Strategic games: Basic definitions and examples

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Basic References non-coop game theory

- Osborne. An Introduction to Game Theory, Oxford University Press, 2004
- Nisan et al. Eds. Algorithmic game theory, Cambridge University Press, 2007

Where to use game theory?

Game theory studies decisions made in an environment in which players interact.
game theory studies choice of optimal behavior when personal costs and benefits depend upon the choices of all participants.

What for?
Game theory looks for states of equilibrium sometimes calles solutions
and analyzes interpretations/properties of such states

Game Theory for CS?

- Framework to analyze equilibrium states of protocols used by rational agents.
  Price of anarchy/stability.
- Tool to design protocols for internet with guarantees.
  Mechanism design.
- New concepts to analyze/justify behavior of on-line algorithms
  Give guarantees of stability to dynamic network algorithms.
- Source of new computational problems to study.
  Algorithmic game theory

Types of games

- Non-cooperative games
  - strategic games
  - extensive games
  - repeated games
  - Bayesian games
- Cooperative games
  - simple games
  - weighted games
  - ...

1. Game theory and CS
2. Strategic games
3. Congestion games

Strategic game

A strategic game Γ (with ordinal preferences) consists of:

- A finite set $N = \{1, \ldots, n\}$ of players.
- For each player $i \in N$, a nonempty set of actions A_i.
- Each player chooses his action once. Players choose actions simultaneously.
  No player is informed, when he chooses his action, of the actions chosen by others.
- For each player $i \in N$, a preference relation (a complete, transitive, reflexive binary relation) $\preceq_i$ over the set $A = A_1 \times \cdots \times A_n$.

It is frequent to specify the players' preferences by giving utility functions $u_i(a_1, \ldots a_n)$. Also called pay-off functions.

Example: Prisoner's Dilemma

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The story

- Two suspects in a major crime are held in separate cells.
- Evidence to convict each of them of a minor crime.
- No evidence to convict either of them of a major crime unless one of them finks.

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The penalties

- If both stay quiet, be convicted for a minor offense (1 year).
- If only one finks, he will be freed (and used as a witness) and the other will be convicted for a major offense (4 years).
- If both fink, each one will be convicted for a major offense with a reward for coperation (3 years each).

Prisoner's Dilemma: Benefits?

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The Prisoner's Dilemma models a situation in which

- there is a gain from cooperation,
- but each player has an incentive to free ride.

Prisoner's Dilemma: rules and preferences

Rules

- Players $N = \{\text{Suspect 1}, \text{Suspect 2}\}$.
- Actions $A_1 = A_2 = \{\text{Quiet}, \text{Fink}\}$.
- Action profiles $A = A_1 \times A_2 =$
  $\{(\text{Quiet}, \text{Quiet}), (\text{Quiet}, \text{Fink}), (\text{Fink}, \text{Quiet}), (\text{Fink}, \text{Fink})\}$

Preferences

- Player 1

  $$(\text{Fink}, \text{Quiet}) \preceq_1 (\text{Quiet}, \text{Quiet}) \preceq_1 (\text{Fink}, \text{Fink}) \preceq_1 (\text{Quiet}, \text{Fink})$$

- Player 2

  $$(\text{Quiet}, \text{Fink}), \preceq_2 (\text{Quiet}, \text{Quiet}) \preceq_2 (\text{Fink}, \text{Fink}) \preceq_2 (\text{Fink}, \text{Quiet})$$

Prisoner's Dilemma: rules and utilities

Rules

- Players $N = \{\text{Suspect 1}, \text{Suspect 2}\}$.
- Actions $A_1 = A_2 = \{\text{Quiet}, \text{Fink}\}$.
- Action profiles $A = A_1 \times A_2$
  $\{(\text{Quiet}, \text{Quiet}), (\text{Quiet}, \text{Fink}), (\text{Fink}, \text{Quiet}), (\text{Fink}, \text{Fink})\}$

profile	u_1	u_2
(Fink, Quiet)	3	0
(Quiet, Quiet)	2	2
(Fink, Fink)	1	1
(Quiet, Fink)	0	3

Prisoner's Dilemma: rules and utilities

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Rationality: Players choose actions in order to maximize personal utility (minimize cost)

Prisoner's Dilemma: rules and costs

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- Players $N = \{\text{Suspect } 1, \text{Suspect } 2\}$.
- Actions $A_1 = A_2 = \{\text{Quiet}, \text{Fink}\}$.
- Action profiles $A = A_1 \times A_2$
  $\{(\text{Quiet}, \text{Quiet}), (\text{Quiet}, \text{Fink}), (\text{Fink}, \text{Quiet}), (\text{Fink}, \text{Fink})\}$

profile	c_1	c_2
(Fink, Quiet)	0	3
(Quiet, Quiet)	1	1
(Fink, Fink)	2	2
(Quiet, Fink)	3	0

Prisoner's Dilemma: rules and costs

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(Quiet, Fink)	3	0

Rationality: Players choose actions in order to minimize personal cost

Prisoner's Dilemma: bi-matrix representation

We can represent the game in a compact way on a bi-matrix.

utility	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

cost	Quiet	Fink
Quiet	1,1	3,0
Fink	0,3	2,2

Example: Matching Pennies

The story

- Two people choose, simultaneously, whether to show the head or tail of a coin.
- If they show same side, person 2 pays person 1 1eur, otherwise person 1 pays person 2 1eur.
- Payoff are equal to the amounts of money involved.

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This is an example of a zero-sum game

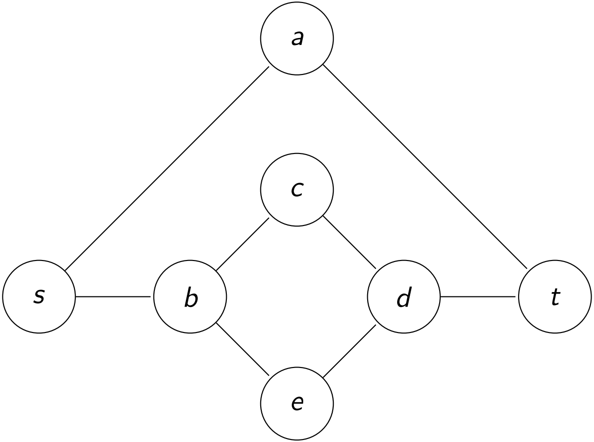
Example: Sending from s to t

The story

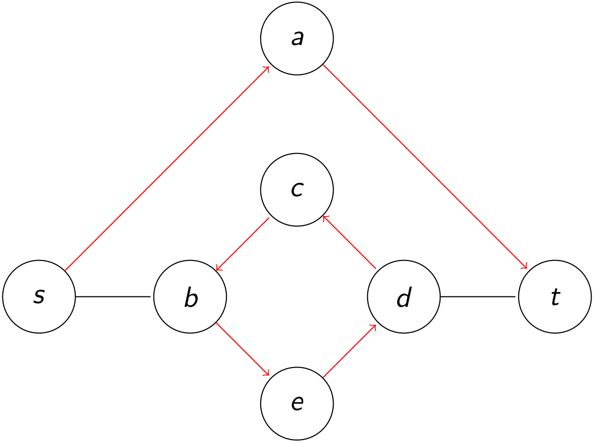
- We have a graph $G = (V, E)$ and two vertices $s, t \in V$.
- There is one player for each vertex $v \in V$, $v \neq t$.
- The set of actions for player u is $N_G(u)$.
- A strategy profile is a set of vertices $(v_1, \ldots, v_{n-1})$.
- Pay-offs are defined as follows:
  player u gets 1 if the shortest path joining s to t in the digraph induced by $v_1, \ldots, v_{n-1}$ contains (u, v_u), otherwise gets 0.

Players are selfish but the system can get a different reward/cost. For example the cost of the shortest path.

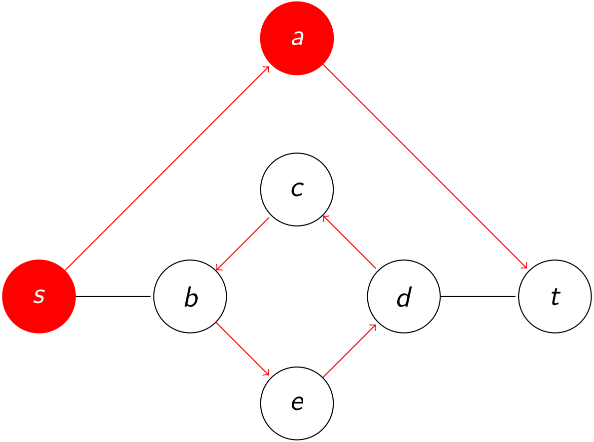
Sending from s to t: example



Sending from s to t: strategy profile (1)

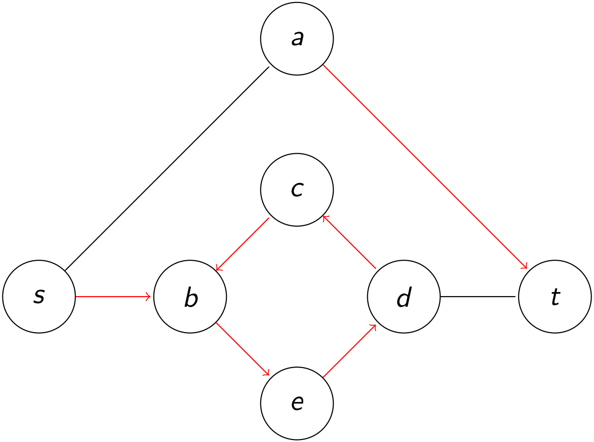


Sending from s to t: pay-offs (1)

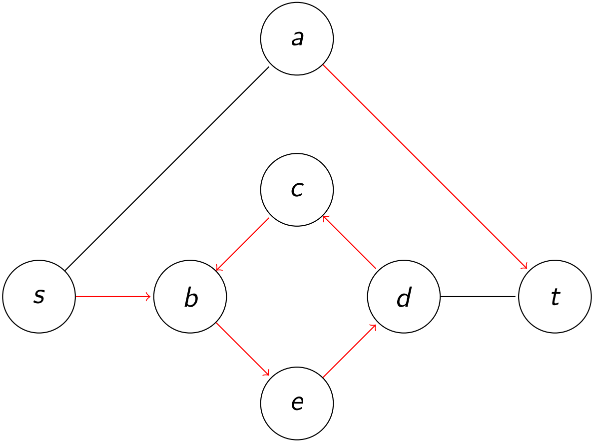


Red nodes get pay-off 1, other nodes get pay-off 0.

Sending from s to t: strategy profile (2)



Sending from s to t: strategy profile (2)



All nodes get pay-off 0.

Strategies: Notation

A strategy of player $i \in N$ in a strategic game Γ is an action $a_i \in A_i$.
A strategy profile $s = (s_1, \ldots, s_n)$ consists of a strategy for each player.

For each $s = (s_1, \ldots s_n)$ and $s_i' \in A_i$ we denote by

$$(s_{-i}, s_i') = (s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$
is not an strategy profile but can be seen as an strategy for the other players.

Best response

Let Γ be an strategic game defined trhough pay-off functions
The set of best responses for player i to s_{-i} is

$$BR(s_{-i}) = \{a_i \in A_i \mid u_i(s_{-i}, a_i) = \max_{a_i' \in A_i} u_i(s_{-i}, a_i')\}$$

Those are the actions that give maximum pay-off provided the other players do not change their strategies.

Solution concepts

- Pure Nash equilibrium
- (Mixed) Nash equilibrium
- Dominant strategies
- Strong Nash equilibrium
- Correlated equilibrium

Dominant strategies

A dominant strategy for player i is an strategy s_i^* if regardless of what other players do the outcome is better for player i.
Formally, for every strategy profile $s = (s_1, \dots, s_n)$,
$u_i(s) \leqslant u_i(s_{-i}, s_i^*)$.

Pure Nash equilibrium

A pure Nash equilibrium is an strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ such that
no player i can do better choosing an action different from s_i^*, given that every other player j adheres to s_j^*:

> *for every player i and for every action $a_i \in A_i$ it holds $u_i(s_{-i}^*, s_i^*) \geqslant u_i(s_{-i}^*, a_i)$.*
>
> *Equivalently, for every player i and for every action $a_i \in A_i$ it holds $s_i^* \in BR(s_{-i}^*)$.*

Pure Nash Equilibrium

- Is a strategy profile in which all players are happy.
- Identified with a fixed point of an iterative process of computing a best response.
- However, the game is played only once!
- GT deals with the existence and analysis of equilibria assuming rational behavior.
  players try to maximize their benefit
- GT does not provide algorithmic tools for computing such equilibrium if one exists.

More games

utility	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

utility	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

utility	swerve	don't sw
swerve	3,3	2,4
don't sw	4,2	1,1

Dominant strategies? Nash equilibria?

Examples of Nash equilibrium

- Prisoner's Dilemma, (Fink, Fink).
- Bach or Stravinsky, (Bach, Bach), (Stravinsky, Stravinsky).
- Matching Pennies, none.
- Chicken, (swerve, don't sw), (don't sw, swerve).

Example: Sending from s to t

The story

- We have a graph $G = (V, E)$ and two vertices $s, t \in V$.
- There is one player for each vertex $v \in V$, $v \neq t$.
- The set of actions for player u is $N_G(u)$.
- A strategy profile is a set of vertices $(v_1, \ldots, v_{n-1})$.
- Pay-offs are defined as follows:
  player u gets 1 if the shortest path joining s to t in the digraph induced by $v_1, \ldots, v_{n-1}$ contains (u, v_u), otherwise gets 0.

Exercise: Dominant strategies? Nash equilibria?

Pure Nash equilibrium

- First notion of equilibrium for non-cooperative games.
- There are strategic games with no pure Nash equilibrium.
- There are games with more than one pure Nash equilibrium.
- How to compute a Nash equilibrium if there is one?

Mixed strategies

Until now players were selecting as strategy an action.

A mixed strategy for player i is a distribution (lottery) σ_i on the set of actions A_i.

The utility function for player i is the expected utility under the joint distribution $\sigma = (\sigma_1, \dots, \sigma_n)$ assuming independence.

$$U_i(\sigma) = \sum_{(a_1,\dots,a_n)\in A} \sigma_1(a_1)\cdots\sigma_n(a_n)u_i(a_1\dots, a_n)$$

Mixed Nash equilibrium

A mixed Nash equilibrium is a profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ such that no player i can get better utility choosing a distribution different from σ_i^*, given that every other player j adheres to σ_j^*.

Theorem (Nash)

Every strategic game has a mixed Nash equilibrium.

From a computational point of view, mixed strategies present an additional representation problem.
In CS we can store only rational numbers. It is known

- For two player game there are always a mixed Nash equilibrium with rational probabilities.
- There are three player games without rational mixed Nash equilibrium.

[Schoenebeck and Vadhan: eccc 51, 2005]

NE in the Matching pennies game

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

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- We know that the game has no PNE

NE in the Matching pennies game

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- We know that the game has no PNE
- Is $((.2, .8), (.4, .6))$ a NE?
- Is $((.5, .5), (.5, .5))$ a NE?

Checking for a Nash equilibrium

Given a distribution σ_i on A_i define the support of σ_i to be the set

$$\mathrm{supp}(\sigma) = \{a_i \mid \sigma_i(a_i) \neq 0\}$$

Theorem

A mixed strategy profile σ is a Nash equilibrium iff, for any player i and any action $a_i \in supp(\sigma)$, a_i is a best response to σ_{-i}

Basic problems

Is (pure) Nash (ISN/ISPN)
Given a strategic game Γ and a mixed (pure) strategy profile s, decide whether s is a Nash equilibrium of Γ.

Exists pure Nash? (EPN)
Given a strategic game Γ, decide whether Γ has a Pure Nash equilibrium.

Compute (pure) Nash (CN,CPN)
Given a strategic game Γ, compute a (pure) Nash equilibrium (if it exists).

Congestion games

Congestion games

A congestion game

- is defined on a finite set E of resources and
- has n players
- using a delay function d mapping $E \times \mathbb{N}$ to the integers.
- The actions for each player are subsets of E.
- The pay-off functions are the following:

$$u_i(a_1, \dots, a_n) = -\left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e))\right)$$

being $f(a_1, \dots, a_n, e) = |\{i \mid e \in a_i\}|$.

Network congestion games

Network congestion games

A network congestion game

- is defined on a directed graph $G = (V, E)$ resources are the edges
- has n players
- using a delay function d mapping $E \times \mathbb{N}$ to the integers.
- The actions for each player are paths from s_i to t_i, for some $s_i, t_i \in V(G)$.
- The pay-off functions are the following:

$$u_i(a_1, \dots, a_n) = -\left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e))\right)$$

being $f(a_1, \dots, a_n, e) = |\{i \mid e \in a_i\}|$.