Network Creation Games

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Network creation games

- Creation and maintenance of a network is modeled as a game
- n players, think of them as vertices in an undirected graph
- The players can buy/create edges to other players for a price per edge (usually constant $\alpha > 0$)
- As a result of a strategy profile s a graph $G(s)$ is created.
- The goal of the player u is to minimize a cost function on $G(s)$

$$c_u(s) = \text{creation cost } + \text{usage cost}$$

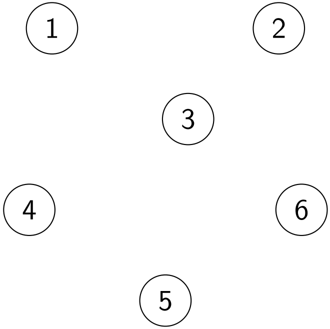
User cost

- Assume that $G = G(s)$ and fix a player u
- Creation cost α (number of edges player u creates)
- Usage cost:
  - SumGame (Fabrikant et al. PODC 2003)
    Sum over all distances $\sum_{v \in V} d_G(u, v)$
    This is an average-case approach to the usage cost
  - MaxGame (Demaine et al. PODC 2007)
    Maximum over all distances $\max_{v \in V} d_G(u, v)$
    A worst-case approach to the usage cost

Social cost

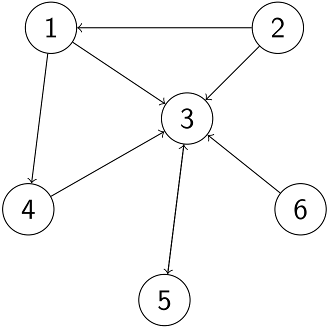
- Assume that $G = G(s)$
- Creation cost $\alpha|E(G)|$
- Usage cost:
  - SumGame
    Sum over all distances $\sum_{u,v \in V} d_G(u, v)$
  - MaxGame (Demaine et al. PODC 2007)
    Maximum over all distances $\max_{u,v \in V} d_G(u, v)$

An example



An example

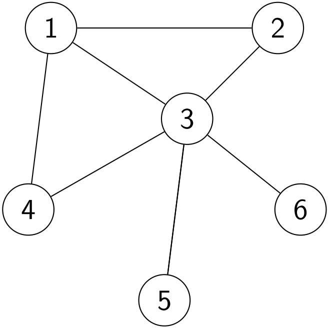
$$s = (\{3,4\}, \{1,3\}, \{5\}, \{3\}, \{3\}, \{3\})$$



An arrow indicates who bought the edge

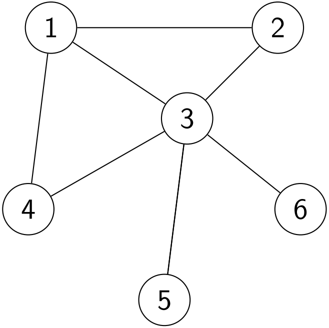
An example

$s = (\{3,4\}, \{1,3\}, \{5\}, \{3\}, \{3\}, \{3\})$ and $G(s)$



An example: SumGame

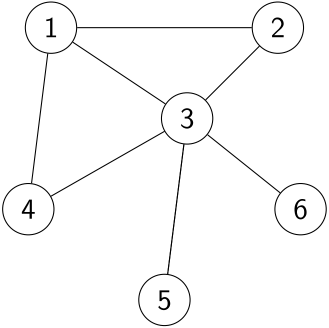
$s = (\{3,4\},\{1,3\},\{5\},\{3\},\{3\},\{3\})$ and $G(s)$



$c_1(s) = 2\alpha + 1 + 1 + 1 + 2 + 2 = 2\alpha + 7 \ldots$
$c(s) = 7\alpha + (7 + 8 + 5 + 8 + 9 + 9) = 7\alpha + 56$

An example: MaxGame

$s = (\{3,4\},\{1,3\},\{5\},\{3\},\{3\},\{3\})$ and $G(s)$



$c_1(s) = 2\alpha + 2 = 2\alpha + 2 \ldots$
$c(s) = 7\alpha + 2$

What to study?

- Are there PNE?
- What are the social optima?
- What network topologies are formed? What families of equilibrium graphs can one construct for a given α?
- How efficient are they? Price of Anarchy/Stability?

We will cover some results on SumGames under some cost variants

Optimal/Equilibrium topologies

$$c_u(s) = \alpha|s_u| + \sum_{v \in V} d_G(u, v)$$

$$c(s) = \alpha|E| + \sum_{u,v \in V} d_G(u, v)$$

- Can an edge be created by more than two players? NO
- We have to study them as a function of α
- When is it better to add/remove an edge?
- Can the graph be disconnected? NO

Add an edge?

$$c_u(s) = \alpha|s_u| + \sum_{v \in V} d_G(u, v)$$

- When is it better to add an edge?
- Set $d = d_G(u, v) > 1$ and let $s'_u = s_u \cup \{v\}$

$$c_u(s_{-u}, s'_u) - c_u(s) = \alpha + 1 - d + \sum_{w \in V, w \neq u} (d_{G'}(u, w)) - d_G(u, w))$$

$$\leq \alpha + 1 - d \leq 0$$

- $d > \alpha$ which implies Nash topologies have diameter $\leq \alpha$.

Computing a Best Response

- Given a game $(1^n, \alpha)$, a strategy profile s and a player i, compute $s_i \in BR_i(s_{-i})$
- We relate the BR with a graph parameter.
- Given a graph G, with $V(G) = \{v_1, \ldots, v_n\}$, consider the following instance for the BR proble,:
  - The game has $n+1$ players, choose α so that $1 < \alpha < 2$, the player will be player v_0. The strategy is defined as follows:
  - Compute an orientation of G and define s_{-0} accordingly. Set $s_0 = V(G)$.
- As $1 < \alpha < 2$, v_0 will like to buy edges to link to any vertex at distance > 2.
- So, in the BR graphs the radius of v_0 must be ≤ 2.
- On such graphs, $c_0(s_{-0}, s'_0) = (\alpha + 1)|s'_0| + 2(n - |s'_0|)$

Computing a Best Response

- So, in the BR graphs the radius of v_0 must be ≤ 2.
- On such graphs, $c_0(s_{-0}, s'_0) = (\alpha + 1)|s'_0| + 2(n - |s'_0|)$
- c_0 is minimized when $|s'_0|$ has minimum cardinality, provided radius of v_0 is ≤ 2.
- To get radius ≤ 2, $|s'_0|$ must be a dominating set.
- The BR strategies are the dominating sets of G having minimum size.
- Computing a minimum size dominating set is NP-hard, so
- Computing a BR in the sum game is NP-hard

Optimal topologies

$$c(s) = \alpha|E| + \sum_{u,v \in V} d_G(u,v)$$

- When two vertices u, v are not connected $d_G(u,v) \geq 2$.
- When two vertices u, v are connected $d_G(u,v) = 1$.
- Therefore

$$\begin{aligned} c(s) = \alpha|E| + \sum_{u,v \in V} d_G(u,v) &\geq \alpha|E| - 2|E| + \sum_{u,v \in V} 2 \\ &\geq \alpha|E| - 2|E| + 2n(n-1) = 2n(n-1) - (\alpha - 2)|E| \end{aligned}$$

- Holds with equality on graphs with diameter ≤ 2.

Optimal topologies

- If $G(s)$ has diameter ≤ 2,

$$c(s) = 2n(n-1) - (\alpha - 2)|E|$$

- This function has different minima depending on whether $(\alpha - 2)$ is positive or negative.
- When $\alpha = 2$, the optimal cost is independent of the number of edges in the graph. So,
- Any graph with diameter ≤ 2 has optimal cost.

Optimal topologies

- If $G(s)$ has diameter ≤ 2,

$$c(s) = 2n(n-1) - (\alpha - 2)|E|$$

- When $\alpha > 2$, to make the cost minimum we have to take the minimum number of edges in G. Of course the graph must be connected. So,
- Only trees with diameter 2 have optimal cost.
- S_n is the unique optimal topology.

Optimal topologies

- If $G(s)$ has diameter ≤ 2,

$$c(s) = 2n(n-1) - (\alpha - 2)|E|$$

- When $\alpha < 2$, to make the cost minimum we have to take the maximum number of edges in G. So,
- K_n is the unique optimal topology.

Nash topologies

$$c_u(s) = \alpha|s_u| + \sum_{v \in V} d_G(u, v)$$

- The star S_n is a Nash equilibrium?
- Vertices $v_1, \ldots, v_n$. Let v_1 be the center of the star.
- Consider s: $s_1 = \emptyset$ and $s_i = \{s_1\}$, for $i > 1$. ($G(s) = S_n$)
- For v_1,
  - $c_1(s) = n - 1$.
  - v_1 is getting the smallest possible cost.

Nash topologies

- The star S_n is a Nash equilibrium?
- Consider s: $s_1 = \emptyset$ and $s_i = \{s_1\}$, for $i > 1$. ($G(s) = S_n$)
- For v_i, $i \geq 1$
  - $c_i(s) = \alpha + 1 + 2(n-2)$.
  - If v_i changes $s_i = \{v_1\}$ for $s'_i = A \cup \{v_1\}$, $v_1 \notin A$,

$$c_i(s_{-i}, s'_i) = \alpha + 1 + (\alpha + 1)|A| + 2(n - 2 - |A|)$$

$$c_i(s) - c_i(s_{-i}, s'_i) = (1 - \alpha)|A|$$

The cost do not decrease for $\alpha \geq 1$

Nash topologies

- The star S_n is a Nash equilibrium?
- Consider s: $s_1 = \emptyset$ and $s_i = \{s_1\}$, for $i > 1$. $(G(s) = S_n)$
- For v_i, $i \geq 1$
  - $c_i(s) = \alpha + 1 + 2(n-2)$.
  - If v_i changes $s_i = \{v_1\}$ for $s'_i = A$, $v_1 \notin A$,

$$c_i(s_{-i}, s'_i) = (\alpha + 1)|A| + 2 + 3(n - 2 - |A|))$$

$$c_i(s) - c_i(s_{-i}, s'_i) = (\alpha + 1)(1 - |A|) - n - 3|A|$$

Which never increases.

Nash topologies

- K_n is the unique Nash topology for $\alpha < 1$
- S_n is a Nash topology for $\alpha \geq 1$
  although they might be other PNE

PoA: $\alpha < 1$

- K_n is the unique Nash topology
- K_n is also an optimal topology
- $PoA = PoS = 1$

PoA: $1 \leq \alpha < 2$

- K_n is an optimal topology
- Any Nash equilibrium must have diameter ≤ 2, so S_n is a Nash topology with the worst social cost.

$$PoA = \frac{c(S_n)}{c(K_n)} = \frac{(n-1)(\alpha - 2 + 2n)}{n(n-1)\frac{\alpha-2}{2} + 2}$$
$$= \frac{4}{2+\alpha} - \frac{4-2\alpha}{n(2+\alpha)} < \frac{4}{2+\alpha} \leq \frac{4}{3}$$

PoA: $\alpha > n^2$

$$c_u(s) = \alpha|s_u| + \sum_{v \in V} d_G(u,v)$$

- When $\alpha > n^2$, unless the distance is infinity, no player has incentive to buy an edge.
- The NE topologies are spanning trees
- The optimal topology is S_n

$$PoA = \frac{c(T_n)}{c(S_n)} = \frac{\alpha(n-1) + \dots}{\alpha(n-1) + 1 + 2n(n-1)} = O(1)$$

PoA: $\alpha < n^2$

- for a worst NE topology G

$$PoA = \left(\frac{\alpha|E| + \sum_{u,v \in V} d_G(u,v)}{\alpha n + n^2} \right)$$

- $d_G(u,v) < 2\sqrt{\alpha}$, otherwise u will be willing to connect to the node in the center of the shortest path from u to v to be closer by $-\sqrt{\alpha}$ to $\sqrt{\alpha}$ nodes.
- Furthermore, $|E| = O(\frac{n^2}{\sqrt{\alpha}})$ (see [Fabrikant et al. 2003])
- Thus $PoA = O(\sqrt{\alpha})$

PoA: Conjectures

PoA on trees ≤ 5 [Fabrikant et al. 2003]

Constant PoA conjecture: For all α, $PoA = O(1)$.

Tree conjecture: for all $\alpha > n$, all NE are trees.

$O(1)$ PoA conjecture: large α

$PoA = O(1)$	
$\alpha > n^{\frac{3}{2}}$	[Lin 2003]
$\alpha > 12n \log n$	[Albers et al. 2014]
$\alpha > 273n$	[Mihalak, Schlegel, 2013]
$\alpha > 65n$	[Mamageishivii et al. 2015]
$\alpha > 17n$	[Alvarez, Messegue 2017]
$\alpha > 4n - 13$	[Bilo, Lezner 2018]
$\alpha > (1 + \epsilon)n$	[Alvarez, Messegue 2019]

[Alvarez, Messegue 2019 arxiv.org/abs/1909.09799]

$O(1)$ PoA conjecture: small α

$PoA = O(1)$	
$\alpha = O(1)$	[Fabrikant et al. 2003]
$\alpha = O(\sqrt{n})$	[Lin 2003]
$\alpha = O(n^{1-\delta})$, $\delta \geq 1/\log n$	[Demaine et al. 2007]