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Lie theory: Lie group and Lie algebra

Why we need Lie theory ?

1. The set of rotations is not a vectorspace in the sense of linear algebra. However, rotations do form another mathematical object called a non-commutative group, which possesses some, but not all, of the usual vectorspace properties.
2. 长话短说：控制算法、机器学习算法，往往依赖于定义两个状态间的"差值"。对于R^n中的两个向量x1, x2来说很简单，x1和x2各元素相减就行了。但刚体运动所构成的集合SE(3)不是向量空间，没办法直接做减法，所以需要重新定义差值；因此，距离、导数、梯度、积分等等运算也需要重新定义。了解李群李代数就能知道怎么正确地定义这些量。

Why we discard other rotation representations (parameterizations) ?

There are several ways to parameterize rotations (e.g., rotation ma- trix, Euler angles, unit-length quaternions). They all have advan- tages and disadvantages; some have singularities while the others have constraints.

What we want is a unconstrained optimization problem, so the ones with constraints (e.g. rotations matrix, quaternion) and the ones whose space is local in that it cannot cover the range of all rotations (e.g. Euler angles). Hence, we choose to use Lie group and Lie algebra which can bypass the issues listed above.

What is Lie theory ?

Adjoint representation / action

Applications of Lie theory

1. 运用在机器人轨迹插值中，实现连续(光滑)插值

Miscellaneous notes

Edge and corner in feature detection

Edge

Corner

An interesting and insightful video about topology

manifold

main cf. Manifolds: A gentle introduction

A compact and visualizing elaboration for manifold

和上面这个视频差不多意思的解释，不过解释了更多的名词，并且是系列视频，还讲了其他的, 该系列视频2，3，5有价值。

What does "smoothness" mean wrt. a manifold ?

“天地有正气，杂然赋流形” -- 文天祥。

其实流形并不是什么神秘的东西，它的英文是"Manifold","流形"这一翻译主要源于文天祥的诗句"天地有正气，杂然赋流形","流形"实际就是意指世界万物。我们过去研究的几何对象总是在标准的欧氏空间中进行的，但世界并不是我们所期待地那么规则，这个时候就有了"流形"的概念。从直观上说，"流形"是可以在局部等同于欧氏空间的一些对象。

Atlas

补充材料

What is atlas on Earth?

TODO

Why an n-sphere?

Quora discussion: Is a sphere 3D or 2D

implicit and explicit representation

"Modern Robotics: Mechanics, Planning, and Control" series course.

Configuration Space topology

Configuration Space configuration

Tensor

A very brief and low-level explanation about tensor, 有点错误！

一篇看起来不错的博客，不过感觉只是从一个角度解释了张量

一个非常有趣从物理的角度解释张量的系列视频, 这里还有一个与该视频作者 consistent 的张量的第三种定义方式的文字解释

另外一个系列视频，看起来好像是从 multidimensional array 的角度解释张量

Topology

An illustration about topology based on a concrete problem

surjective mapping and onto mapping

They are the same.

同构、同态、同胚

它们之间的区别概述, 以及这篇回答的 English counterpart

群中的同构和同态

bilinear map

TODO

对数学的整体的把握

某MIT学生对现代数学的思考与略述

二十世纪的数学

数学的四个角度观察一个数学对象

参考 a micro Lie theory ... paper. geometry algebra TODO

Visualizaing quaternions

Countable set and countably infinite set

A complete elaboration for injection, surjection and bijection

homeomorphism is a bijection.

Why figure-8 is not a manifold?

From definition and from tangent space.

A comprehensive elaboration of group homomorphism

Group

A very low-level but (perhaps) useful introduction to group theory

> The easy answer is that group theory is a perfect tool for studying symmetry, but group theory is not confined to the study of symmetry. It's useful for other things like abstract algebra and dynamical systems and just lots of things.

An element of a group is just like an element of a set.

Group action

More about group action

Group of transformations

Group theory is all about symmetry

A mathexchange discussion

A 3Blue1Brown video, this video illustrates the idea of symmetry and group action.

This video states that "**a group is a collection of symmetric actions on some mathematical object**." 并以 实数线 这一简单的例子解释了group theory 的很多概念，包括 group homomorphism that preserves the group structure.

It states: homo = same, morph = shape, homomorphism = same-shape-ism.

Symmetry group

Isometry group

Isometry

Symmetry

What is symmetry?

A symmetry is a transformation that preserves some structure.

Rotating a hexagon by one-sixth of a full turn puts it back where it started, preserving the placement of vertices and edges. That's a symmetry.

Shifting every real number up by some fixed amount preserves the order of the numbers, as well as their relative distances. That's a symmetry. Shifting the numbers by 1 is a special case where more structure is preserved: the property of being an integer.

Symmetries can be combined: if you have a symmetry A, and a symmetry B, you can perform A and then perform B to obtain another symmetry which will (often) be different from A and B. Rotating a hexagon twice by 1/6 of a turn yields a rotation by 1/3 of a turn, which is also a symmetry of the hexagon.

The set of all symmetries of any object forms a group, in which the successive combination of symmetries forms the operation of multiplication.

You should not assume that the group itself is symmetric (though it is, too, in an appropriate sense). Group theory is the study of symmetry not because groups are nice and symmetric, but because groups describe how the symmetries of any object can be combined together, how they interplay and coexist.

multiplicative group

additive group

metric space and topological space

metric space

A metric on a space induces topological properties like open and closed sets, which lead to the study of more abstract topological spaces.

laws

commutative law
associative law distributive law

Twist in Robotics

An illustrative short video

ESKF相关

A blog

相关的论文 TODO

n-sphere

3-sphere

How to visualize the 3-sphere(n-sphere)?

A 3Blue1Blown video for visualizing the quaternion space(a 3-sphere)

A interesting discusion: Do we live in 3D space?

degree of freedom

dof of manifold = dim of manifold

degree of freedom in mechanics, generalizable to math

In physics, the **degrees of freedom** (**DOF**) of a mechanical system is the number of independent parameters that define its configuration or state. It is important in the analysis of systems of bodies in mechanical engineering,

Motions and dimensions [edit]

The position of an *n*-dimensional rigid body is defined by the rigid transformation, [*T*] = [*A*, *d*], where *d* is an *n*-dimensional translation and *A* is an $n \times n$ rotation matrix, which has *n* translational degrees of freedom and $n(n-1)/2$ rotational degrees of freedom. The number of rotational degrees of freedom comes from the dimension of the rotation group SO(n).

A non-rigid or deformable body may be thought of as a collection of many minute particles (infinite number of DOFs), this is often approximated by a finite DOF system. When motion involving large displacements is the main objective of study (e.g. for analyzing the motion of satellites), a deformable body may be approximated as a rigid body (or even a particle) in order to simplify the analysis.

The degree of freedom of a system can be viewed as the minimum number of coordinates required to specify a configuration. Applying this definition, we have:

1. For a single particle in a plane two coordinates define its location so it has two degrees of freedom;
2. A single particle in space requires three coordinates so it has three degrees of freedom;
3. Two particles in space have a combined six degrees of freedom;
4. If two particles in space are constrained to maintain a constant distance from each other, such as in the case of a diatomic molecule, then the six coordinates must satisfy a single constraint equation defined by the distance formula. This reduces the degree of freedom of the system to five, because the distance formula can be used to solve for the remaining coordinate once the other five are specified.

Rotation formalisms in three dimensions

Axis-angle representation

Solving differential equations (DE)

Math is fun series:

1. Differential Equations
2. Differential Equations Solution Guide

Eigen notes

Exponential of a general square matrix is defined by its Taylor expansion. You don't want to symbolically represent the whole series. And, Eigen is a numeric library anyway, its algorithms are designed to compute to numbers.

A Python symbolic calculation package: SymPy

Use this to calculate the matrices operation with the entries substituted with the symbols instead of the exact numbers.

The reason why we can set ω as a constant in exponential map derivation

General formula of Commonly-used series

How to derive these series from Taylor series?

Prove Taylor's Theorem

在不同点展开，及令 x_0 等于不同的值，其逼近真值的效果(即accuracy)是不一样的。这个逼近的 error 用 remainder term 或称 error term 来描述，即 $R_N(x)$.

Why 3D rotation matrices are not commutative while 2D can?

Derivation of general rotation matrices

Another useful material

4D rotation matrices

4D rotations keep a plane fixed, that is rotating around a plane.

A simple rotation R about a rotation centre O leaves an entire plane A through O (axis-plane) fixed.

A article series illustrating Calculus