x,y) = \hat{a}_y\phi_0(x,y) = 0$, because $\hat{a}_x\psi_0(x) = 0$ and $\hat{a}_y\psi_0(y) = 0$.

Note: the 2D Hilbert space can be viewed as the tensor product of two 1D Hilbert spaces (for x- and y-directions respectively), then $|\varphi_0\rangle = |\psi_0\rangle_x \otimes |\psi_0\rangle_y$. Then $\hat{a}_x$ should be understood as $\hat{a}_x \otimes \hat{\mathbb{1}}_y$, $\hat{a}_y$ should be understood as $\hat{\mathbb{1}}_x \otimes \hat{a}_y$.

Note: ϕ_0 may also be viewed, by pure analogy, as the "vacuum" of two orthonormal "boson annihilation operators" $\hat{a}_{x,y}$.

Eigenstates can be uniquely labeled by eigenvalues of $\hat{n}_x \equiv \hat{a}_x^\dagger\hat{a}_x$ and $\hat{n}_y \equiv \hat{a}_y^\dagger\hat{a}_y$, similar to boson occupation basis, $|n_x,n_y\rangle = \frac{1}{\sqrt{n_x!n_y!}}(\hat{a}_x^\dagger)^{n_x}(\hat{a}_y^\dagger)^{n_y}|\varphi_0\rangle$, with eigenvalue $\hbar\omega\cdot(n_x+n_y+1)$. Here n_x, n_y are non-negative integers.

(b) $\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_H(t) = \frac{\mathbb{i}}{\hbar}[\hat{H}_H(t), \hat{O}_H(t)]$.

Here $\hat{H}_H(t) = \frac{1}{2m}([\hat{p}_{x,H}(t)]^2 + [\hat{p}_{y,H}(t)]^2) + \frac{m\omega^2}{2}([\hat{x}_H(t)]^2 + [\hat{y}_H(t)]^2)$.

The commutation relations are preserved in the Heisenberg picture (for equal time operators), $[\hat{x}_H(t), \hat{p}_{x,H}(t)] = [\hat{y}_H(t), \hat{p}_{y,H}(t)] = \mathbb{i}\hbar$.

(steps omitted)

The Heisenberg equations of motion are,

$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H(t) = \frac{1}{m}\hat{p}_{x,H}(t)$,

$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_{x,H}(t) = -m\omega^2\hat{x}_H(t)$,

$\frac{\mathrm{d}}{\mathrm{d}t}\hat{y}_H(t) = \frac{1}{m}\hat{p}_{y,H}(t)$,

$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_{y,H}(t) = -m\omega^2\hat{y}_H(t)$.

They are just two decoupled equations of motion for 1D harmonic oscillators.

(steps omitted)

The solution to the Heisenberg equations of motion is,

$\hat{x}_H(t) = \hat{x}_H(0)\cos(\omega t) + \frac{1}{m\omega}\hat{p}_{x,H}(0)\sin(\omega t)$,

$\hat{p}_{x,H}(t) = \hat{p}_{x,H}(0)\cos(\omega t) - m\omega\hat{x}_H(0)\sin(\omega t)$,

$\hat{y}_H(t) = \hat{y}_H(0)\cos(\omega t) + \frac{1}{m\omega}\hat{p}_{y,H}(0)\sin(\omega t)$,

$\hat{p}_{y,H}(t) = \hat{p}_{y,H}(0)\cos(\omega t) - m\omega\hat{y}_H(0)\sin(\omega t)$.

(c) Compute the norm $\langle\psi(t=0)|\psi(t=0)\rangle$.

Method #1: expand into orthonormal "occupation number basis" in (a),

Note that $\hat{a}_x^\dagger$ and $\hat{a}_y^\dagger$ commute, then $\exp(z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger) = \exp(z_1\hat{a}_x^\dagger)\exp(z_2\hat{a}_y^\dagger)$, (Homework #1 Problem 2).

$|\psi(t=0)\rangle = A\cdot\sum_{n_x,n_y=0}^{\infty}\frac{z_1^{n_x}z_2^{n_y}}{n_x!n_y!}(\hat{a}_x^\dagger)^{n_x}(\hat{a}_y^\dagger)^{n_y}|\varphi_0\rangle = A\cdot\sum_{n_x,n_y=0}^{\infty}\frac{z_1^{n_x}z_2^{n_y}}{\sqrt{n_x!n_y!}}|n_x,n_y\rangle$.

Then $\langle\psi(t=0)|\psi(t=0)\rangle = |A|^2\sum_{n_x,n_y}\frac{|z_1|^{2n_x}|z_2|^{2n_y}}{n_x!n_y!} = |A|^2\exp(|z_1|^2)\exp(|z_2|^2)$.

Method #2: see Homework #2 Problem 3(a),

$\langle\psi(t=0)|\psi(t=0)\rangle = |A|^2\cdot\langle\varphi_0|\exp(z_1^*\hat{a}_x + z_2^*\hat{a}_y)\exp(z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger)|\varphi_0\rangle$.

Note that $[z_1^*\hat{a}_x + z_2^*\hat{a}_y, z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger] = |z_1|^2 + |z_2|^2$ is a c-number, then (see page 1), $\exp(z_1^*\hat{a}_x + z_2^*\hat{a}_y)\exp(z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger) = \exp(z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger)\exp(z_1^*\hat{a}_x + z_2^*\hat{a}_y)\exp(|z_1|^2 + |z_2|^2)$

$\exp(z_1^*\hat{a}_x + z_2^*\hat{a}_y)|\varphi_0\rangle = [1 + \sum_{n=1}^{\infty}\frac{1}{n!}(z_1^*\hat{a}_x + z_2^*\hat{a}_y)^n]|\varphi_0\rangle = |\varphi_0\rangle$.

$\langle\psi(t=0)|\psi(t=0)\rangle = |A|^2\exp(|z_1|^2 + |z_2|^2)\langle\varphi_0|\exp(z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger)\exp(z_1^*\hat{a}_x + z_2^*\hat{a}_y)|\varphi_0\rangle = |A|^2\exp(|z_1|^2 + |z_2|^2)\cdot\langle\varphi_0|\varphi_0\rangle = |A|^2\exp(|z_1|^2 + |z_2|^2)$.

So we can choose $A = \exp(-\frac{|z_1|^2 + |z_2|^2}{2})$.

To evaluate the expectation values, it will be convenient to rewrite those operators in terms of ladder operators, $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_x + \hat{a}_x^\dagger)$, $\hat{y} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_y + \hat{a}_y^\dagger)$, $\hat{p}_x = -\mathbb{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_x - \hat{a}_x^\dagger)$, $\hat{p}_y = -\mathbb{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_y - \hat{a}_y^\dagger)$.

Method #1: Heisenberg picture.

Evaluate $t = 0$ expectation values, plug them into solution to equations of motion.

The coherent state is eigenstate of lowering operators:

by the Baker-Hausdorff formula,

$\exp(-(z_1a_x^\dagger + z_2a_y^\dagger))\cdot\hat{a}_x\cdot\exp(z_1a_x^\dagger + z_2a_y^\dagger) = \hat{a}_x + z_1$, namely $\hat{a}_xe^{z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger} = e^{z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger}(\hat{a}_x + z_1)$;

$\exp(-(z_1a_x^\dagger + z_2a_y^\dagger))\cdot\hat{a}_y\cdot\exp(z_1a_x^\dagger + z_2a_y^\dagger) = \hat{a}_x + z_2$, namely $\hat{a}_ye^{z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger} = e^{z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger}(\hat{a}_y + z_2)$

then $\hat{a}_x|\psi(t=0)\rangle = A\exp(z_1a_x^\dagger + z_2a_y^\dagger)\cdot(\hat{a}_x + z_1)|\varphi_0\rangle = z_1|\psi(t=0)\rangle$, and

$\hat{a}_y|\psi(t=0)\rangle = A\exp(z_1a_x^\dagger + z_2a_y^\dagger)\cdot(\hat{a}_y + z_2)|\varphi_0\rangle = z_2|\psi(t=0)\rangle$. Therefore,

$\langle\psi(0)|\hat{a}_x|\psi(0)\rangle = z_1$, $\langle\psi(0)|\hat{a}_x^\dagger|\psi(0)\rangle = z_1^*$, $\langle\psi(0)|\hat{a}_y|\psi(0)\rangle = z_2$, $\langle\psi(0)|\hat{a}_y^*|\psi(0)\rangle = z_2^*$.

The expectation values at $t = 0$ are,

$\langle\psi(t=0)|\hat{x}|\psi(t=0)\rangle = \sqrt{\frac{\hbar}{2m\omega}}(z_1 + z_1^*)$, $\langle\psi(t=0)|\hat{p}_x|\psi(t=0)\rangle = -\mathbb{i}\sqrt{\frac{\hbar m\omega}{2}}(z_1 - z_1^*)$, $\langle\psi(t=0)|\hat{y}|\psi(t=0)\rangle = \sqrt{\frac{\hbar}{2m\omega}}(z_2 + z_2^*)$, $\langle\psi(t=0)|\hat{p}_y|\psi(t=0)\rangle = -\mathbb{i}\sqrt{\frac{\hbar m\omega}{2}}(z_2 - z_2^*)$.

Plug these into the results of (b),
$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}(z_1 e^{-\mathrm{i}\omega t} + z_1^* e^{\mathrm{i}\omega t})$, $\langle\psi(t)|\hat{p}_x|\psi(t)\rangle = -\mathrm{i}\sqrt{\frac{\hbar m\omega}{2}}(z_1 e^{-\mathrm{i}\omega t} - z_1^* e^{\mathrm{i}\omega t})$, $\langle\psi(t)|\hat{y}|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}(z_2 e^{-\mathrm{i}\omega t} + z_2^* e^{\mathrm{i}\omega t})$, $\langle\psi(t)|\hat{p}_y|\psi(t)\rangle = -\mathrm{i}\sqrt{\frac{\hbar m\omega}{2}}(z_2 e^{-\mathrm{i}\omega t} - z_2^* e^{\mathrm{i}\omega t})$.

These results can be more conveniently obtained by first simply promoting the relation between position/momentum operators and ladder operators to Heisenberg picture, *e.g*, $\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}[\hat{a}_{x,H}(t) + \hat{a}_{x,H}(t)^\dagger]$, and then solve $\hat{a}_{x,H}(t) = e^{-\mathrm{i}\omega t}\hat{a}_{x,H}(t=0) = e^{-\mathrm{i}\omega t}\hat{a}_x$, $\hat{a}_{y,H}(t) = e^{-\mathrm{i}\omega t}\hat{a}_{y,H}(t=0) = e^{-\mathrm{i}\omega t}\hat{a}_y$.

Method #2: Schrödinger picture,

Solve the explicit form of $|\psi(t)\rangle$, $|\psi(t)\rangle = e^{-\mathrm{i}\frac{\hat{H}_0 t}{\hbar}} \cdot A e^{z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger}|\varphi_0\rangle$
$= e^{-\mathrm{i}\frac{\hat{H}_0 t}{\hbar}} \cdot A e^{z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger} \cdot e^{\mathrm{i}\frac{\hat{H}_0 t}{\hbar}} \cdot e^{-\mathrm{i}\frac{\hat{H}_0 t}{\hbar}} \cdot |\varphi_0\rangle = A\exp(e^{-\mathrm{i}\hat{H}_0 t/\hbar} \cdot (z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger) \cdot e^{\mathrm{i}\hat{H}_0 t/\hbar}) \cdot e^{-\mathrm{i}\omega t}|\varphi_0\rangle$,
here we have used the fact that $\hat{A} \cdot f(\hat{B}) \cdot \hat{A}^{-1} = f(\hat{A}\hat{B}\hat{A}^{-1})$, and $\hat{H}_0|\varphi_0\rangle = \hbar\omega|\varphi_0\rangle$.

$[-\mathrm{i}\hat{H}_0 t/\hbar, z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger] = -\mathrm{i}\omega \cdot (z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger)$.

Then by Baker-Hausdorff formula, $e^{-\mathrm{i}\hat{H}_0 t/\hbar} \cdot (z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger) \cdot e^{\mathrm{i}\hat{H}_0 t/\hbar} = e^{-\mathrm{i}\omega t}(z_1\hat{a}_x^\dagger + z_2\hat{a}_y^\dagger)$.

Therefore $|\psi(t)\rangle = Ae^{-\mathrm{i}\omega t} \cdot \exp(z_1 e^{-\mathrm{i}\omega t}\hat{a}_x^\dagger + z_2 e^{-\mathrm{i}\omega t}\hat{a}_y^\dagger)|\varphi_0\rangle$ is still a coherent state.

The evaluation of expectation values then proceeds as the evaluation of $t = 0$ expectation values in Method #1.

(d) you just need to show that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}_H(t) = \frac{\mathrm{i}}{\hbar}[\hat{H}_{0,H}(t), \hat{L}_H(t)] = 0$.

Method #1: directly compute the commutator, (steps omitted)

Method #2: use the solution of equations of motion for position/momentum operators,

$\hat{L}_H(t) = \frac{1}{\hbar}(\hat{x}_H(t)\hat{p}_{y,H}(t) - \hat{y}_H(t)\hat{p}_{x,H}(t))$.

Plug in the results of (b), $\hat{L}_H(t)$

$$= \frac{1}{\hbar}[(\hat{x}\cos(\omega t) + \frac{\hat{p}_x}{m\omega}\sin(\omega t))(\hat{p}_y\cos(\omega t) - m\omega\hat{y}\sin(\omega t)) - (\hat{y}\cos(\omega t) + \frac{\hat{p}_y}{m\omega}\sin(\omega t))(\hat{p}_x\cos(\omega t) - m\omega\hat{x}\sin(\omega t))]$$

$= \frac{1}{\hbar}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) = \hat{L}_H(t=0)$, independent of t.

Method #3: use the equations of motion for position/momentum operators,

For notation simplicity, the argument t for Heisenberg operators are omitted here,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}_H = \frac{1}{\hbar}(\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H \cdot \hat{p}_{y,H} + \hat{x}_H \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_{y,H} - \frac{\mathrm{d}}{\mathrm{d}t}\hat{y}_H \cdot \hat{p}_{x,H} - \hat{y}_H \cdot \frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_{x,H})$$
$$= \frac{1}{\hbar}[\frac{\hat{p}_{x,H}}{m} \cdot \hat{p}_{y,H} + \hat{x}_H \cdot (-m\omega^2\hat{y}_H) - \frac{\hat{p}_{y,H}}{m} \cdot \hat{p}_{x,H} - \hat{y}_H \cdot (-m\omega^2\hat{x}_H)] = 0.$$

(e) Use the previous results about rewriting $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ in terms of ladder operators, $\hat{L} = -\mathbb{i}\hat{a}_x^\dagger \hat{a}_y + \mathbb{i}\hat{a}_y^\dagger \hat{a}_x = (\hat{a}_x^\dagger, \hat{a}_y^\dagger) \cdot \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{a}_x \\ \hat{a}_y \end{pmatrix}$, similar to Homework #2 Problem 2(e).

Define new orthonormal ladder operators, $\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{a}_x - \mathbb{i}\hat{a}_y)$, $\hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{a}_x + \mathbb{i}\hat{a}_y)$.

Then $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$, $\hat{a}_i|\varphi_0\rangle = 0$. And $\hat{L} = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2$.

So the "occupation basis states" under the $\hat{a}_i$ basis are normalized eigenstates of $\hat{L}$, $|n_1, n_2\rangle = \frac{1}{\sqrt{n_1! n_2!}}(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}|\varphi_0\rangle$, with eigenvalue $n_1 - n_2$, where n_1, n_2 are non-negative integers.

(f) this is similar to Homework #1 Problem 5.

Given $[\hat{A}, \hat{B}] = \hat{C}$, $[\hat{A}, \hat{C}] = -\hat{B}$, then by Baker-Hausdorff formula, $e^{\theta\hat{A}}\hat{B}e^{-\theta\hat{A}} = \cos(\theta)\hat{B} + \sin(\theta)\hat{C}$, $e^{\theta\hat{A}}\hat{C}e^{-\theta\hat{A}} = \cos(\theta)\hat{C} - \sin(\theta)\hat{B}$.

Here $[\mathbb{i}\hat{L}, \hat{y}] = \hat{x}$, $[\mathbb{i}\hat{L}, \hat{x}] = -\hat{y}$.

$e^{\mathbb{i}\theta\hat{L}}\hat{y}e^{-\mathbb{i}\theta\hat{L}} = \hat{y}\cos\theta + \hat{x}\sin\theta$, $e^{\mathbb{i}\theta\hat{L}}\hat{x}e^{-\mathbb{i}\theta\hat{L}} = \hat{x}\cos\theta - \hat{y}\sin\theta$.

Problem 3. (30pts) The single fermion Hilbert space has complete orthonormal basis $|1\rangle$ and $|2\rangle$. Denote the corresponding creation operators by $\hat{f}_1^\dagger$ and $\hat{f}_2^\dagger$. Denote the vacuum state by $|\text{vac}\rangle$. Then $\hat{f}_i|\text{vac}\rangle = 0$ for $i = 1, 2$, and $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{i,j}$.

(a) (5pts) *Write down a complete orthonormal basis for the entire Fock space, in terms of creation operators and* $|vac\rangle$.

(b) (5pts) Define $\hat{S}_x \equiv \hat{f}_2^\dagger \hat{f}_1^\dagger + \hat{f}_1 \hat{f}_2$, $\hat{S}_y \equiv -\mathbb{i}\hat{f}_2^\dagger \hat{f}_1^\dagger + \mathbb{i}\hat{f}_1 \hat{f}_2$, $\hat{S}_z \equiv \hat{f}_1^\dagger \hat{f}_1 + \hat{f}_2^\dagger \hat{f}_2 - 1$. *Compute the commutators* $[\hat{S}_x, \hat{S}_y]$, $[\hat{S}_y, \hat{S}_z]$, $[\hat{S}_z, \hat{S}_x]$. Results should be linear combinations of $\hat{S}_{x,y,z}$.

(c) (10pts) *Represent* $\hat{S}_{x,y,z}$ *by* 4×4 *matrices under the basis in (a).* [Hint: be careful about signs, results should be consistent with the commutation relations in (b)]

(d) (5pts) *Compute* $\exp(\mathbb{i}\frac{\theta}{2}\hat{S}_x) \cdot (a\hat{S}_x + b\hat{S}_y + c\hat{S}_z) \cdot \exp(-\mathbb{i}\frac{\theta}{2}\hat{S}_x)$. Here θ, a, b, c are c-numbers. Results should be a finite-degree polynomial of $\hat{S}_{x,y,z}$. [Hint: some previous results may help]

(e) (5pts) *Solve all the eigenvalues of* $\hat{S}_z + \hat{S}_y$ *in the Fock space.*

Solution: this is similar to Homework #2 Problem 4.

(a) The choice and ordering of these basis are not unique. For later convenience, I choose the occupation basis ordered in the following way, $|\text{vac}\rangle \equiv |n_1 = 0, n_2 = 0\rangle$, $|n_1 = 1, n_2 = 1\rangle = \hat{f}_1^\dagger \hat{f}_2^\dagger |\text{vac}\rangle$, $|n_1 = 1, n_2 = 0\rangle = \hat{f}_1^\dagger |\text{vac}\rangle$, $|n_1 = 0, n_2 = 1\rangle = \hat{f}_2^\dagger |\text{vac}\rangle$.

(b) $[\hat{S}_x, \hat{S}_y] = 2\mathbb{i}\hat{S}_z$, $[\hat{S}_y, \hat{S}_z] = 2\mathbb{i}\hat{S}_x$, $[\hat{S}_z, \hat{S}_x] = 2\mathbb{i}\hat{S}_y$.

Method #1: directly computation,

Use $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}\{\hat{B}, \hat{C}\}\hat{D} - \{\hat{A}, \hat{C}\}\hat{B}\hat{D} + \hat{C}\hat{A}\{\hat{B}, \hat{D}\} - \hat{C}\{\hat{A}, \hat{D}\}\hat{B}$, given on page 1. (steps omitted)

Method #2: do a particle-hole transformation,

Define $\hat{f'}_1 = \hat{f}_1^\dagger$, $\hat{f'}_2 = \hat{f}_2$. Then $\{\hat{f'}_i, \hat{f'}^\dagger_j\} = \delta_{ij}$. And $\hat{f'}^\dagger_1 \hat{f'}_1 = \hat{f}_1 \hat{f}_1^\dagger = 1 - \hat{f}_1^\dagger \hat{f}_1$.

$$\hat{S}_x = (\hat{f'}^\dagger_1, \hat{f'}^\dagger_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{f'}_1 \\ \hat{f'}_2 \end{pmatrix} = (\hat{f'}^\dagger_1, \hat{f'}^\dagger_2) \cdot \sigma_1 \cdot \begin{pmatrix} \hat{f'}_1 \\ \hat{f'}_2 \end{pmatrix},$$

$$\hat{S}_y = (\hat{f'}^\dagger_1, \hat{f'}^\dagger_2) \begin{pmatrix} 0 & \mathbb{i} \\ -\mathbb{i} & 0 \end{pmatrix} \begin{pmatrix} \hat{f'}_1 \\ \hat{f'}_2 \end{pmatrix} = (\hat{f'}^\dagger_1, \hat{f'}^\dagger_2) \cdot (-\sigma_2) \cdot \begin{pmatrix} \hat{f'}_1 \\ \hat{f'}_2 \end{pmatrix},$$

$$\hat{S}_z = (\hat{f'}^\dagger_1, \hat{f'}^\dagger_2) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{f'}_1 \\ \hat{f'}_2 \end{pmatrix} = (\hat{f'}^\dagger_1, \hat{f'}^\dagger_2) \cdot (-\sigma_3) \cdot \begin{pmatrix} \hat{f'}_1 \\ \hat{f'}_2 \end{pmatrix}.$$

Use a fact given in the solution to Homework #2 Problem 4, about the commutator of two "bilinear operators", $[\sum_{i,j} \hat{f}_i^\dagger P_{ij} \hat{f}_j, \sum_{k,\ell} \hat{f}_k^\dagger Q_{k\ell} \hat{f}_\ell] = \sum_{i,j} \hat{f}_i^\dagger ([P, Q])_{ij} \hat{f}_j$. And then use the commutation relations of Pauli matrices.

(c) $\hat{S}_{x,y,z}$ are all hermitian. $\hat{S}_{x,y}$ changes particle number by ± 2, and $\hat{S}_z$ does not change particle number in $\hat{f}$ basis.

The first two basis in (a) have even particle number, last two basis in (a) have odd particle number, so $\hat{S}_{x,y,z}$ are block-diagonalized into these two subspaces.

Under the basis in (a),

$$\hat{S}_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{S}_y = \begin{pmatrix} 0 & -\mathbb{i} & 0 & 0 \\ \mathbb{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{S}_z = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\hat{f}_2^\dagger \hat{f}_1^\dagger |\text{vac}\rangle = -\hat{f}_1^\dagger \hat{f}_2^\dagger |\text{vac}\rangle = -|n_1 = 1, n_2 = 1\rangle$, and

$\hat{f}_1\hat{f}_2|n_1=1,n_2=1\rangle = \hat{f}_1\hat{f}_2\hat{f}_1^\dagger\hat{f}_2^\dagger|\text{vac}\rangle = -\hat{f}_1\hat{f}_1^\dagger\cdot\hat{f}_2\hat{f}_2^\dagger|\text{vac}\rangle$
$= -(1-\hat{n}_1)(1-\hat{n}_2)|n_1=0,n_2=0\rangle = -(1-0)(1-0)|n_1=0,n_2=0\rangle = -|n_1=0,n_2=0\rangle.$

(d) this is similar to Homework #2 Problem 4(c), and Problem 2(f) here.

Note that $[\mathbb{i}\hat{S}_x/2,\hat{S}_z]=\hat{S}_y$, $[\mathbb{i}\hat{S}_x/2,\hat{S}_y]=-\hat{S}_z$.

Then $e^{\theta\cdot\mathbb{i}\hat{S}_x/2}\hat{S}_z e^{-\theta\cdot\mathbb{i}\hat{S}_x/2}=\hat{S}_z\cos\theta+\hat{S}_y\sin\theta$, $e^{\theta\cdot\mathbb{i}\hat{S}_x/2}\hat{S}_y e^{-\theta\cdot\mathbb{i}\hat{S}_x/2}=\hat{S}_y\cos\theta-\hat{S}_z\sin\theta$,

Finally, $e^{\theta\cdot\mathbb{i}\hat{S}_x/2}(a\hat{S}_x+b\hat{S}_y+c\hat{S}_z)e^{-\theta\cdot\mathbb{i}\hat{S}_x/2}=a\hat{S}_x+(b\cos\theta+c\sin\theta)\hat{S}_y+(-b\sin\theta+c\cos\theta)\hat{S}_z$.

(e) Method #1: brute-force diagonalization,

Use the result of (c), $\hat{S}_z+\hat{S}_y=\begin{pmatrix}-1 & -\mathbb{i} & 0 & 0\\ \mathbb{i} & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$.

The top-left 2×2 diagonal block is $-\sigma_3+\sigma_2$, and has eigenvalues $\pm\sqrt{2}$ [see Homework #1 Problem 6(a)], so all the eigenvalues are $\sqrt{2}$, $-\sqrt{2}$, 0, 0.

Method #2: use unitary transformation, see also Homework #2 Problem 4(d),

Use the result of (d), $e^{-\frac{\pi}{4}\mathbb{i}\hat{S}_x/2}(\hat{S}_z+\hat{S}_y)e^{\frac{\pi}{4}\mathbb{i}\hat{S}_x/2}=\sqrt{2}\hat{S}_z$.

$e^{-\frac{\pi}{4}\mathbb{i}\hat{S}_x/2}$ is a unitary operator, so $\sqrt{2}\hat{S}_z$ has the same eigenvalues with $\hat{S}_z+\hat{S}_y$.

The occupation basis $|n_1,n_2\rangle$ in (a) are eigenstates of $\sqrt{2}\hat{S}_z$, with eigenvalues $\sqrt{2}(n_1+n_2-1)$, for $n_1,n_2=0$ or 1.

Problem 4. (5pts) $\mathcal{H}_1$ and $\hat{H}_2$ are both 2-dimensional Hilbert spaces. $\mathcal{H}_1$ has complete orthonormal basis $|e_1\rangle$ and $|e_2\rangle$, $\mathcal{H}_2$ has complete orthonormal basis $|e_1'\rangle$ and $|e_2'\rangle$.

(a) (4pts). Define operators $\hat{\sigma}_1=|e_1\rangle\langle e_2|+|e_2\rangle\langle e_1|$ and $\hat{\sigma}_2=-\mathbb{i}|e_1\rangle\langle e_2|+\mathbb{i}|e_2\rangle\langle e_1|$ in $\mathcal{H}_1$, and $\hat{\sigma'}_1=|e_1'\rangle\langle e_2'|+|e_2'\rangle\langle e_1'|$ and $\hat{\sigma'}_2=-\mathbb{i}|e_1'\rangle\langle e_2'|+\mathbb{i}|e_2'\rangle\langle e_1'|$ in $\mathcal{H}_2$. *Solve all the eigenvalues of* $\hat{O}\equiv\hat{\sigma}_1\otimes\hat{\sigma'}_1+\hat{\sigma}_2\otimes\hat{\sigma'}_2$ *in* $\mathcal{H}_1\otimes\mathcal{H}_2$. [Hint: represent $\hat{O}$ by a 4×4 matrix]

(b) (1pts) *Show that* $\hat{O}$ *in (a) cannot be represented as* $\hat{O}_1\otimes\hat{O}_2$*, where* $\hat{O}_{1,2}$ *are some operators in* $\mathcal{H}_{1,2}$ *respectively.*

Solution

(a) Under the given basis, $\hat{\sigma}_1$ and $\hat{\sigma'}_1$ are both represented by Pauli matrix σ_1 in their respective 2-dim'l Hilbert spaces, and $\hat{\sigma}_2$ and $\hat{\sigma'}_2$ are both represented by Pauli matrix σ_2.

Under the tensor product basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$, $\hat{O}$ is $\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The 2×2 central diagonal block is $2\sigma_1$ and has eigenvalues ± 2. So all the eigenvalues are $0, +2, -2, 0$.

(b) this is similar to Homework #1 Problem 7(b).

Proof by contradiction, suppose $\hat{O} = \hat{O}_1 \otimes \hat{O}_2$. Represent them by matrices.

Method #1: direct computation,

$(O_1)_{1,2} \cdot (O_2)_{1,2} = 0$, $(O_1)_{1,2} \cdot (O_2)_{2,1} = 2$, $(O_1)_{2,1} \cdot (O_2)_{1,2} = 2$, $(O_1)_{2,1} \cdot (O_2)_{2,1} = 0$.

There is no solution to these four matrix elements, $(O_1)_{1,2}$, $(O_1)_{2,1}$, $(O_2)_{1,2}$, $(O_2)_{2,1}$.

Method #2: expand 2×2 matrices into Pauli matrices

Pauli matrices are complete linearly independent basis for 2×2 complex matrices, namely any 2×2 complex matrix can be uniquely expanded into a superposition of Pauli matrices.

Suppose $O_1 = \sum_{i=0}^{3} c_i \sigma_i$, $O_2 = \sum_{i=0}^{3} d_i \sigma_i$, then we must have $c_1 d_1 = 1$, $c_1 d_2 = 0$, $c_2 d_1 = 0$, $c_2 d_2 = 1$. There is no solution to these equations.

Problem 5. (5pts) Consider a projection operator $\hat{P}$, satisfying $\hat{P} \cdot \hat{P} = \hat{P}$. *Prove that if the inner product* $\langle (\hat{\mathbb{1}} - \hat{P})\psi \,|\, \hat{P}\psi \rangle = 0$ *for any state* ψ*, then* $\hat{P}$ *is a hermitian operator.* [Hint: try to show that $\langle \hat{P}\psi_1 \,|\, \psi_2 \rangle = \langle \psi_1 \,|\, \hat{P}\psi_2 \rangle$ for any states ψ_1, ψ_2]

Solution

From $\langle (\hat{\mathbb{1}} - \hat{P})\psi \,|\, \hat{P}\psi \rangle = 0$, we have $\langle \psi \,|\, \hat{P}\psi \rangle = \langle \hat{P}\psi \,|\, \hat{P}\psi \rangle$.

Take complex conjugate, $\langle \psi \,|\, \hat{P}\psi \rangle^* = \langle \hat{P}\psi \,|\, \psi \rangle = \langle \hat{P}\psi \,|\, \hat{P}\psi \rangle^* = \langle \hat{P}\psi \,|\, \hat{P}\psi \rangle = \langle \psi \,|\, \hat{P}\psi \rangle$.

Therefore $\langle \hat{P}\psi \,|\, \psi \rangle = \langle \psi \,|\, \hat{P}\psi \rangle$ for any ψ. This is the condition for $\hat{P}$ to be hermitian. This condition is equivalent to $\langle \psi_1 | \hat{P}\psi_2 \rangle = \langle \hat{P}\psi_1 | \psi_2 \rangle$ for any ψ_1, ψ_2.

The latter condition can be derived from the former, similar to Homework#2 Problem 1.

From $\langle (c_1\psi_1 + c_2\psi_2)\,|\,\hat{P}(c_1\psi_1 + c_2\psi_2)\rangle = \langle \hat{P}(c_1\psi_1 + c_2\psi_2)\,|\,(c_1\psi_1 + c_2\psi_2)\rangle$, expand both sides, we have $c_1^* c_2 \langle \psi_1 | \hat{P}\psi_2\rangle + c_2^* c_1 \langle \psi_2 | \hat{P}\psi_1\rangle = c_1^* c_2 \langle \hat{P}\psi_1 | \psi_2\rangle + c_2^* c_1 \langle \hat{P}\psi_2 | \psi_1\rangle$, or $c_1^* c_2 (\langle \psi_1 | \hat{P}\psi_2\rangle - \langle \hat{P}\psi_1 | \psi_2\rangle) + c_2^* c_1 (\langle \psi_2 | \hat{P}\psi_1\rangle - \langle \hat{P}\psi_2 | \psi_1\rangle) = 0$. By choosing $c_1 = c_2 = 1$ and $c_1 = \mathbb{i}, c_2 = 1$, we can show that $\langle \psi_1 | \hat{P}\psi_2\rangle - \langle \hat{P}\psi_1 | \psi_2\rangle = 0$ and $\langle \psi_2 | \hat{P}\psi_1\rangle - \langle \hat{P}\psi_2 | \psi_1\rangle = 0$.

The fact that $\hat{P}$ is a projection operator, $\hat{P}^2 = \hat{P}$, is not used in this proof, and can actually be derived from the fact that $\langle (\hat{\mathbb{1}} - \hat{P})\psi\,|\,\hat{P}\psi\rangle = 0$ for any ψ.

But the converse is not true, a projection operator may not be hermitian, and may not satisfy $\langle (\hat{\mathbb{1}} - \hat{P})\psi\,|\,\hat{P}\psi\rangle = 0$ for any ψ. For example, $\hat{P} = |e_1\rangle\langle e_1| + |e_1\rangle\langle e_2|$ with orthonormal $|e_1\rangle$ and $|e_2\rangle$.