Truthmakers[1]

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[1] Slides marked with $*$ were not covered during the lecture, but they are included for completeness. You can skip them for what concerns the exam material of the course.

Readings

Suggested:

- Chapter 5 in lecture notes.
- Rodriguez-Pereyra, G. (2006). Truthmakers. *Philosophy Compass*, 1(2), 186-200.
- van Fraassen, B. C. (1969). Facts and tautological entailments. *The Journal of Philosophy*, 66(15), 477-487.
- Fine, K. (2017). A theory of truthmaker content I: Conjunction, disjunction and negation. *Journal of Philosophical Logic*, 46(6), 625-674.
- Fine, K. (2017). Truthmaker semantics. *A Companion to the Philosophy of Language*, 556-577.
- Fine, K., & Jago, M. (2019). Logic for exact entailment. *The review of symbolic logic*, 12(3), 536-556.

Plan

1. Truthmakers
2. Truthmaker Semantics
3. State Spaces*
4. Exact Truthmaking

Outline

1. Truthmakers
2. Truthmaker Semantics
3. State Spaces*
4. Exact Truthmaking

What is a truthmaker?

- Consider some ordinary sentences:
  1. *The grass is green.*
  2. *I am here.*
  3. *Dinosaurs existed.*
  4. *Unicorns do not exist.*
- All of these are (or at least seem to be) true.
- But what in reality **makes** them true?
- Is 'I am here' true because I am here in the world, or does my being here somehow depend on the truth of the sentence?
- Truthmaker theory takes seriously the idea that **the world makes our true sentences true**: the direction of explanation goes from world to word.

Truthbearers and truthmakers

- **Truthbearer**: something that can be true or false (sentences, beliefs, judgements, propositions, . . .).
- **Truthmaker**: something in the world *in virtue of which* a truthbearer is true.
- Example:
  - Truthbearer: 'The cat is on the mat.'
  - Truthmaker: that very cat's being on that very mat.
- Truthmaker theorists hold (roughly): whenever a truthbearer is true, there is some portion of reality that makes it true.
- What kinds of things can be truthmakers, and do *all* truths have them?

Why care about truthmakers?

- Truthmaker theory sharpens the old correspondence idea: true thoughts *match* reality.
- It forces us to connect:
  - claims about truth (logic, semantics),
  - with claims about what there is (metaphysics, ontology).
- Often used as an *argument for* or *against* certain entities:
  - If moral truths are real, what in the world makes them true?
  - Do modal truths (‘It could have been otherwise’) require possible worlds as truthmakers?

Truthmaking as entailment

- A tempting idea: connect truthmaking with logical entailment.

> If x makes p true and $p \models q$, then x also makes q true.

- Consider $p =$ 'Amsterdam is in the Netherlands', and let x be *the fact that you are at the lecture.*
- $p \vee \neg p$ is always true.
- On a naive view, *any* existing x could serve as a truthmaker for $p \vee \neg p$.
- If we also assume that a disjunction is made true only by one of its disjuncts, we are pushed to say that your merely being here makes it true that p.
- That seems wrong: your presence is irrelevant to whether Amsterdam is in the Netherlands.
- This motivates more refined notions such as *relevant* entailment (more on this in Robert's lecture). [Restall 1996]

Truthmaking as necessitation

- From a more metaphysical point of view, we can characterise truthmaking via **necessitation**:

> An entity T is a truthmaker for a proposition p iff
> T *necessitates* p being true.

- So once the truthmaker is in place, the truth could not have failed.
- This builds into truthmaking an explanatory order: truthmakers are prior to (and more fundamental than) the truths they make.
- It also gives us a handle on what kinds of entities can plausibly serve as truthmakers.

Truthmakers as objects

- A first, naive view is to identify truthmakers with ordinary objects.
- Example: ‘Socrates exists’ is made true by Socrates himself.
- But under a strict necessitation condition, a bare particular rarely suffices:
  - The rose could exist without being red.
  - So the rose alone does not necessitate the truth of ‘The rose is red’.
- Conclusion: if we want necessitating truthmakers, we usually need entities *more fine-grained* than just objects.

Truthmakers as states of affairs

- Many truthmaker theorists prefer states of affairs as truthmakers.
- Rough idea: a state of affairs has the form

$$\langle \text{object(s)}, \text{property/relation} \rangle$$

  actually instantiating that property or relation.
- The truth of 'The rose is red' is made true by the state of affairs:

  the rose's being red.

- The truth of 'John greeted Mary' is made true by the relational state of affairs:

  John's greeting Mary.

- Unlike bare objects, such states of affairs cannot exist without the corresponding sentence being true, so they are good candidates for necessitating truthmakers.

Other candidates

- Besides states of affairs, philosophers have proposed:
  - **Events**: particularly for truths about what happens or changes.
  - **Tropes**: particularised properties (e.g. this very redness) as truthmakers for predicative truths.
  - ...
- For today we will mostly talk in the neutral vocabulary of states of affairs/facts.

Which truths have truthmakers?

- Two broad positions:
- **Maximalism**: every truth has a truthmaker (Armstrong 2004).
- **Non-maximalism**: some truths lack truthmakers; there are *truthmaker gaps* (e.g. Cameron 2008).
- Maximalists get a very tight connection between truth and reality, but they also inherit demanding cases:
  - negative truths,
  - general and universal truths,
  - modal, mathematical, moral truths,
  - liar-style sentences.
- Non-maximalists avoid some metaphysical costs by denying that all truths need truthmakers.

Challenges to maximalism: hard truths

- Maximalism has intuitive appeal: truth is always grounded in how things are.
- But some kinds of truths look especially problematic:
- **Negative truths**
  - 'Amsterdam is not in Italy.'
- **Non-existence truths**
  - 'There are no unicorns.'
  - 'Pegasus does not exist.'
- What in the world could *make* such truths true?

Strategies for negative truths

- Two influential maximalist strategies:
- **Negative facts**
  - For each true negative sentence, there is a corresponding negative fact.
  - Example: the fact that there are no unicorns.
  - Objection: this populates our ontology with many strange absence-like entities.
- **Totality facts**
  - Start with all the positive facts about what exists and how it is.
  - Add a *totality* fact saying: and that is everything.
  - From this, it follows that there are no further things (such as unicorns).
  - Objection: totality facts are metaphysically heavyweight.

A liar-style challenge to maximalism

Let M be the sentence:

'This sentence has no truthmaker.'

- Milne (2005) argues:
- Suppose, for reductio, that M has a truthmaker T.
- Then M is true. So what M says is the case is the case: M has no truthmaker.
- On the supposition that M has a truthmaker, we deduce that M has *no* truthmaker.
- By reductio ad absurdum, M has no truthmaker.
- But that is exactly what M says. So M is true and yet has no truthmaker.
- Hence M is (apparently) a truth without a truthmaker: a counterexample to maximalism, analogous to liar-style paradoxes for truth.

Outline

From metaphysics to semantics

Bas van Fraassen

Kit Fine

- So far, we have treated truthmaker theory as a **metaphysical** project:
  - what in reality makes a given sentence, belief, or proposition true?
  - which entities we must admit to be truthmakers?
- But there is also a more **formal** way of thinking about truthmakers:
  - build truthmakers *directly into the semantics* of a language,
  - so that the semantic value of a sentence tells us not just *when* it is true, but *what makes it true*.
- This brings us to what is now known as **truthmaker semantics**, first introduced by Bas van Fraassen and recently further developed and studied by Kit Fine.

Hyperintensionality

- In an **extensional** semantics, the meaning of a sentence is just its truth value in a model (relative to an assignment).
- **Intensional** semantics improves on this: meanings are functions from possible worlds to truth values (sets of worlds).
- But this is still not *fine-grained* enough.
- Consider:

  (1) Venus is Venus.

  (2) Venus is the morning star.

  (1) and (2) are necessarily equivalent (true in exactly the same worlds).
- Intensional semantics cannot distinguish them, but they clearly differ in *informational content* and in what would count as a truthmaker.
- We need a **hyperintensional** semantics: one that distinguishes necessarily equivalent sentences.

[Berto and Nolan 2023]

A note on formalisms and notation

- We will see a variety of ways to formalize truthmaker semantics.
- The point is not to confuse you. Rather, you should get used to the fact that the literature (not just on truthmaker semantics) exhibits:
  - different notations,
  - slightly different but equivalent formalizations,
  - different names for the same underlying notions.
- A useful skill:
  - be able to navigate between these presentations,
  - recognize when they amount to the *same* formalism,
  - and see which parts are notational choice vs. substantive.
- We start with the truthmaker semantics proposed by Bas van Fraassen (1969)[2]

[2] The presentation that follows is an adaptation of Bas van Fraassen (1969) as well as some 2020 posts in Bas van Fraassen blog.

Facts and their combination

We work with a countable set of *facts* Fct.

- Some facts are *atomic*.
- Other facts are *complex*, formed by *combination*.

There is a binary operation '·' on facts:

$$b, c \in \mathsf{Fct} \;\Rightarrow\; b \cdot c \in \mathsf{Fct}.$$

It satisfies:

$$b \cdot b = b \quad \text{(idempotent)}$$
$$b \cdot c = c \cdot b \quad \text{(commutative)}$$
$$b \cdot (c \cdot d) = (b \cdot c) \cdot d \quad \text{(associative)}$$

Atomic and complex facts

Facts: Every fact is a (possibly infinite) combination of atomic facts, and two facts are identical iff they are combinations of exactly the same atomic facts.

Atomic fact: A fact b is *atomic* iff for every fact c,

$$b = b \cdot c \ \Rightarrow \ b = c$$

- Complex facts are like mereological sums or “bundles” of atomic facts.
- The order of components in a bundle is irrelevant.

Subordination of facts

For facts $b, c \in \mathsf{Fct}$:

$$b \leq c \quad \text{iff} \quad \exists d \in \mathsf{Fct} \text{ such that } b = d \cdot c$$

$b \leq c$ means: b is at least as *informative* as c, because b contains all the atomic facts that c contains (and maybe more).[3]

- $\leq$ is reflexive:
$$b = b \cdot b \;\Rightarrow\; b \leq b$$
- $\leq$ is transitive: if $b \leq c$ and $c \leq e$ then $b \leq e$.
- $\leq$ is antisymmetric (recall identity of facts): if $a \leq b$ and $b \leq a$, then $a = b$.
- So it is a partial order.

[3] Keep in mind that we will reverse the conceptual order of $b \leq c$ when moving to Fine's formalization.

Propositions as closed sets of facts

For $X \subseteq \mathsf{Fct}$, we define its closure under $\leq$ as:

$$[X] \;=\; \{y \in \mathsf{Fct} : \exists x \in X \text{ with } y \leq x\}$$

A set $Q \subseteq \mathsf{Fct}$ is a *proposition* iff it is closed under $\leq$, i.e.

$$Q = [Q]$$

The relation $e \triangleright Q$ (*e makes Q true*) is defined by:

$$e \triangleright Q \quad \text{iff} \quad e \in Q$$

If $e \triangleright Q$ and $f \leq e$, then $f \triangleright Q$ as well, by closure.

The lattice of propositions*

Let Prop be the set of all propositions.

- Partial order: $Q \leq R$ iff $Q \subseteq R$
- Meet: $Q \wedge R = Q \cap R$
- Join: $Q \vee R = [Q \cup R]$

Any intersection of propositions is a proposition, and any union has a closure. Thus $(\mathsf{Prop}, \leq)$ is a complete lattice.

Special structure of this lattice*

In this specific construction, something stronger holds.

For any propositions Q, R:

$$Q \cap R = \{e \cdot f : e \in Q,\ f \in R\}$$

For any propositions Q, R:

$$[Q \cup R] = Q \cup R$$

So *every union of two propositions is already closed*.

Thus, for propositions:

$$Q \wedge R = Q \cap R \qquad Q \vee R = Q \cup R$$

Hence the lattice Prop is distributive.

Complementation on atomic facts

On the set of atomic facts there is an operation $b \mapsto \bar{b}$ such that:

- $\bar{b}$ is atomic,
- $\bar{\bar{b}} = b$ for every atomic fact b.

Think of b as "the apple is red", and $\bar{b}$ as "the apple is non-red".

Complex facts *do not* in general have complements (there are no disjunctive facts in the ontology).

Hence, there is no global Boolean complement on the set of propositions Prop. Negation must be handled indirectly, via the propositions expressed by sentences.

Language and primary bases

Fix a propositional language with connectives $\neg, \wedge, \vee$.

For each atomic sentence p, pick an atomic fact e:

$$\mathsf{T}(p) = \{e\} \qquad \mathsf{F}(p) = \{\overline{e}\}$$

For $X, Y \subseteq \mathsf{Fct}$ we define:

$$X \cdot Y \;=\; \{e \cdot f : e \in X,\ f \in Y\}$$

Recursive definition of T- and F-bases

For all sentences A, B:

$$\mathsf{T}(\neg A) = \mathsf{F}(A) \qquad \mathsf{F}(\neg A) = \mathsf{T}(A)$$

$$\mathsf{T}(A \wedge B) = \mathsf{T}(A) \cdot \mathsf{T}(B) \qquad \mathsf{F}(A \wedge B) = \mathsf{F}(A) \cup \mathsf{F}(B)$$

$$\mathsf{T}(A \vee B) = \mathsf{T}(A) \cup \mathsf{T}(B) \qquad \mathsf{F}(A \vee B) = \mathsf{F}(A) \cdot \mathsf{F}(B)$$

- $\mathsf{T}(A)$: the set of primary facts that make A true.
- $\mathsf{F}(A)$: the set of primary facts that make A false.

Examples

Fix atomic sentences p, q, r with basic facts e_p, e_q, e_r and complements $\overline{e_p}, \overline{e_q}, \overline{e_r}$. Then

$$\mathsf{T}(p) = \{e_p\},\ \mathsf{F}(p) = \{\overline{e_p}\}, \quad \ldots$$

(For sets X, Y of facts, $X \cdot Y = \{e \cdot f : e \in X, f \in Y\}$.)

1. $p \wedge \neg q$ and $\neg p \vee q$

$$\mathsf{T}(p \wedge \neg q) = \{e_p \cdot \overline{e_q}\} \quad \mathsf{F}(p \wedge \neg q) = \{\overline{e_p},\ e_q\}$$

$$\mathsf{T}(\neg p \vee q) = \{\overline{e_p},\ e_q\} \quad \mathsf{F}(\neg p \vee q) = \{e_p \cdot \overline{e_q}\}$$

2. $A = (p \wedge q) \vee \neg r$

$$\mathsf{T}(A) = \{e_p \cdot e_q,\ \overline{e_r}\} \qquad \mathsf{F}(A) = \{\overline{e_p} \cdot e_r,\ \overline{e_q} \cdot e_r\}$$

3. $B = (p \wedge q) \vee (p \wedge r)$**,** $C = p \wedge (q \vee r)$

$$\mathsf{T}(B) = \{e_p \cdot e_q,\ e_p \cdot e_r\}\ =\ \mathsf{T}(C)$$

From bases to propositions of sentences

For each sentence A, we determine two propositions:

$$\mathsf{T}^*(A) \;=\; [\mathsf{T}(A)] \qquad \mathsf{F}^*(A) \;=\; [\mathsf{F}(A)]$$

So $\mathsf{T}^*(A)$ is the proposition (closed set of facts) that A is true, and $\mathsf{F}^*(A)$ the proposition that A is false.

(Later we will identify $\mathsf{T}(A)$ with the set of *exact* truthmakers of A, and $\mathsf{T}^*(A)$ with the set of *inexact* truthmakers of A)

We have:

$$\mathsf{T}^*(A \wedge B) = \mathsf{T}^*(A) \cap \mathsf{T}^*(B) \qquad \mathsf{T}^*(A \vee B) = \mathsf{T}^*(A) \cup \mathsf{T}^*(B)$$

$$\mathsf{F}^*(A \wedge B) = \mathsf{F}^*(A) \cup \mathsf{F}^*(B) \qquad \mathsf{F}^*(A \vee B) = \mathsf{F}^*(A) \cap \mathsf{F}^*(B)$$

Examples

Fix atomic sentences p, q, r with basic facts e_p, e_q, e_r and complements $\overline{e_p}, \overline{e_q}, \overline{e_r}$. For facts f, g, recall $f \leq g$ for "f is subordinate to g". Then:

1. Atoms and their negations

$$\mathsf{T}^*(p) = [\{e_p\}] = \{f : f \leq e_p\} \qquad \mathsf{T}^*(\neg p) = [\{\overline{e_p}\}] = \{f : f \leq \overline{e_p}\}$$

2. Conjunction and disjunction

$$\mathsf{T}(p \wedge q) = \{e_p \cdot e_q\} \;\Rightarrow\; \mathsf{T}^*(p \wedge q) = [\{e_p \cdot e_q\}] = \{f : f \leq e_p \cdot e_q\}$$

$$\mathsf{T}(p \vee q) = \{e_p, e_q\} \;\Rightarrow\; \mathsf{T}^*(p \vee q) = [\{e_p, e_q\}] = \{f : f \leq e_p \text{ or } f \leq e_q\}$$

3. A more complex formula

$$A = (p \wedge q) \vee \neg r$$

From $\mathsf{T}(A) = \{e_p \cdot e_q,\ \overline{e_r}\}$ we get

$$\mathsf{T}^*(A) = [\{e_p \cdot e_q,\ \overline{e_r}\}] = \{f : f \leq e_p \cdot e_q \text{ or } f \leq \overline{e_r}\}$$

Facts about double negation and order*

For every sentence A:

$$\mathsf{T}^*(\neg\neg A) = \mathsf{T}^*(A) \qquad \mathsf{F}^*(\neg\neg A) = \mathsf{F}^*(A).$$

Indeed, $\mathsf{T}(\neg\neg A) = \mathsf{F}(\neg A) = \mathsf{T}(A)$ and $\mathsf{F}(\neg\neg A) = \mathsf{T}(\neg A) = \mathsf{F}(A)$, so their closures coincide:

$$\mathsf{T}^*(\neg\neg A) = [\mathsf{T}(\neg\neg A)] = [\mathsf{T}(A)] \quad \mathsf{F}^*(\neg\neg A) = [\mathsf{F}(\neg\neg A)] = [\mathsf{F}(A)]$$

If $\mathsf{T}^*(A) \subseteq \mathsf{T}^*(B)$, then

$$\mathsf{T}^*(\neg B) \subseteq \mathsf{T}^*(\neg A)$$

If every truth-maker for A is also a truth-maker for B, then every truth-maker for $\neg B$ is also a truth-maker for $\neg A$.

Outline

State Spaces*

van Fraassen formulation identifies a proposition with a closed set of facts. Fine (2017) instead takes a proposition to be any set of facts and further discusses different notions of propositions.

There is a non-empty set S of *states* (facts, events, conditions, ...), partially ordered by parthood $\leq$ and closed under *fusion*:

$$s_1, s_2, \dots \in S \;\Rightarrow\; s_1 \sqcup s_2 \sqcup \dots \in S$$

We work with a structured state space

$$\langle S, \leq, \sqcup \rangle$$

Distinguished extremal states:

- *null* state 0: fusion of no states; $0 \leq s$ for all $s \in S$,
- *full* state 1: fusion of all states; $s \leq 1$ for all $s \in S$.

Propositions and constraints*

A (unilateral) **proposition** is a set of verifying states:

$$P \subseteq S$$

Two propositions are identical iff they have the same verifiers.

Fine imposes structural conditions on such P:

- *Verifiability*: every proposition has at least one verifier:

$$P \neq \varnothing$$

- *Closure*: if $s_1, s_2, \dots \in P$, then their fusion is in P:

$$s_1 \sqcup s_2 \sqcup \dots \in P$$

- *Convexity*: if $s, u \in P$ and $s \leq t \leq u$, then $t \in P$

Different choices of constraints give different **domains** of propositions:

- **Full** domain: no structural constraints.
- **Semi-regular** domain: convexity only.
- **Regular** domain: closure + convexity (often also verifiability).

Regular propositions and subject-matter*

In a **regular** domain, each non-empty proposition $P \subseteq S$ is:

- *closed*: closed under (finite / arbitrary) fusions.
- *convex*: if $s, u \in P$ and $s \leq t \leq u$, then $t \in P$.

Such a P has a **maximal verifier**

$$p = \bigsqcup P$$

the fusion of all its verifiers. This p is the *subject-matter* of P: the largest state within which P is evaluated.

Let the *base verifiers* of P be the $\leq$-minimal elements of P:

$$L(P) = \{p_i \in P : \neg \exists t \in P \ (t < p_i)\}$$

In a regular domain we can recover P from $L(P)$ and p:

$$P = \{t \in S : \exists p_i \in L(P) \ (p_i \leq t \leq p)\}$$

This gives a restricted **monotonicity inside the subject-matter**:

$$s \in P \ \& \ s \leq t \leq p \ \Rightarrow \ t \in P$$

Verifiers must be "stable": not below any p_i, and not above p.

Boolean operations on unilateral propositions*

We work first with *unilateral* propositions, identified with sets of verifiers $P, Q \subseteq S$.

Conjunction: exact fusion of verifiers:

$$P \wedge Q \;=\; \{p \sqcup q : p \in P,\; q \in Q\}$$

A state verifies $P \wedge Q$ iff it is (exactly) a fusion of a verifier of P and a verifier of Q.

Disjunction: pool verifiers and then close as required by the chosen domain:

$$\begin{aligned} P \vee Q &= P \cup Q && \text{(full domain)} \\ P \vee Q &= (P \cup Q)^{*} && \text{(semi-regular: convex closure)} \\ P \vee Q &= (P \cup Q)^{**} && \text{(regular: convex + fusion closure)} \end{aligned}$$

So $\wedge$ corresponds to *mereological fusion* of truthmakers. $\vee$ corresponds to *choice* between them, plus structural closure.

Bilateral propositions and negation*

- A **bilateral proposition** is an ordered pair

$$P = (P^+, P^-)$$

where $P^+ \subseteq S$ are its verifiers and $P^- \subseteq S$ its falsifiers.

- We extend the structural constraints componentwise (P is regular, semi-regular, etc. iff both P^+ and P^- are.)
- **Negation** just swaps positive and negative content:

$$\neg(P^+, P^-) = (P^-, P^+)$$

- For conjunction and disjunction we combine the unilateral operations on both sides:

$$(P^+, P^-) \wedge (Q^+, Q^-) = (P^+ \wedge Q^+,\ P^- \vee Q^-)$$
$$(P^+, P^-) \vee (Q^+, Q^-) = (P^+ \vee Q^+,\ P^- \wedge Q^-)$$

- This yields a compositional truthmaker semantics: the semantic value of a sentence A is a (bi)proposition. The value of complex sentences is obtained by these operations.

States, possibility, and worlds*

- We distinguish a set $S^{\Diamond} \subseteq S$ of *possible* (consistent) states, assumed **downward closed**:

$$s \in S^{\Diamond} \;\&\; t \leq s \;\Rightarrow\; t \in S^{\Diamond}$$

- **Compatibility**: two states s, t are (jointly) *compatible* iff they can co-exist in some possible state

$$s \text{ and } t \text{ compatible} \iff \exists u \in S^{\Diamond}\ (s \leq u \;\&\; t \leq u)$$

  Thus: s and t can both obtain together without inconsistency.

- A *world* is a **maximal possible state**: a state $w \in S^{\Diamond}$ such that any possible state compatible with w is already part of w:

  w is a world $\iff$ $w \in S^{\Diamond}$ and $\forall s \in S^{\Diamond}(s$ compatible with $w \Rightarrow s \leq w)$

Bilateral propositions and truth / falsity at a world*

Let w be a world (maximal possible state).

Truth at w:

$$P \text{ is true at } w \iff \exists s \in P^+ \text{ with } s \leq w$$

Falsity at w:

$$P \text{ is false at } w \iff \exists s \in P^- \text{ with } s \leq w$$

Exclusivity, exhaustivity, and bivalence*

- **Exclusivity**: no verifier of P is compatible with any falsifier of P.

  $$\forall s \in P^+ \; \forall t \in P^- \; \neg(s \text{ compatible with } t)$$

  So at no world w is P both true and false (no gluts).

- **Exhaustivity**: every possible state is compatible with some verifier or some falsifier of P.

  $$\forall s \in S^\Diamond \; \exists t \in P^+ \cup P^- \; (s \text{ compatible with } t)$$

  So at every world w, P is at least true or false (no gaps).

- Hence, if P satisfies both conditions, then for every world w exactly one of P *is true at* w and P *is false at* w holds: P is bivalent at worlds.

Extremal propositions*

From the extremal states 0 and 1 we get four **extremal unilateral propositions**:

$$T_0 = \{0\} \quad F_0 = \emptyset \quad T_1 = S \quad F_1 = \{1\}$$

- T_0: trivially true, *nothing* beyond 0 is required.
- T_1: trivial truth verified by *any* state.
- F_0: trivial falsehood, no state can verify it.
- F_1: trivial falsehood verified only by the "impossible" full state 1.

Two kinds of consequence: entailment and containment*

For verifiable *unilateral* propositions $P, Q \subseteq S$ Fine defines two distinct consequence relations:

- P **entails** Q (notation: $P \leq_d Q$) iff

  $$P \subseteq Q$$

  Every verifier of P is also a verifier of Q.

  Disjunction $P \vee Q$ is the *least upper bound* of P and Q w.r.t. $\leq_d$: it is the strongest proposition entailed by both.

- Q **contains** P (notation: $P \leq_c Q$) iff:

  $$\begin{array}{ll} \text{(i)} & \forall q \in Q\ \exists p \in P\ (p \leq q) \\ \text{(ii)} & \forall p \in P\ \exists q \in Q\ (p \leq q) \end{array}$$

  Conjunction $P \wedge Q$ is the *greatest lower bound* of P and Q w.r.t. $\leq_c$ (in a regular domain).

Consequences for bilateral propositions*

For bilateral $P = (P^+, P^-)$ and $Q = (Q^+, Q^-)$, we define two consequence relations:

- [Containment] P is **contained in** Q iff

$$P^+ \leq_c Q^+ \quad \text{and} \quad P^- \leq_d Q^-$$

- [Entailment] P **entails** Q iff

$$P^+ \leq_d Q^+ \quad \text{and} \quad P^- \leq_c Q^-$$

These two consequence relations interact nicely with negation:

$$P \leq_d Q \Rightarrow \neg Q \leq_c \neg P$$
$$P \leq_c Q \Rightarrow \neg Q \leq_d \neg P$$

Fine: three notions of verification*

Let $\mathbf{A}$ be the proposition corresponding to A. Depending on which closure conditions we impose, $\mathbf{A}$ may be any of

$$|A|, \quad |A|^*, \quad |A|_*, \quad |A|^{**}$$

(no closure; fusion-closed; convex; regular = fusion-closed + convex).

Relative to a model $\mathcal{M}$, Fine distinguishes three verification relations on states s:

Exact: $s \models_{\text{ex}} A \iff s \in \mathbf{A}$

Inexact: $s \models_{\text{in}} A \iff \exists s' \leq s\ (s' \models_{\text{ex}} A)$

Loose (classical): $s \models A \iff \forall u(u\ comp\ s \Rightarrow \exists t \in \mathbf{A}\ (u\ comp\ t))$

where $u\ comp\ s$ means: u is *compatible* with s in the underlying modal state space.

Fine: notions of consequence*

From these verification relations we obtain corresponding consequence relations (between formulas A and C):

Exact	$A \models_{\mathsf{e}} C \iff \forall s(s \models_{\mathsf{ex}} A \Rightarrow s \models_{\mathsf{ex}} C)$
Inexact	$A \models_{\mathsf{i}} C \iff \forall s(s \models_{\mathsf{in}} A \Rightarrow s \models_{\mathsf{in}} C)$
Analytic (containment)	$A > C \iff [A] \geq_c [C]$
Loose / classical	$A \models_{\mathsf{l}} C \iff \forall s(s \models A \Rightarrow s \models C)$

Each induces a different logic:

- $\models_{\mathsf{l}}$: classical consequence.
- $>$: Angell's analytic entailment.
- $\models_{\mathsf{i}}$: first-degree entailment.
- $\models_{\mathsf{e}}$: exact truthmaker logic (Correia, Fine & Jago).

Outline

Exact truthmaking

- On the *necessitation* picture, any state that guarantees A counts as a truthmaker for A.
- Often this is too coarse: we want to exclude states with *irrelevant* parts.
- **Exact truthmaker**: a state s which
  - guarantees that A is true, and
  - is wholly relevant to A's truth: no proper part of s is "irrelevant" w.r.t. A.
- If s is an exact truthmaker for A, then every part of s either belongs to a truthmaker for A or is needed to fuse such parts together.
- Fine & Jago (2019) study the logic of *exact truthmaker semantics*.

Frames and exact truthmaking models

- A **frame** is a structure

$$\langle S, \leq \rangle$$

where:

  - S is a non-empty set of *states*
  - $\leq$ is a partial order (parthood / inclusion)
  - every pair $s, t \in S$ has a least upper bound (fusion) $s \sqcup t$

- An **exact truthmaking model** is

$$\mathcal{M} = \langle S, \leq, I \rangle$$

where $I = (I^+, I^-)$ is a pair of functions $S \times P \to \{0, 1\}$ satisfying $\sqcup$-closure:

If $I^+(s, p) = 1$ and $I^+(s', p) = 1$, then $I^+(s \sqcup s', p) = 1$
If $I^-(s, p) = 1$ and $I^-(s', p) = 1$, then $I^-(s \sqcup s', p) = 1$

Exact truthmaking and falsemaking: clauses

Fix a model $\mathcal{M} = \langle S, \leq, I \rangle$ We define, by recursion on φ, two relations $\models^+$ and $\models^-$ for positive and negative exact truthmaking:

$s \models^+ p$ iff $I^+(s, p) = 1$
$s \models^- p$ iff $I^-(s, p) = 1$

$s \models^+ \neg\varphi$ iff $s \models^- \varphi$
$s \models^- \neg\varphi$ iff $s \models^+ \varphi$

$s \models^+ \varphi \wedge \psi$ iff there are $s', s'' \in S$ with $s' \sqcup s'' = s$ and $s' \models^+ \varphi$ and $s'' \models^+ \psi$
$s \models^- \varphi \wedge \psi$ iff $s \models^- \varphi$ or $s \models^- \psi$ or there are $s', s'' \in S$ with $s' \sqcup s'' = s$ and $s' \models^- \varphi$ and $s'' \models^- \psi$

$s \models^+ \varphi \vee \psi$ iff $s \models^+ \varphi$ or $s \models^+ \psi$ or there are $s', s'' \in S$ with $s' \sqcup s'' = s$ and $s' \models^+ \varphi$ and $s'' \models^+ \psi$
$s \models^- \varphi \vee \psi$ iff there are $s', s'' \in S$ with $s' \sqcup s'' = s$ and $s' \models^- \varphi$ and $s'' \models^- \psi$

Logical Consequence[4]

$\Gamma \models_{\mathsf{TM}} \varphi$ iff all models M and states s, if $s \models^{+} \psi$ for all $\psi \in \Gamma$, then $s \models^{+} \varphi$.

[4]We discuss here the *distributive* notion of logical consequence. Fine & Jago (2019) also consider a collective one, which takes the conjunction of all the premises.

Exactness and the failure of $\wedge$-elimination

- Exactness is built into the clauses for $\models^+$: from

$$s \models^+ p$$

  we do *not* infer that $s \sqcup t \models^+ p$ for arbitrary t. Adding irrelevant parts does not preserve exact truthmaking.

- Example: $\mathcal{M} = \langle S, \leq, I \rangle$:
  - States $S = \{s_1, s_2, s_3\}$ with $s_3 = s_1 \sqcup s_2$.
  - Atomic valuation:

$$I^+(s_1, p) = 1 \qquad I^+(s_2, q) = 1 \qquad I^+(s_3, p) = 0$$

- By the conjunction clause for $\models^+$:

$$s_3 \models^+ (p \wedge q)$$

- But $s_3 \not\models^+ p$
- Hence we **lose** $\wedge$-elimination for exact entailment:

$$p \wedge q \not\models_{\mathsf{TM}} p$$

Exact disjunction: “exclusive” vs “inclusive”

For conjunction, exact truthmaking works in a purely mereological way:

$$s \models^+ p \wedge q \;\text{ iff }\; \exists t, u(s = t \sqcup u,\ t \models^+ p,\ u \models^+ q)$$

For disjunction there are two natural options.

- **Exclusive disjunction:**

  $$s \models^+_{\text{exc}} p \vee q \;\text{ iff }\; s \models^+ p \text{ or } s \models^+ q$$

  Only states that already make one disjunct true themselves count as exact truthmakers for $p \vee q$.

  $p \wedge q \;\not\models_{\text{TM}}\; p \vee q$

- **Inclusive disjunction:**

  $$s \models^+_{\text{inc}} p \vee q \;\text{ iff }$$
  $$s \models^+ p \text{ or } s \models^+ q$$
  $$\text{or } \exists t, u(s = t \sqcup u,\ t \models^+ p,\ u \models^+ q)$$

  $p \wedge q \;\models_{\text{TM}}\; p \vee q$

In-class examples

- Examples and models on blackboard
- $(p \vee q) \wedge (p \vee r) \not\models_{\mathsf{TM}} p \vee (q \wedge r)$
- $(p \wedge q) \vee p \not\models_{\mathsf{TM}} p$

The latter also shows that an exact verifier does not have to be minimal:

s minimally verifies φ if $s \models^+ \varphi$ and for any $s' \leq s$ s.t. $s' \models^+ \varphi$, $s' = s$.

From exact truthmaking to van Fraassen's primary bases

For each formula A define its exact truthmaking and falsemaking sets:

$$|A|^+ := \{s \in S : s \models^+ A\} \qquad |A|^- := \{s \in S : s \models^- A\}$$

Using the clauses and fusion-closure of $I^\pm$, one shows by induction on A that $|A|^+$ and $|A|^-$ are closed under finite fusions $s \sqcup t$.

We take the semantic clauses in their **exclusive** version, and we now recover van Fraassen's primary T- and F-bases from $|A|^+$ and $|A|^-$.

Fact-like assumption: For each propositional atom p there are distinguished *atomic* states $e_p, \overline{e_p} \in S$ such that $|p|^+ = \{e_p\}$ and $|p|^- = \{\overline{e_p}\}$

From exact truthmaking to van Fraassen's primary bases

For every formula A set

$$T(A) := |A|^+ \qquad F(A) := |A|^-$$

By induction on A, the $T(A), F(A)$ just defined satisfy van Fraassen's recursive primary bases:

$$\begin{aligned} T(p) &= \{e_p\} & F(p) &= \{\overline{e_p}\} \\ T(\neg A) &= F(A) & F(\neg A) &= T(A) \\ T(A \wedge B) &= \{s \sqcup t : s \in T(A),\ t \in T(B)\} & F(A \wedge B) &= F(A) \cup F(B) \\ T(A \vee B) &= T(A) \cup T(B) & F(A \vee B) &= \{s \sqcup t : s \in F(A),\ t \in F(B)\} \end{aligned}$$

van Fraassen's (inexact) propositions are the closures $\mathsf{T}^*(A) = \mathrm{cl}(T(A))$, $\mathsf{F}^*(A) = \mathrm{cl}(F(A))$ as before.

Metalogical features

- Although the exact truthmaker consequence relation is unusual, it is very well behaved:
  - **Compact**: if $\Gamma \models_{\mathsf{TM}} \varphi$, then some finite $\Delta \subseteq \Gamma$ already satisfies $\Delta \models_{\mathsf{TM}} \varphi$.
  - **Decidable**: there is an effective procedure to determine, for finite Γ, whether $\Gamma \models_{\mathsf{TM}} \varphi$.
- Fine & Jago (2019) also provide a sequent calculus for $\models_{\mathsf{TM}}$, which is sound, complete, and admits cut-elimination.

Application: Imperatives (Fine 2017)*

- There is an intuitive notion of *imperative consequence*:
  - From *Turn on the light and shut the door* one can infer *Turn on the light*.
- Naive idea: reduce imperative consequence to indicative consequence:
  - Let X correspond to an indicative A, and Y to B.
  - Say: Y follows from X iff B follows (classically) from A.
- But this gives Ross's paradox:
  - Indicative: from *You turn on the light* we can infer *You turn on the light or burn the building down*.
  - Imperative: from *Turn on the light* we *cannot* infer *Turn on the light or burn the building down*.
- Truthmaker semantics suggests a different route: give a *direct* semantics for imperatives in terms of actions, parallel to the semantics for indicatives in terms of states.

Imperatives: exact compliance*

- Extend the state space so that **actions** are a special kind of state.
- For an imperative sentence X:
  - An action α is in *exact compliance* with X (intuitively: exactly doing what X requires, no more and no less).
  - An action α is in *exact contravention* of X (intuitively: exactly violating X).
- Analogy with indicatives:
  - Indicative: s exactly verifies *You shut the door* iff s is *just* your shutting the door.
  - Imperative: α is in exact compliance with *Shut the door* iff α is *just* your shutting the door (shutting the door *and* turning on the light is not exact compliance).

Imperative content*

- For a conjunctive imperative $X \wedge Y$:

  $$\alpha \text{ is in exact compliance with } X \wedge Y$$

  iff there are actions β, γ such that:

  1. $\alpha = \beta \sqcup \gamma$
  2. β exactly complies with X
  3. γ exactly complies with Y

- **Content of an imperative** X:

  $$|X|^{\mathsf{imp}} = \{\alpha : \alpha \text{ is in exact compliance with } X\}$$

Imperative consequence as necessary means*

- Given contents $|X|^{\text{imp}}$ and $|Y|^{\text{imp}}$, Fine defines **imperative consequence** by a *containment* (or necessary-means) condition.
- Obeying X always involves, and can be decomposed into, obeying Y plus (possibly) more.
- Formally, Y follows from X iff the content of Y is a *conjunctive part* of the content of X:
  (i) For every $\alpha \in |X|^{\text{imp}}$, there is some $\beta \in |Y|^{\text{imp}}$ with $\beta \leq \alpha$.
  (ii) For every $\beta \in |Y|^{\text{imp}}$, there is some $\alpha \in |X|^{\text{imp}}$ with $\beta \leq \alpha$.
- Then Y is a *necessary means* to X.
- This dissolves Ross's paradox:
  - The content of *Turn on the light or burn the building down* is not a conjunctive part of the content of *Turn on the light*. So the imperative inference fails.
  - But the content of *Turn on the light is* a conjunctive part of the content of *Turn on the light and shut the door*. So that inference is validated.