The mathematics of *BSML*: expressive power and axiomatization

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Overview

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State-semantic closure properties

- *BSML* as an extension of classical modal logic
- *BSML* vs modal dependence logic

Expressive power

Natural deduction axiomatization

Closure properties

ϕ is *downward closed*: $[M, s \models \phi \text{ and } t \subseteq s] \implies M, t \models \phi$

ϕ is *union closed*: $[M, s \models \phi \text{ for all } s \in S \neq \varnothing] \implies M, \bigcup S \models \phi$

ϕ has the *empty state property*: $M, \varnothing \models \phi$ for all M

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Classical formulas (those composed of $p_i, \wedge, \vee, \Diamond$) have all of these properties.

Which properties does $\textsc{ne}$ have?
Union closure.

Dependence atoms

$$s \models =\!(p_1, \ldots, p_n, q) \quad \text{iff}$$

$$\forall w, w' \in s : \bigwedge_{i=1}^{n} (w \models p_i \iff w' \models p_i) \implies (w \models q \iff w' \models q)$$

	p	q
w_1	1	1
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Which properties does $=(p, q)$ have?
Empty state property and downward closure.

Non-dependence (variation) atoms

$$s \models \neq(p_1, \dots, p_n, q) \quad \text{iff}$$

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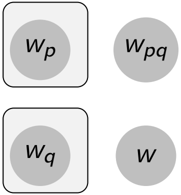
The inquisitive disjunction

$$s \vDash \phi \mathbin{\vee\!\!\!\vee} \psi \quad \text{iff} \quad s \vDash \phi \text{ or } s \vDash \psi$$



$$\begin{array}{lcl} \{w_p\} & \vDash & p \mathbin{\vee\!\!\!\vee} q \\ \{w_q\} & \vDash & p \mathbin{\vee\!\!\!\vee} q \end{array}$$

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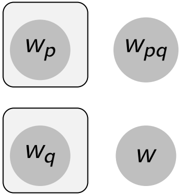

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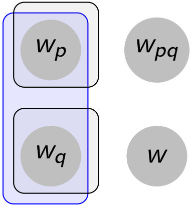

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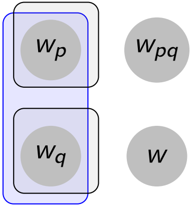

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All three.

$\oslash\phi$ always has the empty state property, regardless of whether ϕ has it.
Note in particular that if α is classical, $\oslash(\alpha \wedge \mathrm{NE}) \equiv \alpha$.

We consider *BSML* and extensions of *BSML* with $\mathbb{W}$ and $\oslash$. In this setting:

- All formulas without NE are downward closed and have the empty state property.
- All formulas without $\mathbb{W}$ are union closed.

Classical formulas and flatness

ϕ is *flat*: $\qquad M, s \vDash \phi \iff M, \{w\} \vDash \phi$ for all $w \in s$

flat $\iff$ downward closed & union closed & empty state property

So classical formulas are flat.

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Also: the classical world-based semantics coincides with state-based semantics on the level of singleton states—for classical α:

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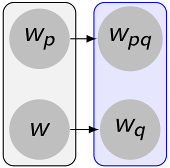
So a classical formula α is supported by a state iff it is true in all worlds in the state:

$$s \vDash \alpha \iff \forall w \in s : w \vDash \alpha$$

The modal dependence logic modalities $\diamondsuit$ and $\boxdot$



t is a successor state of s

$sRt :\iff t \subseteq R[s]$ and $R[w] \cap t \neq \varnothing$ for all $w \in s$

$R[s] = \{v \in W \mid \exists w \in s : wRv\}$

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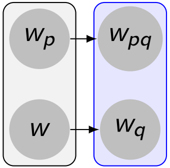

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$$s \models \Diamonddot\phi \qquad\iff\qquad \exists t : sRt \text{ and } t \models \phi$$

$$s \models \boxdot\phi \qquad\iff\qquad R[s] \models \phi$$

$$s \models \Diamond\phi \qquad\iff\qquad \forall w \in s : \exists t \subseteq R[w] : t \models \phi \text{ and } t \neq \varnothing$$

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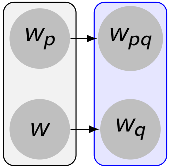

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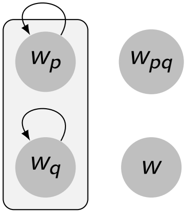
$$s \models \Box\phi \iff \forall w \in s : R[w] \models \phi$$

If ϕ is downward closed, $\Diamond\!\!\!\!\diamond\phi \models \Diamond\phi$ and $\boxdot\phi \models \Box\phi$

If ϕ is union closed and has the empty state property, $\Diamond\phi \models \Diamond\!\!\!\!\diamond\phi$ and $\Box\phi \models \boxdot\phi$

If ϕ is flat, $\Diamond\phi \equiv \Diamond\!\!\!\!\diamond\phi$ and $\Box\phi \equiv \boxdot\phi$

Aloni's free choice explanation does not work with $\diamondsuit$:



$$sRs \text{ and } s \vDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$$
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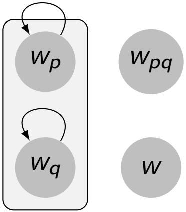

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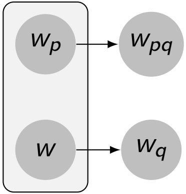
$$\text{Therefore } s \models \diamondsuit((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}))$$

$$s \text{ is the the only successor state of } s \text{ and } s \not\models p$$
$$\text{Therefore } s \not\models \diamondsuit p \text{ so } \diamondsuit((p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})) \not\models \diamondsuit p \wedge \diamondsuit q$$

On the other hand, dependence atoms do not function as intended with $\Diamond$:



$s \vDash \Diamond =(q, p)$

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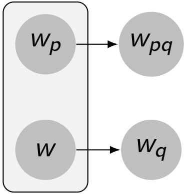

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For any $w' \in R[w]$, $\{w'\} \vDash =(q,p)$.

Expressive Power

We consider $BSML^{\mathbb{W}}$—the extension of $BSML$ with $\mathbb{W}$:

$$\begin{aligned} s \models \phi \mathbin{\mathbb{W}} \psi \quad &\iff \quad s \models \phi \text{ or } s \models \psi \\ s \dashv \phi \mathbin{\mathbb{W}} \psi \quad &\iff \quad s \dashv \phi \text{ and } s \dashv \psi \end{aligned}$$

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Fix a finite set of proposition symbols Φ.

Pointed state model: (M, s) where M is a model over Φ; s is a state on M

State property: set of pointed state models

$||\phi|| := \{(M, s) \mid M, s \models \phi\}$

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$||\phi|| := \{(M, s) \mid M, s \vDash \phi\}$

$\mathbb{P} := \{\text{property } P \mid P \text{ is invariant under state } k\text{-bisimulation for some } k \in \mathbb{N}\}$

Theorem

BSML$^{\mathbb{W}}$ is expressively complete for $\mathbb{P}$:

$$\{||\phi|| \mid \phi \in BSML^{\mathbb{W}}\} = \mathbb{P}$$

Modal depth of ϕ ($md(\phi)$): measure of the deepest nesting of modalities in ϕ.
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State k-bisimulation $\leftrightarrows_k$:
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Property P is *invariant under state k-bisimulation*:

$$[(M,s)\in P \text{ and } M,s \leftrightarroweq_k M',s'] \implies (M',s')\in P$$

Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\mathbb{W}}\}$$
$$=$$
$$\mathbb{P} = \{\text{property } P \mid P \text{ is invariant under state } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

So for instance, there are formulas equivalent to $=(p_1,\ldots,p_n,q)$ and $p_1,\ldots,p_n \subseteq q_1,\ldots,q_n$ in $BSML^{\mathbb{W}}$.

This theorem is crucial for our completeness proof strategy.

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$M,s \leftrightarroweq_k M',s' \iff M,s \equiv_k M',s'$ gives us the left-to-right inclusion. For the right-to-left inclusion, we construct characteristic formulas.

Characteristic formulas for worlds (Hintikka formulas):

$$\chi^0_{M,w} \quad := \quad \bigwedge\{p \mid w \in V(p)\} \wedge \bigwedge\{\neg p \mid w \notin V(p)\} \quad (p \in \Phi)$$

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$$\chi^{k+1}_{M,w} \quad := \quad \chi^k_{M,w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_{M,v} \wedge \Box \bigvee_{v \in R[w]} \chi^k_{M,v}$$

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$$\chi^{k+1}_{M,w} \quad := \quad \chi^k_{M,w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_{M,v} \wedge \Box \bigvee_{v \in R[w]} \chi^k_{M,v}$$

$$w' \vDash \chi^k_w \iff w \leftrightarrows_k w'$$

Characteristic formulas for worlds
(Hintikka formulas):

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Characteristic formulas for states:

$$\Theta^k_{M,s} := \bot \qquad \text{if } s = \varnothing \quad (\bot := p \wedge \neg p)$$
$$\Theta^k_{M,s} := \bigvee_{w \in s} (\chi^k_{M,w} \wedge \mathrm{NE}) \qquad \text{if } s \neq \varnothing$$

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(disjunctive normal form):

for P invariant under k-bisimulation:

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Theorem

$$\{||\phi|| \mid \phi \in BSML^{\mathbb{W}}\}$$
$$=$$
$$\{\text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Property P is *union closed*:

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Consider $||(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})|| \cup ||\bot|| \in \mathbb{U}$. $(\bot := p \wedge \neg p)$
Assume $||\psi|| = ||(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})|| \cup ||\bot||$ for $\psi \in BSML$.

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$||\bot|| \subseteq ||\psi||$, so ψ has the empty state property and hence is downward closed.

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Lemma

For $\phi \in BSML$: ϕ has the empty state property $\implies$ ϕ is downward closed.

Consider $\|(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})\| \cup \|\bot\| \in \mathbb{U}$. $(\bot := p \wedge \neg p)$
Assume $\|\psi\| = \|(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})\| \cup \|\bot\|$ for $\psi \in BSML$.
If $\{w_p, w_{\neg p}\} \vDash (p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})$, then $\{w_p, w_{\neg p}\} \vDash \psi$.
$\|\bot\| \subseteq \|\psi\|$, so ψ has the empty state property and hence is downward closed. Then $\{w_p\} \vDash \psi$.
$\{w_p\} \in \|(p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})\| \cup \|\bot\|$, a contradiction.

$BSML^{\oslash}$—$BSML$ with $\oslash$:

$$
\begin{array}{lll}
s \models \oslash\phi & \Longleftrightarrow & s \models \phi \text{ or } s = \varnothing \\
s \dashv \oslash\phi & \Longleftrightarrow & s \dashv \phi
\end{array}
$$

Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\oslash}\}$$

$$=$$

$$\mathbb{U} = \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Characteristic formulas: $\bigvee_{(M,s)\in P} \oslash\Theta^k_s \qquad (\bigvee_{(M,s)\in P} \oslash\Theta^k_s) \wedge \mathrm{NE}$

Natural deduction axiomatizations

Formulas in *BSML* and extensions may not be closed under uniform substitution:

$p \vee p \vDash p$	but	$(p \mathbin{\mathbb{W}} \neg p) \vee (p \mathbin{\mathbb{W}} \neg p) \not\vDash (p \mathbin{\mathbb{W}} \neg p)$
$\vDash p \vee \neg p$	but	$\not\vDash (p \wedge \mathrm{NE}) \vee \neg(p \wedge \mathrm{NE})$

Natural deduction axiomatizations

Formulas in *BSML* and extensions may not be closed under uniform substitution:

$$p \vee p \models p \qquad \text{but} \qquad (p \mathbin{\mathbb{W}} \neg p) \vee (p \mathbin{\mathbb{W}} \neg p) \not\models (p \mathbin{\mathbb{W}} \neg p)$$

$$\models p \vee \neg p \qquad \text{but} \qquad \not\models (p \wedge \mathrm{NE}) \vee \neg(p \wedge \mathrm{NE})$$

This means that the deduction systems will not admit the usual substitution rule:

$$\frac{\phi(p_1, \ldots, p_n)}{\phi(\psi_1/p_1, \ldots, \psi_n/p_n)}\ Sub$$

It also means that we may formulate rules which are only applicable to certain types of formulas. In particular, whenever α or β occur in a rule, only classical formulas (no NE or $\mathbb{W}$) may be substituted for these formulas. For instance:

$$\frac{\begin{array}{cc} D_1 & D_2 \\ \alpha & \neg\alpha \end{array}}{\beta}\ \neg E$$

This rule only applies to classical formulas α and β.

$BSML^{\mathbb{W}}$ axiomatization

Non-modal portion (adapted from the system for PT^{+}):

¬ introduction

$$\dfrac{\begin{array}{c}[\alpha]\\ D^*\\ \bot\end{array}}{\neg\alpha}\,\neg I\,(*)$$

¬ elimination

$$\dfrac{\begin{array}{cc}D_1 & D_2\\ \alpha & \neg\alpha\end{array}}{\beta}\,\neg E$$

$(*)$ The undischarged assumptions in D^* do not contain NE.

$\wedge$ introduction

$$\dfrac{\begin{matrix} D_1 & & D_2 \\ \phi & & \psi \end{matrix}}{\phi \wedge \psi} \wedge I$$

$\wedge$ elimination

$$\dfrac{\begin{matrix} D \\ \phi \wedge \psi \end{matrix}}{\phi} \wedge E \qquad \dfrac{\begin{matrix} D \\ \phi \wedge \psi \end{matrix}}{\psi} \wedge E$$

$\mathbin{⩛}$ introduction

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\phi \mathbin{⩛} \psi} \mathbin{⩛} I \qquad \dfrac{\begin{matrix} D \\ \psi \end{matrix}}{\phi \mathbin{⩛} \psi} \mathbin{⩛} I$$

$\mathbin{⩛}$ elimination

$$\dfrac{\begin{matrix} & [\phi] & [\psi] \\ D & D_1 & D_2 \\ \phi \mathbin{⩛} \psi & \chi & \chi \end{matrix}}{\chi} \mathbin{⩛} E$$

$\vee$ weak introduction

$$\dfrac{\begin{array}{c} D \\ \phi \end{array}}{\phi \vee \psi} \vee I(**)$$

$\vee$ weakening

$$\dfrac{\begin{array}{c} D \\ \phi \end{array}}{\phi \vee \phi} \vee W$$

$\vee$ weak elimination

$$\dfrac{\begin{array}{c} D \\ \phi \vee \psi \end{array} \quad \begin{array}{c} [\phi] \\ D_1^* \\ \chi \end{array} \quad \begin{array}{c} [\psi] \\ D_2^* \\ \chi \end{array}}{\chi} \vee E(*, \dagger)$$

$\vee$ weak substitution

$$\dfrac{\begin{array}{c} D \\ \phi \vee \psi \end{array} \quad \begin{array}{c} [\psi] \\ D_1^* \\ \chi \end{array}}{\phi \vee \chi} \vee Sub(*)$$

$(*)$ The undischarged assumptions in D_1^*, D_2^* do not contain NE.
$(**)$ ψ may not contain NE.
$(\dagger)$ χ may not contain $\mathbb{W}$ outside the scope of a $\Diamond$.

$\vee$ commutativity

$$\frac{\begin{array}{c} D \\ \phi \vee \psi \end{array}}{\psi \vee \phi}\, Com\vee$$

$\vee$ associativity

$$\frac{\begin{array}{c} D \\ (\phi \vee \psi) \vee \chi \end{array}}{\phi \vee (\psi \vee \chi)}\, Ass\vee$$

$\vee\mathbb{W}$ distributivity

$$\frac{\begin{array}{c} D \\ \phi \vee (\psi \mathbin{\mathbb{W}} \chi) \end{array}}{(\phi \vee \psi) \mathbin{\mathbb{W}} (\phi \vee \chi)}\, Distr \vee \mathbb{W}$$

$\bot$ Elimination

$$\dfrac{\begin{matrix} D \\ \phi \vee \bot \end{matrix}}{\phi} \; \bot E$$

$\amalg := \bot \wedge \mathrm{NE}$

$\amalg$ elimination

$$\dfrac{\begin{matrix} D \\ \amalg \end{matrix}}{\phi} \; \amalg E$$

$\amalg$ contraction

$$\dfrac{\begin{matrix} D \\ \amalg \vee \phi \end{matrix}}{\psi} \; \amalg Ctr$$

NE introduction

$$\frac{}{\bot \mathbin{\mathbb{W}} \mathrm{NE}}\ \mathrm{NE}I$$

$\vee$NE elimination

$$\dfrac{\begin{matrix} \\ D \\ \phi \vee \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ D_1 \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ D_2 \\ \chi \end{matrix} \qquad \begin{matrix} [(\phi \wedge \mathrm{NE}) \vee (\psi \wedge \mathrm{NE})] \\ D_3 \\ \chi \end{matrix}}{\chi}\ \vee\mathrm{NE}E$$

New rules for $\neg$:

$\neg\mathrm{NE}$ elimination

$$\dfrac{\begin{matrix} D \\ \neg\mathrm{NE} \end{matrix}}{\bot}\ \neg\mathrm{NE}E$$

Double $\neg$ elimination

$$\dfrac{\begin{matrix} D \\ \neg\neg\phi \end{matrix}}{\phi}\ DN$$

De Morgan 1

$$\dfrac{\begin{matrix} D \\ \neg(\phi \wedge \psi) \end{matrix}}{\neg\phi \vee \neg\psi}\ DM_1$$

De Morgan 2

$$\dfrac{\begin{matrix} D \\ \neg(\phi \vee \psi) \end{matrix}}{\neg\phi \wedge \neg\psi}\ DM_2$$

De Morgan 3

$$\dfrac{\begin{matrix} D \\ \neg(\phi \mathbin{\mathbb{W}} \psi) \end{matrix}}{\neg\phi \wedge \neg\psi}\ DM_3$$

Modal portion—basic rules:

$\Diamond$ monotonicity

$$\dfrac{\begin{array}{c}[\phi]\\ D'\\ \psi\end{array} \qquad \begin{array}{c} D\\ \Diamond\phi\end{array}}{\Diamond\psi}\ \Diamond Mon(*)$$

$\Box$ monotonicity

$$\dfrac{\begin{array}{c}[\phi_1]\dots[\phi_n]\\ D'\\ \psi\end{array} \qquad \begin{array}{c} D_1\\ \Box\phi_1\end{array} \quad \dots \quad \begin{array}{c} D_n\\ \Box\phi_n\end{array}}{\Box\psi}\ \Box Mon(*)$$

$\Diamond\Box$ interaction

$$\begin{array}{c} D\\ \neg\Diamond\phi\\ \hline\hline \Box\neg\phi\end{array}\ Inter\Diamond\Box$$

$(*)$ D' does not contain undischarged assumptions.

New modal rules:

$\Diamond \mathbb{W} \vee$ conversion	$\Box \mathbb{W} \vee$ conversion
$$\frac{\begin{matrix} D \\ \Diamond(\phi \mathbb{W} \psi) \end{matrix}}{\Diamond\phi \vee \Diamond\psi}\ \mathit{Conv} \Diamond \mathbb{W} \vee$$	$$\frac{\begin{matrix} D \\ \Box(\phi \mathbb{W} \psi) \end{matrix}}{\Box\phi \vee \Box\psi}\ \mathit{Conv} \Box \mathbb{W} \vee$$

$\Diamond$ separation

$$\frac{\begin{array}{c} D \\ \Diamond(\phi \vee (\psi \wedge \mathrm{NE})) \end{array}}{\Diamond\psi}\ \Diamond Sep$$

$\Diamond$ join

$$\frac{\begin{array}{cc} D_1 & D_2 \\ \Diamond\phi & \Diamond\psi \end{array}}{\Diamond(\phi \vee \psi)}\ \Diamond Join$$

$\Box$ instantiation

$$\frac{\begin{array}{c} D \\ \Box(\phi \wedge \mathrm{NE}) \end{array}}{\Diamond\phi}\ \Box Inst$$

$\Box\Diamond$ join

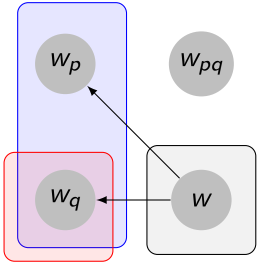
$$\frac{\begin{array}{cc} D_1 & D_2 \\ \Box\phi & \Diamond\psi \end{array}}{\Box(\phi \vee \psi)}\ \Box\Diamond Join$$

$s \models \Diamond\phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing$ and $t \models \phi$

$s \models \Box\phi \iff \forall w \in s : R[w] \models \phi$

$$\frac{\diamond(\phi \vee (\psi \wedge \mathrm{NE}))}{\diamond\psi} \diamond Sep$$

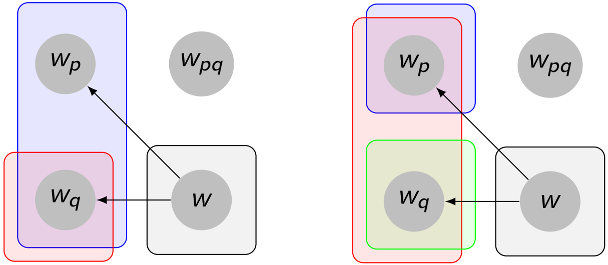
D



$s \vDash \diamond(p \vee (q \wedge \mathrm{NE}))$
$s \vDash \diamond q$

$s \vDash \diamond\phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing$ and $t \vDash \phi$
$s \vDash \Box\phi \iff \forall w \in s : R[w] \vDash \phi$

$$\frac{\Diamond(\phi \vee (\psi \wedge \mathrm{NE}))}{\Diamond\psi}\, D \quad \Diamond Sep \qquad\qquad \frac{\Diamond\phi \qquad \Diamond\psi}{\Diamond(\phi\vee\psi)}\, D_1 \; D_2 \quad \Diamond Join$$



$s \models \Diamond(p \vee (q \wedge \mathrm{NE}))$

$s \models \Diamond q$

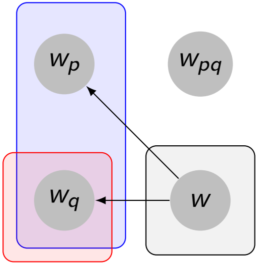
$s \models \Diamond p \wedge \Diamond q$

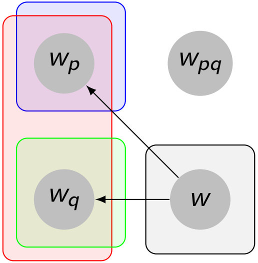
$s \models \Diamond(p \vee q)$

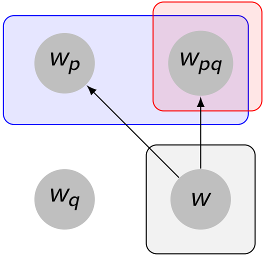
$s \models \Diamond\phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing$ and $t \models \phi$

$s \models \Box\phi \iff \forall w \in s : R[w] \models \phi$

$$\dfrac{\begin{array}{c} D \\ \Diamond(\phi \vee (\psi \wedge \mathrm{NE})) \end{array}}{\Diamond\psi}\ \Diamond Sep \qquad \dfrac{\begin{array}{cc} D_1 & D_2 \\ \Diamond\phi & \Diamond\psi \end{array}}{\Diamond(\phi \vee \psi)}\ \Diamond Join \qquad \dfrac{\begin{array}{cc} D_1 & D_2 \\ \Box\phi & \Diamond\psi \end{array}}{\Box(\phi \vee \psi)}\ \Box\Diamond Join$$







$s \models \Diamond(p \vee (q \wedge \mathrm{NE}))$

$s \models \Diamond q$

$s \models \Diamond p \wedge \Diamond q$

$s \models \Diamond(p \vee q)$

$s \models \Box p \wedge \Diamond q$

$s \models \Box(p \vee q)$

$s \models \Diamond\phi \iff \forall w \in s : \exists t \subseteq R[w] : t \neq \emptyset$ and $t \models \phi$

$s \models \Box\phi \iff \forall w \in s : R[w] \models \phi$

Example derivation:
$\Diamond\phi \vdash \Diamond(\phi \wedge \mathrm{NE})$ (taking $\Diamond\bot \vdash \bot$ as proven)

$$
\begin{prooftree}
\AxiomC{$\Diamond\bot$}
\UnaryInfC{$\bot$}
\AxiomC{$\Diamond\phi$}
\AxiomC{}
\RightLabel{$\mathrm{NE}I$}
\UnaryInfC{$\bot \text{ ⩔ } \mathrm{NE}$}
\AxiomC{$[\bot]$}
\RightLabel{⩔I}
\UnaryInfC{$\bot \text{ ⩔ } (\phi \wedge \mathrm{NE})$}
\AxiomC{$[\phi]$}
\AxiomC{$[\mathrm{NE}]$}
\RightLabel{$\wedge I$}
\BinaryInfC{$(\phi \wedge \mathrm{NE})$}
\RightLabel{⩔I}
\UnaryInfC{$\bot \text{ ⩔ } (\phi \wedge \mathrm{NE})$}
\RightLabel{⩔E}
\TrinaryInfC{$\bot \text{ ⩔ } (\phi \wedge \mathrm{NE})$}
\RightLabel{$\Diamond Mon$}
\BinaryInfC{$\Diamond(\bot \text{ ⩔ } (\phi \wedge \mathrm{NE}))$}
\RightLabel{$\Diamond$ ⩔ $\vee Conv$}
\UnaryInfC{$\Diamond\bot \vee \Diamond(\phi \wedge \mathrm{NE})$}
\RightLabel{$\vee Sub$}
\BinaryInfC{$\bot \vee \Diamond(\phi \wedge \mathrm{NE})$}
\RightLabel{$\bot E$}
\UnaryInfC{$\Diamond(\phi \wedge \mathrm{NE})$}
\end{prooftree}
$$

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Use the disjunctive normal form and a strategy developed in [7]:

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Lemma: $\boxed{\phi \in BSML^{\mathbb{W}} \implies \forall k \geq \text{modal depth}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee\!\!\!\!\bigvee_{(M,s) \in P} \Theta^k_s}$

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$$\phi \vDash \psi$$

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$$\phi \vDash \psi \implies \bigvee_{(M,s)\in P} \Theta^k_s \vDash \bigvee_{(N,t)\in Q} \Theta^k_t$$

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Lemma: $\boxed{\phi \in BSML^{\mathbb{W}} \quad \Longrightarrow \quad \forall k \geq \text{modal depth}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee\!\!\!\!\bigvee_{(M,s)\in P} \Theta^k_s}$

$$\phi \vDash \psi \quad \Longrightarrow \quad \bigvee\!\!\!\!\bigvee_{(M,s)\in P} \Theta^k_s \vDash \bigvee\!\!\!\!\bigvee_{(N,t)\in Q} \Theta^k_t$$

$$\Longrightarrow \forall (M,s) \in P : \exists (N,t) \in Q : \quad s \leftrightarroweq_k t$$

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Lemma: $\boxed{\phi \in BSML^{\mathbb{W}} \implies \forall k \geq \text{modal depth}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee\!\!\!\!\bigvee_{(M,s) \in P} \Theta_s^k}$

$$\phi \vDash \psi \quad \implies \quad \bigvee\!\!\!\!\bigvee_{(M,s) \in P} \Theta_s^k \vDash \bigvee\!\!\!\!\bigvee_{(N,t) \in Q} \Theta_t^k$$

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$$\implies \bigvee_{(M,s)\in P} \Theta_s^k \vdash \bigvee_{(N,t)\in Q} \Theta_t^k \implies \phi \vdash \psi$$

BSML axiomatization

We exclude the $\mathbb{W}$-rules (and $\vee\mathrm{NE}E$) from $BSML^{\mathbb{W}}$ but add rules which simulate the $\mathbb{W}$-rules:

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$$\models \bot \mathbb{W} \mathrm{NE}$$

BSML

$\bot\mathrm{NE}$ translation

$$\dfrac{\begin{array}{c} \\ D \\ \phi \end{array} \qquad \begin{array}{c} [\phi(\psi \wedge \bot/[\psi, m])] \\ D_1 \\ \chi \end{array} \qquad \begin{array}{c} [\phi(\psi \wedge \mathrm{NE}/[\psi, m])] \\ D_2 \\ \chi \end{array}}{\chi}\ \bot\mathrm{NE}\,Trs(*)$$

$(*)$ The occurrence at index m is not within the scope of $\neg$ or $\Diamond$.

$BSML^{\mathbb{W}}$

NE introduction

$$\dfrac{}{\bot \mathbb{W} \mathrm{NE}}\ \mathrm{NE}I$$

$BSML$

$\Diamond \bot \mathrm{NE}$ translation

$$\dfrac{\begin{matrix} D \\ \Diamond\phi \end{matrix} \qquad \begin{matrix} [\phi(\psi \wedge \bot/[\psi, m])] \\ D_1 \\ \chi_1 \end{matrix} \qquad \begin{matrix} [\phi(\psi \wedge \mathrm{NE}/[\psi, m])] \\ D_2 \\ \chi_2 \end{matrix}}{\Diamond\chi_1 \vee \Diamond\chi_2} \; \Diamond\bot\mathrm{NE}\, Trs(*)$$

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D_1, D_2 do not contain undischarged assumptions.

$BSML^{\mathbb{W}}$

$\Diamond \mathbb{W} \vee$ conversion

$$\dfrac{\begin{matrix} D \\ \Diamond(\phi \mathbin{\mathbb{W}} \psi) \end{matrix}}{\Diamond\phi \vee \Diamond\psi} \; Conv \Diamond \mathbb{W} \vee$$

BSML

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$\Box\bot\mathrm{NE}$ translation

$$\dfrac{\begin{matrix} D \\ \Box\phi \end{matrix} \qquad \begin{matrix} [\phi(\psi\wedge\bot/[\psi,m])] \\ D_1 \\ \chi_1 \end{matrix} \qquad \begin{matrix} [\phi(\psi\wedge\mathrm{NE}/[\psi,m])] \\ D_2 \\ \chi_2 \end{matrix}}{\Box\chi_1 \vee \Box\chi_2}\ \Box\bot\mathrm{NE}\,Trs(*)$$

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$\Box \mathbb{W} \vee$ conversion

$$\dfrac{\begin{matrix} D \\ \Box(\phi \mathbin{\mathbb{W}} \psi) \end{matrix}}{\Box\phi \vee \Box\psi}\ Conv\,\Box\mathbb{W}\vee$$

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$BSML^{\oslash}$	$BSML^{\mathbb{W}}$
$\oslash$ introduction	$\mathbb{W}$ introduction
$\dfrac{\begin{matrix} D \\ \bot \end{matrix}}{\oslash\phi}\,\oslash I$ $\quad$ $\dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\oslash\phi}\,\oslash I$	$\dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\phi \mathbin{\mathbb{W}} \psi}\,\mathbb{W} I$ $\quad$ $\dfrac{\begin{matrix} D \\ \psi \end{matrix}}{\phi \mathbin{\mathbb{W}} \psi}\,\mathbb{W} I$

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$\oslash\mathrm{NE}$ introduction	NE introduction
$\dfrac{}{\oslash\mathrm{NE}}\ \oslash\mathrm{NE}I$	$\dfrac{}{\bot \mathbin{\mathbb{W}} \mathrm{NE}}\ \mathrm{NE}I$

$BSML^{\oslash}$ axiomatization

Exclude $\mathbin{\mathbb{V}}$-rules and $\vee\mathrm{NE}E$; and add:

$$\oslash\phi \equiv \phi \mathbin{\mathbb{V}} \bot \qquad\qquad \neg\oslash\phi \equiv \neg\phi$$

$BSML^{\oslash}$	$BSML^{\mathbin{\mathbb{V}}}$
$\oslash$ introduction	$\mathbin{\mathbb{V}}$ introduction
$\dfrac{\begin{matrix} D \\ \bot \end{matrix}}{\oslash\phi}\,\oslash I \qquad \dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\oslash\phi}\,\oslash I$	$\dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\phi \mathbin{\mathbb{V}} \psi}\,\mathbin{\mathbb{V}} I \qquad \dfrac{\begin{matrix} D \\ \psi \end{matrix}}{\phi \mathbin{\mathbb{V}} \psi}\,\mathbin{\mathbb{V}} I$
$\oslash\mathrm{NE}$ introduction	NE introduction
$\dfrac{}{\oslash\mathrm{NE}}\,\oslash\mathrm{NE}I$	$\dfrac{}{\bot \mathbin{\mathbb{V}} \mathrm{NE}}\,\mathrm{NE}I$
$\neg\oslash$ introduction	
$\begin{matrix} D \\ \neg\phi \\ \hline\hline \neg\oslash\phi \end{matrix}\;\neg\oslash I$	

$BSML^{\oslash}$

$\oslash$ elimination

$$\dfrac{\begin{matrix} \\ D \\ \phi \end{matrix} \qquad \begin{matrix} [\phi(\bot/[\oslash\psi, m])] \\ D_1 \\ \chi \end{matrix} \qquad \begin{matrix} [\phi(\psi/[\oslash\psi, m])] \\ D_2 \\ \chi \end{matrix}}{\chi} \oslash E(*)$$

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$$\dfrac{\begin{matrix} \\ D \\ \phi \mathbin{\mathbb{W}} \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ D_1 \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ D_2 \\ \chi \end{matrix}}{\chi} \mathbb{W} E$$

$BSML^{\oslash}$

$\Diamond\oslash$ elimination

$$\cfrac{\begin{matrix} D \\ \Diamond\phi \end{matrix} \qquad \begin{matrix} [\phi(\bot/[\oslash\psi, m])] \\ D_1 \\ \chi_1 \end{matrix} \qquad \begin{matrix} [\phi(\psi/[\oslash\psi, m])] \\ D_2 \\ \chi_2 \end{matrix}}{\Diamond\chi_1 \vee \Diamond\chi_2}\ \Diamond\oslash E(*)$$

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$BSML^{\oslash}$	$BSML^{\mathbb{W}}$

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$\Box \mathbb{W} \vee$ conversion

$$\dfrac{\begin{array}{c}D\\ \Box(\phi \mathbin{\mathbb{W}} \psi)\end{array}}{\Box\phi \vee \Box\psi}\ Conv\,\Box\mathbb{W}\vee$$

Rules for $\diamondsuit$:

$\diamondsuit\mathbb{W}$ distributivity

$$\dfrac{\dfrac{D}{\diamondsuit(\phi \mathbin{\mathbb{W}} \psi)}}{\diamondsuit\phi \mathbin{\mathbb{W}} \diamondsuit\psi}\ \textit{Distr}\ \diamondsuit\ \mathbb{W}$$

$\boxdot\mathbb{W}$ distributivity

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$\diamondsuit\vee$ distributivity

$$\dfrac{\dfrac{D}{\diamondsuit(\phi \vee \psi)}}{\diamondsuit\phi \vee \diamondsuit\psi}\ \textit{Distr}\ \diamondsuit\ \vee$$

$\mathrm{NE}\diamondsuit$ distributivity

$$\dfrac{\dfrac{D}{\diamondsuit\phi \wedge \mathrm{NE}}}{\diamondsuit(\phi \wedge \mathrm{NE})}\ \textit{Distr}\mathrm{NE}\diamondsuit$$

$\boxdot$ instantiation

$$\dfrac{\dfrac{D}{\boxdot\phi}}{\diamondsuit\phi \vee \boxdot\bot}\ \boxdot\textit{Inst}$$

$\boxdot\diamondsuit$ join

$$\dfrac{\dfrac{D_1}{\boxdot(\phi \vee \psi)} \qquad \dfrac{D_2}{(\diamondsuit\psi \wedge \mathrm{NE}) \vee \chi}}{\boxdot(\phi \vee (\psi \wedge \mathrm{NE}))}\ \boxdot\diamondsuit\ \textit{Join}$$

Completeness for *BSML*

The idea: simulate the disjunctive normal forms using "instantiations" [7]. In an instantiation ϕ_f for a formula ϕ, each atom η is replaced by some Θ^0_s such that $s \vDash \psi$:

$$\begin{array}{ccc} \phi & \Longrightarrow & \phi_f \\ p \vee (q \wedge \mathrm{NE}) & \Longrightarrow & \Theta^0_{s_p} \vee (\Theta^0_{s_q} \wedge \Theta^0_{\mathrm{NE}}) \end{array}$$

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Since for each atom η we have $\psi \equiv \mathbb{W}_{(M,s)\in P} \Theta^0_s$, where $P = \|\eta\| = \{(M,s) \mid M, s \vDash \psi\}$, then assuming that $\mathbb{W}$ distributes over everything:

$$\phi \equiv \mathbb{W} F_\phi$$

And given rules that simulate $\mathbb{W}$:

$$\forall \phi_f \in F_\phi : \phi_f \vdash \phi$$
$$\text{if } \forall \phi_f \in F_\phi : \Gamma, \phi_f \vdash \psi, \text{ then } \Gamma, \phi \vdash \psi$$

Problem: in *BSML*, $\mathbb{W}$ does not distribute over $\Diamond$. For instance $\Diamond(p \mathbb{W} q) \not\models \Diamond p \mathbb{W} \Diamond q$.

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Lemma ($\mathbb{W}$-distributive form): $\phi \in \text{BSML}$ implies $\phi \dashv\vdash \phi'$ where
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An instantiation ϕ_f of ϕ in $\mathbb{W}$-distributive form:

each NE is replaced by some $\Theta^0_{s_f}$ where $s_f \models \mathrm{NE}$

each $\eta \in \{p, \neg p, \Diamond\psi, \Box\psi\}$ is replaced by some $\chi^k_{s_f} = \bigvee_{w \in s_f} \chi^k_w$ $(s_f \models \eta;\ k = md(\eta))$

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$$\text{if } \forall \phi_f \in F_\phi : \Gamma, \phi_f \vdash \psi, \text{ then } \Gamma, \phi \vdash \psi$$

$$
\begin{aligned}
& \phi \vDash \psi \\
\Longrightarrow \quad & \mathbb{W} F_\phi \vDash \mathbb{W} F_\psi
\end{aligned}
$$

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$$\boxed{\begin{array}{l} \forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 : \\ \phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2} \end{array}}$$

$$\phi \models \psi$$

$$\implies \bigvee\!\!\!\!\bigvee F_\phi \models \bigvee\!\!\!\!\bigvee F_\psi$$

$$\implies \bigvee\!\!\!\!\bigvee_{(sf_1,sf_2)\in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \models \bigvee\!\!\!\!\bigvee_{(sg_1,sg_2)\in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2}$$

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$$\boxed{\begin{array}{l} \Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \mathbb{W}_{t \subseteq s_2} \Theta^k_{s_1 \uplus t} \\ \forall t \subseteq s_2 : \Theta^k_{s_1 \uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2} \\ [\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1 \uplus t} \vdash \psi] \\ \qquad \implies \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi \end{array}}$$

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$$\implies \bigvee F_\phi \vDash \bigvee F_\psi$$

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$$\implies \quad \mathop{\mathbb{W}}_{(sf_1, sf_2) \in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vDash \mathop{\mathbb{W}}_{(sg_1, sg_2) \in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2}$$

$$\boxed{\forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 : \phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2}}$$

$$\implies \quad \mathop{\mathbb{W}}_{(sf_1, sf_2) \in A} \mathop{\mathbb{W}}_{t \subseteq sf_2} \Theta^k_{sf_1 \uplus t} \vDash \mathop{\mathbb{W}}_{(sg_1, sg_2) \in B} \mathop{\mathbb{W}}_{u \subseteq sg_2} \Theta^k_{sg_1 \uplus u}$$

$$\implies \quad \forall (sf_1, sf_2) \in A, t \subseteq sf_2 : \exists (sg_1, sg_2) \in B, u \subseteq sg_2 :$$
$$\Theta^k_{sf_1 \uplus t} \Leftrightarrow_k \Theta^k_{sg_1 \uplus u}$$
$$\Theta^k_{sf_1 \uplus t} \dashv\vdash \Theta^k_{sg_1 \uplus u} \vdash \Theta^k_{sg_1} \vee \chi^k_{sg_2} \vdash \psi_g \vdash \psi$$

$$\implies \quad \forall (sf_1, sf_2) \in A : \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vdash \psi$$

$$\boxed{\begin{array}{l} \Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \mathop{\mathbb{W}}_{t \subseteq s_2} \Theta^k_{s_1 \uplus t} \\ \forall t \subseteq s_2 : \Theta^k_{s_1 \uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2} \\ [\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1 \uplus t} \vdash \psi] \\ \qquad \implies \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi \end{array}}$$

$$
\begin{aligned}
& \phi \vDash \psi \\
\Longrightarrow\quad & \bigvee\!\!\!\!\bigvee F_\phi \vDash \bigvee\!\!\!\!\bigvee F_\psi \\
\Longrightarrow\quad & \bigvee\!\!\!\!\bigvee_{(sf_1, sf_2)\in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vDash \bigvee\!\!\!\!\bigvee_{(sg_1, sg_2)\in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2} \\
\Longrightarrow\quad & \bigvee\!\!\!\!\bigvee_{(sf_1, sf_2)\in A} \bigvee\!\!\!\!\bigvee_{t\subseteq sf_2} \Theta^k_{sf_1\uplus t} \vDash \bigvee\!\!\!\!\bigvee_{(sg_1, sg_2)\in B} \bigvee\!\!\!\!\bigvee_{u\subseteq sg_2} \Theta^k_{sg_1\uplus u} \\
\Longrightarrow\quad & \forall (sf_1, sf_2) \in A, t \subseteq sf_2 : \exists (sg_1, sg_2) \in B, u \subseteq sg_2 : \\
& \Theta^k_{sf_1\uplus t} \leftrightarrows_k \Theta^k_{sg_1\uplus u} \\
& \Theta^k_{sf_1\uplus t} \dashv\vdash \Theta^k_{sg_1\uplus u} \vdash \Theta^k_{sg_1} \vee \chi^k_{sg_2} \vdash \psi_g \vdash \psi \\
\Longrightarrow\quad & \forall (sf_1, sf_2) \in A : \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vdash \psi \\
\Longrightarrow\quad & \forall \phi_f \in F_\phi : \phi_f \vdash \psi
\end{aligned}
$$

$$
\boxed{\begin{array}{l} \forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 : \\ \phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2} \end{array}}
$$

$$
\boxed{\begin{array}{l} \Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \bigvee\!\!\!\!\bigvee_{t\subseteq s_2} \Theta^k_{s_1\uplus t} \\ \forall t \subseteq s_2 : \Theta^k_{s_1\uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2} \\ [\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1\uplus t} \vdash \psi] \\ \qquad \Longrightarrow \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi \end{array}}
$$

$$\phi \models \psi$$

$$\implies \quad \mathbb{W} F_\phi \models \mathbb{W} F_\psi$$

$$\implies \quad \mathbb{W}_{(sf_1, sf_2) \in A} \Theta^k_{sf_1} \vee \chi^k_{sf_2} \models \mathbb{W}_{(sg_1, sg_2) \in B} \Theta^k_{sg_1} \vee \chi^k_{sg_2}$$

$$\implies \quad \mathbb{W}_{(sf_1, sf_2) \in A} \mathbb{W}_{t \subseteq sf_2} \Theta^k_{sf_1 \uplus t} \models \mathbb{W}_{(sg_1, sg_2) \in B} \mathbb{W}_{u \subseteq sg_2} \Theta^k_{sg_1 \uplus u}$$

$$\implies \quad \forall (sf_1, sf_2) \in A, t \subseteq sf_2 : \exists (sg_1, sg_2) \in B, u \subseteq sg_2 :$$
$$\Theta^k_{sf_1 \uplus t} \rightleftharpoons_k \Theta^k_{sg_1 \uplus u}$$
$$\Theta^k_{sf_1 \uplus t} \dashv\vdash \Theta^k_{sg_1 \uplus u} \vdash \Theta^k_{sg_1} \vee \chi^k_{sg_2} \vdash \psi_g \vdash \psi$$

$$\implies \quad \forall (sf_1, sf_2) \in A : \Theta^k_{sf_1} \vee \chi^k_{sf_2} \vdash \psi$$

$$\implies \quad \forall \phi_f \in F_\phi : \phi_f \vdash \psi$$

$$\implies \quad \phi \vdash \psi$$

$$\boxed{\begin{array}{l} \forall \phi_f : \forall k \geq md(\phi) : \exists sf_1, sf_2 : \\ \phi_f \dashv\vdash \Theta^k_{sf_1} \vee \chi^k_{sf_2} \end{array}}$$

$$\boxed{\begin{array}{l} \Theta^k_{s_1} \vee \chi^k_{s_2} \equiv \mathbb{W}_{t \subseteq s_2} \Theta^k_{s_1 \uplus t} \\ \forall t \subseteq s_2 : \Theta^k_{s_1 \uplus t} \vdash \Theta^k_{s_1} \vee \chi^k_{s_2} \\ [\forall t \subseteq s_2 : \Gamma, \Theta^k_{s_1 \uplus t} \vdash \psi] \\ \qquad \implies \Gamma, \Theta^k_{s_1} \vee \chi^k_{s_2} \vdash \psi \end{array}}$$

Completeness for $BSML^{\oslash}$

Lemma:

$$\boxed{\phi \in BSML^{\oslash} \implies \forall k \geq \mathrm{md}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s)\in P} \oslash\Theta_s^k \quad \text{or} \quad \phi \dashv\vdash (\bigvee_{(M,s)\in P} \oslash\Theta_s^k) \wedge \mathrm{NE}}$$

Completeness for $BSML^{\oslash}$

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$\phi \vDash \psi$

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$$\phi \vDash \psi \quad \implies \quad \bigvee_{(M,s)\in P} \oslash\Theta^k_s \vDash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

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$$\implies \forall (M,s) \in P : \exists R \subseteq Q : \quad s \leftrightarrows_k \biguplus R$$

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$$\phi \models \psi \quad \implies \quad \bigvee_{(M,s)\in P} \oslash\Theta^k_s \models \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

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$$\boxed{\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \equiv \mathbb{W}_{R\subseteq Q} \Theta^k_{\biguplus R}}$$

Completeness for $BSML^{\oslash}$

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$$\boxed{\phi \in BSML^{\oslash} \implies \forall k \geq \mathrm{md}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s)\in P} \oslash\Theta^k_s \quad \text{or} \quad \phi \dashv\vdash (\bigvee_{(M,s)\in P} \oslash\Theta^k_s) \wedge \mathrm{NE}}$$

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$$\Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

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Completeness for $BSML^{\oslash}$

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$$\phi \vDash \psi \quad \implies \quad \bigvee_{(M,s)\in P} \oslash\Theta^k_s \vDash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

$$\implies \forall (M,s) \in P : \exists R \subseteq Q : \quad \begin{array}{l} s \leftrightarrows_k \biguplus R \\ \Theta^k_s \vdash \bigvee_{(N,t)\in R} \oslash\Theta^k_t \\ \Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t \\ \oslash\Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t \end{array}$$

$$\boxed{\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \equiv \mathop{\mathbb{W}}_{R\subseteq Q} \Theta^k_{\biguplus R}}$$

Completeness for $BSML^{\oslash}$

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$$\phi \in BSML^{\oslash} \implies \forall k \geq \mathrm{md}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s)\in P} \oslash\Theta^k_s \quad \text{or} \quad \phi \dashv\vdash (\bigvee_{(M,s)\in P} \oslash\Theta^k_s) \wedge \mathrm{NE}$$

$$\phi \vDash \psi \quad \implies \quad \bigvee_{(M,s)\in P} \oslash\Theta^k_s \vDash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

$$\implies \forall (M,s) \in P : \exists R \subseteq Q : \quad s \leftrightarrows_k \biguplus R$$

$$\Theta^k_s \vdash \bigvee_{(N,t)\in R} \oslash\Theta^k_t$$

$$\Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

$$\oslash\Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

$$\boxed{\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \equiv \mathop{\mathbb{W}}_{R\subseteq Q} \Theta^k_{\biguplus R}}$$

$$\implies \bigvee_{(M,s)\in P} \oslash\Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

Completeness for $BSML^{\oslash}$

Lemma:

$$\boxed{\phi \in BSML^{\oslash} \implies \forall k \geq \mathrm{md}(\phi) : \exists P : \quad \phi \dashv\vdash \bigvee_{(M,s)\in P} \oslash\Theta^k_s \quad \text{or} \quad \phi \dashv\vdash \Big(\bigvee_{(M,s)\in P} \oslash\Theta^k_s\Big) \wedge \mathrm{NE}}$$

$$\phi \models \psi \quad \implies \quad \bigvee_{(M,s)\in P} \oslash\Theta^k_s \models \bigvee_{(N,t)\in Q} \oslash\Theta^k_t$$

$$\implies \forall (M,s) \in P : \exists R \subseteq Q : \quad s \leftrightarrows_k \biguplus R$$

$$\Theta^k_s \vdash \bigvee_{(N,t)\in R} \oslash\Theta^k_t$$

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$$\boxed{\bigvee_{(N,t)\in Q} \oslash\Theta^k_t \equiv \mathbb{W}_{R\subseteq Q} \Theta^k_{\biguplus R}}$$

$$\implies \bigvee_{(M,s)\in P} \oslash\Theta^k_s \vdash \bigvee_{(N,t)\in Q} \oslash\Theta^k_t \quad \implies \quad \phi \vdash \psi$$

Recall that for classical α: $\alpha \equiv \oslash(\alpha \wedge \mathrm{NE})$

Using $\oslash$ we can define a function which cancels pragmatic enrichment:

$$
\begin{array}{lcl}
p^{-} & := & \oslash\, p \\
\mathrm{NE}^{-} & := & \oslash\, \mathrm{NE} \\
(\neg\phi)^{-} & := & \oslash\, \neg\phi^{-} \\
(\phi \wedge \psi)^{-} & := & \oslash\, \phi^{-} \wedge \oslash\psi^{-} \\
(\phi \vee \psi)^{-} & := & \oslash\, \phi^{-} \vee \oslash\psi^{-} \\
(\Diamond\phi)^{-} & := & \oslash\, \Diamond\phi^{-}
\end{array}
$$

For classical $\alpha : (\alpha^{+})^{-} \equiv \alpha$.

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