COURSE 3

The Aitken's algorithm

Let $[a,b] \subset \mathbb{R},\ x_i \in [a,b],\ i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a,b] \to \mathbb{R}$.

Usually, for a practical approximation problem, for a given function $f : [a,b] \to \mathbb{R}$ we have to find the approximation of $f(\alpha)$, $\alpha \in [a,b]$ with an error not greater than a given $\varepsilon > 0$.

If we have enough information about f and its derivatives, we use the inequality $|(R_m f)(x)| \leq \varepsilon$ to find m such that $(L_m f)(\alpha)$ approximates $f(\alpha)$ with the given precision.

We may use the condition $\frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty \leq \varepsilon$, but it should be known $\left\| f^{(m+1)} \right\|_\infty$ or a majorant of it.

A practical method for computing the Lagrange polynomial is **the Aitken's algorithm.** This consists in generating the table:

$$
\begin{array}{c|c|c|cccc}
x_0 & f_{00} & & & & & \\
x_1 & f_{10} & f_{11} & & & & \\
x_2 & f_{20} & f_{21} & f_{22} & & & \\
x_3 & f_{30} & f_{31} & f_{32} & f_{33} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
x_m & f_{m0} & f_{m1} & f_{m2} & f_{m3} & \cdots & f_{mm}
\end{array}
$$

where

$$f_{i0} = f(x_i), \quad i = 0, 1, ..., m,$$

and

$$f_{i,j+1} = \frac{1}{x_i - x_j} \begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}, \qquad i = 0, 1, ..., m; j = 0, ..., i-1.$$

For example,

$$f_{11} = \frac{1}{x_1 - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix}$$
$$= \frac{1}{x_1 - x_0} [f_{00}(x_1 - x) - f_{10}(x_0 - x)]$$
$$= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) = (L_1 f)(x),$$

so f_{11} is the value in x of Lagrange polynomial for the nodes x_0, x_1. We have

$$f_{ii} = (L_i f)(x),$$

$L_i f$ being Lagrange polynomial for the nodes $x_0, x_1, ..., x_i$.

So $f_{11}, f_{22}, ..., f_{ii}, ..., f_{mm}$ is a sequence of approximations of $f(x)$.

If the interpolation procedure is convergent then the sequence is also convergent, i.e., $\lim_{m \to \infty} f_{mm} = f(x)$. By Cauchy convergence criterion it follows

$$\lim_{i \to \infty} |f_{ii} - f_{i-1,i-1}| = 0.$$

This could be used as a stopping criterion, i.e.,

$$\left|f_{ii} - f_{i-1,i-1}\right| \leq \varepsilon, \quad \text{for a given precision } \varepsilon > 0.$$

Recommendation is to sort the nodes $x_0, x_1, ..., x_m$ with respect to the distance to x, such that

$$|x_i - x| \leq \left|x_j - x\right| \text{ if } i < j, \quad i, \quad j = 1, ..., m.$$

Example 1 *Approximate* $\sqrt{115}$ *with precision* $\varepsilon = 10^{-3}$, *using Aitken's algorithm.*

Newton interpolation polynomial

A useful representation for Lagrange interpolation polynomial is

$$(L_m f)(x) := (N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})(D^i f)(x_0) \tag{1}$$

$$= f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f],$$

which is called **Newton interpolation polynomial;** where $(D^i f)(x_0)$(or denoted $[x_0, ..., x_i; f]$) is the i-th order divided difference of the function f at x_0, given by the table

	f	$\mathcal{D}f$	$\mathcal{D}^2 f$	...	$\mathcal{D}^{\mathbf{m-1}} f$	$\mathcal{D}^m f$
x_0	f_0	$\mathcal{D}f_0$	$\mathcal{D}^2 f_0$	...	$\mathcal{D}^{m-1} f_0$	$\mathcal{D}^m f_0$
x_1	f_1	$\mathcal{D}f_1$	$\mathcal{D}^2 f_1$		$\mathcal{D}^{m-1} f_1$	
x_2	f_2	$\mathcal{D}f_2$	$\mathcal{D}^2 f_2$			
...	...	...				
x_{m-2}	f_{m-2}	$\mathcal{D}f_{m-2}$	$\mathcal{D}^2 f_{m-2}$			
x_{m-1}	f_{m-1}	$\mathcal{D}f_{m-1}$				
x_m	f_m					

Newton interpolation formula is

$$f = N_m f + R_m f,$$

where $R_m f$ denotes the remainder.

Assume that we add the point $(x, f(x))$ at the top of the table of divided differences:

	f	Df	$\ldots$	$D^{m+1}f$
x	$f(x)$	$(Df)(x) = [x, x_0; f]$		$[x, x_0, \ldots, x_m; f]$
x_0	$f(x_0)$	$(Df)(x_0) = [x_0, x_1; f]$	$\ldots$	
x_1	$f(x_1)$	$(Df)(x_1) = [x_1, x_2; f]$		
$\ldots$	$\ldots$	$\ldots$		
x_{m-1}	$f(x_{m-1})$	$(Df)(x_{m-1}) = [x_{m-1}, x_m; f]$		
x_m	$f(x_m)$			

For obtaining the interpolation polynomial we consider

$$[x, x_0; f] = \frac{f(x_0) - f(x)}{x_0 - x} \Longrightarrow f(x) = f(x_0) + (x - x_0)[x, x_0; f] \quad (2)$$

$$[x, x_0, x_1; f] = \frac{[x_0, x_1; f] - [x, x_0; f]}{x_1 - x} \quad (3)$$

$$\Longrightarrow [x, x_0; f] = [x_0, x_1; f] + (x - x_1)[x, x_0, x_1; f].$$

Inserting (3) in (2) we get

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1; f] + (x - x_0)(x - x_1)[x, x_0, x_1; f].$$

If we continue eliminating the divided differences involving x in the same way, we get

$$f(x) = (N_m f)(x) + (R_m f)(x)$$

with

$$(N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f]$$

and the remainder (the error) given by

$$(R_m f)(x) = (x - x_0)...(x - x_m)[x, x_0, ..., x_m; f]. \tag{4}$$

Remark 2 *The remainder for Lagrange interpolation formula is also given by*

$$(R_m f)(x) = \frac{(x - x_0)...(x - x_m)}{(m+1)!} f^{(m+1)}(\xi),$$

with ξ between $x, x_0, ..., x_m$, so, by (4), it follows that **the divided differences are approximations of the derivatives**

$$[x, x_0, ..., x_m; f] = \frac{f^{(m+1)}(\xi)}{(m+1)!}.$$

Remark 3 *We notice that*

$$(N_i f)(x) = (N_{i-1} f)(x) + (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f]$$

so the Newton polynomials of degree $2, 3, \ldots,$ *can be iteratively generated, similarly to Aitken's algorithm.*

Example 4 *Find* $L_2 f$ *for* $f(x) = \sin \pi x$, *and* $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{2}$, *in both forms.*

Sol. *a) We have* $u(x) = x(x-\frac{1}{6})(x-\frac{1}{2})$; $u_0(x) = (x-\frac{1}{6})(x-\frac{1}{2})$; $u_1(x) = x(x-\frac{1}{2})$; $u_2(x) = x(x-\frac{1}{6})$

$$\begin{aligned}(L_2f)(x) &= \sum_{i=0}^{2} l_i(x)f(x_i) = \sum_{i=0}^{2} \frac{u_i(x)}{u_i(x_i)}f(x_i)\\ &= \frac{(x-\frac{1}{6})(x-\frac{1}{2})}{(-\frac{1}{6})(-\frac{1}{2})}0 + \frac{x(x-\frac{1}{2})}{\frac{1}{6}(-\frac{1}{3})}\frac{1}{2} + \frac{x(x-\frac{1}{6})}{\frac{1}{2}\cdot\frac{1}{3}}1\\ &= -3x^2 + \frac{7}{2}x.\end{aligned}$$

b)

$$\begin{aligned}(N_2f)(x) &= f(0) + \sum_{i=1}^{2}(x-x_0)...(x-x_{i-1})(D^if)(x_0)\\ &= f(0) + (x-x_0)(Df)(x_0) + (x-x_0)(x-x_1)(D^2f)(x_0)\\ &= x(Df)(x_0) + x(x-\frac{1}{6})(D^2f)(x_0)\end{aligned}$$

The table of divided differences:

x	f	Df	D^2f
0	0	3	-3
$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	
$\frac{1}{2}$	1		

so

$$(N_2f)(x) = 3x - 3x(x - \frac{1}{6}) = -3x^2 + \frac{7}{2}x.$$

2.3. Hermite interpolation

Example 5 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is* $t = 10$ *using Hermite interpolation.*

Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

Let $x_k \in [a, b]$, $k = 0, 1, ..., m$ be such that $x_i \neq x_j$, for $i \neq j$ and let $r_k \in \mathbb{N}$, $k = 0, 1, ..., m$. Consider $f : [a, b] \to \mathbb{R}$ such that there exist $f^{(j)}(x_k)$, $k = 0, 1, ..., m$; $j = 0, 1, ..., r_k$ and $n = m + r_0 + ... + r_m$.

The Hermite interpolation problem (HIP) consists in determining the polynomial P of the smallest degree for which

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, ..., m; \ j = 0, ..., r_k.$$

Definition 6 *A solution of (HIP) is called* **Hermite interpolation polynomial**, *denoted by $H_n f$.*

Hermite interpolation polynomial, $H_n f$, satisfies the interpolation conditions:

$$(H_n f)^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, ..., m; \; j = 0, ..., r_k.$$

Hermite interpolation polynomial is given by

$$(H_n f)(x) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) \in \mathbb{P}_n, \tag{5}$$

where $h_{kj}(x)$ denote **the Hermite fundamental interpolation polynomials.** They fulfill the relations:

$h_{kj}^{(p)}(x_\nu) = 0, \; \nu \neq k, \quad p = 0, 1, ..., r_\nu$

$h_{kj}^{(p)}(x_k) = \delta_{jp}, \; p = 0, 1, ..., r_k, \quad$ for $j = 0, 1, ..., r_k$ and $\nu, k = 0, 1, ..., m,$

with $\delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$

We denote by

$$u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k+1} \qquad \text{and } u_k(x) = \frac{u(x)}{(x - x_k)^{r_k+1}}.$$

We have

$$h_{kj}(x) = \frac{(x - x_k)^j}{j!} u_k(x) \sum_{\upsilon=0}^{r_k-j} \frac{(x - x_k)^\nu}{\nu!} \left[\frac{1}{u_k(x)}\right]_{x=x_k}^{(\nu)}. \qquad (6)$$

Example 7 *Find the Hermite interpolation polynomial for a function f for which we know $f(0) = 1, f'(0) = 2$ and $f(1) = -3$ (equivalent with $x_0 = 0$ multiple node of order 2 or double node, $x_1 = 1$ simple node).*

Sol. We have $x_0 = 0, x_1 = 1, m = 1, r_0 = 1, r_1 = 0, n = m + r_0 + r_1 = 2$

$$\begin{aligned}(H_2 f)(x) &= \sum_{k=0}^{1} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) \\ &= h_{00}(x) f(0) + h_{01}(x) f'(0) + h_{10}(x) f(1).\end{aligned}$$

We have h_{00}, h_{01}, h_{10}. These fulfills relations:

$$h_{kj}^{(p)}(x_\nu) = 0,\ \ \nu \neq k, \quad p = 0, 1, ..., r_\nu$$
$$h_{kj}^{(p)}(x_k) = \delta_{jp},\ \ p = 0, 1, ..., r_k, \quad \text{for } j = 0, 1, ..., r_k \text{ and } \nu, k = 0, 1, ..., m.$$

We have $h_{00}(x) = a_1 x^2 + b_1 x + c_1 \in \mathbb{P}_2$, with $a_1, b_1, c_1 \in \mathbb{R}$, and the system

$$\begin{cases} h_{00}(x_0) = 1 \\ h_{00}'(x_0) = 0 \\ h_{00}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} h_{00}(0) = 1 \\ h_{00}'(0) = 0 \\ h_{00}(1) = 0 \end{cases}$$

that becomes

$$\begin{cases} c_1 = 1 \\ b_1 = 0 \\ a_1 + b_1 + c_1 = 0. \end{cases}$$

Solution is: $a_1 = -1, b_1 = 0, c_1 = 1$ so $h_{00}(x) = -x^2 + 1$.

We have $h_{01}(x) = a_2 x^2 + b_2 x + c_2 \in \mathbb{P}_2$, with $a_2, b_2, c_2 \in \mathbb{R}$. The system is

$$\begin{cases} h_{01}(x_0) = 0 \\ h'_{01}(x_0) = 1 \\ h_{01}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} h_{01}(0) = 0 \\ h'_{01}(0) = 1 \\ h_{01}(1) = 0 \end{cases}$$

and we get $h_{01}(x) = -x^2 + x$.

We have $h_{10}(x) = a_3x^2 + b_3x + c_3 \in \mathbb{P}_2$, with $a_3, b_3, c_3 \in \mathbb{R}$. The system is

$$\begin{cases} h_{10}(x_0) = 0 \\ h'_{10}(x_0) = 0 \\ h_{10}(x_1) = 1 \end{cases} \Leftrightarrow \begin{cases} h_{10}(0) = 0 \\ h'_{10}(0) = 0 \\ h_{10}(1) = 1 \end{cases}$$

and we get $h_{10}(x) = x^2$.

The Hermite polynomial is

$$(H_2f)(x) = -x^2 + 1 - 2x^2 + 2x - 3x^2 = -6x^2 + 2x + 1.$$