Introduction to vector and matrix norms

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Vector norms

- A vector norm is a scalar quantity that reflects the “size” of a vector $\mathbf{x}$.
- The norm of a vector $\mathbf{x}$ is denoted as $\|\mathbf{x}\|$.
- There are many ways to define the size of a vector. If $\mathbf{x} \in \mathbb{R}^n$, the three most popular are

$$\text{one-norm}: \quad \|\mathbf{x}\|_1 = \sum_{k=1}^{n} |x_k|,$$

$$\text{two-norm}: \quad \|\mathbf{x}\|_2 = \sqrt{\sum_{k=1}^{n} |x_k|^2},$$

$$\infty\text{-norm}: \quad \|\mathbf{x}\|_\infty = \max_{1\le k\le n} |x_k|$$

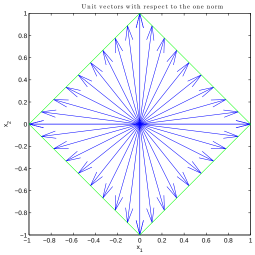
Vector norms

However a vector norm is defined, it must satisfy the following three properties to be called a norm:

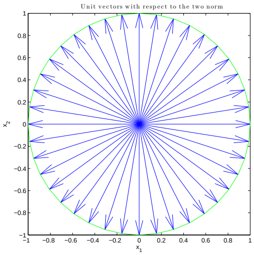
1. $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$ (i.e. $\mathbf{x}$ contains all zeros as its entries).
2. $\|\alpha\mathbf{x}\|$ = $|\alpha|\|\mathbf{x}\|$, for any constant α.
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, where $\mathbf{y} \in \mathbb{R}^n$. This is called the *triangle inequality*.

Unit vectors

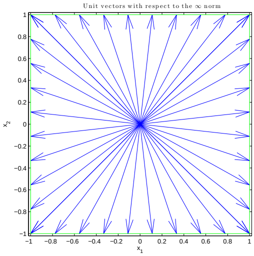
- A vector $\mathbf{x}$ is called a unit vector if its norm is one, i.e. $\|\mathbf{x}\| = 1$.
- Unit vectors will be different depending on the norm applied.
- Below are several unit vectors in the one, two, and ∞ norms for $\mathbf{x} \in \mathbb{R}^2$.



(a) One-norm



(b) Two-norm



(c) ∞-norm

Matrix norms

- A matrix norm is a scalar quantity that reflects the "size" of a matrix $A \in \mathbb{R}^{m \times n}$.
- The norm of A is denoted as $\|A\|$.
- Any matrix norm must satisfy the following four properties:
  1. $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$ (i.e. A contains all zeros as its entries).
  2. $\|\alpha A\|$ = $|\alpha|\|A\|$, for any constant α.
  3. $\|A + B\| \leq \|A\| + \|B\|$, where $B \in \mathbb{R}^{m \times n}$.
  4. $\|AB\| \leq \|A\|\|B\|$, where $B \in \mathbb{R}^{n \times p}$. This is called the submultiplicative inequality.

Matrix norms

Each vector norm induces a matrix norm according to the following definition:

$$\|A\|_p = \max_{\|\mathbf{x}\|_p \neq 0} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p = 1} \|A\mathbf{x}\|_p,$$

where $\mathbf{x} \in \mathbb{R}^n$ and $p = 1, 2, \ldots$.

Induced norms describe how the matrix stretches unit vectors with respect to that norm.

Induced matrix norms

Two popular and easy to define induced matrix norms are

$$\text{One-norm}: \quad \|A\|_1 = \max_{1 \le k \le n} \sum_{j=1}^{m} |a_{jk}|,$$

$$\infty\text{-norm}: \quad \|A\|_\infty = \max_{1 \le j \le m} \sum_{k=1}^{n} |a_{jk}|.$$

- The one-norm corresponds to the maximum of the one norm of every column.
- The ∞-norm corresponds to the maximum of the one norm of every row.

The two-norm of A is defined as the *largest eigenvalue* of the matrix $A^T A$. This is computationally expensive to compute.

Non-induced matrix norms

The most popular matrix norm that is not an induced norm is the *Frobenius* norm:

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_{j=1}^{m}\sum_{k=1}^{n}|a_{jk}|^2}.$$

Important results on matrix norms

The following are some useful inequalities involving matrix norms. Here $A \in \mathbb{R}^{m\times n}$:

- $\|A\mathbf{x}\| \le \|A\|\|\mathbf{x}\|$
- $\dfrac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1$
- $\dfrac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty$
- $\|A\|_2 \le \sqrt{\|A\|_1\|A\|_\infty}$