Linear arrangement of vertices

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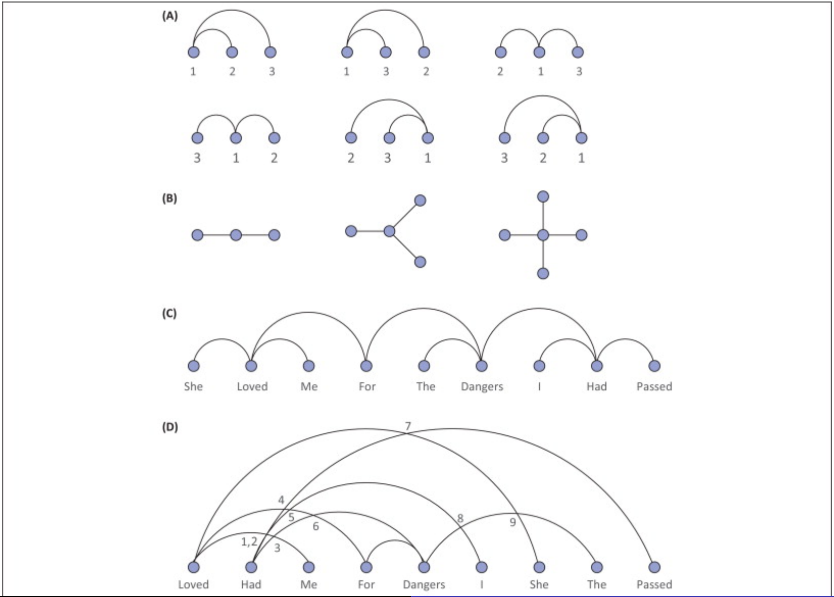

(A)
1 2 3
1 3 2
2 1 3
3 1 2
2 3 1
3 2 1
(B)
(C)
She Loved Me For The Dangers I Had Passed
(D)
7
4
5
6
8
9
1,2
3
Loved Had Me For Dangers I She The Passed

Two interesting properties:

- The linear (euclidean) distance between connected words is "small".
- The number of crossings is "small".

An statistical challenge:

- Are they significantly small?
- What would be a suitable null hypothesis?

A scientific question: if they are significantly small, then why?

Focus on trees

A linear arrangement of vertices

- Vertices are labelled with numbers $1, 2, 3, ..., n$ being n the number of vertices of the network.
- $s, t, u, v, ...$ designate vertices.
- A linear arrangement of vertices is one of the $n!$ possible orderings of n vertices.
- A linear arrangement can be defined by $\pi(v)$, the position of vertex v in the ordering ($\pi(v) = 1$ if v is the first vertex, $\pi(v) = 2$ if v is the second vertex and so on...).
- For a linear arrangement of a tree, the mean edge length is defined as

$$\langle d \rangle = \frac{D}{n-1} = \frac{1}{n-1} \sum_{u \sim v} |\pi(u) - \pi(v)| \tag{1}$$

Edge crossings

Two edges $u \sim v$ and $s \sim t$ such that $\pi(u) < \pi(v)$ and $\pi(s) < \pi(t)$ cross if and only if

- $\pi(u) < \pi(s) < \pi(v) < \pi(t)$ or
- $\pi(s) < \pi(u) < \pi(t) < \pi(v)$

Example with 4 vertices.
The number of crossings is

$$C = \frac{1}{2} \sum_{u \sim v} C_{u,v}, \tag{2}$$

where $C_{u,v}$ is the number of edge crossings involving $u \sim v$.
$C \geq 0$, but what is the maximum value of C?

Degrees in trees

- Mean degree is constant, i.e.

$$\langle k \rangle = \frac{1}{n} \sum_{v=1}^{n} k_v = 2 - 2/n. \quad (3)$$

- Degree variance is fully determined by the 2nd moment, i.e.

$$V[k] = \left\langle k^2 \right\rangle - \langle k \rangle^2 = \left\langle k^2 \right\rangle - (2 - 2/n)^2 \quad (4)$$

- The 2nd moment is minimized by a linear tree and maximized by a star tree, i.e. [Ferrer-i-Cancho, 2013]

$$\text{(linear tree) } 4 - \frac{6}{n} \leq \left\langle k^2 \right\rangle \leq n - 1 \text{ (star tree)} \quad (5)$$

Mean edge length in trees

- Real syntactic dependency trees: sublinear growth (Fig. of [Ferrer-i-Cancho, 2004]).
- Some theoretical bounds [Ferrer-i-Cancho, 2013]
  - In a random linear arrangement, $E[\langle d \rangle] = \frac{n+1}{3}$.
  - In a non-crossing tree, $\langle d \rangle \leq n/2$.
  - $\langle d \rangle \geq \frac{n}{8(n-1)} \left\langle k^2 \right\rangle + \frac{1}{2}$

Length in random linear arrangements

- The number of pairs of edges at distance d is $N(d) = n - d$.
- The probability that an edge has length d is [Ferrer-i-Cancho, 2004]

$$p(d) = \frac{N(d)}{\sum_{d=1}^{n-1} N(d)} = \frac{2(n-d)}{n(n-1)} \tag{6}$$

- $E[\langle d \rangle] = E[d] = \frac{n+1}{3}$. Hint: $\sum_{d=1}^{n-1} d^2 = \frac{(n-1)n(2n-1)}{6}$
- $V[d] = \frac{(n+1)(n-2)}{18}$ [Ferrer-i-Cancho, 2013]

Upper bound of $\langle d \rangle$ on non-crossing trees

Outline

- Examples of non-crossing linear arrangements with $\langle d \rangle = n/2$ (star tree and linear tree).
- Prove that $\langle d \rangle = n/2$ is maximum for a non-crossing tree (proof by induction on n). Idea: decomposition of a non-crossing tree into smaller non-crossing trees.

Lower bounds of $\langle d \rangle$ on trees I

The degree method [Petit, 2003]

$$\langle d \rangle = \frac{1}{2(n-1)} \sum_{v=1}^{n} D_v \tag{7}$$

Idea to bound $\langle d \rangle$ below: minimize each D_v (each node v forms a star tree of $n = k_v + 1$ nodes).

If k_v is even

$$D_v \geq \frac{k_v}{2}\left(\frac{k_v}{2} + 1\right) = \frac{k_v^2}{4} + \frac{k_v}{2} \tag{8}$$

If k_v is odd

$$D_v \geq \left(\frac{k_v + 1}{2}\right)^2 = \frac{k_v^2}{4} + \frac{k_v}{2} + \frac{1}{4} \tag{9}$$

Lower bounds of $\langle d \rangle$ on trees II

$$\begin{aligned}
\langle d \rangle &\geq \frac{1}{4(n-1)} \sum_{v=1}^{n} \left(\frac{k_v^2}{2} + k_v \right). & (10) \\
&= \frac{1}{8(n-1)} \sum_{v=1}^{n} k_v^2 + \frac{1}{4(n-1)} \sum_{v=1}^{n} k_v & (11) \\
&= \frac{n}{8(n-1)} \left\langle k^2 \right\rangle + \frac{1}{2}. & (12)
\end{aligned}$$

The importance of star trees: $\langle d \rangle_{min} \leq \langle d \rangle_{min}^{star}$ [Esteban et al., 2016].
More methods to bound $\langle d \rangle$ below [Petit, 2003].

Why is $\langle d \rangle$ below chance in real dependency networks?

A hypothesis on the limited resources of the human brain [Ferrer-i-Cancho, 2004]

- Two linked vertices u and v, such that $\pi(u) < \pi(v)$, the distance $d = \pi(v) - \pi(u)$ can be seen as the time that is needed to keep the open or unresolved dependency in online memory once u has appeared [Morrill, 2000].
- $d = \pi(u) < \pi(v)$ is being minimized, but how exactly?

A family of models to consider:

- minimum linear arrangement problem (sum of dependency lengths)
- minimum bandwidth problem (minimize maximum dependency length)
- ...

The minimum linear arrangement problem [Díaz et al., 2002]

- $u \sim v$ indicates an edge between vertices u and v.
- Find π such that

$$D = \sum_{u \sim v} |\pi(u) - \pi(v)| \tag{13}$$

  is minimum.
- $D = \langle d \rangle / E$. In a tree: $D = \langle d \rangle / (n-1)$.
- Computational complexity:
  - NP-complete for an unconstrained graph [Garey and Johnson, 1979].
  - Polynomial time for a tree.

Minimum linear arrangements of trees

Unconstrained [Petit, 2011]:

- $O(n^3)$ [Goldberg and Klipker, 1976]
- $O(n^{2.2})$ [Shiloach, 1979]
- $O(n^\lambda)$, with $\lambda > \frac{\log 3}{\log 2} = 1.585...$ [Chung, 1984]

Constrained:

- Non-crossing trees: $O(n)$ [Hochberg and Stallmann, 2003].
- Complete k-level 3-ary trees: $O(n)$ [Chung, 1981].
- More examples... [Petit, 2011].

Big question: is a linear time algorithm for unrestricted trees possible?

Experiment

For a given n,

- Produce many random (labelled) trees.
- Arrange the vertices linearly in an arbitrary order and obtain $\langle d \rangle_0$.
- Arrange the vertices linearly solving the minimum linear arrangement problem to obtain $\langle d \rangle_{mla}$.
- What predictions can we make about $\langle d \rangle_0$ and $\langle d \rangle_{mla}$?

An example: Fig. 2 a) of [Ferrer-i-Cancho, 2006].

- Power-laws? $\rightarrow$ Model selection.
- Producing uniformly distributed random trees: the Aldous-Brother algorithm [Aldous, 1990, Broder, 1989].
- What is the mathematical form of $\langle d \rangle_{mla}$? Theoretical and **empirical** approach.

Interest of crossings

- Computational efficiency (m.l.a. without crossings in linear time [Hochberg and Stallmann, 2003]).
- Theoretical linguistics, computational linguistics and cognitive science.
  - Projectivity = planarity + uncovered root (context-freeness) [Mel'čuk, 1988]
  - Mild context-sensitivity [Joshi, 1985]
- ...

The maximum number of crossings I

- Q: the set of pairs of edges that may potentially cross.
- C: the number of edge crossings, $C \leq |Q|$

$$|Q| = \frac{1}{4} \sum_{u=1}^{n} \sum_{v=1}^{n} a_{uv} C_{pairs}(u, v) \tag{14}$$

The number of crossings in which the edge $u \sim v$ is involved cannot exceed

$$C_{pairs}(u, v) = n - k_u - k_v, \tag{15}$$

being k_v the degree of vertex v.

C defines the number of pairs of edges that can cross.

Notice the 1/4 factor of Eq. 14.

The maximum number of crossings II

$$|Q| = \frac{n}{2}\left(n - 1 - \langle k^2 \rangle\right) \tag{16}$$

- Given n, $|Q|$ is determined by $\langle k^2 \rangle$.
- $|Q| \geq 0$ yields $\langle k^2 \rangle \leq n - 1$. What are the trees for which $\langle k^2 \rangle = n - 1$?
- What are the trees minimizing $\langle k^2 \rangle$?

The expected number of crossings I

$p_c(u, v; s, t)$ is the probability that the edges $u \sim v$ and $s \sim t$ cross.

- $p_c(u, v; s, t) = 0$ if $u \sim v$ and $s \sim t$ share at least one vertex.
- $p_c(u, v; s, t) = 1/3$ otherwise. Outline:
  - Generate four different random numbers from 1 to n.
  - Sort them increasingly.
  - Choose the position of the vertices of one the edges.
  - Then

$$p_c(u, v; s, t) = \frac{2}{\binom{4}{2}} = \frac{1}{3} \tag{17}$$

The expected number of crossings II

Decomposition of C as a sum of indicator variables

$$C = \frac{1}{4} \sum_{u=1}^{n} \sum_{v=1}^{n} a_{uv} C(u, v) \tag{18}$$

and

$$C(u, v) = \frac{1}{2} \sum_{\substack{s = 1 \\ s \neq u, v}}^{n} \sum_{\substack{t = 1 \\ t \neq u, v}}^{n} a_{st} C(u, v; s, t) \tag{19}$$

with $C(u, v; s, t) \in \{0, 1\}$.

The expected number of crossings III

The expectation of the sum is the sum of expectations

$$E[C] = \frac{1}{4} \sum_{u=1}^{n} \sum_{v=1}^{n} a_{uv} E[C(u,v)] \tag{20}$$

and

$$E[C(u,v)] = \frac{1}{2} \sum_{\substack{s=1 \\ s \neq u,v}}^{n} \sum_{\substack{t=1 \\ t \neq u,v}}^{n} a_{st} E[C(u,v;s,t)] \tag{21}$$

with $E[C(u,v;s,t)] = p_c(u,v;s,t)$.

The expected number of crossings IV

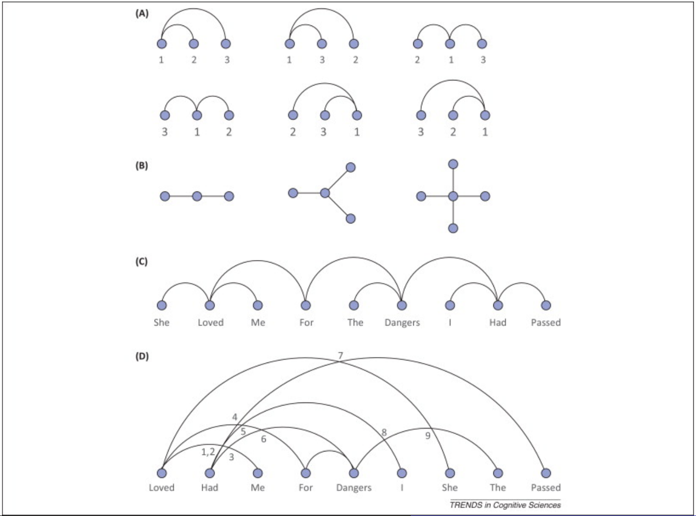
Thus,

$$E[C] = |Q| p_c(u, v; s, t) \tag{22}$$

$$= \frac{|Q|}{3} \tag{23}$$

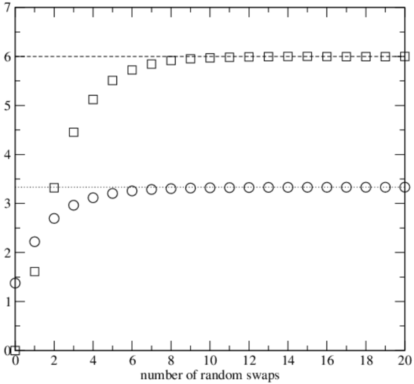
$$= \frac{n}{6}\left(n - 1 - \left\langle k^2 \right\rangle\right) \tag{24}$$

Does the theory work? I



Does the theory work? II

Progressive randomization of the vertex sequence [Ferrer-i-Cancho, 2017]



- Initial example of an English sentence ($n = 9$) starting at
  - $\langle d \rangle = 11/8 = 1.375$
  - $C = 0$.
- Circles: $\langle d \rangle \to \frac{n+1}{3} = 10/3$.
- Squares: $C \to \frac{n}{6}\left(n - 1 - \langle k^2 \rangle\right) = 6$.

Positive correlation between $\langle d \rangle$ and C!

Crossings in uniformly random trees I

$$E[C] = \frac{n}{6}\left(n - 1 - \left\langle k^2 \right\rangle\right) \quad (25)$$

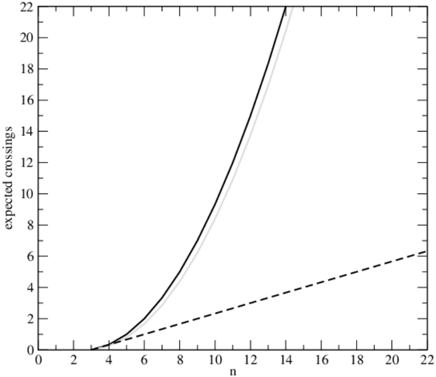
The degree variance for uniformly random labelled trees [Moon, 1970, Noy, 1998]

$$V[k] = \left\langle k^2 \right\rangle - \left\langle k \right\rangle^2 = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \quad (26)$$

Applying $\langle k \rangle = 2 - 2/n$ yields

$$\left\langle k^2 \right\rangle = \frac{n-1}{n}\left(5 - \frac{6}{n}\right) \quad (27)$$

Crossings in uniformly random trees II



There must be a hidden constraint for the scarcity of crossings in real sentences [Ferrer-i-Cancho, 2016]

- Linear trees.
- Uniformly random labelled trees.
- (quasi-star trees)
- Star trees?

An more precise null hypothesis predicts the actual number of crossings with a relative error that is not greater than about 5% (on average)! [Ferrer-i-Cancho, 2014, Gómez-Rodríguez and Ferrer-i-Cancho, 2016].

To conclude

- Real data suggest that $\langle d \rangle$ is been minimized in real trees.
- The small values of C in real dependency trees might be a side-effect of the minimization of $\langle d \rangle$. Figs. 2 c) and d) of [Ferrer-i-Cancho, 2006]
- It is not known how this optimization actually works (but sentence production is not a batch process [Christiansen and Chater, 2016]).
- A mathematical description of $\langle d \rangle$ and C as a function of n in real dependency trees or optimized (mla) trees is not forthcoming.

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