In the name of beauty
The 4th problem set solution of Optical Networks course

Question 1)
a. Given n distinct frequencies $f_1, f_2, \cdots, f_n$, the FWM frequencies may be produced through the relation $f_1 + f_2 - f_3$ when all f_1, f_2 and f_3 are not simultaneously equal. Picking up two distinct frequencies f_1 and f_2 from the set in $\binom{n}{2}$ ways, the possible FWM cases are

$$2f_1 - f_2$$
$$2f_2 - f_1$$

which yields $2\binom{n}{2}$ different cases. When all the three frequencies are different in $\binom{n}{3}$, we obtain the following cases for FWM:

$$f_1 + f_2 - f_3$$
$$f_1 + f_3 - f_2$$
$$f_3 + f_2 - f_1$$

with a total of $3\binom{n}{3}$ different cases. Summing up, leaves us with $2\binom{n}{2}+3\binom{n}{3} = \frac{n^2(n-1)}{2}$ total possible FWM frequency components.
b.

$$f_1 = 193.2THz \quad f_2 = 193.2THz \quad f_3 = 193.1THz \quad : \quad f_{\text{FWM}} = 193.3THz$$
$$f_1 = 193.2THz \quad f_2 = 193.2THz \quad f_3 = 193.0THz \quad : \quad f_{\text{FWM}} = 193.4THz$$
$$f_1 = 193.1THz \quad f_2 = 193.2THz \quad f_3 = 193.0THz \quad : \quad f_{\text{FWM}} = 193.3THz$$
$$f_1 = 193.1THz \quad f_2 = 193.1THz \quad f_3 = 193.2THz \quad : \quad f_{\text{FWM}} = 193.0THz$$
$$f_1 = 193.1THz \quad f_2 = 193.1THz \quad f_3 = 193.0THz \quad : \quad f_{\text{FWM}} = 193.2THz$$
$$f_1 = 193.0THz \quad f_2 = 193.2THz \quad f_3 = 193.1THz \quad : \quad f_{\text{FWM}} = 193.1THz$$
$$f_1 = 193.0THz \quad f_2 = 193.1THz \quad f_3 = 193.2THz \quad : \quad f_{\text{FWM}} = 192.9THz$$
$$f_1 = 193.0THz \quad f_2 = 193.0THz \quad f_3 = 193.2THz \quad : \quad f_{\text{FWM}} = 192.8THz$$
$$f_1 = 193.0THz \quad f_2 = 193.0THz \quad f_3 = 193.1THz \quad : \quad f_{\text{FWM}} = 192.9THz$$

Question 2)
a. By multiplying the first equation in A_x^*, the second one in A_y^* and considering their complex conjugates, we obtain four equations:

$$
\begin{aligned}
A_x^H \frac{\partial A_x}{\partial z} + \frac{\alpha}{2} A_x^H A_x - j\gamma P A_x^H A_x = 0 \\
A_x^T \frac{\partial A_x^*}{\partial z} + \frac{\alpha}{2} A_x^T A_x^* + j\gamma P A_x^T A_x^* = 0 \\
A_y^H \frac{\partial A_y}{\partial z} + \frac{\alpha}{2} A_y^H A_y - j\gamma P A_y^H A_y = 0 \\
A_y^T \frac{\partial A_y^*}{\partial z} + \frac{\alpha}{2} A_y^T A_y^* + j\gamma P A_y^T A_y^* = 0
\end{aligned}
$$

where $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian (transpose+complex conjugate) operators.

By summing up all the equations and substituting $P = |A_x|^2 + |A_y|^2$, the imaginary parts of the PDEs vanish and we finally obtain what we want:

$$
\frac{\partial P}{\partial z} = -\alpha P
$$

with the following solution:

$$
P = P(z,t) = P(0,t)e^{-\alpha z}
$$

b. The PDE can be re-written as

$$
\begin{aligned}
\frac{\partial A_x}{A_x \cdot \partial z} + \frac{\alpha}{2} - j\gamma P(0,t)e^{-\alpha z} = 0 \\
\frac{\partial A_y}{A_y \cdot \partial z} + \frac{\alpha}{2} - j\gamma P(0,t)e^{-\alpha z} = 0
\end{aligned}
$$

which by integration w.r.t. x and y respectively yields

$$
\begin{aligned}
\ln A_x + \frac{\alpha}{2} z + j\frac{\gamma}{\alpha} P(0,t)e^{-\alpha z} + C_1 = 0 \\
\ln A_y + \frac{\alpha}{2} z + j\frac{\gamma}{\alpha} P(0,t)e^{-\alpha z} + C_2 = 0
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
A_x = e^{-\frac{\alpha}{2}z - j\frac{\gamma}{\alpha}P(0,t)e^{-\alpha z} + C_1} \\
A_y = e^{-\frac{\alpha}{2}z - j\frac{\gamma}{\alpha}P(0,t)e^{-\alpha z} + C_2}
\end{aligned}
$$

Substituting $z = 0$ leads to

$$
\begin{aligned}
A_x(0,t) = e^{-j\frac{\gamma}{\alpha}P(0,t) + C_1} \\
A_y(0,t) = e^{-j\frac{\gamma}{\alpha}P(0,t) + C_2}
\end{aligned}
$$

By finding and replacing the constants C_1 and C_2, the result is immediately concluded $\blacksquare$.

Question 3)

The linear value of α is given by

$$\alpha_{\text{Linear}} = \frac{\alpha_{dB}}{4.343} = 4.61 \times 10^{-5} \frac{1}{\text{m}}$$

hence

$$L_{\text{eff}} = \frac{1 - \exp(-\alpha L)}{\alpha} = 14.85\text{km}$$

and by substituting, we obtain

$$\phi_{\text{NL}} = \gamma L_{\text{eff}} P = 0.59\text{rad} = 34.03°$$

Question 4)

a.

$$\gamma = 2.63 \frac{\text{W}^{-1}}{\text{km}}$$
$$L_{\text{eff}} = 18.27\text{km}$$
$$\phi_{\text{NL}} = \pi$$

therefore

$$P = 65.38\text{mW} \equiv 18.15\text{dBm}$$

b.

$$\gamma = 2.11 \frac{\text{W}^{-1}}{\text{km}}$$
$$L_{\text{eff}} = L$$
$$\phi_{\text{NL}} = 2\pi$$
$$P = 6\text{dBm} \equiv 3.98\text{mW}$$

which yield

$$L = 748\text{km}$$