ECE 667

Spring 2013

Synthesis and Verification of Digital Circuits

Scheduling Algorithms
Analytical approach - ILP

Scheduling – a Combinatorial Optimization Problem

- NP-complete Problem
- Optimal solutions for special cases and for ILP
  - Integer linear program (ILP)
  - Branch and bound
- Heuristics
  - iterative Improvements, constructive
- Various versions of the problem
  - Minimum latency, unconstrained (ASAP)
  - Latency-constrained scheduling (ALAP)
  - Minimum latency under resource constraints (ML-RC)
  - Minimum resource schedule under latency constraint (MR-LC)
- If all resources are identical, problem is reduced to *multiprocessor* scheduling (Hu's algorithm)
  - In general, minimum latency multiprocessor problem is intractable under resource constraint
  - Under certain constraints (G(VE) is a tree), greedy algorithm gives optimum solution

Integer Linear Programming (ILP)

- Given:
  - integer-valued matrix $A_{m\ x\ n}$
  - *variables:* $\boldsymbol{x} = (x_1, x_2, \ldots , x_n)^T$
  - constants: $\boldsymbol{b} = (b_1, b_2, \ldots , b_m)^T$ and $\boldsymbol{c} = (c_1, c_2, \ldots , c_n)^T$
- Minimize: $c^T \boldsymbol{x}$

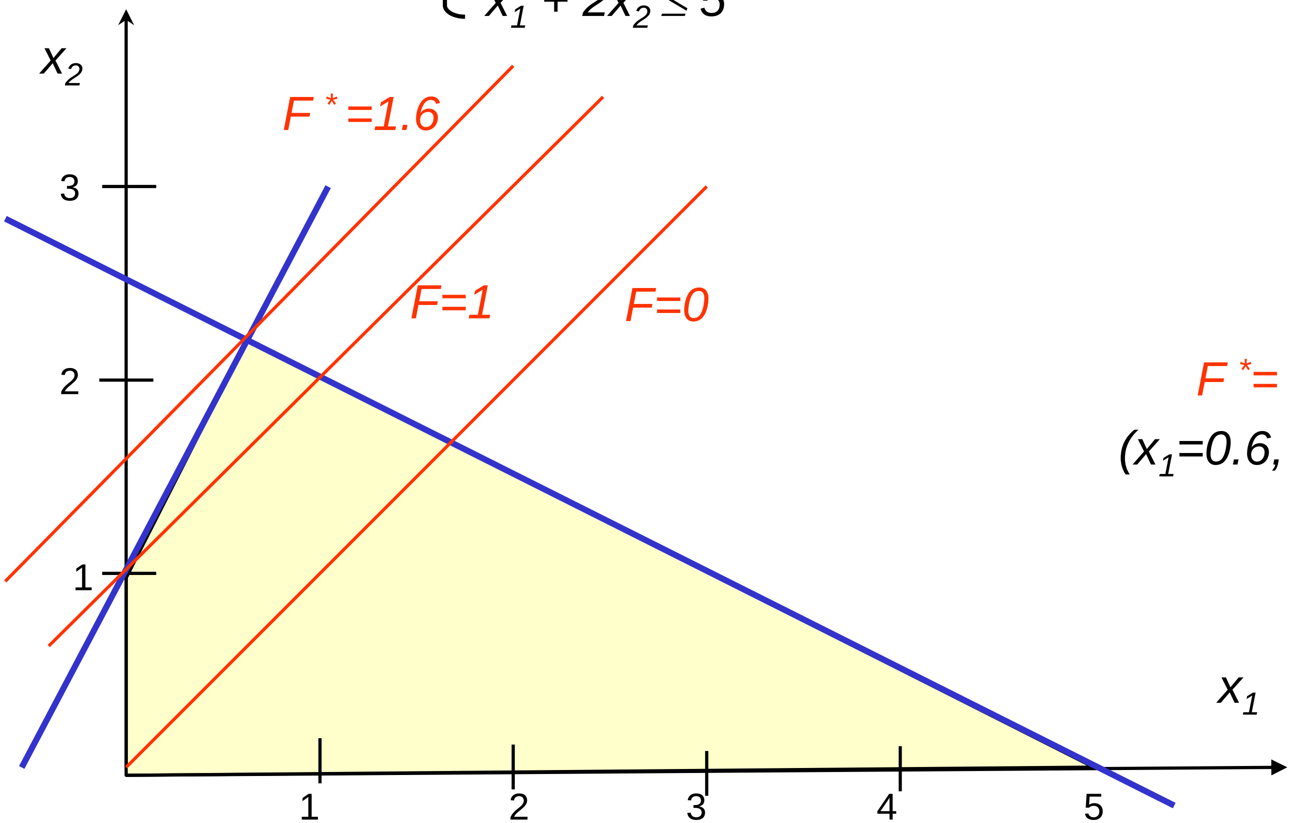
  subject to:

$$\begin{cases} A\,\boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{x} = (x_1, x_2, \ldots , x_n) \ \text{ is an integer-valued vector} \end{cases}$$

- If all variables are *continuous*, the problem is called linear (LP)
- Problem is called *Integer* LP (ILP) if some variables *x* are integer
  - special case: 0,1 (binary) ILP

Linear Programming – example

- Variables: $x = [x_1, x_2]^T$
- Objective function: max $F = -x_1 + x_2 = [-1\ 1]\ [x_1, x_2]^T$
- Constraints: $\begin{cases} -2x_1 + x_2 \le 1 \\ x_1 + 2x_2 \le 5 \end{cases}$



$F^* = 1.6$

$(x_1=0.6,\ x_2=2.2)$

ILP Model of Scheduling

- Binary decision variables x_{il}

  $x_{il} = 1$ if operation v_i starts in step l,
  otherwise $x_{il} = 0$
  $i = 0, 1, \ldots, n$ (operations)
  $l = 1, 2, \ldots \lambda+1$ (steps, with limit λ)

- Start time of each operation vi is unique:

$$\sum_{l} x_{il} = 1, \quad i = 0, 1, \ldots, n$$

Note: $\sum_l x_{il} = \sum_{l=t_i^S}^{l=t_i^L} x_{il}$

where:

t_i^S = time of operation I computed with *ASAP*

t_i^L = time of operation I computed with *ALAP*

ILP Model of Scheduling - constraints

- Start time for v_i:

$$t_i = \sum_l l \cdot x_{il}$$

- Precedence relationships must be satisfied

$$\sum_l l \cdot x_{il} \geq \sum_l l \cdot x_{jl} + d_j, \quad i, j = 0, 1, \ldots, n \quad : (v_j, v_i) \in E$$

- Resource constraints must be met
  - let upper bound on number of resources of type k be a_k

$$\sum_{i:\mathcal{T}(v_i)=k} \quad \sum_{m=l-d_i+1}^{l} x_{im} \leq a_k, \quad k = 1, 2, \ldots, n_{res}, \quad l = 1, 2, \ldots, \bar{\lambda} + 1$$

Latency Minimization - Objective Function

- Function to be minimized: $F = c^T t$, *where* $t_i = \sum_l l \cdot x_{il}$
- Minimum latency schedule: $\mathbf{c} = [0, 0, \ldots, 1]^T$
  - $F = t_n = \sum_l l \, x_{nl}$
  - if sink has no mobility ($x_{n,s} = 1$), any feasible schedule is optimum
- ASAP: $\mathbf{c} = [1, 1, \ldots, 1]^T$
  - finds earliest start times for all operations $\sum_i \sum_l x_{il}$
  - or equivalently:

$$x_{6,1} + 2x_{6,2} + 2x_{7,2} + 3x_{7,3} + x_{8,1} + 2x_{8,2} + 3x_{8,3} + 2x_{9,2} + 3x_{9,3} + \\ + 4x_{9,4} + x_{10,1} + 2x_{10,2} + 3x_{10,3} + 2x_{11,2} + 3x_{11,3} + 4x_{11,4}$$

Minimum-Latency Scheduling under Resource Constraints (ML-RC)

- Let ***t*** be the vector whose entries are *start times*

  $t = [\, t_0, t_1, \ldots, t_n \,]$

- Formal ILP model

minimize $\mathbf{c}^T \mathbf{t}$ such that

$$\sum_l x_{il} = 1, \quad i = 0, 1, \ldots, n$$

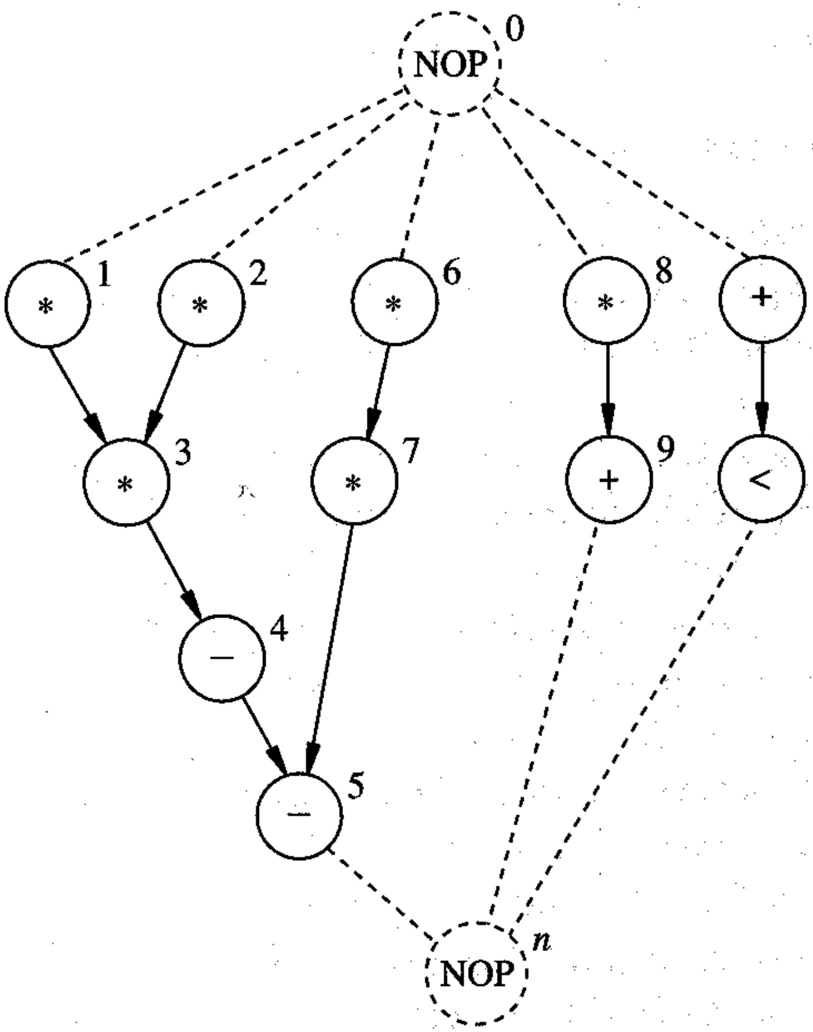
$$\sum_l l \cdot x_{il} - \sum_l l \cdot x_{jl} - d_j \geq 0, \quad i, j = 0, 1, \ldots, n \ : (v_j, v_i) \in E$$

$$\sum_{i:\mathcal{T}(v_i)=k} \ \sum_{m=l-d_i+1}^{l} x_{im} \leq a_k, \quad k = 1, 2, \ldots, n_{res}, \quad l = 1, 2, \ldots, \bar{\lambda} + 1$$

$$x_{il} \in \{0, 1\}, \quad i = 0, 1, \ldots, n, \quad l = 1, 2, \ldots, \bar{\lambda} + 1$$

Example 1 – multiple resources

- Two types of resources
  - MULT
  - ALU
    - Adder, Subtractor
    - Comparator
- Each take 1 cycle of execution time
- Assume upper bound on latency, L = 4
- Use ALAP and ASAP to derive bounds on start times for each operator



Example 1 (cont'd.)

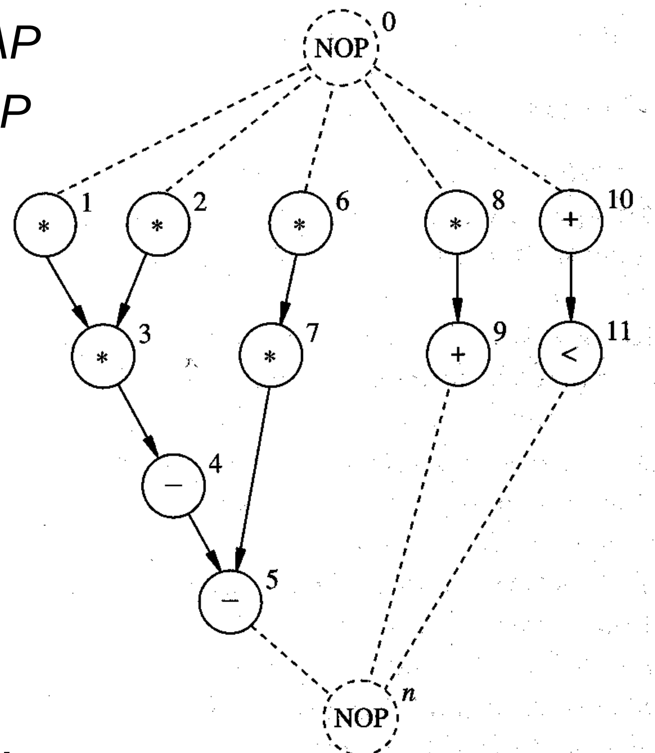
- Start time must be unique

Recall: $\sum_l x_{il} = \sum_{l=t_i^S}^{l=t_i^L} x_{il}$

where:

$t_i^S = t_i$ *computed with ASAP*

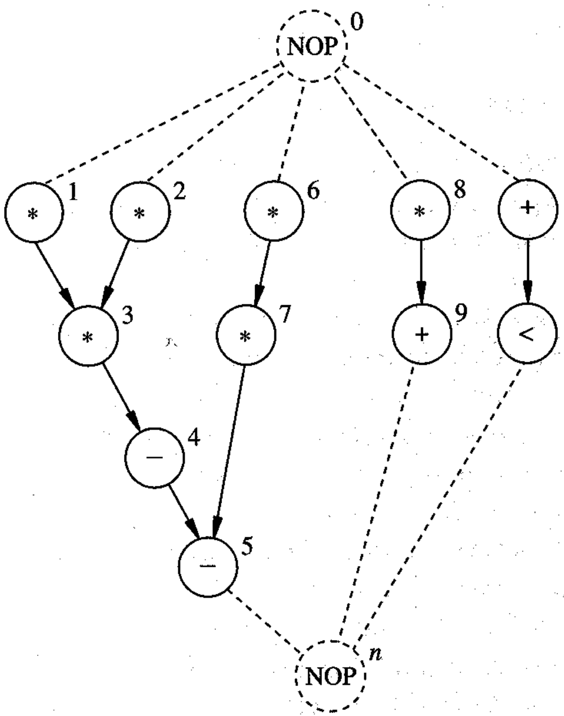
$t_i^L = t_i$ *computed with ALAP*



$$
\begin{aligned}
x_{0,1} &= 1\\
x_{1,1} &= 1\\
x_{2,1} &= 1\\
x_{3,2} &= 1\\
x_{4,3} &= 1\\
x_{5,4} &= 1\\
x_{6,1} + x_{6,2} &= 1\\
x_{7,2} + x_{7,3} &= 1\\
x_{8,1} + x_{8,2} + x_{8,3} &= 1\\
x_{9,2} + x_{9,3} + x_{9,4} &= 1\\
x_{10,1} + x_{10,2} + x_{10,3} &= 1\\
x_{11,2} + x_{11,3} + x_{11,4} &= 1\\
x_{n,5} &= 1
\end{aligned}
$$

Example 1 (cont'd.)

- Precedence constraints
  - Note: only non-trivial ones listed



$$2x_{7,2} + 3x_{7,3} - x_{6,1} - 2x_{6,2} - 1 \geq 0$$

$$2x_{9,2} + 3x_{9,3} + 4x_{9,4} - x_{8,1} - 2x_{8,2} - 3x_{8,3} - 1 \geq 0$$

$$2x_{11,2} + 3x_{11,3} + 4x_{11,4} - x_{10,1} - 2x_{10,2} - 3x_{10,3} - 1 \geq 0$$

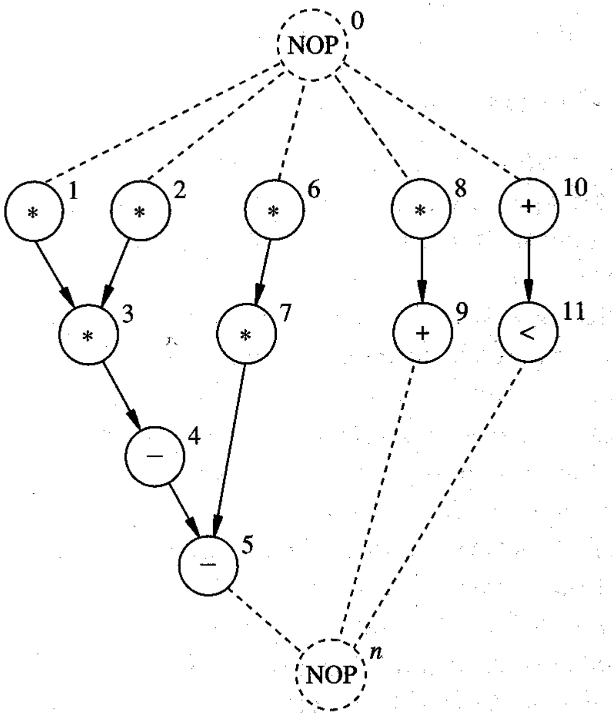
$$4x_{5,4} - 2x_{7,2} - 3x_{7,3} - 1 \geq 0$$

$$5x_{n,5} - 2x_{9,2} - 3x_{9,3} - 4x_{9,4} - 1 \geq 0$$

$$5x_{n,5} - 2x_{11,2} - 3x_{11,3} - 4x_{11,4} - 1 \geq 0$$

Example 1 (cont'd.)

- Resource constraints



MULT

a1=2

$$\begin{cases} x_{1,1} + x_{2,1} + x_{6,1} + x_{8,1} \leq 2 \\ x_{3,2} + x_{6,2} + x_{7,2} + x_{8,2} \leq 2 \\ x_{7,3} + x_{8,3} \leq 2 \end{cases}$$

ALU

a2=2

$$\begin{cases} x_{10,1} \leq 2 \\ x_{9,2} + x_{10,2} + x_{11,2} \leq 2 \\ x_{4,3} + x_{9,3} + x_{10,3} + x_{11,3} \leq 2 \\ x_{5,4} + x_{9,4} + x_{11,4} \leq 2 \end{cases}$$

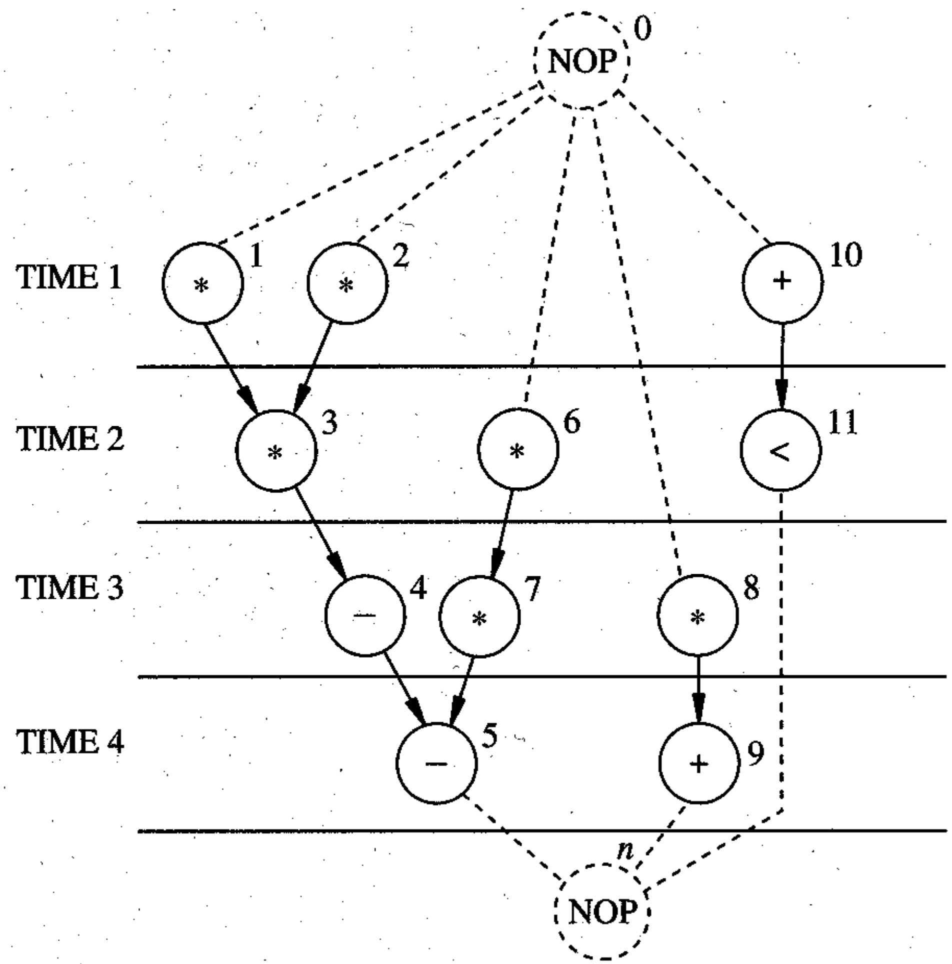
Example 1 (cont'd.)

- Objective function (some possibilities): $F = c^T t$
- $F1$: $\mathbf{c} = [0, 0, \ldots, 1]^T$
  - Minimum latency schedule
  - since sink has no mobility ($x_{n,5} = 1$), any feasible schedule is optimum
- $F2$: $\mathbf{c} = [1, 1, \ldots, 1]^T$
  - finds earliest start times for all operations $\sum_i \sum_l x_{il}$
  - or equivalently:

$$x_{6,1} + 2x_{6,2} + 2x_{7,2} + 3x_{7,3} + x_{8,1} + 2x_{8,2} + 3x_{8,3} + 2x_{9,2} + 3x_{9,3} + \\ + 4x_{9,4} + x_{10,1} + 2x_{10,2} + 3x_{10,3} + 2x_{11,2} + 3x_{11,3} + 4x_{11,4}$$

Example Solution 1:

Min. Latency Schedule Under Resource Constraint



Minimum Resource Scheduling under Latency Constraint (MR-LC)

- Special case
  - Identical operations, each executing in one cycle time
- Given a set of operations $\{v_1, v_2, \ldots, v_n\}$,
  - find the *minimum number* of operation units needed to complete the execution in k control steps (*MR-LC problem*)
- Integer Linear Programming (ILP):
  - Let y_0 be an integer variable (# units to be minimized)
  - for each control step $l = 1, \ldots, k$, define variable x_{il} as

$$x_{il} = \begin{cases} 1, & \text{if computation } v_i \text{ is executed in the } l\text{-th control step} \\ 0, & \text{otherwise} \end{cases}$$

  - define variable y_l (number of units in control step l)

$$y_l = x_{1l} + x_{2l} + \ \ldots + x_{nl} = \Sigma_i \, x_{il}$$

ILP Scheduling – simple MR-LC

- Minimize: y_0

Subject to:

- Each computation v_i can start only once: $\sum_l x_{il} = 1, \quad i = 0, 1, \ldots, n$

  $x_{il} = 1$ *for only one value of* l *(control step)* ("vertical" constraint)
- For each precedence relation:
  - *If* v_j has to be executed *after* v_i

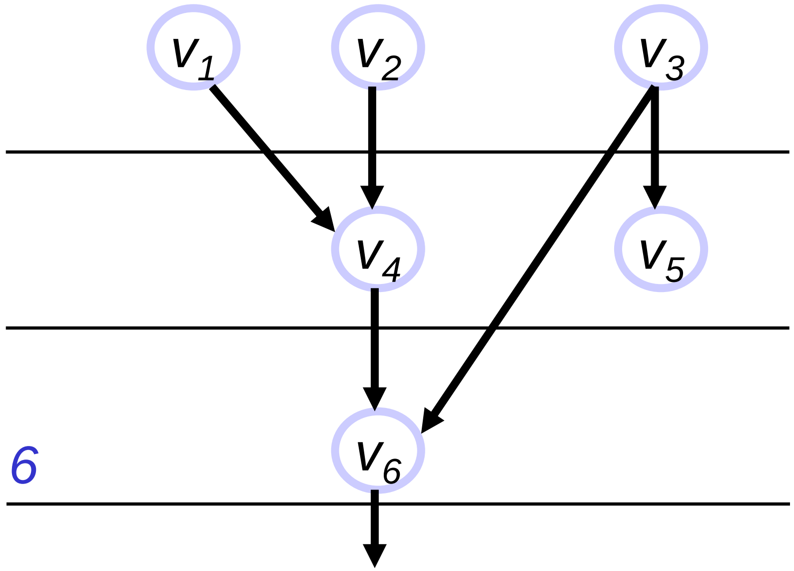
    $x_{j1} + 2\, x_{j2} + \ldots + k\, x_{jk} \geq x_{i1} + 2\, x_{i2} + \ldots + k\, x_{ik} + d(i)$
- $y_l \leq y_0$ for all $l = 1, \ldots, k$ (steps)

- Meaning of $y0$:

  upper bound on the number of units, to be minimized

Example 2 - Formulation

n = 6 computations
k = 3 *control steps*
d(i) = 1



- Execution constraints:

$x_{i1} + x_{i2} + x_{i3} = 1$ for $i = 1, \ldots, 6$

- Resource constraints:

$y_l = x_{1l} + x_{2l} + x_{3l} + x_{4l} + x_{5l} + x_{6l}$ for $l = 1, \ldots, 3$ *(steps)*

- Dependency constraints: e.g. v_4 executes after v_1

$$x_{41} + 2x_{42} + 3x_{43} \geq x_{11} + 2x_{12} + 3x_{13} + 1$$

. . . . *etc.*

Example 2 - Solution

- Minimize: y_0
- Subject to:

  $y_l \leq y_0$ for $l = 1,\ldots, 3$

  - *Starting time constraints …*
  - *Precedence constraints …*
- One possible solution:

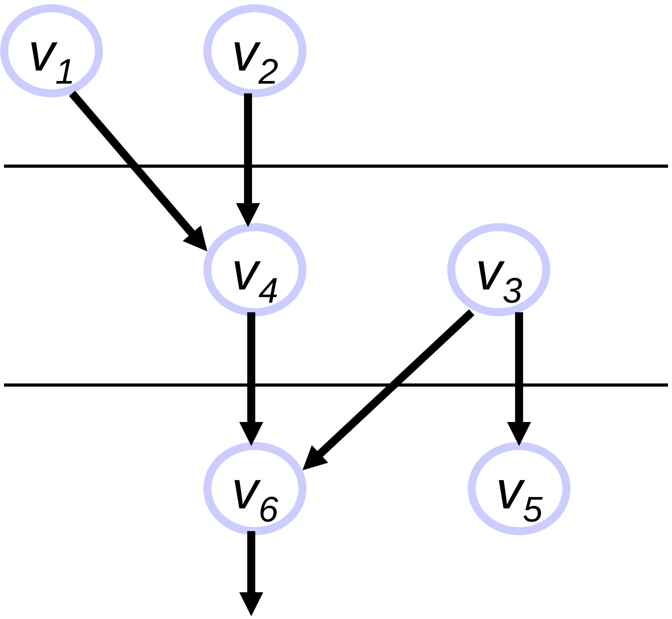
  $y_0 = 2$

  $x_{11} = 1, x_{21} = 1,$

  $x_{32} = 1, x_{42} = 1,$

  $x_{53} = 1, x_{63} = 1.$

all other $x_{il} = 0$



Minimum Resource Scheduling under Latency Constraint – general MR-LC

- General case: several operation units (resources)
- Given
  - vector $c = [c_1, \ldots, c_r]$ of resource costs (areas)
  - vector $a = [a_1, \ldots, a_r]$ of number of resources (unknown)
- Minimize total cost of resources

$$\min \mathbf{c}^T\mathbf{a}$$

- Resource constraints are expressed in terms of variables a_k = number of operators of type k

Example 3 – Min. Resources under Latency Constraint

- Let $c = [5, 1]$
  - MULT costs = 5 units of area, $c_1 = 5$
  - ALU costs = 1 unit of area, $c_2 = 1$
- Starting time constraint – as before
- Sequencing constraints - as before

- Resource constraints – similar to ML-RC, but expressed in terms of *unknown* variables a_1 and a_2

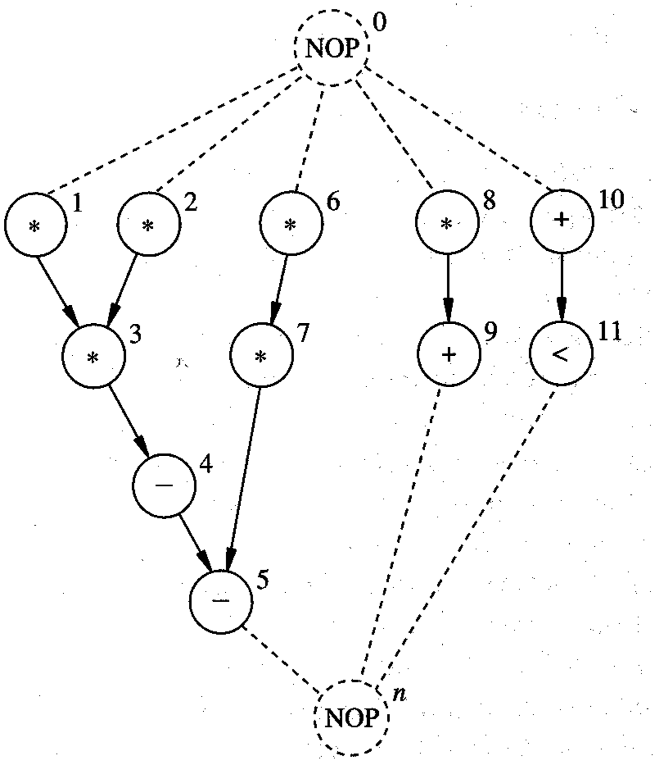
  a_1 = number of multipliers

  a_2 = number of ALUs (add/sub)
- Objective function:

  $$\mathbf{c}^T\mathbf{a} = 5 \cdot a_1 + 1 \cdot a_2$$

Example 3 (contd.)

- Resource constraints



MULT

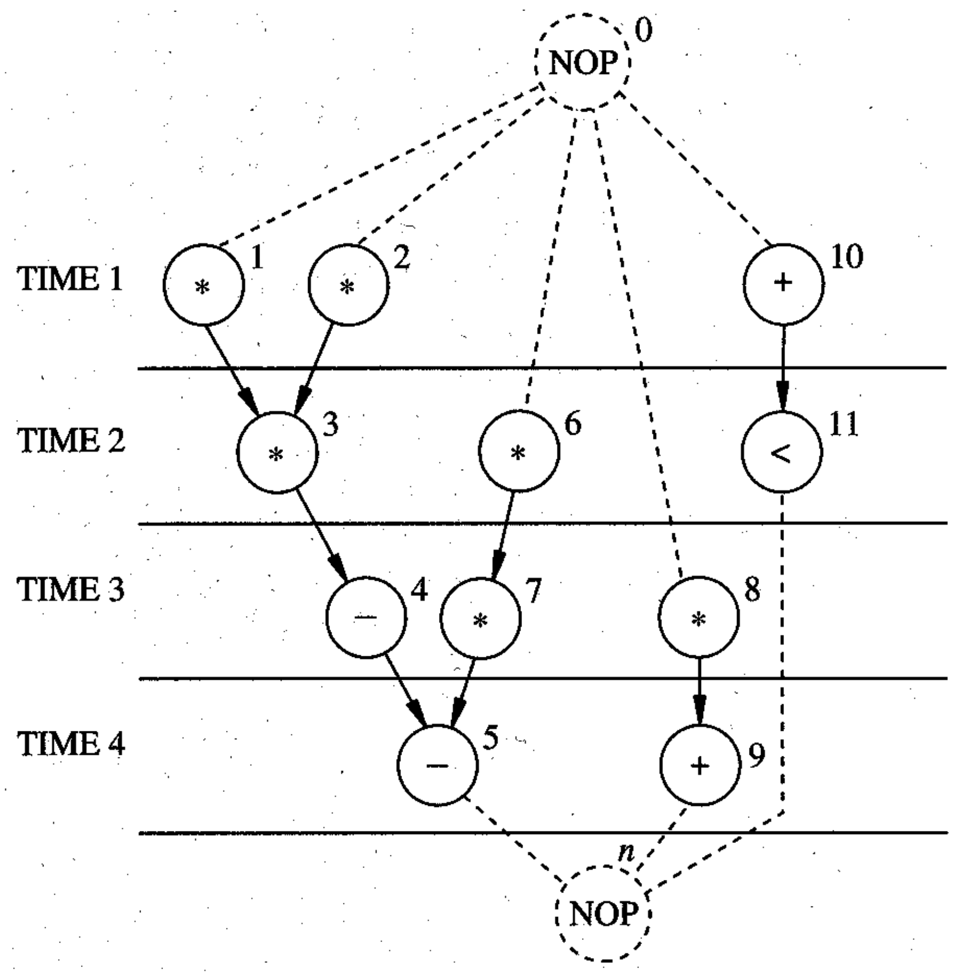
$$\begin{cases} x_{1,1} + x_{2,1} + x_{6,1} + x_{8,1} - a_1 \leq 0 \\ x_{3,2} + x_{6,2} + x_{7,2} + x_{8,2} - a_1 \leq 0 \\ x_{7,3} + x_{8,3} - a_1 \leq 0 \end{cases}$$

ALU

$$\begin{cases} x_{10,1} - a_2 \leq 0 \\ x_{9,2} + x_{10,2} + x_{11,2} - a_2 \leq 0 \\ x_{4,3} + x_{9,3} + x_{10,3} + x_{11,3} - a_2 \leq 0 \\ x_{5,4} + x_{9,4} + x_{11,4} - a_2 \leq 0 \end{cases}$$

Example 3 - Solution

- Minimize
  $\boldsymbol{c}^T\boldsymbol{a} = 5 \cdot a_1 + 1 \cdot a_2$
- Solution with *cost = 12*
  a1 = 2
  a2 = 2



Precedence-constrained Multiprocessor Scheduling

- All operations performed by the same type of resource
  - intractable problem; even if operations have unit delay
  - except when the G_c is a tree (then it is optimal and *O(n)*)
    - *Hu's algorithm*

$$\text{minimize } \mathbf{c}^T\mathbf{t} \quad \text{such that}$$

$$\sum_l x_{il} = 1, \quad i = 0, 1, \ldots, n$$

$$\sum_l l \cdot x_{il} - \sum_l l \cdot x_{jl} \geq 1, \quad i, j = 0, 1, \ldots, n \; : (v_j, v_i) \in E$$

$$\sum_i x_{il} \leq a, \quad l = 1, 2, \ldots, \bar{\lambda} + 1$$

$$x_{il} \in \{0, 1\}, \quad i = 0, 1, \ldots, n, \quad l = 1, 2, \ldots, \bar{\lambda} + 1$$