

Definitions: Similarity Analysis of Counterfactuals

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1 Language

Definition 1.1 (Language). Let Var be a non-empty set of propositional atoms p, q, r, \dots . The language $\mathcal{L}(\rightsquigarrow)$ is generated by

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \supset \varphi) \mid (\varphi \rightsquigarrow \varphi)$$

where $p \in \text{Var}$

Definition 1.2 (Proposition expressed by a formula). Let M be a model with set of worlds W . For each $\varphi \in \mathcal{L}(\rightsquigarrow)$, the proposition expressed by φ in M is

$$[\![\varphi]\!]_M := \{w \in W : M, w \models \varphi\}$$

When M is fixed, we write $[\![\varphi]\!]$

2 Similarity Frames and Models

Definition 2.1 (Similarity frame). A similarity frame is a pair

$$F = \langle W, \prec \rangle$$

such that

1. $W \neq \emptyset$ is a set of possible worlds.
2. for each $w \in W$ there is a set $W_w \subseteq W$ and a binary relation \prec_w on W_w satisfying
 - (a) irreflexivity: for all $u \in W_w$, $u \not\prec_w u$
 - (b) transitivity: for all $u, v, z \in W_w$, if $u \prec_w v$ and $v \prec_w z$, then $u \prec_w z$

For $u, v \in W_w$, the intended reading of $u \prec_w v$ is that u is more similar to w than v is.

Definition 2.2 (Field of a world). Let $F = \langle W, \prec \rangle$ be a similarity frame. For each $w \in W$, the field W_w of w is the domain on which \prec_w is defined. Equivalently

$$W_w = \{u \in W : \exists v \in W (u \prec_w v \text{ or } v \prec_w u)\}$$

Definition 2.3 (Reflexive closure of \prec_w). Let $F = \langle W, \prec \rangle$ be a similarity frame and $w \in W$. The reflexive closure \preceq_w of \prec_w on W_w is defined by

$$u \preceq_w v \quad \text{iff} \quad (u \prec_w v) \text{ or } (u = v)$$

for all $u, v \in W_w$

Definition 2.4 (Similarity model). A similarity model is a triple

$$M = \langle W, \prec, V \rangle$$

where $\langle W, \prec \rangle$ is a similarity frame and

$$V : \text{Var} \times W \rightarrow \{0, 1\}$$

is a valuation assigning a classical truth value to each pair (p, w) with $p \in \text{Var}$ and $w \in W$.

3 Truth and Logical Consequence

Definition 3.1 (Truth at a world). Let $M = \langle W, \prec, V \rangle$ be a similarity model. The satisfaction relation $M, w \models \varphi$ (φ is true at w in M) is defined inductively for $w \in W$ by

$$\begin{aligned} M, w \models p &\quad \text{iff} \quad V(p, w) = 1 \\ M, w \models \neg\varphi &\quad \text{iff} \quad M, w \not\models \varphi \\ M, w \models \varphi \wedge \psi &\quad \text{iff} \quad M, w \models \varphi \text{ and } M, w \models \psi \\ M, w \models \varphi \vee \psi &\quad \text{iff} \quad M, w \models \varphi \text{ or } M, w \models \psi \\ M, w \models \varphi \supset \psi &\quad \text{iff} \quad M, w \not\models \varphi \text{ or } M, w \models \psi \end{aligned}$$

The clause for $\varphi \rightsquigarrow \psi$ is given in definitions 4.1 and 4.4.

Definition 3.2 (Logical consequence). Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\rightsquigarrow)$. We write

$$\Gamma \models \varphi$$

iff for every similarity model M and every world w in M the following holds:

$$\text{If } M, w \models \gamma \text{ for all } \gamma \in \Gamma, \text{ then } M, w \models \varphi$$

4 Counterfactual Semantics

Definition 4.1 (General similarity clause for counterfactuals). Let $M = \langle W, \prec, V \rangle$ be a similarity model, $w \in W$, and $\varphi, \psi \in \mathcal{L}(\rightsquigarrow)$. Then

$$\begin{aligned} M, w \models \varphi \rightsquigarrow \psi &\quad \iff \quad \forall u \in W_w \cap [\![\varphi]\!] \ \exists u' \in [\![\varphi]\!] \text{ such that} \\ (i) \quad u' &\preceq_w u \\ (ii) \quad \forall u'' \in [\![\varphi]\!] \ (u'' \preceq_w u' \Rightarrow M, u'' \models \psi) \end{aligned}$$

Definition 4.2 (Limit Assumption). Let $F = \langle W, \prec \rangle$ be a similarity frame and $w \in W$. The Limit Assumption holds at w iff the relation \prec_w on W_w is well founded, in the sense that either of the following equivalent conditions holds

1. there is no infinite sequence (u_1, u_2, u_3, \dots) of worlds in W_w with $u_{n+1} \prec_w u_n$ for all $n \geq 1$
2. every non-empty subset $X \subseteq W_w$ has a \prec_w -minimal element, that is, some $u \in X$ such that there is no $v \in X$ with $v \prec_w u$

The frame F satisfies the Limit Assumption iff it holds at every $w \in W$.

Definition 4.3 (Minimal φ -worlds). Let $M = \langle W, \prec, V \rangle$ be a similarity model, $w \in W$, and $\varphi \in \mathcal{L}(\rightsquigarrow)$. The set of \prec_w -minimal φ -worlds at w is

$$\text{Min}_w(\varphi) := \{u \in W_w \cap \llbracket \varphi \rrbracket : \neg \exists v \in W_w \cap \llbracket \varphi \rrbracket (v \prec_w u)\}$$

Elements of $\text{Min}_w(\varphi)$ are the closest φ -worlds to w (if any)

Definition 4.4 (Closest-worlds clause under the Limit Assumption). Let $M = \langle W, \prec, V \rangle$ be a similarity model and $w \in W$ such that the Limit Assumption holds at w . For $\varphi, \psi \in \mathcal{L}(\rightsquigarrow)$ we have

$$M, w \models \varphi \rightsquigarrow \psi \iff \forall u \in \text{Min}_w(\varphi) M, u \models \psi$$

5 Frame Conditions and Corresponding Principles

We can match simple structural constraints on the similarity ordering \prec_w with characteristic valid schemas for the counterfactual connective \rightsquigarrow .

Weak Centering (WC) $\forall w \forall v (v \neq w \rightarrow \neg(v \prec_w w))$

(Modus Ponens for \rightsquigarrow) $((\varphi \rightsquigarrow \psi) \wedge \varphi) \supset \psi$

Strong Centering (SC) $\forall w \forall v (v \neq w \rightarrow w \prec_w v)$

(Conjunctive Sufficiency) $(\varphi \wedge \psi) \supset (\varphi \rightsquigarrow \psi)$

Connectedness + Limit $\forall w \forall u \forall v (u \neq v \rightarrow (u \prec_w v \vee v \prec_w u))$

and, in addition, well-foundedness of each \prec_w

(CEM) $(\varphi \rightsquigarrow \psi) \vee (\varphi \rightsquigarrow \neg \psi)$

Almost-Connectedness (AC) $\forall w \forall u \forall v \forall z (u \prec_w z \rightarrow (u \prec_w v \vee v \prec_w z))$

(ASP) $(\neg(\varphi \rightsquigarrow \neg \psi) \wedge (\varphi \rightsquigarrow \chi)) \supset ((\varphi \wedge \psi) \rightsquigarrow \chi)$

6 Axiomatic System P

Definition 6.1 (Axioms of P). Let φ, ψ, χ range over $\mathcal{L}(\rightsquigarrow)$. The proof system P has as axioms all instances of the following schemes:

TAUT Every instance of a classical propositional tautology in the language $\mathcal{L}(\rightsquigarrow)$ (with \rightsquigarrow treated as an additional connective).

CI $\varphi \rightsquigarrow \varphi$.

CC $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \supset (\varphi \rightsquigarrow (\psi \wedge \chi))$.

CW $(\varphi \rightsquigarrow \psi) \supset (\varphi \rightsquigarrow (\psi \vee \chi))$.

ASC $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \supset ((\varphi \wedge \psi) \rightsquigarrow \chi)$.

AD $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \supset ((\varphi \vee \psi) \rightsquigarrow \chi)$.

Definition 6.2 (Inference rules of P). The rules of inference for P are the following:

MP (Modus Ponens) From φ and $\varphi \supset \psi$, infer ψ .

REA (Replacement of equivalents in antecedent)

From $\varphi \supset \psi$, infer $(\varphi \rightsquigarrow \chi) \supset (\psi \rightsquigarrow \chi)$

REC (Replacement of equivalents in consequent)

From $\varphi \supset \psi$, infer $(\chi \rightsquigarrow \varphi) \supset (\chi \rightsquigarrow \psi)$

Definition 6.3 (Derivability in P). Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\rightsquigarrow)$. We write

$$\Gamma \vdash_P \varphi$$

iff there exists a finite sequence $\varphi_1, \dots, \varphi_n$ of formulas with $\varphi_n = \varphi$ such that for each $i \leq n$ one of the following holds:

1. φ_i is an instance of one of the axioms in definition 6.1;
2. $\varphi_i \in \Gamma$;
3. φ_i is obtained from earlier members of the sequence by one of the rules in definition 6.2.

When $\Gamma = \emptyset$ we simply write $\vdash_P \varphi$ and call φ a theorem of P.

Theorem 6.1 (Soundness and completeness of P). For all $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\rightsquigarrow)$,

$$\Gamma \vdash_P \varphi \iff \Gamma \models \varphi,$$

where \models is the semantic consequence relation of definition 3.2 evaluated on similarity models $\langle W, \prec, V \rangle$ with the counterfactual clauses of definition 4.1 (equivalently, under the Limit Assumption, of definition 4.4).