No Odd Perfect Numbers Exist

Zengyong LIANG
March 18, 2022

Abstract: The existence of odd perfect numbers is a well- known problem in number theory . This paper proved that no prefect number exist in the natural numbers .

Key words: perfect number; odd perfect number; combination; function
CLC number: O156. 1

1 Introduction

A perfect number is a number n such that _(n) = 2n. In other words a number is perfect number if it is the sum of its divisors other than itself . We only known a few of even perfect numbers[1][2][3] . It seems probable that there are no odd perfect numbers , bat this has not been proved. It is unknown whether there are any odd perfect numbers, though various results have been obtained. Carl Pomerance has presented a heuristic argument which suggests that no odd perfect numbers exist[5]

Any odd perfect number N must satisfy the following conditions:

- $N > 101500$, result published in 2012[4].
- N is of the form

$$N = q_1^{\alpha} p_2^{2e_1} ... p_k^{2e_k}$$

  where:
  $q, p_1, ..., p_k$ are distinct primes (Euler).
  $q \equiv \alpha \equiv 1 (mod 4)$ (Euler).
  The smallest prime factor of N is less than (2k + 8) / 3 [4].
- The largest prime factor of N is greater than 108 [4].
- The second largest prime factor is greater than 104, and the third largest prime factor is greater than 100 [4].
- $N > 10^{200}$, which it must have at least 8 different prime factors and that its largest prime factor must be greater than 100110 [1].
- N has at least three prime factors. There is another argument [5].
- $N > 10^{200}$ and with at least 15 different prime factors [5].
- If N is odd perfect and $\omega(N) < k$, then $N < 2^{4^k}$ [6].

1.1 The Function _(*N*)

Definition: Function _(n) is the sum of combination of its divisors [1] .

Let $n = q_1^{\beta_1} q_2^{\beta_2} ... q_i^{\beta_t}$. Using mathematical induction is very easy to prove $\sigma(n)$, which the expression of the product:

$$\sigma(n) = (1 + q_1 + ... + q_1^{\beta_1})(1 + q_2 + ... + q_2^{\beta_2})...(1 + q_i + ... + q_i^{\beta_t}) \tag{1}$$

where q_i is anyone prime factor ,which $q_1 < q_2 < ... < q_i$.

In this and next we reserve the letter p for primes, as $p_1, p_2, p_3, ..., p_i$ are primes of $2, 3, 5, ..., p_i$.

If $\sigma(n)$ is a even prefect number that must be

$$\sigma(n) = 2n \tag{2}$$

2 The Function k_x of $\sigma(n)$

We obtain

$\sigma(n) = (1+q_1+...+q_1^{\beta_1})(1+q_2+...+q_2^{\beta_2})...(1+q_i+...+q_i^{\beta_i}) = 2q_1^{\beta_1}q_2^{\beta_2}...q_i^{\beta_i}$

by (1)and (2).

then

$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_i+...+q_i^{\beta_i}}{q_i^{\beta_i}} = 2 \tag{3}$$

Let $k_x = \frac{1+q_x+...+q_x^{\beta_x}}{q_x^{\beta_x}}$, so

$$k_1k_2...k_i = 2 \tag{4}$$

Clearly, for any n ,if (4)is true, so $\sigma(n) = 2n$,the n is a prefect number.

2.1 The values of k_x

Theorem 1. . $q_x^{\beta_x} + q_x^{\beta_x-1} + ... + q_x + 1 < (1 + \frac{1}{q_x-1})q_x^{\beta_x}$

Proof. By algebraic formula we obtain:

$$q_x^{\beta_x} - 1 = (q_x - 1)(q_x^{\beta_x-1} + q_x^{\beta_x-2} + ... + q_x + 1)$$

$$q_x^{\beta_x-1} + q_x^{\beta_x-2} + ... + q_x + 1 = \frac{q_x^{\beta_x}-1}{q_x-1} < \frac{q_x^{\beta_x}}{q_x-1}$$

so

$$q_x^{\beta_x}+(q_x^{\beta_x-1}+...+q_x+1) < (1+\frac{1}{q_x-1})q_x^{\beta_x}$$

□

It is clear that allowing Theorem 2:

Theorem 2. . For any odd $q_x, have$ $1 < k_x < 1.5$.

Proof. Let $n = q_x^{\beta_x} + q_x^{\beta_x-1} + ... + q_x + 1$,we obtain

$$q_x^{\beta_x} + q_x^{\beta_x-1} + ... + q_x + 1 < (1 + \frac{1}{q_x-1})q_x^{\beta_x}$$

by Theorem 1.

i.e.

$$k_x = \frac{q_x^{\beta_x}+q_x^{\beta_x-1}+...+q_x+1}{q_x^{\beta_x}} < 1 + \frac{1}{q_x-1}$$

It is clear that $k_k > 1$.

When $q_x = 3$, the

$$max\ k_x < (1 + \frac{1}{q_x-1}) = 1 + \frac{1}{3-1} = 1.5$$

but $q_x \to \infty$, the

$$min\ k_x > \lim(1 + \frac{1}{q_x-1}) = 1 \quad .$$

Hence , $1 < k_x < 1.5$.

□

2.2 The Table of Values of k_x

Table 1. . A fell values of k_x

i	p_i	$k(1+p_i)$	$k(1+p_i+p_i^2)$	$k(1+p_i+p_i^2+p_i^3)$	max $k_i <$
1	2	1.5	1.75	1.875	2
2	3	1.3333	1.4444	1.4814	1.5
3	5	1.2	1.24	1.248	1.25
4	7	1.1428	1.1632	1.1661	1.167
5	11	1.0909	1.0991	1.0999	1.1
6	13	1.0769	1.0828	1.0832	1.084

This shows that k is a monotone decreasing function with q_x, when q_x is larger, the smaller the k_x. In contrast, when k is constant, it is monotonous litres of function with $\_x$;when $\_x$ is the larger, the larger the k_x.

3 Proof of proposition

Theorem 3. . Let k_y be a decimal , c be a integer . To make ck_y is integer ,at lest that either of the following conditions:

(1)The bottom digit of decimal of k_y is 5 , and a is even;for example, $k_y = 1.25, a = 4$, $1.25 \times 4 = 5$.

(2)The bottom digit of decimal of k_u is a even , and bottom digit of the a is 5, for example, $k_y = 1.2, a = 15, 1.2 \times 15 = 18$.

Proof. Assume they bottom digits with a and b, if have neither of (1),(2) two cases, the product of ab must be a decimal,which con't carry.

Theorem 4. . If n is odd ,$n = q_1^{\beta_1} q_2^{\beta_2} ... q_i^{\beta_i}$, then $k_1 k_2 ... k_i \neq s$.

Proof. Suppose the n is a odd prefect number, then

$$k_1 k_2 ... k_i = \frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_i+...+q_i^{\beta_i}}{q_i^{\beta_i}} = s,$$

where s is odd by (3)and (4).

Set $1 + q_i + ... + q_i^{\beta_i} = sb_i$,

$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{i-1}+...+q_{i-1}^{\beta_{i-1}}}{q_{i-1}^{\beta_{i-1}}} \times sb_i \times \frac{1}{q_i^{\beta_i}} = s$$

so it needs
$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{i-1}+...+q_{i-1}^{\beta_{i-1}}}{q_{i-1}^{\beta_{i-1}}} \times b_i = a$$

where a is an integer, and $a = q_i^{\beta_i}$.

I.e., $k_1 k_2 ... k_{i-1} b_i = a$.

Since that :(1)when the bottom digit of k_x is 5 , but b_i con't be even (because $a = q_i^{\beta_i}$, it is odd);

(2)when bottom digit of k_x is even , but the bottom digit of b_i con't be 5 ;

Neither of (1),(2)can't take $k_1 k_2 ... k_{i-1} b_i$ (i.e. a)to be a integer. So

$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{i-1}+...+q_{i-1}^{\beta_{i-1}}}{q_{i-1}^{\beta_{i-1}}} \times b_i \times \frac{1}{q_i^{\beta_i}} \neq 1$$

and
$$\frac{1+q_1+...+q_1^{\beta_1}}{q_1^{\beta_1}} \bullet \frac{1+q_2+...+q_2^{\beta_2}}{q_2^{\beta_2}} \bullet ... \bullet \frac{1+q_{i-1}+...+q_{i-1}^{\beta_{i-1}}}{q_{i-1}^{\beta_{i-1}}} \times 2b_i \times \frac{1}{q_i^{\beta_i}} \neq s$$

□

Hence the $k_1 k_2 ... k_i \neq s$.

Proposition . *If n is odd,and $n = q_1^{\beta_1} q_2^{\beta_2} ... q_i^{\beta_i}$,* it is impossible which n is an odd perfect number .

Proof 1. Let n be odd, and $n = q_1^{\beta_1} q_2^{\beta_2} ... q_i^{\beta_i}$, then:

(1)When $i = 2$, $n = q_1^{\beta_1} q_2^{\beta_2}$, let $q_1 = 3$, $q_2 = 5$,then $\max k_1 k_2 = 1.5 \times 1.25 = 1.875 < 2$. Since all $k'_x < k_2$, when $p_x > p_3 = 5$;any $k'_1 k'_2 < 1.875$. So, i.e.,when $i \leq 2$ the $k_1 k_2 < 2$.

(2)When $i = 7$, $k_1 k_2 ... k_7 = 2.92$;When $i = 8$, $k_1 k_2 ... k_8 = 3.0564$ by Tabel 1. Hence $k_1 k_2 ... k_8 \neq 3$.

Proof 2. Let n be odd, by we nkow that fomustor mast be thrue

$$(1 + q_1 + ... + q_1^{\beta_1})(1 + q_2 + ... + q_2^{\beta_2})...(1 + q_i + ... + q_i^{\beta_i}) = sq_1^{\beta_1} q_2^{\beta_2} ... q_i^{\beta_i}$$

$$\frac{sq_1^{\beta_1} q_2^{\beta_2} ... q_i^{\beta_i}}{q_1^{\beta_1} q_2^{\beta_2} ... q_i^{\beta_i}} = s$$

But by Theorem 4,

So the n can not be an odd perfect number .

Table 2.

i	p_i	k_i	$f(i)$
1	3	1.5	1.5
2	5	1.25	1.875
3	7	1.16666666666667	2.1875
4	11	1.1	2.40625
5	13	1.08333333333333	2.60677083333333
6	17	1.0625	2.76969401041667
7	19	1.05555555555556	2.92356589988426
8	23	1.04545454545455	3.05645525896991
9	29	1.03571428571429	3.16561437536169
10	31	1.03333333333333	3.27113485454041
11	37	1.02777777777778	3.36199971161098
12	41	1.025	3.44604970440125
13	43	1.02380952380952	3.52809850688 7
14	47	1.02173913043478	3.60479630051498

where $f(i) = k_1 h_2 ... k_i$.

Theorem 4. .

$$\frac{sq_1^{\beta_1}q_2^{\beta_2}...q_i^{\beta_i}}{q_1^{\beta_1}q_2^{\beta_2}...q_i^{\beta_i}} \neq s$$

Proof 2. If

$$\frac{sq_1^{\beta_1}q_2^{\beta_2}...q_i^{\beta_i}}{q_1^{\beta_1}q_2^{\beta_2}...q_i^{\beta_i}} = s \tag{5}$$

must be that the numerator and denominator must be the same and can be approximately divided. That is, all factors (except s) are the same However,

n	n²-1	factor			
1	1	1			
2	3	3			
3	8	2			
4	15	3	5		
5	24	3	2		
6	35	5	7		
7	48	3	2		
8	63	3	7		
9	80	5	2		
10	99	3	11		
11	120	3	5	2	
12	143	11	13		
13	168	3	7	2	
14	195	3	5	13	
15	224	7		2	
16	255	3	5	17	
17	288	3		2	
18	323	17	19		
19	360	3	5	2	
20	399	3	7	19	
21	440	5	11	2	
22	483	3	7	23	
23	528	3	11	2	
24	575	5	23		
25	624	3	13	2	
26	675	3	5		
27	720	7	13	2	2
28	783	3	29		
29	840	3	5	7	2
30	899	29	31		

Table 3. Prime factor of integer $n^2 - 1$

According to the following analysis, this situation is unlikely to happen.

Table 2 shows the case of $n^2 - 1$integer prime factor, and table 3 shows the case of $n^3 - 1$ integer prime factor When n is greater than 5, each integer contains two or more prime factors. Then, it has been proved that 2 knows that the prime factor ratio must be more than 8 primes. In this way, it is impossible to make the numerator and denominator must be the same (the difference will only be greater in the past month).

n	n³-1	factor			
1	1	1			
2	7	7			
3	26	13	2		
4	63	3	7		
5	124	31	2		
6	215	5	43		
7	342	3	19	2	
8	511	7	73		
9	728	7	13	2	
10	999	3	37		
11	1330	5	7	19	2
12	1727	11	157		
13	2196	3	61	2	
14	2743	13	211		
15	3374	7	2	241	
16	4095	3	5	7	
17	4912		2	307	
18	5831	7	17		
19	6858	3	127	2	
20	7999	19	421		
21	9260	5	2	463	
22	10647	3	7	13	
23	12166	7	11	79	2
24	13823	23	601		

Table 4. Prime factor of integer $n^3 - 1$

Figure 1 illustrates this situation with the tree structure of graph theory. The bifurcation of branches only returns more and more.

So the same numerator and denominator won't happen. That is, equation (5) does not hold.

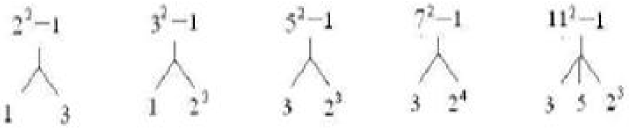

Figure 1. Prime factor tree structure of integer n^2-1

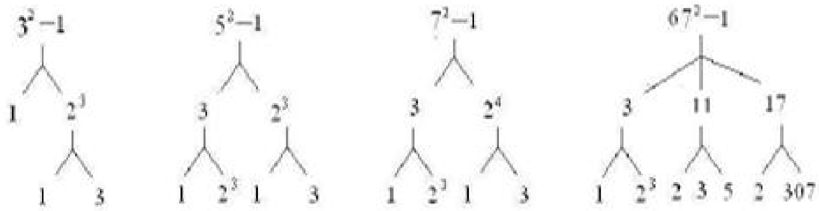

Figure 2. Tree graph of 8 prime factors .

When i of $K_i, i=8$, the prime factor tree structure starting from n^2-1 is shown in Figure 2,8 prime factor are 2,3,5,7,11,17,67,307. For example, each new prime factor leads to at least another new prime factor; Moreover, the number of each prime factor can not be exactly the same (only a few of them can be the same). In this way, the prime factor of the numerator and denominator of the formula can not be the same forever.

Therefore, considering the above reasons ,the equation (5) cannot be established.

Wherefore , if n is odd, the $k_1k_2...k_i \neq s$. So $\sigma(n) \neq sn$,and the n can not be a prefect number by definition. □

4 Conclusion

Now, we can say that may no odd perfect numbers.Above from various as pects have been proved that can't have odd perfect number existence in any case.Perfect number is only even a characteristic, will appear even perfect num ber in certain circumstances ; at the same time, even prefect number is infinite

References

[1] G.H.Harly E.M.Wright,*An Introduction to the Theorem of Numbers* , Posts and telecom press, Beijing, 2007,19-256.

[2] Zhou Xiaorao, Yu Mo, *Interesting Famous Quistions of Number Theorem* , Hunan uni- versity press, Changsha, 2012,115-116.

[3] (U.A.S)Du De Li, (Zhou Zhongliang translated)*Basic Number Theory,* Harbin institute of technology press , Harbin, 2011,47-50.

[4] From Wikipedia, the free encyclopedia *Perfect Number* available at http://en.wikipedia.org/wiki/Perfect number

[5] San Zun ,*The Knowledge and Question of Elementary Number Theory,* Harbin institute of technology press , Harbin, 2011,54-55.

[6] Paul Pollack , *Finiteness Theorems for Perfect Numbers and Their Kin* , Amer. Math. Monthly 119 (2012),670-681 ; available at http://www.maa.org/.