Definitions: Truthmakers

Philosophical Logic 2025/2026

1 Van Fraassen: Truthmaking Sets

Definition 1.1 (Facts and their combination). *Let* Fct *be a non-empty set whose elements are called* facts. *There is a binary operation*

$$\cdot : \mathsf{Fct} \times \mathsf{Fct} \to \mathsf{Fct}, \qquad (b, c) \mapsto b \cdot c$$

such that for all $b, c, d \in \mathsf{Fct}$*:*

$$\begin{aligned} b \cdot b &= b && \textit{(idempotence)} \\ b \cdot c &= c \cdot b && \textit{(commutativity)} \\ b \cdot (c \cdot d) &= (b \cdot c) \cdot d && \textit{(associativity)} \end{aligned}$$

Definition 1.2 (Atomic and complex facts). *A fact* $b \in \mathsf{Fct}$ *is* atomic *iff for every* $c \in \mathsf{Fct}$

$$b = b \cdot c \quad \Rightarrow \quad b = c$$

A fact is complex *iff it is not atomic.*

Definition 1.3 (Subordination of facts). *For* $b, c \in \mathsf{Fct}$ *define the* subordination *relation* $\leq$ *by*

$$b \leq c \quad \textit{iff} \quad \exists d \in \mathsf{Fct} \textit{ such that } b = d \cdot c$$

The relation $\leq$ *is a partial order on* Fct

Definition 1.4 (Closure under subordination). *For any* $X \subseteq \mathsf{Fct}$ *its* closure under subordination *is*

$$\mathrm{cl}(X) \;=\; \{y \in \mathsf{Fct} : \exists x \in X \textit{ with } y \leq x\}$$

So $\mathrm{cl}(X)$ *contains every fact that is subordinate to some member of* X

Definition 1.5 (Primary truthmaking and falsemaking bases). *Fix a propositional language with connectives* $\neg, \wedge, \vee$ *For each atomic sentence* p *choose an atomic fact* e_p *and a distinct atomic fact* $\overline{e_p}$ *(its* complement*). The* primary truthmaking *and* falsemaking *bases* $T(A)$ *and* $F(A)$ *for any sentence* A *are defined recursively as follows.*

Atomic case.

$$T(p) = \{e_p\}, \qquad F(p) = \{\overline{e_p}\}$$

Combination of sets of facts. *For $X, Y \subseteq \mathsf{Fct}$ define*

$$X \cdot Y := \{b \cdot c : b \in X, \ c \in Y\}$$

Boolean connectives. *For all sentences A, B:*

$$\begin{aligned} T(\neg A) &= F(A) & F(\neg A) &= T(A) \\ T(A \wedge B) &= T(A) \cdot T(B) & F(A \wedge B) &= F(A) \cup F(B) \\ T(A \vee B) &= T(A) \cup T(B) & F(A \vee B) &= F(A) \cdot F(B) \end{aligned}$$

Definition 1.6 (Truthmaking and falsemaking sets of a sentence). *For any sentence A its* truthmaking set *and* falsemaking set *are the propositions*

$$\mathsf{T}^*(A) := \mathrm{cl}(T(A)) \qquad \mathsf{F}^*(A) := \mathrm{cl}(F(A))$$

2 Exact Truthmaking (semantic clauses)

Definition 2.1 (Frame). *A* frame *for exact truthmaking is a structure*

$$\langle S, \leq \rangle$$

where:

- *S is a non-empty set of* states*;*
- *$\leq$ is a partial order on S*
- *for any $s, t \in S$ there exists a least upper bound (fusion) $s \sqcup t \in S$ such that*

  $$s \leq s \sqcup t, \quad t \leq s \sqcup t$$

  and whenever u is a state with $s \leq u$ and $t \leq u$ we have $s \sqcup t \leq u$

Definition 2.2 (Exact truthmaking model). *Fix a set P of propositional atoms. An* exact truthmaking model *is a triple*

$$\mathcal{M} = \langle S, \leq, I \rangle$$

where:

- *$\langle S, \leq \rangle$ is a frame as in definition 2.1;*
- *$I = (I^+, I^-)$ where*

  $$I^+, I^- : S \times P \to \{0, 1\}$$

  assign to each state $s \in S$ and atom $p \in P$ whether s is a positive *(truthmaking) or* negative *(falsemaking) exact verifier of p.*

- I^+ *and* I^- *are* fusion-closed*: for all* $s, t \in S$ *and* $p \in P$

$$I^+(s,p) = I^+(t,p) = 1 \Rightarrow I^+(s \sqcup t, p) = 1$$
$$I^-(s,p) = I^-(t,p) = 1 \Rightarrow I^-(s \sqcup t, p) = 1$$

Definition 2.3 (Positive and negative exact satisfaction)**.** *Let* $\mathcal{M} = \langle S, \leq, I \rangle$ *be an exact truthmaking model. We define, by recursion on formulas* φ *two relations* $s \vDash^+ \varphi$ *and* $s \vDash^- \varphi$ *between states* $s \in S$ *and formulas* φ*:*

- ***Atoms.*** *For* $p \in P$*:*

$$s \vDash^+ p \iff I^+(s,p) = 1, \qquad s \vDash^- p \iff I^-(s,p) = 1$$

- ***Negation.***

$$s \vDash^+ \neg\varphi \iff s \vDash^- \varphi, \qquad s \vDash^- \neg\varphi \iff s \vDash^+ \varphi$$

- ***Conjunction.***

$$s \vDash^+ (\varphi \wedge \psi) \iff \exists s', s'' \in S\,(s' \sqcup s'' = s,\ s' \vDash^+ \varphi,\ s'' \vDash^+ \psi)$$

$$s \vDash^- (\varphi \wedge \psi) \iff s \vDash^- \varphi \text{ or } s \vDash^- \psi \text{ or } \exists s', s'' \in S\,(s' \sqcup s'' = s,\ s' \vDash^- \varphi,\ s'' \vDash^- \psi)$$

- ***Disjunction (inclusive clause).***

$$s \vDash^+ (\varphi \vee \psi) \iff s \vDash^+ \varphi \text{ or } s \vDash^+ \psi \text{ or } \exists s', s'' \in S\,(s' \sqcup s'' = s,\ s' \vDash^+ \varphi,\ s'' \vDash^+ \psi)$$

$$s \vDash^- (\varphi \vee \psi) \iff \exists s', s'' \in S\,(s' \sqcup s'' = s,\ s' \vDash^- \varphi,\ s'' \vDash^- \psi)$$

Definition 2.4 (Exact truthmakers, falsemakers, and consequence)**.** *Let* $\mathcal{M} = \langle S, \leq, I \rangle$ *be an exact truthmaking model.*

1. *A state* $s \in S$ *is an* exact truthmaker *for a formula* φ *(in* $\mathcal{M}$*) iff*

$$s \vDash^+ \varphi$$

   A state s *is an* exact falsemaker *for* φ *iff* $s \vDash^- \varphi$

2. *For a set of formulas* Γ *and a formula* φ *we say that* φ *is an* exact truthmaking consequence *of* Γ *and write*

$$\Gamma \vDash_{\mathrm{TM}} \varphi$$

   iff for every exact truthmaking model $\mathcal{M}$ *and every state* $s \in S$*:*

$$(\forall \psi \in \Gamma\,(s \vDash^+ \psi)) \Rightarrow s \vDash^+ \varphi$$