Games with pure equilibria

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Best response dynamics

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- PNE are defined as the fix point among mutually best responses.
- It seems natural to consider variants of the process of local changes to try to get a PNE.
- Consider the algorithm:
  - choose $s \in A_1 \times \cdots \times A_n$
  - while s is not a NE do

    choose $i \in \{1, \ldots, n\}$ such that $s_i \notin BR(s_{-i})$

    Set s_i to be an action in $BR(s_{-i})$
- The process looks similar to local search algorithms. Is there any difference?

Best response graph

- The Nash dynamics or Best response graph has
  - $V = A_1 \times \cdots \times A_n$
  - An edge $(s, (s_{-i}, s_i'))$ for $i \in N$, $s_i \notin BR(s_{-i})$ and $s_i' \in BR(s_{-i})$.
- Performing local search on the best response graph
  - Does it produce a PNE?
  - If so, how much time?
  - Let's look to some examples.

Prisonner's dilemma

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Quiet	2,2	0,3
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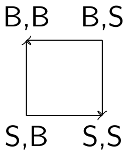

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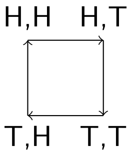

Other games

- sending from s to t?
- congestion games?

In those games we cannot get the best response graph in polynomial time.
However we can perform a local improvement step in polynomial time.
Although, even assuring convergence, it might take exponential time to reach a NE.

Best response graph: properties

- A NE is a sink (a node with out-degree 0) in the best response graph.
- The existence of a cycle in the best response graph does not rule out the existence of a PNE.
- If the best response graph is acyclic, the game has a PNE.

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- The existence of a cycle in the best response graph does not rule out the existence of a PNE.
- If the best response graph is acyclic, the game has a PNE. Furthermore, best response dynamics converges to a PNE, maybe with a lot of time.

Potential games

(Monderer and Shapley 96)

- Consider a strategic game $\Gamma = (N, A_1, \ldots, A_n, u_1, \ldots, u_n)$. Let $S = A_1 \times \cdots \times A_n$.

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- A function $\Phi : S \to \mathbb{R}$ is an exact potential function for Γ if

$$\forall\, i \in N\, \forall\, s \in S\, \forall\, s_i' \in A_i \;\; u_i(s) - u_i(s_{-i}, s_i') = \Phi(s) - \Phi(s_{-i}, s_i')$$

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- Γ is a potential game if it admits a potential function.

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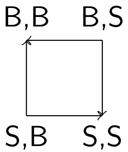

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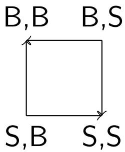

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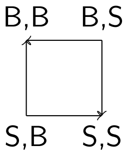

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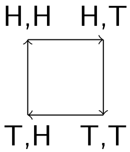

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This is not a potential game

The property on Φ cannot hold along a cycle in the best response graph.

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Theorem

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- The existence of a potential function Φ and the fact that, for each pair of connected strategy profiles in G, at least one player improves, implies the non existence of cycles in G.
- If G is acyclic, a topological sort of the graph provides a potential function for Γ.

$\square$

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We have a way to show that a game has a PNE by showing that it is a potential game.

Congestion games

Congestion games

A congestion game

- is defined on a finite set E of resources.
- There is a delay function d mapping $E \times \mathbb{N}$ to the integers.
- Player's actions are subsets of E (all or some).
- The cost functions are the following:

$$c_i(a_1, \dots, a_n) = \left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e)) \right)$$

being $f(a_1, \dots, a_n, e) = |\{i \mid e \in a_i\}|$.

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  being $f(a_1, \ldots, a_n, e) = |\{i \mid e \in a_i\}|$.
- A singleton congestion game has $A_i = \{\{r\} \mid e \in E\}$.

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- We have 3 orders that have to be send to one of the end production lines.
- The cutting machine on the first line takes 1 hour to process a single order, 2 hours to process 2 and 4 hours to process 3. The cutting machine on the second line takes 4, 5 and 9 hours respectively.
- The packing machine on the first line takes 2 additional hours to pack a single order, 3 hours to pack 2 and 7 hours to pack 3. The packing machine on the second line takes instead 0, 2 and 9 hours respectively.

An example of a congestion game

- We have 4 resources C_1, C_2, P_1, P_2 and 3 players $N = \{1, 2, 3\}$
- $A_i = \{\{C_1, P_1\}, \{C_2, P_2\}\}$, $i = 1, 2, 3$
- Delay functions are defined by the processing times.

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Does this game have a PNE?

Rosenthal's theorem

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- For a strategy profile $s = (a_1, \ldots, a_n)$, define

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Let us show that Φ is a potential function.

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$$\Phi(s) - \Phi(s') = \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k) - \sum_{e' \in r(s')} \sum_{k=1}^{f(s',e')} d(e', k)$$

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$$= \sum_{e \in a_i, e \notin a_i'} d(e, f(s, e)) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s', e'))$$

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&= \sum_{e \in a_i, e \notin a_i'} [\sum_{k=1}^{f(s',e)+1} d(e,k) - \sum_{k=1}^{f(s',e)} d(e,k)] \\
&+ \sum_{e \notin a_i, e \in a_i'} [\sum_{k=1}^{f(s,e)} d(e,k) - \sum_{k=1}^{f(s,e)+1} d(e,k)]
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$$= \sum_{e \in a_i, e \notin a_i'} [\sum_{k=1}^{f(s',e)+1} d(e,k) - \sum_{k=1}^{f(s',e)} d(e,k)]$$

$$+ \sum_{e \notin a_i, e \in a_i'} [\sum_{k=1}^{f(s,e)} d(e,k) - \sum_{k=1}^{f(s,e)+1} d(e,k)]$$

$$= \sum_{e \in a_i, e \notin a_i'} d(e, f(s', e) + 1) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s, e) + 1)$$

$$= \sum_{e \in a_i, e \notin a_i'} d(e, f(s, e)) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s', e))$$

$$= c_i(s) - c_i(s_{-i}, s_i')$$

Network congestion games

- A network congestion game is a congestion game defined by a directed graph G and a collection of pairs of vertices (s_i, t_i).
  - The set of resources are the arcs in G.
  - The acrions, for player i, are the $s_i \to t_i$ paths on G.
- A network congestion game is symmetric when $s_i = s$ and $t_i = t$, for $i \in N$.

An example of a network congestion game

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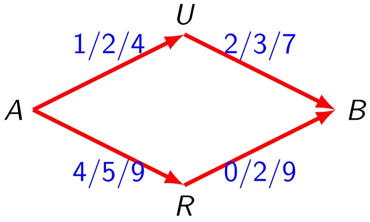

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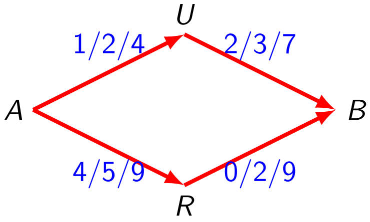

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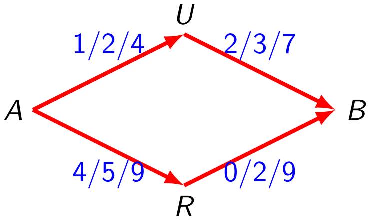

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- A NE?

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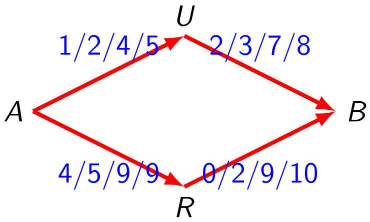

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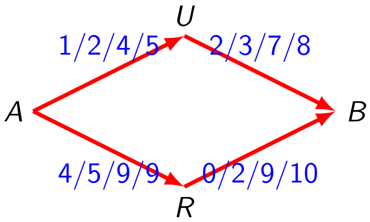

- Player's objective: send w_i units from $s = A$ to $t = B$ as fast as possible.
- Strategy profiles: paths from A to B.
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Results on convergence time

Theorem (Fabrikant, Papadimitriou, Talwar (STOC 04))

There exist network congestion games with an initial strategy profile from which all better response sequences have exponential length.

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Theorem (Ieong, McGrew, Nudelman, Shoham, Sun (AAAI 05))

In singleton congestion games all best response sequences have length at most $n^2 m$.

Complexity classification?

Optimization problem

An optimization problem is a structure $\Pi = (\mathrm{I}, \mathrm{sol}, \mathrm{m}, \mathrm{goal})$, where

- C is the input set to Π;
- $\mathrm{sol}(x)$ is the set of feasible solutions for an input x.
- m is an integer measure defined over pairs (x, y), $x \in I$ and $y \in \mathrm{sol}(x)$.
- goal is the optimization criterium MAX or MIN.

An optimization problem is a function problem whose goal, with respect to an instance x, is to find an optimum solution, that is, a feasible solution y such that

$$y = \mathrm{goal}\{(m(x, y') \mid y' \in \mathrm{sol}(x)\}.$$

Example: Given a graph and two vertices, obtain a path joining them with minimum length.

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A local search problems belongs toPLS (Polynomial Local Search)if polynomial time algorithms exist for

- finding initial feasible solution $s \in \mathrm{sol}(x)$,
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A local search problems belongs toPLS (Polynomial Local Search)if polynomial time algorithms exist for

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- checking whether a solution is a local optimum and if not finding a better solution in the neighborhood.

PLS reductions

A PLS reduction from $(\Pi_1, \mathcal{N}_1)$ to $(\Pi_1, \mathcal{N}_1)$ is

- a polynomial time computable function $f : \mathrm{I}_{\Pi_1} \rightarrow \mathrm{I}_{\Pi_2}$ and
- a polynomial time computable function $g : \mathrm{sol}(f(x)) \rightarrow \mathrm{sol}(x)$, for $x \in \mathrm{I}_{\Pi_1}$ such that
- if $s_2 \in \mathrm{sol}(f(x))$ locally optimal then $g(s_2)$ is locally optimal.

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A PLS problem $(\Pi, \mathcal{N})$ is PLS-complete if every problem in PLS is PLS-reducible to $(\Pi, \mathcal{N})$.

PLS complete problems

- MAX-SAT (maximum satisfiability) problem
  - Given a Boolean formula in conjunctive normal form with a positive integer weight for each clause.
  - A solution is an assignment of the value 0 or 1 to all variables.
  - Its weight, to be maximized, is the sum of the weights of all satisfied clauses.
  - As neighborhood consider the Flip-neighborhood, where two assignments are neighbors if one can be obtained from the other by fliipping the value of a single variable.

PLS complete problems

- MaxCut problem.
  - Given a graph $G = (V, E)$ with non-negative edge weights.
  - A feasible solution is a partition of V into two sets A and B.
  - The objective is to maximize the weight of the edges between the two sets A and B.
  - In the Flip-neighborhood two solutions are neighbors if one can be obtained from the other by moving a single vertex from one set to the other.

PLS completeness

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- We provide a reduction from MaxCut under the Flip-neigborhood.

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  - The players correspond to the nodes in V, $v \in V$ has strategies $S_v^a = \{e^a \mid v \text{ incident to } e\}$ and $S_v^b = \{e^b \mid v \text{ incident to } e\}$

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  - Solutions (A, B) of MaxCut corresponds to strategy S_v^a for $v \in A$ and S_v^b for $v \in B$.

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  - Furthermore, the local optima of the MaxCut instance coincide with the Nash equilibria of the congestion game.
- We have a PLS-reduction from MaxCut.

1. Best response dynamics
2. Potential games
3. Congestion games
4. References

Reference

B. Vöcking, Congestion Games: Optimization in Competition