

Definitions: Non-monotonic Logic

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Throughout, let L be a propositional language based on a set of atoms Prop , and let \models denote classical consequence on L . A valuation on a non-empty set of worlds W is a function $V : W \times \text{Prop} \rightarrow \{0, 1\}$, and classical truth at worlds in a model, $\mathcal{M}, w \models \alpha$, is defined as usual.

1 System C: Cumulative Consequence

Definition 1.1 (System C and cumulative consequence relation). *A (single-conclusion) non-monotonic consequence relation \vdash on L is cumulative if it satisfies the following principles for all formulas $\varphi, \psi, \chi \in L$:*

1. **Reflexivity:** $\varphi \vdash \varphi$.
2. **Left logical equivalence:** if $\varphi \models \psi$ and $\psi \models \varphi$, and $\varphi \vdash \chi$, then $\psi \vdash \chi$.
3. **Right weakening:** if $\varphi \models \psi$ and $\chi \vdash \varphi$, then $\chi \vdash \psi$.
4. **Cut:** if $\varphi \wedge \psi \vdash \chi$ and $\varphi \vdash \psi$, then $\varphi \vdash \chi$.
5. **Cautious monotonicity:** if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \wedge \psi \vdash \chi$.

2 Cumulative Models

Definition 2.1 (Cumulative model). *Let W be a non-empty set of worlds. A cumulative model is a quadruple*

$$\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$$

such that:

1. S is a non-empty set of states;
2. $\ell : S \rightarrow \mathcal{P}(W) \setminus \{\emptyset\}$ labels each state with a non-empty set of worlds;
3. \prec is a binary relation on S ;
4. $V : W \times \text{Prop} \rightarrow \{0, 1\}$ is a classical valuation on W ;
5. for every formula $\alpha \in L$, the set $[\alpha]_{\mathcal{M}_C} = \{s \in S \mid \mathcal{M}_C, s \models \alpha\}$ is smooth with respect to \prec .

Definition 2.2 (State satisfaction). Let $\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$ be a cumulative model. For a state $s \in S$ and a formula $\alpha \in L$ we define:

$$\mathcal{M}_C, s \models \alpha \iff \forall w \in \ell(s) (\mathcal{M}_C, w \models \alpha)$$

Definition 2.3 (Truth set of a formula at states). Let $\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$ be a cumulative model. The truth set of a formula $\alpha \in L$ at the level of states is $[\![\alpha]\!]^{\mathcal{M}_C} = \{s \in S \mid \mathcal{M}_C, s \models \alpha\}$.

Definition 2.4 (Smoothness (for states)). Let (S, \prec) be a set of states equipped with a binary relation \prec . A subset $A \subseteq S$ is smooth (with respect to \prec) iff for every $s \in A$:

- either s is \prec -minimal in A , i.e. there is no $s' \in A$ such that $s' \prec s$;
- or there exists a \prec -minimal $s' \in A$ with $s' \prec s$.

Definition 2.5 (Cumulative consequence in a model). Let $\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$ be a cumulative model. The associated cumulative consequence relation $\vdash_{\mathcal{M}_C}$ is defined by:

$$\alpha \vdash_{\mathcal{M}_C} \beta \iff \text{for every } \prec\text{-minimal } s \in [\![\alpha]\!]^{\mathcal{M}_C}, \mathcal{M}_C, s \models \beta.$$

2.1 Soundness and completeness of system C

Definition 2.6 (Satisfaction in a cumulative model). Let K be a set of conditionals of the form $\alpha \succsim \beta$, and $\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$ a cumulative model. We say that \mathcal{M}_C satisfies K iff for every $\alpha \succsim \beta \in K$ we have

$$\alpha \vdash_{\mathcal{M}_C} \beta.$$

Definition 2.7 (Cumulative entailment). Let K be a set of conditionals. For formulas $\alpha, \beta \in L$ we write

$$K \models_C \alpha \succsim \beta$$

iff for every cumulative model \mathcal{M}_C , if \mathcal{M}_C satisfies K , then

$$\alpha \vdash_{\mathcal{M}_C} \beta.$$

In the special case $K = \emptyset$ we say that $\alpha \succsim \beta$ is valid in all cumulative models and write $\models_C \alpha \succsim \beta$.

Definition 2.8 (Derivability in system C). Let K be a set of conditionals. We write

$$K \vdash_C \alpha \succsim \beta$$

iff $\alpha \succsim \beta$ can be obtained from premises in K by a finite number of applications of the rules of system C. We write $\vdash_C \alpha \succsim \beta$ for $\emptyset \vdash_C \alpha \succsim \beta$.

Theorem 2.1 (Soundness of system C). If $K \vdash_C \alpha \succsim \beta$, then $K \models_C \alpha \succsim \beta$. In particular, every rule of system C preserves validity in cumulative models.

Theorem 2.2 (Completeness / representation for system C). Let \succsim be a non-monotonic consequence relation on L . Then the following are equivalent:

1. \vdash is cumulative (i.e. it satisfies all rules of system C).
2. There exists a cumulative model \mathcal{M}_C such that for all formulas $\alpha, \beta \in L$,

$$\alpha \vdash \beta \iff \alpha \vdash_{\mathcal{M}_C} \beta$$

As a consequence, for every set K of conditionals and all $\alpha, \beta \in L$:

$$K \vdash_C \alpha \vdash \beta \iff K \models_C \alpha \vdash \beta$$

3 Preferential Models and System P

Definition 3.1 (Preferential model). A preferential model is a triple

$$\mathcal{M}_P = \langle W, \prec, V \rangle$$

such that:

1. W is a non-empty set of worlds;
2. \prec is a strict partial order on W (irreflexive and transitive);
3. $V : W \times \text{Prop} \rightarrow \{0, 1\}$ is a classical valuation on W ;
4. for every formula $\alpha \in L$, the truth set

$$[\alpha]^{M_P} = \{w \in W \mid M_P, w \models \alpha\}$$

is smooth with respect to \prec .

Definition 3.2 (Smoothness (for worlds)). Let (W, \prec) be as above. A subset $A \subseteq W$ is smooth (with respect to \prec) iff for every $w \in A$:

- either w is \prec -minimal in A ,
- or there exists a \prec -minimal $w' \in A$ with $w' \prec w$.

Definition 3.3 (Preferential consequence). Let $\mathcal{M}_P = \langle W, \prec, V \rangle$ be a preferential model. The associated preferential consequence relation $\vdash_{\mathcal{M}_P}$ is defined by:

$$\alpha \vdash_{\mathcal{M}_P} \beta \iff \text{for every } \prec\text{-minimal } w \in [\alpha]^{M_P}, M_P, w \models \beta.$$

Thus from α we defeasibly infer β iff every most normal α -world satisfies β .

Definition 3.4 (System P). A non-monotonic consequence relation \vdash on L satisfies system P (preferential consequence) iff:

- it is cumulative in the sense of Definition 1.1, and
 - in addition it satisfies the rule:
- (Or)** If $\varphi \vdash \chi$ and $\psi \vdash \chi$, then $\varphi \vee \psi \vdash \chi$.

3.1 Knowledge bases and preferential entailment

Definition 3.5 (Default knowledge base). A (default) knowledge base is a set K of conditional assertions of the form

$$\alpha \vdash \beta$$

where $\alpha, \beta \in L$.

Definition 3.6 (Satisfaction of a knowledge base). Let K be a default knowledge base and $\mathcal{M}_P = \langle W, \prec, V \rangle$ a preferential model. We say that \mathcal{M}_P satisfies K iff for every conditional $\alpha \vdash \beta \in K$ we have

$$\alpha \vdash_{\mathcal{M}_P} \beta.$$

Definition 3.7 (Preferential entailment). Let K be a default knowledge base. For formulas $\alpha, \beta \in L$ we write

$$K \models_{\text{pref}} \alpha \vdash \beta$$

iff for every preferential model \mathcal{M}_P , if \mathcal{M}_P satisfies K , then

$$\alpha \vdash_{\mathcal{M}_P} \beta.$$

3.2 Soundness and completeness of system P

Definition 3.8 (Derivability in system P). Let K be a default knowledge base. We write

$$K \vdash_P \alpha \vdash \beta$$

iff $\alpha \vdash \beta$ can be obtained from premises in K by a finite number of applications of the rules of system P (i.e. the rules of system C together with rule (Or)). We write $\vdash_P \alpha \vdash \beta$ for $\emptyset \vdash_P \alpha \vdash \beta$.

Theorem 3.1 (Soundness of system P). If $K \vdash_P \alpha \vdash \beta$, then $K \models_{\text{pref}} \alpha \vdash \beta$. Equivalently, every rule of system P is valid in all preferential models.

Theorem 3.2 (Completeness / representation for system P). Let \vdash be a non-monotonic consequence relation on L . Then the following are equivalent:

1. \vdash satisfies all rules of system P.
2. There exists a preferential model \mathcal{M}_P such that for all formulas $\alpha, \beta \in L$,

$$\alpha \vdash \beta \iff \alpha \vdash_{\mathcal{M}_P} \beta$$

In particular, for every default knowledge base K and formulas $\alpha, \beta \in L$:

$$K \vdash_P \alpha \vdash \beta \iff K \models_{\text{pref}} \alpha \vdash \beta$$