MAT110 - Obligatorisk innlevering 1

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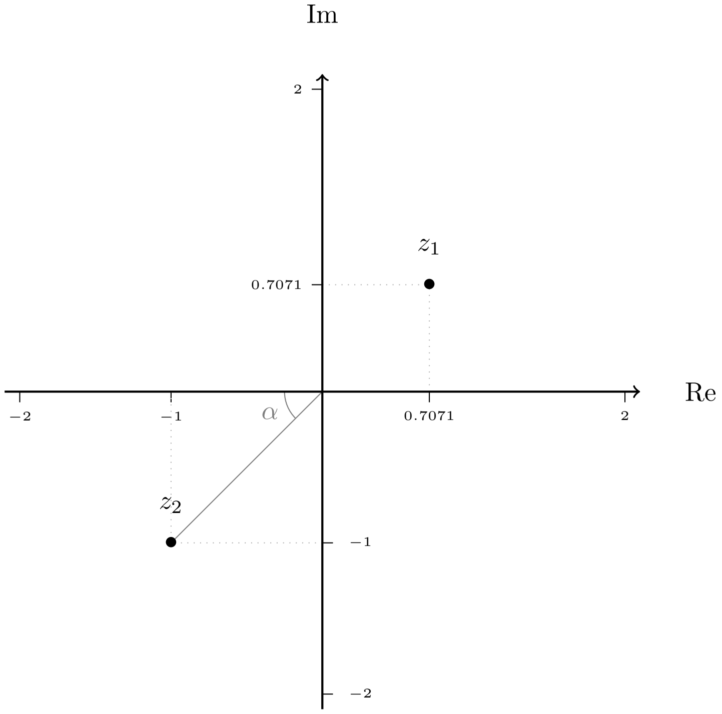
Oppgave 1

a)



b)

$$
\begin{aligned}
z_1 &= e^{i\frac{\pi}{4}} \quad \text{(eksponentialform)} \rightarrow r = 1, \quad \theta = \frac{\pi}{4} \\
&= 1(\cos\theta + i \cdot \sin\theta) \\
&= cos\frac{\pi}{4} + i \cdot sin\frac{\pi}{4} \\
&= x_1 + i \cdot y_1 \\
\Rightarrow x_1 &= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\Rightarrow y_1 &= \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
z_1 &= \frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2} \quad \text{(standardform)}
\end{aligned}
$$

$$
\begin{aligned}
z_2 &= -1 - i \quad \text{(standardform)} \\
&= x_2 + i \cdot y_2 \rightarrow x_2 = -1, \quad y_2 = -1 \\
& r = \sqrt{x_2^2 + y_2^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \\
& \theta = \tan^{-1}(|\frac{y}{x}|) \quad \text{(Generelt)} \\
& \text{Men ettersom det komplekse tallet er i 3. kvadrant} \rightarrow \theta = \alpha + \pi \\
& \alpha = \tan^{-1}(|\frac{-1}{-1}|) = \tan^{-1}(1) = \frac{\pi}{4} \\
& \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \\
z_2 &= r(\cos\theta + i \cdot \sin\theta) \\
z_2 &= \sqrt{2}(\cos\frac{5\pi}{4} + i \cdot \frac{5\pi}{4}) \\
z_2 &= \sqrt{2} \cdot e^{i\frac{5\pi}{4}} \quad \text{(eksponentialform)}
\end{aligned}
$$

Oppgave 1

c)

$$\begin{aligned}
z_1 + z_2 &= (\frac{\sqrt{2}}{2} + \sqrt{2}2 \cdot i) + (-1 - i) \\
&= (\frac{\sqrt{2}}{2} - 1) + (\frac{\sqrt{2}}{2} - 1) \cdot i \\
&= (0.7071 - 1) + (0.7071 - 1) \cdot i \\
&= -0.29 - 0.29i
\end{aligned}$$

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}e^{i\frac{5\pi}{4}}} = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4} - i\frac{t\pi}{4}} = \frac{1}{\sqrt{2}}e^{-i\pi} \\
&= \frac{1}{\sqrt{2}}(\cos(-\pi) + i\sin(-\pi)) = \frac{1}{\sqrt{2}}(-1 + i(0)) = -\frac{1}{\sqrt{2}} + i0 \\
&= -\frac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
z_1 &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\
\overline{z_1} &= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\
-\overline{z_1} &= -1(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}
\end{aligned}$$

d) $z^6 = 1 + i$. La $w = 1 + i$, da må vi finne den 6. roten av w.
Først konverterer vi w til eksponentialform.
$r = \sqrt{1^2 + 1^2} = \sqrt{2}$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$
$w = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

Videre tar vi i bruk De Moivres thm. for røtter:
Hvis $z^n = w = re^{i\theta}$, da er røttene gitt ved $z_k = \sqrt[n]{r}e^{i(\frac{\theta + 2\pi k}{n})}$, for $k = 0, 1, 2, \ldots, n-1$.

I dette tilfellet blir $n = 6, r = \sqrt{2}$ og $\theta = \frac{\pi}{4}$.
$\sqrt[6]{r} = \sqrt[6]{\sqrt{2}} = \sqrt[6]{2^{\frac{1}{2}}} = 2^{(\frac{1}{2})(\frac{1}{6})} = 2^{\frac{1}{12}} = \sqrt[12]{2}$.
Videre finner vi argumentene for $k = 0, 1, 2, 3, 4, 5$:
Det generelle argumentet er $\phi_k = \frac{\frac{\pi}{4} + 2\pi k}{6} = \frac{\pi}{24} + \frac{2\pi k}{6} = \frac{\pi}{24} + \frac{\pi k}{3}$.

$$\begin{aligned}
k = 0 : \phi_0 &= \frac{\pi}{24} \\
k = 1 : \phi_1 &= \frac{\pi}{24} + \frac{\pi}{3} = \frac{9\pi}{24} \\
k = 2 : \phi_2 &= \frac{\pi}{24} + \frac{2\pi}{3} = \frac{17\pi}{24} \\
k = 3 : \phi_3 &= \frac{\pi}{24} + \pi = \frac{25\pi}{24} \\
k = 4 : \phi_4 &= \frac{\pi}{24} + \frac{4\pi}{3} = \frac{33\pi}{24} \\
k = 5 : \phi_5 &= \frac{\pi}{24} + \frac{5\pi}{3} = \frac{41\pi}{24}
\end{aligned}$$

Det er seks løsninger:

$z_k = 2^{\frac{1}{12}} \cdot e^{i\phi_k}$, for $k = 0, 1, 2, 3, 4, 5$.

Oppgave 2

a) $\begin{bmatrix} 1 & 0 & 2k & k & 2 \\ 1 & 1 & (2k-1) & 0 & 1 \\ -2 & 0 & -4k & k & 5 \end{bmatrix} \sim R_2 - R_1 \begin{bmatrix} 1 & 0 & 2k & k & 2 \\ 0 & 1 & -1 & -k & -1 \\ -2 & 0 & -4k & k & 5 \end{bmatrix} \sim \underset{R_3 + 2R_1}{} \begin{bmatrix} 1 & 0 & 2k & k & 2 \\ 0 & 1 & -1 & -k & -1 \\ 0 & 0 & 0 & 3k & 9 \end{bmatrix}$

$$\begin{aligned} &(1)\quad x_1 + 2kx_3 + kx_4 = 2 \\ &(2)\quad x_2 - x_3 - kx_4 \quad = -1 \\ &(3)\quad 3kx_4 \qquad\qquad = 9 \end{aligned}$$

b) Fra (3) $3kx_4 = 9$

Case 1:
$k \neq 0 \Rightarrow x_4 = \frac{3}{k}$

Vi setter $x_4 = \frac{4}{k}$ inn i (2):
$x_2 = x_3 + k\frac{3}{k} - 1 = x_3 + 2$
Så vi trenger verdien til x_3 for å beregne x_2.

Vi setter $x_4 = \frac{3}{k}$ inn i (1):
$x_1 + 2kx_3 + k\frac{3}{k} = x_1 + 2kx_3 + 3 = 2$
$\Rightarrow x_1 = -1 - 2kx_3$

For $x \neq 0$, x_1 og x_2 er uttrykt med bruk av den frie variablen x_3.
Derfor er det uendelig mange løsninger for alle $k \neq 0$.

Case 2:
$k = 0 \Rightarrow$ (3) blir $0 \cdot x_4 = 9 \Rightarrow 0 = 9$.

Dette er umulig og vi kan konkludere at det er ingen løsning for $k = 0$.

Utifra dette kan vi slå fast at det kan ikke være akkurat en eller akkurat to løsninger i dette tilfelle, enten uendelig mange løsninger (for $k \neq 0$) eller ingen løsning (for $k = 0$).

c)

$$\begin{aligned} &(3)\quad (3)(6)x_4 = 9 \Rightarrow 18x_4 = 9 \Rightarrow x_4 = \frac{1}{2} \\ &(2)\quad x_2 - x_3 - 6(\frac{1}{2}) = -1 \Rightarrow x_3 - x_3 - 3 = -1 \Rightarrow x_2 = x_3 + 2 \\ &(1)\quad x_1 + 2(6)x_3 + 6(\frac{1}{2}) = 2 \Rightarrow x_1 + 12x_3 + 3 = 2 \Rightarrow x_1 = -1 - 12x_3 \\ &\therefore\quad (x_1, x_2, x_3, x_4) = (-1 - 12t, t + 2, t, \frac{1}{2}), t \in \mathbb{R} \quad \text{(Uendelig mange løsninger)} \end{aligned}$$

Oppgave 3

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}$$

a) $A + B$ er ikke mulig ettersom matrisene har ulike dimensjoner.

$$3A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 6 & -3 \\ 6 & 9 & 9 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$3A - B^T$ er ikke mulig ettersom matrisene har ulike dimensjoner.

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1*1+1*1+1(-1) & 1*1+1(-1)+1*2 \\ 1*1+2*1-1(-1) & 1*1+2(-1)-1*2 \\ 2*1+3*1+3(-1) & 2*1+3(-1)+3*2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -3 \\ 2 & 5 \end{bmatrix}$$

BA er ikke mulig ettersom dimensjonene på matrisene må være $m \times p \cdot p \times n$ for å gjennomføre multiplikasjon.

$$\mathrm{B}^T A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)+1(1)-1(2) & 1(1)+1(2)-1(3) & 1(1)+1(-1)-1(3) \\ 1(1)-1(1)+2(2) & 1(1)-1(2)+2(3) & 1(1)-1(-1)+2(3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 4 & 5 & 8 \end{bmatrix}$$

b) $$\det(A) = 1 \cdot \begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1 \cdot (6+3) - 1(3+2) + 1(3-4) = 9 - 5 - 1 = 3$$

c) $$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \sim \quad R_2 - R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \quad R_3 - 2R_1 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right] \sim \quad R_3 - R_2 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 3 & -1 & -1 & 1 \end{array}\right]$$

$$\sim \quad \tfrac{1}{3}R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \sim \quad R_2 + 2R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{5}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\sim \quad R_1 - R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{8}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{5}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \sim \quad R_1 - R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & -\frac{5}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\mathrm{A}^{-1} = \begin{bmatrix} 3 & 0 & -1 \\ -\frac{5}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Oppgave 3

d)

$$
\begin{aligned}
&A\vec{x} = \vec{b} \\
&\vec{b} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \\
&\Rightarrow \vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 3 & 0 & -1 \\ -\frac{5}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&3(1) + 0(4) - 1(2) = 3 - 2 && = 1 \\
&-\frac{5}{3}(1) + \frac{1}{3}(4) + \frac{2}{3}(2) = -\frac{5}{3} + \frac{4}{3} + \frac{4}{3} && = 1 \\
&-\frac{1}{3}(1) - \frac{1}{3}(4) + \frac{1}{3}(2) = -\frac{1}{3} - \frac{4}{3} + \frac{2}{3} && = -1 \\
&\Rightarrow x_1 = 1, \quad x_2 = 1, \quad x_3 = -1
\end{aligned}
$$

e)

```
from sympy import Matrix

A = Matrix([
    [ 1,  1,  1],
    [ 1,  2, -1],
    [ 2,  3,  3]
])

b = Matrix([
    [1],
    [4],
    [2]
])

x = A.inv() * b

print(f'Løsning: x_1 = {x[0]}, x_2 = {x[1]}, x_3 = {x[2]}')

# Dette gir utputt: "Løsning: x_1 = 1, x_2 = 1, x_3 = -1"
```

Oppgave 4

$$
\vec{u} = \left[\frac{1}{\sqrt{2}},\ \frac{1}{2},\ \frac{1}{2}\right], \quad \vec{v} = \left[-\frac{1}{\sqrt{2}},\ \frac{1}{2},\ \frac{1}{2}\right], \quad \vec{w} = \left[-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right]
$$

a)

$$
\begin{aligned}
|\vec{u}| &= \sqrt{(1/\sqrt{2})^2 + (1/2)^2 + (1/2)^2} &&= \sqrt{1/2 + 1/4 + 1/4} = \sqrt{1} = 1 \\
|\vec{v}| &= \sqrt{(-1/\sqrt{2})^2 + (1/2)^2 + (1/2)^2} &&= \sqrt{1/2 + 1/4 + 1/4} = \sqrt{1} = 1 \\
|\vec{w}| &= \sqrt{(-1/\sqrt{2})^2 + 0^2 + (1/sqrt2)^2} &&= \sqrt{1/2 + 1/2} = \sqrt{1} = 1
\end{aligned}
$$

Oppgave 4

b)

$$\begin{aligned}
&\vec{u}\cdot\vec{v} = {}^1/\sqrt{2}\cdot(-{}^1/\sqrt{2}) + {}^1/2\cdot{}^1/2 + {}^1/2\cdot{}^1/2 = -{}^1/2 + {}^1/4 + {}^1/4 = 0\\
&\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cos\theta = 1\cdot 1\cdot\cos\theta = \cos\theta = 0\\
&\cos\theta = 0 \Rightarrow \theta = 90^\circ \quad (\vec{u}\perp\vec{v})
\end{aligned}$$

$$\begin{aligned}
&\vec{u}\cdot\vec{w} = \frac{1}{\sqrt{2}}\cdot(\frac{-1}{\sqrt{2}}) + \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = -\frac{1}{2} + \frac{1}{2\sqrt{2}}\\
&\Rightarrow \cos\theta = \frac{1}{2\sqrt{2}} - \frac{1}{2}\\
&\Rightarrow \theta = \cos^{-1}(\frac{1}{2\sqrt{2}} - \frac{1}{2}) \approx 98,4^\circ
\end{aligned}$$

c)

$$\vec{u}\times\vec{v} = \begin{bmatrix} i & j & k\\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2}\\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$i:\ \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} \qquad = 0$$

$$j:\ -\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{2} - \frac{1}{2}\cdot(-\frac{1}{\sqrt{2}})\right) = -\frac{1}{\sqrt{2}}$$

$$k:\ \frac{1}{\sqrt{2}}\cdot\frac{1}{2} - \frac{1}{2}\cdot(-\frac{1}{\sqrt{2}}) \qquad = \frac{1}{\sqrt{2}}$$

$$\vec{u}\times\vec{v} = \left[0,\quad -\frac{1}{\sqrt{2}},\quad \frac{1}{\sqrt{2}}\right]$$

d) $\vec{u}\times\vec{v}$ er vinkelrett til både $\vec{a}$ og $\vec{b}$, så vinkelen mellom $\vec{a}$ og $\vec{u}\times\vec{v}$ er 90°.

e)

$$\vec{a}\left[5\vec{a} + \vec{b} + (\vec{a}\times\vec{b})\right] = 5(\vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot(\vec{a}\times\vec{b}))$$

Vi vet at:

(i) $\vec{a}\cdot\vec{a} = |\vec{a}|^2 = 2^2 \qquad = 4$

(ii) $\vec{a}\cdot\vec{b} = 0$ (ettersom $\vec{a}$ og $\vec{b}$ står vinkelrett på hverandre)

(iii) $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$ (ettersom kryssproduktet til to vektorer står vinkelrett på begge vektorer)

$$\therefore\ \vec{a}\left[5\vec{a} + \vec{b} + (\vec{a}\times\vec{b})\right] = 5(4) + 0 + 0 = 20$$

Oppgave 5

a)

$$
\begin{aligned}
50x_1 + 25x_2 + 75x_3 + 100x_4 &= 4200 \\
75x_1 + 100x_2 + 50x_3 + 25x_4 &= 3800 \\
75x_1 + 75x_2 + 75x_3 + 50x_4 &= 4400 \\
50x_1 + 50x_2 + 50x_3 + 75x_4 &= 3600
\end{aligned}
$$

x_1 er antall produserte Floral Fusion blandinger.
x_2 er antall produserte Burgundy Bonanza blandinger.
x_3 er antall produserte Morgensol blandinger.
x_4 er antall produserte Ahh Svart! blandinger.

b) $A = \begin{bmatrix} 50 & 25 & 75 & 100 \\ 75 & 100 & 50 & 25 \\ 75 & 75 & 75 & 50 \\ 50 & 50 & 50 & 75 \end{bmatrix}$

```
# Utregning av determinant (Python):

from sympy import Matrix

A = Matrix([
    [ 50, 25, 75, 100],
    [ 75, 100, 50, 25],
    [ 75, 75, 75, 50],
    [ 50, 50, 50, 75]
])

print(f'det(A) = {A.det()}')

# Dette gir utputt: "det(A) = 0"
```

Hvis A har en invers A^{-1} kan inversen brukes for å løse likningssystemet ($\vec{b} = A^{-1}\vec{x}$), men $\det(A) = 0$ forteller oss at A ikke har invers.

c)

```
# Gauss-Jordan-eliminasjon for å løse likningssystemet (Python):

from sympy import Matrix

A = Matrix([
    [ 50, 25, 75, 100],
    [ 75, 100, 50, 25],
    [ 75, 75, 75, 50],
    [ 50, 50, 50, 75]
])

b = Matrix([
    [4200],
    [3800],
    [4400],
    [3600]
])

x, params, free_vars = A.gauss_jordan_solve(b, freevar=True);
x = [str(row).replace('tau0', 'x_'+str(free_vars[0]+1)) for row in x]
print(f'Løsning: (x_1, x_2, x_3, x_4) = ({x[0]}, {x[1]}, {x[2]}, {x[3]})')

# Dette gir utputt: "Løsning: (x_1, x_2, x_3, x_4) = (56 - 2*x_3, x_3 - 8, x_3, 16)"
```

Ettersom x_1, x_2, x_3, x_4 representerer antall produserte blandinger må verdiene være 0 eller positivt heltall.
Løsningen forteller oss at x_3 er en fri variabel, i tillegg til at $x_3 >= 8$ for å hindre at x_2 blir negativ, og $x_3 <= 26$ for å hindre at x_1 blir negativ.
Dette gir $x_1 = 56 - 2t, \quad x_2 = t - 8, \quad x_3 = t, \quad x_4 = 16, \quad t \in \{8\,..\,26\}$.