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Algorithms for Game Theory
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# First Problem Assignment

## Problem 17

### Statement

Consider the sending from $s$ to $t$ game. Compute (or provide bounds for) the PoA and the PoS for the following social cost/utility functions:

(a) $C(s) = \begin{cases} \sum_{i \in N} u_i(s) & \text{if there is a path from } s \text{ to } t \text{ in } G[s]. \\ n^2 & \text{otherwise.} \end{cases}$

(b) $U(s) = \max_{i \in N} u_i(s)$.

(c) $U(s) = \sum_{i \in N} u_i(s)$.

### Solution

In order to ease the explanation, I define the following functions:

- $LP(G)$ = "number of vertices that are inside the longest path of $G$".
- $SP(G)$ = "number of vertices that are inside the shortest path of $G$".

Both situations (longest and shortest paths) leads to a NE.

- Longest path: all vertices that are on the way between $s$ and $t$ (i.e., all possible vertices that can be included in the path) are included in the (longest) path. Hence, all vertices are "happy".
- Shortest path: no vertex outside the (shortest) path can be part of it with only changing its own action.

With this, I can start with the computation of the bounds.

(a) Here I use the formulas of PoA and PoS for the cost functions, which are:

$$\text{PoA}(\Gamma) = \frac{max_{s \in \text{NE}(\Gamma)} C(s)}{min_{s \in A} C(s)} \qquad \text{PoS}(\Gamma) = \frac{min_{s \in \text{NE}(\Gamma)} C(s)}{min_{s \in A} C(s)} \tag{1}$$

I know that $min_{s \in \text{NE}(\Gamma)} C(s) = min_{s \in A} C(s) = min(SP(G), n^2)$ since the minimum path will be always a NE and the shortest possible one. If $G$ is unconnected, $SP(G) = \infty$ and $\forall s \in A$, $C(s) = n^2$.

By the statement, I also know that if there is not a path from $s$ to $t$ in G[s], then $max_{s \in \text{NE}(\Gamma)} C(s) = n^2$.

If $G$ is connected

$$\frac{LP(G)}{SP(G)} \leq \text{PoA}(\Gamma) \leq \frac{n^2}{SP(G)} \tag{2}$$

Moreover, the largest value of $SP(G)$ is $SP(G) = LP(G)$, and the smallest value of it is $SP(G) = 1$, therefore

$$1 \leq \text{PoA}(\Gamma) \leq n^2 \tag{3}$$

If $G$ is unconnected, it is possible that no path from $s$ to $t$ exists. This implies that $min_{s \in A} C(s) = max_{s \in \text{NE}(\Gamma)} C(s) = n^2$ can happen. However, it does not affect the bounds.

$$\frac{n^2}{n^2} = 1 \leq \text{PoA}(\Gamma) \leq n^2 \tag{4}$$

As I explained before, $min_{s \in \text{NE}(\Gamma)} C(s) = min_{s \in A} C(s) = min(SP(G), n^2)$, so $\text{PoS} = 1$.

(b) Here, and in the following point, I use the formulas of PoA and PoS for the utility functions, which are:

$$\text{PoA}(\Gamma) = \frac{max_{s \in A} U(s)}{min_{s \in \text{NE}(\Gamma)} U(s)} \qquad \text{PoS}(\Gamma) = \frac{max_{s \in A} U(s)}{max_{s \in \text{NE}(\Gamma)} U(s)} \tag{5}$$

In this case $\forall s$, $0 \leq U(s) \leq 1$.

If $G$ is connected, there always exists a path from $s$ to $t$, so $max_{s \in A} U(s) = 1$. It can happen that a NE exists with no path from $s$ to $t$, if there is no single action from any of the vertices that connects $s$ to $t$. In this case, $min_{s \in \text{NE}(\Gamma)} U(s) = 0$. Then, PoA can be bounded as follows:

$$\frac{1}{1} = 1 \leq \text{PoA} \leq \infty = \frac{1}{0} \tag{6}$$

Because there always exists a path between $s$ and $t$, $\text{PoS} = 1$.

If $G$ is unconnected, $max_{s \in A} U(s) = 1$ is not true because maybe a path from $s$ to $t$ does not exist. Then, the PoA and PoS can be equal to the indeterminate form $\frac{0}{0}$.

(c) If $G$ is connected, there always exists a path from $s$ to $t$, so $max_{s \in A} U(s) = LP(G)$. As in (b), it can happen that a NE exists with no path from $s$ to $t$, so $min_{s \in \text{NE}(\Gamma)} U(s)$ can be 0, otherwise it is equal to $SP(G)$. Then

$$\frac{LP(G)}{SP(G)} \leq \text{PoA} \leq \infty = \frac{LP(G)}{0} \tag{7}$$

As in (b), if $G$ is unconnected, PoA and PoS can be equal to the indeterminate form $\frac{0}{0}$.

# Problem 28

### Statement

Consider a GSP auction for $n$ players. Recall that in such an auction each bid profile $b$ defines an allocation $\pi$ mapping slots to players. We say that an allocation is reasonable if for each pair $i$, $j$ of slots

$$\frac{\alpha_j}{\alpha_i} + \frac{\gamma_{\pi(i)} v_{\pi(i)}}{\gamma_{\pi(j)} v_{\pi(j)}} \geq 1.$$

(a) Prove that when $b$ is a NE, the corresponding allocation $\pi$ is reasonable.

(b) Use the previous fact to show that the price of anarchy, on pure strategies, of the GSP auction is at most 2.

### Solution

(a) I first divided the prove in two parts, when $i \geq j$ and $i < j$.

- **i ≥ j**: Using the fact that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$:
$$\alpha_j \geq \alpha_i \Rightarrow \frac{\alpha_j}{\alpha_i} \geq 1$$

- **i < j**: Using the fact that $b$ is a NE, I can use the NE property for $i < j$, which is:
$$\alpha_j(\gamma_{\pi(j)}v_{\pi(j)} - \gamma_{\pi(j+1)}b_{\pi(j+1)}) \geq \alpha_i(\gamma_{\pi(j)}v_{\pi(j)} - \gamma_{\pi(i)}b_{\pi(i)})$$
Since $0 \leq b_k \leq v_k$ (for all $k$), it is easy to see that $-\gamma_{\pi(i)}b_{\pi(i)} \geq -\gamma_{\pi(i)}v_{\pi(i)}$, hence
$$\alpha_i(\gamma_{\pi(j)}v_{\pi(j)} - \gamma_{\pi(i)}b_{\pi(i)}) \geq \alpha_i(\gamma_{\pi(j)}v_{\pi(j)} - \gamma_{\pi(i)}v_{\pi(i)})$$
As I said, $0 \leq b_k$, so
$$\alpha_j(\gamma_{\pi(j)}v_{\pi(j)}) \geq \alpha_j(\gamma_{\pi(j)}v_{\pi(j)} - \gamma_{\pi(j+1)}b_{\pi(j+1)})$$
All this leads to
$$\begin{aligned}
\alpha_j(\gamma_{\pi(j)}v_{\pi(j)}) &\geq \alpha_i(\gamma_{\pi(j)}v_{\pi(j)} - \gamma_{\pi(i)}v_{\pi(i)}) \\
\frac{\alpha_j}{\alpha_i} &\geq \frac{\gamma_{\pi(j)}v_{\pi(j)} - \gamma_{\pi(i)}v_{\pi(i)}}{\gamma_{\pi(j)}v_{\pi(j)}} \\
\frac{\alpha_j}{\alpha_i} &\geq \frac{\gamma_{\pi(j)}v_{\pi(j)}}{\gamma_{\pi(j)}v_{\pi(j)}} - \frac{\gamma_{\pi(i)}v_{\pi(i)}}{\gamma_{\pi(j)}v_{\pi(j)}} \\
\frac{\alpha_j}{\alpha_i} &\geq 1 - \frac{\gamma_{\pi(i)}v_{\pi(i)}}{\gamma_{\pi(j)}v_{\pi(j)}}
\end{aligned}$$
Finally, I obtain the initial inequality.
$$\frac{\alpha_j}{\alpha_i} + \frac{\gamma_{\pi(i)}v_{\pi(i)}}{\gamma_{\pi(j)}v_{\pi(j)}} \geq 1.$$

## Problem 35

### Statement

Consider a the following decision process which runs on a leader-follower society with $n$ members. The interaction among the society is represented by a bipartite directed graph $G = (L, F, E)$ having the following property: all the vertices in $L$ have in-degree 0 and all the vertices in $F$ have out-degree 0. The decision process is defined by two parameters $\alpha$, $0 \leq \alpha \leq 1$ and $q$, $0 \leq q \leq n$.

When the society has to reach a decision about some topic, each member takes an initial position. We model this situation as an initial decision vector $x \in \{0, 1\}^n$. Then, each $i \in F$ looks at the values $p_{i1} = |\{(j, i) \in E \mid x_j = 1\}|$ and $p_{i0} = |\{(j, i) \in E \mid x_j = 0\}|$ and reconsiders its position according to the following algorithm.

- If $p_{i1} > \alpha(p_{i1} + p_{i0})$ and $p_{i0} < \alpha(p_{i1} + p_{i0})$, $x_i = 1$
- If $p_{i0} > \alpha(p_{i1} + p_{i0})$ and $p_{i1} < \alpha(p_{i1} + p_{i0})$, $x_i = 0$

Finally, the society reaches a "yea" (1) when $\sum_{i=1}^{n} x_i \geq q$, and a "nay" (0) otherwise.

(a) Assuming that a coalition $S$ is represented as the initial decision vector $x \in \{0,1\}^n$ defined as $x_i = 1$ iff $i \in S$, the decision system process defines a cooperative game assigning to a coalition $S$ a value in $v(S) \in \{0,1\}$. Is this game simple?

(b) Provide a characterization of the games in the family with non-empty core.

(c) Can the Banzhaf value of player $i$ be computed in polynomial time?

## Solution

(a) Yes, it is a WVG to be exact. We can represent it as $\Gamma = (q; x)$, and also as $\Gamma = (N, S)$, being S either the winning or the losing coalition (depending on the problem). I assume that $x$ has already passed through the algorithm.

(b) To know if a simple game has a non-empty core, you only have to ensure that it has at least one veto player. A player $p$ is a veto player if $v(C) = 0$, for any $C \subseteq N \setminus \{p\}$. The only scenario where a veto player exists is when $\mathbf{\sum_{i=i}^{n} x_i = q}$. In this case $v(C) = 1$. Then, the $q$ nodes $x_i = 1$ are all veto players because if I remove one from the game, then $v(C) = 0$.