Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

SP Signal Processing

Prof. Dr.-Ing. Timo Gerkmann

Digital Media Signal Processing

Outline

Medien

- Ein **Medium** (lat.: medium = Mitte, Mittelpunkt; auch Öffentlichkeit, Gemeinwohl, öffentlicher Weg) ist nach neuerem Verständnis ein Vermittelndes im ganz allgemeinen Sinn. Das Wort „Medium" in der Alltagssprache lässt sich oft mit **Kommunikationsmittel** gleichsetzen.
- Der Plural **Medien** wird etwa seit den 1980er-Jahren für die **Gesamtheit aller Kommunikationsmittel** und **Kommunikationsorganisationen** verwendet.

- Unter **digitalen Medien** versteht man elektronische Medien, die mit digitalen Codes arbeiten. Digitale Medien sind **Kommunikationsmedien**, die auf der Grundlage digitaler Informations- und Kommunikationstechnologie funktionieren. Als digitale Medien werden zum anderen **technische Geräte zur Digitalisierung, Berechnung, Aufzeichnung, Speicherung, Verarbeitung, Distribution und Darstellung von digitalen Inhalten (Content)** bezeichnet.
- Beispiel: Aufzeichnung und Speicherung von medialen Inhalten als digitale Daten (Musikstück, Bild, Videosequenz) ist in der Regel ein technisch hochkomplexer Vorgang und gehört zum Gebiet der digitalen Signalverarbeitung.
- In dieser Vorlesung werden die Grundlagen der digitalen Signalverarbeitung vermittelt.
- Anwendungen insbesondere aus der Audio und Bildverarbeitung

Signalverarbeitung ist allgegenwärtig

- Bildverarbeitung
  - Weichzeichner
  - Scharfzeichner
  - Kompression (PNG, JPEG)
  - Erkennung von Gesichtern und Objekten in Fotos
  - Rauschunterdrückung
- Audioverarbeitung
  - Mastering: Equalizer, Kompressor, Limiter
  - Signalverbesserung
  - Datenkompression
  - Audio: FLAC, MP3, AAC, MPEG-H
  - Sprache: CELP, A-law, LPC-10, ...

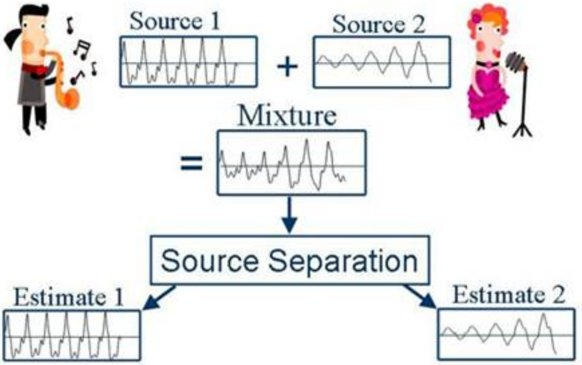
Literatur

- **John G. Proakis, Dimitris K. Manolakis, Digital Signal Processing, Pearson 2014.**
- Alan V. Oppenheim, Ronald W. Schafer. Discrete-Time Signal Processing, 2014.
- Martin Meyer, Signalverarbeitung, Springer Vieweg, 2014.
- Karl-Dirk Kammeyer, Kristian Kroschel, Digitale Signalverarbeitung, Springer Vieweg, 2012
- Digital Signal Processing – Fundamentals and Applications, Academic Press, 2019
- Fracis F. Li, Trevor J. Cox, Digital Signal Processing in Audio and Acoustical Engineering, CRC Press, 2019

➡ Numerierte Beispiele aus den Folien lassen sich im Proakis/Manolakis vertiefen.

Was wir machen

- Signalverarbeitung mit dem Schwerpunkt Audio-, Sprach-, und audiovisuelle Signalverarbeitung
- Insbesondere probabilistische Verfahren (Statistical Signal Processing) und maschinelles Lernen



1. Introduction of Basic Concepts

Outline

What is Digital Media Signal Processing

- **Signal**: Any quantity that varies with time, space, or other variables.
  - Audio signals (varies with time)
  - Speech signals (varies with time)
  - Image (varies with space)
  - Video (varies with time and space)
- **Digital Signal**: The signal is presented in machine code. This requires the signal to be discretized in time (sampling) and amplitude (quantization)
- **Digital Media Signals**: We apply the general and powerful tools specifically to digital media signals such as audio and images.
- **Processing**: Processing of signals includes
  - Resampling, changing the temporal and spectral content, compression (lossy and lossless)

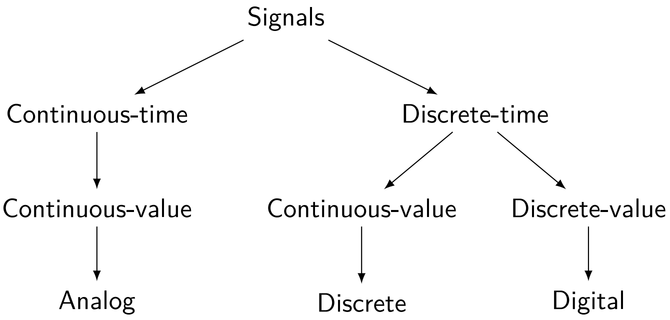
- **Signal**: variable x whose value varies with time (audio: $x(n)$) or space (image $x(n, m)$)
- The value of the signal may represent physical quantities such as
  - The sound pressure level (audio)
  - The color value (image)
- When the value of the signal is available over a continuum of times, it is referred to as **continuous-time signal**.
- Continuous-time signals whose amplitudes also vary over a continuous range are called **analog signal**.
- If the value of the signal is available only at discrete instants of time, it is called a **discrete-time signal**

Motivation II

- Some signals (e.g. economic data) are discrete in nature. However media signals, like audio, need to be discretized by taking time samples of the analog signal $x_a(t)$, referred to as **sampling**

$$x(n) = x_a(nT), \quad |n| = 0, 1, 2, ...$$

  - With T the sampling period
- When finite precision is used to represent the value of $x(n)$, the sequence of quantized values is called a **digital signal**
- A system or algorithm which processes one digital signal $x(n)$ to produce a digital output signal $y(n)$ is called digital signal processor


Signals
Continuous-time
Discrete-time
Continuous-value
Continuous-value
Discrete-value
Analog
Discrete
Digital

Applications of Digital Signal Processing

- Speech and audio enhancement, coding, and transmission
- Processing of music, images, and video
- Detection of targets (audio, image, video, radar, sonar),
- Medical signal processing (EEG, EKG, ultrasound)

Applications of Digital Signal Processing

- **Speech and audio enhancement, coding, and transmission**
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- Detection of targets (**audio, image, video**, radar, sonar),
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Rapid Development of DSP Algorithms

- Digital Signal Processing has developed rapidly over the past 50 years
- The development is a result of significant advances in digital computer technology and integrated-circuit fabrication.
- Algorithms do not require dedicated devices, but run on small portable devices
  - Laptops, smartphones, hearing aids
- Along with very-large-scale integration of electronic circuits, the hardware is getting
  - More powerful, smaller, faster, and cheaper

➡ Highly sophisticated algorithms are enabled.

- In the past, signal processing was done in analog circuits

Inputs: $x_a(t)$ → Analog proc. circuit → Outputs: $y_a(t)$

- Advantages
  - Extremely low processing latencies can be achieved
  - Can be a requirement to process extremely wideband signals

- Today signal processing is mostly done digitally as

- Three components are needed
  - Analog-to-digital converter
  - Actual processing
  - Digital-to-analog converter
- Advantages
  - Flexible programmable operations
  - Often a higher order of precision

Outline

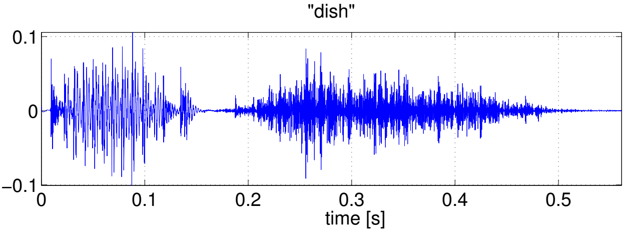
- A *signal* is defined as any physical quantity that varies with time (e.g. audio), space (e.g. image), or any other independent variable(s).
- Mathematically, a signal is a function of these independent variables, e.g.

$$\begin{aligned} s_1(t) &= 5t \\ s_2(t) &= 20t^2 \end{aligned} \tag{1}$$

or

$$s(x, y) = 3x + 2xy + 10y^2 \tag{2}$$

- Many signals, like speech or images cannot be described functionally by an expression as given in (1)



- However, a *segment* of speech can be represented to a high degree of accuracy as a sum of several sinusoids of different amplitudes, frequencies, and phases

$$\sum_{i=1}^{N} A_i(t) \sin[2\pi F_i(t) t + \theta_i] \tag{3}$$

Systems

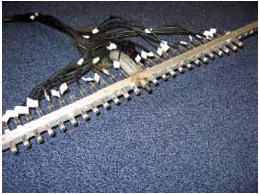
- Signal generation is usually associated with a *system*
  - Speech is generated by forcing (vibrating) air through the vocal tract
  - Images are obtained by exposing a photographic sensor/film to a scene or object
- A system may also be defined as a physical device that performs an operation on a signal
  - A room is a system (adds reverberation)
- Also a software realization of operations on a signal is referred to as a system
  - A noise reduction filter is a system
  - An image-blurring filter is a system
- When passing a signal through a system, as in filtering, we say we have *processed* the signal

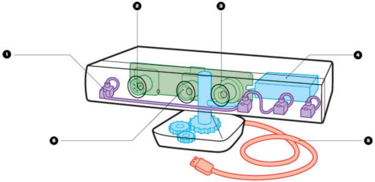
Outline

Classification of Signals

- The methods chosen for processing and/or analyzing a signal heavily depend on the signals characteristics
- Any investigation in signal processing thus starts with a classification of the targeted signals
- Signals can e.g. be
  - real-valued, e.g. $s_1(t) = A\sin(2\pi t)$
  - complex-valued, e.g. $s_2(t) = Ae^{j3\pi t} = A\cos(3\pi t) + jA\sin(3\pi t)$
  - scalar / vector
- Real-world physical signals are always real-valued
- Signals recorded by multiple sources or sensors can be represented by vectors

Multichannel and Multimodal Signals





- Vector representation of multichannel signal (3 dimensions)

$$\mathbf{S}_3 = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix}$$

Multidimensional Signals

- The dimensionality of a signal is determined by the number of independent variables
  - Audio is 1-dimensional in time-domain, e.g. $s(t)$ (but frequently processed in a 2-D time-frequency space)
  - B/W Image is 2-dimensional, e.g. $I(x, y)$
  - B/W Video is 3-dimensional, e.g. $I(x, y, t)$
- A color TV RGB signal is a 3-dimensional and 3-channel signal

$$\mathbf{I}(x, y, t) = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t) \end{bmatrix}$$

Deterministic Versus Random Signals

- A **Signal Model** is used to describe a signal mathematically
- Any signal that can be uniquely described by an explicit mathematical expression, a table of data, or a well-defined rule is called **deterministic**
  - ➡ all past, present, and future values are known precisely, i.e. without any uncertainty
- In practice the most interesting signals (i.e. those that carry information) are per se not fully predictable and cannot be fully described by explicit mathematical formulas. These signals are referred to as **random** signals.
  - ➡ Mathematical framework: theory of probability and stochastic processes.
- In practice both a deterministic signal model and a random signal model can be reasonable to solve a specific DSP problem

Outline

Logarithms and Exponentials

- Exponentials

$$
\begin{aligned}
e^x e^y &= e^{x+y} \\
e^x / e^y &= e^{x-y} \\
(e^x)^y &= (e^y)^x = e^{xy}
\end{aligned}
$$

- Logarithms

$$
\begin{aligned}
\log(xy) &= \log x + \log y \\
\log \frac{x}{y} &= \log x - \log y \\
\log x^n &= n \log x
\end{aligned}
$$

Complex Numbers

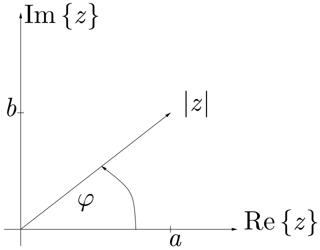
- For many spectral transformations, complex numbers are needed
- A complex number z is composed of a real a and an imaginary part b

$$z = \mathrm{Re}\left\{z\right\} + \mathrm{j}\mathrm{Im}\left\{z\right\} = a + \mathrm{j}b$$

- j is the **imaginary unit** and **separates** the real and imaginary parts

Visualization

- To understand the different representations of complex numbers, draw a diagram



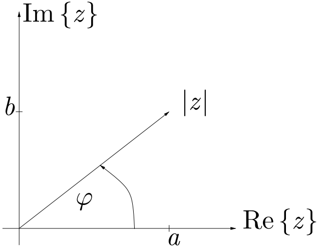
- Complex numbers can be represented by
  - real and imaginary parts (Cartesian coordinates)

$$z = a + \mathrm{j}b$$

  - Absolute value and phase (polar coordinates)

$$z = |z|e^{\mathrm{j}\phi}$$

Different Representations



- Simple geometry reveals the relations
  - Pythagoras: $|z| = \sqrt{a^2 + b^2}$
  - Trigonometry:
    - $\varphi = \arctan(\frac{b}{a})$
    - $a = |z| \cos \varphi$
    - $b = |z| \sin \varphi$

Euler relation (memorize!)

$$e^{j\varphi} = \cos \varphi + j \sin(\varphi)$$

Calculus

Addition is most easily done in Cartesian coordinates

$$z_1 + z_2 = (a_1 + a_2) + \mathrm{j}(b_1 + b_2)$$

➡ Real and imaginary parts are added separately

Multiplication is most easily done in polar coordinates

$$z_1 z_2 = |z_1||z_2| e^{\mathrm{j}(\varphi_1 + \varphi_2)}$$

➡ Absolute values are multiplied, the phases add

Conjugate For a **complex conjugate** z^* of z, the sign of the imaginary part is flipped

$$z = a + \mathrm{j}b = |z| e^{\mathrm{j}\varphi}$$

$$z^* = a - \mathrm{j}b = |z| e^{-\mathrm{j}\varphi}$$

➡ mirror complex vector along Re-axis

Some conclusions

1. Transform $e^{\mathrm{j}\pi}$ to Cartesian coordinates
2. Transform $e^{\mathrm{j}\frac{\pi}{2}}$ to Cartesian coordinates
3. Express $\frac{1}{j}$ in terms of polar coordinates
4. Express $\frac{1}{j}$ in terms of Cartesian coordinates
5. Express $\sqrt{-1}$ in terms of polar coordinates
6. Express $\sqrt{-1}$ in terms of Cartesian coordinates

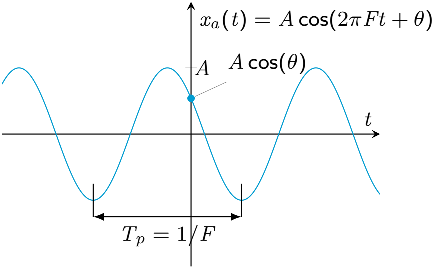
Outline

- Frequency is related to periodic motion and has the physical dimension $[1/\mathrm{s}] = [\mathrm{Hz}]$
- A simple harmonic oscillation is mathematically described by a sinusoidal signal

$$x_a(t) = A\cos(\Omega t + \theta)\,, \quad t \in \mathbb{R} \tag{4}$$

- completely characterized by three parameters
  - A: the signal *amplitude*
  - Ω: the *angular frequency* in radians per second $[\mathrm{rad/s}]$
  - θ: is the *phase* in radians
- Instead of the angular frequency Ω, we often use the frequency F in cycles per second or hertz $[1/\mathrm{s}] = [\mathrm{Hz}]$, where

$$\Omega = 2\pi F$$



- This signal $x_a(t)$ in (4) has the following properties
  - For a fixed frequency F, $x_a(t)$ is periodic, i.e. $x_a(t + T_p) = x_a(t)$, with $T_p = 1/F$, the **fundamental period**
  - Increasing the frequency F increases the rate of oscillation of the signal, i.e. more periods are included in a given time interval.
  - There is no principal limit to $F = 1/T_p$

- These relations carry over to the class of complex exponential signals given by

$$x_a(t) = Ae^{j(\Omega t+\theta)}$$

- This can be seen by using Euler's equation

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \tag{5}$$

- By definition, frequency is an inherently positive quantity, namely the number of cycles per unit time in a periodic signal
- Negative frequencies are used for mathematical convenience. A real-valued cosine can be represented using (5) as

$$x_a(t) = A\cos(\Omega t+\theta) = \frac{A}{2}e^{j(\Omega t+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)}$$

Continuous-Time Complex Exponentials II

$$x_a(t) = A\cos(\Omega t + \theta) = \frac{A}{2}e^{j(\Omega t+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)}$$

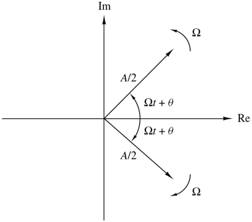


Fig: Representation of a cosine function by a pair of complex-conjugate exponentials (phasors)

Discrete-Time Sinusoidal Signals I

- A discrete-time sinusoidal signal may be expressed as

$$x(n) = A\cos(\omega n + \theta)\,, \quad n \in \mathbb{Z}$$

  - n: integer variable, called *sample number*
  - A: *Amplitude*
  - ω: *angular frequency* in radians per sample
- Instead of the angular frequency ω we often use the frequency variable f, as

$$\omega = 2\pi f$$

  - f has unit cycles per sample

Discrete-Time Sinusoidal Signals II

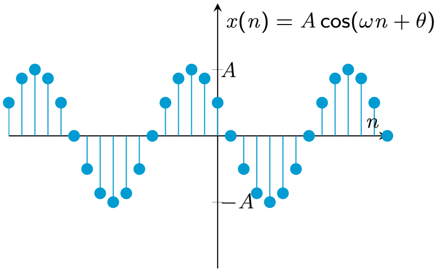


Fig: Example of a discrete-time sinusoidal signal ($\omega = \pi/6$, $f = 1/12$ and $\theta = \pi/3$)

- $f = 1/12$, means 12 samples per cycle

It holds that

1. *Discrete-time sinusoids are periodic with period N only if their frequency f is a rational number, i.e. $f \in \mathbb{Q}$ (only then we have an integer number of samples per cycle), and*

$$x(n+N) = x(n)$$

2. *Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical*

$$\cos[(\omega + 2k\pi)n + \theta] = \cos(\omega n + \theta), \quad k, n \in \mathbb{N}$$
$$\cos[2\pi(f+k)n + \theta] = \cos(2\pi f n + \theta)$$

- Any sequence with a frequency $|\omega| > \pi$ or $|f| > \frac{1}{2}$ is identical to a sequence obtained from a sinusoidal signal with frequency $|\omega| < \pi$

➡ Frequencies in the range $-\pi \leq \omega \leq \pi$ or $-\frac{1}{2} \leq f \leq \frac{1}{2}$ are unique

➡ Frequencies in the range $|\omega| > \pi$ or $|f| > \frac{1}{2}$ are **aliases**

Discrete-Time Sinusoidal Signals IV

3. *The highest rate of oscillation in a discrete-time sinusoid is attained when* $|\omega| = \pi$ *or* $|f| = \frac{1}{2}$*)*

➡ Usually we choose the *fundamental range* $0 \leq \omega \leq 2\pi$ $(0 \leq f \leq 1)$ or $-\pi \leq \omega \leq \pi$ $(-\frac{1}{2} \leq f \leq \frac{1}{2})$, as these ranges constitute all existing sinusoids.

Continuous-time Exponentials

- **Continuous-time Exponentials** with harmonically related frequencies

$$s_k(t) = e^{jk2\pi F_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{6}$$

  - $F_0 = 1/T_p$ is the fundamental frequency
- From (6) a linear combination of harmonically related complex coefficients can be obtained

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k s_k(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi F_0 t} \tag{7}$$

  - c_k are arbitrary complex constants
- The signal $x_a(t)$ is periodic with fundamental period $T_p = 1/F_0$
- (7) is called the **Fourier series expansion of $x_a(t)$**, with
  - Fourier coefficients c_k and
  - Harmonics $s_k(t)$

- **Discrete-time Exponentials** with harmonically related frequencies

$$s_k(n) = e^{jk2\pi f_0 n} = e^{j2\pi k \frac{n}{N}}, \quad k = 0, \pm 1, \pm 2, ... \tag{8}$$

  - $N = 1/f_0$ is the number of samples per fundamental period
- In contrast to continuous-time, we note that

$$s_{k+N}(n) = e^{j2\pi(k+N)\frac{n}{N}} = s_k(n)e^{j2\pi n} = s_k(n) \tag{9}$$

  - ➡ There are only N distinct periodic complex exponentials
  - ➡ Typically we choose $k = 0, 1, 2, ..., N-1$

Harmonically Related Exponentials II

Discrete-time Exponentials

- Similar to the continuous case, we have

$$x(n) = \sum_{k=0}^{N-1} c_k s_k(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi k \frac{n}{N}} \tag{10}$$

  - c_k are arbitrary complex constants
- The signal $x(n)$ is periodic with fundamental period $N = 1/f_0$
- (10) is called the **Fourier series expansion of $x(n)$**, with
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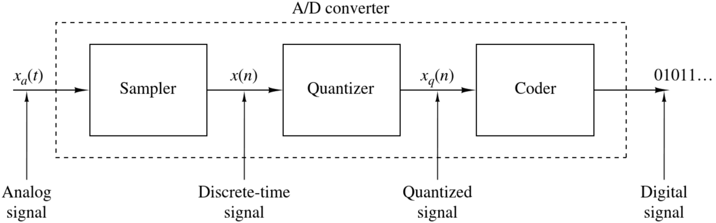
Outline

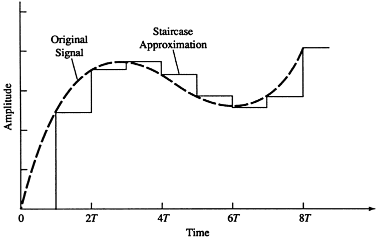
Analog-to-Digital Conversion



1. **Sampling:** Continuous-time $\rightarrow$ discrete-time
   - $x_a(nT) \equiv x(n)$, where T is the *sampling interval*
2. **Quantization:** Continuous-valued signal $\rightarrow$ discrete-valued signal
   - Introduces quantization error $x(n) - x_q(n)$
3. **Coding:** Each discrete value x_q is represented by a b-bit binary sequence

Digital-to-Analog Conversion

- To be able to reproduce an analog signal, e.g. for audio-playback, the discrete samples must be interpolated
- This could be a zeroth-order interpolation, a linear (first order) interpolation, quadratic interpolation, ...

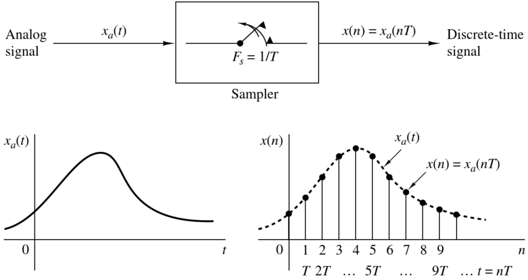


- Is there an optimal interpolator?
  - For bandlimited signals (i.e. limited frequency content) the sampling theorem yields the optimum form of interpolation

Sampling of Analog Signals I

- We limit ourselves to periodic uniform sampling

$$x(n) = x_a(nT), \quad n \in \mathbb{Z}$$



- T is the time interval between samples
- $F_S = 1/T$ is the sampling rate [samples per second]

Sampling of Analog Signals II

- Continuous time and discrete time are related, as $t = nT = \frac{n}{F_\mathrm{S}}$

➔ Also the frequencies F, Ω of analog signals and f, ω of discrete signals are related!

  - $f = \frac{F}{F_\mathrm{S}} = FT \rightarrow$ aka. normalized frequency $\qquad \omega = \Omega T$

- However, recall that
  - $-\infty < F < \infty, \qquad -\infty < \Omega < \infty$
  - $-\frac{1}{2} \leq f \leq \frac{1}{2}, \qquad -\pi \leq \omega \leq \pi$

- Combining these equations reveals that the frequency of an analog sinusoid sampled at F_S must fall in
  - $-\frac{F_\mathrm{S}}{2} \leq F \leq \frac{F_\mathrm{S}}{2} \qquad -\pi F_\mathrm{S} \leq \Omega \leq \pi F_\mathrm{S}$

Relations Among Frequency Variables

TABLE 1 Relations Among Frequency Variables

Continuous-time signals		Discrete-time signals
$\Omega = 2\pi F$		$\omega = 2\pi f$
$\frac{\text{radians}}{\text{sec}}$ Hz		$\frac{\text{radians}}{\text{sample}}$ $\frac{\text{cycles}}{\text{sample}}$
	$\xrightarrow{\omega = \Omega T,\ f = F/F_s}$	$-\pi \le \omega \le \pi$
	$\xleftarrow{\Omega = \omega/T,\ F = f \cdot F_s}$	$-\frac{1}{2} \le f \le \frac{1}{2}$
$-\infty < \Omega < \infty$		$-\pi/T \le \Omega \le \pi/T$
$-\infty < F < \infty$		$-F_2/2 \le F \le F_s/2$

- The fundamental difference between continuous and discrete time is the range of frequency variables
- Periodic sampling with F_S implies a mapping of the infinite frequency range into a finite range with limit $F_\mathrm{max} = F_\mathrm{S}/2$

Example 1.4.1

- Consider the sampling of the following two signals sampled at $F_\mathrm{S} = 40\,\mathrm{Hz}$

$$\begin{aligned} x_1(t) &= \cos(2\pi\ 10\,\mathrm{Hz}\ t) \\ x_2(t) &= \cos(2\pi\ 50\,\mathrm{Hz}\ t) \end{aligned}$$

Example 1.4.1

- Consider the sampling of the following two signals sampled at $F_\mathrm{S} = 40\,\mathrm{Hz}$

$$
\begin{aligned}
x_1(t) &= \cos(2\pi\ 10\,\mathrm{Hz}\ t) \\
x_2(t) &= \cos(2\pi\ 50\,\mathrm{Hz}\ t)
\end{aligned}
$$

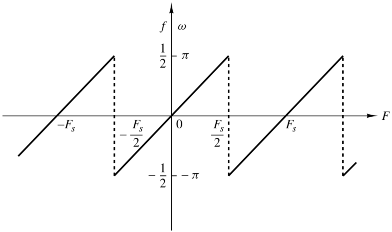
- Substitute $t = nT = n/F_\mathrm{S}$

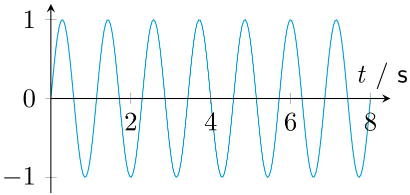
$$
\begin{aligned}
x_1(n) &= \cos\left(2\pi\ \frac{10\,\mathrm{Hz}}{40\,\mathrm{Hz}}\ n\right) = \cos\frac{\pi}{2}n \\
x_2(n) &= \cos\left(2\pi\ \frac{50\,\mathrm{Hz}}{40\,\mathrm{Hz}}\ n\right) = \cos\frac{5\pi}{2}n = \cos\frac{\pi}{2}n
\end{aligned}
$$

➡ The two sampled cosines are identical and thus indistinguishable

- $F_2 = 50\,\mathrm{Hz}$ is an **alias** of the frequency $F_1 = 10\,\mathrm{Hz}$ at the sampling rate $F_\mathrm{S} = 40\,\mathrm{Hz}$

- Sampling is unique iff $x_a(t)$ contains frequencies $\leq F_\mathrm{S}/2$
- Frequencies $F_k = F_0 + kF_\mathrm{S}$ are indistinguishable from F_0 after sampling. We say, they are **aliased** to F_0
- The frequency $F_\mathrm{S}/2$ is also called **folding frequency** or Nyquist frequency
- Given a discrete signal $x(n)$ an ambiguity exists to which continuous-time signal $x_a(t)$ it represents





Aliasing

- Sampling is unique iff $x_a(t)$ contains frequencies $\leq F_\mathrm{S}/2$
- Frequencies $F_k = F_0 + kF_\mathrm{S}$ are indistinguishable from F_0 after sampling. We say, they are **aliased** to F_0
- The frequency $F_\mathrm{S}/2$ is also called **folding frequency** or Nyquist frequency
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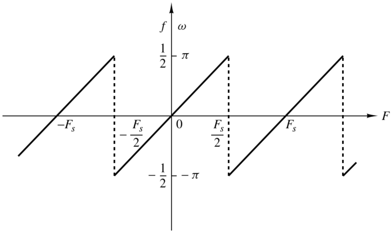

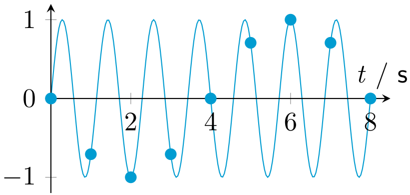

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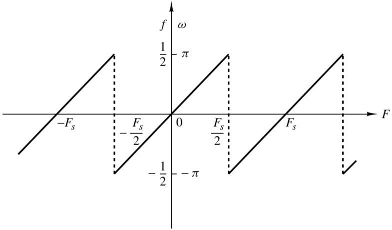
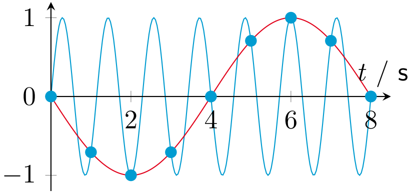




Example 1.4.2

SP
Signal Processing

- Consider the analog signal

$$x_a(t) = 3\,\mathrm{cos}\left(100\frac{1}{\mathrm{s}}\;\pi\;t\right)$$

(a) Determine the minimum sampling rate required to avoid aliasing
(b) Suppose that the signal is sampled at the rate $F_\mathrm{S} = 200\,\mathrm{Hz}$. What is the discrete-time signal obtained after sampling?
(c) Suppose that the signal is sampled at the rate $F_\mathrm{S} = 75\,\mathrm{Hz}$. What is the discrete-time signal obtained after sampling?
(d) What is the frequency $0 < F < F_\mathrm{S}/2$ of a sinusoid that yields samples identical to those obtained in part (c)?

The Sampling Theorem

- Given an analog signal, how should we set the sampling period T?

→ Depends on the frequency content
  - Speech has most important content up to $4\mathrm{kHz}$ (fricatives up to $8\mathrm{kHz}$)
  - Music has content up to $20\mathrm{kHz}$
  - Analog TV signals up to $5\mathrm{MHz}$

- Alias frequencies are avoided if

$$F_{\mathrm{S}} > 2F_{\max}$$

→ If no alias frequencies exist, the analog signal can be reconstructed perfectly from the sampled signal

The Sampling Theorem

Sampling Theorem

- If the highest frequency contained in an analog signal $x_a(t)$ is $F_{\max} = B$ and the signal is sampled at a rate

$$F_\mathrm{S} > 2F_{\max} \equiv 2B$$

  then $x_a(t)$ can be **perfectly reconstructed** from its sample values.
- The sampling rate $F_\mathrm{N} = 2B = 2F_{\max}$ is called the **Nyquist rate**

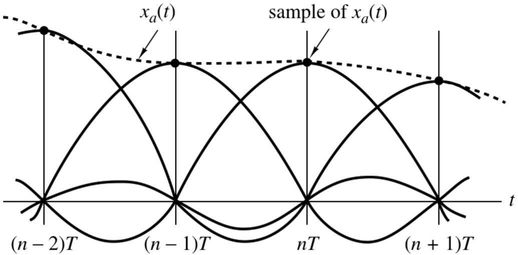
Perfect reconstruction is then obtained using the interpolation function

$$g(t) = \frac{\sin(2\pi Bt)}{2\pi Bt}$$

as

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) g\left(t - \frac{n}{F_\mathrm{S}}\right)$$

Reconstruction using ideal interpolation



- The ideal reconstruction

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) g\left(t - \frac{n}{F_S}\right)$$

involves an infinite sum

➡ In practice, approximations are used

Example 1.4.3

- Consider the analog signal

$$x_a(t) = 3\cos\left(50\frac{1}{\mathrm{s}}\ \pi\ t\right) + 10\sin\left(300\frac{1}{\mathrm{s}}\ \pi\ t\right) - \cos\left(100\frac{1}{\mathrm{s}}\ \pi\ t\right)$$

- What is the Nyquist rate for this signal?

Example 1.4.3

- Consider the analog signal

$$x_a(t) = 3\cos\left(50\frac{1}{\mathrm{s}}\ \pi\ t\right) + 10\sin\left(300\frac{1}{\mathrm{s}}\ \pi\ t\right) - \cos\left(100\frac{1}{\mathrm{s}}\ \pi\ t\right)$$

- What is the Nyquist rate for this signal?

- Solution:
  - The frequencies in this signal are $F_1 = 25\,\mathrm{Hz}, F_2 = 150\,\mathrm{Hz}, F_3 = 50\,\mathrm{Hz}$
  - $F_{\mathrm{max}} = 150\,\mathrm{Hz}$
  - $F_{\mathrm{S}} > 2F_{\mathrm{max}} = 300\,\mathrm{Hz}$
  - Nyquist rate: $F_{\mathrm{N}} = 2F_{\mathrm{max}} = 300\,\mathrm{Hz}$

- Consider the analog signal

$$x_a(t) = 3\cos\left(50\frac{1}{\mathrm{s}}\ \pi\ t\right) + 10\sin\left(300\frac{1}{\mathrm{s}}\ \pi\ t\right) - \cos\left(100\frac{1}{\mathrm{s}}\ \pi\ t\right)$$

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- Solution:
  - The frequencies in this signal are $F_1 = 25\,\mathrm{Hz}, F_2 = 150\,\mathrm{Hz}, F_3 = 50\,\mathrm{Hz}$
  - $F_{\mathrm{max}} = 150\,\mathrm{Hz}$
  - $F_{\mathrm{S}} > 2F_{\mathrm{max}} = 300\,\mathrm{Hz}$
  - Nyquist rate: $F_{\mathrm{N}} = 2F_{\mathrm{max}} = 300\,\mathrm{Hz}$

→ Observe that sampling at the Nyquist rate results in sampling only zeros from the sinusoid above. Solution: sample slightly higher than at Nyquist rate

Example 1.4.4

- Consider the analog signal

$$x_a(t) = 3\cos\left(2000\frac{1}{\mathrm{s}}\;\pi\;t\right) + 5\sin\left(6000\frac{1}{\mathrm{s}}\;\pi\;t\right) - 10\cos\left(12000\frac{1}{\mathrm{s}}\;\pi\;t\right)$$

(a) What is the Nyquist rate for this signal?
(b) Assume now that we sample this signal using a sampling rate of $F_\mathrm{S} = 5000\,\mathrm{Hz}$. What is the discrete-time signal obtained after sampling
(c) What is the analog signal $y_a(t)$ that we can reconstruct from the samples if we use ideal interpolation?

Quantization of Continuous-Amplitude Signals

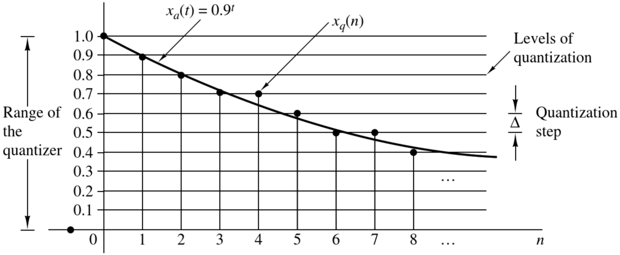
- A **digital signal** is a discrete-time signal where each sample is represented by a finite number of digits
- The process of converting a discrete-time continuous amplitude signal into a digital signal is called **quantization**
- As opposed to sampling, quantization always introduces an error also referred as **quantization noise**
- We denote the process of quantization as

$$x_q(n) = Q[x(n)]$$

- The quantization noise denotes the introduced quantization error

$$e_q(n) = x_q(n) - x(n)$$

Illustration of Quantization



- Important parameters of a quantization scheme are
  - Quantization levels (here: $0.1, 0.2, 0.3, \ldots$)
  - Quantization step-size Δ (distance between two levels)
  - Range of the quantizer should match range of signal amplitudes
  - Quantization can be achieved by
    - Rounding (take the closer line), e.g. $0.86 \rightarrow 0.9$
    - Truncation (take the next lower line), e.g. $0.86 \rightarrow 0.8$

Quantification of Quantization Noise

- The power of the quantization noise is dominated by the step size Δ, and given by

$$P_q = \frac{\Delta^2}{12}$$

- The signal-to-noise (SNR) level is proportional to the number of bits b spent

$$\mathrm{SNR[dB]} \propto 6\mathrm{dB} \cdot b$$

Coding of Quantized Samples

- If we need L quantization levels, we need at least L different binary numbers
- With a word length of b bits we can create 2^b different numbers
- Hence we need $b \geq \log_2 L$ bit.
- Examples
  - ISDN telephone speech: $b = 8$ bit (non-uniformly quantized)
  - Audio CD: $b = 16$ bit (uniformly)
  - Audio recordings: $b = 24$ bit (uniformly)
- Q: Why does it make sense to use a higher bitrate for audio recordings?
- Q: Would it be reasonable to also use higher sampling rates?

2. Discrete-Time Signals and Systems

Motivation

- While sinusoids are very elementary signals, here we introduce more elementary signals that facilitate signal processing
  - Elementary signals are used as building blocks for more complex signals
- Introduce and analyze the concept of Discrete-Time Systems
- What are the advantages of linear time-invariant (LTI) systems?
  - Large collection of mathematical techniques to elegantly describe LTI systems
  - Many practical systems are (approximately) LTI
    - Rooms that introduce reverberation
    - Many filters (lowpass, highpass, bandpass)
- Introduction of measures for signal correlation
  - Autocorrelation, crosscorrelation

Elementary Discrete-Time Signals

- The *unit sample sequence* / *unit impulse* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

related to time domain unit impulse $\delta(t)$ (but much simpler).

- The *unit sample sequence* / *unit impulse* is denoted as $\delta(n)$ and is defined as

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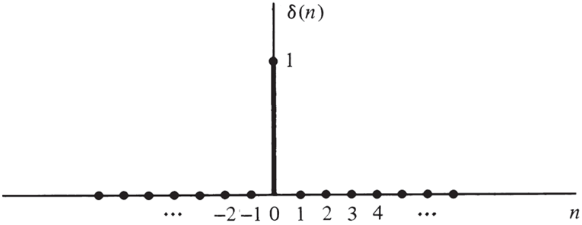

Figure 2.1.2 Graphical representation of the unit sample signal.

- The *unit step signal*

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

- The *unit step signal*

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

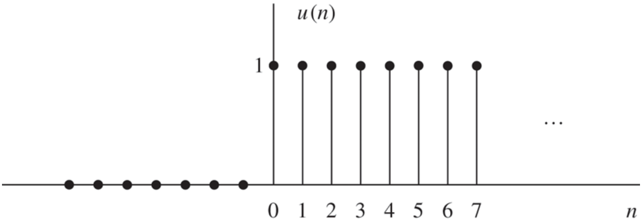

Figure 2.1.3 Graphical representation of the unit step signal.

Elementary Discrete-Time Signals

- The *unit ramp* signal

$$u_{\mathrm{r}}(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

- The *unit ramp* signal

$$u_{\mathrm{r}}(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

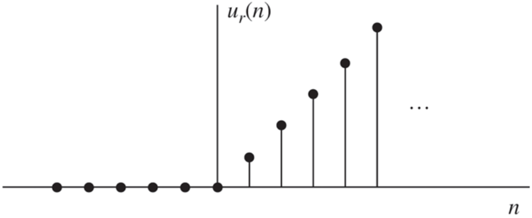

Figure 2.1.4 Graphical representation of the unit ramp signal.

Elementary Discrete-Time Signals

- The *exponential signal* has the form

$$x(n) \equiv a^n \quad \text{for all } n$$

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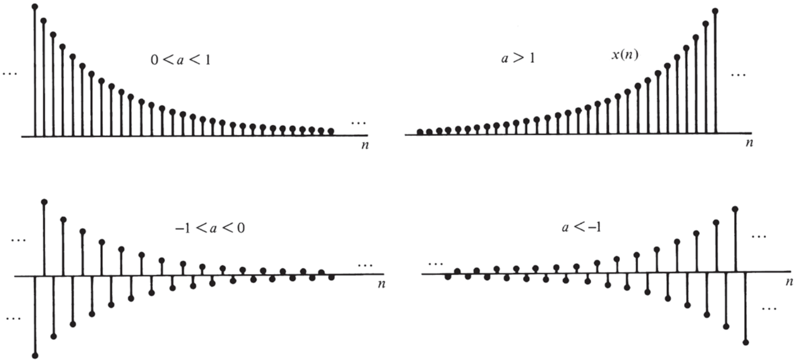


Figure 2.1.5 Graphical representation of exponential signals.

- The *exponential signal* has the form

$$x(n) \equiv a^n \quad \text{for all } n$$

  - If a is real-valued $\rightarrow x(n)$ is real-valued
  - When complex-valued we have $a \equiv re^{j\theta}$, and

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ x(n) &= r^n (\cos\theta n + j \sin\theta n) \end{aligned}$$

Representation of complex numbers

Cartesian $x(n) = x_R(n) + jx_I(n)$ $(= r^n \cos\theta n + jr^n \sin\theta n)$

Polar $x(n) = |x(n)| e^{j\angle x(n)}$ $(= r^n e^{j\theta n})$

- Cartesian representation (left) and polar representation (right)
  $x(n) = (0.9e^{j\frac{\pi}{10}})^n$

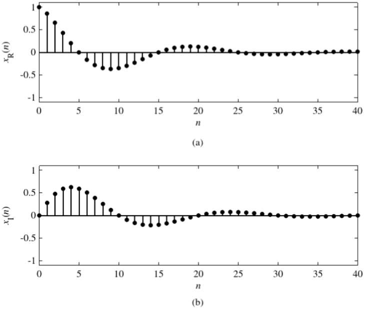
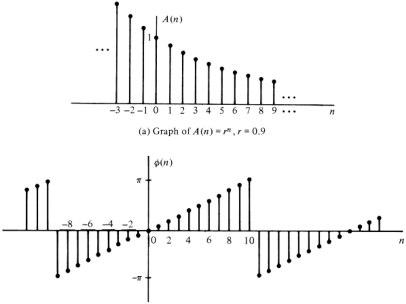


Figure: Graph of (a) $x_R(n) = (0.9)^n \cos \frac{\pi n}{10}$,
(b) $x_I(n) = (0.9)^n \sin \frac{\pi n}{10}$



(a) Graph of $A(n) = r^n$, $r = 0.9$

(b) Graph of $\phi(n) = \frac{\pi}{10}n$, modulo 2π plotted in the range $(-\pi, \pi)$

Energy and power signals

- The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2$$

  ➡ If $0 < E < \infty$ (i.e. E is finite), $x(n)$ is called **energy signal**

- The average power of a signal is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

  - If $0 < E < \infty$ (i.e. E is finite), then $P = 0$
  - If E is infinite, P can either be finite or infinite

  ➡ If P is finite, $x(n)$ is called **power signal**

Periodic and aperiodic signals

- A signal $x(n)$ is periodic with period N $(N > 0)$, iff

$$x(n+N) = x(n) \quad \text{for all } n \tag{11}$$

- The smallest N for which (11) holds is called **fundamental period**
- If there is no N for which (11) holds, $x(n)$ is called **nonperiodic**
- Periodic signals with $x(n) < \infty$ $(\forall n)$, are power signals, i.e. they have finite power

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

Classification of Discrete-Time Signals I

Even and odd signals

- A real-valued signal $x(n)$ is **even** if

$$x(-n) \equiv x(n)$$

- A real-valued signal $x(n)$ is **odd** if

$$x(-n) \equiv -x(n)$$

  - implies $x(0) = 0$

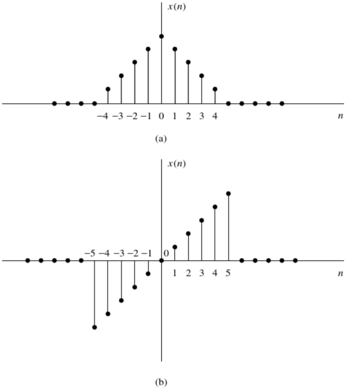


Figure: even (a) and odd (b) signals

Even and odd signals

In general, a signal $x(n)$ can be decomposed

- into an even part $x_\mathrm{e}(-n) = x_\mathrm{e}(n)$, as

$$x_\mathrm{e}(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

- and an odd part $x_\mathrm{o}(-n) = -x_\mathrm{o}(n)$, as

$$x_\mathrm{o}(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

such that $x(n) = x_\mathrm{e}(n) + x_\mathrm{o}(n)$

Time shift A signal delay by $k > 0$ samples is obtained by replacing n by $n - k$,

$$\mathrm{TD}_k\,[x(n)] = x(n-k)$$

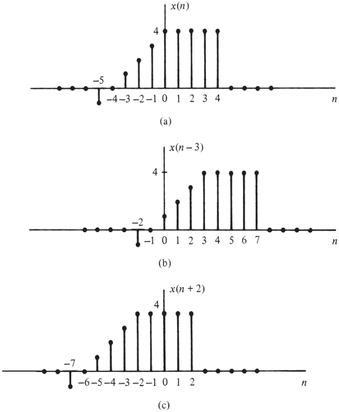


Figure 2.1.9 Graphical representation of a signal, and its delayed and advanced versions.

Simple Modifications of Signals II

Folding A folding at origin $n = 0$ is obtained as $\mathrm{FD}\left[x(n)\right] = x(-n)$

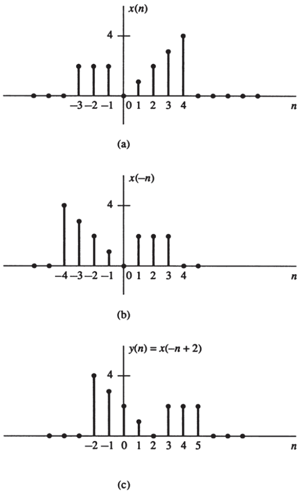


Figure 2.1.10 Graphical illustration of the folding and shifting operations.

- Note that the time dependent operations of folding and delaying are not commutative. I.e.

$$\mathrm{TD}_k \left\{ \mathrm{FD}\left[x(n) \right] \right\} = \mathrm{TD}_k \left[x(-n) \right] = x(-(n-k)) = x(-n+k)$$

while

$$\mathrm{FD} \left\{ \mathrm{TD}_k \left[x(n) \right] \right\} = \mathrm{FD}\left[x(n-k) \right] = x(-n-k)$$

Down-sampling A down-sampling is obtained by

$$\mathrm{TS}_\mu\left[x(n)\right] = x(\mu n), \quad \mu = 2, 3, 4, \ldots$$

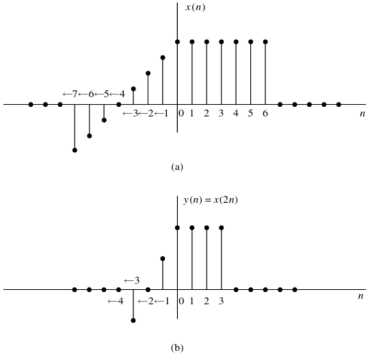


Figure: Graph of (a) $A(n) = r^n$, $r = 0.9$, (b) $(n) = \frac{\pi}{1}0n$, modulo 2π, plotted in the range $(-\pi, \pi)$

➡ Be careful with sampling theorem!

Simple Modifications of Signals VI

Amplitude Scaling of a signal with constant A

$$y(n) = Ax(n), \quad -\infty < n < \infty$$

Summation of two signals

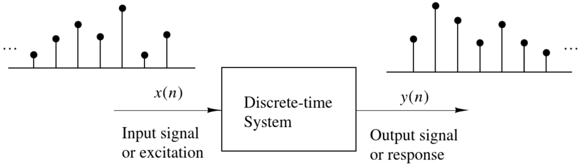
$$y(n) = x_1(n) + x_2(n), \quad -\infty < n < \infty$$

Product of two signals is defined on a sample by sample basis

$$y(n) = x_1(n)x_2(n), \quad -\infty < n < \infty$$

Outline

- A system is a device or an algorithm that performs some prescribed operation on a signal
- The system processes the system input signal (excitation) $x(n)$ to obtain the processed output signal (response) $y(n)$



- This relation is described as

$$y(n) \equiv \mathcal{T}\left[x(n)\right]$$

Input-Output Description

- The magic trick is, to treat the system as a black box, but to still find a mathematical rule which explicitly defines the relation between the input and the output

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

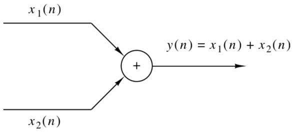
Example 2.2.1: Determine the output of the following systems given the input signal

$$x(n) = \begin{cases} |n|, & -3 \leq n \leq 3 \\ 0, & \text{else} \end{cases}$$

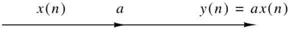
1. $y(n) = x(n)$ (identity)
2. $y(n) = x(n-1)$ (unit delay system)
3. $y(n) = \frac{1}{3}\left[x(n+1) + x(n) + x(n-1)\right]$ (moving average filter)
4. $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \ldots$ (accumulator)

Block Diagram Representation I

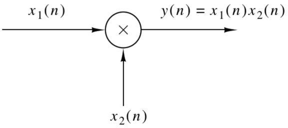
Adder No storing is necessary. The addition is *memoryless*



Constant Multiplier (memoryless)

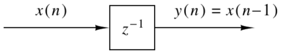


Signal Multiplier (memoryless)

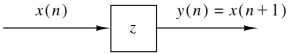


Block Diagram Representation II

Unit delay requires a memory. The notation z^{-1} becomes clear when discussing the z-Transform



Unit advance Moves the input ahead by one sample $\rightarrow$ impossible to realize in real-time



Example 2.2.3

- Sketch the block diagram of the system using the basic building blocks

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$

Example 2.2.3

- Sketch the block diagram of the system using the basic building blocks

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$

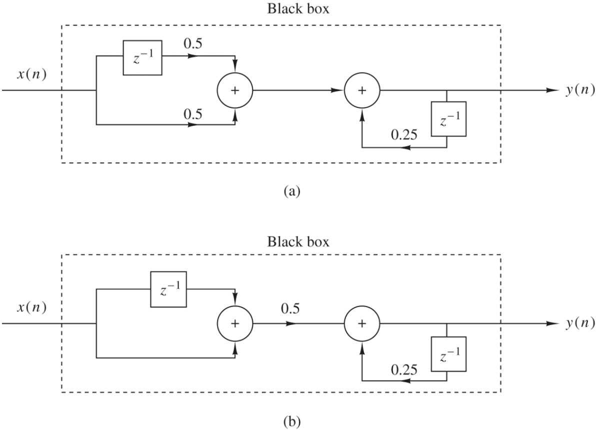


Figure 2.2.7 Block diagram realizations of the system $y(n) = 0.25y(n-1) + 0.5x(n) + 0.5x(n-1)$.

Static versus dynamic systems

Static The system is memoryless and depends only on the current sample (but not past or future samples). Can be described by

$$y(n) = \mathcal{T}\left[x(n), n\right]$$

Example:

$$y(n) = nx(n) + bx^3(n)$$

Dynamic A system with memory.

Example:

$$y(n) = x(n) + 2x(n-1)$$

Relaxed A dynamic system is relaxed, if it received no prior excitation, i.e. the output signal depends only on the input signal

Signal Processing

Time-variant versus time-invariant systems

Time Invariant System

A relaxed system $\mathcal{T}$ is **time invariant** (a.k.a. **shift invariant**) iff

$$\mathrm{TD}_k\{\mathcal{T}[x(n)]\} \overset{!?}{=} \mathcal{T}\{\mathrm{TD}_k[x(n)]\}$$

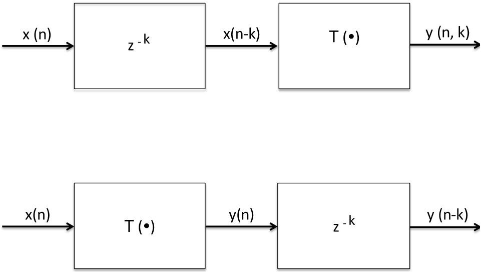
i.e. with $y(n) = \mathcal{T}\{x(n)\}$ it must hold that

$$\mathcal{T}\{\mathrm{TD}_k[x(n)]\} = \mathcal{T}\{x(n-k)\} = y(n,k) \overset{!?}{=} y(n-k) = \mathrm{TD}_k[y(n)]$$

for every input signal $x(n)$ and every time shift k.

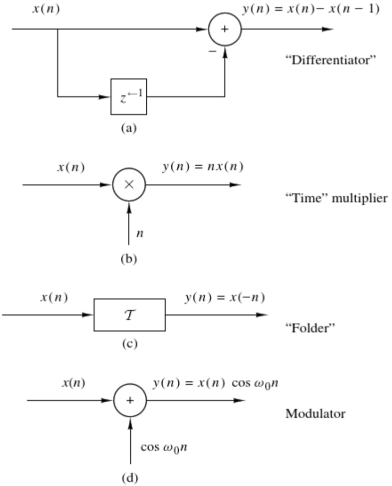
Time-variant versus time-invariant systems

To check, we excite a system with a delayed input sequence $x(n-k)$ which produces the output $y(n,k)$. Next, we compute the output for the input signal $x(n)$ and delay the output $y(n)$ by k samples to obtain $y(n-k)$. If $y(n,k) = y(n-k)$, the system is shift invariant.



Example 2.2.4

Determine if the following systems are time invariant



Linear versus nonlinear systems

Linear System

A system is linear iff

$$\mathcal{T}\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 \mathcal{T}\left[x_1(n)\right] + a_2 \mathcal{T}\left[x_2(n)\right]$$

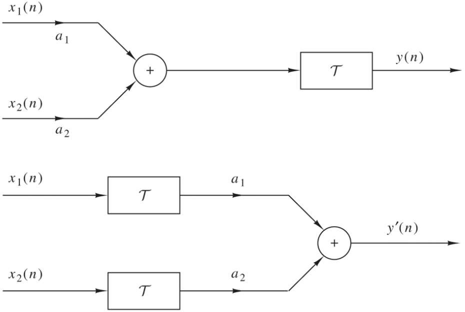


Figure 2.2.9 Graphical representation of the superposition principle. $\mathcal{T}$ is linear if and only if $y(n) = y'(n)$.

Example 2.2.5:

1. $y(n) = nx(n)$
2. $y(n) = x(n^2)$
3. $y(n) = x^2(n)$

Causal versus noncausal systems

Causal System

A system is **causal** if the output of the system at any time n does not depend on future samples $x(n+k)$ with $k>0$.

- Noncausal systems cannot be physically realized (we cannot look into the future)
- Noncausal processing is possible on pre-recorded data.

Example 2.2.6:

1. $y(n) = x(n) - x(n-1)$
2. $y(n) = \sum_{k=-\infty}^{n} x(k)$
3. $y(n) = x(n+2)$
4. $y(n) = x(2n)$
5. $y(n) = x(-n)$

Classification of Discrete-Time Systems

Stable versus unstable systems

Bounded-input bounded output (BIBO) stability

An arbitrary relaxed system is said to be BIBO stable, iff every bounded input $|x(n)| < \infty$ results in a bounded output $|y(n)| < \infty$ for all n.
If for any n the output is infinite, the system is classified unstable

Example 2.2.7:

- System: $y(n) = y^2(n-1) + x(n)$
- Input: $x(n) = C\delta(n)$

Outline

Linear Time-Invariant (LTI) Systems

- While previously we classified systems w.r.t. linearity, causality, stability, and time-invariance, now we particular consider the important class of systems that are at the same time linear and time-invariant, so-called LTI-systems
- We will learn that
  - LTI-systems are fully characterized by their response to a unit sample sequence $\delta(n)$ (for continuous systems: the response to an impulse $\delta(t)$)
  - An arbitrary signal can be represented as a weighted sum of unit sample sequences
  - ➡ The response of a system to any arbitrary input signal can be expressed in terms of the unit sample response (impulse response) of the system

Resolve Discrete-Time Signal into Impulses

➡ An arbitrary signal $x(n)$ can be represented as a weighted sum of unit sample sequences

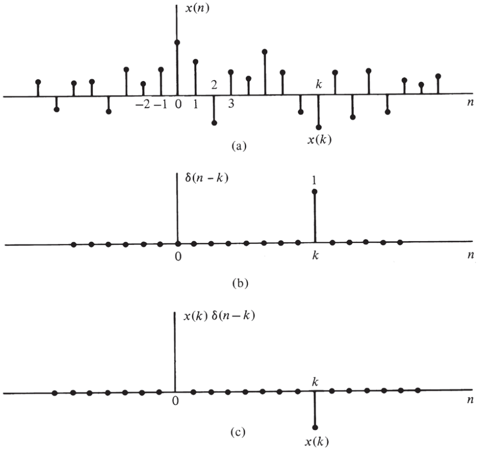


Figure 2.3.1 Multiplication of a signal $x(n)$ with a shifted unit sample sequence.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Resolve Discrete-Time Signal into Impulses

Example 2.3.1

- Consider the special case of a finite-duration sequence given as

$$x(n) = \left\{2, \underset{\uparrow}{4}, 0, 3\right\}$$

Resolve the sequence $x(n)$ into a sum of weighted impulse sequences

Resolve Discrete-Time Signal into Impulses

Example 2.3.1

- Consider the special case of a finite-duration sequence given as

$$x(n) = \left\{2, \underset{\uparrow}{4}, 0, 3\right\}$$

  Resolve the sequence $x(n)$ into a sum of weighted impulse sequences

- Solution:

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

Response of Linear Systems to Arbitrary Inputs

- Unit sample response:

$$h(n,k) \equiv \mathcal{T}\left[\delta(n-k)\right]$$

- Decompose input into sum of unit samples:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

- Response to arbitrary signal

$$y(n) = \mathcal{T}\left[x(n)\right] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

- For a relaxed **linear system**:

$$\begin{aligned} y(n) &= \mathcal{T}\left[x(n)\right] = \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}\left[\delta(n-k)\right] \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n,k) \end{aligned}$$

Response of LTI Systems to Arbitrary Inputs I

The Convolution Sum

- For *time-invariant* systems we have:

$$\mathcal{T}\left[\delta(n-k)\right] = h(n,k) = h(n-k)$$

Response of an LTI-System to Arbitrary Inputs

$$y(n) = \mathcal{T}\left[x(n)\right] = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{2.3.17}$$

- System response: $y(n) = \mathcal{T}\left[x(n)\right]$
- Unit sequence response (impulse response): $h(n) \equiv \mathcal{T}\left[\delta(n)\right]$

- A relaxed LTI system is completely characterized by a single function $h(n)$, which is the response to the unit sample sequence $\delta(n)$
- The input-output relation (2.3.17) is called the **convolution sum**

The Convolution Sum

- The convolution sum $y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0 - k)$ at time n_0 is obtained by
  1. *Folding.* Fold $h(k)$ about $k = 0$ to obtain $h(-k)$
  2. *Shifting.* Shift $h(-k)$ by n_0 (to the right if n_0 is positive) to obtain $h(n_0 - k)$
  3. *Multiplication.* Multiply $x(k)$ by $h(n_0 - k)$ to obtain $v_{n_0} \equiv x(k)h(n_0 - k)$
  4. *Summation.* Sum all the values of the product sequence v_{n_0} to obtain $y(n_0)$

Example 2.3.2

- The impulse response of an LTI system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

Determine the response of the system to the input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

Example 2.3.2

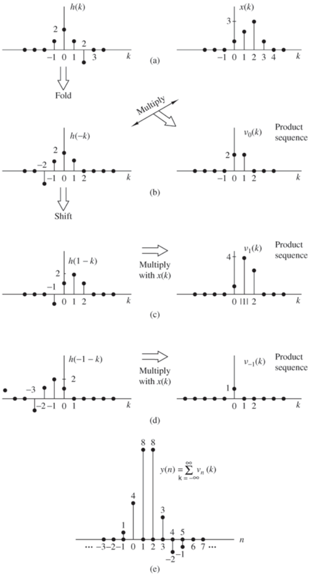


Figure 2.3.2 Graphical computation of convolution.

Properties of Convolution I

Notation The convolution is denoted by an asterisk "$*$"

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Identity and shifting The unit sample sequence $\delta(n)$ is the identity element of convolution, i.e.

- $x(n) * \delta(n) = x(n)$,
- $x(n) * \delta(n-k) = x(n-k)$

Properties of Convolution II

Commutative With a change of variable $k \leftarrow n-k$, we see that

$$y(n) = x(n) * h(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Associative

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

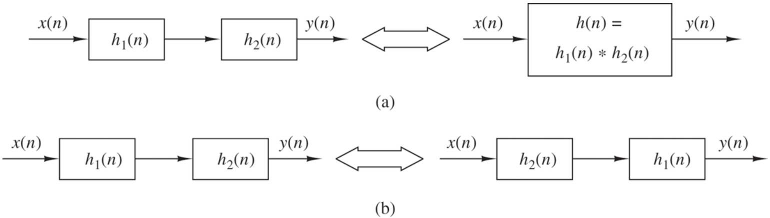


Figure 2.3.5 Implications of the associative (a) and the associative and commutative (b) properties of convolution.

Distributive

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

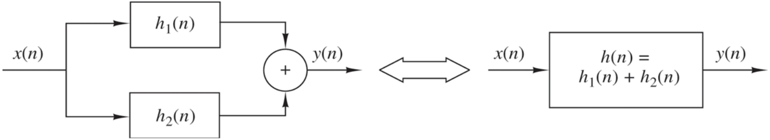


Figure 2.3.6 Interpretation of the distributive property of convolution: two LTI systems connected in parallel can be replaced by a single system with $h(n) = h_1(n) + h_2(n)$.

➡ Conversely, also means that: any LTI system can be decomposed into a parallel interconnection of subsystems

Causal Linear Time-Invariant Systems

- For a causal system, the output at time n_0 is independent of future observations $n > n_0$
- From the convolutional sum,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

we follow that for an LTI-system, this means that

$$h(n) = 0, \quad n < 0$$

An LTI system is causal, iff its impulse response is zero for negative n

- Then

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{n} x(k)h(n-k) = \sum_{k=0}^{\infty} x(n-k)h(k)$$

- For LTI-systems we have

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
|y(n)| &= \left| \sum_{k=-\infty}^{\infty} x(k)h(n-k) \right| \\
|y(n)| &\leq \sum_{k=-\infty}^{\infty} \underbrace{|x(k)|}_{\text{bounded}} \, |h(n-k)|
\end{aligned}
$$

- Consequently, an LTI-system is BIBO stable, if $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

An LTI-system is stable if its impulse response is absolutely summable

FIR and IIR Systems

It will be convenient to distinguish between systems with Finite Impulse Responses (**FIR**) and Infinite Impulse Responses (**IIR**).

- For causal FIR Systems, we have $h(n) = 0$, for $n < 0$ and $n \geq M$

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

  - Here the system acts as a windows which only views the most recent M samples.
  - Finite memory of M samples

- For causal IIR Systems, we have

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

Systems Described by Difference Equations

- For an FIR filter it is clear that it can be realized by means of a finite amount of memory, additions and multiplications

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

- But is it even possible to realize an IIR filter in practice?

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

Systems Described by Difference Equations

- For an FIR filter it is clear that it can be realized by means of a finite amount of memory, additions and multiplications

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

- But is it even possible to realize an IIR filter in practice?

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

  - Certainly not using a convolution
  - But are there other means?

Example

- Take the cumulative average

$$y(n) = \frac{1}{n+1} \sum_{k=0}^{n} x(k), \quad n = 0, 1, ...$$

  → requires storage of all input samples
- Instead, we can compute $y(n)$ more efficiently in a recursive manner

Example

- Take the cumulative average

$$y(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k), \quad n = 0, 1, \ldots$$

→ requires storage of all input samples

- Instead, we can compute $y(n)$ more efficiently in a recursive manner
- With a simple rearrangement we obtain

$$\begin{aligned}(n+1)y(n) &= \sum_{k=0}^{n-1} x(k) + x(n) \\ &= ny(n-1) + x(n) \\ y(n) &= \frac{n}{n+1} y(n-1) + \frac{1}{n+1} x(n)\end{aligned}$$

→ In a recursive implementation, only one memory element is needed

- Infinite impulse responses result by adding a recursive component, e.g.

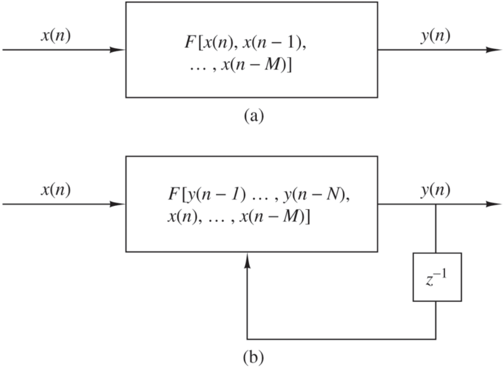


Figure 2.4.3 Basic form for a causal and realizable (a) nonrecursive and (b) recursive system.

- Systems containing recursive and non-recursive parts are described by **difference equations** with **finite** N, M

$$y(n) = \underbrace{-\sum_{k=1}^{N} a_k y(n-k)}_{\text{recursive part}} + \underbrace{\sum_{k=0}^{M} b_k x(n-k)}_{\text{nonrecursive part}}, \tag{2.5.6}$$

- N is called the **order** of the difference equation / system
- Solving the difference equations (determining the output for a given input) is similar to solving differential equations (see Sec. 2.4.3 in Proakis' Book)
  - ➡ However, solving the difference equations in a spectral domain (the z-domain) is much simpler!

BIBO Stability

➔ The impulse response of such a system is exponential, i.e. **IIR**, e.g. of the form

$$h(n) = \sum_{k=1}^{N} C_k \lambda_k^n$$

- To be BIBO stable, it must hold that $\sum_{n=0}^{\infty} |h(n)| < \infty$
  - If $|\lambda_k| < 1$ for all k then $\sum_{n=0}^{\infty} |\lambda_k|^n < \infty$ and the system is stable
  - If one or more $|\lambda_k| \geq 1$, the system is not BIBO stable

Outline

Direct Form I and Direct Form II

$$y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$$

Direct Form I and Direct Form II

$$y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$$

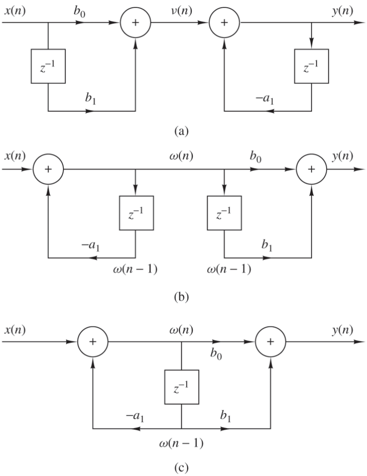


Figure 2.5.1 Steps in converting from the direct form I realization in (a) to the direct form II realization in (c).

Direct Form I for Difference Equation (2.5.6)

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

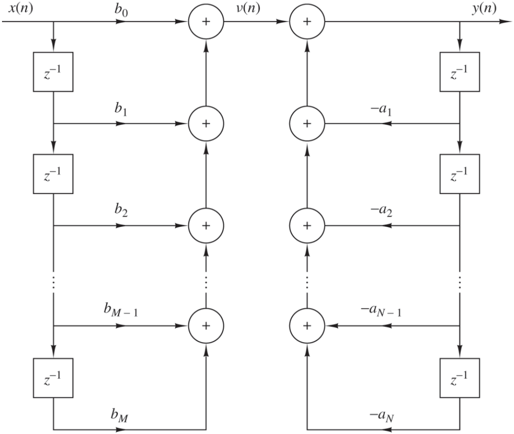


Figure 2.5.2 Direct form I structure of the system described by (2.5.6).

Direct Form II (Canonical Form) for Difference Equation (2.5.6)

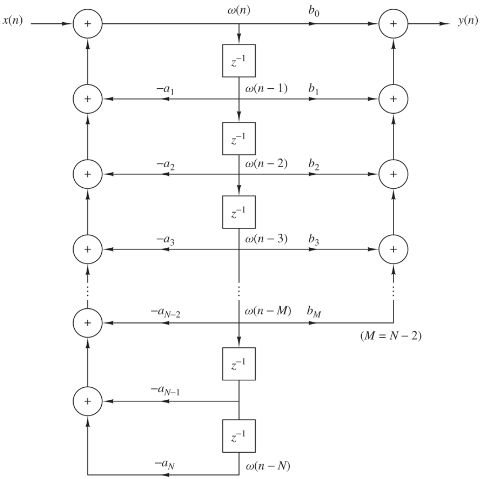


Figure 2.5.3 Direct form II structure for the system described by (2.5.6).

Examples of second order systems

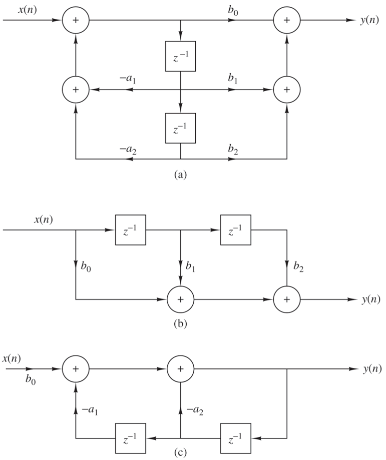


Figure 2.5.4 Structures for the realization of second-order systems: (a) general second-order system; (b) FIR system; (c) "purely recursive system."

Special cases

- Recall that the difference equation is given by

$$y(n) = \underbrace{-\sum_{k=1}^{N} a_k y(n-k)}_{\text{recursive part}} + \underbrace{\sum_{k=0}^{M} b_k x(n-k)}_{\text{nonrecursive part}}, \qquad (2.5.6)$$

  - If $a_k = 0$ for all k, we only have a non-recursive filter with finite impulse response (FIR) $h(k) = b_k$. Such as system is also referred to as **Moving Average (MA)** system
  - For $M = 0$ we have a purely recursive filter with infinite impulse response (IIR). Such as system is also referred to as **Auto Regressive (AR)** system

Outline

- Correlation measures the *similarity* between two signals
  - e.g. to find objects in images (e.g. faces)
- Correlation is mathematically similar to convolution, the difference is that for correlation, the signals are not folded

Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

Correlation

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = x(l) * y(-l)$$

- l is called the signal **lag**
- If a convolution module is available, it can also be used to compute correlation (and vice versa)

- While the cross correlation $r_{xy}(l)$ measures the similarity between signals $x(n)$ and $y(n)$, the *autocorrelation* $r_{xx}(l)$ measures the self-similarity of a signal

Autocorrelation

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) = x(l) * x(-l)$$

- $r_{xx}(0)$ is the energy of signal $x(n)$
- For signals periodic in N we have

$$r_{xx}(0) = r_{xx}(N) = \max_{l} r_{xx}(l)$$

  - Can be used to find the fundamental frequency in speech or music signals

Example of Autocorrelation

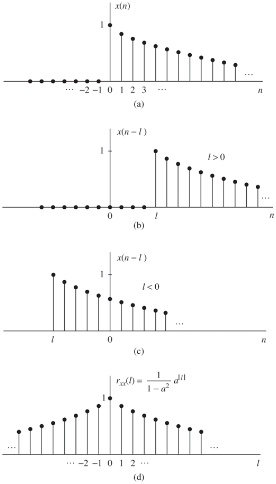
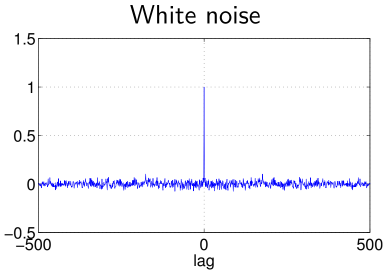
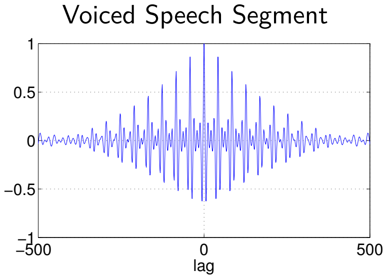


Figure 2.6.2 Computation of the autocorrelation of the signal $x(n) = a^n$, $0 < a < 1$.

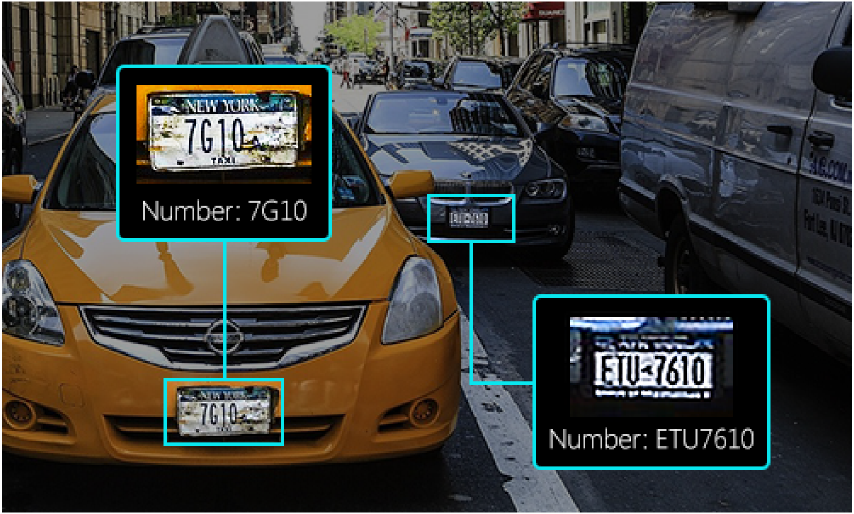




White noise $r_{xx}(l) = E_x\delta(n)$: Successive samples are uncorrelated

Voiced speech The peak next to the lag $l = 0$ of the autocorrelation function corresponds to the fundamental period N.

Convolutional Neural Networks



Source: http://rnd.azoft.com/convolutional-neural-networks-object-detection/

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SP
Signal Processing

3. The z-Transform and Its Applications

Outline

Motivation

- The z-transform is a specific spectral transform
- It is particularly well suited to analyze discrete-time LTI-systems
- The convolution of two time-domain signals corresponds to their multiplication in z-domain
- The z-transform plays the same role for discrete-time signals as the Laplace transform for continuous-time signals.

Outline

The z-Transform

The z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) \equiv Z\{x(n)\} \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{12}$$

with $z \in \mathbb{C}$ a complex-valued variable

- z-transform exists only for those values of z where the power series converges
- The **region of convergence (ROC)** of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value
- Thus, whenever we cite a z-transform, we should also indicate the ROC.

Examples

$$X(z) \equiv Z\{x(n)\} \equiv \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

1. $x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$
2. $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$
3. $x(n) = \delta(n)$
4. $x(n) = \delta(n-k), k > 0$
5. $x(n) = \delta(n+k), k > 0$

Examples

$$X(z) \equiv Z\{x(n)\} \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

1. $x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$
2. $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$
3. $x(n) = \delta(n)$
4. $x(n) = \delta(n-k), k > 0$
5. $x(n) = \delta(n+k), k > 0$

Solution

1. $X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire z-plane, except $z = 0$
2. $X(z) = 1z^{2} + 2z^{1} + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane, except $z = 0, z = \infty$
3. $\delta(n) \overset{z}{\circ\!\!-\!\!\bullet} 1$, ROC: entire z-plane
4. $\delta(n-k) \overset{z}{\circ\!\!-\!\!\bullet} z^{-k}$, ROC: entire z-plane, except $z = 0$
5. $\delta(n+k) \overset{z}{\circ\!\!-\!\!\bullet} z^{k}$, ROC: entire z-plane, except $z = \infty$

The Region of Convergence (ROC)

- The ROC of a finite-duration signal is the entire z-plane, except possibly $z = 0$ and $z = \infty$
  - z^k $(k > 0)$ becomes unbounded for $z = \infty$
  - $z^{-k} = \frac{1}{z^k}$ $(k > 0)$ becomes unbounded for $z = 0$
- Mathematically speaking, the z-transform is simply an alternative representation of a signal.
  - The coefficient of z^{-n} is the signal value at time n

$$
\begin{aligned}
x(n) \quad &= \quad \{\ldots, x(0), x(1), x(2), x(3), \ldots\} \\
&\updownarrow z \\
X(z) \quad &= \quad \ldots + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \ldots
\end{aligned}
$$

Example 3.1.2 I

- Determine the z-transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

- Solution:

$$x(n) = \left\{1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \ldots, \left(\frac{1}{2}\right)^n, \ldots\right\}$$

$$X(z) = 1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \ldots + \left(\frac{1}{2}\right)^n z^{-n} + \ldots$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n$$

- Note that in general the *geometric series* converges as

$$1 + A + A^2 + A^3 + \ldots = \frac{1}{1-A}, \quad \text{if } |A| < 1$$

- Consequently for $|\frac{1}{2}z^{-1}| < 1$, i.e. $|z| > \frac{1}{2}$, $X(z)$ converges as

$$x(n) = \left(\frac{1}{2}\right)^n u(n) \quad \overset{z}{\circ\!\!-\!\!\bullet} \quad X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

z in polar form

- Let us represent the complex variable $z \in \mathbb{C}$ in polar form

$$z = re^{j\theta}$$

with $r = |z|$ and $\theta = \angle z$ Then, the z-transform results in

$$X(z)\Big|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}$$

- To determine the ROC, it must hold that $|X(z)| < \infty$, with

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n} \right| \leq \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} e^{-j\theta n} \right| = \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right|$$

➡ ROC only depends on $r = |z|$

Region of Convergence (ROC) with z in polar form

$$
\begin{aligned}
|X(z)| &\leq \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| \\
&\leq \sum_{n=1}^{\infty} \left| x(-n) r^{n} \right| \\
&\quad + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \\
&\stackrel{!}{\leq} \infty
\end{aligned}
$$

- from the first summand it follows that $r \stackrel{!}{<} r_1$
- from the second summand it follows that $r \stackrel{!}{>} r_2$

➡ ROC: $r_2 < r < r_1$

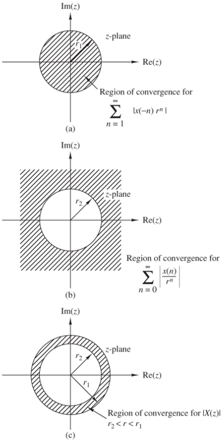


Figure 3.1.1 Region of convergence for $X(z)$ and its corresponding causal and anticausal components.

Example 3.1.3

- Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Example 3.1.3

- Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- From the z-transform definition we get

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \left(\alpha z^{-1}\right)^n$$

- if $|\alpha z^{-1}| < 1$ (i.e. $|z| > |\alpha|$) this power series converges to

$$x(n) = \alpha^n u(n) \;\overset{z}{\circ\!\!-\!\!\bullet}\; X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{ROC: } |z| > |\alpha|$$

Visualization

$$x(n) = \alpha^n u(n) \quad \overset{z}{\circ\!\!-\!\!\bullet} \quad X(z) = \frac{1}{1 - \alpha z^{-1}}, \qquad \text{ROC: } |z| > |\alpha|$$

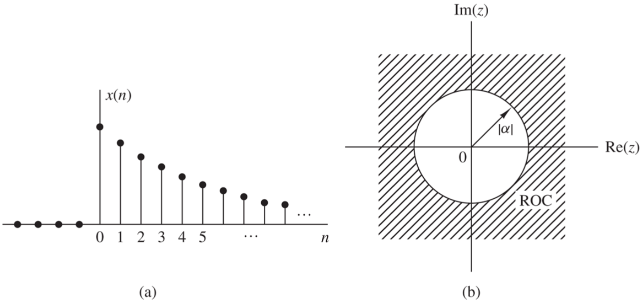


Figure 3.1.2 The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

Example 3.1.4

- Determine the z-transform of the signal

$$x(n) = -\alpha^n u(-n-1)$$

- Determine the z-transform of the signal

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

$$x(n) = -\alpha^n u(-n-1) \overset{z}{\circ\!\!-\!\!\bullet} X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \text{ROC: } |z| < |\alpha|$$

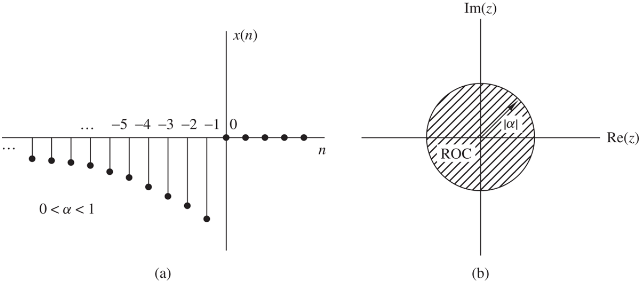


Figure 3.1.3 Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

Discussion of Examples 3.1.3 and 3.1.4

- The causal signal $\alpha^n u(n)$ and the anticausal signal $x(n) = -\alpha^n u(-n-1)$ have the exact same z-transform!

➡ The closed-form expression of the z-transform does not uniquely specify the time-domain signal

➡ The ambiguity can be resolved if the ROC is specified!

Example 3.1.5

- Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

Example 3.1.5

- Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}}$$

with ROC $|\alpha| < |z| < |b|$

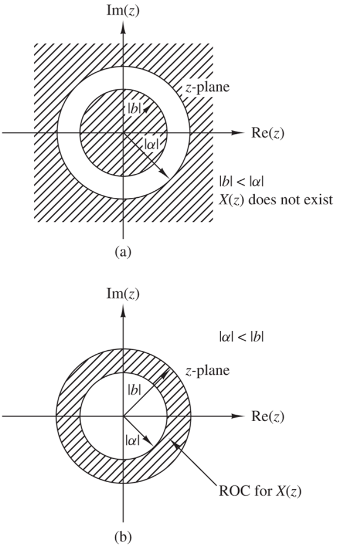


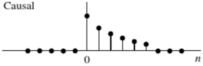
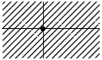
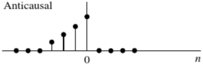
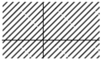
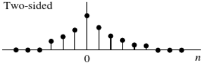
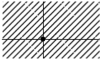
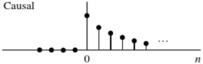
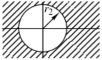
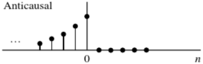
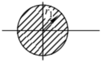
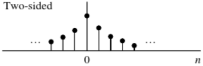
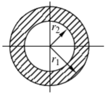
Figure 3.1.4 ROC for z-transform in Example 3.1.5.

Conclusions from the Examples

- A discrete time-domain signal is uniquely determined by its z-transform $X(z)$ and the region of convergence (ROC) of $X(z)$
- The ROC of a causal signal is the exterior of a circle of some radius r_2, while
- the ROC of an anticausal signal is the interior of a circle of some radius r_1
- If there exists a ROC for an infinite-duration two-sided signal, it is a ring in the z-plane

Characteristic Families of Signals with their ROCs

TABLE 1 Characteristic Families of Signals with Their Corresponding ROCs

	Signal	ROC	
Finite-Duration Signals			
Causal			Entire z-plane except $z = 0$
Anticausal			Entire z-plane except $z = \infty$
Two-sided			Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals			
Causal			$\|z\| > r_2$
Anticausal			$\|z\| < r_1$
Two-sided			$r_2 < \|z\| < r_1$

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Cauchy Integral Theorem

- The inverse z-Transform is based on the *Cauchy Integral Theorem*

$$\frac{1}{2\pi j}\oint_C z^{n-1-k}\,dz = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

The $\oint_C$ denotes an integration over a closed contour within the ROC of $X(z)$

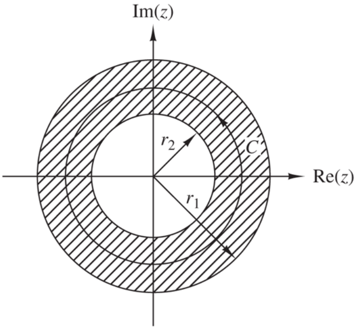

Figure 3.1.5 Contour C for integral in (3.1.13).

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Derivation

- Start with definition of z-transform $X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$
- Multiply both sides with z^{n-1} and integrate over closed contour within ROC

$$\begin{aligned} \oint_C X(z) z^{n-1} dz &= \oint_C \sum_{k=-\infty}^{\infty} x(k) z^{n-1-k} dz \\ &= \sum_{k=-\infty}^{\infty} x(k) \oint_C z^{n-1-k} dz \end{aligned}$$

- The using the Cauchy integral theorem

$$\frac{1}{2\pi j} \oint_C z^{n-1-k} dz = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

- Leads to

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

The z-Transform and Its Inverse

SP Signal Processing

The z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The inverse z-Transform

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- Typically, we deal with signals which have a rational z-transforms (i.e. z-transforms that are a ratio of two polynomials)
- Simpler method: Use a table lookup!

Some Common z-Transform Pairs, Table 3.3

	Signal, $x(n)$	z -Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{\left(1-a_2^{-1}\right)^2}$	$\|z\| < \|a\|$
7	$(\cos\omega_0 n)\, u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin\omega_0 n)\, u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n\cos\omega_0 n)\, u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n\sin\omega_0 n)\, u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Outline

Properties of the z-Transform

- The properties of the z-Transform can all be derived using the definition of the z-transform
- See Section 3.2 in Proakis Manolakis book

Time Shifting

$$x(n-k) \overset{z}{\circ\!\!-\!\!\bullet} z^{-k}X(z)$$

Time Shifting

$$x(n-k) \overset{z}{\circ\!\!-\!\!\bullet} z^{-k}X(z)$$

- Derivation

$$
\begin{aligned}
X(z) &= \sum_{n'=-\infty}^{\infty} x(n')z^{-n'} \\
&\downarrow \quad n' = n-k \\
&= \sum_{n=-\infty}^{\infty} x(n-k)z^{-n}z^{k} \\
X(z)z^{-k} &= \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \\
x(n-k) &\overset{z}{\circ\!\!-\!\!\bullet} z^{-k}X(z)
\end{aligned}
$$

Properties of the z-Transform

Time Reversal

$$x(n) \overset{z}{\circ\!\!-\!\!\bullet} X(z), \qquad \text{ROC: } r_2 < |z| < r_1$$

$$x(-n) \overset{z}{\circ\!\!-\!\!\bullet} X(z^{-1}), \qquad \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

Properties of the z-Transform

Time Reversal

$$x(n) \circ\!\!\stackrel{z}{-}\!\!\bullet\ X(z), \qquad \text{ROC: } r_2 < |z| < r_1$$

$$x(-n) \circ\!\!\stackrel{z}{-}\!\!\bullet\ X(z^{-1}), \qquad \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

- Derivation

$$\begin{aligned}
Z\{x(-n)\} &= \sum_{n=-\infty}^{\infty} x(-n) z^{-n} \\
&\quad \downarrow l = -n \\
&= \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l} \\
&= X(z^{-1})
\end{aligned}$$

Convolution of Two Sequences

$$x_1(n) * x_2(n) \overset{z}{\circ\!\!-\!\!\bullet} X_1(z)X_2(z)$$

Properties of the z-Transform

Convolution of Two Sequences

$$x_1(n) * x_2(n) \overset{z}{\circ\!\!-\!\!\bullet} X_1(z)X_2(z)$$

Derivation

- Definition of convolution: $x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$
- Take z-transform

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x_1(k) \underbrace{\left[\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}\right]}_{X_2(z)z^{-k}} \\
&= X_2(z) \sum_{k=-\infty}^{\infty} x_1(k)z^{-k} = X_2(z)X_1(z)
\end{aligned}$$

SP Signal Processing

Convolution of Two Sequences

$$x_1(n) * x_2(n) \overset{z}{\circ\!\!-\!\!\bullet} X_1(z)X_2(z)$$

- One of the most powerful properties!

Computation of the convolution $x(n) = x_1(n) * x_2(n)$

1. Compute z-transform of the signals to be convolved

$$\begin{aligned} X_1(z) &= Z\{x_1(n)\} \\ X_2(z) &= Z\{x_2(n)\} \quad \text{(time domain} \longrightarrow z\text{-domain)} \end{aligned}$$

2. Multiply the two z-transforms

$$X(z) = X_1(z)X_2(z) \quad (z\text{-domain})$$

3. Find the inverse z-transform of $X(z)$

$$x(n) = Z^{-1}\{X(z)\} \quad (z\text{-domain} \longrightarrow \text{time domain})$$

Properties of the z-Transform

Property	Time Domain	z-Domain	ROC
Notation	$x(n), x_1(n), x_2(n)$	$X(z), X_1(z), X_2(z)$	ROC: $r_2 < \lvert z \rvert < r_1$, ROC$_1$, ROC$_2$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	At least the intersection of ROC$_1$ and ROC$_2$
Time shifting	$x(n-k)$	$z^{-k} X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1} z)$	$\lvert a \rvert r_2 < \lvert z \rvert < \lvert a \rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z \rvert < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$Re\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$Im\{x(n)\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-Domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z \rvert < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z) X_2(z)$	At least the intersection of ROC$_1$ and ROC$_2$
Correlation	$r_{x_1 x_2}(l) = x_1(l) * x_2(l)$	$R_{x_1 x_2}(z) = X_1(z) X_2(z^{-1})$	At least the intersection of the ROCs of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n) x_2(n)$	$\frac{1}{2\pi j} \oint_C X_1(v) X_2(\frac{z}{v}) v^{-1} dv$	At least $r_{1l} r_{2l} < \lvert z \rvert < r_{1u} r_{2u}$
Parseval's relation	$\sum^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*(\frac{1}{v^*}) v^{-1} dv$		

Some Common z-Transform Pairs

	Signal, $x(n)$	z -Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{\left(1-a_2^{-1}\right)^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)\,u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)\,u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)\,u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)\,u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Rational z-Transforms

- As we saw from previous table, typical z-transforms are *rational functions*, i.e. ratios of polynomials in z^{-1}
- We will now show that rational z-transforms are also encountered when characterizing discrete-time LTI-systems described by difference equations

Fundamental theorem of algebra

- "Every non-zero single-variable polynomial has at least one root"
- A polynomial in z can be divided by $(z - a)$

$$\begin{aligned} \frac{p(z)}{z-a} &= q(z) + \frac{R}{z-a} \\ \rightarrow p(z) &= (z-a)q(z) + R \end{aligned}$$

- $q(z)$ is a polynomial with one degree less than $p(z)$
  The residual R is not a function of z
- if $a = z_0$ is a root, then

$$p(z_0) \stackrel{!}{=} 0 = (z_0 - z_0)q(z_0) + R \quad \rightarrow R = 0!$$

$\Rightarrow$ If $a = z_0$ is a root, the residual is zero!

$$p(z) = (z - z_0)q(z)$$

➡ Every polynomial can be factorized by its roots

Rational z-Transforms I

Poles and zeros

- The **zeros** of a z-transform $X(z)$ are the values of z for which $X(z) = 0$
- The **poles** of a z-transform $X(z)$ are the values of z for which $X(z) = \infty$
- If $X(z)$ is a rational function then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_K z^{-k}}{\sum_{k=0}^{N} a_K z^{-k}}$$

- If $a_0 \neq 0$ and $b_0 \neq 0$ we can avoid negative powers of z by factorizing as

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^{-M}}{a_0 z^{-N}} \cdot \frac{z^M + (b_1/b_0) z^{M-1} + \ldots + b_M/b_0}{z^N + (a_1/a_0) z^{N-1} + \ldots + a_M/a_0}$$

Rational z-Transforms II

Poles and zeros

- Since $B(z)$ and $A(z)$ are polynomials, they can be expressed in factored form as

$$
\begin{aligned}
X(z) &= \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\dots(z-z_M)}{(z-p_1)(z-p_2)\dots(z-p_N)} \\
&= \underbrace{G}_{\frac{b_0}{a_0}} z^{N-M} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}
\end{aligned}
$$

$X(z)$ has

- M finite zeros z_k (the roots of the numerator polynomial)
- N finite poles p_k (the roots of the denominator polynomial)
- $|N-M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin $z = 0$
- A zero (pole) exists at $z = \infty$ if $X(\infty) = 0$ ($X(\infty) = \infty$)
- $X(z)$ has exactly as many poles as zeros!

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Pole-zero plot

- $X(z)$ can be represented graphically by a *pole-zero plot* in the complex plane
- poles are represented by crosses $\times$
- zeros are represented by circles $\circ$
- Obviously, by definition, the ROC should not contain any poles!

Pole-zero plot

- $X(z)$ can be represented graphically by a *pole-zero plot* in the complex plane
- poles are represented by crosses $\times$
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➡ Exercise 3.3.1: Determine pole-zero plot for $x(n) = a^n u(n)$

Pole-zero plot

- $X(z)$ can be represented graphically by a *pole-zero plot* in the complex plane
- poles are represented by crosses $\times$
- zeros are represented by circles $\circ$
- Obviously, by definition, the ROC should not contain any poles!

➡ Exercise 3.3.1: Determine pole-zero plot for $x(n) = a^n u(n)$

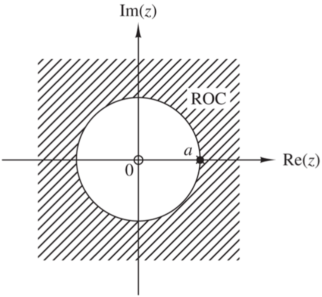

Figure 3.3.1 Pole–zero plot for the causal exponential signal $x(n) = a^n u(n)$.

Signal Processing

Example 3.3.2

- Determine the pole-zero plot for the signal

$$x(n) = \begin{cases} a^n, & 0 \leq n \leq M-1 \quad \text{where } a > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Example 3.3.2

- Determine the pole-zero plot for the signal

$$x(n) = \begin{cases} a^n, & 0 \leq n \leq M-1 \\ 0, & \text{elsewhere} \end{cases} \quad \text{where } a > 0$$

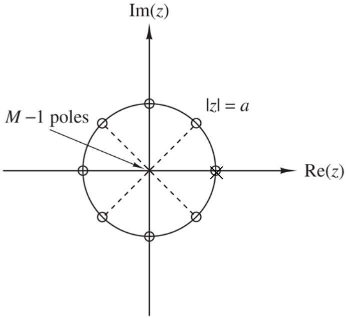

Figure 3.3.2 Pole–zero pattern for the finite-duration signal $x(n) = a^n$, $0 \leq n \leq M-1 (a > 0)$, for $M = 8$.

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Example 3.3.3

- Determine the z-transform and the signal that corresponds to the pole-zero plot of Figure 3.3.3

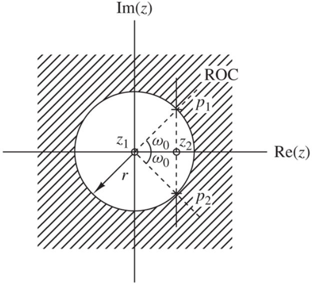

Figure 3.3.3 Pole-zero pattern for Example 3.3.3.

- The product $(z - p_1)(z - p_2)$ results in a polynomial with real-valued coefficients *if* p_1 *and* p_2 *are complex conjugates*
- In general, if a polynomial has real-valued coefficients, its roots are either real-valued or occur in complex-conjugate pairs

Visualization of $|X(z)|$

- Instead of the pole-zero plot, we can also represent the magnitude $|X(z)|$ as a two-dimensional surface in the complex plane, e.g.

$$X(z) = \frac{z^{-1} - z^{-2}}{1 + 1.2732z^{-1} + 0.81z^{-2}} \tag{3.3.3}$$

Visualization of $|X(z)|$

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$$X(z) = \frac{z^{-1} - z^{-2}}{1 + 1.2732z^{-1} + 0.81z^{-2}} \tag{3.3.3}$$

➡ one zero at $z_1 = 1$, two poles at $p_1, p_2 = 0.9e^{\pm j\pi/4}$

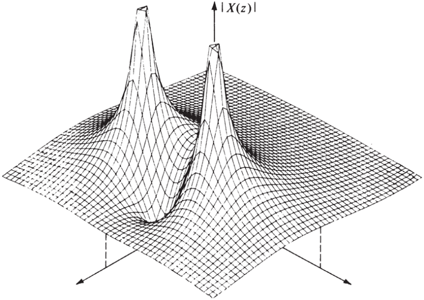

Figure 3.3.4 Graph of $|X(z)|$ for the z-transform in (3.3.3).

Pole Location and Time-Domain Behavior I

Single real-valued pole

- Here, we investigate the relation between pole-pairs and the corresponding time-domain signal
- Here, we exclusively deal with real-valued causal signals
- We will see that the characteristic behavior of causal signals depends on whether the poles are inside ($|z| < 1$) or outside ($|z| > 1$), or *on* ($|z| = 1$) the **unit circle**
- If a real-valued signal has a z-transform with one pole, this pole has to be real-valued. The only such signal is the real-valued exponential!

$$x(n) = a^n u(n) \overset{z}{\circ\!\!-\!\!\bullet} X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$

having one zero at $z_1 = 0$ and one pole at $p_1 = a$ on the real axis

- see illustration on next page

Single real-valued pole

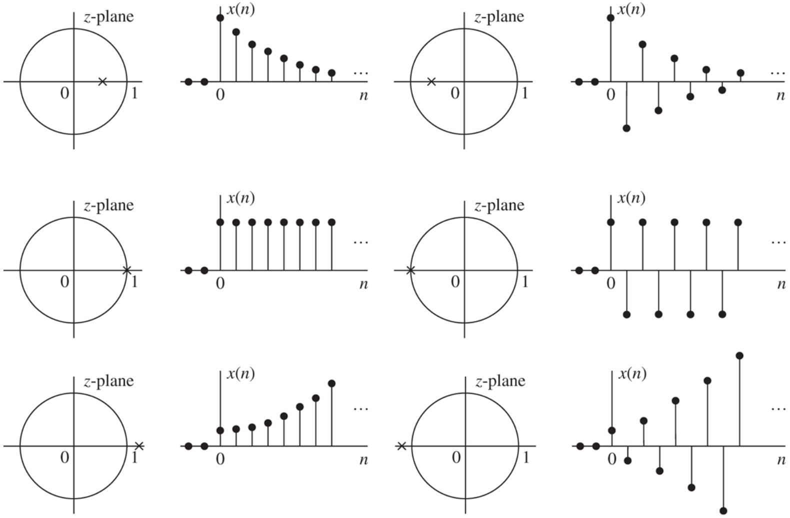

Figure 3.3.5 Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

Pole Location and Time-Domain Behavior III

Single real-valued pole

- Signal is decaying if pole is inside the unit circle
- Signal is fixed if the pole is on the unit circle
- Signal is growing if the pole is outside the unit circle
- Negative poles results in a signal that alternates in sign.

→ Causal signals with poles outside the unit circle become unbounded, cause overflow in digital systems and should generally be avoided

Double real-valued pole

- A causal real-valued signal with a double real-valued pole has the form (see Tables): $x(n) = na^n u(n) \quad \circ\!\!\overset{z}{-\!\!-}\!\!\bullet \quad \frac{az^{-1}}{(1-az^{-1})^2}$

➡ A double real pole on the unit circle results in an unbounded signal

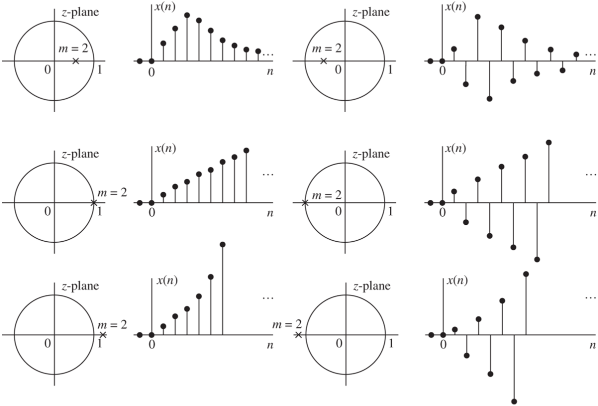

Figure 3.3.6 Time-domain behavior of causal signals corresponding to a double $(m=2)$ real pole, as a function of the pole location.

Complex conjugate poles

- Complex conjugate poles ➡ exponentially weighted sinusoidal signal

$$r^n \cos(\omega_0 n) u(n) \quad \overset{z}{\circ\!\!-\!\!\bullet} \quad \frac{1 - r z^{-1} \cos(\omega_0)}{1 - 2 r z^{-1} \cos(\omega_0) + z^{-2} r^2}$$

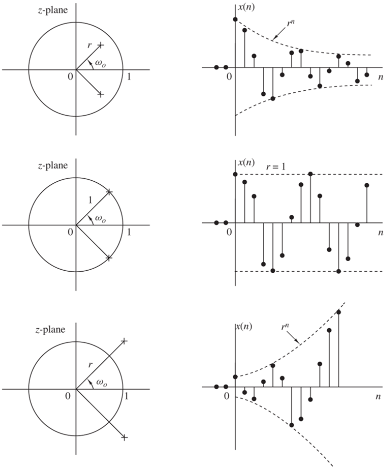

Figure 3.3.7 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

Double conjugate poles

- Careful with double poles on the unit circle

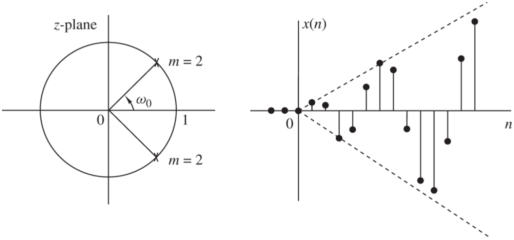

Figure 3.3.8 Causal signal corresponding to a double pair of complex-conjugate poles on the unit circle.

Summary

- Causal real-valued signals with simple real-valued poles or simple complex-conjugate pairs of poles which are inside or on the unit circle are always bounded in amplitude
- The closer poles are to the origin, the more rapidly the signal decays
- ➡ Time-domain behavior of a signal depends strongly on the *location of the poles relative to the unit circle*
- ➡ Zeros affect behavior of a signal not as strongly as poles
  - e.g. for a sinusoidal signal, the presence and location of zeros affects only its phase
- The results found for causal signals also applies to the impulse responses of causal LTI systems
  - ➡ If a system has a pole outside the unit circle, the system is unstable

System Function of an LTI-System

- The output of an LTI-system to an input sequence $x(n)$ is given by

$$y(n) = x(n) * h(n) \quad \overset{z}{\circ\!\!-\!\!\bullet} \quad Y(z) = H(z)X(z)$$

- Impulse response $h(n)$ corresponds to the time-domain characterization of the system
- System function $H(z) = Z\{h(n)\}$ corresponds to the z-domain characterization of the system
- Show that: If the system is described by a linear constant-coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

the system function is a *rational function*

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the system function is a *rational function*

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

All-zero systems

- $a_k = 0$ for $1 \leq k \leq N$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \cancel{\sum_{k=1}^{N} a_k z^{-k}}}$$

$$= \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

$$y(n) = -\sum_{k=1}^{N} \cancel{a_k y(n-k)} + \sum_{k=0}^{M} b_k x(n-k)$$

$$= \sum_{k=0}^{M} b_k x(n-k)$$

- $H(z)$ contains M zeros whose values are determined by the system parameters b_k (and Mth order pole at $z = 0$)
- called **all-zero** system or moving average (**MA**) system
- has finite-duration impulse response (**FIR**)

All-pole systems

- $b_k = 0$ for $1 \le k \le M$

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}}, \qquad a_0 \equiv 1$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + b_0 x(n)$$

  - $H(z)$ contains N poles whose values are determined by the system parameters a_k (and Nth order zero at $z = 0$)
  - called **all-pole** system or autoregressive (**AR**) system
  - has infinite-duration impulse response (**IIR**)

Pole-zero systems

- The general system

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

  - $H(z)$ contains N poles and M zeros b_k
  - Poles and/or zeros at $z = 0$ and $z = \infty$ are implied but not counted explicitly
  - called **pole-zero** system or autoregressive moving average (**ARMA**) system
  - has infinite-duration impulse response (**IIR**)

System Function of an LTI-System

Example 3.3.4

- Determine system function and the unit sample response of the system described by the difference equation

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

Example 3.3.4

- Determine system function and the unit sample response of the system described by the difference equation

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

- Solution:

$$H(z) = \frac{2}{1 - \frac{1}{2}z^{-1}}$$

  - pole at $z = 1/2$ and a zero at the origin.
- Using the table we obtain the inverse and thus the unit sample response

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

Outline

- Recall that if $X(z)$ is a rational function then

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The polynomials $B(z)$ and $A(z)$ can also be represented by their roots

$$X(z) = \underbrace{G}_{\frac{b_0}{a_0}} z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

  ➡ realization by a serial concatenation of sub-filters possible

- For distinct poles $p_1, p_2, \ldots, p_N$ a partial fraction expansion results in

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \ldots + \frac{A_N}{z - p_N}, \quad A_k = \left. \frac{(z - p_k)X(z)}{z} \right|_{z=p_k}$$

  ➡ realization by parallel sub-filters (the summands) possible

  ➡ the **inverse z-transform** can be found by using a partial fraction expansion and looking up each summand in an inversion table

Recent background read: B. Bank (2018): "Converting Infinite Impulse Response Filters to Parallel Form", IEEE Sig. Proc. Mag, May 2018

Examples 3.4.5 and 3.4.8

Determine inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}, \quad \text{ROC: } |z| > 1$$

Examples 3.4.5 and 3.4.8

Solution:

- Determine the partial-fraction expansion of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

- Eliminate negative powers

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

- Find poles $\rightarrow p_1 = 1,\ p_2 = 0.5$
- Approach

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} \stackrel{!}{=} \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

Partial-Fraction Expansion III

Examples 3.4.5 and 3.4.8

- Multiply both sides by $(z-1)(z-0.5)$

$$\begin{aligned} z &= (z-0.5)A_1 + (z-1)A_2 \\ z &= z(A_1+A_2) - 0.5A_1 - A_2 \end{aligned}$$

- For the last equation to hold, it follows

$$\rightarrow A_2 = -1, A_1 = 2$$

- Thus

$$\frac{X(z)}{z} = \frac{2}{z-1} + \frac{-1}{z-0.5}$$

$$X(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

- For ROC $|z| > 1$ both components are causal. The inverse z-transform is

$$\begin{aligned} x(n) &= 2 \cdot 1^n u(n) - 0.5^n u(n) \\ &= (2-0.5^n)u(n) \end{aligned}$$

Outline

- A necessary and sufficient condition for an LTI system to be BIBO stable is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- Since

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

it follows that

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}| \overset{|z|=1}{=} \sum_{n=-\infty}^{\infty} |h(n)|$$

- In the ROC, per definition $|H(z)| < \infty$
- If unit circle $|z| = 1$ is in the ROC, then $|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty$
  - ➡ the system is BIBO stable

Causality and Stability

Causality

An LTI system is causal, iff the ROC of the system is the exterior of a circle of radius $r < \infty$, including $z = \infty$

Stability

An LTI system is BIBO stable, iff the ROC of the system includes the unit circle

Since per definition, the ROC cannot contain any poles:

Stability for Causal Systems

A *causal* LTI system is BIBO stable, iff all poles of $H(z)$ are inside the unit circle

Example 3.5.2

- An LTI system is given by

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

- Specify the ROC, determine $h(n)$, and comment on the causality/stability for the following conditions
  a) The system is stable
  b) The system is causal
  c) The system is anticausal

Stability of Second-Order Systems I

- Second-order recursive linear filters with two poles and two zeros are also called **biquad** or biquadratic filters
- Consider a causal two-pole system described by the 2nd order difference equation

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n)$$

- The system function is then

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{b_0 z^2}{z^2 + a_1 z + a_2}$$

- Poles are given by pq-formula

$$p_{1,2} = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}$$

- To be BIBO-stable it must hold that $p_{1,2} < 1$ (poles inside unit circle)

- $a_1 = -(p_1 + p_2)$
  $a_2 = p_1 p_2$
- ➡ $|a_2| = |p_1 p_2| < 1$
  $|a_1| < 1 + a_2$

➡ This defines a region in the coefficient plane (a_1, a_2) which is in the form of a triangle, the **stability triangle**

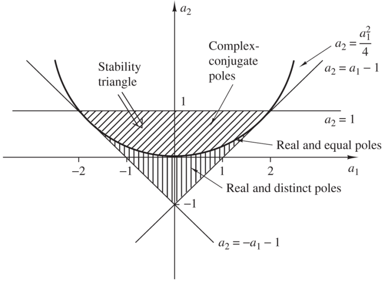

Figure 3.5.1 Region of stability (stability triangle) in the (a_1, a_2) coefficient plane for a second-order system.

Summary

- The convolution of two time-domain signals results in the multiplication of their z-transforms
- For LTI-systems the input output relation is given by $Y(z) = H(z)X(z)$, with the system function $H(z) = Z\{h(n)\}$
- Many signals of practical interest have rational z-transforms
- LTI-systems described by constant-coefficient linear difference equations also possess rational system functions $H(z)$
- Inverses of rational z-transforms can be found by table look-ups
- The inverse z transfrom depends on ROC!
- Stability and causality of a system depends on position of poles!
- Continuous-time equivalent for z-transform is the Laplace transform
- Tool for testing the impact of zeros and poles and frequency: `http://www.micromodeler.com/dsp/`

4. Frequency Analysis of Signals

Outline

Motivation

- When signals are decomposed in terms of sinusoidal (or complex exponential) components, the signals are said to be represented *in the frequency domain*
- For periodic signals this is achieved by the *Fourier series*
- For finite energy signals this is achieved by the *Fourier transform*
- A linear system only modifies amplitude and phase of a sinusoid (no sinusoids are added)

➡ Important technique to analyse signals and systems

- A prism breaks up white light into the colors of the rainbow
- Newton used the term *spectrum* for the resulting bands of colors
- Each color corresponds to a specific frequency of the visible spectrum
- ➜ Prism: frequency decomposition or *spectral analysis* of the light

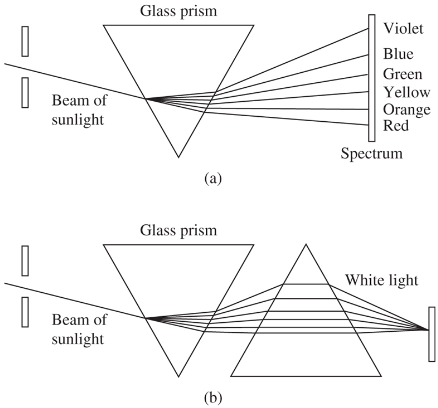

Figure 4.1.1 (a) Analysis and (b) synthesis of the white light (sunlight) using glass prisms.

Fourier Series for Continuous-Time Periodic Signals I

- Examples of periodic signals: sinusoid, complex exponentials, square waves, triangle waves, ...
- Fourier series: decompose signal into linear weighted sum of harmonically related sinusoids (or complex exponentials)
- Jean Baptiste Joseph Fourier (1768–1830)

- Reconstruct periodic signal by combination of harmonically related complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$

  - Fundamental period $T_p = 1/F_0$
- The complex exponentials $e^{j2\pi kF_0 t}$ are the basic “building blocks” to construct a periodic signal
- Question is: how to compute the Fourier series coefficients c_k

- Fourier coefficients c_k for a periodic signal $x(t)$ can be obtained by integrating over the kth exponential (derivation in Sec. 4.1.1)

Frequency Analysis of Continuous-Time Periodic Signals: Fourier series

Synthesis equation: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$

Analysis equation: $c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$

→ Periodic signals have discrete spectra

Real-valued signals

- In general Fourier coefficients c_k are complex-valued
- Show that: If the periodic signal is real-valued, c_k and c_{-k} are complex conjugates, i.e.
  - $c_k = |c_k| e^{j\theta_k}$ ➡ $c_{-k} = |c_k| e^{-j\theta_k}$

Real-valued signals

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  - $c_k = |c_k|e^{j\theta_k}$ ➡ $c_{-k} = |c_k|e^{-j\theta_k}$
- As a consequence for real-valued signals, the Fourier expansion can be written as

$$\begin{aligned} x(t) &= c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(2\pi k F_0 t + \theta_k) \\ &= a_0 + \sum_{k=1}^{\infty} (a_k \cos(2\pi k F_0 t) - b_k \sin(2\pi k F_0 t)) \end{aligned}$$

  - with $a_0 = c_0,\ a_k = 2|c_k| \cos\theta_k,\ b_k = 2|c_k| \sin\theta_k$

➡ Real-valued periodic signals can be represented by a superposition of
  - cosines with different amplitudes and phases, or, alternatively
  - weighted sum of cosines and sinusoids with phase zero

- Recall that a periodic signal has infinite energy and a finite **average power**

$$P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 dt$$

Power Spectral Density of Periodic Signals II

- Recall that a periodic signal has infinite energy and a finite **average power**

$$P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 \, dt$$

- Inserting the Fourier series with $|x(t)|^2 = x(t)x^*(t)$ we obtain

$$\begin{aligned} P_x &= \frac{1}{T_p} \int_{T_p} x(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi k F_0 t} \, dt \\ &= \sum_{k=-\infty}^{\infty} c_k^* \left[\frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} \, dt \right] \\ &= \sum_{k=-\infty}^{\infty} |c_k|^2 \end{aligned}$$

➡ *Parseval's relation* for power signals

Power Spectral Density of Periodic Signals III

- The squared Fourier coefficients $|c_k|^2$ plotted as a function of frequencies kF_0 is called **power spectral density**

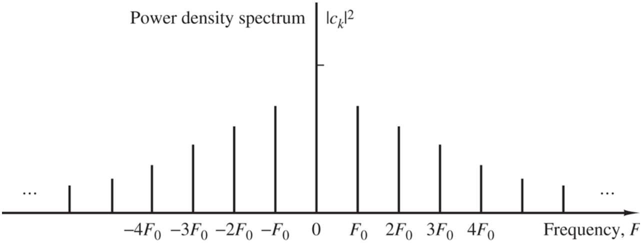

Figure 4.1.2 Power density spectrum of a continuous-time periodic signal.

- for a real-valued signal we have $c_k = c^*_{-k}$, and
  - the **power spectral density** $|c_k|^2$ (and also the magnitude) is an even symmetric function.
  - the phase spectrum $\theta_k = \angle c_k$ is an odd function.

Example 4.1.1

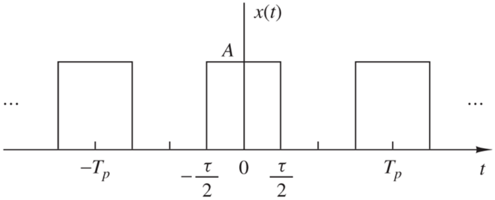

Figure 4.1.3 Continuous-time periodic train of rectangular pulses.

Example 4.1.1

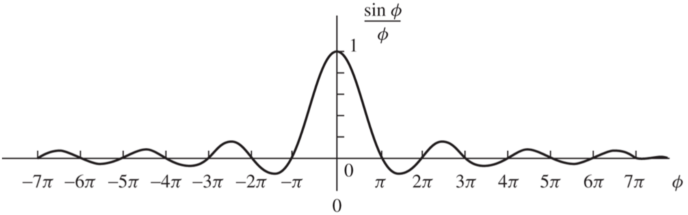

Figure 4.1.4 The function $(\sin\phi)/\phi$.

➡ The function $(\sin\phi)/\phi$ is often referred to as **Sinc function**

Example 4.1.1

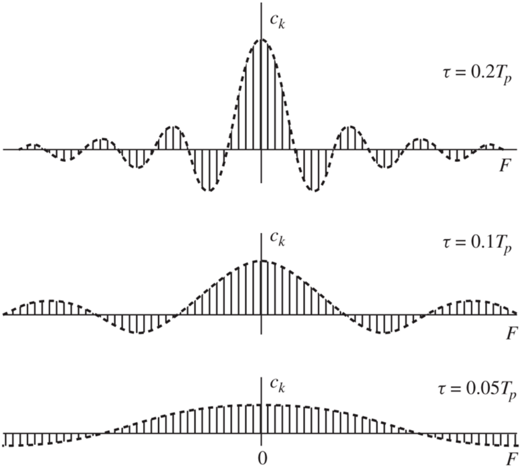

Figure 4.1.5 Fourier coefficients of the rectangular pulse train when T_p is fixed and the pulse width τ varies.

Example 4.1.1

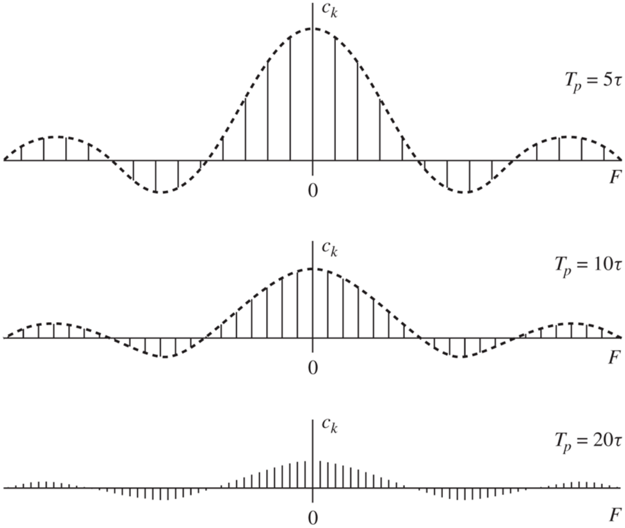

Figure 4.1.6 Fourier coefficient of a rectangular pulse train with fixed pulse width τ and varying period T_p.

Conclusions

- Periodic signals can be represented by a linear combination of harmonically related complex exponentials
- Periodic signals possess line-spectra with equidistant lines.
- The line spacing is equal to the fundamental frequency F_0
- The fundamental frequency is the inverse of the fundamental period $T_p = 1/F_0$

- The case of an aperiodic signal can be seen as a special case of a periodic signal with $T_p \to \infty$
- Then, as $T_p \to \infty$ we have $F_0 = 0$, i.e. the line line-spectra become infinitely close

➡ A continuous-time *aperiodic* signal exhibits a continuous spectrum

- $x(t) = \lim\limits_{T_p \to \infty} x_p(t)$
- Fourier *series*: $c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-2\pi k F_0 t} dt$
- Fourier *transform*: $X(F) = \int_{-\infty}^{\infty} x(t) e^{-2\pi F t} dt$
- Main difference: F is continuous!

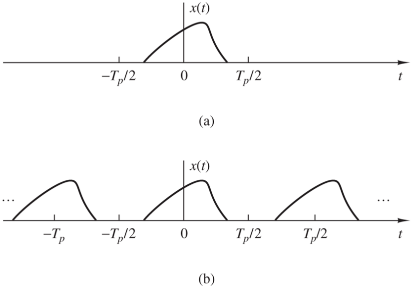

Figure 4.1.7 (a) Aperiodic signal $x(t)$ and (b) periodic signal $x_p(t)$ constructed by repeating $x(t)$ with a period T_p.

- Inverse Fourier series: $x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi kF_0 t}$
- Inverse Fourier transform: $x(t) = \int_{-\infty}^{\infty} X(F) e^{-j2\pi Ft} dF$
- Interpretation: for $T_p \to \infty$, the line spacing kF_0 becomes infinitely small ➡ the sum becomes an integral

Frequency Analysis of Continuous-Time Aperiodic Signals:
Fourier transform

Synthesis equation: $x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$

Analysis equation: $X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$

➡ Aperiodic signals have continuous spectra

- The Fourier transform pair can also be represented in terms of the radian frequency variable $\Omega = 2\pi F$. Since $dF = d\Omega/2\pi$

  Synthesis equation: $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$

  Analysis equation: $X(\Omega) = \displaystyle\int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$

Energy Density Spectrum of Aperiodic Signals

- Show that: the energy of the finite energy signal $x(t)$ with Fourier transform $X(F)$ is

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} |X(F)|^2 \, dF \end{aligned}$$

➡ *Parseval's relation* for aperiodic finite energy signals

- The spectrum is usually represented in polar coordinates, i.e. in terms of magnitude and phase: $X(F) = |X(F)| e^{j\theta(F)}$
- For deterministic (non-random) signals, $S_{xx}(F) = |X(F)|^2$ represents the distribution of energy as a function of frequency, referred to as the **energy density spectrum**

Example 4.1.2

- Determine the Fourier transform and the energy density spectrum of a rectangular pulse

$$x(t) = \begin{cases} A, & |t| \leq \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

Example 4.1.2

- Determine the Fourier transform and the energy density spectrum of a rectangular pulse

$$x(t) = \begin{cases} A, & |t| \leq \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

➡

$$X(F) = \int_{-\tau/2}^{\tau/2} A e^{-j2\pi F t} = A\tau \frac{\sin \pi F \tau}{\pi F \tau}$$

Example 4.1.2

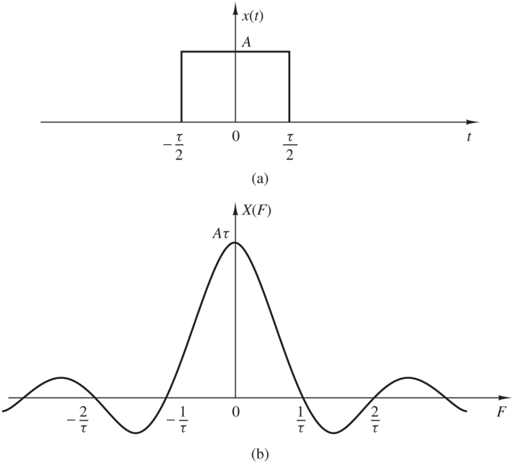

Figure 4.1.8 (a) Rectangular pulse and (b) its Fourier transform.

Fourier Spectrum of Rectangular Function II

Example 4.1.2

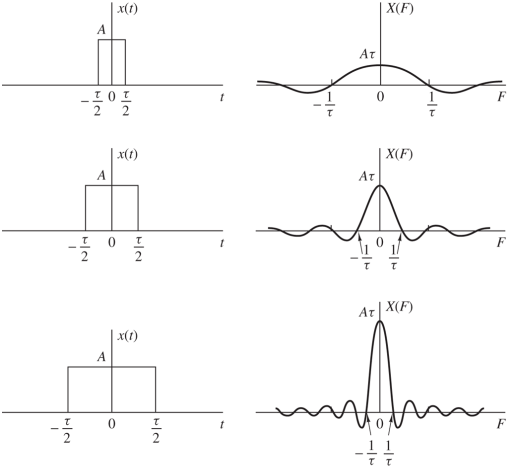

Figure 4.1.9 Fourier transform of a rectangular pulse for various width values.

➡ Uncertainty principle between time and frequency domains

Outline

- The Fourier series may consist of a (possibly infinite) number of frequency components spaced by F_0, i.e. the spectrum may span from $-\infty$ to ∞
- In contrast, for discrete-time signals, the frequency range is unique over the interval $(-\pi, \pi)$ or $(0, 2\pi)$
- A periodic discrete signal of fundamental period N consists of frequency components separated by $\omega = 2\pi/N$ radians or $f = 1/N$ cycles per sample
  - ➡ Fourier series of discrete-time signal will contain at most N frequency components

Frequency Analysis of Discrete-Time Periodic Signals:
Discrete-Time Fourier Series (DTFS)

Synthesis equation: $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$

Analysis equation: $c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

- Given: periodic sequence $x(n)$ with period N, i.e
  $x(n) = x(n+N), \quad$ for all n
- The Fourier coefficients c_k provide a description of $x(n)$ in the frequency domain, in the sense that c_k represents the amplitude and phase associated with the frequency component $e^{j2\pi kn/N} = e^{j\omega_k n}$ with $\omega_k = 2\pi k/N$

- As $e^{-j2\pi kn/N}$ is periodic in N, also the Fourier coefficients c_k are periodic

$$c_{k+N} = c_k$$

→ The spectrum of the periodic signal $x(n)$ is also a periodic sequence with period N

→ We only need to consider the coefficients $k = 0, 1, ..., N-1$

→ Using the sampling frequency F_S, the range $0 \le k \le N-1$ corresponds to the frequency range $0 \le F < F_\mathrm{S}$

SP Signal Processing

- The average power of a discrete-time periodic signal with period N is

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

- From the definition of the Fourier series it follows that

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

  i.e. the average power in the signal is the sum of the powers of the individual frequency components

- For the energy in a single period we have

$$E_x = \sum_{n=0}^{N-1} |x(n)|^2 = N \sum_{k=0}^{N-1} |c_k|^2$$

Example 4.2.2

- Determine the Fourier series coefficients and the power density spectrum of the periodic signal shown in Fig 4.2.2

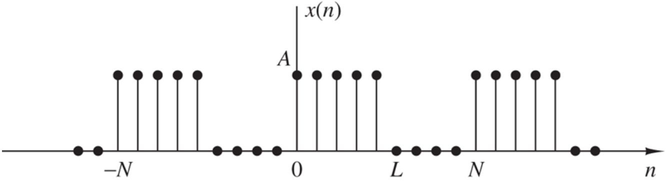

Figure 4.2.2 Discrete-time periodic square-wave signal.

Fourier Series of Discrete-Time Pulse Train II

SP Signal Processing

Example 4.2.2

$$
c_k = \begin{cases} \dfrac{AL}{N}, & k = 0, \pm N, \pm 2N \\ \dfrac{AL}{N} e^{-j\pi k(L-1)/N} \dfrac{\sin(\pi k L/N)}{\sin(\pi k/N)}, & \text{otherwise} \end{cases}
$$

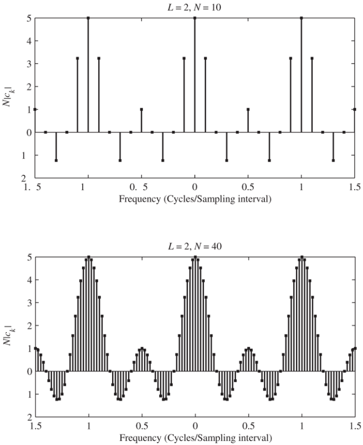

Figure 4.2.3 Plot of the power density spectrum given by (4.2.22).

- Recall that for discrete signals, as $e^{-j2\pi kn/N}$ is periodic in N,
  - $c_{k+N} = c_k$
- Recall that, if the discrete-time signal $x(n)$ is real-valued we have
  - $x^*(n) = x(n)$
  - $c_k^* = c_{-k}$
  - $|c_k| = |c_{-k}|$ (magnitude is even symmetric)
  - $\angle c_k = -\angle c_{-k}$ (phase is odd symmetric)
- For discrete-time real-valued signals we have
  - $|c_k| = |c_{N-k}|$
  - $\angle c_k = -\angle c_{N-k}$
  - More specifically

$$
\begin{array}{lll}
|c_0| = |c_N| & \angle c_0 = -\angle c_N = 0 & \\
|c_1| = |c_{N-1}| & \angle c_1 = -\angle c_{N-1} & \\
|c_{N/2}| = |c_{N/2}| & \angle c_{N/2} = -\angle c_{N/2} = 0 & \text{if } N \text{ is even} \\
|c_{(N-1)/2}| = |c_{(N+1)/2}| & \angle c_{(N-1)/2} = -\angle c_{(N+1)/2} & \text{if } N \text{ is odd}
\end{array}
$$

- For a real-valued signal, **the signal is completely described** by the spectrum c_k with
  - $k = 0, 1, .., N/2$, if N is even
  - $k = 0, 1, .., (N-1)/2$, if N is odd
- Consistent: The unique spectrum is $0 \leq \omega_k = \frac{2\pi k}{N} \leq \pi \quad \longleftrightarrow \quad 0 \leq k \leq N/2$

Frequency Analysis of Discrete-Time Aperiodic Signals:
Discrete-Time Fourier Transform (DTFT)

Synthesis equation: $x(n) = \dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega$

Analysis equation: $X(\omega) = \displaystyle\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$

Frequency Analysis of Discrete-Time Aperiodic Signals:
Discrete-Time Fourier Transform (DTFT)

Synthesis equation: $x(n) = \dfrac{1}{2\pi} \displaystyle\int_{2\pi} X(\omega) e^{j\omega n} d\omega$

Analysis equation: $X(\omega) = \displaystyle\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Derivation of the synthesis equation:

- For discrete-time signals, we observe that the spectrum is periodic,

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega + 2\pi k)n} = X(\omega)$$

- Thus, the periodic spectrum $X(\omega)$ can be analyzed by a Fourier series. This results in the synthesis equation above

Frequency Analysis of Discrete-Time Aperiodic Signals:
Discrete-Time Fourier Transform (DTFT)

Synthesis equation: $$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

Analysis equation: $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Frequency Analysis of Continuous-Time Periodic Signals: **Fourier series**

Synthesis equation: $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

Analysis equation: $$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

SP Signal Processing

Example: ideal low-pass filter

- Ideal low-pass filter with cut-off frequency Ω_c:

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

Example: ideal low-pass filter

- Ideal low-pass filter with cut-off frequency Ω_c:

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

➡

$$x(n) = \begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}, & n \neq 0 \end{cases}$$

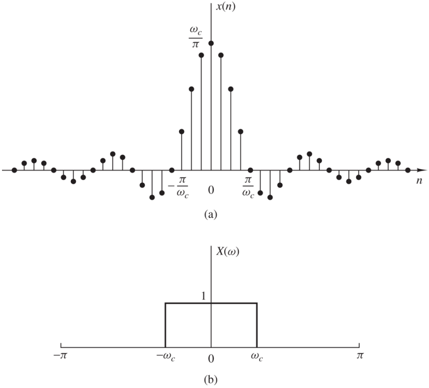

Figure 4.2.4 Fourier transform pair in (4.2.35) and (4.2.36).

Gibbs phenomenon

- Now, let us look at the spectrum of the time-domain signal $x(n) = \frac{\sin \omega_c n}{\pi n}$, $-\infty < n < \infty$, when analyzed over a finite time-window

$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

- Oscillatory overshoot at $\omega = \omega_c$
- For $N \to \infty$ overshoot becomes infinitely narrow, but amplitude remains

➡ **Gibbs phenomenon**

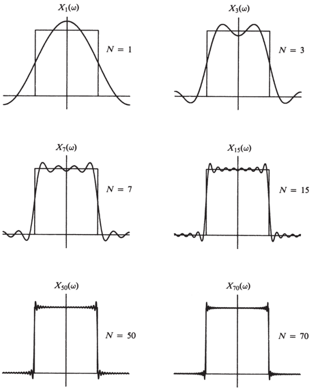

Figure 4.2.5 Illustration of convergence of the Fourier transform and the Gibbs phenomenon at the point of discontinuity.

Energy Density Spectrum of Aperiodic Signals

- The energy of a discrete-time signal is

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

- From the definition of the DTFT it follows that

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega$$

➡ *Parseval's relation* for discrete-time aperiodic signals with finite energy

- The spectrum is usually represented in polar coordinates, i.e. in terms of magnitude and phase: $X(\omega) = |X(\omega)| e^{j\theta(\omega)}$
- For deterministic (non-random) signals, $S_{xx}(\omega) = |X(\omega)|^2$ represents the distribution of energy as a function of frequency, referred to as the **energy density spectrum**

- Suppose that the discrete-time signal $x(n)$ is real-valued, then
  - $x^*(n) = x(n)$
  - $X^*(\omega) = X(-\omega)$
  - $|X(-\omega)| = |X(\omega)|$ (magnitude is even symmetric)
  - $\angle X(-\omega) = -\angle X(\omega)$ (phase is odd symmetric)
  - $S_{xx}(-F) = S_{xx}(F)$ (even symmetry)

➡ A real-valued signal, is completely described by half the spectrum $X(\omega)$ with $0 \leq \omega \leq \pi$, i.e. the spectrum in the range $0 \leq F \leq F_\mathrm{S}/2$

Example 4.2.4

- Determine the Fourier transform and the energy density spectrum of the sequence

$$x(n) = \begin{cases} A, & 0 \le n \le L-1 \\ 0, \text{otherwise} & \end{cases}$$

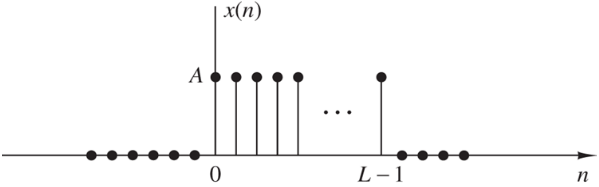

Figure 4.2.7 Discrete-time rectangular pulse.

Example 4.2.4

$$X(\omega) = \sum_{n=0}^{L-1} A e^{-j\omega n} = A e^{-j\omega \frac{L-1}{2}} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

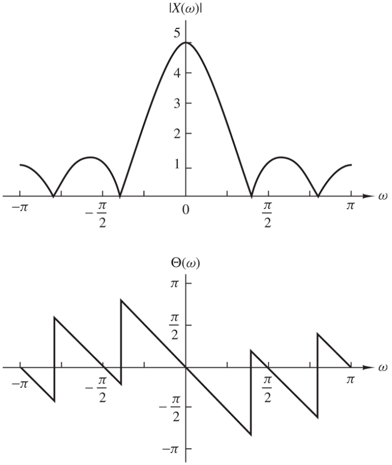

Figure 4.2.8 Magnitude and phase of Fourier transform of the discrete-time rectangular pulse in Fig 4.2.7.

Outline

Introduced analysis tools

- The following frequency analysis tools have been introduced
  1. Fourier series for continuous-time periodic signals (FS)
  2. Fourier transform for continuous-time aperiodic signals (FT)
  3. Fourier series for discrete-time periodic signals (DTFS)
  4. Fourier transform for discrete-time aperiodic signals (DTFT)

A summary of the previous sections

- Continuous-time signals have aperiodic spectra
- Discrete-time signals have periodic spectra
- Periodic signals have discrete spectra
- Aperiodic finite energy signals have continuous spectra

➡ Periodicity with “period” α in one domain implies discretization with “spacing” of $1/\alpha$ in the other domain

- The **energy density spectrum** can be used to characterize finite energy **aperiodic** signals
- The **power density spectrum** can be used to characterize **periodic** signals

Dualities between transforms

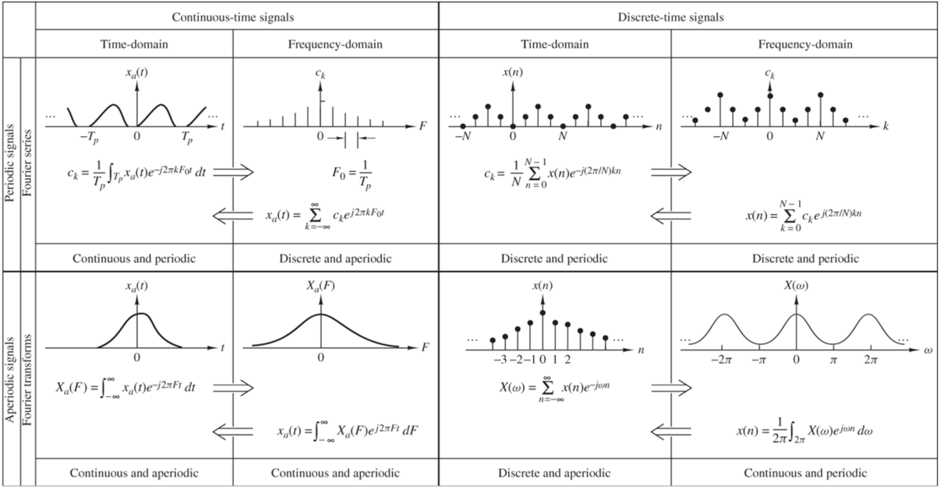

Figure 4.3.1 Summary of analysis and synthesis formulas.

Low-frequency, high-frequency, and bandpass signals

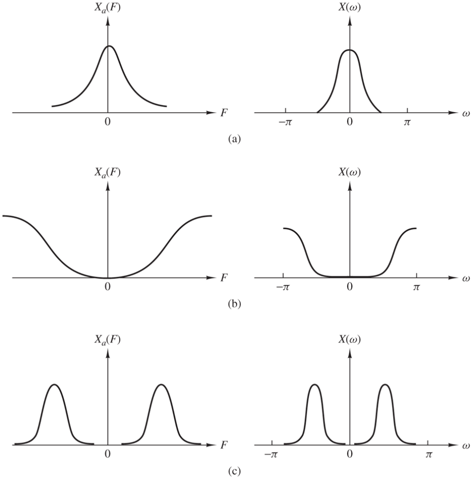

Figure 4.2.10 (a) Low-frequency, (b) high-frequency, and (c) medium-frequency signals.

➡ medium-frequency signals are also referred to as *bandpass signals*

SP Signal Processing

Bandwitdh, narrowband, wideband, bandlimited

- A 95% **bandwidth** signal has 95% of its energy (or power) in a range $F_1 \leq F \leq F_2$
- A **narrowband** signal has a bandwidth $F_2 - F_1$ which is much smaller (e.g. factor 10) than $(F_2 + F_1)/2$. Otherwise the signal is called **wideband**
- A signal is **bandlimited** if its spectrum is zero outside the frequency range $F \geq B$
- A discrete-time finite energy signal $x(n)$ is said to be **(periodically) bandlimited** if $|X(\omega)| = 0$ for $\omega_0 < |\omega| < \pi$

No signal can be time-limited and bandlimited simultaneously!
(e.g. rect $\circ\!\!-\!\!\bullet$ sinc)

Relationship of the Fourier Transform to the z-Transform I

- The z-Transform of a sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad \text{ROC: } r_2 < |z| < r_1$$

- Let us now represent the complex variable z in polar form, i.e. $z = re^{j\omega}$
- Then, within the ROC we can substitute z

$$X(z)\big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} \left[x(n) r^{-n}\right] e^{-j\omega n}$$

➡ $X(z)$ can be interpreted as the Fourier transform of sequence $x(n)r^{-n}$

- The z-Transform of a sequence $x(n)$ is defined as

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➡ $X(z)$ can be interpreted as the Fourier transform of sequence $x(n)r^{-n}$

- Alternatively, if $|z| = 1$ is within the ROC, we obtain

$$X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} [x(n)] e^{-j\omega n}$$

➡ Fourier transform: z-transform evaluated on the unit circle

➡ Fourier transform interpreted as a z-transform evaluated on the unit circle

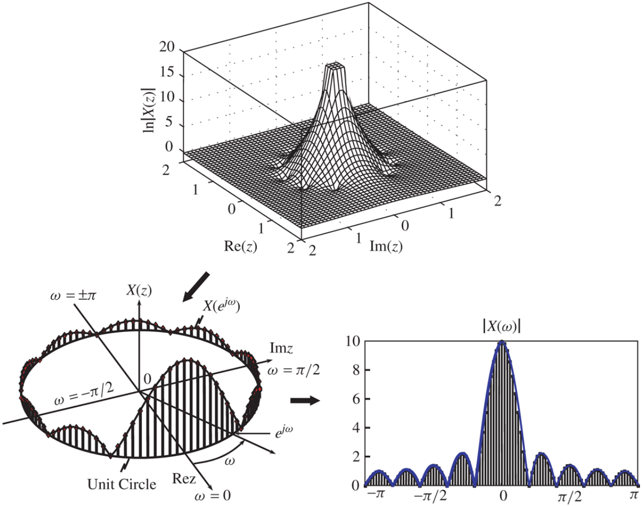

Figure 4.2.9 relationship between $X(z)$ and $X(\omega)$ for the sequence in Example 4.2.4, with $A = 1$ and $L = 10$

Symmetry Properties of the DTFT I

$$
\begin{array}{ccccccccccc}
x(n) & = & x_R^e(n) & + & x_R^o(n) & + & jx_I^e(n) & + & jx_I^o(n) \\
\updownarrow & & \updownarrow & & & & & & \\
X(\omega) & = & X_R^e(\omega) & + & X_R^o(\omega) & + & jX_I^e(\omega) & + & jX_I^o(\omega)
\end{array}
$$

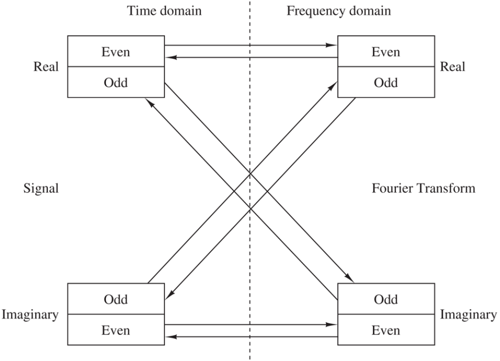

Figure 4.4.2 Summary of symmetry properties for the Fourier transform.

Symmetry Properties of the DTFT II

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}\left[X(\omega) + X^*(-\omega)\right]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}\left[X(\omega) - X^*(-\omega)\right]$
$x_e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right]$	$jX_I(\omega)$
Real signals	
Any real-valued signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\angle X(\omega) = -\angle X(-\omega)$
$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$	$X_I(\omega)$

Example 4.4.1

Determine and sketch $X_R(\omega)$, $X_I(\omega)$, $|X(\omega)|$, and $\angle X(\omega)$

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}, \quad -1 < a < 1$$

SP Signal Processing

Example 4.4.1

$X(\omega) = \frac{1}{1-ae^{-j\omega}}, \quad -1 < a < 1$

Solution:

- Extend by complex conjugate of denominator:

$$X(\omega) = \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \frac{1 - a\cos(\omega) - ja\sin(\omega)}{1 - 2a\cos(\omega) + a^2}$$

- Split into real and imaginary part

$$X_R(\omega) = \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2}, \quad X_I(\omega) = -\frac{a\sin(\omega)}{1 - 2a\cos(\omega) + a^2}$$

- Determine absolute value and phase

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)} = \frac{1}{\sqrt{1 - 2a\cos(\omega) + a^2}}$$

$$\angle X|\omega| = \tan^{-1}\frac{X_I(\omega)}{X_R(\omega)} = -\tan^{-1}\left(\frac{a\sin(\omega)}{1 - a\cos(\omega)}\right)$$

Example 4.4.1

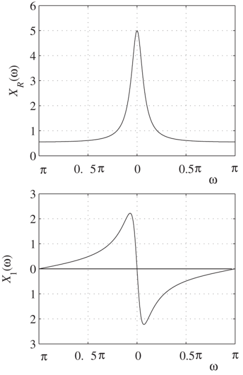

Figure 4.4.3 Graph of $X_R(\omega)$ and $X_I(\omega)$ for the transform in Example 4.4.1.

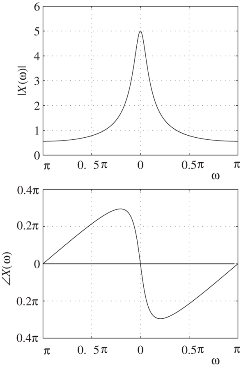

Figure 4.4.4 Magnitude and phase spectra of the transform in Example 4.4.1.

➡ The representation in magnitude and phase is more common.

Example 4.4.2

Represent the Fourier transform of the signal in magnitude and phase

$$x(n) = \begin{cases} A, & -M \leq n \leq M \\ 0, & \text{elsewhere} \end{cases} \quad , n \in \mathbb{Z}$$

Example 4.4.2

Determine the Fourier transform of the signal

$$x(n) = \begin{cases} A, & -M \leq n \leq M \\ 0, & \text{elsewhere} \end{cases} \quad , n \in \mathbb{Z}$$

Solution:

- $$X(\omega) = A \frac{\sin\left[(M + \frac{1}{2})\omega\right]}{\sin(\omega/2)}$$
- $$|X(\omega)| = \left| A \frac{\sin\left[(M + \frac{1}{2})\omega\right]}{\sin(\omega/2)} \right|$$

$$\angle X(\omega) = \begin{cases} 0, & X(\omega) > 0 \\ \pi, & X(\omega) < 0 \end{cases}$$

Example 4.4.2

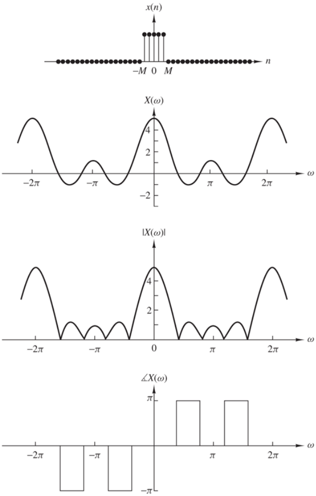

Figure 4.4.5 Spectral characteristics of rectangular pulse in Example 4.4.2.

Representation of Spectra

Example 4.4.3

$$x(n) = a^{|n|}, \quad -1 < a < 1, \quad n \in \mathbb{Z}$$

Example 4.4.3

$$x(n) = a^{|n|}, \quad -1 < a < 1, \quad n \in \mathbb{Z}$$

Solution: $X(\omega) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} = \frac{1-a^2}{1-2a\cos(\omega)+a^2}$

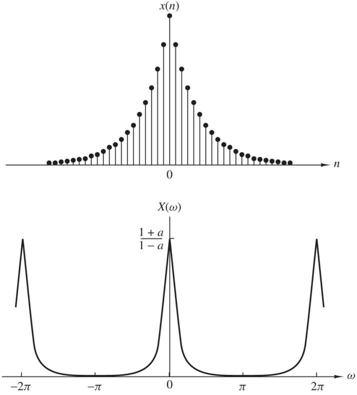

Figure 4.4.6 Sequence $x(n)$ and its Fourier transform in Example 4.4.3 with $a = 0.8$.

Properties of the DTFT

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-w)$
		$= X_1(\omega)X_2^*(\omega)$ if $x_2(n)$ real
Wiener-Khintchine th.	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega-\omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}\left(X(\omega+\omega_0) + X(\omega-\omega_0)\right)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Freq Differentiation	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

Frequency-shifting property

$$e^{j\omega_0 n}x(n) \;\overset{\mathrm{F}}{\circ\!\!-\!\!\bullet}\; X(\omega-\omega_0)$$

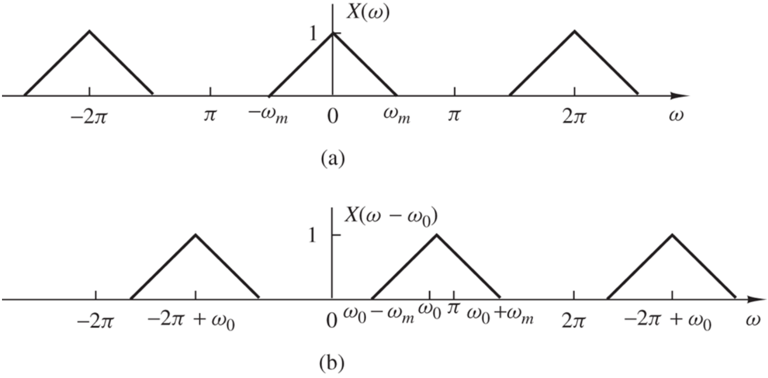

Figure 4.4.8 Illustration of the frequency-shifting property of the Fourier transform ($\omega_0 \leq 2\pi - \omega_m$).

Modulation property

$$x(n)\cos\omega_0 n = x(n)\frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n}) \;\overset{\mathrm{F}}{\circ\!\!-\!\!\bullet}\; \frac{1}{2}\left(X(\omega+\omega_0) + X(\omega-\omega_0)\right)$$

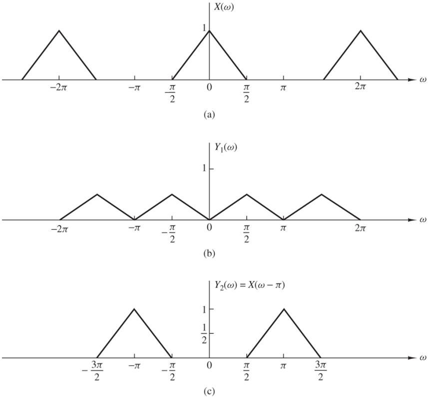

Figure 4.4.9 Graphical representation of the modulation theorem.

Example: $y_1(n) = x(n)\cos(\pi/2\, n)$, $\quad y_2(n) = x(n)\cos(\pi n)$

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Convolution property

$$x_1(n) * x_2(n) \overset{F}{\circ\!\!-\!\!\bullet} X_1(\omega)X_2(\omega)$$

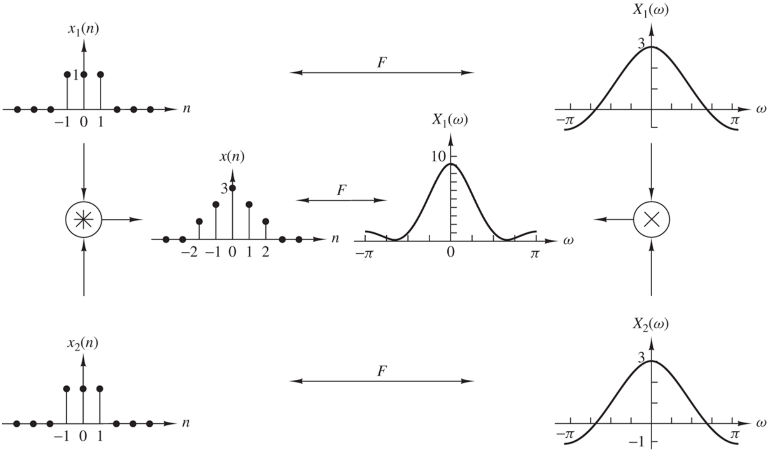
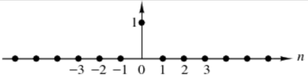
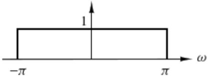
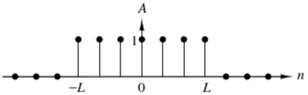
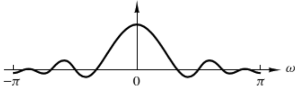
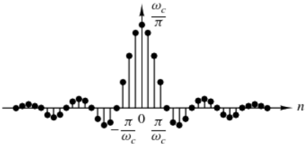
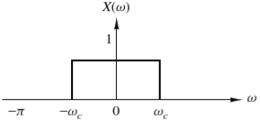


Figure 4.4.7 Graphical representation of the convolution property.

Useful DTFT Transform Pairs

TABLE 6 Some Useful Fourier Transform Pairs for Discrete-Time Aperiodic Signals

Signal $x(n)$	Spectrum $X(\omega)$
 $x(n) = \delta(n)$	 $X(\pi) = 1$
 $x(n) = \begin{cases} A, & \lvert n\rvert \le L \\ 0, & \lvert n\rvert > L \end{cases}$	 $X(\omega) = A \dfrac{\sin\left(L+\frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}$
 $x(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin \omega_c n}{\pi n}, & n \neq 0 \end{cases}$	 $X(\omega) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c \le \lvert\omega\rvert \le \pi \end{cases}$
$x(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n > 0 \end{cases}$	$X(\omega) = \frac{1}{1-ae^{-j\omega}}$

Summary

- The Fourier series and the Fourier transform are tools for analyzing the characteristics of signals in the frequency domain
- The Fourier series is appropriate for representing periodic signals as a weighted sum of harmonically related sinusoidal components. The weighting coefficients represent the strengths of each harmonic
- The Fourier transform is appropriate for representing the spectral characteristics of aperiodic signals with finite energy

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5. Frequency Analysis of LTI Systems

Outline

Summary

- Characterization of linear time-invariant (LTI) systems in the frequency domain.
- Basic excitations are complex exponentials and sinusoidal functions
- LTI systems perform a filtering (attenuation/amplification) on the various frequency components.
- Each frequency is filtered independently of other frequencies. No frequencies are added.

→ Simple description by input-output function (the transfer function / frequency response) possible

- This transfer function is the frequency transform of the response of the LTI system to an impulse in the time-domain (the impulse response)

- The frequency response $H(\omega)$
  - is the Fourier transform of the impulse response $h(n)$ of the system,
  - completely characterizes an LTI system in the frequency domain,
  - allows us to determine the steady-state response of the system to any arbitrary weighted linear combination of sinusoids or complex exponentials
- As, using a Fourier transform, signals can be seen as a superposition of complex exponentials, the response of an LTI system to arbitrary signals can be determined using the frequency response

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The Frequency Response Function I

- Recall that the response of any relaxed linear time-invariant system to an arbitrary input signal $x(n)$ is given by the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

  ➡ Time-domain: system characterized by unit sample response $h(n)$

- To develop a frequency-domain characterization, let us excite a system by a complex exponential

$$x(n) = Ae^{j\omega n}$$

where A: amplitude; $\omega \in [-\pi, \pi]$ an arbitrary frequency

- Then

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)\left[Ae^{j\omega(n-k)}\right] \\
&= A\underbrace{\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]}_{H(\omega)} e^{j\omega n} \\
&= H(\omega)Ae^{j\omega n}
\end{aligned}
$$

- The system response is again given by a complex exponential at the same frequency ω, but multiplicatively altered by $H(\omega)$
- Thus, the exponential signal $x(n) = Ae^{j\omega n}$ is called an *eigenfunction* of the system
- The multiplicative factor $H(\omega)$ is called an *eigenvalue* of the system

Example 5.1.1

- Determine the output sequence of the system with impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

when the input is the complex exponential sequence

$$x(n) = Ae^{j\pi n/2}, \quad -\infty < n < \infty$$

The Frequency Response Function III

- In general, the frequency response is a complex-valued function and is typically expressed in polar form

$$H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$$

  i.e. the system modifies
  - the magnitude multiplicatively
  - the phase additively (results in a phase-shift)
- $H(\omega)$: *frequency response*
  $|H(\omega)|$: *magnitude response*
  $\Theta(\omega)$: *phase response*
- For many real-world systems, the impulse response $h(n)$ is real-valued. Consequently
  - $|H(\omega)|$ is an even function in ω
  - $\Theta(\omega)$ is an odd function in ω
  - ➔ if we know $H(\omega)$ for $0 \leq \omega \leq \pi$, we also know $H(\omega)$ for $-\pi \leq \omega \leq 0$

Example 5.1.2

- Determine magnitude and phase of $H(\omega)$ for the tree-point moving average (MA) system

$$y(n) = \frac{\sqrt{2}}{3}[x(n+1) + x(n) + x(n-1)]$$

Example 5.1.2

- Determine magnitude and phase of $H(\omega)$ for the tree-point moving average (MA) system

$$y(n) = \frac{\sqrt{2}}{3}[x(n+1) + x(n) + x(n-1)]$$

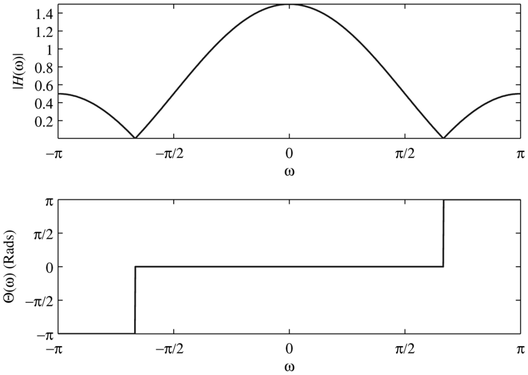


Figure 5.1.1 Magnitude and phase responses for the MA system in Example 5.1.2.

Example 5.1.4 I

- An LTI system is described by the difference equation

$$y(n) = ay(n-1) + bx(n), \quad 0 < a < 1$$

a) Determine the magnitude and phase of the frequency response $H(\omega)$
b) Choose b so that the maximum value of $|H(\omega)|$ is unity
c) Sketch $|H(\omega)|$ and $\Theta(\omega)$ for $a = 0.9$
d) Determine the output of the system to the input signal

$$x(n) = 5 + 12 \sin\left(\frac{\pi}{2} n\right) - 20 \cos\left(\pi n + \frac{\pi}{4}\right)$$

Example 5.1.4 II

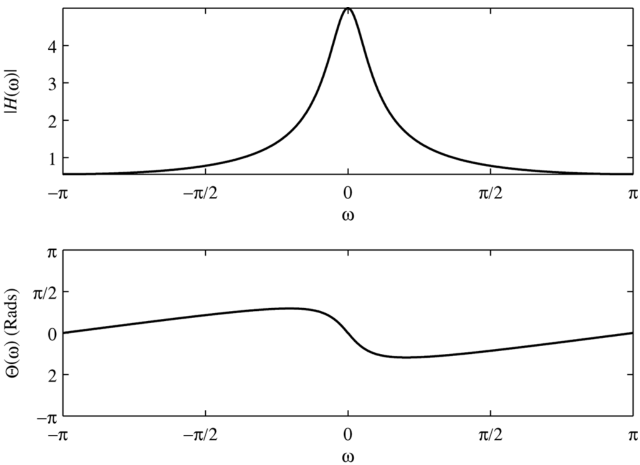


Figure 5.1.2 Magnitude and phase responses for the system in Example 5.1.4 with $a = 0.9$.

Sum of sinusoids

- If the input signal consists of an arbitrary linear combination of sinusoids

$$x(n) = \sum_{i=1}^{L} A_i \cos(\omega_i n + \phi_i)\,, \quad -\infty < n < \infty$$

The response of the system is simply

$$y(n) = \sum_{i=1}^{L} A_i |H(\omega_i)| \cos(\omega_i n + \phi_i + \Theta(\omega_i))\,, \quad -\infty < n < \infty$$

- Different frequencies are affected differently by the system
  - Some frequencies may be set to zero,
  - Others may not be modified at all

Outline

Steady-State and Transient Response I

To Sinusoidal Input Signals

- So far, we considered eternal sinusoids/exponentials, i.e. those applied at time $n = -\infty$. The observed response is the *steady-state response*
- To demonstrate the behavior of a system when the signal is applied at, say, $n = 0$, let us consider the system described by the first-order difference equation

$$y(n) = ay(n-1) + x(n)$$

- The system response to any $x(n)$ applied at $n = 0$ is

$$y(n) = a^{n+1} \underbrace{y(-1)}_{\text{initial condition}} + \sum_{k=0}^{n} a^k x(n-k), \quad n \geq 0$$

To Sinusoidal Input Signals

- Given the complex-exponential input $x(n) = Ae^{j\omega n}$, $n \geq 0$,

$$
\begin{aligned}
y(n) &= a^{n+1}y(-1) + A\sum_{k=0}^{n} a^k e^{j\omega(n-k)} \\
&= a^{n+1}y(-1) + A\left[\sum_{k=0}^{n} a^k e^{-j\omega k}\right] e^{j\omega n} \\
&= a^{n+1}y(-1) + A\left[\sum_{k=0}^{n} \left(ae^{-j\omega}\right)^k\right] e^{j\omega n} \\
&= a^{n+1}y(-1) + A\frac{1 - a^{n+1}e^{-j\omega(n+1)}}{1 - ae^{-j\omega}} e^{j\omega n}, \quad n \geq 0 \\
&= a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1 - ae^{-j\omega}} e^{j\omega n} + \frac{A}{1 - ae^{-j\omega}} e^{j\omega n}, \quad n \geq 0
\end{aligned}
$$

Steady-State and Transient Response III

To Sinusoidal Input Signals

- Steady-state response

$$
\begin{aligned}
y_{\mathrm{ss}}(n) &= \lim_{n\to\infty} y(n) = \frac{A}{1-ae^{-j\omega}} e^{j\omega n} \\
&= AH(\omega) e^{j\omega n}
\end{aligned}
$$

- Transient response

$$
y_{\mathrm{tr}}(n) = a^{n+1} y(-1) - \frac{A a^{n+1} e^{-j\omega(n+1)}}{1-ae^{-j\omega}} e^{j\omega n}, \quad n \geq 0
$$

- For BIBO stable systems (here $|a| < 1$), the transient part goes to zero for $n \to \infty$

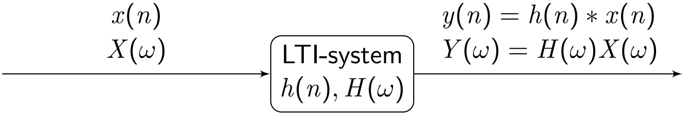
➡ In many practical applications the transient response is unimportant and is usually ingnored in dealing with the response to sinusoidal inputs

Response to Aperiodic Input Signals

- From the convolution theorem it follows that

$$Y(\omega) = H(\omega)X(\omega)$$

- $H(\omega)$ acts as a filter to the frequency components
- $|H(\omega)|$ determines which amplitudes are attenuated or amplified
- $\Theta(\omega) = \angle H(\omega)$ determines the phase shift
- *The output of an LTI-system cannot contain frequency components that are not contained in the input signal*



Outline

Systems with a Rational System Function

- Recall that LTI-systems with rational system functions are described by constant-coefficient difference equations in time-domain.
- If the system function $H(z)$ converges on the unit circle we obtain the frequency response as

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

- If $H(z)$ is a rational function, i.e. of the form $H(z) = B(z)/A(z)$

$$\begin{aligned} H(\omega) &= \frac{B(\omega)}{A(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}} \\ &= b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega})} \end{aligned}$$

  - where a_k, b_k are real-valued, but the zeros z_k and poles p_k are generally complex-valued.

Relation of Complex Conjugate to z-Transform

- $|H(\omega)|^2 = H(\omega)H^*(\omega)$ can be obtained from complex conjugate $H^*(\omega)$
- Merging our knowledge of z-transform and DTFT, we obtain

$$
\begin{aligned}
X(-\omega) &\bullet\!\!-\!\!\circ\; x(-n) \;\circ\!\!-\!\!\bullet\; X(z^{-1}) \\
X^*(-\omega) &\bullet\!\!-\!\!\circ\; x^*(n) \;\circ\!\!-\!\!\bullet\; X^*(z^*) \\
X^*(\omega) &\bullet\!\!-\!\!\circ\; x^*(-n) \;\circ\!\!-\!\!\bullet\; X^*(1/z^*)
\end{aligned}
$$

➡ The complex conjugate can be obtain from the z-transform as

$$
H^*(\omega) = H^*(1/z^*)\Big|_{z=e^{j\omega}}
$$

- $|H(\omega)|^2 = H(\omega)H^*(\omega)$ can be obtained from complex conjugate $H^*(\omega)$
- Merging our knowledge of z-transform and DTFT, we obtain

$$
\begin{aligned}
X(-\omega) &\bullet\!\!-\!\!\circ\; x(-n) \;\circ\!\!-\!\!\bullet\; X(z^{-1}) \\
X^*(-\omega) &\bullet\!\!-\!\!\circ\; x^*(n) \;\circ\!\!-\!\!\bullet\; X^*(z^*) \\
X^*(\omega) &\bullet\!\!-\!\!\circ\; x^*(-n) \;\circ\!\!-\!\!\bullet\; X^*(1/z^*)
\end{aligned}
$$

➡ The complex conjugate can be obtain from the z-transform as

$$H^*(\omega) = H^*(1/z^*)\big|_{z=e^{j\omega}}$$

- For a real-valued $x(n)$, poles and zeros occur in complex-conjugate pairs

$$
\begin{aligned}
X(\omega) = X^*(-\omega) &\bullet\!\!-\!\!\circ\; x^*(n) = x(n) \;\circ\!\!-\!\!\bullet\; X^*(z^*) = X(z) \\
X^*(\omega) = X(-\omega) &\bullet\!\!-\!\!\circ\; x^*(-n) = x(-n) \;\circ\!\!-\!\!\bullet\; X^*(1/z^*) = X(z^{-1})
\end{aligned}
$$

- The magnitude can be obtained by $|H(\omega)|^2 = H(\omega)H^*(\omega)$ with

$$H^*(\omega) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k^* e^{j\omega})}{\prod_{k=1}^{N}(1 - p_k^* e^{j\omega})}$$

- $H^*(\omega)$ is obtained by evaluating $H^*(1/z^*)$ on the unit circle

$$H^*(1/z^*) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k^* z)}{\prod_{k=1}^{N}(1 - p_k^* z)}$$

- If $h(n)$ is real-valued, $H^*(1/z^*) = H(z^{-1})$, such that

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega) = H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$

  - $H(z)H(z^{-1}) \overset{z}{\circ\!\!-\!\!\bullet} r_{hh}(m)$ (autocorrelation sequence of $h(n)$)
  - $|H(\omega)|^2 \overset{\mathrm{F}}{\circ\!\!-\!\!\bullet} r_{hh}(m)$ *Wiener-Khintchine theorem*
- For a given $H(z)$ it is straight-forward to compute $H(z^{-1})$ and thus $|H(\omega)|^2$. The opposite does not hold.

Computation of the Frequency Response

- To determine the frequency response it is convenient to use a representation by means of poles and zeros, i.e.

$$H(\omega) \quad = \quad b_0\, e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

- Let us now represent the complex-valued factors in polar form as

$$e^{j\omega} - z_k \equiv V_k(\omega) e^{j\Theta_k(\omega)}$$
$$e^{j\omega} - p_k \equiv U_k(\omega) e^{j\Phi_k(\omega)}$$

- Then, since $|e^{j\omega(N-M)}| = 1$, the magnitude of $H(\omega)$ is

$$|H(\omega)| = |b_0| \frac{\prod_{k=1}^{M} V_k(\omega)}{\prod_{k=1}^{N} U_k(\omega)}$$

- The phase is given by

$$\angle H(\omega) = \angle b_0 + \omega(N-M) + \left(\sum_{k=1}^{M} \Theta_k(\omega)\right) - \left(\sum_{k=1}^{M} \Phi_k(\omega)\right)$$

➡ Magnitude and phase of $H(\omega)$ can be computed for given z_k, p_k

Computation of the Frequency Response I

Geometric Interpretation

$$|H(\omega)| = |b_0| \frac{\prod_{k=1}^{M} V_k(\omega)}{\prod_{k=1}^{N} U_k(\omega)}$$

- let $\mathbf{CL} = e^{j\omega}, \mathbf{CA} = p_k$, and $\mathbf{CB} = z_k$ denote vectors. Then
- $\mathbf{CL} = \mathbf{CA} + \mathbf{AL}$,
  $\mathbf{CL} = \mathbf{CB} + \mathbf{BL}$,
- ➡ $\mathbf{AL} = e^{j\omega} - p_k = U_k(\omega) e^{j\Phi_k(\omega)}$,
  $\mathbf{BL} = e^{j\omega} - z_k = V_k(\omega) e^{j\Theta_k(\omega)}$

Thus,

- distance between pole p_k and $e^{j\omega}$ corresponds to $U_k(\omega)$
- distance between zero z_k and $e^{j\omega}$ corresponds to $V_k(\omega)$

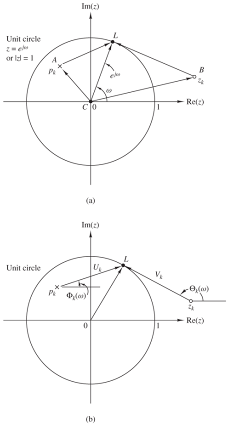


Figure 5.2.1 Geometric interpretation of the contribution of a pole and a zero to the Fourier transform (1) magnitude: the factor V_k/U_k, (2) phase: the factor $\Theta_k - \Phi_k$.

Computation of the Frequency Response II

Geometric Interpretation

$$|H(\omega)| = |b_0| \frac{\prod_{k=1}^{M} V_k(\omega)}{\prod_{k=1}^{N} U_k(\omega)}$$

- Presence of a zero close to the unit circle causes $|H(\omega)|$ to be small close to that zero
- Presence of a pole close to the unit circle causes $|H(\omega)|$ to be large close to that pole

➡ Poles have the opposite effect of zeros

➡ If both poles and zeros are present, a greater variety of shapes of $H(\omega)$ can be realized

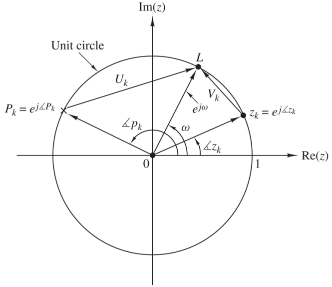


Figure 5.2.2 A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = \measuredangle z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = \measuredangle p_k$.

Recall from Chapter 2

- Correlation measures the *similarity* between two signals
- Correlation is mathematically similar to convolution, the difference is that for correlation, the signals are not folded

Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

Correlation

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = x(l) * y(-l)$$

- l is called the signal **lag**

- Consider the following relationships between the input and output sequences of an LTI system

$$\begin{aligned} r_{yx}(l) &= y(l) * x(-l) = h(l) * r_{xx}(l) \\ r_{yy}(l) &= y(l) * y(-l) = h(l) * h(-l) * r_{xx}(l) \end{aligned}$$

  - $r_{xx}(l)$: autocorrelation sequence of input signal $x(n)$
  - $r_{yy}(l)$: autocorrelation sequence of output signal $y(n)$
  - $r_{yx}(l)$: crosscorrelation sequence of input and output signal
- The z-transform results in

$$\begin{aligned} S_{yx}(z) &= H(z)S_{xx}(z) \\ S_{yy}(z) &= H(z)H(z^{-1})S_{xx}(z) \end{aligned}$$

  - recall: $h(-n) \overset{z}{\circ\!\!-\!\!\bullet} H(z^{-1})$; $\quad r_{hh} = h(n) * h(-n) \overset{z}{\circ\!\!-\!\!\bullet} H(z)H(z^{-1})$

Determining the Frequency Response

- Substituting $z = e^{j\omega}$

$$S_{yx}(\omega) = H(\omega)S_{xx}(\omega)$$
$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

- For a flat (white) spectrum $S_{xx}(w) = S_x =$ constant for all ω thus

$$H(\omega) = S_{yx}(\omega)/S_x$$

or equivalently

$$h(n) = r_{yx}(l)/S_x$$

➡ The impulse response $h(n)$ can be determined by exciting the system by a spectrally flat input signal and cross-correlating the input and the output of the system

- Random process: capital letter, e.g. $X(n)$
- sample sequence (a realization of the random process): lower-case letter, e.g. $x(n)$
- Let $h(n)$ be a real-valued sample response
- Let $x(n)$ be a sample function of a stationary random process $X(n)$
- Then the system response is (still) given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

however, as $x(n)$ is a random signal, also $y(n)$ is a a random signal

➡ for each sample sequence $x(n)$ of the process $X(n)$, there is a sample sequence $y(n)$ of the random process $Y(n)$

Expected Value

- The expected value of the output $y(n)$ is

$$m_{\mathrm{y}} = \mathrm{E}(y(n)) = \mathrm{E}\left(\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right)$$

$$= \mathrm{E}(y(n)) = \sum_{k=-\infty}^{\infty} h(k)\mathrm{E}(x(n-k))$$

$$m_{\mathrm{y}} = m_{\mathrm{x}} \sum_{k=-\infty}^{\infty} h(k)$$

- In the frequency domain

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$H(0) = \sum_{k=-\infty}^{\infty} h(k)$$

$$m_{\mathrm{y}} = m_{\mathrm{x}} H(0)$$

Autocorrelation Sequence

- The autocorrelation sequence of the output random process $y(n)$ is

$$\gamma_{yy}(l) = \mathrm{E}(y^*(n)y(n+l)) = \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} h(k)h(j)\gamma_{xx}(k-j+l)$$

- In the frequency domain we obtain the **power density spectrum**

$$\begin{aligned}\Gamma_{yy}(\omega) &= \sum_{l=-\infty}^{\infty} \gamma_{yy}(l)e^{-j\omega l} \\ &= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} h(k)h(j)\left[\sum_{l=-\infty}^{\infty}\gamma_{xx}(k-j+l)e^{-j\omega l}\right] \\ &= \Gamma_{xx}(\omega)\left[\sum_{k=-\infty}^{\infty} h(k)e^{j\omega k}\right]\left[\sum_{l=-\infty}^{\infty} h(l)e^{-j\omega l}\right] \qquad \text{(Verschiebungssatz)} \\ &= |H(\omega)|^2\Gamma_{xx}(\omega)\end{aligned}$$

Crosscorrelation Sequence

- Crosscorrelation sequence of input and output random processes

$$\mathrm{E}(y(n)x^*(n-l)) = \mathrm{E}\left(\sum_{k=-\infty}^{\infty} h(k)x(n-k)x^*(n-l)\right)$$

$$\gamma_{\mathrm{yx}}(l) = \sum_{k=-\infty}^{\infty} h(k)\mathrm{E}(x^*(n-l)x(n-k))$$

$$= \sum_{k=-\infty}^{\infty} h(k)\gamma_{\mathrm{xx}}(l-k)$$

Crosscorrelation Sequence

- Crosscorrelation sequence of input and output random processes

$$\mathrm{E}(y(n)x^*(n-l)) = \mathrm{E}\left(\sum_{k=-\infty}^{\infty} h(k)x(n-k)x^*(n-l)\right)$$

$$\gamma_{\mathrm{yx}}(l) = \sum_{k=-\infty}^{\infty} h(k)\mathrm{E}(x^*(n-l)x(n-k))$$

$$= \sum_{k=-\infty}^{\infty} h(k)\gamma_{\mathrm{xx}}(l-k)$$

- Note the convolution form. Thus, in the frequency domain we have

$$\Gamma_{\mathrm{yx}}(\omega) = H(\omega)\Gamma_{\mathrm{xx}}(\omega)$$

- For white noise $\Gamma_{\mathrm{xx}}(\omega) = \sigma_x^2$, with σ_x^2 the input noise power and

$$\Gamma_{\mathrm{yx}}(\omega) = H(\omega)\sigma_x^2$$

Input-Output Correlation Functions and Spectra

Determining the Frequency Response

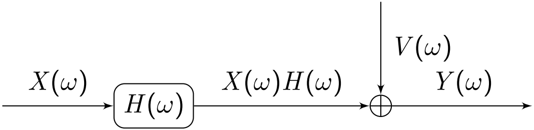
- For white noise $\Gamma_{\mathrm{xx}}(\omega) = \sigma_x^2$, with σ_x^2 the input noise power and

$$\Gamma_{\mathrm{yx}}(\omega) = H(\omega)\sigma_x^2$$

- Uncorrelated measurement noise (e.g. microphone noise) will cancel out:

$$Y(\omega) = X(\omega)H(\omega) + V(\omega)$$

$$\Gamma_{\mathrm{yx}}(\omega) = H(\omega)\sigma_x^2 + \underbrace{\Gamma_{\mathrm{xv}}(\omega)}_{\to 0}$$



➡ A system can be identified by exciting the input with white noise.

➡ Very useful and practically relevant in measuring the impulse response!

Outline

LTI Systems as Frequency-Selective Filters

- A *filter* is a device that discriminates what passes through it.
- An LTI-system discriminates different of various frequencies, described by $H(\omega)$, or a_k, b_k
- $H(\omega)$ acts as a *spectral weighting function* to different frequency components
- "LTI-system" and "filter" can be used interchangeably
- Examples of filtering
  - Removal of undesired noise from target signals
  - Spectral shaping, equalization
  - Signal detection
  - Spectral analysis filters

Ideal Filter Characteristics

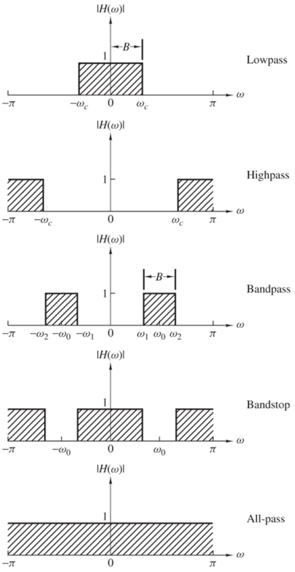


Figure 5.4.1 Magnitude responses for some ideal frequency-selective discrete-time filters.

Phase

- Ideal filters
  - Have a constant (unity-)gain in passband
  - Have a linear phase in passband $\Theta(\omega) = -\omega n_0$

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

- Time-shifting property: a linear phase results in a constant delay

$$y(n) = Cx(n - n_0)$$

  Pure delay is usually not considered a distortion of the signal

- Negative derivative of the phase w.r.t frequency is called **group delay**

$$\tau_g(\omega) = -\frac{d\Theta(\omega)}{d\omega}$$

- The group delay is the time-delay frequency component ω undergoes when passing the filter
- For a linear phase, the group delay is constant $\tau_g = n_0$

Ideal Filter Characteristics

Limitations

- Ideal filters are not physically realizable
  (but serve as a mathematical idealization of practical filters)
- Example: ideal low-pass filter:

Transfer function

$$H_{\mathrm{lp}}(\omega) = \begin{cases} 1, & -\omega_c < \omega < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

Impulse response

$$h_{\mathrm{lp}}(n) = \frac{\sin \omega_c \pi n}{\pi n}, \quad -\infty < n < \infty$$

✗ not causal

✗ infinitely long

➡ physically unrealizable

Design of Filters by Placing Zeros and Poles

- Locate poles near the unit circle to emphasize frequencies
- Place zeros near the unit circle to deemphasize frequencies

Design of Filters by Placing Zeros and Poles

1. All poles should be placed inside the unit circle to generate a stable and causal filter.
   Zeros can be placed anywhere in the z-plane
2. All complex zeros and poles must occur in complex-conjugate pairs in order for the filter coefficients to be real-valued

- Recall that for a given pole-zero pattern, the system function $H(z)$ can be expressed as

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

  - b_0 often chosen such that $H(\omega_0) = 1$ in the passband
  - often $N \geq M$, filter has more (non-trivial) poles than zeros

Lowpass and Highpass Filters

- In the design of lowpass filters,
  - poles: near unit circle at low frequencies (near $\omega = 0$)
  - zeros: near or on the unit circle at high frequencies (near $\omega = \pi$)
- The opposite holds for highpass filters

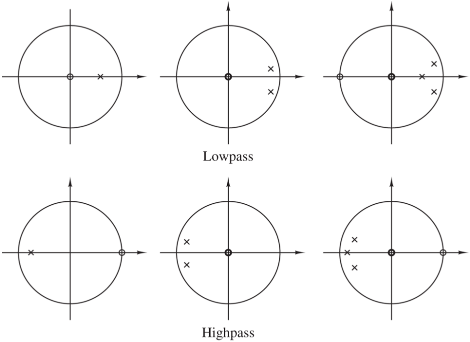


Figure 5.4.2 Pole–zero patterns for several lowpass and highpass filters.

Demo: `http://www.micromodeler.com/dsp/`

Magnitude and Phase of Lowpass Filter

- Single-pole filter

$$H_1(z) = \frac{1-a}{1-az^{-1}}$$

- Additional zero at $z=-1$

$$H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}}$$

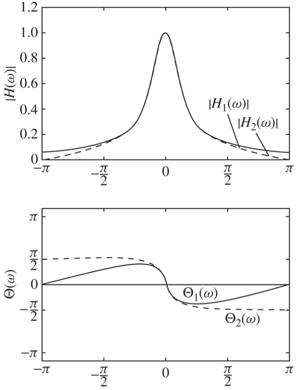


Figure 5.4.3 Magnitude and phase response of (1) a single-pole filter and (2) a one-pole, one-zero filter; $H_1(z) = (1-a)/(1-az^{-1})$, $H_2(z) = [(1-a)/2][(1+z^{-1})/(1-az^{-1})]$ and $a = 0.9$.

Magnitude and Phase of Highpass Filter

- Obtained by reflecting (folding) the pole-zero locations of previous lowpass about the imaginary axis in the z-plane

$$H_3(z) = \frac{1-a}{2} \frac{1-z^{-1}}{1+az^{-1}}$$

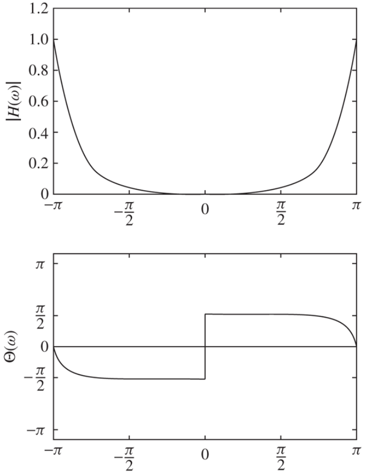


Figure 5.4.4 Magnitude and phase response of a simple highpass filter; $H(z) = [(1-a)/2][(1-z^{-1})/(1+az^{-1})]$ with $a = 0.9$.

Example 5.4.1

- A two-pole lowpass filter has the system function

$$H(z) = \frac{b_0}{(1 - pz^{-1})^2}$$

- Determine the values of b_0 and p such that the frequency response of $H(\omega)$ satisfies the conditions
  - $H(0) = 1$
  - $|H(\frac{\pi}{4})|^2 = \frac{1}{2}$

Example 5.4.1

- A two-pole lowpass filter has the system function

$$H(z) = \frac{b_0}{(1 - pz^{-1})^2}$$

- Determine the values of b_0 and p such that the frequency response of $H(\omega)$ satisfies the conditions
  - $H(0) = 1$
  - $|H(\frac{\pi}{4})|^2 = \frac{1}{2}$
- Solution:

$$H(z) = \frac{0.46}{(1 - 0.32z^{-1})^2}$$

Example 5.4.2

- Design a two-pole bandpass filter that has the center of its passband at $\omega = \pi/2$, zero in its frequency response characteristic at $\omega = 0$ and $\omega = \pi$, and a magnitude response of $1/\sqrt{2}$ at $\omega = 4\pi/9$

- Design a two-pole bandpass filter that has the center of its passband at $\omega = \pi/2$, zero in its frequency response characteristic at $\omega = 0$ and $\omega = \pi$, and a magnitude response of $1/\sqrt{2}$ at $\omega = 4\pi/9$

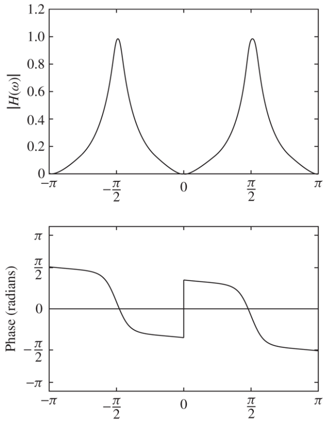


Figure 5.4.5 Magnitude and phase response of a simple bandpass filter in Example 5.4.2; $H(z) = 0.15[(1 - z^{-2})/(1 + 0.7z^{-2})]$.

Simple Lowpass-to-Highpass Filter Transformation

- Given a prototype lowpass, a highpass can be obtained by translation

$$H_{\mathrm{hp}}(\omega) = H_{\mathrm{lp}}(\omega - \pi)$$

- Or due to the frequency shifting property of the Fourier transform

$$h_{\mathrm{hp}}(n) = e^{j\pi n} h_{\mathrm{lp}}(n) = (-1)^n h_{\mathrm{lp}}(n)$$

  and conversely

$$h_{\mathrm{lp}}(n) = (-1)^n h_{\mathrm{hp}}(n)$$

  i.e. sign-change for every other sample

For rational system functions

- If the lowpass filter is described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

- Its frequency response is

$$H_{\mathrm{lp}}(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

- Now, if we replace $\omega \to \omega - \pi$

$$H_{\mathrm{hp}}(\omega) = \frac{\sum_{k=0}^{M} (-1)^k b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} (-1)^k a_k e^{-j\omega k}}$$

- Which corresponds to the difference equation

$$y(n) = -\sum_{k=1}^{N} (-1)^k a_k y(n-k) + \sum_{k=0}^{M} (-1)^k b_k x(n-k)$$

Example 5.4.3

- Convert the following lowpass filter into a highpass

$$y(n) = 0.9y(n-1) + 0.1x(n)$$

Example 5.4.3

- Convert the following lowpass filter into a highpass

$$y(n) = 0.9y(n-1) + 0.1x(n)$$

- Solution: The difference equation for the HP filter is

$$y(n) = -0.9y(n-1) + 0.1x(n)$$

with frequency response

$$H_{\text{hp}}(\omega) = \frac{0.1}{1 + 0.9e^{-j\omega}}$$

- A *digital resonator* is a special two-pole bandpass filter with the pair of complex-conjugate poles located near the unit circle
- The name *resonator* refers to the fact that the filter has a large magnitude response (i.e. it resonates) in the vicinity of the poles
- Useful in many applications including bandpass filtering and speech generation
- For a resonant peak at $\omega = \omega_0$ we select the complex-conjugate poles at

$$p_{1,2} = re^{\pm j\omega_0}, \quad 0 < r < 1$$

- Two placings of zeros are of special interest
  - Two zeros at the origin
  - One zero at $z = 1$ and one at $z = -1$

Two zeros at the origin

$$H(z) = \frac{b_0}{(1 - re^{j\omega_0} z^{-1})(1 - re^{-j\omega_0} z^{-1})} = \frac{b_0}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$$

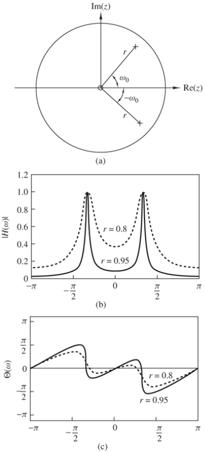


Figure 5.4.6 (a) Pole–zero pattern and (b) the corresponding magnitude and phase response of a digital resonator with (1) $r = 0.8$ and (2) $r = 0.95$.

One zero at $z = 1$ and one at $z = -1$

- Slightly smaller bandwidth
- Very small shift in the resonant frequency due to the presence of the zeros

$$H(z) = \frac{G(1 - z^{-1})(1 + z^{-1})}{(1 - re^{j\omega_0} z^{-1})(1 - re^{-j\omega_0} z^{-1})} = \frac{G(1 - z^{-2})}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}}$$

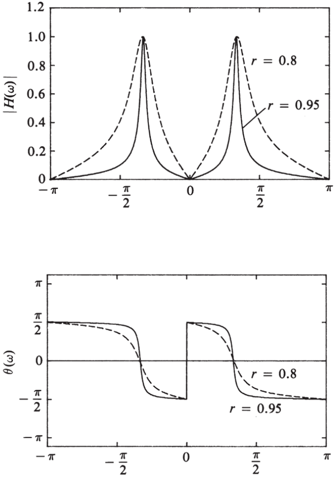


Figure 5.4.7 Magnitude and phase response of digital resonator with zeros at $\omega = 0$ and $\omega = \pi$ and (1) $r = 0.8$ and (2) $r = 0.95$.

- A filter that contains one or more deep notches (or, ideally, perfect nulls) in its frequency response.

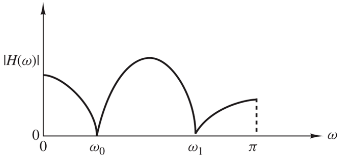


Figure 5.4.8 Frequency response characteristic of a notch filter.

- Useful in many applications where specific frequency components must be eliminated
  - Example: Instrumentation and recording systems require the elimination of the 50Hz power-line frequency
- Simply introduce a pair of complex-conjugate zeros on the unit circle at an angle ω_0

$$z_{1,2} = e^{\pm j w_0}$$

$$H(z) = b_0(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$$

Notch Filters II

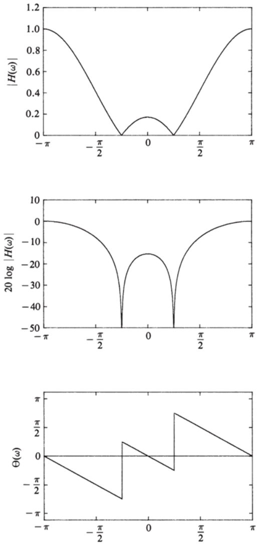


Figure 5.4.9 Frequency response characteristics of a notch filter with a notch at $\omega = \pi/4$ or $f = 1/8$; $H(z) = G[1 - 2\cos\omega_0 z^{-1} + z^{-2}]$.

Notch Filters with Additional Poles

- Problem of an all-zero notch filter: broad bandwidth of notches
- Idea: place poles in vicinity of zeros to reduce bandwidth of notches

$$p_{1,2} = re^{\pm j\omega_0}$$

- Problem: may result in ripples in the passband. Reduction possible by introducing additional poles/zeros (trial-and-error)

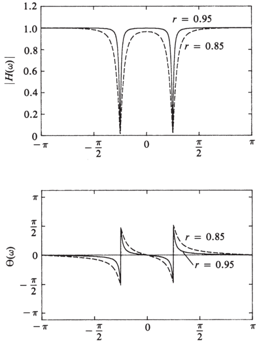


Figure 5.4.10 Frequency response characteristics of two notch filters with poles at (1) $r = 0.85$ and (2) $r = 0.95$; $H(z) = b_0[(1 - 2\cos\omega_0 z^{-1} + z^{-2})/(1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2})]$.

- In its simplest form, a notch filter with periodically occurring notches

→ E.g. suppression of power-line harmonics

- Taking any FIR filter $h(n)$ with system function

$$H(z) = \sum_{k=0}^{M} h(k) z^{-k}$$

an Lth order repetition can be obtained by replacing z by z^L

$$H_L(z) = \sum_{k=0}^{M} h(k) z^{-kL}$$

$$H_L(\omega) = \sum_{k=0}^{M} h(k) e^{-jkL\omega} = H(L\omega)$$

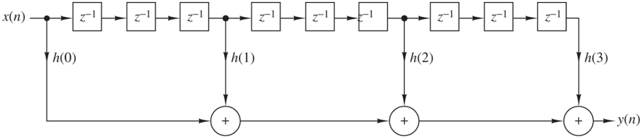


Figure 5.4.13 Realization of an FIR comb filter having $M = 3$ and $L = 3$.

Comb Filters III

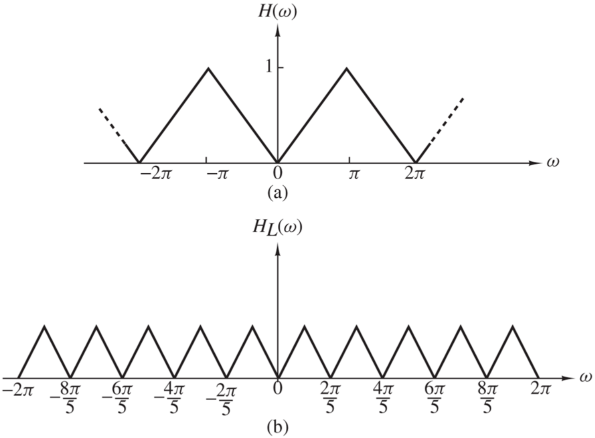


Figure 5.4.12 Comb filter with frequency response $H_L(\omega)$ obtained from $H(\omega)$.

- An all-pass filter is defined as a system that has a constant magnitude response for all frequencies

$$|H(\omega)| = 1, \quad 0 \leq \omega \leq \pi$$

- Simplest example: pure delay system $H(z) = z^{-k}$
- All pass filter obtained when zeros and poles are reciprocal to each other (if z_0 is a pole of $H(z)$, then $1/z_0$ is a zero)

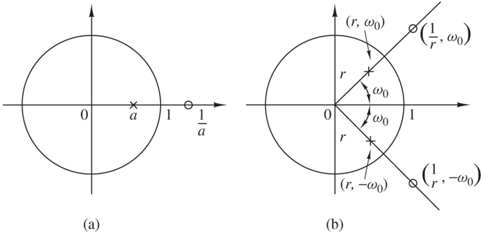


Figure 5.4.16 Pole–zero patterns of (a) a first-order and (b) a second-order all-pass filter.

All-Pass Filters II

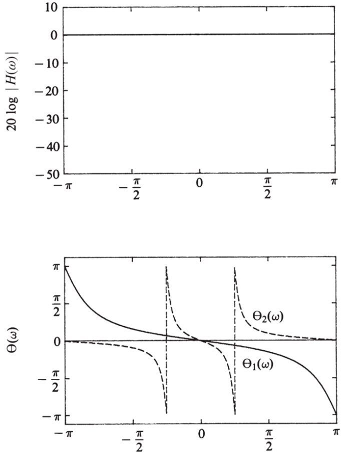


Figure 5.4.17 Frequency response characteristics of an all-pass filter with system functions (1) $H(z) = (0.6 + z^{-1})/(1 + 0.6z^{-1})$, (2) $H(z) = (r^2 - 2r\cos\omega_0 z^{-1} + z^{-2})/\ (1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2})$, $r = 0.9$, $\omega_0 = \pi/4$.

➡ All-pass filters find application as phase equalizers

Digital Sinusoidal Oscillators

- Two-pole resonator with complex-conjugate poles *on* the unit circle

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

with $a_1 = -2\cos\omega_0$ and $a_2 = 1$

- for $b_0 = A\sin\omega_0$ the unit sample response is a *sinusoid*

$$h(n) = A\sin((n+1)\omega_0)\, u(n)$$

- Basic component of a digital frequency synthesizer

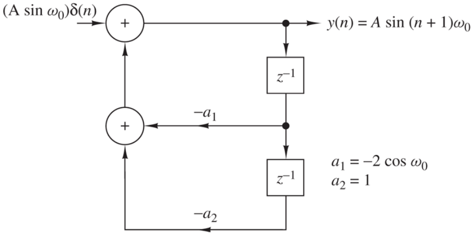


Figure 5.4.18 Digital sinusoidal generator.

- $y(n) = -a_1 y(n-1) - y(n-2) + b_0\delta(n)$

Outline

Invertibility of LTI Systems I

- Often we are interested in inverting the effect of LTI systems, e.g.
  - remove the distortion introduced by a telephone channel
  - remove the reverberance introduced by a room
- However, not every system is invertible
- To be invertible, the inverse of the tranfer function has to be stable
- Think of an ideal low-pass that sets higher frequency components to zero. These frequencies are lost and cannot be reconstructed

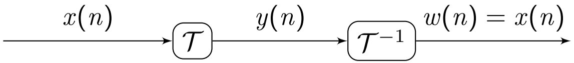
Invertibility of LTI Systems II

- A system is said to be **invertible** if there is a one-to-one correspondence between its input and output signals
  - If we know the output sequence $y(n), -\infty < n < \infty$ of an invertible system $\mathcal{T}$, we can uniquely determine its input $x(n), -\infty < n < \infty$
  - The **inverse system** with input $y(n)$ and output $x(n)$ is denoted by $\mathcal{T}^{-1}$
  - The cascade connection of a system and its inverse is equivalent to the identity system

$$w(n) = \mathcal{T}^{-1}\{y(n)\} = \mathcal{T}^{-1}\{\mathcal{T}\{x(n)\}\} = x(n)$$

- Examples for invertible systems: $y(n) = ax(n)$, $y(n) = x(n-5)$
- Examples for non-invertible systems: $y(n) = x^2(n)$, $y(n) = 0$

Invertibility of LTI Systems III



- The cascade can be written as the convolution

$$w(n) = h_I(n) * h(n) * x(n) = x(n)$$

- Implies that $h(n) * h_I(n) = \delta(n)$
- Solution in time-domain usually difficult. Easier in z-domain, where

$$H(z)H_I(z) = 1$$

and therefore

$$H_I(z) = \frac{1}{H(z)}$$

Invertibility of LTI Systems IV

- Thus, if $H(z)$ has a rational system function

$$H(z) = \frac{B(z)}{A(z)}$$

then

$$H_I(z) = \frac{A(z)}{B(z)}$$

i.e. the poles and zeros switch!

- If $H(z)$ is an FIR system, its inverse $H_I(z)$ is an all-pole system
- If $H(z)$ is an all-pole system, its inverse $H_I(z)$ is an FIR system

Example 5.5.1

- Determine the inverse of the system with impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

- Determine the inverse of the system with impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

Solution:

- The system function corresponding to $h(n)$ is

$$\frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

The system is both causal and stable. Since $H(z)$ is an all-pole system, its inverse is FIR and given by the system function

$$H_I(z) = 1 - \frac{1}{2}z^{-1}$$

- The impulse response is given by

$$h_I(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Example 5.5.2 I

- Determine the inverse of the system with the impulse response

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Example 5.5.2 II

- Determine the inverse of the system with the impulse response

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Solution:

- This is an FIR system with system function $H(z) = 1 - \frac{1}{2}z^{-1}$, ROC: $|z| > 0$
- The inverse system has the system function

$$H_I(z) = \frac{1}{H(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}}$$

  i.e. it has a zero at the origin and a pole at $z = \frac{1}{2}$
- Two possible solutions:
  - If we take ROC as $|z| > \frac{1}{2}$: causal and stable system
  - If we take ROC as $|z| < \frac{1}{2}$: anticausal and unstable system

Example 5.5.2 III

- Determine the inverse of the system with the impulse response

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Solution:

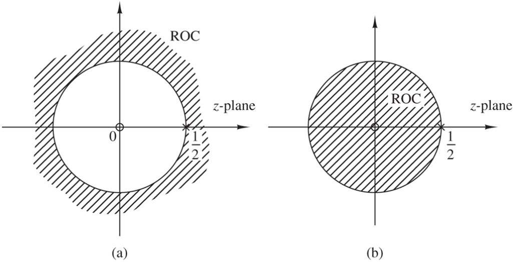


Figure 5.5.2 Two possible regions of convergence for $H(z) = z/(z - \frac{1}{2})$.

- e.g. $h(n) = \left(\frac{1}{2}\right)^n u(n)$

Recursive Inversion of Causal Systems in Time-Domain

- For causal systems we have

$$\sum_{k=0}^{n} h(k)h_I(n-k) = \delta(n)$$

- By assumption, $h_I(n) = 0$ for $n < 0$.
- For $n = 0$ we obtain

$$h_I(0) = \frac{1}{h(0)}$$

- for $n \geq 1$ the inverse can be obtained recursively as

$$h_I(n) = -\sum_{k=1}^{n} \frac{h(k)h_I(n-k)}{h(0)}, \quad n \geq 1$$

- Practical problem: numerical accuracy deteriorates for large n

Example 5.5.3

- Determine the causal inverse of the FIR system with impulse response

$$h(n) = \delta(n) - \alpha\delta(n-1)$$

Example 5.5.3

- Determine the causal inverse of the FIR system with impulse response

$$h(n) = \delta(n) - \alpha\delta(n-1)$$

Solution:

- Since $h(0) = 1, h(1) = -\alpha$, and $h(n) = 0$ for $n \geq 1$, we have

$$h_I(0) = 1/h(0) = 1$$

and

$$h_I(n) = \alpha h_I(n-1), \quad n \geq 1.$$

Consequently

$$h_I(1) = \alpha, h_I(2) = \alpha^2, ..., h_I(n) = \alpha^n$$

$\rightarrow$ the inverse is an IIR system

Motivation

- We will show that the invertibility of an LTI system is intimately related to the characteristics of its spectral phase function.
- Illustration: Consider the following two systems

$$H_1(z) = 1 + \frac{1}{2}z^{-1} = z^{-1}(z + \frac{1}{2})$$

$$H_2(z) = \frac{1}{2} + z^{-1} = z^{-1}(\frac{1}{2}z + 1)$$

  - $H_1(z)$ has a zero at $z = -\frac{1}{2}$ and impulse response $h = [1, \frac{1}{2}]$
  - $H_2(z)$ has a zero at $z = -2$ and impulse response $h = [\frac{1}{2}, 1]$
  - ➡ reciprocal zeros result in time reversal

$$x(-n) \overset{z}{\circ\!\!-\!\!\bullet} X(z^{-1})$$

Motivation

- In the frequency domain the two systems differ only in their phase

$$|H_1(\omega)| = |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos(\omega)}$$

$$\Theta_1(\omega) = \tan^{-1}\left(\frac{-\sin(\omega)}{2 + \cos(\omega)}\right)$$

$$\Theta_2(\omega) = \tan^{-1}\left(\frac{-\sin(\omega)}{\frac{1}{2} + \cos(\omega)}\right)$$

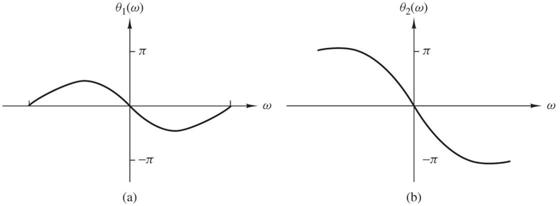


Figure 5.5.3 Phase response characteristics for the systems in (5.5.10). and (5.5.11).

Motivation

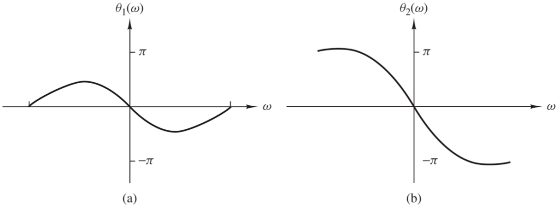


Figure 5.5.3 Phase response characteristics for the systems in (5.5.10). and (5.5.11).

- System one (zero inside unit circle) starts with a phase of zero at $\omega = 0$ and ends with a phase of zero at $\omega = \pi$. As the net phase change $\Theta(\pi) - \Theta(0) = 0$, we refer to this system as **minimum-phase system**
- System two (zero outside unit circle) starts with a phase of zero at $\omega = 0$ and ends with a phase of π at $\omega = \pi$. As the net phase change $\Theta(\pi) - \Theta(0) = \pi$, we refer to this system as **maximum-phase system**

FIR filters of arbitrary lengths

- Extension to FIR filters of arbitrary lengths is straight forward:
  - An FIR system of length $M+1$ has M zeros.

$$
\begin{aligned}
H(\omega) &= b_0 \quad (1 - z_1 e^{-j\omega}) \quad (1 - z_2 e^{-j\omega}) \quad \ldots \quad (1 - z_M e^{-j\omega}) \\
&= b_0 \quad H_1(\omega) \quad H_2(\omega) \quad \ldots \quad H_M(\omega) \\
&= b_0 \quad |H_1(\omega)| e^{j\Theta_1(\omega)} \quad |H_2(\omega)| e^{j\Theta_2(\omega)} \quad \ldots \quad |H_M(\omega)| e^{j\Theta_M(\omega)} \\
\Theta(\omega) &= \angle b_0 \quad + \Theta_1(\omega) \quad + \Theta_2(\omega) \quad + \ldots + \quad \Theta_M(\omega)
\end{aligned}
$$

  - If all zeros are inside the unit circle, each zero will contribute a net phase change of zero ➡ **minimum phase system**
  - If all zeros are outside the unit circle, each zero will contribute a net phase change of π and $\Theta(\pi) - \Theta(0) = M\pi$ ➡ **maximum-phase system**
  - FIR systems with some zeros inside and outside the unit circle are called **mixed-phase systems**

Implications

- For an FIR system with real-valued coefficients we have

$$H(z^{-1})\Big|_{z=e^{j\omega}} = H(e^{-j\omega}) = H(-\omega) \overset{\text{real-valued}}{=} H^*(\omega)$$

➡ For a real-valued FIR system, replacing a zero z_k by its inverse $1/z_k$ does not change the magnitude response

➡ For real-valued FIR systems, we can make a system minimum-phase without changing $|H(\omega)|$ by replacing $z_k \to 1/z_k$ such that all zeros are inside the unit circle

- A minimum phase system has the lowest possible delay for a given magnitude!
- The inverse of a minium phase system is an all-pole system with all poles inside the unit circle ➡ stable causal inverse
- The inverse of mixed-phase system and maximum-phase systems have poles outside the unit circle ➡ not a stable causal inverse

Example 5.5.4

- Determine the zeros and indicate if the system is minimum phase, maximum phase, or mixed phase

$$
\begin{aligned}
H_1(z) &= 6 + z^{-1} - z^{-2} \\
H_2(z) &= 1 - z^{-1} - 6z^{-2} \\
H_3(z) &= 1 - \frac{5}{2}z^{-1} - \frac{3}{2}z^{-2} \\
H_4(z) &= 1 + \frac{5}{3}z^{-1} - \frac{2}{3}z^{-2}
\end{aligned}
$$

Example 5.5.4

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H_4(z) &= 1 + \frac{5}{3}z^{-1} - \frac{2}{3}z^{-2}
\end{aligned}
$$

Solution:

- $H_1(z) \longrightarrow z_{1,2} = -\frac{1}{2}, \frac{1}{3}$

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$$\begin{aligned} H_1(z) &= 6 + z^{-1} - z^{-2} \\ H_2(z) &= 1 - z^{-1} - 6z^{-2} \\ H_3(z) &= 1 - \frac{5}{2}z^{-1} - \frac{3}{2}z^{-2} \\ H_4(z) &= 1 + \frac{5}{3}z^{-1} - \frac{2}{3}z^{-2} \end{aligned}$$

Solution:

- $H_1(z) \longrightarrow z_{1,2} = -\frac{1}{2}, \frac{1}{3} \longrightarrow$ minimum phase
- $H_2(z) \longrightarrow z_{1,2} = -2, 3$

- Determine the zeros and indicate if the system is minimum phase, maximum phase, or mixed phase

$$
\begin{aligned}
H_1(z) &= 6 + z^{-1} - z^{-2} \\
H_2(z) &= 1 - z^{-1} - 6z^{-2} \\
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H_4(z) &= 1 + \frac{5}{3}z^{-1} - \frac{2}{3}z^{-2}
\end{aligned}
$$

Solution:

- $H_1(z) \longrightarrow z_{1,2} = -\frac{1}{2}, \frac{1}{3} \longrightarrow$ minimum phase
- $H_2(z) \longrightarrow z_{1,2} = -2, 3 \longrightarrow$ maximum phase
- $H_3(z) \longrightarrow z_{1,2} = -\frac{1}{2}, 3$

Example 5.5.4

- Determine the zeros and indicate if the system is minimum phase, maximum phase, or mixed phase

$$
\begin{aligned}
H_1(z) &= 6 + z^{-1} - z^{-2} \\
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H_4(z) &= 1 + \frac{5}{3}z^{-1} - \frac{2}{3}z^{-2}
\end{aligned}
$$

Solution:

- $H_1(z) \longrightarrow z_{1,2} = -\frac{1}{2}, \frac{1}{3} \longrightarrow$ minimum phase
- $H_2(z) \longrightarrow z_{1,2} = -2, 3 \longrightarrow$ maximum phase
- $H_3(z) \longrightarrow z_{1,2} = -\frac{1}{2}, 3 \longrightarrow$ mixed phase
- $H_4(z) \longrightarrow z_{1,2} = -2, \frac{1}{3}$

- Determine the zeros and indicate if the system is minimum phase, maximum phase, or mixed phase

$$
\begin{aligned}
H_1(z) &= 6 + z^{-1} - z^{-2} \\
H_2(z) &= 1 - z^{-1} - 6z^{-2} \\
H_3(z) &= 1 - \frac{5}{2}z^{-1} - \frac{3}{2}z^{-2} \\
H_4(z) &= 1 + \frac{5}{3}z^{-1} - \frac{2}{3}z^{-2}
\end{aligned}
$$

Solution:

- $H_1(z) \longrightarrow z_{1,2} = -\frac{1}{2}, \frac{1}{3} \longrightarrow$ minimum phase
- $H_2(z) \longrightarrow z_{1,2} = -2, 3 \longrightarrow$ maximum phase
- $H_3(z) \longrightarrow z_{1,2} = -\frac{1}{2}, 3 \longrightarrow$ mixed phase
- $H_4(z) \longrightarrow z_{1,2} = -2, \frac{1}{3} \longrightarrow$ mixed phase

Rational IIR systems

- The minimum-phase property carries over to rational IIR systems described by

$$H(z) = \frac{B(z)}{A(z)}$$

- ➡ $H(z)$ is **minimum-phase** if all poles and zeros are inside the unit circle
- ➡ For a stable and causal system (i.e. all roots of $A(z)$ are within the unit circle), the system is **maximum-phase** if all zeros are outside the unit circle
- ➡ For a stable and causal system, the system is **mixed-phase** if some but not all zeros are outside the unit circle
- A stable pole-zero system that is minimum phase has a stable inverse which is also minimum phase given by

$$H^{-1}(z) = \frac{A(z)}{B(z)}$$

- Mixed-phase and Maximum-phase systems result in unstable inverse systems!

Decomposition into minimum-phase and all-pass systems

Goal: Make a rational IIR system $H(z) = B(z)/A(z)$ minimum phase, and thus invertible!

- Express numerator as $B(z) = B_1(z)B_2(z)$
  - $B_1(z)$ consists of all zeros inside the unit circle
  - $B_2(z)$ consists of all zeros outside the unit circle
- A minimum phase system with $|H_{\min}(\omega)| \propto |H(\omega)|$ is obtained by mirroring the zeros outside the unit circle to be inside as

$$H_{\min}(z) = \frac{B_1(z)B_2(z^{-1})}{A(z)}$$

- The original system is obtained by multiplying an all-pass system $H_{\mathrm{ap}}(z)$ with $|H_{\mathrm{ap}}(\omega)| = 1$

$$H(z) = H_{\min}(z)H_{\mathrm{ap}}(z)$$

with $H_{\mathrm{ap}}(z) = \frac{B_2(z)}{B_2(z^{-1})}$, i.e. a maximum phase system with reciprocal poles and zeros

System Identification in z-domain

- **System Identification**: Determine the impulse response of unknown systems
- Can be solved by **deconvolution**

$$
\begin{aligned}
y(n) &= h(n) * x(n) \\
&= \sum_{k=-\infty}^{\infty} h(k)x(n-k)
\end{aligned}
$$

- An analytical solution of the deconvolution problem can be found in z-domain

$$Y(z) = H(z)X(z)$$

and hence

$$H(z) = \frac{Y(z)}{X(z)}$$

where $X(z)$ and $Y(z)$ are z-transforms of the available input signal $x(n)$ and the observed output $y(n)$

Example 5.5.5

A causal system produces the output sequence

$$y(n) = \begin{cases} 1, & n = 0 \\ \frac{7}{10}, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

when excited by the input sequence

$$x(n) = \begin{cases} 1, & n = 0 \\ -\frac{7}{10}, & n = 1 \\ \frac{1}{10}, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the input-output difference equation and the impulse response

Solution

- Determine system function by taking the z-transform of $x(n)$ and $y(n)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{7}{10}z^{-1}}{1-\frac{7}{10}z^{-1}+\frac{1}{10}z^{-2}}$$
$$= \frac{1+\frac{7}{10}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})}$$

- Causal system $\rightarrow$ ROC is $|z| > \frac{1}{2}$.
- System is stable since poles lie inside unit circle
- The input-output difference equation for the system is

$$y(n) = \frac{7}{10}y(n-1) - \frac{1}{10}y(n-2) + x(n) + \frac{7}{10}x(n-1)$$

- The impulse response can be determined using partial-fraction expansion of $H(z)$ and inverse z-transformation

$$h(n) = \left[4\left(\frac{1}{2}\right)^n - 3\left(\frac{1}{5}\right)^n\right]u(n)$$

System Identification in z-domain

- We observe that $H(z) = \frac{Y(z)}{X(z)}$ determines the unknown system uniquely, if it is known that the system is causal
- This approach is still artificial, as generally the system response $\{y(n)\}$ is infinite in duration

Recursive System Identification

- Alternatively, we can directly work in time domain. For a causal system we have

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k), \quad n \geq 0$$

and hence

$$h(0) = \frac{y(0)}{x(0)} \tag{13}$$

$$h(n) = \frac{y(n) - \sum_{k=0}^{n-1} h(k)x(n-k)}{x(0)}, \quad n \geq 1 \tag{14}$$

- This results in a **recursive solution** which requires $x(0) \neq 0$
- Again, as before, if $h(n)$ has infinite duration, this approach is not practical (or we need to truncate the result)

Crosscorrelation Approach

- The input-output crosscorrelation function is given by

$$r_{yx}(m) = \sum_{k=0}^{\infty} h(k) r_{xx}(m-k) = h(n) * r_{xx}(m)$$

- In frequency domain, this results in

$$S_{yx}(\omega) = H(\omega) S_{xx}(\omega) = H(\omega) |X(\omega)|^2$$

- Hence

$$H(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} = \frac{S_{yx(\omega)}}{|X(\omega)|^2}$$

→ The transfer function can be obtained from the Fourier transforms of the crosscorrelation and the input

→ If the input sequence is chosen to have a flat spectrum, the values of the impulse response are simply $h(n) = r_{yx}(n)$

- This crosscorrelation method is an effective and practical method for system identification!

6. Sampling and Reconstruction of Signals

Outline

Motivation

- Recall that, if bandlimited continuous-time signals are sampled with a rate twice as high as the highest frequency contained in the signal, a perfect reconstruction is possible
- For digitization, besides sampling (discretization of the time axis) also quantization (discretization of the amplitude axis) is necessary
- As opposed to sampling, quantization is in general a lossy process, i.e. perfect reconstruction is not possible.

Outline

Ideal Sampling of Continuous-Time Signals I

- To process a continuous-time signal using a computer, it is necessary to convert the signal into a sequence of numbers. This is usually done by sampling the analog signal $x_a(t)$ periodically every T seconds to produce a discrete-time signal $x(n)$ given by

$$x(n) = x_a(nT), \quad -\infty < n < \infty$$

- The sampling frequency $F_\mathrm{S} = 1/T$ must be selected large enough such that the sampling does not cause any loss of spectral information (no aliasing)
- Here, we now investigate the sampling process by finding the relationship between the spectra of signals $x_a(t)$ and $x(n)$

- Because $x(n) = x_a(nT)$, t and n are related

$$t = nT = \frac{n}{F_\mathrm{S}}$$

- Furthermore with $t = \frac{n}{F_\mathrm{S}}$, using the definitions of the DTFT and the FT

$$\begin{aligned} x(n) &= x_a(nT) \\ \int_{-1/2}^{1/2} X(f) e^{j2\pi fn}\, df &= \int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_\mathrm{S}}\, dF \end{aligned}$$

- Periodic sampling imposes a relationship between the frequency variables as

$$f = F/F_\mathrm{S}$$

- With a change of variable $f = F/F_\mathrm{S}$ on the left we obtain

$$\frac{1}{F_\mathrm{S}} \int_{-F_\mathrm{S}/2}^{F_\mathrm{S}/2} X(F) e^{j2\pi nF/F_\mathrm{S}}\, dF = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_\mathrm{S}}\, dF$$

- We can split the infinite integral into snippets of length F_S

$$
\begin{aligned}
\frac{1}{F_\mathrm{S}} \int_{-F_\mathrm{S}/2}^{F_\mathrm{S}/2} X(F) e^{j2\pi nF/F_\mathrm{S}} dF &= \int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_\mathrm{S}} dF \\
&= \sum_{k=-\infty}^{\infty} \int_{-F_\mathrm{S}/2}^{F_\mathrm{S}/2} X_a(F - kF_\mathrm{S}) e^{j2\pi n(F-kF_\mathrm{S})/F_\mathrm{S}} dF \\
&= \int_{-F_\mathrm{S}/2}^{F_\mathrm{S}/2} \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_\mathrm{S}) \right] e^{j2\pi nF/F_\mathrm{S}} dF
\end{aligned}
$$

Ideal Sampling of Continuous-Time Signals IV

- From this equation it follows that

$$X(F) = F_\mathrm{S} \sum_{k=-\infty}^{\infty} X_a(F - kF_\mathrm{S})$$

or equivalently

$$X(f) = F_\mathrm{S} \sum_{k=-\infty}^{\infty} X_a((f - k)F_\mathrm{S})$$

DTFT spectrum X

➡ F_S-periodic repetition of the analog spectrum X_a!

Ideal Sampling of Continuous-Time Signals V

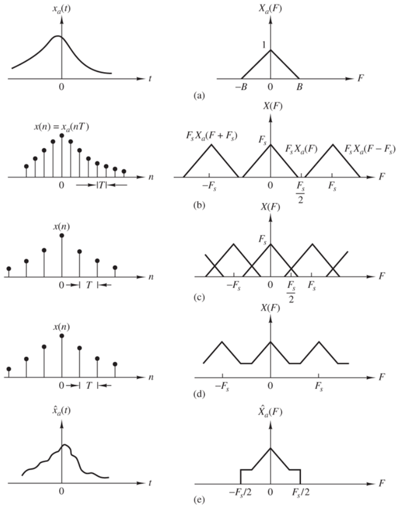

Figure 6.1.1 Sampling of an analog bandlimited signal and aliasing of spectral components.

Ideal Sampling of Continuous-Time Signals VI

- If the spectrum is zero for $|F| > F_\mathrm{S}/2$, the original spectrum can be obtain from the spectrum of the sampled signals using a low-pass filter
- If the spectrum is non-zero for $|F| > F_\mathrm{S}/2$, the periodic repetitions overlap and the original signal cannot be reconstructed!
- From this, the sampling theorem follows
  - ➡ The sampling rate (i.e. the periodic repetition) F_S has to be larger than $2B$ to allow for a reconstruction of the original spectrum. Then

$$X(F) = F_\mathrm{S} X_a(F), \quad |F| \leq F_\mathrm{S}/2$$

  - In this case, no aliasing occurs
- If $F_\mathrm{S} < 2B$, the periodic continuation of $X_a(F)$ results in spectral overlap.
  - ➡ The spectrum $X(F)$ contains aliased frequency components and $x_a(t)$ can not be perfectly recovered

Ideal Reconstruction of Continuous-Time Signals I

- Let us assume that $F_\mathrm{S} \geq 2B$
- Using $X_a(F) = \frac{1}{F_\mathrm{S}} X(F)$ for $|F| \leq F_\mathrm{S}/2$ in $x_a(t) = \int_{-F_\mathrm{S}/2}^{F_\mathrm{S}/2} X_a(F) e^{j2\pi Ft} dF$ follows

$$x_a(t) = \frac{1}{F_\mathrm{S}} \int_{-F_\mathrm{S}/2}^{F_\mathrm{S}/2} \underbrace{\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi Fn/F_\mathrm{S}}\right]}_{X(F)} e^{j2\pi Ft} dF$$

$$= \frac{1}{F_\mathrm{S}} \sum_{n=-\infty}^{\infty} x(n) \int_{-F_\mathrm{S}/2}^{F_\mathrm{S}/2} e^{j2\pi F(t-n/F_\mathrm{S})} dF$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT) \underbrace{\frac{\sin((\pi/T)(t-nT))}{(\pi/T)(t-nT)}}_{g(t-nT)}$$

- Formula for ideal reconstruction of analog signal involves the **ideal interpolation function**

$$g(t) = \frac{\sin(\pi/Tt)}{\pi/Tt}$$

which is shifted by nT and weighted by samples $x_a(nT)$

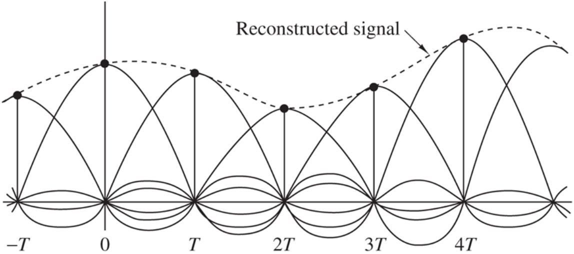

Figure 6.1.2 Reconstruction of a continuous-time signal using ideal interpolation.

Sampling Theorem – Revisited I

- The ideal interpolation function $g(t)$ is an ideal lowpass with cutoff frequency $F_\mathrm{S}/2$
- A bandlimited continuous-time signal with highest frequency (bandwidth) B can be uniquely recovered from its samples iff $F_\mathrm{S} \geq 2B$
- However, the ideal interpolation function requires an infinite number of samples.
- In practice, we reconstruct finite duration signals from a finite number of samples
- In practice, prefiltering with an antialising filter (low-pass filter) is usually employed prior to sampling
  $\Rightarrow$ aliasing frequencies cause negligible distortions on the desired signal

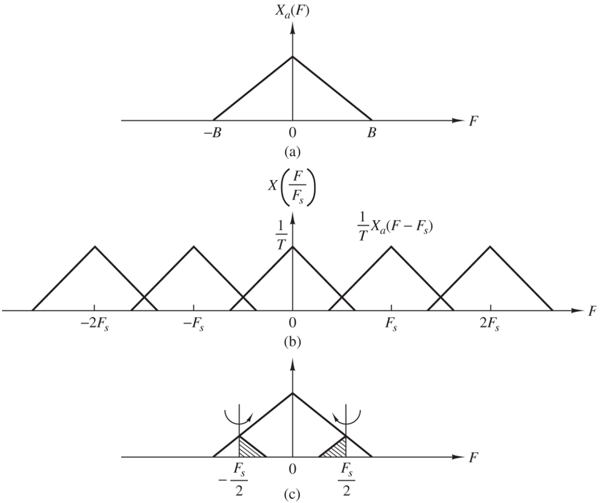

Figure 6.1.3 Illustration of aliasing around the folding frequency.

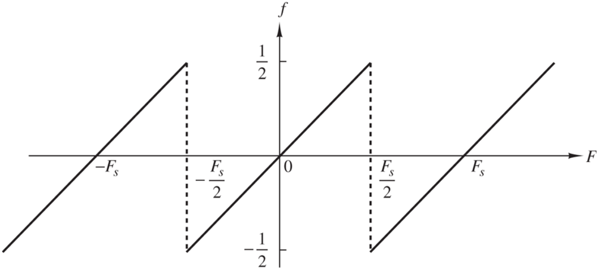

Figure 6.1.4 Relationship between frequency variables F and f.

Example 6.1.1

- Consider the analog signal

$$x_a(t) = \cos 2\pi F_0 t = \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi F_0 t} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi F_0 t}$$

- Its spectrum is depicted in Figure 6.1.6 (a).
- Now we chose two different sampling rates, which lead to the repetitions of the analog spectrum depicted in (b) an (d), respectively:
  1. $F_\mathrm{S}/2 < F_0 < F_\mathrm{S}$ (b)
  2. $F_\mathrm{S} < F_0 < 3F_\mathrm{S}/2$ (d)
- To reconstruct the analog signal, we should select only the components inside the fundamental frequency range $|F| \leq F_\mathrm{S}/2$ ➡ (c) & (e)

➡ The reconstructed signals are
  1. $\hat{x}_a(t) = \cos 2\pi(F_\mathrm{S} - F_0)t$
  2. $\hat{x}_a(t) = \cos 2\pi(F_0 - F_\mathrm{S})t$

➡ Input $\neq$ Output!

Example 6.1.1

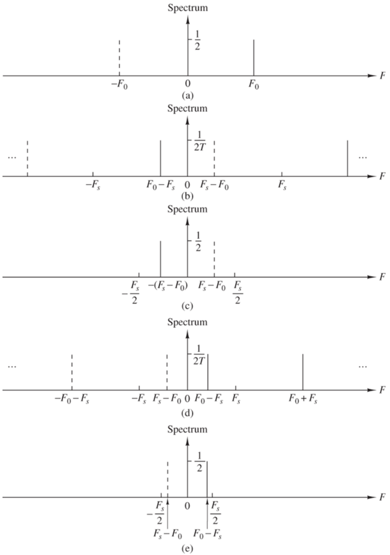

Figure 6.1.6 Aliasing of sinusoidal signals.

Example 6.1.2

- Consider the following continuous-time two-sided exponential signal

$$x_a(t) = e^{-A|t|} \;\overset{\mathrm{F}}{\circ\!\!-\!\!\bullet}\; X_a(F) = \frac{2A}{A^2 + (2\pi F)^2}, \quad A > 0$$

a) Determine the spectrum of the sampled signal $x(n) = x_a(nT)$
b) Plot the signals $x_a(t)$ and $x(n) = x_a(nT)$ for $T = 1/3 \text{ sec}$ and $T = 1 \text{ sec}$, and their spectra
c) Plot the continuous-time signal $\widehat{x_a}(t)$ after reconstruction with ideal bandlimited interpolation

Example 6.1.2

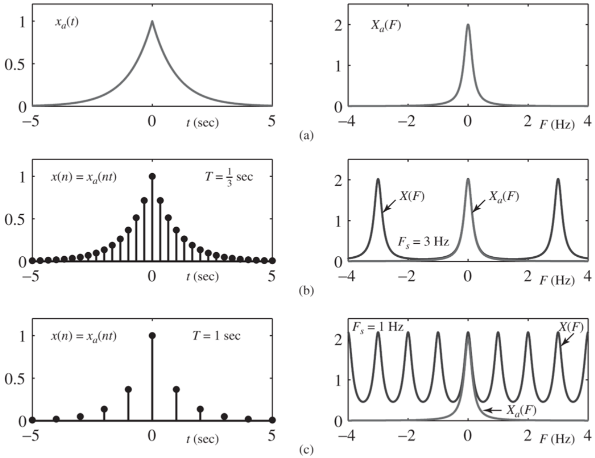

Figure 6.1.7 (a) Analog signal $x_a(t)$ and its spectrum $X_a(F)$; (b) $x(n) = x_a(nT)$ and its spectrum for $F_s = 3$ Hz; and (c) $x(n) = x_a(nT)$ and its spectrum for $F_s = 1$ Hz.

Example 6.1.2

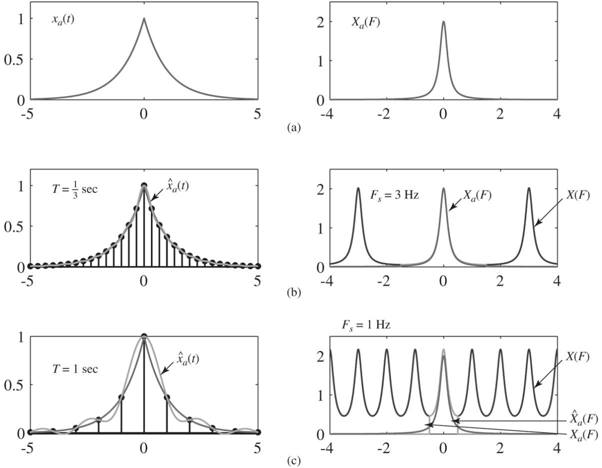

Figure 6.1.8 (a) Analog signal $x_a(t)$ and its spectrum $X_a(F)$; (b) reconstructed signal $\hat{x}_a(t)$ and its spectrum for $F_s = 3$ Hz; and (c) reconstructed signal $\hat{x}_a(t)$ and its spectrum for $F_s = 1$ Hz.

1. Introduction of Basic Concepts
2. Discrete-Time Signals and Systems
3. The z-Transform and Its Applications
4. Frequency Analysis of Signals
5. Frequency Analysis of LTI Systems
6. Sampling and Reconstruction of Signals

6.1 Ideal Sampling and Reconstruction of Continuous-Time Signals
6.2 Discrete-Time Processing of Continuous-Time Signals
6.3 Quantization and Coding
6.4 Analog-to-Digital and Digital-to-Analog Converters
6.5 Sampling and Reconstruction of Continuous-Time Bandpass Signals
6.6 Sampling and Interpolation of Discrete-Time Signals
6.7 Oversampling A/D and D/A Converters

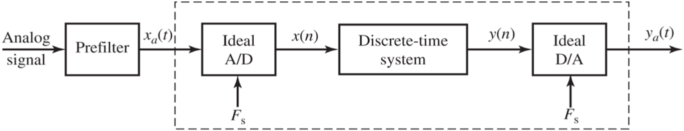

Figure 6.2.1 System for the discrete-time processing of continuous-time signals.

- The prefilter limits bandwidth to desired frequency range
- The ideal A/D converter samples the continuous-time signal $x_a(t)$
- Then the discrete-time signal $x(n)$ can be modified, i.e. filtered
- The ideal D/A converter finally transforms the modified discrete-time signal $y(n)$ into a continuous-time signal $y(t)$

→ Possible application: replace complicated/costly analog filters with simpler/cheaper/more reliable digital filtering.

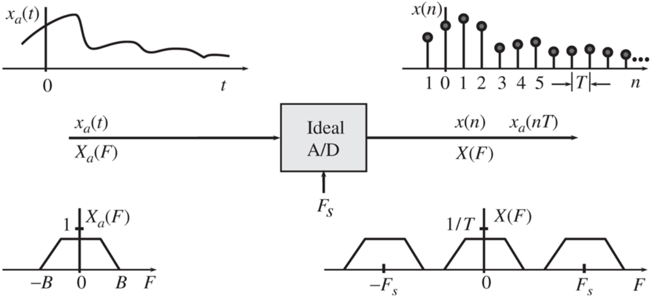

Figure 6.2.2 Characteristics of an ideal A/D converter in the time and frequency domains.

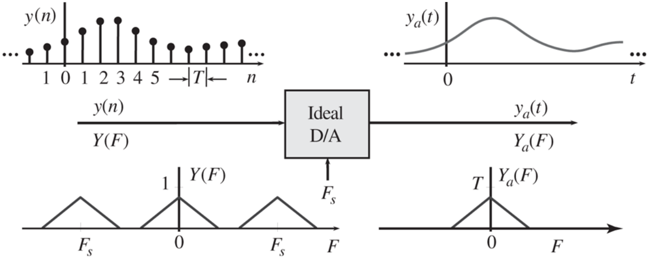

Figure 6.2.3 Characteristics of an ideal D/A converter in the time and frequency domains.

- Apply an ideal lowpass with cutoff frequency $F_S/2$ to consider only the fundamental frequency region

Simulation of an Analog Integrator I

- Consider an analog integrator with input-output relation

$$RC\frac{dy_a(t)}{dt} + y_a(t) = x_a(t)$$

- Using the Fourier transform we obtain

$$H_a(F) = \frac{Y_a(F)}{X_a(F)} = \frac{1}{1 + jF/F_c}, \quad F_c = \frac{1}{2\pi RC}$$

- With the inverse Fourier transform we obtain the infinite impulse response

$$h_a(t) = Ae^{-At}u(t), \quad A = \frac{1}{RC}$$

- Clearly, the system is not bandlimited.

Simulation of an Analog Integrator II

- To be able to apply (an approximation) of such a filter to a *time-discrete* signal $x(n)$, we sample its impulse response and obtain

$$
\begin{aligned}
h(n) &= h_a(nT) = A(e^{-AT})^n u(n) \\
H(z) &= \sum_{n=0}^{\infty} A(e^{-AT})^n z^{-n} = \frac{A}{1 - e^{-AT} z^{-1}} \\
y(n) &= e^{-AT} y(n-1) + Ax(n)
\end{aligned}
$$

- The system is causal and has a pole $p = e^{-AT}$.
- Since $|p| < 1$ and $0 < A < \infty$, the system is always stable

Simulation of an Analog Integrator III

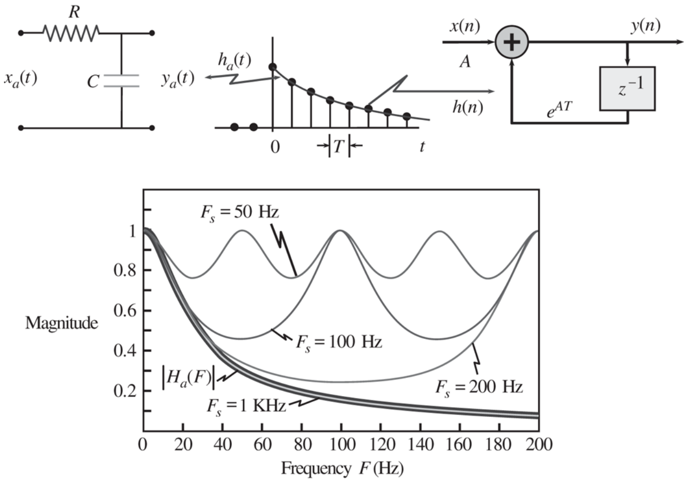

Figure 6.2.4 Discrete-time implementation of an analog integrator using impulse response sampling. The approximation is satisfactory when the bandwidth of the input signal is much less than the sampling frequency.

Fractional Delay

Example 6.2.3

- A continuous-time delay system is defined by

$$y_a(t) = x_a(t - t_d), \quad \text{for any } t_d > 0$$

Fractional Delay

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- While seemingly simple, difficult for discrete-time systems if t_d is not an integer multiple of T, i.e. the time delay between sampling points

$$y(n) = y_a(nT) = x_a(nT - t_d) = x_a[(n - \Delta)T] = x(n - \Delta), \quad \text{with } \Delta = t_d/T$$

SP Signal Processing

Example 6.2.3

- A continuous-time delay system is defined by

$$y_a(t) = x_a(t - t_d), \quad \text{for any } t_d > 0$$

- While seemingly simple, difficult for discrete-time systems if t_d is not an integer multiple of T, i.e. the time delay between sampling points

$$y(n) = y_a(nT) = x_a(nT - t_d) = x_a[(n-\Delta)T] = x(n-\Delta), \quad \text{with } \Delta = t_d/T$$

- Possible solution: consider frequency response

$$H_{\mathrm{id}}(\omega) = e^{-j\omega\Delta}$$

with impulse response

$$h_{\mathrm{id}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{id}}(\omega) e^{j\omega n} d\omega =$$

Example 6.2.3

- A continuous-time delay system is defined by

$$y_a(t) = x_a(t - t_d), \quad \text{for any } t_d > 0$$

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$$H_{\mathrm{id}}(\omega) = e^{-j\omega\Delta}$$

with impulse response

$$h_{\mathrm{id}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{id}}(\omega) e^{j\omega n} d\omega = \frac{\sin(\pi(n - \Delta))}{\pi(n - \Delta)}$$

- For integer Δ, we have $h_{\mathrm{id}} = \delta(n - \Delta)$, because $\sin(\cdot)$ is sampled at zero-crossings
- In general however: h_{id} has an infinite duration. Approximation via

Outline

1. Introduction of Basic Concepts

2. Discrete-Time Signals and Systems

3. The z-Transform and Its Applications

4. Frequency Analysis of Signals

5. Frequency Analysis of LTI Systems

6. Sampling and Reconstruction of Signals

6.1 Ideal Sampling and Reconstruction of Continuous-Time Signals
6.2 Discrete-Time Processing of Continuous-Time Signals
6.3 Quantization and Coding
6.4 Analog-to-Digital and Digital-to-Analog Converters
6.5 Sampling and Reconstruction of Continuous-Time Bandpass Signals
6.6 Sampling and Interpolation of Discrete-Time Signals
6.7 Oversampling A/D and D/A Converters

Quantization and Coding I

- The task of an A/D converter is to convert a continuous range of input amplitudes into a discrete set of digital code words. This involves
  - prefiltering and sampling
  - quantization and coding
- Quantization is a nonlinear and noninvertible (lossy) process that maps a given amplitude $x(n) \equiv x_a(nT)$ at time $t = nT$ into an amplitude x_k taken from a finite set of values

Quantization and Coding II

- The signal amplitude range is divided into L intervals

$$I_k = \{x_k < x(n) \leq x_{k+1}\}\,, \quad k = 1, 2, ..., L$$

  by the $L+1$ **decision levels** $x_1, x_2, ..., x_{L+1}$
- The possible outputs, i,e, the **quantization levels**, are denoted $\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_L$
- The operation of the quantizer is defined by the relation

$$x_q(n) \equiv Q[x(n)] = \widehat{x}_k, \quad \text{if } x(n) \in I_k$$

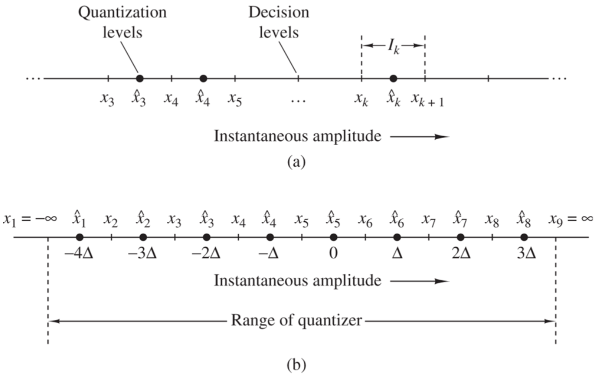


Figure 6.3.2 Quantization process and an example of a midtread quantizer.

- Often, quantization is memoryless and simply denoted as $x_q = Q[x]$

Quantization and Coding IV

- Often, we us **uniform quantizers** (aka linear quantizers) defined by

$$\widehat{x}_{k+1} - \widehat{x}_k = \Delta, \quad k = 1, 2, ..., L-1$$
$$x_{k+1} - x_k = \Delta, \quad \text{for finite } x_k, x_{k+1}$$

  where Δ is the **quantizer step size**

- If zero is assigned to a quantization level, we call it a **midtread** quantizer.
- If zero is assigned to a decision level, we call it a **midrise** quantizer.

Quantization and Coding V

- In the coding process, a binary number is assigned to each quantization level
- With b bits spent per sample, 2^b different levels can be distinguished
- With the range of the quantizer R, the step size (or resolution) of the A/D converter is given by

$$\Delta = \frac{R}{2^b}$$

Quantization and Coding VI

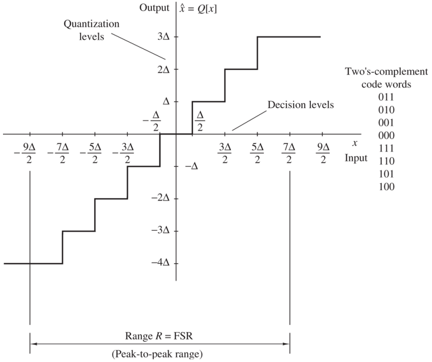


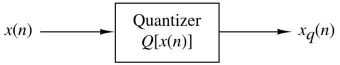
Figure 6.3.3 Example of a midtread quantizer.

- It can be seen that the quantization error $e_q = \widehat{x} - x$ is always in the range

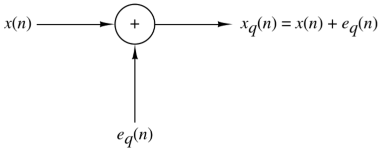
$$-\frac{\Delta}{2} < e_q(n) \leq \frac{\Delta}{2}$$

Quantization Error I

- The quantization error is dependent on the characteristics of the input signal. Thus deterministically intractable.
  - ➡ statistical approach
- We model the error as noise that is added to the original signal



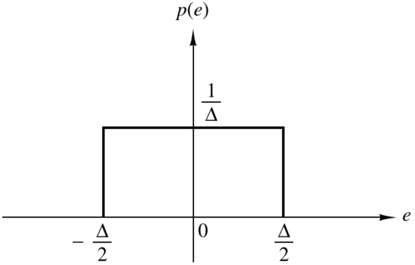
(a) Actual system



(b) Mathematical model

- To carry out the analysis we make the following assumptions

1. The error e_q is uniformly distributed over the range $\Delta/2 < e_q(n) < \Delta/2$
2. The error sequence $e_q(n)$ is a stationary white noise sequence, i.e. $e_q(n)$ and $e_q(m)$ are uncorrelated for $n \neq m$.
3. The error sequence $e_q(n)$ is uncorrelated with the signal sequence $x(n)$
4. The signal sequence $x(n)$ is zero mean and stationary.



- Assumptions are well fulfilled for sufficiently fine quantization.

Quantization Error III

- The effect of additive quantization noise can be quantified by evaluating the signal-to-quantization-noise ratio (SQNR)

$$\mathrm{SQNR} = 10\log_{10}\frac{P_x}{P_n}$$

  where P_x is the signal power, and P_n is the power of the quantization noise
- The quantization noise power is given by

$$P_n = \sigma_e^2 = \int_{-\Delta/2}^{\Delta/2} e^2 p(e)\,de = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^2\,de = \frac{\Delta^2}{12}$$

- With the range R of the quanitizer, $\Delta = R/2^b$ we get

$$\mathrm{SQNR} = \left(6.02b + 10.79 - 20\log_{10}\left(\frac{R}{\sigma_x}\right)\right)\mathrm{dB}$$

➡ **The SQNR increases with 6 dB per bit spent for quantization**

Outline

A/D and D/A Converter I

- After discussing the effects of quantization and coding in the previous section, we will now have a closer look at the details of analog-to-digital converters (part 1) and digital-to-analog converters (part 2)

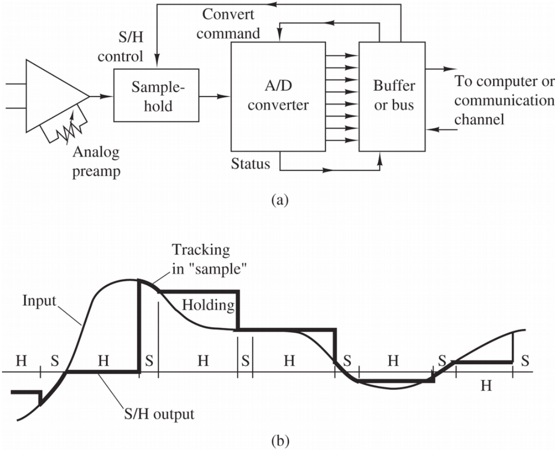


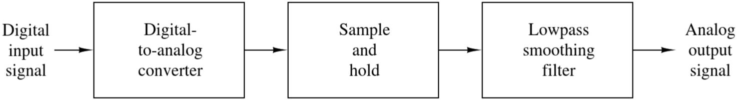
Figure 6.3.1 (a) Block diagram of basic elements of an A/D converter; (b) time-domain response of an ideal S/H circuit.

Analog-to-Digital Converter II

- A/D conversion has two major components:
  1. Sample-and-hold (S/H) circuit
  2. Digitalization / Quantization (see last section)
- Sample-and-hold is a digitally controlled analog circuit with 2 modes:
  1. Sample mode: track the analog input signal
  2. Hold mode: keep the instantaneous value of the analog input signal fixed
     - ➡ Increases the time for the A/D converter circuit to obtain the digital representation (a number of bits) of the analog input
     - ➡ Without S/H, errors would occur when the analog signal changes rapidly during the conversion time.
     - ➡ Hold time should be longer than the time needed for the A/D conversion
     - ➡ S/H and A/D components need to be synchronized

Digital-to-Analog Converter I

- In practice, D/A conversion is usually performed by combining an D/A converter with a sample-and-hold (S/H) followed by a lowpass (smoothing) filter.
- The D/A converter accepts at its input digital signals that correspond to a binary word and produces an output voltage or current that is proportional to the value of the binary word.



Digital-to-Analog Converter II

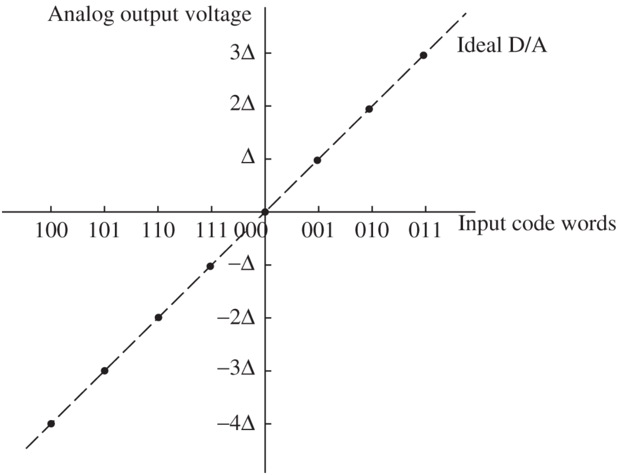


Figure 6.3.7

Ideal D/A converter characteristic for 3-bit bipolar signal

Typical deviations from the ideal characteristic:

- offset
- gain error
- non-linear distortions

Digital-to-Analog Converter III

- Important parameter of a D/A converter: **settling time**
  - Time required to reach and maintain the output within a given fraction (typically $\pm\frac{1}{2}$ LSB) of the final value, after application of the input code word
  - Often, the application of a new code word results in a high amplitude transient ("glitch") $\rightarrow$ more critical when old input and new input differ by several bits
  - Usual way to remedy this problem: sample-and-hold circuit as "deglitcher"
  - S/H holds the output of the D/A converter constant at the previous output value until the new sample at the output of the D/A converter reaches steady-state

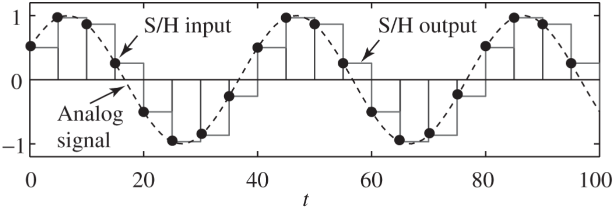


Figure 6.3.8 Response of an S/H interpolator to a discrete-time sinusoidal signal.

- S/H approximates the analog signal by a series of rectangular pulses (staircase function)
- The interpolation function of the S/H system is a square pulse

$$g_{\mathrm{SH}}(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

- The frequency-domain characteristics are obtained by evaluating its Fourier transform

$$G_{\mathrm{SH}}(F) = \int_{-\infty}^{\infty} g_{\mathrm{SH}}(t)\mathrm{e}^{-\mathrm{j}2\pi Ft}\mathrm{d}t = T\frac{\sin \pi FT}{\pi FT}\mathrm{e}^{-\mathrm{j}2\pi F(T/2)}$$

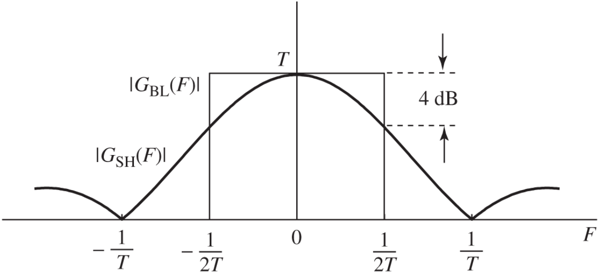


Figure 6.3.9 Frequency responses of sample-and-hold and the ideal bandlimited interpolator.

Digital-to-Analog Converter VI

- The S/H element does not possess a sharp cut-off frequency (mainly due to the sharp transitions of its interpolation function)
- This leads to undesired frequency components above $F_{\mathrm{S}}/2$
  - ➡ *post aliasing*
- Common practice: apply a low pass filter with a sharp cut-off at $F_{\mathrm{S}}/2$.
- In effect, this smooths the output by removing the discontinuities.

Digital-to-Analog Converter VII

- Sometimes the filter is designed such that it compensates the $\frac{\sin x}{x}$ distortion of the S/H (aperture effect):

$$H_a(F) = \begin{cases} \frac{\pi F T}{\sin \pi F T} \mathrm{e}^{\mathrm{j}2\pi F(T/2)}, & |F| \leq F_\mathrm{S}/2 \\ 0, & |F| > F_\mathrm{S}/2 \end{cases}$$

- Aperture effect compensation is usually neglected, but can be introduced with a digital pre-filter
- The half-sample delay of the S/H cannot be compensated, because we cannot design analog filters that can introduce a time advance.

Uniform or First-Order Sampling I

- Sampling with a rate of F_S leads to a repetition of the spectrum of the analog signal

$$X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F - kF_\mathrm{S})$$

- Sampling theorem for lowpass signals: To avoid aliasing, a signal with frequency content in $0 \leq F \leq F_\mathrm{H}$ must be sampled with

$$F_\mathrm{S} \geq 2F_\mathrm{H}$$

- Special case: **band-limited signals**, i.e. signals that only contain frequencies F with $F_\mathrm{L} \leq |F| \leq F_\mathrm{H}$
- For such signals, the sampling rate F_S can often be chosen much lower, while still avoiding aliasing
- Applications: digital communications, radar, sonar, ...
- *Bandwidth:* $B = F_\mathrm{H} - F_\mathrm{L}$ *Band position:* $m = F_\mathrm{H}/B$

Integer Band Positioning ($m \in \mathbb{Z}$): $F_H = mB$

- A sampling frequency of $F_\mathrm{S} = 2B$ is sufficient to avoid aliasing!

➡ For a narrow band signal at high frequencies, $2B$ can be significantly lower than $2F_\mathrm{H}$.

➡ The original analog bandpass signal can be reconstructed from the sampled signal via:

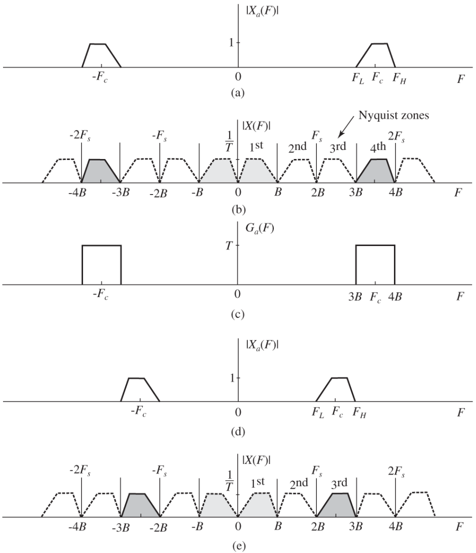
$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) g_a(t - nT)$$

where

$$g_a(t) = \underbrace{\frac{\sin \pi Bt}{\pi Bt}}_{\text{ideal lowpass}} \underbrace{\cos 2\pi F_\mathrm{C} t}_{\text{modulation with } F_\mathrm{C}}$$

with center frequency $F_\mathrm{C} = \frac{F_\mathrm{L} + F_\mathrm{H}}{2}$

Uniform or First-Order Sampling III



$m = 4$
(even band positioning)
Sample at $F_S = 2B$
$\rightarrow$ inverted baseband spectrum

$m = 3$
(odd band positioning)
$\rightarrow$ baseband spectrum is not inverted

Figure 6.4.1 Illustration of bandpass signal sampling for integer band

Arbitrary Band Positioning:

Now, the band position $m = F_{\mathrm{H}}/B$ is not limited to integer values anymore, i.e. the bands can be placed arbitrarily

- To avoid aliasing, the sampling frequency should be such that the $(k-1)$th and the kth shifted replicas of the "negative" spectral band do not overlap with the "positive" spectral band.
- This is the case if $F_{\mathrm{H}} \leq kF_{\mathrm{S}}/2$ and $(k-1)F_{\mathrm{S}}/2 \leq F_{\mathrm{L}}$.
- This leads to a range of acceptable sampling rates:

$$\frac{2F_{\mathrm{H}}}{k} \leq F_{\mathrm{S}} \leq \frac{2F_{\mathrm{L}}}{k-1},$$

  where integer k can be chosen from $1 \leq k \leq \mathrm{floor}\left\{\frac{F_{\mathrm{H}}}{B}\right\}$
- $k = 1 \rightarrow 2F_{\mathrm{H}} \leq F_{\mathrm{S}} \leq \infty \Rightarrow$ Sampling theorem for lowpass signals

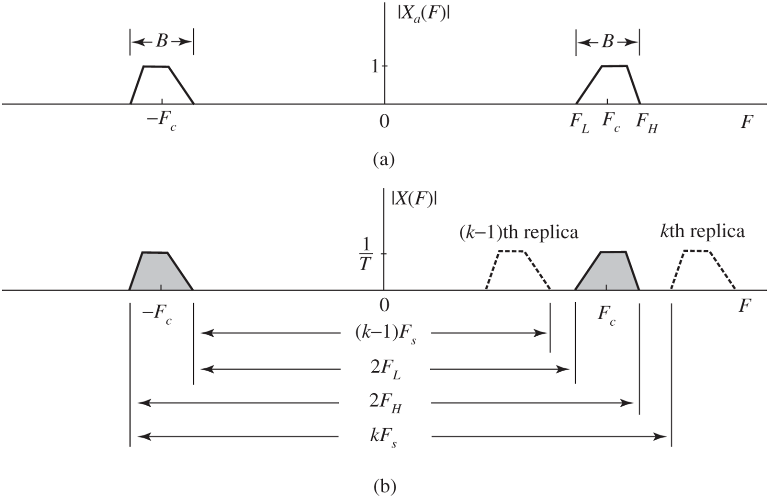


Figure 6.4.2 Illustration of bandpass signal sampling for arbitrary band positioning.

Choosing a Sampling Frequency:

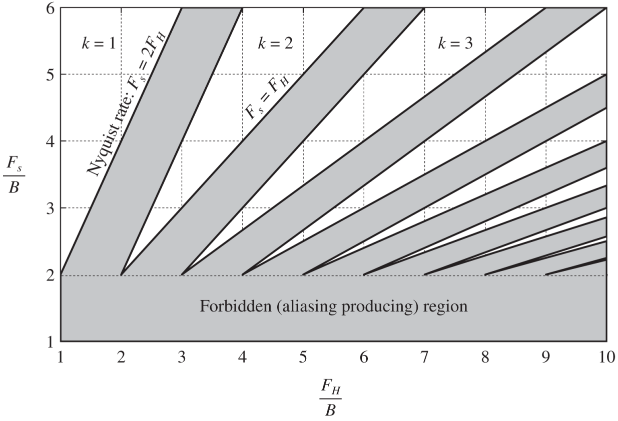


Figure 6.4.3 Allowed (white) and forbidden (shaded) sampling frequency regions for bandpass signals. The minimum sampling frequency $F_s = 2B$, which corresponds to the corners of the alias-free wedges, is possible for integer-positioned bands only.

Guard Bands:

- When choosing the lowest possible sampling rate F_S, even the slightest deviation in F_S or B can lead to aliasing
- To stay on the safe side, a higher F_S can be chosen

➡ This effectively introduces a guard band $\Delta B = \Delta B_L + \Delta B_H$

$$\begin{aligned} F'_\mathrm{L} &= F_\mathrm{L} + \Delta B_L \\ F'_\mathrm{H} &= F_\mathrm{H} + \Delta B_H \\ B' &= B + \Delta B \\ k' &= \mathrm{floor}\left\{ \frac{F'_\mathrm{H}}{B'} \right\} \\ \frac{2F'_\mathrm{H}}{k'} &\leq F_\mathrm{S} \leq \frac{2F'_\mathrm{L}}{k'-1} \end{aligned}$$

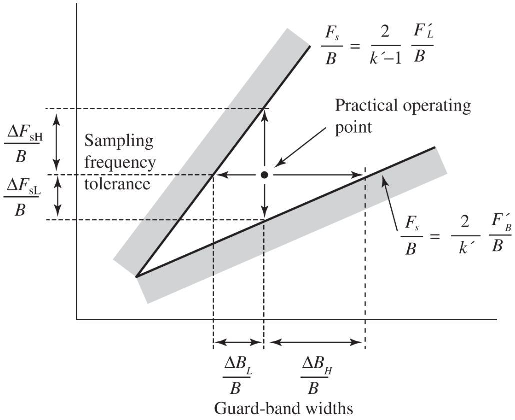


Figure 6.4.4 Illustration of the relationship between the size of guard bands and allowed sampling frequency deviations from its nominal value for the kth wedge.

Whiteboard: Example on using guard bands (Chapter 6, Example 4.1)

Interleaved (or Nonuniform) Second-Order Sampling

➡ Alternative to uniform sampling of bandpass signals

- Sample a continuous-time signal $x_a(t)$ with $F_i = \frac{1}{T_i}$ at $t = nT_i + \Delta_i$, with constant offset Δ_i:

$$x_i(nT_i) = x_a(nT_i + \Delta_i), \quad -\infty < n < \infty$$

- Fourier transform of $x_i(nT_i)$:

$$X_i(F) = \frac{1}{T_i} \sum_{k=-\infty}^{\infty} X_a(F - \frac{k}{T_i}) \mathrm{e}^{\mathrm{j}2\pi(F - \frac{k}{T_i})\Delta_i}$$

Interleaved or Nonuniform Second-Order Sampling II

- Using the sequence $x_i(nT_i)$ and a reconstruction function $g_a^{(i)}(t)$ we obtain the continuous-time signal:

$$y_a^{(i)}(t) = \sum_{n=-\infty}^{\infty} x_i(nT_i) g_a^{(i)}(t - nT_i - \Delta_i).$$

- Fourier transform of $y_a^{(i)}(t)$:

$$\begin{aligned} Y_a^{(i)}(F) &= G_a^{(i)}(F)\ X_i(F)\, \mathrm{e}^{-\mathrm{j}2\pi F \Delta_i} \\ &= G_a^{(i)}(F) \left(\frac{1}{T_i} \sum_{k=-\infty}^{\infty} X_a(F - \frac{k}{T_i}) \mathrm{e}^{\mathrm{j}2\pi(F - \frac{k}{T_i})\Delta_i} \right) \mathrm{e}^{-\mathrm{j}2\pi F \Delta_i} \end{aligned}$$

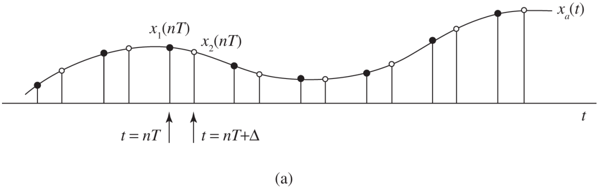
- Then, we take the sum of p such reconstructed signals:

$$y_a(t) = \sum_{i=1}^{p} y_a^{(i)}(t)$$

Interleaved or Nonuniform Second-Order Sampling III

- From now on, we concentrate on the most commonly used second-order sampling, which is defined by:

$$p = 2, \Delta_1 = 0, \Delta_2 = \Delta, T_1 = T_2 = \frac{1}{B} = T$$



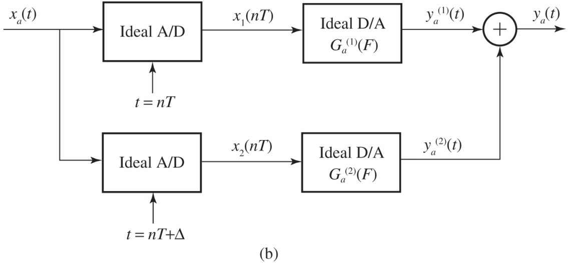


Figure 6.4.5 Illustration of second-order bandpass sampling: (a) interleaved sampled sequences (b) second-order sampling and reconstruction system.

- For this setup, we get:

$$Y_a(F) = BG_a^{(1)}(F) \sum_{k=-\infty}^{\infty} X_a(F - kB) + BG_a^{(2)}(F) \sum_{k=-\infty}^{\infty} X_a(F - kB) \mathrm{e}^{-\mathrm{j}2\pi B \Delta k} \quad (15)$$

- **Goal:** find reconstruction functions $G_a^{(1)}(F)$, $G_a^{(2)}(F)$, and time offset Δ such that $Y_a(F) = X_a(F)$

How to determine $G_a^{(1)}(F)$ and $G_a^{(2)}(F)$ I

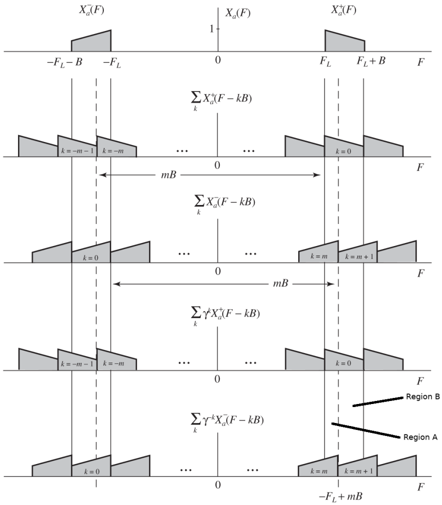


Figure 6.4.6 Illustration of aliasing in second-order bandpass sampling.

How to determine $G_a^{(1)}(F)$ and $G_a^{(2)}(F)$ II

- To avoid undesired new frequency components, we need $G_a^{(1)}(F) \stackrel{!}{=} G_a^{(2)}(F) \stackrel{!}{=} 0$ for $|F| < F_\mathrm{L}$ and $|F| < F_\mathrm{L} + B$
- For simplicity, we then separate the spectrum $X_a(F)$ into a "positive" and "negative" band:
  $X_a^+(F) = \begin{cases} X_a(F), & |F| \geq 0 \\ 0, & |F| < 0 \end{cases}$ $X_a^-(F) = \begin{cases} X_a(F), & |F| \leq 0 \\ 0, & |F| > 0 \end{cases}$
- From Fig. 6.4.6 we see that only components with $k = \pm m$ and $k = \pm(m+1)$ overlap with the original spectrum, with $m = \mathrm{ceil}\left\{\frac{2F_\mathrm{L}}{B}\right\}$
- Region A:

$$Y_a^+(F) = \underbrace{\left[BG_a^{(1)}(F) + BG_a^{(2)}(F)\right]}_{\stackrel{!}{=}1} X_a^+(F) \qquad \text{signal comp.}$$

$$+ \underbrace{\left[BG_a^{(1)}(F) + B\gamma^m G_a^{(2)}(F)\right]}_{\stackrel{!}{=}0} X_a^+(F - mB) \qquad \text{aliasing comp.}$$

How to determine $G_a^{(1)}(F)$ and $G_a^{(2)}(F)$ III

with $\gamma = \mathrm{e}^{-\mathrm{j}2\pi B\Delta}$.

- Solving both equations leads to

$$G_a^{(1)}(F) = \frac{1}{B}\frac{1}{1-\gamma^{-m}}, \quad G_a^{(2)}(F) = \frac{1}{B}\frac{1}{1-\gamma^{m}},$$

which exists for all Δ such that $\gamma^{\pm m} = \mathrm{e}^{\mp \mathrm{j}2\pi B\Delta m} \neq 1$. $\Rightarrow$ Choose Δ accordingly.

- Region B:

$$Y_a^+(F) = \underbrace{\left[BG_a^{(1)}(F) + BG_a^{(2)}(F)\right]}_{\overset{!}{=}1} X_a^+(F) \qquad \text{signal}$$

$$+ \underbrace{\left[BG_a^{(1)}(F) + B\gamma^(m+1)G_a^{(2)}(F)\right]}_{\overset{!}{=}0} X_a^+(F-(m+1)B) \qquad \text{aliasing}$$

- Solving both equations leads to

$$G_a^{(1)}(F) = \frac{1}{B}\frac{1}{1-\gamma^{-(m+1)}},\; G_a^{(2)}(F) = \frac{1}{B}\frac{1}{1-\gamma^{m+1}},$$

which exists for all Δ such that $\gamma^{\pm(m+1)} = \mathrm{e}^{\mp \mathrm{j}2\pi B\Delta(m+1)} \neq 1$.

- Analogously, results can also be found for the "negative" spectrum, which leads to $G_a^{(1)}(F)$ as presented in Figure 6.4.7.
- A similar plot of $G_a^{(2)}(F)$ shows that $G_a^{(2)}(F) = G_a^{(1)}(-F)$.
- Accordingly, in the time domain we have $g_a^{(2)}(t) = g_a^{(1)}(-t) = g_a(-t)$, which finally leads to

$$y_a(t) = x_a(t) = \sum_{n=-\infty}^{\infty} x_a(\frac{n}{B})g_a(t-\frac{n}{B}) + \sum_{n=-\infty}^{\infty} x_a(\frac{n}{B}+\Delta)g_a(-t+\frac{n}{B}+\Delta)$$

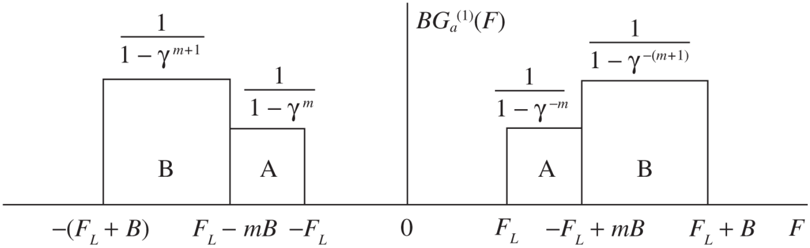


Figure 6.4.7 Frequency domain characterization of the bandpass interpolation function for second-order sampling.

General Solution and Special Cases I

- Perfect reconstruction of a bandpass signal is possible from two interleaved uniformly sampled sequences with an average rate of $F_\mathrm{S} = 2B$!
- Computing the inverse Fourier transform of the function in Figure 6.4.7 yields the interpolation function

$$g_a(t) = a(t) + b(t)$$

$$a(t) = \frac{\cos\left[2\pi(mB - F_\mathrm{L})t - \pi mB\Delta\right] - \cos\left[2\pi F_\mathrm{L} t - \pi mB\Delta\right]}{2\pi Bt \sin\left(\pi mB\Delta\right)}$$

$$b(t) = \frac{\cos\left[2\pi(F_\mathrm{L} + B)t - \pi(m+1)B\Delta\right] - \cos\left[2\pi(mB - F_\mathrm{L})t - \pi(m+1)B\Delta\right]}{2\pi Bt \sin\left(\pi(m+1)B\Delta\right)}$$

General Solution and Special Cases II

- **Special Cases:**
- Integer band positioning $(m = 2F_\mathrm{L}/B) \Rightarrow a(t) = 0$
- Integer band positioning with Δ such that $\gamma^{\pm(m+1)} = -1 \Rightarrow$ direct quadrature sampling:

$$g_Q(t) = \frac{\sin \pi B t}{\pi B t} \cos 2\pi F_\mathrm{C} t$$

- Lowpass signal $(F_\mathrm{L} = 0$ and $m = 0)$ with $\Delta = B/2 \Rightarrow$ well-known interpolation function $g_{LP}(t) = \sin(2\pi B t)/(2\pi B t)$.

Bandpass Signal Representations I

- The main reason of complications in the sampling of a real-valued bandpass signal $x_a(t)$ is the presence of two separate spectral bands ("positive" and "negative").
- However, for real-valued signals we have $X_a(-F) = X_a^*(F)$
  - ➡ The signal can be completely specified by one half of the spectrum.

Bandpass Signal Representations II

- Recall that $\cos 2\pi F_C t = \frac{1}{2}e^{j2\pi F_C t} + \frac{1}{2}e^{-j2\pi F_C t}$
  - ➡ Two spectral lines of magnitude $1/2$, one at F_C and one at $-F_C$.
- Equivalently we have: $\cos 2\pi F_C t = 2\mathrm{Re}\left\{\frac{1}{2}e^{j2\pi F_C t}\right\}$
  - ➡ Represent the real-valued signal as the real part of a complex-valued signal.
- The complex-valued signal has only a single line at $F = F_C$ with double the amplitude of the real-valued signal
  - ➡ Simplifies sampling

- Extension to general signals:

$$x_a(t) = \mathrm{Re}\left\{\psi_a(t)\right\},$$

where $\psi_a(t)$ is the (complex-valued) *analytic signal* of $x_a(t)$.

- The spectrum $\Psi(F)$ of $\psi_a(t)$ can be expressed in terms of the unit step function $V_a(F)$:

$$\Psi(F) = 2X_a(F)\,V_a(F) = \begin{cases} 2X_a(F), & F > 0 \\ 0, & F < 0 \end{cases}$$

with $\Psi(0) = X_a(0)$ if $X_a(0) \neq 0$

- Recall that the inverse FT of $V_a(F)$ is $v_a(t) = \frac{1}{2}\delta(t) + \frac{\mathrm{j}}{2\pi t}$
- Accordingly, with $h_q(t) = \frac{1}{\pi t}$ we have

$$\psi_a(t) = 2x_a(t) * v_a(t) = x_a(t) + \mathrm{j}h_q(t) * x_a(t) = x_a(t) + \mathrm{j}\hat{x}_a(t),$$

  where $\hat{x}_a(t) = h_q(t) * x_a(t)$ is the *Hilbert transform* of $x_a(t)$.
- The Hilbert transform only shifts the spectral phase of $x_a(t)$:

$$H_Q(F) = \begin{cases} -j, & F > 0 \\ j, & F < 0 \end{cases}$$

- Note that $h_q(t)$ is non-causal and thus physically unrealizable. $\Rightarrow$ For a physical realization, we introduce the *complex envelope* of $x_a(t)$ by shifting it to the baseband region:

$$x_{\mathrm{LP}}(t) = \mathrm{e}^{-\mathrm{j}2\pi F_{\mathrm{C}} t}\psi_a(t) \overset{FT}{\leftrightarrow} X_{\mathrm{LP}}(F) = \Psi_a(F + F_{\mathrm{C}})$$

- The complex envelope can be expressed as

$$x_{\mathrm{LP}}(t) = x_I(t) + \mathrm{j}x_Q(t),$$

with in-phase component $x_I(t)$ and *quadrature component* $x_Q(t)$.

- With this definition it follows that:

$$x_a(t) = \mathrm{Re}\left\{\psi_a(t)\right\} = x_I(t)\cos 2\pi F_{\mathrm{C}} t - x_Q(t)\sin 2\pi F_{\mathrm{C}} t,$$

which is physically realizable as depicted in Figure 6.4.8.

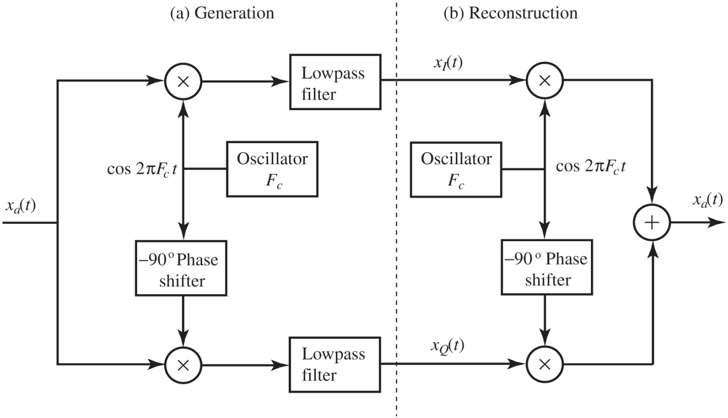


Figure 6.4.8 (a) Scheme for generating the in-phase and quadrature components of a bandpass signal. (b) Scheme for reconstructing a bandpass signal from its in-phase and quadrature components.

Sampling Using Bandpass Representations I

- The in-phase and quadrature components can be sampled according to Figure 6.4.8 (a)
- Alternatively, the two components can also be directly sampled via interleaved sampling with

$$x_a(t_n) = x_I(t_n) \text{ when sampled at } t_n = \frac{n}{2F_\mathrm{C}}$$

  and

$$x_a(t_n) = -x_Q(t_n) \text{ when sampled at } t_n = \frac{n+1/2}{2F_\mathrm{C}},$$

  which avoids the complex demodulation step.
- With fast A/D converters, it is also possible to uniformly sample a bandpass signal and compute $x_I(n)$ and $x_Q(n)$ using the discrete-time approach $\Rightarrow$ Next slides

Outline

Sampling of Discrete-Time Signals I

- Downsampling: only keep every D-th sample of a discrete-time signal
  $\Rightarrow$ Corresponds to sampling at a rate of only F_S/D.

  $$x_d(n) = x(nD), \quad -\infty \leq n \leq \infty$$

  where $x(n)$ has been obtained from sampling an analog signal $x_a(t)$ with $F_\mathrm{S} = 1/T$ without aliasing.
- The DTFT of the original discrete time signal is

  $$X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F - kF_\mathrm{S})$$

- After downsampling we have

$$X(F) = \frac{1}{DT} \sum_{k=-\infty}^{\infty} X_a(F - kF_\mathrm{S}/D)$$
$$= \frac{1}{D} \sum_{k=0}^{D-1} X(F - kF_\mathrm{S}/D),$$

  i.e. the periodic spectrum is repeated D times

- To avoid aliasing we need $F_\mathrm{S}/D \geq 2F_\mathrm{max}$

Sampling of Discrete-Time Signals III

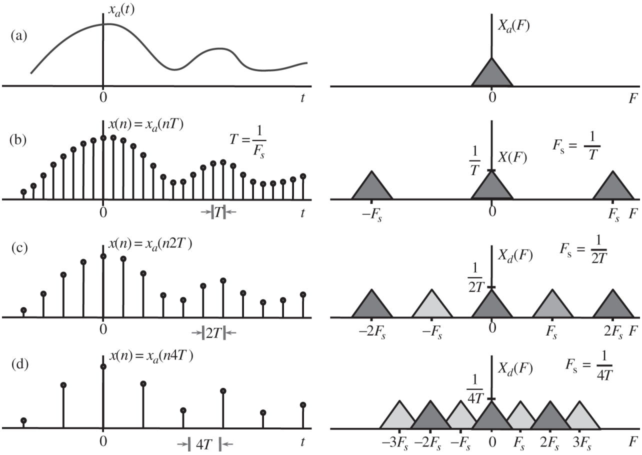


Figure 6.5.1 Illustration of discrete-time signal sampling in the frequency domain.

- Now, assuming that there is no aliasing $(F_\mathrm{S}/D \geq 2F_\mathrm{max})$, we want to reconstruct the original sequence $x(n)$ from the downsampled sequence $x_d(n)$.
- **Ideal interpolation** via ideal lowpass filter with cut-off frequency at $F_\mathrm{S}/D = 1/(TD)$:

$$G_\mathrm{BL}(\omega) = \begin{cases} D, & |\omega| \leq \pi/D \\ 0, & \pi/D < |\omega| \leq \pi \end{cases} \overset{FT}{\leftrightarrow} g_\mathrm{BL}(n) = \frac{\sin\left[\frac{\pi/D}{n}\right]}{\frac{\pi/D}{n}}$$

- With the ideal interpolator we have

$$x(n) = \sum_{m=-\infty}^{\infty} x_d(m) g_\mathrm{BL}(n - mD)$$

- The interpolation involves an infinite number of samples
  - ➡ Can only be approximated!

Interpolation of Discrete-Time Signals II

- Widely used: **linear interpolation**
- Linear interpolation of the value $x_a(t)$ from time instances $(m-1)T_d$ and mT_d with $T_d = DT$:

$$x_{\text{lin}}(t) = \left[1 - \frac{t-(m-1)T_d}{T_d}\right] x(m-1) + \left[1 - \frac{mT_d - t}{T_d}\right] x(m),$$

- This can be formulated more generally as

$$x_{\text{lin}}(t) = \sum_{m=-\infty}^{\infty} x_d(m) g_{\text{lin}}(t - mD),$$

with

$$g_{\text{lin}}(t) = \begin{cases} 1 - \frac{|t|}{T_d}, & |t| \leq T_d \\ 0, & |t| > T_d \end{cases}$$

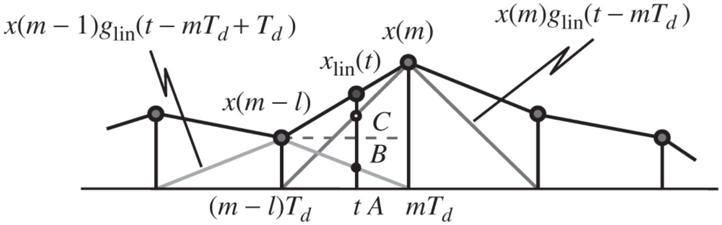


Figure 6.5.2 Illustration of continuous-time linear interpolation.

- If we are only interested in the time-discrete signal $x(n)$ (which we are right now), simply exchange $t = nT$ and $T_d = DT$:

$$x_{\text{lin}}(n) = \sum_{m=-\infty}^{\infty} x_d(m)g_{\text{lin}}(n - mD),$$

with

$$g_{\text{lin}}(n) = \begin{cases} 1 - \frac{|n|}{D}, & |n| \leq D \\ 0, & |n| > D \end{cases}$$

- The Fourier transform of the linear interpolator is depicted in Figure 6.5.3.

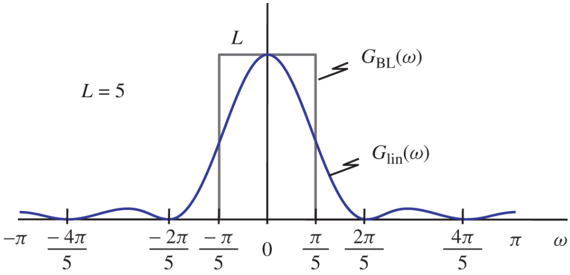


Figure 6.5.3 Frequency response of ideal and linear discrete-time interpolators.

Interpolation of Discrete-Time Signals V

- Linear interpolator: good performance only if the spectrum of $x_{\mathrm{lin}}(n)$ is negligible for $|\omega| > \pi/D$

➡ Only the case if the original continuous-time signal has been oversampled!

- The bandpass representations of discrete-time signals are analogous to those of continuous-time signals (which we discussed before)
- We only need to take into account the periodicity of the spectrum
- Analytic signal:

$$\Psi(\omega) = \begin{cases} 2X(\omega), & 0 \leq \omega < \pi \\ 0, & -\pi \leq \omega < 0 \end{cases}$$

$$= X(\omega) + \mathrm{j}H(\omega)X(\omega),$$

with the ideal discrete-time Hilbert transformer

$$H(\omega) = \begin{cases} -j, & 0 \leq \omega < \pi \\ j, & -\pi \leq \omega < 0 \end{cases}$$

- The impulse response $h(n)$ of the ideal discrete-time Hilbert transformer is depicted in Figure 6.5.5.

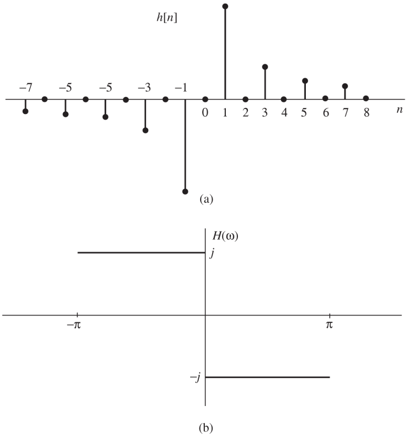


Figure 6.5.5 Impulse response (a) and frequency response (b) of the discrete-time Hilbert transformer.

➡ $h(n)$ is noncausal and unstable

- The time-discrete complex envelope $x_{\mathrm{LP}}(n)$, quadrature $x_Q(n)$, and in-phase component $x_I(n)$ are obtained from the respective continuous-time counterparts by simply replacing t with nT.
- With $x_{\mathrm{LP}}(n) = x_Q(n) + x_I(n)$, it is now possible to obtain a lowpass representation of a bandpass signal from a time-discrete signal $x(n)$
- Advantage: The analog bandpass signal can be uniformly sampled
  - ➡ Avoids hardware for quadrature demodulation
  - ➡ The quadrature demodulation is instead performed digitally on the discrete-time signal

Outline

SP Signal Processing

Oversampling A/D Converters - Motivation I

- Alternative to "conventional" A/D and D/A converters
- Advantages:
  - No analog antialiasing filters
  - No sample-and-hold circuit
  - Robust to variations in the analog circuits
  - Low cost

Oversampling A/D Converters I

- Recall that the quantization error variance is

$$\sigma_e^2 = \frac{\Delta^2}{12} = \frac{R^2}{12\ \ 2^b},$$

with step size Δ, number of bits b, and range R.

- The dynamic range of a signal ($\propto \sigma_x$) should match the range R of the quantizer
- The lower the dynamic range of the analog signal that should be quantized, the lower we can choose R and the lower σ_e^2
- Basic idea of oversampling A/D converters:
  - ➡ Increase the sampling rate to the point that a low-resolution quantizer suffices.

- **Differential quantization:** Instead of quantizing $x(n)$ directly, quantize the difference

$$d(n) = x(n) - x(n-1),$$

which has the variance

$$\begin{aligned}\sigma_d^2 &= \mathrm{E}\big(d^2(n)\big) = \mathrm{E}\Big(\left[x(n) - x(n-1)\right]^2\Big) \\ &= \mathrm{E}\big(x^2(n)\big) - 2\mathrm{E}\big(x(n)x(n-1)\big) + \mathrm{E}\big(x^2(n-1)\big) \\ &= 2\sigma_x^2\left[1 - \gamma_{xx}(1)\right],\end{aligned}$$

with autocorrelation sequence $\gamma_{xx}(m)$ at $m = 1$.

- For $\gamma_{xx}(1) > 0.5$ we have $\sigma_d^2 < \sigma_x^2 \Rightarrow$ reduced variance
- Higher sampling rate $\Rightarrow$ smaller distance between values of successive samples

➡ Higher correlation $\gamma_{xx}(1)$

Oversampling A/D Converters III

- An even better approach is to quantize

$$d(n) = x(n) - \hat{x}(n) = x(n) - ax(n-1)$$

with the optimal coefficient

$$a = \frac{\gamma(1)}{\gamma(0)} = \frac{\gamma(1)}{\sigma_x^2}$$

and

$$\sigma_d^2 = \sigma_x^2 \left[1 - a^2\right] \leq \sigma_x^2$$

- This approach can be further generalized by using

$$\hat{x}(n) = \sum_{k=1}^{p} a_k x(n-k)$$

with a_k chosen to minimize σ_d.

➜ Differential Pulse Code Modulation (DPCM)

- For instance used for speech coding and transmission in cell-phones

Oversampling A/D Converters IV

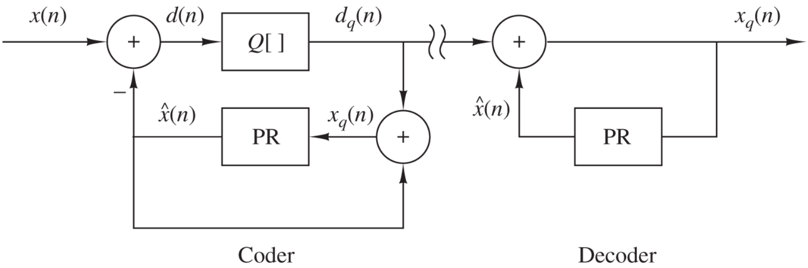


Figure 6.6.1 Encoder and decoder for differential predictive signal quantizer system.

The feedback loop around the quantizer is needed to avoid the accumulation of errors at the decoder. In this setup we have the error:

$$e(n) = d(n) - d_q(n) = x(n) - \hat{x}(n) - d_q(n) \qquad = x(n) - x_q(n)$$

with $\hat{x}(n) + d_q(n) = x_q(n)$

Here we concentrate on the simplest form of DPCM:

Delta Modulation (DM)

- First-order predictor with a 1-bit quantizer
- Staircase approximation of the input signal
- At every sampling instant, the staircase approximation makes a step Δ into the direction of the input signal.

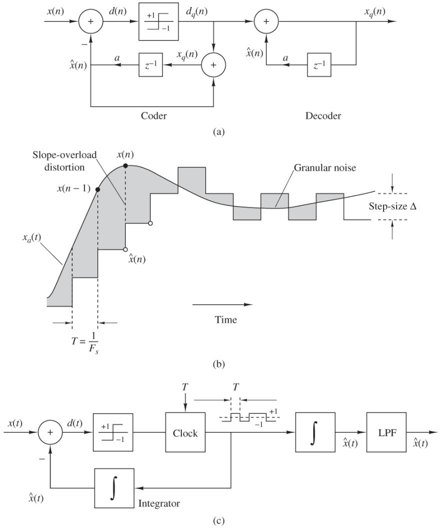


Figure 6.6.2 Delta modulation system and two types of quantization errors.

Oversampling A/D Converters VII

- From Figure 6.6.2(a) we see that

$$x_q(n) = a x_q(n-1) + d_q(n),$$

  which is a discrete time equivalent of an analog integrator
- $a = 1 \rightarrow$ ideal accumulator (integrator)
- $a < 1 \rightarrow$ "leaky integrator"
- Figure 6.6.2(c) depicts an analog model to illustrate the principle for a practical implementation
  - The analog lowpass is necessary to suppress out-of-band components $(B < F <= F_\mathrm{S}/2)$ since $F_\mathrm{S} \gg B$

- Figure 6.6.2(b): analog, time-discrete, and quantized signals
- We observe two types of errors:
  1. Slope-overload distortion $\Rightarrow$ the increase in $x(n)$ is so steep that the quantizer with step size Δ cannot follow
  2. Granular noise $\Rightarrow$ In flat regions, $x_q(n)$ jumps around the true value

➡ Trade-off step size Δ:

  - $\Delta \gg$: little slope-overload, but strong granular noise
  - $\Delta \ll$: strong slope-overload, but low granular noise

One way to reduce these two types of distortions:

Sigma-Delta Modulation (SDM)

- ➡ SDM uses an integrator in front of the DM, which has two advantages:
  1. Emphasizes low frequencies, which increases the correlation in the DM input
  2. Simplifies the DM decoder to a simple analog lowpass, as the inverse operation (differentiator) cancels out the DM integrator

Sigma-Delta Modulation (SDM) II

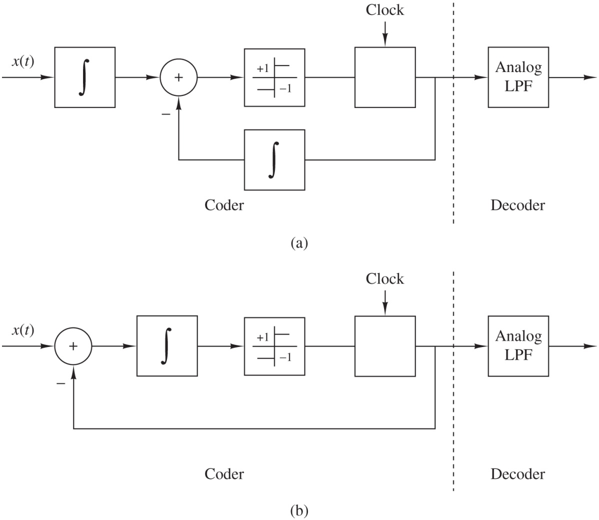


Figure 6.6.3 Sigma-delta modulation system.

- SDM is an ideal candidate for A/D conversion
  - Spreads the quantization noise across the whole spectrum up to $F_\mathrm{S}/2$
  - In the signal-free band $B < F \leq F_\mathrm{S}/2$, this can be removed via digital filtering
  - This is illustrated in the discrete-time SDM model in Fig. 6.6.4, where the comparator is modeled by an additive white noise source with variance $\sigma_e^2 = \Delta^2/12$

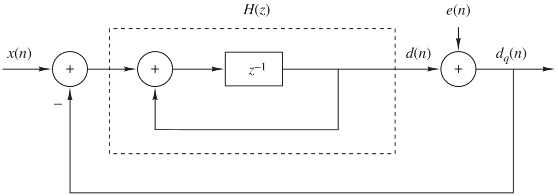


Figure 6.6.4 Discrete-time model of sigma-delta modulation.

Sigma-Delta Modulation (SDM) IV

- For this model we have

$$H(z) = \frac{z^{-1}}{1 - z^{-1}}$$

and

$$\begin{aligned} D_q(z) &= \frac{H(z)}{1+H(z)} X(z) + \frac{1}{1+H(z)} E(z) \\ &= H_s(z) X(z) + H_n(z) E(z) \end{aligned}$$

- A good SDM system has a flat $H_s(\omega)$ and a low $H_n(\omega)$ in the signal band $0 \leq F \leq B$

- For first-order SDM we have

$$H_s(z) = z^{-1} \text{ and } H_n(z) = 1 - z^{-1}$$

➡ $H_s(z)$ does not disturb the signal

➡ It can be shown that for $F_\mathrm{S} \gg 2B$, $H_n(z)$ reduces the quantization noise variance to

$$\sigma_n^2 \approx \frac{1}{3}\pi^2\sigma_e^2\left(\frac{2B}{F_\mathrm{S}}\right)^3$$

➡ **Doubling F_S reduces the noise power by 9 dB**

Sigma-Delta Modulation (SDM) VI

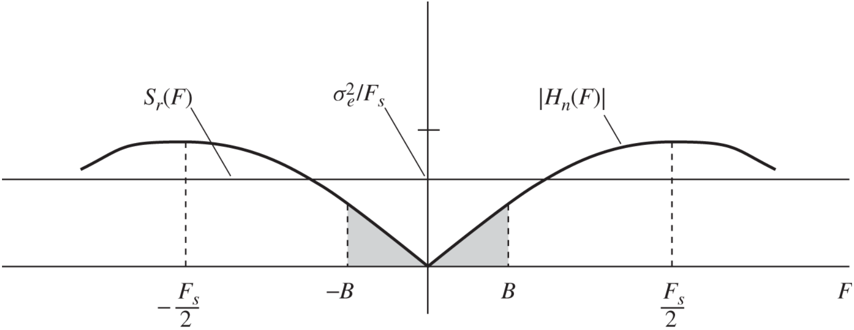


Figure 6.6.5 Frequency (magnitude) response of noise system function.

Sigma-Delta Modulation (SDM) VII

- Now: How to come from the 1-bit SDM output at $F_\mathrm{S} = I2B$ to a b-bit quantized signal at the Nyquist rate
- Oversampling / Interpolation factor I
- Recall that the decoder is an analog lowpass with cutoff frequency B
- Digitally (Example):
  - For an interpolation factor of $I = 256$, average non-overlapping blocks of 256 bits
  - This results in a a digital signal with a range from 0 to 256 ($\approx$ 8 bit) at the Nyquist rate
- Oversampling A/D converters for voiceband (3 kHz) are typically realized in integrated circuits, operating at 2 MHz, with down-sampling to 8kHz and 16 bit accuracy.

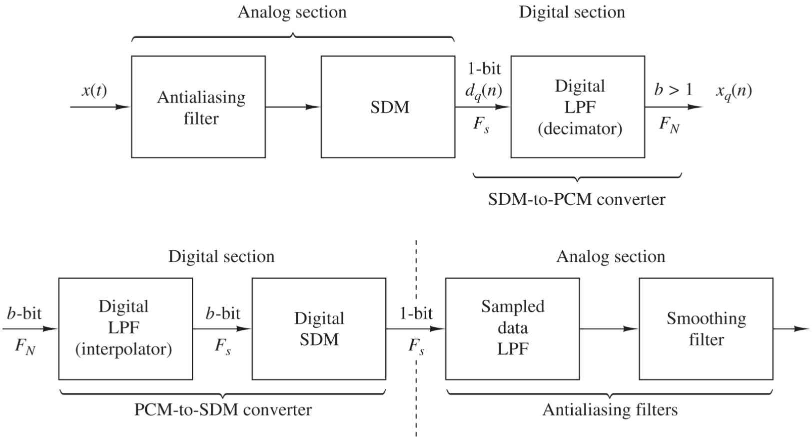


Figure 6.6.6 Basic elements of an oversampling A/D converter.

Oversampling D/A Converters I

- Finally, we have a look at the opposite way, i.e. going from a digital signal to an analog signal via oversampling D/A conversion.
- The D/A converter can be split into a digital front-end and an analog back-end
- Digital front end:
  - Interpolation of the digital signal by a factor I through inserting $I-1$ zeros between successive low-rate samples
  - Low-pass filter to reject the replicas of the spectrum
  - This high-rate signal is fed to a SDM, which outputs 1-bit noise-shaped samples
- Analog back-end:
  - 1-bit D/A
  - antialiasing and smoothing filters with a passband of $0 \leq F \leq B$ that removes the quantization noise at higher frequencies.

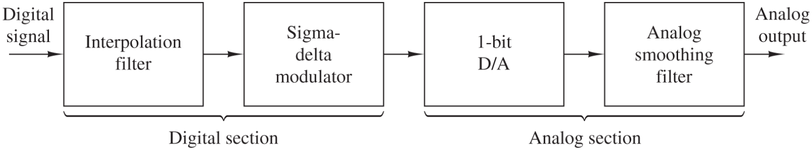


Figure 6.6.7 Elements of an oversampling D/A converter.

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SP
Signal Processing

7. The Discrete Fourier Transform

Outline

Motivation

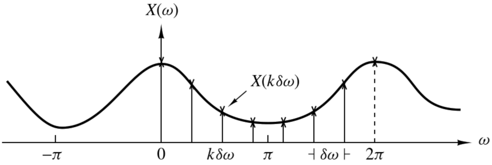
- Frequency analysis of discrete-time signals is most conveniently performed on a digital signal processor, i.e. either a general-purpose digital computer or specially designed hardware
- To perform an analysis of a discrete-time signal $x(n)$ we convert the sequence to the Fourier domain
- However, the DTFT $H(\omega)$ is a continuous function of frequency and thus not a convenient representation for a computer.
- In this chapter we consider a sampled (i.e. discretized) spectrum as presented by the Discrete Fourier Transform (DFT)

Outline

- Consider the DTFT, i.e. the Fourier transform of an aperiodic discrete-time signals given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- The resulting spectrum is a continuous function in ω
- To be able to represent the spectrum in a digital computer, we need to sample the spectrum
- However, realize, that this discretization of the spectrum results in a periodic extension of the time-domain signal!



- Similar to spectral aliasing, also temporal aliasing can occur, if the sampling grid is not taken fine enough
- Seen the other way around, when we periodically extend the time-domain signal, we will receive a discrete spectrum that is representable on a computer

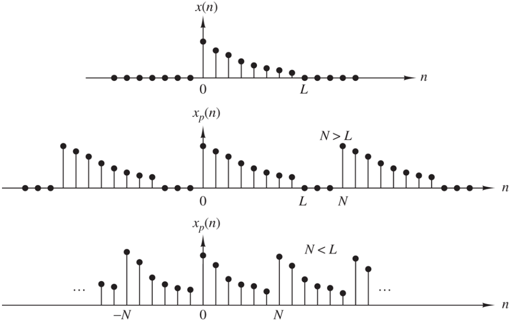


Figure 7.1.2 Aperiodic sequence $x(n)$ of length L and its periodic extension for $N \geq L$ (no aliasing) and $N < L$ (aliasing).

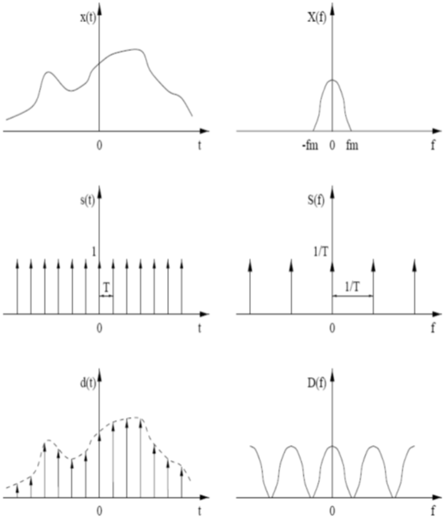
Periodic Extension of the Time-Domain Signal II

- An aperiodic discrete-time signal with finite duration L can be exactly recovered from its samples at frequencies $\omega_k = 2\pi k/N$ if $N \geq L$
- The procedure for reconstruction is

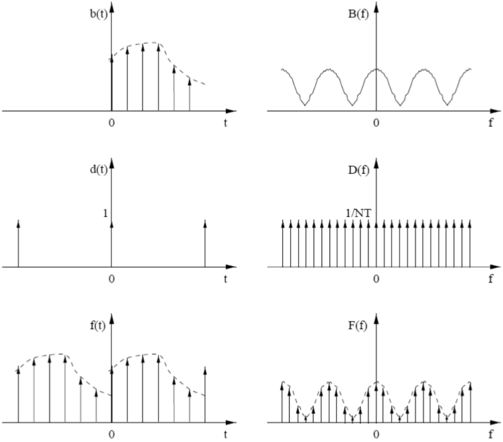
$$x(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases}$$

- The spectrum $X(\omega)$ can then be obtain from the reconstructed $x(n)$
- Analog to continuous-time signals, the continuous spectrum $X(\omega)$ can also be obtained by interpolating the sampled spectrum $X(2\pi k/N)$

Graphical Illustration



Graphical Illustration



SP Signal Processing

Derivation

- We take N equidistant samples in $0 \leq \omega < 2\pi$, i.e. we evaluate the spectrum at $\omega = 2\pi k/N$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, ..., N-1$$

- This infinite sum can be subdivided in to an infinite number of finite summations

$$\begin{aligned} X(2\pi k/N) &= \cdots + \sum_{n=-N}^{-1} x(n) e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \\ &\quad + \sum_{n=N}^{2N-1} x(n) e^{-j2\pi kn/N} + \cdots \\ &= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j2\pi kn/N} \end{aligned}$$

Derivation

- Changing the order of summation and the index from n to $n - lN$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N}, \quad k = 0, 1, 2, ..., N-1$$

- Note that the signal

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

is periodic in N. (Easy to see if $x(n)$ is nonzero for $L < N$ samples, i.e. $x(n)$ is time-limited)

➡ Recall: A periodic signal can be analyzed by Fourier series

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \quad \text{with} \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}$$

Frequency-Domain Sampling III

Derivation

- A comparison with equation for $X\left(\frac{2\pi}{N}k\right)$ reveals similarity

$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right)$$

- Thus, we can reconstruct the periodic signal also from the sampled spectrum

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- However, we would really like to know how to get the original signal $x(n)$ from the sampled spectrum!

Derivation

- If $x(n)$ is non-zero for $L < N$ samples, reconstruction is simple as

$$x(n) = x_p(n), \quad 0 \leq n \leq N-1$$

➡ The spectrum of an aperiodic discrete-time signal with finite duration L can be exactly recovered from its samples at frequencies $\omega_k = 2\pi k/N$ if $N \geq L$

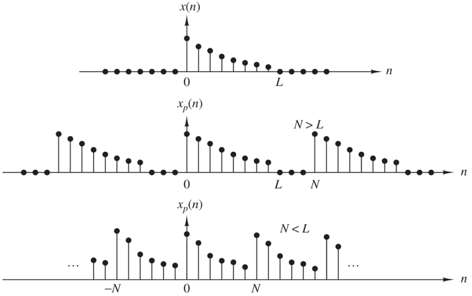


Figure 7.1.2 Aperiodic sequence $x(n)$ of length L and its periodic extension for $N \geq L$ (no aliasing) and $N < L$ (aliasing).

Reconstruction by Interpolation

- As for continuous-time signals, we can obtain $X(\omega)$ perfectly from its samples $X(2\pi k/N)$ by interpolation
- To show this, we begin with the finite signal $x(n)$ with $N \geq L$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(2\pi k/N) e^{j2\pi kn/N}$$

- Computing the DTFT, we obtain

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(2\pi k/N) e^{j2\pi kn/N} \right] e^{-j\omega n} \\ &= \sum_{k=0}^{N-1} X(2\pi k/N) \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{-j(\omega n - 2\pi kn/N)} \right] \\ &= \sum_{k=0}^{N-1} X(2\pi k/N)\, P(\omega - 2\pi k/N) \end{aligned}$$

Reconstruction by Interpolation

- With the interpolation function

$$P(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{\sin(\omega N/2)}{N\sin(\omega/2)} e^{-j\omega(N-1)/2}$$

- Note that at $\omega_k = 2\pi k/N$ the original samples are taken, as

$$P(2\pi k/N) = \begin{cases} 1, & k=0 \\ 0, & k=1,2,\ldots,N-1 \end{cases}$$

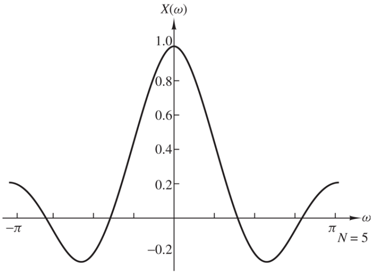


Figure 7.1.3 Plot of the function $[\sin(\omega N/2)]/[N\sin(\omega/2)]$.

Example 7.1.1 I

- Consider the signal

$$x(n) = a^n u(n), \quad 0 < a < 1$$

The spectrum of this signal is sampled at frequencies $\omega_k = 2\pi k/N, \quad k = 0, 1, ..., N-1$

- Determine the reconstructed spectra for $a = 0.8$ when $N = 5$ and $N = 50$

Example 7.1.1 II

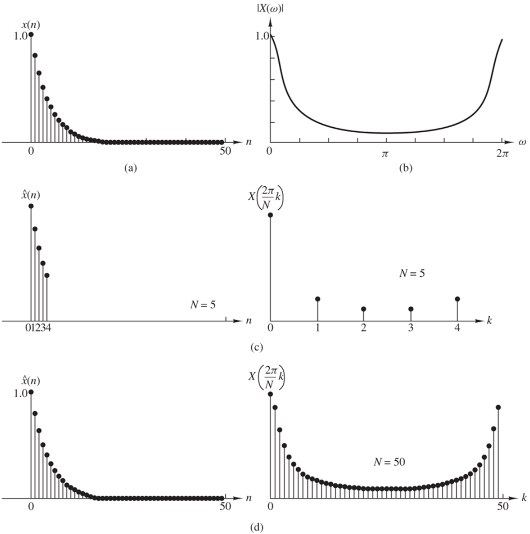


Figure 7.1.4 (a) Plot of sequence $x(n) = (0.8)^n u(n)$; (b) its Fourier transform (magnitude only); (c) effect of aliasing with $N = 5$; (d) reduced effect of aliasing with $N = 50$.

Reconstruction of finite length sequences

- In general, equally spaced frequency samples $X(\frac{2\pi k}{N})$, $k = 0, 1, ..., N-1$, do not uniquely represent the original sequence $x(n)$ when $x(n)$ has *infinite* duration.
- Instead, the frequency samples $X(\frac{2\pi k}{N})$, $k = 0, 1, ..., N-1$ correspond to a periodic sequence $x_p(n)$ of period N

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

- When the sequence $x(n)$ has *finite* duration of length $L \leq N$, then $x(n)$ can be reconstructed from $x_p(n)$ by cutting out the first N samples.

➡ Then, also the finite signal $x(n)$ can be perfectly reconstructed from the sampled spectrum!

Zero-padding

- L: length of sequence; N: spectral samples
- If $L < N$, then $N - L$ samples are filled with zeros in time-domain
- It is important to realize that zero-padding does not provide any additional information, the L coefficients are sufficient to reconstruct $X(\omega)$
- However, padding $x(n)$ with $N - L$ zeros and computing an N-point DFT results in a *better display* of the Fourier transform $X(\omega)$

Summary of relation to DTFT

- A finite-duration sequence $x(n)$ of length L has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}, \quad 0 \le \omega \le 2\pi$$

- When we sample $X(\omega)$ at frequencies $\omega_k = 2\pi k/N$ with $N \ge L$

$$\begin{aligned} X(k) \quad &\equiv \quad X(2\pi k/N) = \sum_{n=0}^{L-1} x(n) e^{-j2\pi kn/N} \\ &= \quad \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \end{aligned}$$

This relation is called the **discrete Fourier transform (DFT)** of $x(n)$

The Discrete Fourier Transform (DFT)

$$\text{DFT:} \quad X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \,, \qquad k = 0, 1, 2, ..., N-1$$

$$\text{IDFT:} \; x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}, \qquad n = 0, 1, 2, ..., N-1$$

- The spectral coefficients are spaced by $\frac{2\pi}{N}$
- Note that both $x(n)$ and $X(k)$ are periodic in N
- Note that the DFT and IDFT are highly related (up to the normalization factor) to the discrete-time Fourier series (DTFS) of periodic discrete-time signals
- The DFT can be efficiently computed by means of the fast Fourier transform (FFT) with only $\mathcal{O}(N \log(N))$ operations

Example 7.1.2 I

A finite-duration sequence of length L is given as

$$x(n) = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the N-point DFT of this sequence for $N \geq L$

Example 7.1.2 II

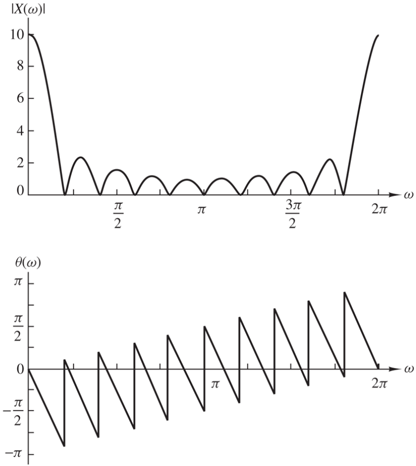


Figure 7.1.5 Magnitude and phase characteristics of the Fourier transform for signal in Example 7.1.2.

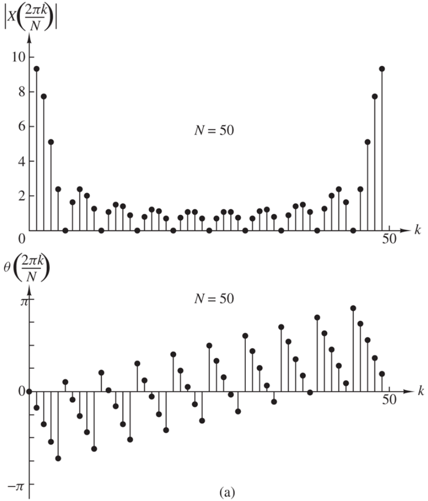


Figure 7.1.6 Magnitude and phase of an N-point DFT in Example 7.1.2; (a) $L = 10$, $N = 50$; (b) $L = 10$, $N = 100$.

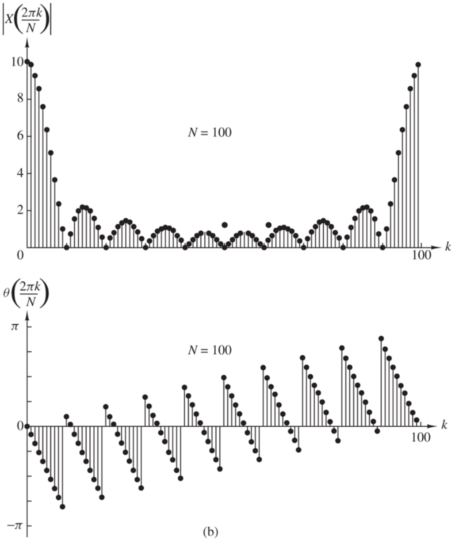


Figure 7.1.6 **Figure** *continued*

Zero-Padding to approximate DTFT

Conclusions from Example 7.1.2

- If $N > L$ zeros are appended to the signal before the periodic extensions
  - This **zero-padding** results in more samples of the spectrum
  - For $N \to \infty$ the DTFT is approximated

The DFT as a Linear Transformation I

- With $W_N = e^{-j2\pi/N}$ the DFT and IDFT can be expressed as

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad k = 0, 1, \ldots, N-1 \\
x(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}, \quad n = 0, 1, \ldots, N-1
\end{aligned}
$$

- The computations for each point need N complex multiplications and $N-1$ complex additions
- For N-point DFT values we need N^2 multiplications and $N(N-1)$ additions
- Practical relevance of DFT came with the ability of a fast implementation, the Fast Fourier Transform (FFT), with only $\mathcal{O}(N \log(N))$ operations

- The DFT can be elegantly written in vector form
- Let us define

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

- Then we can write the DFT and IDFT simply as

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$

The DFT as a Linear Transformation III

- For an invertible $\mathbf{W}_N$ we can also write

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N, \quad \mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N$$

- Vectorizing the definition of the IDFT, we find

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N$$

- These relations imply that

$$\begin{aligned} \mathbf{W}_N^{-1} &= \frac{1}{N} \mathbf{W}_N^* \\ \mathbf{W}_N \mathbf{W}_N^* &= N \mathbf{I}_N \end{aligned}$$

➡ The DFT matrix $\mathbf{W}_N$ is an orthogonal (unitary) matrix

Example 7.1.3

- Compute the DFT of the four-point sequence

$$x(n) = \{0\ 1\ 2\ 3\}$$

Example 7.1.3

- Compute the DFT of the four-point sequence

$$x(n) = \{0\ 1\ 2\ 3\}$$

Solution: Exploiting the symmetry property $W_N^{k+N/2}$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^{2(3)} \\ 1 & W_4^3 & W_4^{2(3)} & W_4^{(3)(3)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\mathbf{X}_4 = \mathbf{W}_4 \mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Relationship to Fourier series

- A periodic sequence $x_p(n)$ with fundamental period N can be represented in the Fourier series

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N}, \quad -\infty < n < \infty$$

with Fourier series coefficients

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N}, \quad k = 0, 1, ..., N-1$$

Comparison to the DFT reveals that for a sequence $x(n) = x_p(n)$ for $0 \leq n \leq N-1$ the DFT of this sequence is simply

$$X(k) = Nc_k$$

- Thus, the DFT provides the exact line spectrum of a sequence periodic in N

Relationship to Fourier transform of an aperiodic sequence

- We have already shown that the DFT corresponds to the DTFT sampled at the N equally spaced frequencies $\omega_k = 2\pi k/N$
- Due to this sampling, the reconstructed time-domain signal is periodic in N
- If the underlying time-domain signal is of duration $L \leq N$, it can be perfectly recovered from the inverse DFT.
- If the underlying time-domain signal is longer, i.e. $L > N$, the reconstructed N-sample time-domain sequence bears no resemblance to the original sequence

Relationship to the z-transform

- Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

with an ROC that includes the unit circle

- If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j2\pi k/N}$, we obtain

$$X(k) \equiv X(z)\Big|_{z_k = e^{j2\pi k/N}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi k/N}$$

- If $x(n)$ has finite duration $L \leq N$ it can be perfectly recovered from its DFT spectrum.
  - ➡ Its z-transform is uniquely determined by its N-point DFT

Outline

Motivation

- In the previous section we introduced the DFT/IDFT and unvealed the relationships to Fourier transforms, Fourier series and the z-transform
- ➡ We expect the DFT to resemble the properties of these other transforms and series
- However, important differences exist which we will discuss now!
- A good understanding of these properties and differences is extremely helpful when applying the DFT to practical problems

Periodicity and Linearity

Periodicity If $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$\begin{aligned} x(n+N) &= x(n) \quad \text{for all } n \\ X(k+N) &= X(k) \quad \text{for all } k \end{aligned}$$

→ Both the time-domain signal and the spectrum are periodic in N!

Linearity As the other transforms, the DFT is linear

$$a_1 x_1(n) + a_2 x_2(n) \overset{\mathrm{DFT}_N}{\circ\!\!-\!\!\bullet} a_1 X_1(k) + a_2 X_2(k)$$

Circular Symmetries of a Sequence

- As we have seen, the N-point DFT of a finite duration sequence $x(n)$ of length $L \leq N$ is equivalent to the N-point DFT of a periodic sequence $x_p(n)$ of period N, obtained by periodically extending $x(n)$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

- Now suppose, we shift the periodic sequence $x_p(n)$ by k units to the right, then

$$x_p'(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n - k - lN)$$

➡ The finite duration sequence

$$x'(n) = \begin{cases} x_p'(n), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

is related to the original sequence by a **circular shift**

Circular Symmetries of a Sequence

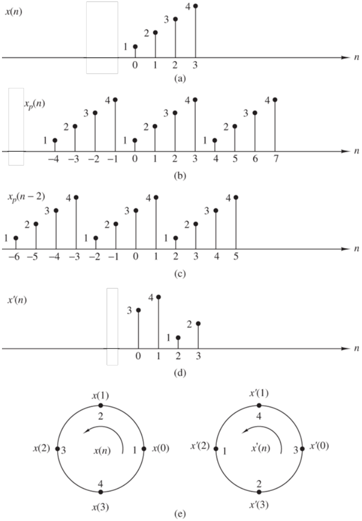


Figure 7.2.1 Circular shift of a sequence.

Circular Symmetries of a Sequence

- The circular shift can be represented as the index modulo N

$$\begin{aligned} x'(n) &= x(n-k, \text{modulo } N) \\ &\equiv x((n-k))_N \end{aligned}$$

- Example: $k = 2$, $N = 4$

$$x'(n) = x((n-2))_4$$

which implies

$$\begin{aligned} x'(0) &= x((-2))_4 = x(2) \\ x'(1) &= x((-1))_4 = x(3) \\ x'(2) &= x((0))_4 = x(0) \\ x'(3) &= x((1))_4 = x(1) \end{aligned}$$

➡ A circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension, and vice versa

Properties of the DFT IV

Circular Symmetries of a Sequence

- An N-point sequence is **circularly even** if it is symmetric about the point zero on the circle

$$x(N-n) = x(n), \quad 1 \leq n \leq N-1$$

- An N-point sequence is **circularly odd** if it is antisymmetric about the point zero on the circle

$$x(N-n) = -x(n), \quad 1 \leq n \leq N-1$$

- The time reversal of an N-point sequence is attained by reversing its samples about the point zero on the circle

$$x((-n))_N = x(N-n). \quad 0 \leq n \leq N-1$$

Properties of the DFT V

Circular Symmetries of a Sequence

- Equivalently we define for the periodic sequence $x_p(n)$

$$\begin{aligned} \text{even:} \quad & x_p(n) = x_p(-n) = x_p(N-n) \\ \text{odd:} \quad & x_p(n) = -x_p(-n) = -x_p(N-n) \end{aligned}$$

- For complex-valued sequences this can be extended to obtain

$$\begin{aligned} \text{conjugate even:} \quad & x_p(n) = x_p^*(N-n) \\ \text{conjugate odd:} \quad & x_p(n) = -x_p^*(N-n) \end{aligned}$$

- We can decompose the sequence into its even and odd part

$$x_p(n) = x_{pe(n)} + x_{po}(n)$$

with

$$\begin{aligned} x_{pe} &= \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right] \\ x_{po} &= \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right] \end{aligned}$$

Properties of the DFT I

Symmetry of real-valued sequences

- If $x(n)$ is real-valued, it follows from the definition of the DFT

$$X(N-k) = X^*(k) = X(-k)$$

- Consequently
  - $|X(N-k)| = |X(k)|$ and
  - $\angle X(N-k) = -\angle X(k)$

SP Signal Processing

Symmetry Properties of the DFT

N-Point Sequence $x(n)$ $0 \leq n \leq N-1$	N-Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
$jX_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$	$jX_I(k)$
Real Signals	
Any real signal $x(n)$	$X(k) = X^*(N-k)$
	$X_R(k) = X_R(N-k)$
	$X_I(k) = -X_I(N-k)$
	$\lvert X(k)\rvert = \lvert X(N-k)\rvert$
	$\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$	$jX_I(k)$

Summary

- Convolution is an important technique in signal processing ➡ output of an LTI system: convolution with the system's impulse response
- For a signal segment of length N and an impulse resonse of length L_h, the resulting length of a convolution is $N + L_h - 1$
- A convolution can be efficiently realized by multiplying the frequency responses and going back to time domain

$$x(n) * h(n) \quad \overset{\mathrm{F}}{\circ\!\!-\!\!\bullet} \quad X(\omega)H(\omega)$$

- But careful: in the DFT the overall length is limited to N. Thus **cyclic convolution** may occur
- Zeros before the DFT (a.k.a. **zero-padding**) can be added to avoid cyclic convolution effects

- Recall the definition of the DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, \quad k = 0, 1, ..., N-1$$

- When multiplying two DFTs $X_1(k)X_2(k) = X_3(k)$, the IDFT results again in a sequence of length N

$$\begin{aligned} x_3(m) &= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2(k) e^{j2\pi km/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N} \right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi lk/N} \right] e^{j2\pi km/N} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} \right] \end{aligned}$$

- With $a = e^{j2\pi(m-n-l)/N}$, the inner bracket has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \frac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

- We observe that
  - $a = 1$ when $m - n - l$ is a multiple of N
  - $a^N = 1$ for any $a \neq 0$

➡ $$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m - n + pN = ((m-n))_N, \quad \text{with some } p \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

- Inserting into the equation for $x_3(n)$ yields final result

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N, \quad m = 0, 1, ..., N-1$$

➡ The multiplication of two DFTs is equivalent to a circular convolution of the two sequences in time-domain

Example 7.2.1

- Perform the circular convolution of the following two sequences

$$\begin{aligned} x_1(n) &= \{\underset{\uparrow}{2}, 1, 2, 1\} \\ x_2(n) &= \{\underset{\uparrow}{1}, 2, 3, 4\} \end{aligned}$$

Solution:

- Illustrate by points on a circle (next slide)
- Starting with $m = 0$ we have

$$x_3(0) = \sum_{n=0}^{3} x_1(n) x_2((-n))_4$$

$x_2((-n))_4$ is the sequence $x_2(n)$ folded (see Fig. 7.2.2(b))

- Then the product sequence can be taken and the values added to obtain $x_3(0) = 14$
- Continue similarly for $m = 1$, $m = 2$, ... yields $x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$

Example 7.2.1

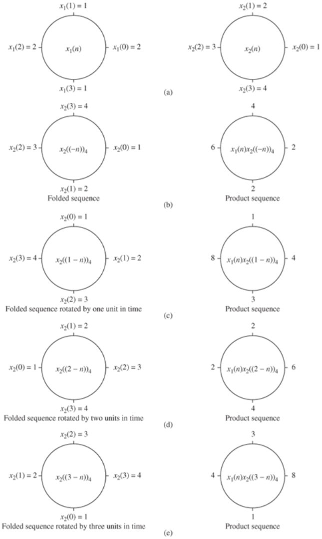


Figure 7.2.2 Circular convolution of two sequences.

Conclusions from Example 7.2.1

- Circular Convolution involves the same four steps as ordinary linear convolution:
  1. Folding (time reversing)
  2. Shifting (here: rotating) the folded sequence
  3. Multiplying the two sequences
  4. Summing the values of the product sequence
- The difference to linear convolution is that folding and shifting (rotating) are performed in a circular fashion
- Recall that in linear convolution there is no modulo operation
- Note that it does not matter which of the two sequences is folded and rotated

Example 7.2.2

- Now, determine the sequence $x_3(n)$ using the DFT and IDFT of $x_1(n)$ and $x_2(n)$ with the same values as in Example 7.2.1

Example 7.2.2

- Now, determine the sequence $x_3(n)$ using the DFT and IDFT of $x_1(n)$ and $x_2(n)$ with the same values as in Example 7.2.1

Solution:

- First compute the DFTs of $x_1(n)$ and $x_2(n)$

$$\begin{aligned} X_1(k) &= \sum_{n=0}^{3} x_1(n) e^{-j2\pi nk/4}, \quad k = 0, 1, 2, 3 \\ &= 2 + e^{-j\pi k/2} + 2e^{-j\pi k} + e^{-j3\pi k/2} \end{aligned}$$

➡ $X_1(k) = \{6, 0, 2, 0\}$; $\quad X_2(k) = \{10, -2 + j2, -2, -2 - j2\}$

➡ $X_3(k) = \{60, 0, -4, 0\}$

- Compute the IDFT of X_3

$$x_3(n) = \frac{1}{4} \sum_{k=0}^{3} X_3(k) e^{j2\pi nk/4} = \frac{1}{4} \left(60 - 4e^{j\pi n}\right)$$

➡ $x_3(n) = \{14, 16, 14, 16\}$, $\quad n = 0, 1, 2, 3$

Properties of the DFT

Property	Time Domain	Frequency Domain
Notation	$x(n)$, $y(n)$	$X(k)$, $Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(k) + a_2 X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-1))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \text{ N } x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \text{ N } y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N} X_1(k) \text{ N } X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Linear Filtering Methods Based on the DFT

- Often, we are interested in performing linear convolutions, e.g. when computing the output of an LTI system (linear convolution of signal with impulse response)
- Using the Fast Fourier Transform (FFT), a convolution can be very efficiently computed by multiplying DFT spectra.
- However, this results in a circular convolution and artifacts!
- Here, we will see that using zero-padding, a linear convolution can be achieved also using DFT multiplication

Use of the DFT in Linear Filtering

- Goal: convolution of signal and impulse response
  - $x(n) = 0, \quad n < 0$ and $n \geq L$
  - $h(n) = 0, \quad n < 0$ and $n \geq M$
- **A linear convolution results in a sequence of length $L + M - 1$**

$$y(n) = x(n) * h(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

➡ A DFT of length $N \geq L + M - 1$ is needed to represent $y(n)$ in the sampled frequency domain

➡ If the lengths of $x(n)$ and $h(n)$ are increased by appending zeros, with this zero-padding a linear convolution is achieved when multiplying their DFTs

Examples 7.3.1 and 7.3.2

Compute the linear and circular convolutions of the signals

$$h(n) = \{\underset{\uparrow}{1}, 2, 3\}$$

$$x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$$

Circular Convolution and Zero-Padding ($N = 10$)

- Here N is chosen such that it equals the sum of nonzero entries minus one, i.e. $N = 7 + 4 - 1 = 10$.

Motivation

- Often the input sequence to a system is very long, think of speech as the input to a telephone channel.
- Taking the DFT of this very long sequence is not reasonable and would also result in a large processing latency
- Thus, signals are processed in blocks.
- We now describe two methods for linear FIR filtering of a long sequence on a block-by-block basis using the DFT

Overlap-add method

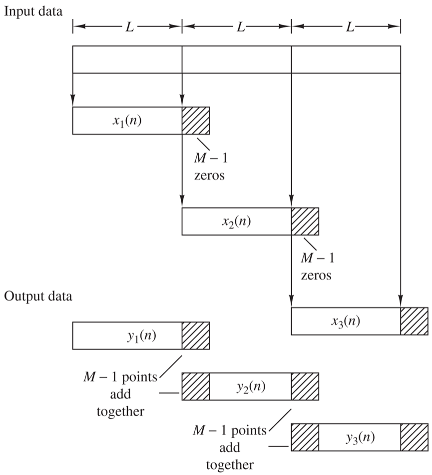


Figure 7.3.2 Linear FIR filtering by the overlap-add method.

Convolution with FIR Filter of Length M

Overlap-save method

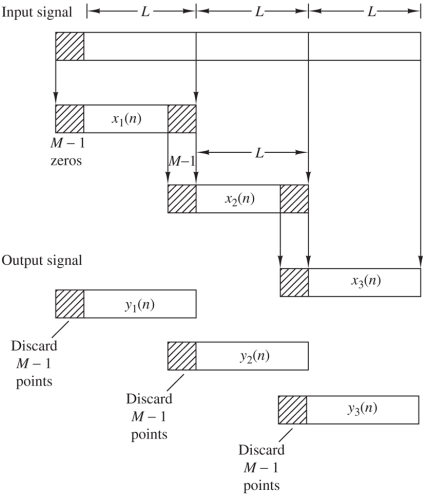


Figure 7.3.1 Linear FIR filtering by the overlap-save method.

Convolution with FIR Filter of Length M

Fast Filtering of Long Sequences

- While a bit more complicated, implementing a linear convolution of long sequences using the DFT on bocks and overlap-add or overlap-save
  - Computationally more efficient (because of FFT)
  - Yields the exact same result as a linear convolution
- The latency of filtering is different
  - For a time domain convolution latency is defined by the group delay (half the filter length for linear-phase filters, i.e. $M/2$)
  - For overlap-add and overlap-save the smallest achievable latency is the block-size L

➡ A time-domain implementation is computationally more demanding but results in a lower latency

Outline

Frequency Analysis of Signals Using the DFT I

- To compute the spectrum of a signal (be it continuous or discrete) the values of the signal for all times are required
- However, in practice, we observe signals for only a finite duration

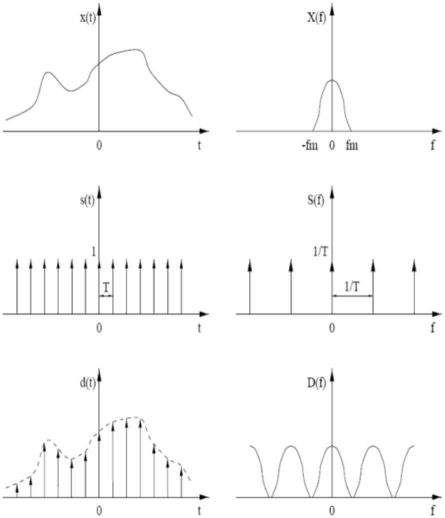
➡ The spectrum can only be approximated from a finite data record

- Here, we examine the implications of a finite data record in frequency analysis using the DFT

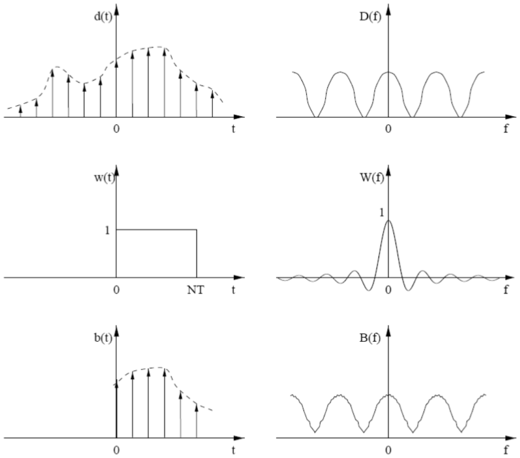
Frequency Analysis of Signals Using the DFT II

- If the signal is analog, the following steps are taken
  - Pass it through antialiasing filter
  - Sample it at rate $F_\mathrm{S} \geq 2B$, with B the bandwidth of the filtered signal
  - Limit length to time interval $T_0 = LT$
- The finite observation interval places a limit on the frequency resolution
  - limits our ability to distinguish two frequencies separated by less than $1/T_0 = 1/LT$ in frequency

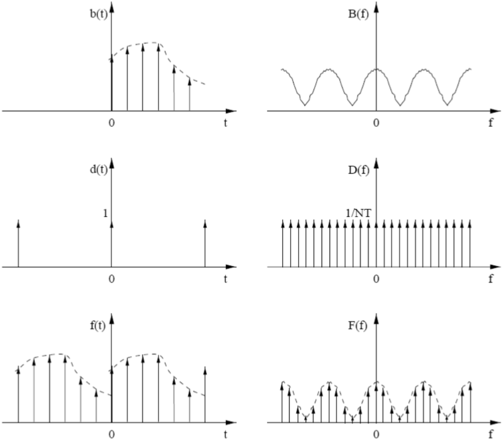
Graphical Illustration



Graphical Illustration



Graphical Illustration



Spectral Leakage (Rectangular Window)

Spectral Leakage (Rectangular Window)

Spectral Leakage (Rectangular Window)

Rectangular window

DFT Using a Hann-Window

Choice of Windowing Function

- The choice of the window function is always a trade-off
- A rectangular spectral analysis window has the most narrow main lobe, but terribly large side-lobes that lead to spectral leakage
- A tapered spectral analysis window (e.g. Hann or Hamming) exhibits a wider main-lobe but lower side-lobes.
  ➡ often the preferred choice!

Rectangular window

$$x(n) = \cos(\omega_0 n) + \cos(\omega_1 n) + \cos(\omega_2 n)$$

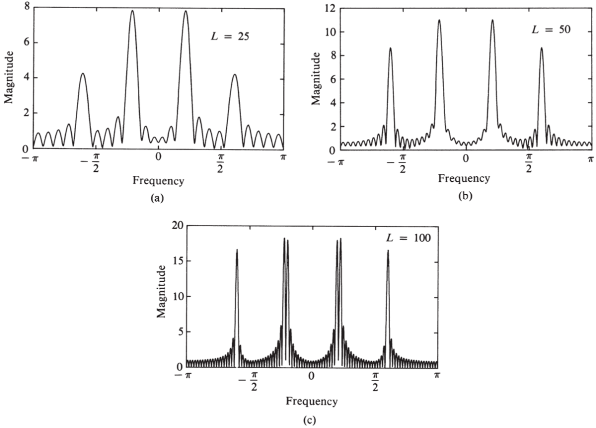


Figure 7.4.2 Magnitude spectrum for the signal given by (7.4.8), as observed through a rectangular window.

➡ The spectral resolution is increased by taking longer data segments

Hann window

$$x(n) = \cos(\omega_0 n) + \cos(\omega_1 n) + \cos(\omega_2 n)$$

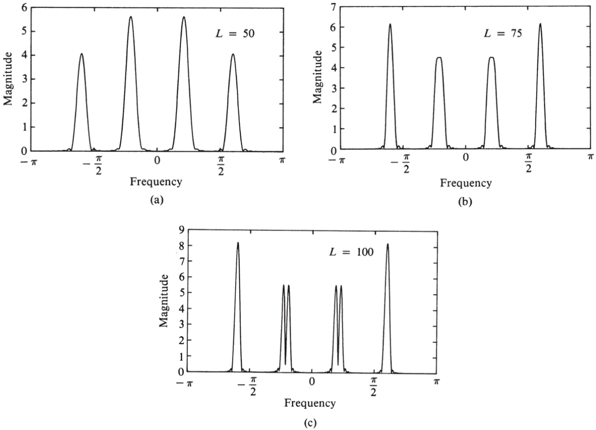


Figure 7.4.4 Magnitude spectrum of the signal in (7.4.8) as observed through a Hanning window.

➡ The spectral resolution is increased by taking longer data segments

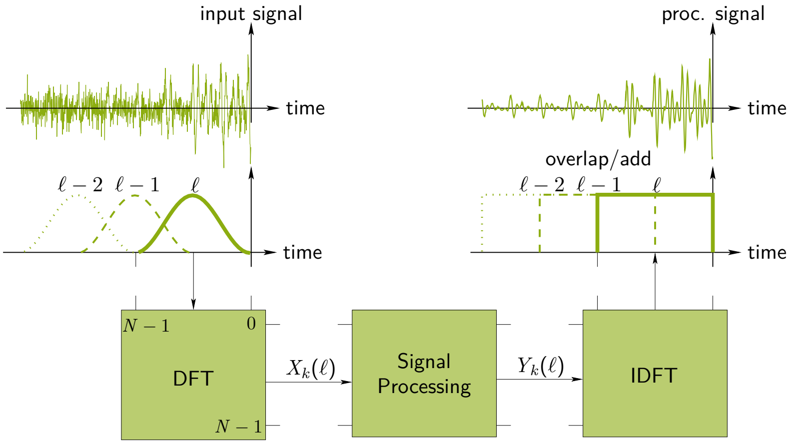
Spectral Resolution and Zero-Padding

- If the DFT length N is larger than the number of nonzero entries L this results in a **zero-padding**
- Zero-padding can be employed to avoid cyclic convolution artifacts
- Zero-padding results in an interpolation of the spectrum.
- Zero-padding *does not* increase the spectral resolution, i.e. the ability to separate two near-by frequencies
- The spectral resolution is increased by increasing the data, i.e. the number of non-zero elements L

Outline

Short-Time Fourier Transformation (STFT)

- Given an (infinitely) long input sequence
- Take overlapping segments
- Weight using spectral analysis window
- Take DFT, do processing, take inverse DFT
- Reconstruct by adding overlapping segments


input signal
time
$\ell-2$ $\ell-1$ ℓ
time
proc. signal
time
overlap/add
$\ell-2$ $\ell-1$ ℓ
time
$N-1$
0
DFT
$N-1$
$X_k(\ell)$
Signal Processing
$Y_k(\ell)$
IDFT

Outline

The Fast Fourier Transform (FFT) I

SP Signal Processing

- The DFT is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad 0 \leq k \leq N-1$$

  Thus, the order of complexity is $\mathcal{O}(N^2)$
- Basic idea of FFT: decompose the DFT in several smaller DFTs
  ➡ This is also called **divide-and-conquer** approach

The Fast Fourier Transform (FFT) II

- The most common decomposition is a radix-2 FFT, where a DFT of length $N = 2^{\nu}$ is decomposed into DFTs of length 2.

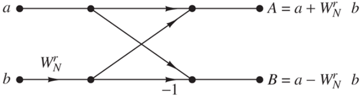


Figure 8.1.7 Basic butterfly computation in the decimation-in-time FFT algorithm.

- The 2-point DFT is denoted by a **butterfly** computation
- The computational complexity of each butterfly is $\mathcal{O}(2)$ (1 complex multiplication and 2 complex additions)

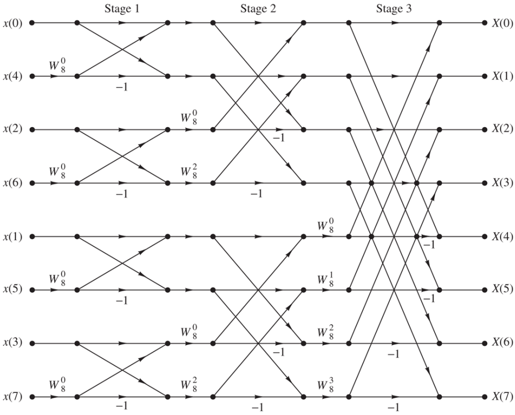


Figure 8.1.6 Eight-point decimation-in-time FFT algorithm.

- For $N = 2^\nu$ there are $N/2$ butterflies in $\nu = \log_2 N$ stages

➡ Total order: $\mathcal{O}(N \log_2(N))$
($N/2 \log_2(N)$ complex multiplications and $N \log_2(N)$ complex additions)

The Fast Fourier Transform (FFT) IV

- While the computational complexity of a DFT is in the order $\mathcal{O}(N^2)$, the FFT is only of order $\mathcal{O}(N \log_2 N)$
- Significant reduction in complexity.
  ➡ For a 512 point DFT 262 144 additions are needed, while for the FFT only 4 608 additions are required