

1 LTD

$$\frac{d_t}{1+R_t} = \beta E_t d_{t+1}$$

$$E_t d_{t+1} = \frac{1}{\beta P_t}$$

$$d_t = \frac{1+R_t}{P_t}$$

$$z_{t,t+1} = \frac{\beta P_t}{P_{t+1}} \frac{1+R_{t+1}}{1+R_t}$$

$$\frac{1+R_t}{P_t(1+R_{2t})} = \beta E_t \frac{1}{P_{t+1}}$$

$$\frac{1}{1+R_{2t}} = \beta E_t \left[\frac{d_{t+1}}{d_t (1+R_{t+1})} \right] = \frac{\beta}{1+R_t} E_t \left[\frac{P_t}{P_{t+1}} \right]$$

2 LTD

$$\frac{d_t}{1+R_{m,t}} = \beta E_t \frac{d_{t+1}}{1+R_{m-1,t+1}}$$

$$\frac{1}{1+R_{m,t}} = \frac{\beta}{\delta_t} E_t \left[\frac{d_{t+1}}{1+R_{m-1,t+1}} \right] = \frac{\beta P_t}{1+R_t} E_t \left[\frac{d_{t+1}}{1+R_{m-1,t+1}} \right]$$

$$= \frac{\beta P_t}{1+R_t} E_t \left[\frac{1+R_{t+1}}{P_{t+1}(1+R_{m-1,t+1})} \right]$$

$$= \frac{\beta P_t}{1+R_t} E_t \left[\frac{(1+R_{t+1})}{P_{t+1}} \frac{\beta P_{t+1}}{1+R_{t+1}} E_{t+1} \left[\frac{1+R_{t+2}}{P_{t+2}(1+R_{m-2,t+2})} \right] \right]$$

$$= \dots = \frac{\beta^{m-1} P_t}{1+R_t} E_t \left[E_{t+m-2} \left[\frac{1+R_{t+m-1}}{P_{t+m-1}(1+R_{1,t+m-1})} \right] \right]$$

$$= \frac{\beta^{m-1}}{1+R_t} E_t \left[\frac{P_t}{P_{t+m-1}} \right]$$

3ZTD

$$BC \text{ time } 0 \quad W_0 = \mu_0 + \frac{B_0}{1+R_0} + E_0[z_{0,1} A_1] + \frac{B_{2,0}}{1+R_{2,0}}$$

(time 1 multiplied by $z_{0,1}$, taking expected value as of time 0)

$$E_0 \left[z_{0,1} \left(B_0 + \mu_0 + P_0 (\gamma_0 - c_{1,0} - c_{2,0}) + A_1 + \frac{B_{2,0}}{1+R_1} - T_1 \right) \right]$$

$$= E_0[z_{0,1} W_1] = E_0 \left[z_{0,1} \left(\frac{B_1}{1+R_1} + \mu_1 + z_{1,2} A_2 + \frac{B_{2,1}}{1+R_{2,1}} \right) \right]$$

Substitute R for $E_0[z_{0,1} A_1]$ from BC time 1 into BC time 0

$$W_0 = \mu_0 \left[1 - E_0 z_{0,1} \right] + B_0 \left[\frac{1}{1+R_0} - E_0 z_{0,1} \right] + E_0 z_{0,1} T_1$$

$$+ B_{2,0} \left[\frac{1}{1+R_{2,0}} - E_0 \left[\frac{z_{0,1}}{1+R_1} \right] \right] +$$

$$+ E_0 \left[z_{0,1} \left(\frac{B_1}{1+R_1} + \mu_1 + z_{1,2} A_2 + \frac{B_{2,1}}{1+R_{2,1}} \right) \right]$$

No arbitrage requires $\frac{1}{1+R_0} = E_0 z_{0,1}$ and

$$E_0 z_{0,1} = \frac{1}{1+R_{2,0}}$$

LTD

Keep iterating on substituting $E_t z_{t+1} A_{t+1}$, imposing no arbitrage, and take limit imposing no Ponzi, get the same PVBC as with one-period debt, except for the definition of nominal wealth

Now, assume fiscal policy is such that

$$T_S = \bar{T} P_{S-1}, \quad s \neq T+1$$

$$\bar{T}_{T+1} = P_T (\bar{T} + \bar{\tau}_{T+1})$$

Realization revealed at $t < T$ and $E_S [\bar{T}_{T+1}] = 0$ for $s < t$

In a competitive equilibrium, compute

$$\sum_{v=s}^{\infty} E_S [z_{sv+1} T_{v+1}]$$

For period $s > T$, get

$$\sum_{v=s}^{\infty} E_S [z_{sv+1} T_{v+1}] = \bar{T} \sum_{v=s}^{\infty} E_S [z_{sv+1} P_v] =$$

$$\bar{T} \sum_{v=s}^{\infty} E_S \left[\frac{z_{sv} P_v}{1+r_v} \right] = \frac{\bar{T} P_s}{1+r_s} \sum_{v=s}^{\infty} \beta^{v-s} = \frac{\bar{T} P_s}{(1-\beta)(1+r_s)}$$

SLID

For period $s \in [t, T]$, get

$$\sum_{v=s}^{\infty} E_S[z_{S,V+1} T_{V+1}] = \frac{\bar{T} P_S}{(1-\beta)(1+r_S)} + E_S\left[\frac{z_{S,T}}{1+r_T} P_T \tilde{T}_{T+1}\right]$$
$$= \frac{\bar{T} P_S}{(1-\beta)(1+r_S)} + \beta \frac{\bar{T}_{T+1}^{T-s} P_S}{1+r_S}$$

For period $s < t$

$$\sum_{v=s}^{\infty} E_S[z_{S,V+1} T_{V+1}] = \frac{\bar{T} P_S}{(1-\beta)(1+r_S)} + E_S\left[\frac{z_{S,T}}{1+r_T} P_T \tilde{T}_{T+1}\right] =$$

$$= \frac{\bar{T} P_S}{(1-\beta)(1+r_S)} + \beta \frac{\bar{T}^{T-s} P_S}{1+r_S} E_S\left[\tilde{T}_{T+1}\right]$$

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6 LTD

Reference steady state

Euler equation

$$1 = E_t \left[\beta / (1 + R_{t+1}) \frac{P_t}{P_{t+1}} \right] \Rightarrow 1 = \frac{\beta(1 + \bar{R})}{\bar{\pi}}$$

Asset pricing kernel

$$z_{t,t+1} = \frac{\beta P_t}{P_{t+1}} \frac{1 + R_{t+1}}{1 + R_t} \Rightarrow \bar{z} = \frac{\beta}{\bar{\pi}}$$

m -period interest rate

$$\frac{1}{1 + R_{m,t}} = \beta E_t \left[\frac{z_{t,t+1}}{1 + R_{m+1,t+1}} \right] \Rightarrow \frac{1}{1 + \bar{R}_m} = \frac{\beta}{\bar{\pi}} \frac{1}{1 + \bar{R}_{m-1}} \Rightarrow 1 + \bar{R}_m = \left(\frac{\bar{\pi}}{\beta} \right)^m$$

Gov't PBC when $\frac{M_t}{P_t} \approx 0$

known
 B_{t-1}

$$B_{1,t-1} + \sum_{v=2}^{\infty} \frac{B_{v,t-1}}{1 + R_{v-1,t}} = E_t \sum_{s=t}^{\infty} z_{ts} T_s$$

$$\Rightarrow \bar{b} P_{t-1} + \sum_{v=2}^{\infty} \phi^{v-1} \bar{b} P_{t-1} \left(\frac{\beta}{\bar{\pi}} \right)^{v-1} = \sum_{s=0}^{\infty} \left(\frac{\beta}{\bar{\pi}} \right)^s \bar{T} P_{t-1} \bar{\pi}^s$$

7CTD

Define $\hat{\phi} := \frac{\beta\phi}{\bar{\pi}}$

$$\Rightarrow \frac{\bar{b}}{1-\hat{\phi}} = \frac{\bar{T}}{1-\beta}$$

Loglinearization

$$E_t \left[\tilde{R}_{t+1} - \tilde{\pi}_{t+1} \right] = 0$$

$$\tilde{z}_{t,t+1} = \tilde{R}_{t+1} - \tilde{R}_t - \tilde{\pi}_{t+1}$$

$$\tilde{R}_{m,t} = E_t \left[-\tilde{z}_{t,t+1} + \tilde{R}_{m-1,t+1} \right] = E_t \left[-\tilde{R}_{t+1} + \tilde{R}_t + \tilde{\pi}_{t+1} + \right.$$

$$\left. - \tilde{z}_{t+1,t+2} + \tilde{R}_{m-2,t+2} \right] = E_t \left[-\tilde{R}_{t+2} + \tilde{R}_t + \tilde{\pi}_{t+1} + \tilde{\pi}_{t+2} + \tilde{R}_{m-2,t+2} \right]$$

$$= \dots = E_t \left[-\tilde{R}_{t+m-1} + \tilde{R}_t + \sum_{s=t+1}^{t+m-1} \tilde{\pi}_s + \tilde{R}_{1,t+m-1} \right] =$$

$$= \tilde{R}_t + E_t \left[\sum_{s=t+1}^{t+m-1} \tilde{\pi}_s \right]$$

loglinearization of BC (holding geometric maturity structure fixed)

$$\text{Preliminary step } z_{t,s} T_s = \frac{\beta P_E}{P_S} \frac{1+r_s}{1+r_t} \left(\frac{T_s}{P_{S-1}} \right) = \frac{\beta P_E}{P_S} \frac{1+r_s}{1+r_t} \frac{P_{S-1}}{P_S}$$

(8CTD)

$$= \beta^{s-t} \frac{P_t}{P_{t-1}} \frac{P_{s-1}}{P_s} \frac{1+R_s}{1+R_t} \frac{T_s}{P_{s-1}} \frac{P_{t-1}}{P_t}$$

Define $b_{m,t} := \frac{B_{m,t}}{P_t}$ and divide equation by P_{t-1} on both sides

prior to linearization

$$\tilde{b}_{1,t-1} - \bar{b} + \sum_{v=2}^{\infty} \phi^{v-1} \tilde{b}_{1,t-1} \bar{b} \left(\frac{\beta}{\pi} \right)^{v-1} =$$

$$- \sum_{v=2}^{\infty} \phi^{v-1} \bar{b} \tilde{R}_{v-1,t} \left(\frac{\beta}{\pi} \right)^{v-1} = E_t \left[\sum_{s=t}^{\infty} \beta^{s-t} \bar{T} \left(\tilde{\pi}_t - \tilde{\pi}_s + \tilde{R}_s - \tilde{R}_t + \tilde{\pi}_s \right) \right]$$

Divide LHS by $\frac{\bar{b}}{1-\phi}$ and RHS by $\frac{\bar{T}}{1-\beta}$, we $E_t[\tilde{R}_{t+s} - \tilde{\pi}_{t+s}] = e_{s,t}$

$$\tilde{b}_{1,t-1} - (1-\hat{\phi}) \sum_{v=2}^{\infty} \hat{\phi}^{v-1} \left[\tilde{R}_t + E_t \left[\sum_{s=1}^{v-1} \tilde{\pi}_{t+s} \right] \right] =$$

$$= (1-\beta) E_t \sum_{s=t}^{\infty} \beta^{s-t} \tilde{T}_s - (\tilde{R}_t - \tilde{\pi}_t)$$

$$\Rightarrow \tilde{b}_{1,t-1} + (1-\hat{\phi}) \tilde{R}_t - (1-\hat{\phi}) \sum_{s=1}^{\infty} E_t \tilde{\pi}_{t+s} \left(\sum_{v=2}^{\infty} \hat{\phi}^{v-1} \right) = (1-\beta) \sum_{s=t}^{\infty} \beta^{s-t} E_t \tilde{T}_s + \tilde{\pi}_t$$

$$\tilde{b}_{1,t-1} + (1-\hat{\phi}) \tilde{R}_t = (1-\beta) \sum_{s=t}^{\infty} \beta^{s-t} E_t \tilde{T}_s + (1-\hat{\phi}) \sum_{s=0}^{\infty} \hat{\phi}^s E_t \tilde{\pi}_{t+s}$$