COURSE 2

Lagrange interpolation (continuation)

Let $[a,b]\subset\mathbb{R}$, $x_i\in[a,b]$, $i=0,1,...,m$ such that $x_i\neq x_j$ for $i\neq j$ and consider $f:[a,b]\to\mathbb{R}$. The Lagrange interpolation polynomial is given by

$$(L_m f)(x)=\sum_{i=0}^{m}\ell_i(x)f(x_i), \tag{1}$$

where by $\ell_i(x)$ denote **the Lagrange fundamental interpolation polynomials.**

We have

$$u(x)=\prod_{j=0}^{m}(x-x_j),$$

$$u_i(x)=\frac{u(x)}{x-x_i}=(x-x_0)...(x-x_{i-1})(x-x_{i+1})...(x-x_m)=\prod_{\substack{j=0\\ j\neq i}}^{m}(x-x_j)$$

and

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)} = \frac{(x-x_0)...(x-x_{i-1})(x-x_{i+1})...(x-x_m)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_m)} = \prod_{\substack{j=0\\j\neq i}}^{m} \frac{x-x_j}{x_i-x_j}, \tag{2}$$

for $i = 0, 1, ..., m$.

Theorem 1 *The operator L_m is linear.*

Proof.

$$\begin{aligned} L_m(\alpha f + \beta g)(x) &= \sum_{i=0}^{m} \ell_i(x)(\alpha f + \beta g)(x_i) = \sum_{i=0}^{m} [\ell_i(x)\alpha f(x_i) + \ell_i(x)\beta g(x_i)] \\ &= \alpha(L_m f)(x) + \beta(L_m g)(x), \end{aligned}$$

so

$$L_m(\alpha f + \beta g) = \alpha L_m f + \beta L_m g, \qquad \forall f, g : [a,b] \to \mathbb{R} \text{ and } \alpha, \beta \in \mathbb{R}.$$

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Example 2 *a) Consider the nodes* x_0, x_1 *and a function* f *to be interpolated.*

b) Find the Lagrange polynomial that interpolates the data in the following table and find the approximative value of $f(-0.5)$.

x	-1	0	3
$f(x)$	8	-2	4

Sol.

a) We have $m = 1$,

$$\begin{aligned} u(x) &= (x - x_0)(x - x_1) \\ u_0(x) &= x - x_1 \\ u_1(x) &= x - x_0 \end{aligned}$$

$$\begin{aligned} (L_1 f)(x) &= l_0(x) f(x_0) + l_1(x) f(x_1) \\ &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1), \end{aligned}$$

which is the line passing through the given points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

b) We have $m = 2$. The Lagrange polynomial is

$$(L_2 f)(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2).$$

$u(x) = (x+1)(x-0)(x-3)$ and it follows

$$l_0(x) = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} = \frac{1}{4}x(x-3)$$
$$l_1(x) = \frac{(x+1)(x-3)}{(0+1)(0-3)} = -\frac{1}{3}(x+1)(x-3)$$
$$l_2(x) = \frac{(x+1)(x-0)}{(3+1)(3-0)} = \frac{1}{12}x(x+1),$$

The polynomial is

$$(L_2 f)(x) = 2x(x-3) + \frac{2}{3}(x+1)(x-3) + \frac{1}{3}x(x+1).$$

and $(L_2 f)(-0.5) = 2.25$.

Remark 3 *Disadvantages of the form (1) of Lagrange polynomial: requires many computations and if we add or substract a point we have to start with a complete new set of computations.*

Some calculations allow us to reduce the number of operations:

$$(L_m f)(x) = \frac{(L_m f)(x)}{1} = \frac{\sum\limits_{i=0}^{m} l_i(x) f(x_i)}{\sum\limits_{i=0}^{m} l_i(x)}.$$

Dividing the numerator and the denominator by

$$u(x) = \prod_{i=1}^{m} (x - x_i)$$

and denoting

$$A_i = \frac{1}{\prod\limits_{j=0, j \neq i}^{m} (x_i - x_j)} = \frac{1}{u_i(x_i)}$$

one obtains

$$(L_m f)(x) = \frac{\sum\limits_{i=0}^{m} \frac{A_i f(x_i)}{x - x_i}}{\sum\limits_{i=0}^{m} \frac{A_i}{x - x_i}}, \tag{3}$$

called **the barycentric form** *of Lagrange interpolation polynomial*.

Remark 4 *Formula (3) needs half of the number of arithmetic operations needed for (1) and it is easier to add or substract a point.*

The Lagrange polynomial generates **the Lagrange interpolation formula**

$$f = L_m f + R_m f,$$

where $R_m f$ denotes **the remainder** (**the error**).

Theorem 5 *Let* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$. *If* $f \in C^m[\alpha, \beta]$ *and* $f^{(m)}$ *is derivable on* (α, β) *then* $\forall x \in (\alpha, \beta)$, *there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi). \tag{4}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix}.$$

From hypothesis it follows that $F \in C^m[\alpha, \beta]$ and there exists $F^{(m+1)}$ on (α, β).

We have

$$F(x) = 0, \;\; F(x_i) = 0, \qquad i = 0, 1, ..., m,$$

as

$$u(x_i) = \prod_{j=0}^{m} (x_i - x_j) = 0$$

and

$$(R_m f)(x_i) = f(x_i) - (L_m f)(x_i) = f(x_i) - f(x_i) = 0,$$

so F has $m+2$ distinct zeros in (α, β). Applying successively the Rolle theorem it follows that: F has $m+2$ zeros in $(\alpha, \beta) \Rightarrow F'$ has at least $m+1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(m+1)}$ has at least one zero in (α, β)

So $F^{(m+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(m+1)}(\xi) = 0$.

We have

$$F^{(m+1)}(z) = \begin{vmatrix} u^{(m+1)}(z) & (R_m f)^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix},$$

with

$$u(z) = \prod_{i=0}^{m} (z - z_i) \Rightarrow u^{(m+1)}(z) = (m+1)!,$$

and

$$\begin{aligned} (R_m f)^{(m+1)}(z) &= (f - (L_m f))^{(m+1)}(z) \\ &= f^{(m+1)}(z) - (L_m f)^{(m+1)}(z) = f^{(m+1)}(z) \end{aligned}$$

(as, $L_m f \in \mathbb{P}_m$).

We have $F^{(m+1)}(\xi) = 0$, for $\xi \in (\alpha, \beta)$, so

$$F^{(m+1)}(\xi) = \begin{vmatrix} (m+1)! & f^{(m+1)}(\xi) \\ u(x) & (R_m f)(x) \end{vmatrix} = 0,$$

i.e., $(m+1)!(R_m f)(x) = u(x) f^{(m+1)}(\xi)$, whence $(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)$. ∎

Corollary 6 *If* $f \in C^{m+1}[a,b]$ *then*

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty, \qquad x \in [a,b]$$

where $\|\cdot\|_\infty$ *denotes the uniform norm, and* $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$.

Example 7 *If we know that* $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699$, *find* $\lg 76$. *Study the approximation error.*

Example 8 *Which is the limit of the error for computing* $\sqrt{115}$ *using Lagrange interpolation formula for the nodes* $x_0 = 100$, $x_1 = 121$ *and* $x_2 = 144$? *Find the approximative value of* $\sqrt{115}$.