

COURSE 4

The Hermite interpolation formula is

$$f = H_n f + R_n f,$$

where $R_n f$ denotes the remainder term (the error).

Theorem 1 *If $f \in C^n[\alpha, \beta]$ and $f^{(n)}$ is derivable on (α, β) , with $\alpha = \min\{x, x_0, \dots, x_m\}$ and $\beta = \max\{x, x_0, \dots, x_m\}$, then there exists $\xi \in (\alpha, \beta)$ such that*

$$(R_n f)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi). \quad (1)$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_n f)(z) \\ u(x) & (R_n f)(x) \end{vmatrix}.$$

$F \in C^n[\alpha, \beta]$ and there exists $F^{(n+1)}$ on (α, β) .

We have

$$F(x) = 0, \quad F^{(j)}(x_k) = 0, \quad k = 0, \dots, m; \quad j = 0, \dots, r_k;$$

because

$$u(x) = \prod_{k=0}^m (x - x_k)^{r_k+1} \Rightarrow u^{(j)}(x_k) = 0, \quad j = 0, \dots, r_k$$

and

$$(R_m f)^{(j)}(x_k) = f^{(j)}(x_k) - (H_n f)^{(j)}(x_k) = f^{(j)}(x_k) - f^{(j)}(x_k) = 0.$$

So, F and its derivatives have $n + 2$ distinct zeros in (α, β) . Applying successively Rolle's theorem it follows that F' has at least $n + 1$ zeros in $(\alpha, \beta) \Rightarrow \dots \Rightarrow F^{(n+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(n+1)}(\xi) = 0$.

We have

$$F^{(n+1)}(z) = \begin{vmatrix} u^{(n+1)}(z) & (R_n f)^{(n+1)}(z) \\ u(x) & (R_n f)(x) \end{vmatrix},$$

with $u(z) = \prod_{k=0}^m (z - z_k)^{r_k+1} \in \mathbb{P}_{n+1} \Rightarrow u^{(n+1)}(z) = (n+1)!$, and $(R_n f)^{(n+1)}(z) = f^{(n+1)}(z) - (H_n f)^{(n+1)}(z) = f^{(n+1)}(z)$ (as, $H_n f \in \mathbb{P}_n$). We get

$$F^{(n+1)}(\xi) = \begin{vmatrix} (n+1)! & f^{(n+1)}(\xi) \\ u(x) & (R_n f)(x) \end{vmatrix} = 0,$$

whence it follows (1). ■

Corollary 2 *If $f \in C^{n+1}[a, b]$ then*

$$|(R_n f)(x)| \leq \frac{|u(x)|}{(n+1)!} \|f^{(n+1)}\|_{\infty}, \quad x \in [a, b]$$

where $\|\cdot\|_{\infty}$ denotes the uniform norm ($\|f\|_{\infty} = \max_{x \in [a, b]} |f(x)|$).

Remark 3 *In case of $m = 0$, i.e., $n = r_0$, (HIP) becomes **Taylor interpolation problem**. Taylor interpolation polynomial is*

$$(T_n f)(x) = \sum_{j=0}^n \frac{(x - x_0)^j}{j!} f^{(j)}(x_0).$$

Example 4 Find the Hermite interpolation formula for the function $f(x) = xe^x$ for which we know $f(-1) = -0.3679$, $f(0) = 0$, $f'(0) = 1$, $f(1) = 2.7183$, (equivalent with $x_0 = -1$ simple, $x_1 = 0$ multiple of order 2 and $x_2 = 1$ simple). Which is the limit of the error for approximating $f(\frac{1}{2})$?

Hermite interpolation with double nodes

Example 5 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is $t = 10$ using Hermite interpolation.*

<i>Time</i>	0	3	5	8	13
<i>Distance</i>	0	225	383	623	993
<i>Speed</i>	75	77	80	74	72

Consider $f : [a, b] \rightarrow \mathbb{R}$, $x_0, x_1, \dots, x_m \in [a, b]$

and the values $f(x_0), f(x_1), \dots, f(x_m), f'(x_0), f'(x_1), \dots, f'(x_m)$.

The Hermite interpolation polynomial with double nodes, H_{2m+1} , satisfies the interpolation properties:

$$\begin{aligned} H_{2m+1}(x_i) &= f(x_i), \quad i = \overline{0, m}, \\ H'_{2m+1}(x_i) &= f'(x_i), \quad i = \overline{0, m}. \end{aligned}$$

It is a polynomial of $n = 2m + 1$ degree.

For computation: use Lagrange polynomial written in Newton form, with divided differences table having each node x_i written twice.

Consider $z_0 = x_0, z_1 = x_0, z_2 = x_1, z_3 = x_1, \dots, z_{2m} = x_m, z_{2m+1} = x_m$.

Form divided differences table: each node appear twice, in the first column write the values of f for each node twice; in the second column, at the odd positions put the values of the derivatives of f ; the other elements are computed using the rule from divided differences.

We obtain the following table:

z_0	$f(z_0)$	$(\mathcal{D}^1 f)(z_0) = f'(x_0)$	$(\mathcal{D}^2 f)(z_0)$		$(\mathcal{D}^{2m} f)(z_0)$	$(\mathcal{D}^{2m+1} f)(z_0)$
z_1	$f(z_1)$	$(\mathcal{D}^1 f)(z_1)$	\vdots		$(\mathcal{D}^{2m} f)(z_1)$	
z_2	$f(z_2)$	$(\mathcal{D}^1 f)(z_2) = f'(x_1)$				
z_3	$f(z_3)$	\vdots				
\vdots	\vdots	$(\mathcal{D}^1 f)(z_{2m-1})$	$(\mathcal{D}^2 f)(z_{2m-1})$	\ddots		
z_{2m}	$f(z_{2m})$	$(\mathcal{D}^1 f)(z_{2m}) = f'(x_m)$		\dots		
z_{2m+1}	$f(z_{2m+1})$			\dots		

Newton interpolation polynomial for the nodes x_0, \dots, x_n is

$$(N_n f)(x) = f(x_0) + \sum_{i=1}^n (x - x_0) \dots (x - x_{i-1}) (\mathcal{D}^i f)(x_0),$$

and similarly, Hermite interpolation polynomial is

$$(H_{2m+1} f)(x) = f(z_0) + \sum_{i=1}^{2m+1} (x - z_0) \dots (x - z_{i-1}) (\mathcal{D}^i f)(z_0),$$

where $(\mathcal{D}^i f)(z_0)$, $i = 1, \dots, 2m + 1$ are the elements from the first line and columns $2, \dots, 2m + 1$.

Example 6 Consider the double nodes $x_0 = -1$ and $x_1 = 1$, and $f(-1) = -3, f'(-1) = 10, f(1) = 1, f'(1) = 2$. Find the Hermite interpolation polynomial, that approximates the function f , in both forms, using the classical formula and using divided differences.

Sol. We present here the method with divided differences. We have $m = 1, r_0 = r_1 = 1 \Rightarrow n = 3$

$z_0 = -1$	$f(-1) = -3$	$f'(-1) = 10$	$\frac{\frac{f(1)-f(-1)}{2} - f'(-1)}{z_2 - z_0} = -4$	$\frac{0 - (-4)}{z_3 - z_0} = 2$
$z_1 = -1$	$f(-1) = -3$	$\frac{f(1)-f(-1)}{z_2 - z_1} = 2$	$\frac{f'(1) - \frac{f(1)-f(-1)}{2}}{z_3 - z_1} = 0$	
$z_2 = 1$	$f(1) = 1$	$f'(1) = 2$		
$z_3 = 1$	$f(1) = 1$			

The Hermite interpolation polynomial is

$$\begin{aligned}
 (H_3 f)(x) &= f(z_0) + \sum_{i=1}^3 (x - z_0) \dots (x - z_{i-1}) (\mathcal{D}^i f)(z_0) \\
 &= f(z_0) + (x - z_0) (\mathcal{D}^1 f)(z_0) + (x - z_0)(x - z_1) (\mathcal{D}^2 f)(z_0) \\
 &\quad + (x - z_0)(x - z_1)(x - z_2) (\mathcal{D}^3 f)(z_0)
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 (H_3 f)(x) &= f(-1) + (x + 1) f'(-1) + (x + 1)^2 \frac{f(1) - f(-1) - 2f'(-1)}{4} \\
 &\quad + (x + 1)^2 (x - 1) \frac{2f'(1) - f(1) + f(-1)}{4} \\
 &= -3 + 10(x + 1) - 4(x + 1)^2 + 2(x + 1)^2 (x - 1) \\
 &= 2x^3 - 2x^2 + 1.
 \end{aligned}$$

Example 7 *Considering the the following data*

x	0	2	3
$f(x)$	0	10	12
$f'(x)$	5	3	7

find the corresponding Hermite interpolation polynomial.

2.4. Birkhoff interpolation

Let $x_k \in [a, b]$, $k = 0, 1, \dots, m$, $x_i \neq x_j$ for $i \neq j$, $r_k \in \mathbb{N}$ and $I_k \subset \{0, 1, \dots, r_k\}$, $k = 0, 1, \dots, m$, $f : [a, b] \rightarrow \mathbb{R}$ s.t. $\exists f^{(j)}(x_k)$, $k = 0, \dots, m$, $j \in I_k$, and denote $n = |I_0| + \dots + |I_m| - 1$, where $|I_k|$ is the cardinal of the set I_k .

The Birkhoff interpolation problem (BIP) consists in determining the polynomial P of the smallest degree such that

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, \dots, m; \quad j \in I_k.$$

Remark 8 If $I_k = \{0, 1, \dots, r_k\}$, $k = 0, \dots, m$, then (BIP) reduces to a (HIP). Birkhoff interpolation is also called lacunary Hermite interpolation.

In order to check if (BIP) has an unique solution, we consider the polynomial $P(x) = a_n x^n + \dots + a_0$ and the $(n + 1) \times (n + 1)$ linear system

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, \dots, m; \quad j \in I_k, \quad (2)$$

having as unknowns the coefficients of the polynomial. If the determinant of the system (2) is nonzero then (BIP) has a unique solution.

Definition 9 *A solution of (BIP), if exists, is called **Birkhoff interpolation polynomial**, denoted by $B_n f$.*

Birkhoff interpolation polynomial is given by

$$(B_n f)(x) = \sum_{k=0}^m \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k), \quad (3)$$

where $b_{kj}(x)$ denote the Birkhoff fundamental interpolation polynomials. They fulfill relations:

$$\begin{aligned} b_{kj}^{(p)}(x_\nu) &= 0, \quad \nu \neq k, \quad p \in I_\nu \\ b_{kj}^{(p)}(x_k) &= \delta_{jp}, \quad p \in I_k, \quad \text{for } j \in I_k \text{ and } \nu, k = 0, 1, \dots, m, \end{aligned} \quad (4)$$

with $\delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$

Remark 10 *Because of the gaps of the interpolation conditions, it is hard to find an explicit expression for b_{kj} , $k = 0, \dots, m$; $j \in I_k$. They are found using relations (4).*

Birkhoff interpolation formula is

$$f = B_n f + R_n f,$$

where $R_n f$ denotes the remainder term.

Example 11 *Let $f \in C^2[0, 1]$, the nodes $x_0 = 0$, $x_1 = 1$ and we suppose that we know $f(0) = 1$ and $f'(1) = \frac{1}{2}$. Find the corresponding interpolation formula.*

Sol. *We have $m = 1$, $I_0 = \{0\}$, $I_1 = \{1\}$, so $n = 1 + 1 - 1 = 1$.*

We check if there exists a solution of the problem.

Consider $P(x) = a_1x + a_0 \in \mathbb{P}_1$ and the system

$$\begin{cases} P(0) = f(0) \\ P'(1) = f'(1) \end{cases} \iff \begin{cases} a_0 = f(0) \\ a_1 = f'(1) \end{cases}.$$

The determinat of the system is

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,$$

so the problem has an unique solution.

The Birkhoff polynomial is

$$(B_1f)(x) = b_{00}(x)f(0) + b_{11}(x)f'(1) \in \mathbb{P}_1.$$

We have $b_{00}(x) = ax + b \in \mathbb{P}_1$ and

$$\begin{cases} b_{00}(x_0) = 1 \\ b'_{00}(x_1) = 0 \end{cases} \iff \begin{cases} b_{00}(0) = 1 \\ b'_{00}(1) = 0 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ a = 0 \end{cases},$$

whence

$$b_{00}(x) = 1.$$

For $b_{11}(x) = cx + d \in \mathbb{P}_1$ we have

$$\begin{cases} b_{11}(x_0) = 0 \\ b'_{11}(x_1) = 1 \end{cases} \iff \begin{cases} b_{11}(0) = 0 \\ b'_{11}(1) = 1 \end{cases} \iff \begin{cases} d = 0 \\ c = 1 \end{cases}$$

whence

$$b_{11}(x) = x.$$

So,

$$(B_1 f)(x) = f(0) + x f'(1) = 1 + \frac{1}{2}x.$$

Example 12 Considering $f'(0) = 1$, $f(1) = 2$ and $f'(2) = 1$. Find the approximative value of $f(\frac{1}{2})$.