Vagueness: other accounts

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Readings

Suggested:

- Cobreros, P. & Tranchini, L. (2019). *Supervaluationism, Subvaluationism and the Sorites Paradox*. In Sergi Oms & Elia Zardini (eds.), *The Sorites Paradox*. New York, NY: Cambridge University Press. pp. 38–62.
- Cobreros, P., Egré, P., Ripley, D., van Rooij, R. (2015). *Vagueness, Truth and Permissive Consequence*. In: Achourioti, T., Galinon, H., Martínez Fernández, J., Fujimoto, K. (eds) Unifying the Philosophy of Truth. Logic, Epistemology, and the Unity of Science, vol 36. Springer, Dordrecht.

Further reading:

- Williamson, T. (2002). *Vagueness*. Routledge.

Outline

Supertrue vs Subtrue

- **Supervaluationism** takes a sentence to be true just in case it is true in **all** of its possible precisifications.
- **Subvaluationism** takes a sentence to be true just in case it is true in **some** of its possible precisifications.
- Supervaluationism: vagueness as *underdetermination*. Borderline cases are neither supertrue nor superfalse.
- Subvaluationism: vagueness as *overdetermination*. Borderline cases are both subtrue and subfalse.

Subtrue and Subfalse

Definition (Subtruth & Subfalsity)

Let $V \neq \varnothing$ be a set of classical valuations. Using the relation $V, v \models \varphi$ from last lecture, we define for any formula φ:

$$\textbf{(Subtruth)} \quad V \models^{\exists 1} \varphi \iff \exists v \in V : \; V, v \models \varphi$$

$$\textbf{(Subfalsity)} \quad V \models^{\exists 0} \varphi \iff \exists v \in V : \; V, v \not\models \varphi$$

Global and Local consequence

Definition (Global subvaluationist consequence)

$\Gamma \models_g^{\exists} \varphi$ iff for all non-empty V, if $V \models^{\exists 1} \gamma$ for all $\gamma \in \Gamma$, then $V \models^{\exists 1} \varphi$.

Definition (Local subvaluationist consequence)

$\Gamma \models_l^{\exists} \varphi$ iff for all non-empty V, if for all $\gamma \in \Gamma$ there exists $v \in V$ with $V, v \models \gamma$, then there exists $v' \in V$ with $V, v' \models \varphi$.

Yet another logical consequence

Definition (Another consequence)

$\Gamma \models_{\forall}^{\exists} \varphi$ iff for all non-empty V, if there exists $v \in V$ with $V, v \models \gamma$ for all $\gamma \in \Gamma$, then there exists $v' \in V$ with $V, v' \models \varphi$.

$$\Gamma \models_{\forall}^{\exists} \varphi \iff \Gamma \models_{CL} \varphi$$

The latter holds over the base language. For the $(\Rightarrow)$ direction take a singleton $V = \{v\}$.

Global consequence

- Global subvaluationism and classical consequence do not coincide:

$$\Gamma \models_g^{\exists} \varphi \Rightarrow \Gamma \models_{CL} \varphi$$

$$\Gamma \models_g^{\exists} \varphi \not\Leftarrow \Gamma \models_{CL} \varphi$$

- Global subvaluationism is paraconsistent:

$$\{p, \neg p\} \not\models q$$

- Note: consequence is different from LP: $p \wedge \neg p \models_g^{\exists} q$ and $\{p, q\} \not\models_g^{\exists} p \wedge q$

Sets of Conclusions

- So far we worked with single-conclusion consequence $\Gamma \models \varphi$
- **What about a set of conclusions $\Gamma \models \Phi$?**

$$\Gamma \models \Phi \quad \Longleftrightarrow \quad \neg\exists\, v \,\big(v \models \Gamma \text{ and } v \models \neg\Phi\big), \quad \text{where } \neg\Phi := \{\neg\varphi : \varphi \in \Phi\}$$

Intuition: when all premises hold, at least one member of Φ must hold (no joint countermodel).

- Multiple-conclusion arguments may not mirror ordinary inference (cf. Steinberger 2011)
- But they yield a useful connection: global sup/sub-valuationist consequence are the dual of each other:

$$\Gamma \models_g \Phi \iff \neg\Phi \models_g^{\exists} \neg\Gamma$$

Bivalence fails in sup.s: $\not\models_g \{\varphi, \neg\varphi\}$
Split-consistency fails in sub.s: $\{\varphi, \neg\varphi\} \not\models_g^{\exists}$

The Sorites

V	$\varphi(1)$	$\varphi(2)$	$\varphi(3)$	$\varphi(4)$
v_1	1	0	0	0
v_2	1	1	0	0
v_3	1	1	1	0

Conditionals $C_k : \varphi(k) \to \varphi(k+1)$*:* each C_k is true at all $v \neq v_k$ and false at v_k.

$$\forall k : V \models^{\exists 1} C_k \quad \text{but} \quad V \not\models^{\exists 1} \bigwedge_k C_k$$

- Hence every step holds somewhere (it is subtrue), but not all together at one v.
- But the *argument is invalid* since **Modus Ponens can fail** for a conditional that is both subtrue and subfalse

MP is not valid under SbV

Let $V = \{v_1, v_2\}$ with $v_1(p) = 1,\ v_1(q) = 0$ and $v_2(p) = 0,\ v_2(q) = 0$.

$$\underbrace{V \models^{\exists 1} p}_{v_1} \quad \underbrace{V \models^{\exists 1} (p \to q)}_{v_2} \quad \underbrace{V \not\models^{\exists 1} q}_{\text{no } v}$$

So $p,\ (p \to q)\ \not\models^{\exists}_{g} q$.

Each conditional C_k is subtrue (witness $v \neq v_k$) and $\varphi(1)$ is subtrue, but $\varphi(N)$ need not be subtrue. The chain of MP steps can break at the world where some C_k is false.

Universal form of the Sorites

- Consider a single line form of the argument:
  $\big(p_1 \wedge (p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \ldots\big) \Rightarrow p_n$
- Then the argument is valid for subvaluationism.
- But there is no *single* v which makes all the steps true and hence the universal $\forall n\,(p_{n-1} \to p_n)$ is **not even subtrue.**
- Thus the Sorites is here blocked, even though the argument is valid.
- Subvaluationism is thus committed to different answers, depending on the form of the paradox.
- To be fair, also supervaluationism displays this asymmetry for sets of conclusions:

Supervaluationism ($\models_g$)

$$p \vee q \;\not\models_g\; \{p, q\}$$
$$\{p, q\} \;\models_g\; \{p, q\}$$

Subvaluationism ($\models_g^{\exists}$)

$$\{p, q\} \;\not\models_g^{\exists}\; p \wedge q$$
$$\{p, q\} \;\models_g^{\exists}\; \{p, q\}$$

Outline

The epistemic solution

Timothy Williamson

- **Sharp but unknowable.** Vague expressions have *sharp* extension/cutoffs. Our ignorance makes them appear vague.
- There is a precise number distinguishing *bald* from *not bald*, but **we do not know** it.
- Vagueness = **epistemic limitation**, not semantic indeterminacy.

Inexact knowledge & margins of error

- (Williamson, *Vagueness* 1994) Knowledge is **safe**: it must be stable under small changes.
- **Margin-for-error schema** (for a vague predicate P and metric d):

$$Know_\alpha(P(x)) \;\Rightarrow\; \forall y\big(d(x,y) \leq \alpha \;\Rightarrow\; P(y)\big)$$

- Intuition: if you *know* that x is P, then anything α-close to x must also be P. Near the sharp cutoff, this *forces* ignorance: neither *know P* nor *know not P* can hold.

Fixed margin models

A fixed-margin (Kripke) model $M = \langle W, R, V \rangle$ with metric d and error parameter $\alpha > 0$:

$$R(x,y) \text{ iff } d(x,y) \leq \alpha$$

Reading: $R(x,y)$ = "y is within the α-margin of x"

Assume for all $x, y, z \in W$:

$$d(x,y) = 0 \Leftrightarrow x = y, \qquad d(x,y) = d(y,x), \qquad d(x,z) \leq d(x,y) + d(y,z)$$

Consequences for R:

- **Reflexive** ($d(x,x) = 0 \leq \alpha$)
- **Symmetric**
- **Not necessarily transitive** ($d(x,z)$ may be 2α)

[Égré and Bonnay 2010; Williamson 2002]

Knowledge operator

$$M, x \models \Box\varphi \quad \text{iff} \quad \forall y \, (R(x, y) \Rightarrow M, y \models \varphi)$$

- $\Box\varphi$ at x: "I **know** φ throughout the α-neighborhood of x".
- We work with a simple line model: $W = \mathbb{N}$, $d(m, n) = |m - n|$, $\alpha = 1$.

[Égré and Bonnay 2010]

Example (line model, $\alpha = 1$)

$$\begin{tikzcd}[row sep=tiny]
0 \arrow[loop above] \arrow[r, leftrightarrow] & 1 \arrow[loop above] \arrow[r, leftrightarrow] & 2 \arrow[loop above] \arrow[r, leftrightarrow] & 3 \arrow[loop above] \arrow[r, leftrightarrow] & 4 \arrow[loop above] \arrow[r, leftrightarrow] & 5 \arrow[loop above] \arrow[r] & \cdots \\
p & p & p & p & \neg p & \neg p &
\end{tikzcd}$$

Take p as a vague predicate (e.g., “not a heap”). So small n satisfy p, large n satisfy $\neg p$.

- At 2: $\Box p$ holds (both $1, 2, 3$ satisfy p). At 5: $\Box\neg p$ holds.
- At the boundary $3, 4$: $\neg\Box p \wedge \neg\Box\neg p$ (ignorance).

Unknown sharp cutoff

- There is a **sharp cutoff** n^*. For $n < n^*$: p (not a heap). For $n \geq n^*$: $\neg p$ (heap).
- **Margin-for-error** forbids knowledge at the α-neighbors of n^*.
- At a clear case like 0, we naturally want very strong epistemic security about p: ideally $\Box p, \Box^2 p, \Box^3 p, \ldots$ all hold.
- Fixed-margin models block this ideal: they validate $\Box p$ at clear cases but *do not* generally validate all iterations $\Box^n p$. Epistemicist reply: as we iterate the knowledge operator, knowledge **"erodes"**.

Knowledge axioms

(1) $\Box\varphi \to \varphi$

Factivity: if I know φ, then φ is true.

(2) $\Box\varphi \to \Box\Box\varphi$

Positive introspection: if I know φ, then I know that I know φ.

(3) $\neg\Box\varphi \to \Box\neg\Box\varphi$

Negative introspection: if I don't know φ, then I know that I don't know φ.

- In the fixed-margin semantics, (1) is valid: knowledge is factive.
- If we also require (2) or (3) to be valid at all worlds, then the accessibility relation R must be **transitive**.
- But non-transitivity of R is exactly what allows the model to block the Sorites.
  So, on the epistemicist picture, (2) and (3) are *not* generally valid for inexact knowledge.

Assessing the epistemic response

- **Counterintuitive sharpness:** It posits **precise** cutoffs for *tall*, *heap*, etc.
- **Ignorance challenge:** If there *is* a cutoff, why can't competent speakers know it?
- **Higher-order ignorance:** Do we *know* the margin α? If not, do margins themselves admit margins? (Iterated ignorance.)

Outline

1. Subvaluationism

2. Epistemicism

3. Logic of ST

4. Contextualism

Strict vs. tolerant truth

Fix the Strong Kleene truth-functions on $\{0, i, 1\}$.

- **Strict truth** (s): $v(\varphi) = 1$.
- **Tolerant truth** (t): $v(\varphi) \neq 0$ (i.e. 1 or i).
- Duality: φ is t-true iff $\neg\varphi$ is not s-true (and vice versa).
- "strict" tracks full truth
- "tolerant" tracks non-falsity (or permissive assertability).

[Abdallah 1995; Bennett 1998; Cobreros et al. 2015]

Mixed consequence (four variants)

For $n, m \in \{\mathsf{s}, \mathsf{t}\}$:
$\Gamma \models_{nm} \varphi$ iff there is no Strong Kleene valuation v with $[\forall \gamma \in \Gamma,\ v \models_n \gamma] \,\&\, v \not\models_m \varphi$.

- $\models_{\mathsf{ss}}$ = K3 (preserves 1).
- $\models_{\mathsf{tt}}$ = LP (preserves $\neq 0$).
- $\models_{\mathsf{ts}}$ is *empty* (premises weaker than conclusions).
- $\models_{\mathsf{st}}$ = **ST**: from strict premises to tolerant conclusions.

$$\Gamma \models_{ST} \varphi \text{ iff } [\forall \gamma \in \Gamma,\ v(\gamma) = 1] \ \Rightarrow\ v(\varphi) \neq 0$$

Conditionals and why ST keeps Modus Ponens

$$v(A \to B) = \max(1 - v(A),\ v(B))$$

$$v(A \to B) \neq 0 \quad \Leftrightarrow \quad [v(A) \neq 1] \text{ or } [v(B) \neq 0].$$

So the object-language $\to$ mirrors the meta-pattern s$\Rightarrow$t:

if A is strictly true, then B is tolerantly true.

- ST validates **Modus Ponens** and the **Deduction Theorem.**

Classicality of ST (base language)

Theorem (ST = Classical Consequence)

For any Γ, φ in the base language,

$$\Gamma \models_{ST} \varphi \text{ iff } \Gamma \models_{\mathrm{CL}} \varphi$$

Prove the contrapositive:

- ($\Leftarrow$) Any classical countermodel is an ST countermodel.
- ($\Rightarrow$) Any ST countermodel can be *refined* by replacing each $\frac{1}{2}$ with 0 or 1 so as to yield a classical countermodel.

(Write out the proof in full: you may use a refinement argument from SK to classical valuations, which you need to prove by induction on formulas.)

Adding vagueness: ST with indifference (STVP)

- We extend the language with a crisp similarity predicate I_P for each vague P. With the *closeness proviso* on all valuations v:

$$v \models_{\mathsf{s}} aI_Pb \quad \text{iff} \quad v \models_{\mathsf{t}} aI_Pb \quad \text{iff} \quad |v(Pa) - v(Pb)| < 1$$

- Hence I_P is reflexive symmetric, but *not necessarily transitive*.
- **Tolerance (valid in STVP):**

$$Pa,\, aI_Pb \models_{STVP} Pb \qquad\qquad \models_{STVP} \forall xy\,(Px \wedge xI_Py \rightarrow Py)$$

Intuition: if Pa holds strictly and b is P-indistinguishable from a, then Pb holds at least tolerantly.

STVP: non-transitive consequence (Sorites blocked)

$$a \overset{I_P}{\text{———}} b \overset{I_P}{\text{———}} c$$

$$P(a) = 1 \qquad P(b) = \tfrac{1}{2} \qquad P(c) = 0$$

$\{Pa,\ aI_Pb\} \models_{STVP} Pb$ (valid)

$\{Pb,\ bI_Pc\} \models_{STVP} Pc$ (valid)

$\{Pa,\ aI_Pb,\ bI_Pc\} \not\models_{STVP} Pc$ (not valid)

- STVP validates *local* Tolerance but consequence is *non-transitive*, so the Sorites chain fails.

ST and the Sorites

- Keeps the intuitive *Tolerance* principle in conditional form.
- Blocks the paradox via *non-transitive* consequence once I_P is present.
- Preserves classical behavior (Modus Ponens, Deduction Theorem, ...) on the base language.

Outline

1. Subvaluationism

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Contextualism

- Vague predicates are **context-sensitive** (comparison class / standard shifts).
- The conditional premises in Sorites are *intuitively compelling*. Contextualism explains this pull via **context shifts.**

Indistinguishability Relation I

We can state the Sorites premise as:

If x is P and x is indistinguishable from y, then y is P.

$$(P(x) \;\wedge\; xI_Py) \;\rightarrow\; P(y)$$

What properties should I_P have? In particular, can I_P be *transitive*?

From “significant difference” $\succ_P$ to indistinguishability I_P

Define “significantly P-er than” by $x \succ_P y$. Then set:

$$xI_Py \ := \ \neg(x \succ_P y) \ \wedge \ \neg(y \succ_P x)$$

- If $\succ_P$ is a *strict weak order* (irreflexive, transitive, almost-connected)[1], then I_P is an **equivalence relation** (check this as an exercise). Hence transitive.
- To avoid transitivity, we use a more realistic **“just noticeable difference”** ordering.

[1] **Almost connectedness**: $\forall x \forall y \forall z \ \big(R(x,y) \rightarrow (R(x,z) \ \vee \ R(z,y))\big)$.

Semi orders

We define $\succ_P$ as a **semi-order**:

Irreflexive: $\forall x : \neg x \succ x$
Interval-order: $\forall x, y, v, w : (x \succ y \wedge v \succ w) \rightarrow (x \succ w \vee v \succ y)$
Semi-transitive $\forall x, y, z, v : (x \succ y \wedge y \succ z) \rightarrow (x \succ v \vee v \succ z)$

- Irreflexive: nothing is *noticeably* more P than itself.
- Interval-order: if we can tell x is better than y and v is better than w, then at least one "better" element also beats the other pair's "worse" element (there are no two completely independent just-noticeable differences)
- Semi-transitive: if x is better than y and y is better than z, then any fourth option v must line up with one of the extremes, x or z

Hence I_P is reflexive and symmetric, but need not be transitive (check this as an exercise).

Context-dependent I

Contextualist move: I_P is **context-dependent** and the context *changes* along a Sorites sequence.

Similarity relativized to a comparison class $c \subseteq D$:

$$xI_P^c y \quad \text{iff} \quad \forall z \in c:\ xI_P z \leftrightarrow yI_P z$$

"x and y are not (even indirectly) distinguishable relative to c."

Local validity vs. global invalidity

Conditionals are safe *in isolation* at their own context c:

$$\big(P(x,c) \ \wedge \ xI_P^c y\big) \ \rightarrow \ P(y,c)$$

where here $c = \{x, y\}$.

But we cannot conjoin premises across *different* contexts:

(1) $P(x,c)$ with $c = \{x, y, z\}$
(2) $\big(P(x,c) \wedge xI_P^c y\big) \rightarrow P(y,c)$ with $c = \{x, y\}$
(3) $\big(P(y,c) \wedge yI_P^c z\big) \rightarrow P(z,c)$ with $c = \{y, z\}$
(4) $P(z,c)$ with $c = \{x, y, z\}$

From (1)-(3) *you cannot derive* (4): we would need (2) and (3) on $c = \{x, y, z\}$

Contextualism

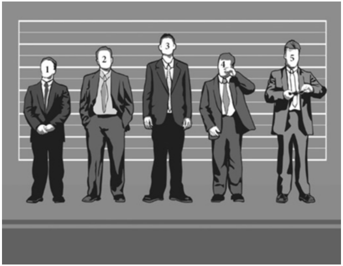
Experimental angle:

- Forced-march tasks (stepwise “still P?”) often show tolerance
- Presentation/manipulation of *comparison class* and *stimulus variation* shift judgments.

Philosophical concerns:

- Equivocation worry: Are we merely “changing the subject”?
- Fixing c: Even if we stipulate a fixed c, the paradox can still feel compelling: what explains that pull?
- Higher-order vagueness: Standards themselves seem vague. Can contextualism iterate the story?

Vagueness and Experiments



- Alxatib & Pelletier (2011): 5 men of increasing height.
- Participants judged, for each man:
  - "He is tall"
  - "He is not tall"
  - "He is tall and not tall"
  - "He is neither tall nor not tall"
- Results for man #2:
  - "He is **tall and not tall**": True 44.7%, False 40.8%, Can't tell 14.5%
  - "He is **neither tall nor not tall**": True 53.9%, False 42.1%, Can't tell 4.0%.