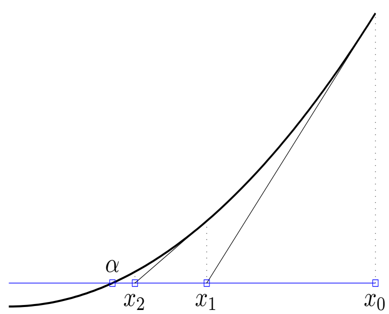
Numerical methods for solving nonlinear equations in $\mathbb{R}$

Let $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$. Consider the equation

$$f(x) = 0, \quad x \in \Omega. \tag{1}$$

Newton's method

$$x_{i+1} = F_2^T(x_i) = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0, 1, ...,$$



The algorithm:

Let x_0 be the initial approximation.

for $n = 0, 1, ..., ITMAX$

$$x_{n+1} \leftarrow x_n - \frac{f(x_n)}{f'(x_n)}.$$

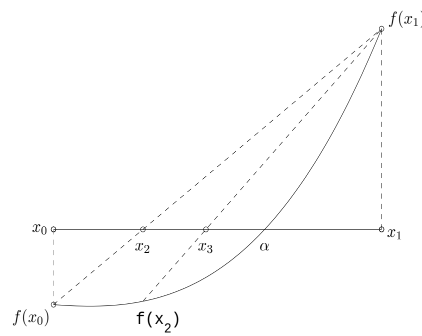
A stopping criterion is:

$$|f(x_n)| \leq \varepsilon \text{ or } \left|x_{n+1} - x_n\right| \leq \varepsilon \text{ or } \frac{\left|x_{n+1} - x_n\right|}{\left|x_{n+1}\right|} \leq \varepsilon,$$

where ε is a specified tolerance value.

The secant method

$$x_{i+1} := F_1^L\left(x_{i-1}, x_i\right) = x_i - \frac{\left(x_i - x_{i-1}\right) f\left(x_i\right)}{f\left(x_i\right) - f\left(x_{i-1}\right)}, \qquad i = 1, 2, \ldots$$



The algorithm:

Let x_0 and x_1 be two initial approximations.

for $n = 1, 2, \ldots, ITMAX$

$$x_{n+1} \leftarrow x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right].$$

A suitable stopping criterion is

$$|f(x_n)| \leq \varepsilon \text{ or } \left|x_{n+1} - x_n\right| \leq \varepsilon \text{ or } \frac{\left|x_{n+1} - x_n\right|}{\left|x_{n+1}\right|} \leq \varepsilon,$$

where ε is a specified tolerance value.

THE BISECTION METHOD

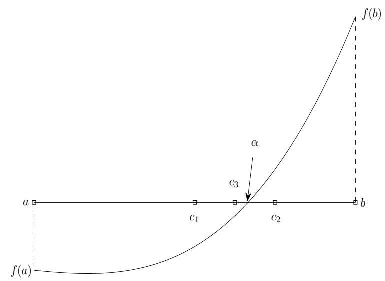
Let f be a given function, continuous on an interval $[a, b]$, such that

$$f(a)f(b) < 0. \tag{2}$$

By Mean Value Theorem, it follows that there exists at least one zero α of f in (a, b).

The bisection method is based on halving the interval $[a, b]$ to determine a smaller and smaller interval within α must lie.

First we give the midpoint of $[a, b]$, $c = (a + b)/2$ and then compute the product $f(c)f(b)$. If the product is negative, then the root is in the interval $[c, b]$ and we take $a_1 = c$, $b_1 = b$. If the product is positive, then the root is in the interval $[a, c]$ and we take $a_1 = a$, $b_1 = c$. Thus, a new interval containing α is obtained.



Bisection method

The algorithm:

Suppose $f(a)f(b) \leq 0$. Let $a_o = a$ and $b_o = b$.

for $n = 0, 1, ...,$ITMAX

$c \leftarrow \frac{a_n + b_n}{2}$

if $f(a_n)f(c) \leq 0$, set $a_{n+1} = a_n, b_{n+1} = c$

else, set $a_{n+1} = c, b_{n+1} = b_n$

The process of halving the new interval continues until the root is located as accurately as desired, namely

$$|a_n - b_n| < \varepsilon, \tag{3}$$

where a_n and b_n are the endpoints of the n-th interval $[a_n, b_n]$ and ε is a specified precision. The approximation of the solution will be $\frac{a_n+b_n}{2}$.

Some other stopping criterions: $\frac{|a_n-b_n|}{|a_n|} < \varepsilon$ or $|f(a_n)| < \varepsilon$.

THE METHOD OF FALSE POSITION

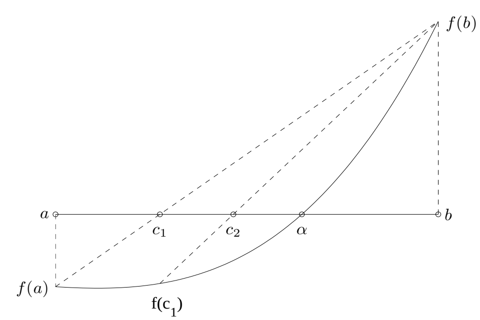
This method is also known as *regula falsi*, is similar to the Bisection method but has the advantage of being slightly faster than the latter. The function have to be continuous on $[a, b]$ with

$$f(a)f(b) < 0.$$

The point c is selected as point of intersection of the Ox-axis, and the straight line joining the points $(a,\ f(a))$ and $(b, f(b))$. From the equation of the secant line, it follows that

$$c = b - f(b)\frac{b-a}{f(b)-f(a)} = \frac{af(b)-bf(a)}{f(b)-f(a)} \tag{4}$$

Compute $f(c)$ and repeat the procedure between the values at which the function changes sign, that is, if $f(a)f(c) < 0$ set $b = c$, otherwise set $a = c$. At each step we get a new interval that contains a root of f and the generated sequence of points will eventually converge to the root.



Method of false position.

The algorithm:

Given a function f continuous on $[a_0, b_0]$, with $f(a_0)f(b_0) < 0$,

input: a_0, b_0

for $n = 0, 1, ..., ITMAX$

$c \leftarrow \frac{f(b_n)a_n - f(a_n)b_n}{f(b_n) - f(a_n)}$

if $f(a_n)f(c) < 0$, set $a_{n+1} = a_n, b_{n+1} = c$ **else** set $a_{n+1} = c, b_{n+1} = b_n$.

Stopping criterions: $|f(a_n)| \leq \varepsilon$ or $|a_n - a_{n-1}| \leq \varepsilon$, where ε is a specified tolerance value.

One of the main disadvantages of this method is that if the sequence of points generated by its algorithm is one-sided, the convergence of the method is slow.