16. Compute the PoA and the PoS under the egalitarian and the utilitarian utility of the following game:

		Player 2 A	Player 2 B
Player 1	A	6,6	2,7
	B	7,2	0,0

17. Consider the sending from s to t game. Compute (or provide bounds for) the PoA and the PoS for the following social cost/utility functions:

- $C(s) = \begin{cases} \sum_{i \in N} u_i(s) & \text{if there is a path from } s \text{ to } t \text{ in } G[s] \\ n^2 & \text{otherwise} \end{cases}.$
- $U(s) = \max_{i \in N} u_i(s)$.
- $U(s) = \sum_{i \in N} u_i(s)$.

18. In the **cover game** the players are the vertices in an undirected graph $G = (V, E)$ on a set of n vertices. The goal of the game is to select a set of vertices X that covers a lot of edges. An edge is covered by a set X if at least one of its ends points belongs to X.

    Formally, the set of actions allowed to player i is $A_i = \{0, 1\}$. Those players playing 1 will form the set. Let $s = (s_1, \ldots, s_n)$, $s_i \in \{0, 1\}$, be an strategy profile, and let $X(s) = \{i \mid s_i = 1\}$.

    The cost function for player $i \in V$ is defined as follows

    $$c_i(s) = s_i + |\{(i, j) \in E \mid i, j \notin X(s)\}|.$$

    Assuming that the social cost of a strategy profile is defined as

    $$c(s) = |\{(i, j) \in E \mid i, j \notin X(s)\}|.$$

    What can you say about PoA and PoS?

19. The *cooperation* game is defined as follows. There is a group of n people and a task to be performed. To perform correctly the task requires that exactly k persons cooperate. Each player can decide whether to cooperate (1) or not (0). The utility of a strategy profile $x \in \{1, 0\}^i$ for player i is defined as

    $$u_i(x) = \begin{cases} 1 & \text{the task is performed and } x_i = 1. \\ 0 & \text{otherwise} \end{cases}$$

    Consider as social utility the value 1 if the task is performed and 0 otherwise. What can you say about PoA and PoS?

20. The *matching* game is played in a bipartite graph $G = (V_1, V_2, E)$ in which edges connect only vertices in V_1 to vertices in V_2. The players are the vertices in the graph that is $V_1 \cup V_2$. Each player has to select one of its neighbors. Player i gets utility 1 when the selection is mutual (player i selects j and player j selects i) otherwise he gets 0.

    Compute the PoA and the PoS when the social utility is the number of matched pairs.

21. Consider a set of n players that must be partitioned into two groups. However, there is a set of bad pairings and the two players in such a pair do not want to be in the same group. Moreover, each player is free to choose which of the two groups to be in. We can model this by a graph $G = (V, E)$ where each player i is a vertex. There is an edge (i, j) if i and j form a bad pair. The private objective of player i is to maximize the number of its neighbors that are in the other group.

    Assuming that the social utility is the number of edges among the two groups, what can you say about PoA and PoS?

22. The Max 2SAT game is defined by a weighted 2-CNF formula on n variables. In a weighted formula each clause has a weight. The game has n players. Player i controls the i-th variable and can decide the value assigned to this variable. A strategy profile is a truth assignment $x \in \{0, 1\}^n$. Player i gets proportionally the weight of the clauses that are satisfied due to its bit selection.

    - Taking as social utility the sum of the player's utilities, provide bounds for the PoA and the PoS.

23. Assume that we have fixed a finite set K of k colors. Consider a graph $G = (V, E)$ with a labeling function $\ell : V \to 2^K$ and define an associated *coloring game* $\Gamma(G, \ell)$ as follows

    - the players are $V(G)$,
    - the set of strategies for player v is $\ell(v)$,
    - the payoff function of player v is $u_v(s) = |\{u \in N(v) \mid s_u = s_v\}|$.

    Assuming that the social utility is $|\{(u, v) \in E \mid s_u = s_v\}|$, what can you say about the PoA and the PoS.

24. Consider the *local diameter* network creation game in which we have a set V of n. A player can select strategically its set of neighbors. For a given profile $s = (s_1, \ldots, s_n)$, let $G[s]$ be the undirected graph induced by the player's decisions. The cost for player $u \in V$ is defined by means of three parameters $\alpha, \beta, w \geq 0$ according to the following expression:

    $$c_u(s) = \alpha|s_u| + w\,\#\{v \mid d_{G[s]}(u, v) > \beta\}.$$

    Analyze under which parameter assignments the following graphs are NE.

    - S_n a star graph.
    - K_n a complete graph.
    - I_n an independen set.
    - A tree with diameter β.

25. Assume that we have a graph with edge and node non-negative weights, i.e. $G = (V, E, w, b)$ where $w : E \to \mathbb{Z}^+$ and $b : V \to \mathbb{Z}^+$. An *information gathering* game is defined on top of G as $\Gamma = (I, G)$ where $I \subseteq V$. The game has $m = |I|$ players. Player $i \in I$ can select any path in G starting at node i. The players use the selected path to gather the information hidden at the nodes of the graph. In order to traverse a path they have to pay the toll fees represented by the edge weights. The value of the information hidden in a node u is $b(u)$. However, the value of the information degrades proportionally to the number of players that discover such a piece of information.

   - Provide a formal definition of the cost function for the information gathering game.
   - Does this family of games have always a PNE?

26. Consider a keyword auction in the model and social welfare described at class. Provide a way to compute the payments of the VCG mechanism and illustrate it with some examples.

27. Consider a GSP auction for n players.

   - Show that any bid $b_i > v_i$ is dominated by bid $b_i = v_i$.
   - Show that every envy-free equilibrium is efficient.
   - Show that the PoS of the GSP mechanism is 1.

28. Consider a GSP auction for n players. Recall that in such an auction each bid profile b defines an allocation π mapping slots to players. We say that an allocation is *reasonable* if for each pair i, j of slots

$$\frac{\alpha_j}{\alpha_i} + \frac{\gamma_{\pi(i)} v_{\pi(i)}}{\gamma_{\pi(j)} v_{\pi(j)}} \geq 1.$$

   - Prove that when b is a NE, the corresponding allocation π is reasonable.
   - Use the previous fact to show that the price of anarchy, on pure strategies, of the GSP auction is at most 2.

29. Consider a k unit auction in which the seller has a specific revenue target R. The participants have private values and submit bids to the seller. In the Revenue target auction, given a set of bids b as input, the following algorithm determines the allocating rule defining the winning bidders and their payments.

```
function REVENUE TARGET(R, b)
    S = the top k bidders
    while there is  i ∈ S  with  b_i < R/|S| do
        remove an arbitrary such bidder from S
    end while
    allocate an item to each bidder in S at a price of R/|S|
end function
```

   (a) Give an explicit description of the allocation rule in the revenue target auction. Think of an allocation rule as a vector $\mathbf{x}(b)$ with n components so that $x_i(b) = 1$ iff bidder i gets a unit.

(b) We say that an allocation rule $\mathbf{x}(b)$ is *monotone* if, for every bidder i and bids b_{-i} for the other players, the allocation $x_i(b_{-i}, z)$ to i is non decreasing in z. Prove that the allocation rule in the revenue target auction is monotone.

(c) Is the revenue target auction truthful?

(d) Prove that whenever the VCG mechanism for the k-unit auction that we saw in class obtains revenue at least R, the revenue target auction obtains revenue R.