Introduction to Linear Algebra

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Chapter 1

Introduction to Vectors

1.1 Vectors and Linear Combinations

linear combination = vector addition + scalar multiplication
col / row vector
three vector representations

1.2 Lengths and Dot products

dot / inner prod.
length / norm of a vector
unit vector
angle between two vectors, cosine formula, perp.
Schwarz Inequlity, Triangle Inequlity; Geometric mean, Arithmatic mean

1.3 Matrices[1]

Matrix times vector Ax = comb. of cols of A or multiplication a row at a time
$Ax = b, x = A^{-1}b$, if A is inv.
indep. / dep.

[1]This sec is looking ahead the key ideas, not fully explained yet.

Chapter 2

Solving Linear Equations

2.1 Vectors and linear equations

a system of equs	vector equ	mat equ
row pic	col pic	matrix pic
dot prod. with rows	comb. of cols	matrix-vector multiplication

2.2 The idea of elimination

Gauss elimination $\rightarrow$ Upper triangular U
pivot, multiplier
forward elimination, back substitution, exchange rows
#pivots, singular

2.3 Elimination using matrices

elementary (elimination) matrix, permutation matrix
augmented matrix
matrix multiplication by rows / cols

2.4 Rules for matrix operations

inner (dot) / outer prod.
mnp multiplications
four ways to matrix multi., the fourth way!
The laws of matrix operations
powers of A
block matrices and block multi., block elimination $\rightarrow$ Schur complement

2.5 Inverse Matrices

inv. def
#pivots and invty; if $Ax = 0$ has a nonzero sol'n, then A is not inv.
inverse of a prod. of mats
Gauss-Jordan elim. $\rightarrow$ rref.
diagonally dominant mats are inv.

2.6 Elimination = Factorization: $A = LU$

Gauss elim. without row exchanges $\rightarrow A = LU$
triangular factorization: $A = LU$ or $A = LDU$, for better symmetry. Here D is the pivot matrix
predict zeros in L and U according to the first entry of rows and cols of A
L stores the multiplier l_{ij}
factor and solve

2.7 Transposes and Permutations

tranpose of sum, prod. and inverses
inner prod. and outer prod. with the introduction of tranpose T
symmetric mats S, $A^\top A$ and $AA^\top$
$A = LDU \rightarrow$ symmetric factorization $S = LDL^\top$ (with no row exchanges, U is exactly $L^\top$)
P^{-1} is also permutation mat if P is a permutation mat; $P^\top = P^{-1}$ because both come from the prod. of row exchanges *in reverse order*

Chapter 3

Vector Spaces and Subspaces

3.1 Spaces of Vectors

vector spaces $\mathbb{R}^2$, $\mathbb{C}^2$, $\mathbb{Z}$, all real 2 by 2 matrices, all real functions, ...
eight rules for vector addition and scalar multiplication a vector space must obey
subspace, linear comb. requirement, every subspace contains the zero "vector"
cols of A span the col space $C(A)$, attainable right side $\mathbf{b}$

3.2 The Nullspace of A: Solving $Ax = 0$ and $Rx = 0$

nullspace $N(A)$
pivot / free cols $\to$ pivot / free vars $\to$ special sol'ns s
adding extra equs (giving extra rows), imposing more conditions, $N(A)$ certainly cannot go larger; adding extra unknowns (giving extra cols), more dofs., $N(A)$ goes larger (more #components of vector $\mathbf{v}$ in $N(A)$
rref. R reveals pivot cols and free cols, pivot rows and cols contain I
with $n > m$ there's at least one free var, then $Ax = 0$ has nonzero sol'ns
every "free col" is a comb. of earlier pivot cols. It's the special sol'ns s that tell us those combinations (with signs reversed)
dimension of a vector space, #components of a vector
rank r, rank one matrix $A = uv^\top$, geometry of rank 1 matrices

3.3 The Complete Solution to $Ax = b$

particular sol'n x_p (free vars = 0) $\rightarrow Ax_p = b$
special sol'ns x_n (free vars $= 1, 0, 0, \dots$) $\rightarrow Ax_n = 0$
complete sol'n $\rightarrow x = x_p + x_n$ = one particular sol'n + all special sol'ns
x_p comes directly from d on the right side, x_n comes from the free cols of R
full col rank $r = n \leq m$, $R = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} mxnidentitymatrix \\ m - nrowsofzeros \end{bmatrix}$, A is overdetermined, $Ax = b$ has at most one sol'n
full row rank $r = m \leq n$, $R = \begin{bmatrix} I & F \end{bmatrix}$, F is the free part of R, A is underdetermined, $Ax = b$ has one or ∞ sol'ns
inv mat $r = m = n$, it always has one sol'n

3.4 Indepence, Basics and Dimension

linear indep / dep: put a sequence of vecs into a mat and consider its shape and rank
row space $C(A^\top)$ in $\mathbb{R}^n$, col space $C(A)$ in $\mathbb{R}^m$
span = fill, no need to be indep; basis = span + indep
basis of a space is not unique; every vector of the space is a unique comb. of the basis
find basis from a set of vecs: put them into the rows (cols) of a mat, and eliminate to find the pivot rows of A or R (pivot cols of A, not R)
standard basis for $\mathbb{R}^n$: $n \times n\ I$
$C(A) \neq C(R)$, their bases are different, their dims are the same
dimension = #vectors in any and every basis of the space = "dofs" of the space
basis and dim for matrix spaces

3.5 Dimensions of the Four Subspaces

The Big Picture, four fundamental subspaces:

	dim	basis
row space $C(A^\top)$ in $\mathbb{R}^n$	r	pivot rows
col space $C(A)$ in $\mathbb{R}^m$	r	pivot cols
nullspace $N(A)$ in $\mathbb{R}^n$	$n - r$	special sol'ns
left nullspace $N(A^\top)$ in $\mathbb{R}^m$	$m - r$	special sol'ns for $A^\top y = \mathbf{0}$ or $y^\top A = \mathbf{0}^\top$

Fundamental Theorem of Linear Algebra, Part I:

- Rank Theorem: $dim(C(A)) = dim(C(A^\top)) = rank \quad r$

- Counting Theorem: $dim(C(A)) + dim(N(A)) = dim(\mathbb{R}^n) \to r + (n - r) = n$,
  or replace A for $A^\top$, $dim(C(A^\top)) + dim(N(A^\top)) = dim(\mathbb{R}^m) \to m + (m - r) = m$

$A \xrightarrow{elim.} R$:

- $C(A) \neq C(R)$, the cols of A don't often end in zero rows while R does

- $C(A^\top) = C(R^\top)$, every row of A is a comb. of the rows of R, and vice versa

- $N(A) = N(R)$, elim. doesn't change sol'ns

incidence matrix, nodes and edges (ideas in Graph Theory), loops $\to$ dependence, trees $\to$ independence, Kirchhoff's Voltage Law, Kirchhoff's Current Law
rank one mats $A = \mathbf{u}\mathbf{v}^\top$ = a basis for $C(A)$ $\times$ a basis for $C(A^\top)$. (they are both single vector)
rank two mats = rank one mat + rank one mat
every rank r mat is a sum of r rank one mats. (The original mat A is separated into E^{-1} and R, and the cols of E^{-1} $\times$ rows of R get rank one mats, and then sum these mats to recover A)

Chapter 4

Orthogonality

4.1 Orthogonality of the Four Subspaces

orthogonal subspaces $\mathbf{V}$ and $\mathbf{W}$, $\mathbf{v}^\top \mathbf{w} = \mathbf{0}$ for all $\mathbf{v}$ in $\mathbf{V}$ and all $\mathbf{w}$ in $\mathbf{W}$
when a vector is in two subspaces, it must be the zero vector
$C(A^\top) \perp N(A)$, proved using $Ax = 0$, rows of $A \times \; x \to 0$
$C(A) \perp N(A^\top)$, proved using $A^\top y = 0$, rows of $A^\top \times \; y \to 0$
orthogonal complements $V^\top$ contains every vector that is perp. to V
Fundamental Theorem of Linear Algebra, Part II:

- $C(A^\top)^\perp = N(A)$
- $C(A)^\perp = N(A^\top)$

every $\mathbf{x}$ can be split into a row space component $\mathbf{x_r}$ (which goes to the col space, $Ax_r = Ax$), and a nullspace component $\mathbf{x_n}$ (which goes to zero, $Ax_n = 0$)
a basis has two properties, when the #vectors is right, one property implies the other
when a matrix has the right #vectors, it's inv.

4.2 Projections

To find projection $p = \hat{x}_1 a_1 + \cdots + \hat{x}_n a_n$:

1. certain multiple $\hat{x}$: $A^\top(b - A\hat{x}) = 0$ or $A^\top A\hat{x} = A^\top b \Rightarrow \hat{x} = A(A^\top A)^{-1}A^\top$, A is comb. of a's. We're projecting onto the $C(A)$. When $n = 1$, $A = a$, we're projecting onto a line.
   This normal equation can be derived by geometry (perp.) or by linear algebra (nullspace).

2. projection p: $p = A\hat{x}$

3. projection matrix $P = A(A^\top A)^{-1}A^\top$

Projecting b onto a subspace leaves the error vector $e = b - p$ perp. to the subspace. b can be split into two components: p in $C(A)$ and e in $N(A^\top)$.
$(I - P)b = b - p = e \Rightarrow I - P$ projects b onto the $C(A)^\perp$.
Note that $A^\top A$ generally cannot be split into $A^{-1} \times (A^\top)^{-1}$, because there's no A^{-1} if A is rectangular.
$A^\top A$ is inv. iff A has indep. cols (no need for A to be inv.)
$A^\top A$ has the same nullspace as A
$P^\top = P$; $P^2 = P$, projecting a second time doesn't change anything.

4.3 Least Squares Approximations

4.4 Orthogonal Bases and Gram-Schmidt

Chapter 5

Determinants

5.1 The Properties of Determinants

5.2 Permutations and Cofactors

5.3 Cramer's Rule, Inverses, and Volumes