Data-Driven Nests in Discrete Choice Models

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Motivation

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Models of discrete choice are the workhorse in demand estimation with random utility.

If idiosyncratic shocks are $\sim$ Type I EV $\implies$ Multinomial logit:

- Closed form solutions of choice probability.
- Low number of parameters.
- Generates unrealistic substitution patterns (more on this later).

Motivation (cont)

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  - Flexible substitution patterns.
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  - Distributional assumptions on heterogeneity.

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- Random Coefficients (RC): Logit with heterogeneity in preferences across consumers
  - Flexible substitution patterns.
  - Computationally expensive: non-linear optimization, no closed-form demand.
  - Distributional assumptions on heterogeneity.
- Nested Logit (NL): Natural extension of Multinomial Logit
  - Closed form solutions, parsimony (linear IV regression), interpretability.
  - Less flexible substitution patterns.
  - Nests need to be specified ex-ante.

This Paper: Estimate Groups in Nested Logit

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Monte Carlo Simulations and Empirical Application: US Automobile Sale Data.

Related Literature

- **Discrete Choice Models:** McFadden (1978), Cardell (1997), Kovach & Tserenjigmid (2020)
- **Empirical Models with Nesting Structures:** Goldberg (1995), Einav (2007), Grennan (2013), Ciliberto & Williams (2014), Conlon & Rao (2016), Miller & Weinberg (2017)
- **Group Fixed Effect Estimator:** Han & Moon (2010), Bonhomme & Manresa (2015)
- **Alternative Grouping Structure:** Fosgerau, Monardo, & De Palma (2021), Hortacsu, Lieber, Monardo & de Paula (ongoing)

Preview of Results

Monte Carlo simulations:

- Correctly match $\sim$ 90% with their true groups with only 10 markets.
- When number of markets increase to 100, match $\sim$ 100% products.
- Biases in parameters also decrease as number of markets increases, going from from 11% to 3%.

BLP Application:

- Seven groups with separation in price, car characteristics, and market trends.
- Parameters consistent with utility maximization.
- Different groups show different substitution patters
- Range from very low to very high levels of substitution

Consumer choice model

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In Nested Logit with groups $B_1, \ldots, B_K$, when errors $(\varepsilon_{i1}, \ldots, \varepsilon_{iJ})$ have cumulative distribution:

$$\sim \exp\Big(- \sum_{k=1}^{K} \big(\sum_{j \in B_K} e^{-\frac{\epsilon_j}{\sigma^{k(j)}}} \big)^{\sigma^{k(j)}} \Big),$$

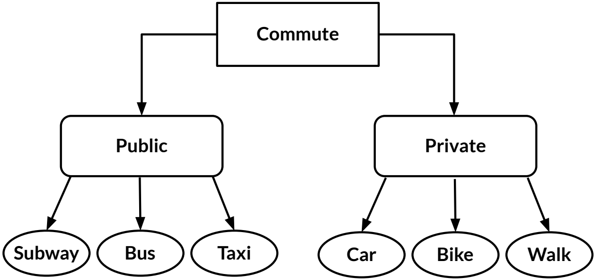
choice probabilities are given by:

$$\mathbb{P}_j = \frac{e^{\frac{\delta_j}{\sigma^{k(j)}}} \Big(\sum_{d \in B_k} e^{\frac{\delta_d}{\sigma^{k(d)}}} \Big)^{\sigma^{k(j)}-1}}{\sum_{l=1}^{K} \Big(\sum_{d \in B_{l(j)}} e^{\frac{\delta_d}{\sigma^{l(d)}}} \Big)^{\sigma^l}}$$

Nested Logit as Sequential Choice

Choose nest, then alternative within nest:

$$\mathbb{P}_j = \frac{e^{\frac{\delta_j}{\sigma^{k(j)}}}}{\sum_{d \in B_k} e^{\frac{\delta_d}{\sigma^{k(d)}}}} \frac{\left(\sum_{d \in B_k} e^{\frac{\delta_d}{\sigma^{k(d)}}}\right)^{\sigma^{k(j)}}}{\sum_{l=1}^{K}\left(\sum_{d \in B_l} e^{\frac{\delta_d}{\sigma^{l(d)}}}\right)^{\sigma^l}} = \mathbb{P}_{j|k(j)}\mathbb{P}_{k(j)}$$



Nested Logit: Substitution Patterns

Elasticities:

- For Logit:

$$E_j^q = -\mathbb{P}_q \frac{\partial \delta_q}{\partial p_q} p_q$$

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- For Nested Logit:

$$E_j^q = \begin{cases} -\mathbb{P}_q \frac{\partial \delta_q}{\partial p_q} p_q & \text{if } q \in B_{k'} \neq B_k \\ (\sigma^{k(j)} - 1)\mathbb{P}_{q|k(j)} \frac{\partial \delta_q}{\partial p_q} p_q - \mathbb{P}_q \frac{\partial \delta_q}{\partial p_q} p_q & \text{if } q \in B_k \end{cases}$$

$\Longrightarrow$ Elasticity Multinomial Logit $\leq$ Elasticity Nested Logit (within nest).

$\Longrightarrow$ Products within same nest closer substitutes than across nests.

$\Longrightarrow$ More substitution as σ^k decreases.

$\Longrightarrow$ $(1 - \sigma^{k(j)}) \in [0, 1]$ can be interpreted as correlation within nest.

Blue Bus - Red Bus

Identification

Identification: Groups

Recall

$$\mathbb{P}_j = \frac{e^{\frac{\delta_j}{\sigma^{k(j)}}} \left(\sum_{d \in B_k} e^{\frac{\delta_d}{\sigma^{k(d)}}} \right)^{\sigma^{k(j)} - 1}}{\sum_{l=1}^{K} \left(\sum_{d \in B_l} e^{\frac{\delta_d}{\sigma^{l(d)}}} \right)^{\sigma^l}}$$

Let

$$\text{IV}^k \equiv \sum_{d \in B_k} e^{\frac{\delta_d}{\sigma^{k(d)}}} \quad \text{and} \quad \text{IV} \equiv \sum_{l=1}^{K} \left(\sum_{d \in B_l} e^{\frac{\delta_d}{\sigma^{k(d)}}} \right)^{\sigma^l}$$

Applying logs to choice probabilities

$$\log \mathbb{P}_j = \frac{\delta_j}{\sigma^{k(j)}} + (\sigma^{k(j)} - 1) \log \text{IV}^{k(j)} - \log \text{IV}$$

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Applying logs to choice probabilities

$$\log \mathbb{P}_j = \frac{\delta_j}{\sigma^{k(j)}} + (\sigma^{k(j)} - 1) \log \mathrm{IV}^{k(j)} - \log \mathrm{IV}$$

For simplicity, we assume $\delta_0 = 0$ and $k(0) = \{0\}$ as in Berry (1994). It follows:

$$\log \frac{\mathbb{P}_j}{\mathbb{P}_0} = \frac{\delta_j}{\sigma^{k(j)}} + (\sigma^{k(j)} - 1) \log \mathrm{IV}^{k(j)}$$

Identification: Groups

For the ease of exposition, assume linear utility in one observable component:

$$\delta_j = \beta x_j$$

Substituting inside choice probabilities

$$\log \frac{\mathbb{P}_j}{\mathbb{P}_0} = \frac{\beta}{\sigma^{k(j)}} x_j + (\sigma^{k(j)} - 1) \log IV^{k(j)}$$

Denote

$$\beta^{k(j)} = \frac{\beta}{\sigma^{k(j)}} \quad \text{and} \quad \lambda^{k(j)} = (\sigma^{k(j)} - 1) \log IV^{k(j)}$$

We obtain the following equation:

$$\log \frac{\mathbb{P}_j}{\mathbb{P}_0} = \beta^{k(j)} x_j + \lambda^{k(j)}$$

$\implies \beta^k$ and λ^k common to all products within the same nest!

Identification: Groups

$$\log \frac{\mathbb{P}_j}{\mathbb{P}_0} = \beta^{k(j)} \mathbf{x}_j + \lambda^{k(j)}$$

Intuition: Assume products j and j' have same x's

1. If $\log \frac{\mathbb{P}_j}{\mathbb{P}_0}$ is equal to $\log \frac{\mathbb{P}_j'}{\mathbb{P}_0} \implies j$ and j' are in the same group.
2. If $\log \frac{\mathbb{P}_j}{\mathbb{P}_0}$ is different from $\log \frac{\mathbb{P}_j'}{\mathbb{P}_0} \implies j$ and j' are **not** in the same group.

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2. If $\log \frac{\mathbb{P}_j}{\mathbb{P}_0}$ is different from $\log \frac{\mathbb{P}'_j}{\mathbb{P}_0} \implies j$ and j' are **not** in the same group.

Remarks:

1. We think of $\beta^{k(j)}$ and $\lambda^{k(j)}$ as group-specific slope and intercept in a regression equation.
2. Identification of the groups is hence obtained without fully imposing the structure of the model.

Identification: Structural Parameteres

The first step not only recovers groups but also $\{\beta^1, \ldots, \beta^K, \lambda^1, \ldots, \lambda^K\}$.

Recall

$$\log \frac{\mathbb{P}_j}{\mathbb{P}_0} = \frac{\delta_j}{\sigma^{k(j)}} + \lambda^{k(j)}, \qquad \lambda^k = (\sigma^k - 1) \log IV_k = (\sigma^k - 1) \log \sum_{j \in B_k} e^{\frac{\delta_j}{\sigma^k}} \qquad \text{and} \qquad \beta^k = \frac{\beta}{\sigma^k}$$

Then, σ^k and β are jointly identified from the following equations:

$$\lambda^k = \frac{\sigma^k - 1}{\sigma^k} \log\Big(\sum_{j \in B_k} \frac{\mathbb{P}_j}{\mathbb{P}_0} \Big) \qquad \text{and} \qquad \beta^k = \frac{\beta}{\sigma^k}$$

Estimation

Relevant Empirical Models

In what follows, we assume that covariates x are exogenous but allow for endogeneity of prices p.

We consider two different models of indirect utility:

1. A panel data framework with product fixed effects:

$$\delta_{jm} = \beta_p p_{jm} + x_{jm}\beta_x + \xi_j + \nu_{jm},$$

   where

$$\mathbb{E}[\nu_{jm}|p_{j1}, \ldots, p_{jM}, x_{j1}, \ldots, x_{jM}, \lambda_1^1, \ldots, \lambda_M^K, \xi_j] = 0$$

2. Panel data with exogenous shifters:

$$\delta_{jm} = \beta_p p_{jm} + x_{jm}\beta_x + \nu_{jm},$$

   where there exists z_{jm} such that

$$\mathbb{E}[\nu_{jm}|x_{jm}, z_{jm}, \lambda_1^1, \ldots, \lambda_M^K] = 0$$

Outline

Panel Data: Estimation

Recall, indirect utility is defined as:

$$\delta_{jm} = \beta_p p_{jm} + x_{jm}\beta_x + \xi_j + \nu_{jm},$$

where

$$\mathbb{E}[\nu_{jm}|p_{j1},\ldots,p_{jM},x_{j1},\ldots,x_{jM},\lambda_1^1,\ldots,\lambda_M^K,\xi_j] = 0$$

Therefore,

$$\log\frac{\mathbb{P}_{jm}}{\mathbb{P}_{0m}} = \frac{\beta_p p_{jm} + x_{jm}\beta_x + \xi_j + \nu_{jm}}{\sigma^{k(j)}} + \lambda^{k(j),m} = \beta_p^{k(j)} p_{jm} + x_{jm}\beta_x^{k(j)} + \tilde{\xi}_j + \tilde{\nu}_{jm} + \lambda_m^{k(j)},$$

where $\tilde{\xi}_j = \frac{\xi_j}{\sigma^{k(j)}}$ and $\tilde{\nu}_{jm} = \frac{\nu_{jm}}{\sigma^{k(j)}}$

We demean data to remove fixed effects $\tilde{\xi}_j$

$$\overline{\log\frac{\mathbb{P}_{jm}}{\mathbb{P}_{0m}}} = \beta_p^{k(j)}\bar{p}_{jm} + \bar{x}_{jm}\beta_x^{k(j)} + \bar{\lambda}_m^{k(j)} + \bar{\nu}_{jm},$$

where $\bar{y}$ indicates demeaned variables.

First Step: Classification Algorithm

We propose the following classification algorithm based on Bonhomme and Manresa (2015):

1. Let $(\beta^{1,0}, \ldots, \beta^{K,0}, \lambda_1^{K,0}, \ldots, \lambda_M^{K,0})$ be a starting value.

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2. For $(\beta^{1,s}, \ldots, \beta^{K,s}, \lambda_1^{K,s}, \ldots, \lambda_M^{K,s})$, compute for all $j \in J$:

$$k(j)^{s+1} = \underset{k\in\{1,\ldots,K\}}{\arg\min} \sum_{m=1}^{M} \left(\overline{\log \frac{\mathbb{P}_{jm}}{\mathbb{P}_{0m}}} - \left(\bar{x}_{jm}\beta^{k,s} + \lambda_m^{k,s}\right)\right)^2,$$

to recover grouping structure $\mathcal{B}^{s+1}$.

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3. Compute:

$$(\beta^{1,s+1}, \ldots, \beta^{K,s+1}, \lambda_1^{K,s+1}, \ldots, \lambda_M^{K,s+1}) =$$

$$\underset{\beta^1,\ldots,\beta^K,\lambda_1^1,\ldots,\lambda_M^K}{\arg\min} \sum_{j=1}^{J} \sum_{m=1}^{M} \left(\overline{\log \frac{\mathbb{P}_{jm}}{\mathbb{P}_{0m}}} - \left(\bar{x}_{jm}\beta^{k(j),s+1} + \lambda_m^{k(j),s+1}\right)\right)^2$$

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4. Repeat until convergence of parameters.

Second Step: Linear Regression

Based on the estimated classification from the first step, we follow Berry (1994).

Under normalization $\delta_0 = 0$ and $k(0) = \{0\}$

$$\log \frac{\mathbb{P}_{jm}}{\mathbb{P}_0} = \delta_{jm} + (1 - \sigma^{k(j)}) \log \mathbb{P}_{jm|k(j)} + \nu_{jm}$$

Substituting the expression for δ_j:

$$\log \frac{\mathbb{P}_{jm}}{\mathbb{P}_{0m}} = \beta_p p_{jm} + x_{jm}\beta + (1 - \sigma^{k(j)}) \log \mathbb{P}_{jm|k(j)} + \nu_{jm}$$

Linear regression equation on x_{jm} and $\log \mathbb{P}_{jm|k(j)}$:

- Construct $\mathbb{P}_{j|k(j)}$ based on estimated groups from first step $\{\hat{B}_k\}_{k=1}^{K}$:

$$\hat{\mathbb{P}}_{jm|k} = \frac{\mathbb{P}_{jm}}{\sum_{j \in \hat{B}_k} \mathbb{P}_j}$$

- Simultaneity problem: $\Longrightarrow$ Instrument $\hat{\mathbb{P}}_{jm|k(j)}$ using second order moments of exogenous x_{jm}.

Outline

Control Function Approach: Estimation

Assume indirect utility model is described as:

$$\delta_{jm} = \beta_p p_{jm} + x_{jm}\beta_x + \nu_{jm}$$

If $\mathbb{E}[\nu_{jm} p_{jm}] \neq 0$, the algorithm outlined before does not consistently recover the groups.

To overcome this issue, we use a Control Function Approach defined in Petrin and Train (2010).

We require the existence of z_{jm} such that

$$\mathbb{E}[\nu_{jm} | x_{jm}, z_{jm}, \lambda_1^1, \ldots, \lambda_M^K] = 0.$$

Control Function Approach

Concretely, there is an unobservable confounder μ_{jm} such that:

$$p_{jm} = f(x_{jm}, z_{jm}, \mu_{jm}) \qquad \text{and} \qquad \nu_{jm} = g(\mu_{jm}, \varepsilon_{jm}),$$

for which we assume that

$$p_{jm} \perp\!\!\!\perp \nu_{jm} | \mu_{jm}.$$

For simplicity we also assume:

$$p_{jm} = f(x_{jm}, z_{jm}; \gamma) + \mu_{jm} \qquad \text{and} \qquad \nu_{jm} = CF(\mu_{jm}) + \varepsilon_{jm}$$

Include $CF(\mu_{jm})$ as part of indirect utility :

$$\begin{aligned}
\delta_{jm} &= \beta_p p_{jm} + x_{jj}\beta_x + \nu_{jm} \\
&= \beta_p p_{jm} + x_{jm}\beta_x + CF(\mu_{jm}) + \varepsilon_{jm},
\end{aligned}$$

with $\mathbb{E}[\varepsilon_{jm} | p_{jm}, x_{jm}, \mu_{jm}] = 0$.

Control Function Approach (cont)

Substituting δ_{jm} inside choice probabilities:

$$\log \frac{\mathbb{P}_{jm}}{\mathbb{P}_{0m}} = \beta_p^{k(j)} p_{jm} + x_j \beta_x^{k(j)} + \widetilde{CF}(\mu_{jm}) + \tilde{\lambda}_m^{k(j)} + \tilde{\varepsilon}_{jm},$$

which is a known expression with p_{jm}, x_{jm} and μ_{jm} as observable covariates.

This expression motivates the following steps:

1. Project p_{jm} on exogenous variables (z_{jm}, x_{jm}) to estimate μ_{jm}

$$\hat{\mu}_{jm} = p_{jm} - \hat{f}(x_{jm}, z_{jm})$$

2. Include $\hat{\mu}_{jm}$ in our classification algorithm as a control for the confounder between ν_{jm} and p_{jm}.
3. Follow step 2 as if the groups are known.

Statistical Properties

Statistical Properties: First Step

- Let us consider the following simplified model:

$$\log \mathbb{P}_{jm} - \log \mathbb{P}_{0m} = \beta_p^{k(j)} p_{jm} + x_{jm}\beta_x^{k(j)} + \lambda_m^{k(j)} + \nu_{jm}$$

with $\mathbb{E}[\nu_{jm}|p_{j1}, \ldots, p_{jM}, x_{j1}, \ldots, x_{jM}, \lambda_1^1, \ldots \lambda_M^K] = 0$.

- Build upon results in Bonhomme and Manresa (2015).
- Work in progress: allow for product fixed effects and projection of prices.

Two key assumptions

- **Group separation**. For simplicity, assume simplest model:

$$\log \frac{\mathbb{P}_{jm}}{\mathbb{P}_{0m}} = \lambda^k(j) + \nu_{jm}, \qquad \text{with } k \in \{1,2\},\ \lambda^2 > \lambda^1,\ \nu_{jm} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$$

It follows

$$\mathbb{P}(\hat{k}(j) = 2 | k(j) = 1) = \mathbb{P}\Big(\sum_{m=1}^{M} (\lambda^1 + \nu_{jm} - \lambda^2)^2 < \sum_{m=1}^{M} (\lambda^1 + \nu_{jm} - \lambda^1)^2 \Big) = \mathbb{P}(\bar{\nu}_j > \lambda^2 - \lambda^1) = 1 - \Phi\Big(\sqrt{M}\big(\frac{\lambda^2 - \lambda^1}{2}\big)\Big)$$

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- **Rank condition**: Variation in x at the intersection of any group with true groups $\implies$ separate β^k from λ^k

Super consistency

- It can be shown:
$$\mathbb{P}\Big(\sup_{j \in \{1,2,\ldots,J\}} |\widehat{k}(j) - k(j)| > 0 \Big) = o(1) + o(JM^{-\delta})$$
for any $\delta > 0$, as J and M go to infinity.
- Both J and M grow to infinity, but M can grow at a much lower rate!
- "Super consistency" of group estimation $\implies$ standard inference in the second step.

Choosing the Number of Groups

Choosing K: Cross-Validation with Elbow Method

So far we have assumed the number of groups is known.

In practice, we can also estimate the number of groups using a *N*-fold cross-validation procedure.

For all $k \in \mathcal{K}$:

- Divide products into n equal parts, $P_1, \ldots, P_N$.
- Fix one part P_n and estimate grouping structure and grouping parameters in the other $N - 1$ parts.
- Classify products across estimated groups in part P_n and compute out-of-sample MSE

$$MSE_n(k) = \frac{1}{J \cdot M} \sum_{m=1}^{M} \sum_{j \in P_n} (y_j - \beta_{m,-n}^{k(j)} x_j - \lambda_{m,-n}^{k(j)})^2$$

- Take average across N folds:

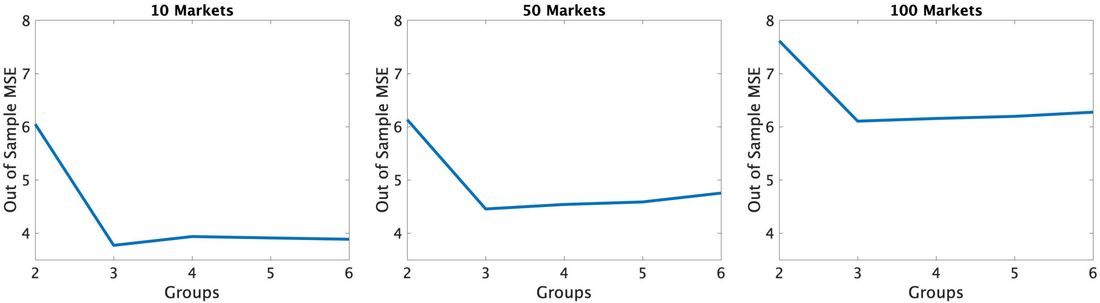
$$MSE(k) = \frac{1}{N} \sum_{n=1}^{N} MSE_n(k)$$

- Choose k according to

$$k^* = \{k(j) | \text{where slope of } MSE(k) \text{ changes}\}$$

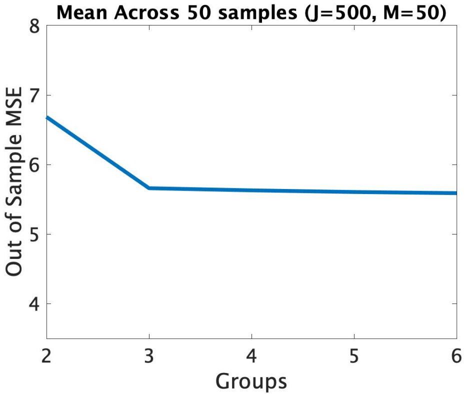
Cross Validation: Results

groups = 3, # products = 100, # folds = 5



Cross Validation: Monte Carlo

groups = 3, # folds = 5, # MC samples = 50



Elbow is at $G = 3$ across all samples, but minimum only once.

Monte Carlo

Monte Carlo Design: Nest Structure

Fix the number of groups K and classify products randonmly $k(j) \sim \mathcal{U}\{1, \ldots, K\}$.

We fix the vector of $(\sigma^1, \ldots, \sigma^K) \in [0, 1]^K$.

Monte Carlo Design: Data

Indirect utility δ_{jm} is given by

$$\delta_{jm} = \beta_p p_{jm} + \beta_1 x_{jm,1} + \beta_2 x_{jm,2} + \xi_j + \nu_{jm},$$

where $p_{jm,1}$ are prices and $(x_{jm,1}, x_{jm,2})$ are exogenous covariates. We set:

- $p_{jm,1} = \tilde{p}_{jm} + \xi_{j,p}$, with:
  - $\tilde{p}_{jm,1} \overset{i.i.d.}{\sim} \mathcal{N}\Big(k(j) \cdot \arctan(m+1), 1\Big)$
  - $\begin{bmatrix} \xi_{j,p} \\ \xi_j \end{bmatrix} \sim \mathcal{N}\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}\Big)$
- $x_{jm,1}, x_{jm,2} \overset{i.i.d.}{\sim} \mathcal{N}\Big(k(j) \cdot (-1)^{k(j)} \cdot \arctan(m+1), 1\Big)$
- $\mathbb{E}[\nu_{jm} | p_{j1}, x_{j1,1}, x_{j1,2}, \ldots, p_{jM}, x_{jM,1}, x_{jM,2}, \xi_j] = 0 \quad$ with $\quad \nu_{jm} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$
- $\beta_p = -1 \quad$ and $\quad \beta_1 = \beta_2 = 1$

Monte Carlo Design: Outcome variables

We leverage the closed form solution of Nested Logit models.

Construct IV_m^k as follows:

$$IV_{k,m} = \Big(\sum_{d \in B_k} e^{\frac{\delta_{dm}}{\sigma^k}} \Big)$$

Finally, log probabilities are given by:

$$\log \mathbb{P}_{jm} - \log \mathbb{P}_{0m} = \frac{1}{\sigma^{k(j)}} \delta_{jm} + (\sigma^{k(j)} - 1) \log IV_m^{k(j)}$$

Results: K = 3, B = 500

				β_p	β_1	β_2	σ_1	σ_2	σ_3
Markets	Products	Matched	True	-1	1	1	0.3	0.5	0.7
10	100	0.911	Mean β	-0.898	0.896	0.894	0.254	0.425	0.627
			Std β	0.122	0.131	0.131	0.065	0.063	0.063
50	100	1.000	Mean β	-0.956	0.957	0.958	0.286	0.476	0.669
			Std β	0.058	0.059	0.059	0.030	0.031	0.030
100	100	1.000	Mean β	-0.971	0.970	0.971	0.291	0.485	0.679
			Std β	0.046	0.046	0.046	0.024	0.024	0.024
10	500	0.879	Mean β	-0.912	0.904	0.903	0.264	0.429	0.629
			Std β	0.078	0.080	0.080	0.038	0.037	0.039
50	500	0.996	Mean β	-0.959	0.959	0.958	0.287	0.478	0.671
			Std β	0.047	0.047	0.047	0.024	0.024	0.024
100	500	1.000	Mean β	-0.967	0.967	0.967	0.290	0.483	0.677
			Std β	0.044	0.044	0.044	0.023	0.023	0.022
10	1000	0.870	Mean β	-0.903	0.898	0.897	0.267	0.427	0.625
			Std β	0.054	0.056	0.056	0.026	0.027	0.026
50	1000	0.988	Mean β	-0.963	0.963	0.963	0.289	0.478	0.673
			Std β	0.038	0.038	0.038	0.019	0.019	0.019

Application: US Automobile Data

US Automobile Data

We use US Automobile data from BLP (1995).[1]

Information on (essentially) all models marketed between 1971 and 1990.

Models both enter and exit over this period $\Longrightarrow$ unbalanced panel.

Total sample size is 2217 model/years representing 557 distinct models.

We set different years as different markets.

[1] We use data from the `R`-package `hdm` developed by Chernozhukov, Hansen & Spindler (2019)

Product Characteristics

Description of product characteristics:

- *log share*: log of market shares
- *price*: deflated price to 1983 dollars using CPI
- *mpd*: miles per dollar
- *air*: air conditioning
- *mpg*: miles per gallon rating
- *space*:size (measured as length times width)
- *hpwt*: the ratio of horsepower to weight (in HP per 10 lbs)

Panel Construction

Unbalanced panel of products:

- We consider an unbalanced panel of cars with:
  - At least five years of data.
  - At least three consecutive years.
- We are left with 82 products.
- We adapt our classification algorithm to allow for “missing data”:

  $\Longrightarrow$ Products can enter and exit over time.

  $\Longrightarrow$ Group of products can also enter and exit over time!

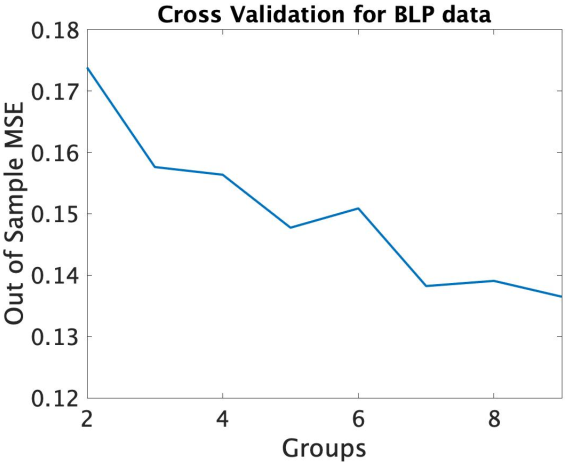
Statistics

Statistics of subsample of cars (N=82)

	Mean	Std. Dev.	Median	Min	Max	t-stat
Price	-.147	7.911	-2.532	-6.601	43.351	-1.06
Miles per Dollar	2.349	.513	2.376	1.352	3.805	2.78
AC	.299	.409	0	0	1	0.49
Miles per Gallon	2.214	.46	2.195	1.38	3.42	1.45
Space	1.266	.187	1.223	.951	1.711	0.13
Horse Power	.407	.069	.386	.308	.727	-0.23
Market Share	.001	.001	.001	0	.004	0.00
Yearly Observations	9.085	4.264	7	5	20	10.42
Year Entry	1980	5.261	1983	1971	1986	-4.62
Year Exit	1989	.88	1990	1988	1990	20.41

Full sample

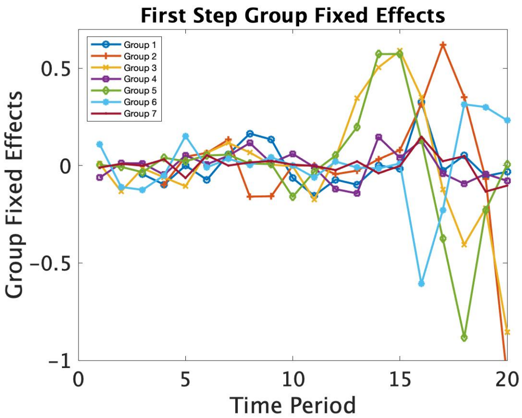
BLP Application: Choosing the number of groups

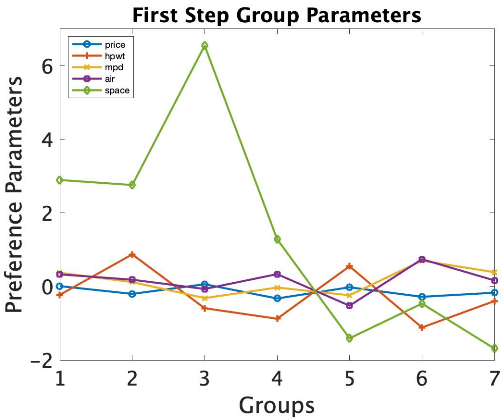


BLP Application: First-step Group Characteristics

K = 7	Mean	Std	1	2	3	4	5	6	7
Shares	0.001	0.001	0.002	0.001	0.001	0.001	0.001	0.001	0.002
Price	-0.741	6.893	-2.920	-2.776	-2.346	-0.590	0.150	0.161	1.212
HP	0.398	0.080	0.383	0.385	0.391	0.384	0.421	0.414	0.408
Miles per $	2.261	0.698	2.546	2.847	2.027	2.063	2.140	2.288	2.375
AC	0.277	0.447	0.227	0.239	0.214	0.283	0.340	0.419	0.236
Space	1.288	0.218	1.185	1.123	1.372	1.339	1.284	1.325	1.269
Type of car			Subcomp.	Compact Subcomp.	Mid-size	Compact Luxury	Sport	Mid-size Luxury	Luxury
# of Products	82		11	7	8	17	13	8	18

BLP Application: First Step Estimates





BLP Application: Second-step Results

Estimates Preference Parameters

	$\hat{\beta}$	$\sigma_{\hat{\beta}}$
Price	-0.086***	(0.033)
Horse Power	-0.347	(0.410)
Miles per $	0.039	(0.084)
AC	0.209	(0.130)
Space	0.425	(0.648)

Estimates Within-Nest Correlation

	Group						
	1	2	3	4	5	6	7
$\hat{\sigma}$	0.917***	0.840***	0.794***	0.363*	0.832***	0.772***	0.943***
$\sigma_{\hat{\sigma}}$	0.342	0.144	0.284	0.189	0.252	0.149	0.127

Conclusion

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- Monte Carlo simulations:
    - $\sim$ 90% match rate with only 10 and $\sim$ 100% with 100 markets.
    - Biases in preference parameters decrease as number of market increases.
- BLP application:
    - Seven groups with separation in prices, car characteristics, and market trends.
    - Wide range of substitution patterns, from very independent to highly correlated.

Appendix

IIA is not always realistic: Red-bus-Blue-bus problem

A traveler has a choice of commuting by **car** or taking a **blue bus**

Assume indirect utility from the two is the same so

$$\mathbb{P}_c = \mathbb{P}_{bb} = \frac{1}{2} \implies \frac{\mathbb{P}_c}{\mathbb{P}_{bb}} = 1$$

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Given IIA, $\frac{\mathbb{P}_c}{\mathbb{P}_{bb}} = 1$. The only consistent model with both is

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Is $\mathbb{P}_c = \mathbb{P}_{bb} = \mathbb{P}_{rb} = \frac{1}{3}$ realistic? Not really.

If **blue** and **red** only differ in color, we should expect

$$\mathbb{P}_c = \frac{1}{2} \qquad \mathbb{P}_{bb} = \mathbb{P}_{rb} = \frac{1}{4}$$

The ratio $\frac{\mathbb{P}_c}{\mathbb{P}_{bb}}$ should actually change with the introduction of the **red bus**!

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Statistics for Full Sample

Table 1: Average characteristics of all cars, (N = 557)

	Mean	Std. Dev.	Median	Min	Max	t-stat
Price	.862	8.983	-2.516	-8.368	43.351	1.06
Miles per Dollar	2.175	.641	2.094	1.055	6.437	-2.78
AC	.275	.424	0	0	1	-0.49
Miles per gallon	2.133	.552	2.07	1	5.3	-1.45
Space	1.263	.216	1.223	.79	1.888	-0.13
Horse Power	.409	.098	.385	.207	.888	0.23
Market Share	.001	.001	0	0	0.006	0.00
Yearly Observations	3.899	3.857	2	1	20	-10.42
Entry Year	1980	6.511	1981	1971	1990	4.62
Exit Year	1984	6.101	1986	1971	1990	-20.41

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