Final Exam (Quantum Statistical Physics; 2015-2016 Fall Sem.; @PKU)

By Ryuichi Shindou

I. NOTATION

k_F: Fermi wavelength, c: speed of light, e: eletron charge (positive; $-e$ is the electron charge). m: electron mass. $\hbar$: Plank constant. A step function $\theta(x)$ is defined as

$$\theta(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x < 0) \end{cases} \tag{1}$$

II. PROBLEM.1

A. Problem.1A

Consider a following Lagrangian for longitudinal phonon in a solid;

$$\mathcal{L}_{\text{phonon}} = \int d\boldsymbol{x}^3 \big[\rho_m \frac{\partial d_i}{\partial t}\frac{\partial d_i}{\partial t} - B\frac{\partial d_i}{\partial x_j}\frac{\partial d_i}{\partial x_j}\big]. \tag{2}$$

with the rotation free condition on the displacement field $\boldsymbol{d}(\boldsymbol{x},t)$;

$$\boldsymbol{\nabla} \times \boldsymbol{d} = 0. \tag{3}$$

B denotes the bulk modulus and ρ_m denotes the mass density of nuclues ions; $\rho_m \equiv M/V$ (M: total mass of all ions, V: total volume of the system). **Second-quantize** the displacement field; **obtain** the corresponding Hamiltonian which takes a quadratic form of a boson field; $\sum_\alpha \hbar\omega_\alpha b^\dagger_\alpha b_\alpha$ and **express** the displacement field $\boldsymbol{d}(\boldsymbol{x})$ in terms of the boson field (b_α and $b^\dagger_\alpha$). **Explain** what is the Debye frequency $\hbar\omega_D$?

B. Problem.IB

In the jellium model (the simplest model for metal), a periodic lattice of postively charged nucleus ions is treated as uniformly-distributed postively-charged 'background'. Quantum dynamics of the background is described by that of the displacement field $\boldsymbol{d}(\boldsymbol{x},t)$. The divergence of the displacement field is propotional to extra postive charge density;

$$\rho_b(\boldsymbol{x}) = Ze\rho_0\big(1 - \boldsymbol{\nabla}\cdot\boldsymbol{d}(\boldsymbol{x})\big) \equiv Ze\rho_0 + \delta\rho_b(\boldsymbol{x}).$$

where ρ_0 stands for the uniform density of the ions; $\rho_0 \equiv N/V$ (N: the number of nucleus ions). Ze is the strength of the positive charge carried by each nucleus ion. Electron and nucleus are coupled with each other via the Coulombic interaction,

$$\mathcal{H}_{\text{el}-\text{p}} = \int d^3\boldsymbol{x} \int d^3\boldsymbol{x}' \frac{\rho_e(\boldsymbol{x})\delta\rho_b(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \equiv \int d^3\boldsymbol{x}\rho_e(\boldsymbol{x})\delta\phi_b(\boldsymbol{x}), \tag{4}$$

$$\delta\phi_b(\boldsymbol{x}) \equiv \int d^3\boldsymbol{x}' \frac{\delta\rho_b(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{5}$$

$$\tag{6}$$

Here an electron density $\rho_e(\boldsymbol{x})$ being given as

$$\rho_e(\boldsymbol{x}) = -e\psi^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x}) \tag{7}$$

with $\alpha = \pm\frac{1}{2}$ (spin-1/2). In Eq. (4), the coupling between the electron density and the uniform background ($Ze\rho_0$) is excluded due to the charge neutrality. Correspondingly, we will exmploy the following hamiltonian for electrons;

$$\begin{aligned}\mathcal{H}_{\rm el} &= \sum_{\boldsymbol{k},\alpha} \frac{\hbar^2 k^2}{2m} a^{\dagger}_{\boldsymbol{k},\alpha} a_{\boldsymbol{k},\alpha} + \frac{e^2}{2V} \sum_{\alpha,\alpha'} \sum_{\boldsymbol{k},\boldsymbol{k}'} \sum_{\boldsymbol{q}\neq 0} \frac{4\pi}{q^2} a^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\alpha} a^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q},\alpha'} a_{\boldsymbol{k}',\alpha'} a_{\boldsymbol{k},\alpha} \\ &\equiv \sum_{\boldsymbol{k},\alpha} \epsilon_{\boldsymbol{k}}\, a^{\dagger}_{\boldsymbol{k},\alpha} a_{\boldsymbol{k},\alpha} + \frac{1}{2V} \sum_{\alpha,\alpha'} \sum_{\boldsymbol{k},\boldsymbol{k}'} \sum_{\boldsymbol{q}\neq 0} V_c(q)\, a^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\alpha} a^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q},\alpha'} a_{\boldsymbol{k}',\alpha'} a_{\boldsymbol{k},\alpha}. \end{aligned} \tag{8}$$

Here the electron creation operator in eq. (7) is related with its Fourier transform $a^{\dagger}_{\boldsymbol{k},\alpha}$ as

$$\psi^{\dagger}_{\alpha}(\boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\boldsymbol{x}} a^{\dagger}_{\boldsymbol{k},\alpha}.$$

In metals, characteristic time scale of phonon dynamics is usually much slower than that of electron dynamics ($\hbar\omega_D \ll \epsilon_F$). Thus, the positive charge induced by finite displacement $\delta\rho_b(\boldsymbol{r})$ is immediately screened by 'fast' electrons. As a result, an electron density $\rho_e(\boldsymbol{x})$ is coupled not only with $\delta\rho_b(\boldsymbol{x})$ but also with the screening electrons induced by finite $\delta\rho_b(\boldsymbol{x})$, say $\delta\rho_e(\boldsymbol{x})$;

$$\mathcal{H}_{\rm el-ph} \rightarrow \mathcal{H}'_{\rm el-p} = \int d^3\boldsymbol{x} \int d^3\boldsymbol{x}' \frac{\rho_e(\boldsymbol{x})\delta\rho_b(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} + \int d^3\boldsymbol{x} \int d^3\boldsymbol{x}' \frac{\rho_e(\boldsymbol{x})\delta\rho_e(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \equiv \int d^3\boldsymbol{x} \rho_e(\boldsymbol{x}) \delta\phi_{\rm eff}(\boldsymbol{x}). \tag{9}$$

with

$$\delta\phi_{\rm eff}(\boldsymbol{x}) \equiv \delta\phi_b(\boldsymbol{x}) + \int d\boldsymbol{x}' \frac{\delta\rho_e(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{10}$$

The induced electron density can be evaluated from the linear response theory;

$$\delta\rho_e(\boldsymbol{x}) = \frac{e^2}{\hbar} \int d\boldsymbol{x}' \int_{-\infty}^{\infty} dt' D^R(\boldsymbol{x},t;\boldsymbol{x}',t') \phi_b(\boldsymbol{x}') = \frac{e^2}{\hbar} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} D^R(\boldsymbol{k},\omega=0) \delta\phi_b(\boldsymbol{k}).$$

where $D^R(\boldsymbol{x},t;\boldsymbol{x}',t')$ denotes the retarded density-density correlation function and $D^R(\boldsymbol{k},\omega)$ is its Fourier transform;

$$iD^R(\boldsymbol{x},t;\boldsymbol{x}',t') = \theta(t-t') \frac{\langle\Psi_0|\big[\tilde{\rho}_H(\boldsymbol{x},t),\tilde{\rho}_H(\boldsymbol{x}',t')\big]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}. \tag{11}$$

with

$$\tilde{\rho}_H(\boldsymbol{x},t) \equiv e^{i\frac{\mathcal{H}t}{\hbar}} \psi^{\dagger}_{\alpha}(\boldsymbol{x})\psi_{\alpha}(\boldsymbol{x}) e^{-i\frac{\mathcal{H}t}{\hbar}} - \frac{\langle\Psi_0|\psi^{\dagger}_{\alpha}(\boldsymbol{x})\psi_{\alpha}(\boldsymbol{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}, \tag{12}$$

and

$$D^R(\boldsymbol{k},\omega) \equiv \int d^3\boldsymbol{x} \int dt e^{-i\boldsymbol{k}\boldsymbol{x}+i\omega t} D^R(\boldsymbol{x},t;\boldsymbol{0},0). \tag{13}$$

$|\Psi_0\rangle$ is the ground state wavefunction of $\mathcal{H}_{\rm el}$ in eq. (8). $\delta\phi_b(\boldsymbol{k})$ is a Fourier transform of $\delta\phi_b(\boldsymbol{x})$ introduced in eq. (5). With this in mind, the Fourier transform of $\delta\phi_{\rm eff}(\boldsymbol{x})$ is given by

$$\delta\phi_{\rm eff}(\boldsymbol{k}) = \delta\phi_b(\boldsymbol{k}) + \frac{1}{\hbar} D^R(\boldsymbol{k},\omega=0) V_c(k) \delta\phi_b(\boldsymbol{k}) \equiv V_{\rm eff}(\boldsymbol{k}) \delta\rho_b(\boldsymbol{k}), \tag{14}$$

with $V_c(k) \equiv \frac{4\pi e^2}{k^2}$ and

$$V_{\rm eff}(\boldsymbol{k}) \equiv \frac{4\pi}{k^2} \Big(1 + \frac{1}{\hbar} D^R(\boldsymbol{k},\omega=0) V_c(k)\Big). \tag{15}$$

With $V_{\rm eff}(\boldsymbol{x}) \equiv \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} V_{\rm eff}(\boldsymbol{k})$, we have the following from eqs. (9,14)

$$\mathcal{H}'_{\rm el-ph} = \int d^3\boldsymbol{x} d^3\boldsymbol{x}' \rho_e(\boldsymbol{x}) V_{\rm eff}(\boldsymbol{x}-\boldsymbol{x}') \delta\rho_b(\boldsymbol{x}'). \tag{16}$$

Using the Lehmann representation, the Fourier-series of the retarded function are related with those of the time-ordered function;

$$\mathrm{Re}D^R(\boldsymbol{k},\omega) = \mathrm{Re}D(\boldsymbol{k},\omega) \tag{17}$$
$$\mathrm{Im}D^R(\boldsymbol{k},\omega) = \mathrm{sgn}\omega\, \mathrm{Im}D(\boldsymbol{k},\omega), \tag{18}$$

where the time-ordered density-density correlation function is defined as

$$iD(\boldsymbol{x},t;\boldsymbol{x}',t') = \frac{\langle\Psi_0|T\{\tilde{\rho}_H(\boldsymbol{x},t)\tilde{\rho}_H(\boldsymbol{x}',t')\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}. \tag{19}$$

with

$$D(\boldsymbol{k},\omega) \equiv \int d^3\boldsymbol{x} \int dt e^{-i\boldsymbol{k}\boldsymbol{x}+i\omega t} D(\boldsymbol{x},t;\boldsymbol{0},0). \tag{20}$$

Within the ring diagram approximation, the Fourier series of the time-ordered correlation function is given by

$$\frac{1}{\hbar}D(\boldsymbol{k},\omega) = \frac{\frac{1}{\hbar}D^{\star}_{(r)}(\boldsymbol{k},\omega)}{1-\frac{1}{\hbar}D^{\star}_{(r)}(\boldsymbol{k},\omega)V_c(k)} \equiv \frac{\Pi_{(0)}(\boldsymbol{k},\omega)}{1-\Pi_{(0)}(\boldsymbol{k},\omega)V_c(k)} \tag{21}$$

The bare polarization part $\Pi_0(\boldsymbol{k},\omega) \equiv \frac{1}{\hbar}D^{\star}_{(r)}(\boldsymbol{k},\omega)$ is given by the non-interacting single-particle Green's function in eqs. (73,74). Using eqs. (75,76), **calculate**

$$A \equiv V_{\text{eff}}(\boldsymbol{k}=\boldsymbol{0}). \tag{22}$$

Due to the screening effect, $V_{\text{eff}}(\boldsymbol{x})$ becomes a short-range interaction; for simplicity, assume that $V_{\text{eff}}(\boldsymbol{x})$ takes a form of the delta function,

$$V_{\text{eff}}(\boldsymbol{x}-\boldsymbol{x}') = A\delta^3(\boldsymbol{x}-\boldsymbol{x}').$$

The effective electron-phonon coupling Hamiltonian is then given by

$$\mathcal{H}' = A\int d^3\boldsymbol{x}\int d^3\boldsymbol{x}'\rho_e(\boldsymbol{x})\delta\rho_b(\boldsymbol{x}). \tag{23}$$

According to the BCS theory of superconductivity, the critical temperature of superconductivity can be evaluated as

$$k_BT_c \simeq 1.13\hbar\omega_D \exp\Big[-\frac{1}{N(0)g}\Big] \tag{24}$$

where $\hbar\omega_D$ is the Debye frequency, $N(0)$ is the density of states of the metal at the Fermi level and g is the square of the electron-phonon coupling strength;

$$g = \gamma^2$$
$$\gamma \equiv \frac{Ze^2}{c}A\Big(\frac{\rho_0}{M}\Big)^{\frac{1}{2}}. \tag{25}$$

where c is the velocity of longitudinal phonon. **Explain** why T_c of BCS superconductors is usually higher in those metals with lighter nucleus ions.

III. PROBLEM II

A. problem IIA

Suppose that a magnetic impurity spin is introduced into a non-interacting electron gas;

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{26}$$

$$\mathcal{H}_0 = \sum_{\alpha=\uparrow,\downarrow} \int d\boldsymbol{x}^3 \psi^\dagger_\alpha(\boldsymbol{x}) \Big(- \frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} \Big) \psi_\alpha(\boldsymbol{x}), \tag{27}$$

$$\mathcal{H}_1 = J \sum_{\mu=x,y,z} \boldsymbol{s}_\mu(\boldsymbol{x}_{\rm i}) S_\mu(\boldsymbol{x}_{\rm i}), \tag{28}$$

$$2\boldsymbol{s}_\mu(\boldsymbol{x}) \equiv \sum_{\alpha,\beta=\uparrow,\downarrow} \psi^\dagger_\alpha(\boldsymbol{x}) \big[\boldsymbol{\sigma}_\mu\big]_{\alpha,\beta} \psi_\beta(\boldsymbol{x}) \tag{29}$$

where $\boldsymbol{x}_{\rm i}$ stands for the position of the magnetic impurity, and $\boldsymbol{\sigma}_\mu$ denotes the 2 by 2 Pauli matrices ($\mu = x, y, z$). We assumed that the electron spin density and the impurity spin interact with each other via a short-range potential (a delta function; 'point-contact' interaction). Regarding the impurity spin as an external perturbation, one can expect that the impurity spin induces spin polariation in an electron gas. Based on the linear response theory, **show** that the induced spin density is given as follows (use eq. (75) in 'Hint for the Problem III'),

$$\langle \boldsymbol{s}_\mu(\boldsymbol{x}) \rangle = J V(\boldsymbol{x} - \boldsymbol{x}_{\rm i}) S_\mu(\boldsymbol{x}_{\rm i}). \tag{30}$$

$$V(\boldsymbol{y}) \equiv -\frac{m k_F}{8\pi^2 \hbar^2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{y}} \Big\{ 1 + \frac{1-x^2}{2x} \ln \frac{|1+x|}{|1-x|} \Big\}_{|x=\frac{|\boldsymbol{k}|}{2k_F}}. \tag{31}$$

B. problem IIB

Consider that two magnetic impurity spins are introduced at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively. The non-interacting electron gas is spin polarized by the impurity spin at $\boldsymbol{x}_1$,

$$\langle \boldsymbol{s}(\boldsymbol{x}) \rangle = J V(\boldsymbol{x} - \boldsymbol{x}_1) \boldsymbol{S}(\boldsymbol{x}_1). \tag{32}$$

Such a spin polarization will interact with the other impurity spin at $\boldsymbol{x}_2$ via the short-range potential. Thus, the effective interaction between these two impurity spins can be calculated as;

$$\mathcal{H}_{\rm RKKY} = J^2 V(\boldsymbol{x}_1 - \boldsymbol{x}_2) \boldsymbol{S}(\boldsymbol{x}_1) \cdot \boldsymbol{S}(\boldsymbol{x}_2). \tag{33}$$

Show that

$$V(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \frac{m k_F}{\hbar^2} \Big(\frac{k_F}{\pi} \Big)^3 \bigg\{ \frac{\cos\big(2k_F|\boldsymbol{x}_1 - \boldsymbol{x}_2|\big)}{\big(2k_F|\boldsymbol{x}_1 - \boldsymbol{x}_2|\big)^3} - \frac{\sin\big(2k_F|\boldsymbol{x}_1 - \boldsymbol{x}_2|\big)}{\big(2k_F|\boldsymbol{x}_1 - \boldsymbol{x}_2|\big)^4} \bigg\} \tag{34}$$

C. problem IIC

Consider that the non-interacting electron gas is under the magnetic field (Zeeman magnetic field H_z);

$$\mathcal{H}_0 = \textstyle\sum_{\alpha=\uparrow,\downarrow} \int d\boldsymbol{x}^3 \psi^\dagger_\alpha(\boldsymbol{x}) \Big(- \frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} + h_{\rm z} \big[\sigma_z\big]_{\alpha\alpha} \Big) \psi_\alpha(\boldsymbol{x}), \tag{35}$$

where $h_{\rm z} \equiv \frac{1}{2} g \mu_B H_{\rm z}$, μ_B is the Bohr magneton and g is called as the g-factor ($\mathcal{H}_1$ is same as above). Since the system possesses the U(1) spin rotational symmetry around the z-direction, the effective spin-exchange interaction between magnetic impurity spins respects the same $U(1)$ symmetry;

$$\mathcal{H}_{\rm RKKY} = J^2 V_{ZZ}(\boldsymbol{x}_1 - \boldsymbol{x}_2) S_z(\boldsymbol{x}_1) S_z(\boldsymbol{x}_2) + J^2 V_{XY}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \big(S_x(\boldsymbol{x}_1) S_x(\boldsymbol{x}_2) + S_y(\boldsymbol{x}_1) S_y(\boldsymbol{x}_2) \big). \tag{36}$$

Explain how $V_{ZZ}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ depends on the Zeeman mangetic field h_z.

IV. PROBLEM III

A. Problem IIIA

Suppose an electron-electron interaction potential comprises of spin-independent and spin-dependent parts;

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{37}$$

$$\mathcal{H}_0 = \sum_{\alpha=\uparrow,\downarrow} \int d\boldsymbol{x}^3 \psi^\dagger_\alpha(\boldsymbol{x})\Big(-\frac{\hbar^2\boldsymbol{\nabla}^2}{2m}\Big)\psi_\alpha(\boldsymbol{x}), \tag{38}$$

$$\mathcal{H}_1 = \frac{1}{2}\int d^3\boldsymbol{x}d^3\boldsymbol{x}' V_{\alpha\beta,\gamma\delta}(\boldsymbol{x}-\boldsymbol{x}')\psi^\dagger_\alpha(\boldsymbol{x})\psi^\dagger_\gamma(\boldsymbol{x}')\psi_\delta(\boldsymbol{x}')\psi_\beta(\boldsymbol{x}), \tag{39}$$

$$V_{\alpha\beta,\gamma\delta}(\boldsymbol{x}-\boldsymbol{x}') = V_0(\boldsymbol{x}-\boldsymbol{x}')\,\delta_{\alpha\beta}\delta_{\gamma\delta} + V_1(\boldsymbol{x}-\boldsymbol{x}')\,[\boldsymbol{\sigma}_\mu]_{\alpha\beta}[\boldsymbol{\sigma}_\mu]_{\gamma\delta}. \tag{40}$$

The summation over the repeated spin indices (α,β, γ,δ=$\uparrow,\downarrow$) were and will be omitted henceforth unless dictated otherwise. The same convention for the repeated spin component indices $\mu,\nu = x,y,z$. Suppose that the system is subjected under an impulsive scalar potential field;

$$\mathcal{H}_{\rm ext} = \int d^3\boldsymbol{x}\rho(\boldsymbol{x})\phi^{\rm ex}(\boldsymbol{x},t) \tag{41}$$

with

$$\rho(\boldsymbol{x}) \equiv \psi^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x}),\quad \phi^{\rm ex}(\boldsymbol{x},t) = e^{i\boldsymbol{q}\boldsymbol{x}}\phi\delta(t), \tag{42}$$

the induced charge density is given by the retarded density correlation function;

$$\langle\rho(\boldsymbol{x},t)\rangle = \frac{1}{\hbar}\int_{-\infty}^{+\infty} dt'd^3\boldsymbol{x}' D^R(\boldsymbol{x},t;\boldsymbol{x}',t')\phi^{\rm ex}(\boldsymbol{x}',t'), \tag{43}$$

with eqs.(11,12,13). Using the Lehmann representation, one can relate the Fourier series of the retarded function with the corresponding time-ordered correlation function as in eqs. (17,18,19,20). **Show** that, within the ring approximation, the Fourier series of the time-ordered density correlation function (eqs. (19,20)) is given by

$$\frac{1}{\hbar}D(\boldsymbol{k},\omega) = \frac{\frac{1}{\hbar}D^\star_{(r)}(\boldsymbol{k},\omega)}{1-\frac{1}{\hbar}D^\star_{(r)}(\boldsymbol{k},\omega)V_0(\boldsymbol{k})}. \tag{44}$$

$V_0(\boldsymbol{k})$ is the Fourier series of spin-independent interaction potential;

$$V_0(\boldsymbol{k}) = \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{k}\boldsymbol{x}}\, V_0(\boldsymbol{x})$$

The bare polarization part $\frac{1}{\hbar}D^\star_{(r)}(\boldsymbol{k},\omega)$ is given by the non-interacting single-particle Green's function as

$$\frac{1}{\hbar}D^\star_{(r)}(\boldsymbol{k},\omega) = -\frac{2i}{\hbar}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\int\frac{d\omega'}{2\pi}G^0(\boldsymbol{k}+\boldsymbol{k}',\omega+\omega')G^0(\boldsymbol{k}',\omega'), \tag{45}$$

$$G^0(\boldsymbol{k},\omega) = \frac{\theta(|\boldsymbol{k}|-k_F)}{\omega-\omega_{\boldsymbol{k}}+i\eta} + \frac{\theta(k_F-|\boldsymbol{k}|)}{\omega-\omega_{\boldsymbol{k}}-i\eta}, \tag{46}$$

with $\hbar\omega_{\boldsymbol{k}} = \frac{\hbar^2\boldsymbol{k}^2}{2m}$ and infinitesimally small positive η.

B. Problem IIIB

Without loss of generality, we can assume that $V_0(\boldsymbol{k})$ is real-valued ($V_0(\boldsymbol{x}) = V_0(-\boldsymbol{x})$). The Fourier series of the retarded function is immediately obtained from eqs. (17,18) as;

$$\frac{1}{\hbar}D^R(\boldsymbol{k},\omega) = \frac{\frac{1}{\hbar}D^{\star,R}_{(r)}(\boldsymbol{k},\omega)}{1-\frac{1}{\hbar}D^{\star,R}_{(r)}(\boldsymbol{k},\omega)V_0(\boldsymbol{k})}, \tag{47}$$

with

$$\mathrm{Re}D^{\star,R}_{(r)}(\boldsymbol{k},\omega) = \mathrm{Re}D^{\star}_{(r)}(\boldsymbol{k},\omega), \tag{48}$$

$$\mathrm{Im}D^{\star,R}_{(r)}(\boldsymbol{k},\omega) = \mathrm{sign}(\omega)\mathrm{Im}D^{\star}_{(r)}(\boldsymbol{k},\omega). \tag{49}$$

Show that, within this approximation, the spin density induced by the impulsive magnetic field eq. (41) is given by

$$\langle \rho(\boldsymbol{x},t)\rangle \propto \phi e^{i\boldsymbol{q}\boldsymbol{x}-i\Omega_{\boldsymbol{q}}t-\gamma_q t} \tag{50}$$

where $\Omega_{\boldsymbol{q}}$ and $\gamma_{\boldsymbol{q}}$ are obtained as a pole of eq. (47);

$$1 = V_0(\boldsymbol{q})\frac{1}{\hbar}D^{\star,R}_{(r)}(\boldsymbol{q},\Omega_{\boldsymbol{q}}-i\gamma_{\boldsymbol{q}}). \tag{51}$$

In the longe wavelength limit ($|\boldsymbol{q}|\to 0$), we can assume that $\Omega_{\boldsymbol{q}}$ thus determined is porportional to $|\boldsymbol{q}|$;

$$\Omega_{\boldsymbol{q}} = c_c|\boldsymbol{q}|. \tag{52}$$

Show that, in this limit, $\gamma_{\boldsymbol{q}} = 0$ when $\frac{mc_c}{\hbar k_F} > 1$. Assuming further that $\frac{mc_c}{\hbar k_F} > 1$ and $V_0(\boldsymbol{q}=0) > 0$, **derive** an expression for c_c in the following two limiting cases;

$$V_0(\boldsymbol{q}=0) \ll \frac{\pi^2\hbar^2}{mk_F} \quad \text{(weak coupling)} \tag{53}$$

$$V_0(\boldsymbol{q}=0) \gg \frac{\pi^2\hbar^2}{mk_F} \quad \text{(strong coupling)} \tag{54}$$

C. problem IIIC

Consider that the system is under the Zeeman field (the other parts of the Hamiltonian is same as above);

$$\mathcal{H}_0 = \sum_{\alpha=\uparrow,\downarrow}\int d\boldsymbol{x}^3\psi^\dagger_\alpha(\boldsymbol{x})\Big(-\frac{\hbar^2\boldsymbol{\nabla}^2}{2m} + h_z\big[\sigma_z\big]_{\alpha\alpha}\Big)\psi_\alpha(\boldsymbol{x}), \tag{55}$$

The time-ordered density correlation function comprises of four contributions;

$$iD(\boldsymbol{x},t;\boldsymbol{x}',t') = iD_{\uparrow\uparrow}(\cdots) + iD_{\uparrow\downarrow}(\cdots) + iD_{\downarrow\uparrow}(\cdots) + iD_{\downarrow\downarrow}(\cdots). \tag{56}$$

$iD_{\alpha\beta}(\cdots)$ with $\alpha,\beta =\uparrow,\downarrow$ are defined as follows

$$iD_{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{x}',t') = \frac{\big\langle\Psi_0\big|T\big\{\tilde{\rho}_{H,\alpha}(\boldsymbol{x},t)\tilde{\rho}_{H,\beta}(\boldsymbol{x}',t')\big\}\big|\Psi_0\big\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{57}$$

and

$$\tilde{\rho}_{H,\alpha}(\boldsymbol{x},t) \equiv e^{i\frac{\mathcal{H}t}{\hbar}}\psi^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x})e^{-i\frac{\mathcal{H}t}{\hbar}} - \frac{\langle\Psi_0|\psi^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{58}$$

for $\alpha =\uparrow,\downarrow$ respectively (the summation over α is not assumed above). **Show** that the Dyson equation for these four generally takes the following form in the moment space;

$$\frac{1}{\hbar}\begin{pmatrix} D_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix} = \frac{1}{\hbar}\begin{pmatrix} D^\star_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D^\star_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix}$$
$$+\frac{1}{\hbar^2}\begin{pmatrix} D^\star_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D^\star_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix}\begin{pmatrix} V_0+V_1 & V_0-V_1 \\ V_0-V_1 & V_0+V_1\end{pmatrix}\begin{pmatrix} D_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix}. \tag{59}$$

where $V_0 \pm V_1 \equiv V_0(\boldsymbol{k}) \pm V_1(\boldsymbol{k})$ and $D^\star_{\alpha\beta}(\boldsymbol{k},\omega)$ $(\alpha,\beta =\uparrow,\downarrow)$ denotes the proper part of the density correlation function.

Within the ring diagram approximation, one may approxiate the proper part by its lowest order in the interactions;

$$\frac{1}{\hbar}\begin{pmatrix} D^\star_{\uparrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\uparrow\downarrow}(\boldsymbol{k},\omega) \\ D^\star_{\downarrow\uparrow}(\boldsymbol{k},\omega) & D^\star_{\downarrow\downarrow}(\boldsymbol{k},\omega)\end{pmatrix} = \begin{pmatrix} \Pi_{(0),\uparrow}(\boldsymbol{k},\omega) & 0 \\ 0 & \Pi_{(0),\downarrow}(\boldsymbol{k},\omega)\end{pmatrix} + \mathcal{O}(V_0,V_1).$$

where

$$\Pi_{(0),\alpha}(\boldsymbol{k},\omega) = -\frac{i}{\hbar}\int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\int \frac{d\omega'}{2\pi} G^0(\boldsymbol{k}+\boldsymbol{k}',\omega+\omega',\alpha)G^0(\boldsymbol{k}',\omega',\alpha), \tag{60}$$

$$G^0(\boldsymbol{k},\omega,\alpha) = \frac{\theta(|\boldsymbol{k}|-k_{F,\alpha})}{\omega-\omega_{\boldsymbol{k}}-(\sigma_z)_{\alpha\alpha}h_z+i\eta} + \frac{\theta(k_{F,\alpha}-|\boldsymbol{k}|)}{\omega-\omega_{\boldsymbol{k}}-(\sigma_z)_{\alpha\alpha}h_z-i\eta}, \tag{61}$$

with $\alpha =\uparrow,\downarrow$ respectively. $k_{F,\alpha}$ denotes the Fermi momentum for α-spin in the presence of the Zeeman field;

$$\frac{\hbar^2 k^2_{F,\uparrow}}{2m} + h_z = \frac{\hbar^2 k^2_{F,\downarrow}}{2m} - h_z \equiv \mu. \tag{62}$$

Show that $\Pi_{0,\alpha}(\boldsymbol{k},\omega)$ thus introduced takes exactly the same form as $\Pi_0(\boldsymbol{k},\omega)$ with k_F *being replaced by* $k_{F,\alpha}$ *and the overall factor* 2 *being dropped*. Sovling eq. (59), **show** that $D^T(\boldsymbol{k},\omega)$ is given by the following,

$$D^T(\boldsymbol{k},\omega) = \frac{\hbar\big(\Pi_{(0),\uparrow}+\Pi_{(0),\downarrow}-4\Pi_{(0),\uparrow}\Pi_{(0),\downarrow}V_1\big)}{1-(\Pi_{(0),\uparrow}+\Pi_{(0),\downarrow})(V_0+V_1)+4\Pi_{(0),\uparrow}\Pi_{(0),\downarrow}V_0V_1} \tag{63}$$

Noting that $V_0(\boldsymbol{k})$ and $V_1(\boldsymbol{k})$ are real-valued, one obtain the retarded density correlation function as

$$D^R(\boldsymbol{k},\omega) = \frac{\hbar\big(\Pi^R_{(0),\uparrow}+\Pi^R_{(0),\downarrow}-4\Pi^R_{(0),\uparrow}\Pi^R_{(0),\downarrow}V_1\big)}{1-(\Pi^R_{(0),\uparrow}+\Pi^R_{(0),\downarrow})(V_0+V_1)+4\Pi^R_{(0),\uparrow}\Pi^R_{(0),\downarrow}V_0V_1} \tag{64}$$

with

$$\mathrm{Re}\Pi^R_{(0),\alpha}(\boldsymbol{k},\omega) = \mathrm{Re}\Pi_{(0),\alpha}(\boldsymbol{k},\omega),$$
$$\mathrm{Im}\Pi^R_{(0),\alpha}(\boldsymbol{k},\omega) = \mathrm{sgn}\omega\,\mathrm{Im}\Pi_{(0),\alpha}(\boldsymbol{k},\omega).$$

In limit of small ω and $\boldsymbol{q}$, the retarded density correlation function has two poles in the complex ω plane, say $\omega = \Omega_{\boldsymbol{q},1}-i\gamma_{\boldsymbol{q},1}$ and $\omega = \Omega_{\boldsymbol{q},2}-i\gamma_{\boldsymbol{q},2}$. They are the zeros of the denominator of the right hand side of eq. (64). To see this, we may regard again that $\Omega_{\boldsymbol{q},j}$ becomes proportional to $|\boldsymbol{q}|$ in the long wavelength limit ($|\boldsymbol{q}|\rightarrow 0$);

$$\Omega_{\boldsymbol{q},1} = c_1|\boldsymbol{q}|, \quad \Omega_{\boldsymbol{q},2} = c_2|\boldsymbol{q}|. \tag{65}$$

with $c_1 > c_2$. **Show** that, in this limit, $\gamma_{\boldsymbol{q},1}=\gamma_{\boldsymbol{q},2}=0$ when $\frac{mc_1}{\hbar k_{F,\uparrow}} > \frac{mc_2}{\hbar k_{F,\uparrow}} > 1$ and $\frac{mc_1}{\hbar k_{F,\downarrow}} > \frac{mc_2}{\hbar k_{F,\downarrow}} > 1$. Assuming that c_1 and c_2 is sufficiently large that all these conditions are satisfied, **evaluate** c_1 and c_2 from the dynamic limit of eq. (64); $\lim_{\omega\rightarrow 0}\lim_{|\boldsymbol{q}|\rightarrow 0}$ ($V_0(\boldsymbol{k}=0)$ and $V_1(\boldsymbol{k}=0)$ are finite postive values).

D. Hint for problem IIIA, IIIC

In order to derive eqs. (44,45,46), consider the following function (temporally called 'connected' two-point Green's function);

$$G^{(c)}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) \equiv G_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) - G_{\alpha\delta}(x,x+)G_{\beta\gamma}(x',x'+). \tag{66}$$

where the time-ordered single-particle Green's function and two-particle Green's function are defined as usual;

$$iG_{\alpha\beta}(x,x') \equiv \frac{\langle\Psi_0|T\{\psi_{H,\alpha}(x)\psi^\dagger_{H,\beta}(x')\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}. \tag{67}$$

$$i^2 G_{\alpha\beta,\delta\gamma}(x_1,x_2,;x'_1,x'_2) \equiv \frac{\langle\Psi_0|T\{\psi_{H,\alpha}(x_1)\psi_{H,\beta}(x_2)\psi^\dagger_{H,\gamma}(x'_2)\psi^\dagger_{H,\delta}(x'_1)\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{68}$$

$|\Psi_0\rangle$ is the ground state wavefunction of the interacting Hamiltonian $\mathcal{H}$. $x\equiv(\boldsymbol{x},t)$, $x'\equiv(\boldsymbol{x}',t')$, $x+\equiv(\boldsymbol{x},t+)$ and so on. The time-ordered density correlation function is given by this 'connected' two-point Green's function as

$$\begin{aligned} iD_{\alpha\beta}(x,x') &= -G^{(c)}_{\alpha\beta,\alpha\beta}(x,x';x+,x'+). \\ iD(x,x') &\equiv \sum_{\alpha,\beta} iD_{\alpha\beta}(x,x'). \end{aligned} \tag{69}$$

To derive eq. (44), one can begin with Fourier transform of the following Dyson equation for the 'connected' two-point Green's function;

$$G^{(c)}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) = G^{(c),\star}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) + \frac{i}{\hbar}\int d^4x_1 \int d^4x_2$$
$$G^{(c),\star}_{\alpha\gamma_1,\delta\gamma_2}(x,x_1;x+,x_1+)\, V_{\gamma_2\gamma_1,\delta_2\delta_1}(\boldsymbol{x}_1-\boldsymbol{x}_2)\delta(t_1-t_2)\, G^{(c)}_{\delta_1\beta,\delta_2\gamma}(x_2,x';x_2+,x'+). \tag{70}$$

where $G^{(c),\star}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+)$ denotes the so-called 'proper part' of the connected two-point Green's function. In the ring approximation, we replace this proper part as its bare contribution (lowest order in the interaction potential);

$$G^{(c),\star}_{\alpha\beta,\delta\gamma}(x,x';x+,x'+) = -G^0_{\alpha\gamma}(x,x'+)G^0_{\beta\delta}(x',x+) + \mathcal{O}(V). \tag{71}$$

where $G^0_{\alpha\beta}(x,x')$ denotes the bare (non-interacting) single-particle Green's function;

$$iG^0_{\alpha\beta}(x,x') = \frac{\langle\Phi_0|T\{\psi_{I,\alpha}(x)\psi^\dagger_{I,\beta}(x')\}|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle} = \delta_{\alpha\beta}\frac{\langle\Phi_0|T\{\psi_I(x)\psi^\dagger_I(x')\}|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle} \tag{72}$$

$|\Phi_0\rangle$ is the ground state wavefunction of the non-interacting Hamiltonian $\mathcal{H}_0$. The fourier transform of eq. (72) is calculated to be eq. (46). The minus sign in the r.h.s. of eq. (71) comes from an exchange of the fermion field.

E. Hint for problem IIIB,IIIC

With $h_Z = 0$, the bare polarization part is given by

$$\Pi_0(\boldsymbol{k},\omega) \equiv \frac{1}{\hbar}D^\star_{(r)}(\boldsymbol{k},\omega) = -\frac{2i}{\hbar}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\int\frac{d\omega'}{2\pi}G^0(\boldsymbol{k}+\boldsymbol{k}',\omega+\omega')G^0(\boldsymbol{k}',\omega'), \tag{73}$$
$$G^0(\boldsymbol{k},\omega) = \frac{\theta(|\boldsymbol{k}|-k_F)}{\omega-\omega_{\boldsymbol{k}}+i\eta} + \frac{\theta(k_F-|\boldsymbol{k}|)}{\omega-\omega_{\boldsymbol{k}}-i\eta}. \tag{74}$$

With a proper normalization, $\boldsymbol{k}\equiv k_F\boldsymbol{q}$ and $\omega \equiv \frac{\hbar k_F^2}{m}\nu$, its real part and imaginary part are calculated as follows,

$$\mathrm{Re}\Pi_0(k_F\boldsymbol{q},\frac{\hbar k_F^2}{m}\nu) = \frac{2mk_F}{\hbar^2}\frac{1}{4\pi^2}\times\Big\{-1+\frac{1}{2q}\Big(1-\Big(\frac{\nu}{q}-\frac{q}{2}\Big)^2\Big)\ln\Big[\frac{|\nu-\frac{q^2}{2}+q|}{|\nu-\frac{q^2}{2}-q|}\Big]$$
$$-\frac{1}{2q}\Big(1-\Big(\frac{\nu}{q}+\frac{q}{2}\Big)^2\Big)\ln\Big[\frac{|\nu+\frac{q^2}{2}+q|}{|\nu+\frac{q^2}{2}-q|}\Big]\Big\} \tag{75}$$

and

$$\mathrm{Im}\Pi_0(k_F\boldsymbol{q},\frac{\hbar k_F^2}{m}\nu)$$
$$= \begin{cases} -\frac{mk_F}{\hbar^2}\frac{1}{4\pi q}\Big[1-\Big(\frac{\nu}{q}-\frac{q}{2}\Big)^2\Big] & (q>2\ \&\ \frac{q^2}{2}-q\le\nu\le\frac{q^2}{2}+q)\ \text{ or }\ (q<2\ \&\ -\frac{q^2}{2}+q\le\nu\le\frac{q^2}{2}+q) \\ -\frac{mk_F}{\hbar^2}\frac{\nu}{2\pi q} & (q<2\ \&\ 0\le\nu\le-\frac{q^2}{2}+q) \\ 0 & (\text{otherwise}) \end{cases} \tag{76}$$

V. PROBLEM IV

In the presence of the external magnetic gauge potential $\boldsymbol{A}(\boldsymbol{x})$, the BCS Hamiltonian takes the form of

$$\mathcal{H} = \int d^3\boldsymbol{x}\psi^\dagger_\alpha(\boldsymbol{x})\frac{1}{2m}\Big(-i\hbar\nabla+\frac{e\boldsymbol{A}}{c}\Big)^2\psi_\alpha(\boldsymbol{x}) - \frac{g}{2}\int d^3\boldsymbol{x}\psi^\dagger_\alpha(\boldsymbol{x})\psi^\dagger_\beta(\boldsymbol{x})\psi_\beta(\boldsymbol{x})\psi_\alpha(\boldsymbol{x}).$$

Show that the continuity equation with $\boldsymbol{A}\neq 0$;

$$\partial_t\rho(\boldsymbol{x}) + \boldsymbol{\nabla}\cdot\boldsymbol{j}^{(0)}(\boldsymbol{x}) + \boldsymbol{\nabla}\cdot\boldsymbol{j}^{(1)}(\boldsymbol{x}) = 0$$

The electron density, paramagnetic and diamagnetic current densities are defined as

$$\begin{aligned}
\rho(\boldsymbol{x}) &\equiv -e\psi^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x}) \\
\boldsymbol{j}^{(0)}_\mu(\boldsymbol{x}) &\equiv \frac{i\hbar e}{2m}\Big\{\psi^\dagger_\alpha(\boldsymbol{x})\nabla_\mu\psi_\alpha(\boldsymbol{x}) - \big(\nabla_\mu\psi^\dagger_\alpha(\boldsymbol{x})\big)\psi_\alpha\Big\} \\
\boldsymbol{j}^{(1)}_\mu(\boldsymbol{x}) &\equiv -\frac{e^2}{mc}\boldsymbol{A}_\mu(\boldsymbol{x})\psi^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x}),
\end{aligned}$$

respectively. Based on the linear response theory at finite temperature, **express** the expectation value of the current density, $\boldsymbol{j}(\boldsymbol{x}) \equiv \boldsymbol{j}^{(0)}(\boldsymbol{x}) + \boldsymbol{j}^{(1)}(\boldsymbol{x})$, up to the linear order in the vector potential $\boldsymbol{A}(\boldsymbol{x})$.

VI. PROBLEM V

Derive $V_{XY}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ in eq. (36) and **explain** how $V_{XY}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ depends on h_z.