# Combinatorial Optimization Games

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- In the game $\Gamma(G, w) = (N, v)$ the set of players is $N$, and the value $v$ of a coalition $C \subseteq N$ is

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- Observe that $v(\emptyset) = 0$ and $v(N) = w(E)$.

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- Weights can be exponential in $n$ and still have polynomial size.

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$$v(C) = \begin{cases} 0 & if |C| \leq 1 \\ 1 & if |C| = 2 \\ 6 & if |C| = 3 \end{cases}$$

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  - By the first condition all self-loops must have weight 0.
  - By the second condition any pair of different vertices must be connected by an edge with weight 1. So $G$ must be a triangle.
  - But then $v(\{1, 2, 3\}) = 3 \neq 6$.

# Properties of valuations

- monotone if $v(C) \leq v(D)$ for $C \subseteq D \subseteq N$.
- superadditive if $v(C \cup D) \geq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
- supermodular $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$.

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- A game $(N, v)$ is convex iff $v$ is supermodular.

- Since we allow for negative edge weights, induced subgraph games are not necessarily monotone.
- However, when all edge weights are non-negative, induced subgraph games are convex.

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The core of $\Gamma(N, v)$ is the set of all imputations $x$ such that $v(S) \leq x(S)$, for each coalition $S \subseteq N$.

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- Let us show that $(x_1, ..., x_n)$ is in the core of $\Gamma$.
  - For $C \subseteq N$, we can assume that $C = \{i_1, \ldots, i_s\}$ where $\pi(i_1) < \cdots < \pi(i_s)$.
  - So, $v(C) = v(\{i_1\}) - v(\emptyset) + v(\{i_1, i_2\}) - v(\{i_1\}) + \cdots + v(C) - v(C \setminus \{i_s\})$.
  - By supermodularity we have, $v(\{i_1, \ldots, i_{j-1}, i_j\}) - v(\{i_1, \ldots, i_{j-1}\}) \leq v(\{1, \ldots, i_j\}) - v(\{1, \ldots, i_{j-1}\})$.
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  - Therefore $v(C) \leq x(C)$ and $v(N) = x(N)$.
- Observe that we have shown that the vector formed by the Shapley value is in the core of a convex game.

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- For $C \subseteq N$, let $\delta_i(C) = v(C \cup \{i\}) - v(C)$
- The Shapley value of player $i$ in a game $\Gamma = (N, v)$ with $n$ players is

$$\Phi_i(\Gamma) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \delta_i(S_\pi(i))$$

# Shapley value: Axiomatic Characterization

Properties of the Shapley value:

- Efficiency: $\Phi_1 + \ldots + \Phi_n = v(N)$
- Dummy: if $i$ is a dummy, $\Phi_i = 0$
- Symmetry: if $i$ and $j$ are symmetric, $\Phi_i = \Phi_j$
- Additivity: $\Phi_i(\Gamma_1 + \Gamma_2) = \Phi_i((\Gamma_1) + \Phi_i(\Gamma_2)$

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**Theorem**

*The Shapley value is the only payoff distribution scheme that has properties (1) - (4)*

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**Theorem**

*The Shapley value of player $i$ in $\Gamma(G, w)$ is*

$$\Phi(i) = \frac{1}{2} \sum_{(i,j) \in E} w_{i,j}.$$

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- When $e_j = (i, \ell)$ for some $\ell \in N$, players $i$ and $\ell$ are symmetric in $\Gamma_j$.
- Since the value of the grand coalition in $\Gamma_j$ equals $w(i, \ell)$, by efficiency and symmetry we get $\Phi_i(\Gamma_j) = w(i, \ell)/2$.

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## Corollary

*The Shapley values of induced subgraph games can be computed in polynomial time.*

# Can the core be empty?

**Theorem**

*Consider a game $\Gamma(G, w)$, the following are equivalent*

- *The vector of Shapley values is in the core*
- *$(G, w)$ has no negative cut*
- *The core is non-empty*

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- Let $e(S, x) = v(S) - x(S)$ be the excess of coalition $S$ at the imputation $x$.
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- For the Shapley values, $e(S,\Phi)$ is $-\frac{1}{2}$ times the weight of the edges going from $S$ to $N \setminus S$.

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- Thus, $x$ is in the core iff $e(x,S) \leq 0\ \forall S \subseteq N$.
- For the Shapley values, $e(S,\Phi)$ is $-\frac{1}{2}$ times the weight of the edges going from $S$ to $N \setminus S$.
- Hence the Shapley value is in the core if and only if there is no negative cut $(S, N \setminus S)$.

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- We have seen that if the core is non-empty, then the vector of Shapley values is in the core.

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- Let $(G, w)$ with non-negative weights and an integer $k$. $G'$ is obtained as the disjoint union of $G$ and the graph $(\{a, b\}, \{(a, b)\})$. Define $w'$ as $w'(e) = w(e)$ for $e \in E(G)$ and $w((a, b)) = -k$.

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- $G$ has a a cut of size at least $k$ iff $G'$ has a negative cut.

# Complexity of core related problems

**Theorem**

*The following problems are NP-complete:*

- *Given $(G, w)$ and an imputation $x$, is it not in the core of $\Gamma(G, w)$?*
- *Given $(G, w)$, is the vector of Shapley values of $\Gamma(G, w)$ not in the core of $\Gamma(G, w)$?*
- *Given $(G, w)$, is the core of $\Gamma(G, w)$ empty?*

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**Theorem**

*Given $(G, w)$, when all weights are non-negative, we can test in polynomial time*

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The first question is trivial as the vector of Shapley values belong to the core. The second problem can be solved by a reduction to MAX-FLOW.

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  We assume $w_{i,j} \geq 0$
- In the game $\Gamma(G, w) = (N, c)$ the set of players is $N = \{v_1, \dots, v_n\}$, and the cost $c$ of a coalition $C \subseteq N$ is

  $$c(C) = \text{ the weight of a minimum spanning tree of } G[S \cup \{v_0\}]$$

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- The cost of a singleton coalition $\{i\}$ is $c(\{i\}) = w_{0,i}$.

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- The cost of a singleton coalition $\{i\}$ is $c(\{i\}) = w_{0,i}$.
- Observe that $v(\emptyset) = 0$ and $v(N) = w(T)$ where $T$ is a MST of $G$.

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- The representation is succinct as long as the number of bits required to encode edge weights is polynomial in $|N|$: using an adjacency matrix to represent the graph requires only $n^2$ entries.

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  - Thus, a coalition with $|C| = 2$ has a MST with zero cost and the second condition cannot be met.

# Properties of valuations

- monotone if $v(C) \leq v(D)$ for $C \subseteq D \subseteq N$.
- superadditive if $v(C \cup D) \geq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
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- $c$ is subadditive.

# Can the core be empty?

**Theorem**

*Consider a MST game $\Gamma(G, w)$. Let $T^*$ be a MST of $(G, w)$ obtained using Prim's algorithm. The vector $x = (x_1, \ldots, x_n)$ that allocates to player $i \in N$ the weight of the first edge $i$ encounters on the (unique path) from $v_i$ to $v_0$ in $T^*$ belongs to the core of $\Gamma$.*

Such an allocation is called standard core allocation

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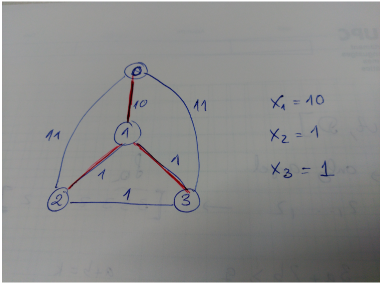
- The selected edge corresponds to the point in which Prim's algorithm connects the vertex to the component including $v_0$, i.e., it is a minimum weight edge in the allowed cut.
- Analyzing carefully both executions it can be shown that $x_j \leq y_j$ as the edges considered in one partition are a subset of the other.

# How fair are standard core allocations?




- Most of the cost is charged to player 1.
- How to find more appropriate core allocations?

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- Norde, Moretti and Tijs [2001] show how to find a population monotonic allocation scheme (PMAS), which is an allocation scheme that provides a core element for the game and all its subgames and which, moreover, satisfies a monotonicity condition in the sense that players have to pay less in larger coalitions.

# Complexity of core related problems

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*The following problem is NP-complete:*

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The proof follows by a reduction from EXACT COVER BY 3-SETS [Faigle et al., Int. J. Game Theory 1997]

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