

# Supervaluationism

Marco Degano

Philosophical Logic 2025  
3 November 2025

# Readings

## Suggested:

- Cobreros, Pablo & Tranchini, Luca (2019). *Supervaluationism, Subvaluationism and the Sorites Paradox*. In Sergi Oms & Elia Zardini (eds.), *The Sorites Paradox*. New York, NY: Cambridge University Press. pp. 38–62.

# Outline

## 1. Supervaluationism

## 2. Modalized Supervaluationism

## 3. Higher-order Vagueness

## 4. Truth-functionality

# Making things precise



Bas van Fraassen

- ▶ **Supervaluationism** (van Fraassen 1966; Fine 1975): handle vagueness by evaluating formulas over a range of *admissible precisifications*.
- ▶ A *precisification* is a classical “sharpening” of the vocabulary that preserves clear positives and clear negatives.
- ▶ Example: the predicate *heap* may be sharpened to “has at least  $n$  grains of sand,” for many choices of  $n$  (e.g.  $n = 1000, 1001, \dots$ ).
- ▶ Thus, multiple precisifications are admissible : supervaluationism does not single out a *unique* cutoff.



Kit Fine

# Semantic Indecision

*The reason it's vague where the outback begins is not that there's this thing, the outback, with imprecise borders; rather there are many things, with different borders, and nobody has been fool enough to try to enforce a choice of one of them as the official referent of the word "outback."*

*Vagueness is semantic indecision.*

(Lewis 1986: *On the Plurality of Worlds*, p. 213)

# Precisification

## Definition (Precisification)

Let  $v : P \rightarrow \{1, i, 0\}$  be a three-valued valuation. We say that a classical valuation  $v'$  is a *precisification* of  $v$ , and we write  $v \leq v'$  iff:

$$v(p) = 1 \Rightarrow v'(p) = 1$$

$$v(p) = 0 \Rightarrow v'(p) = 0$$

$$v(p) = i \Rightarrow v'(p) \in \{0, 1\}$$

	$p$	$q$		$p$	$q$
$v$	$i$	$0$		$v'_1$	$0$
				$v'_2$	$0$

# Supertrue and Superfalse

A formula is *supertrue* when it is true on *all* its precisifications; *superfalse* when it is false on all of them.

## Definition (Supertruth & Superfalsity)

Let  $v : P \rightarrow \{1, i, 0\}$  be three-valued and write  $\text{Prec}(v) := \{v' : v \leq v'\}$  for its set of classical precisifications. For any formula  $\varphi$ :

$$\text{(Supertruth)} \quad v \models^1 \varphi \iff \forall v' \in \text{Prec}(v) : v'(\varphi) = 1$$

$$\text{(Superfalsity)} \quad v \models^0 \varphi \iff \forall v' \in \text{Prec}(v) : v'(\varphi) = 0$$

# Logical Consequence

We can define both a *global* and a *local* notion of consequence.

## Definition (Global consequence)

$\Gamma \models_g \varphi$  iff for all three-valued valuations  $v$ , if  $v \models^1 \gamma$  for all  $\gamma \in \Gamma$ , then  $v \models^1 \varphi$ .

## Definition (Local consequence)

$\Gamma \models_l \varphi$  iff for all three-valued valuations  $v$ , for all  $v' \in \text{Prec}(v)$ , if  $v'(\gamma) = 1$  for all  $\gamma \in \Gamma$ , then  $v'(\varphi) = 1$ .



# Local and Global

- ▶ Over the base propositional language  $\{\neg, \wedge, \vee, \rightarrow\}$ , global and local consequence coincide.
- ▶ In fact, supervaluationist consequence is equivalent to classical consequence.

Fact (Consequence equivalence)

$$\Gamma \models_g \varphi \text{ iff } \Gamma \models_l \varphi \text{ iff } \Gamma \models_{\text{CL}} \varphi$$

# Consequence equivalence (proof sketch)

$$\Gamma \models_g \varphi \Rightarrow \Gamma \models_l \varphi \Rightarrow \Gamma \models_{CL} \varphi \Rightarrow \Gamma \models_g \varphi$$

For any classical  $w : P \rightarrow \{0, 1\}$ , we can view  $w$  as three-valued (no  $i$ ), so  $\text{Prec}(w) = \{w\}$ . We write  $v'(\Gamma) = 1$  for “ $\forall \gamma \in \Gamma, v'(\gamma) = 1$ ”.

**(1)**  $\Gamma \models_g \varphi \Rightarrow \Gamma \models_l \varphi$ . Let  $v$  be arbitrary and let  $v' \in \text{Prec}(v)$  with  $v'(\Gamma) = 1$ . Then  $v' \models^1 \Gamma$  (since  $\text{Prec}(v') = \{v'\}$ ). By global consequence,  $v' \models^1 \varphi$ , hence  $v(\varphi) = 1$ .

**(2)**  $\Gamma \models_l \varphi \Rightarrow \Gamma \models_{CL} \varphi$ . Let  $w$  be classical with  $w(\Gamma) = 1$ . Viewing  $w$  as three-valued gives  $\text{Prec}(w) = \{w\}$ . By local consequence,  $w(\varphi) = 1$ .

**(3)**  $\Gamma \models_{CL} \varphi \Rightarrow \Gamma \models_g \varphi$ . Fix any three-valued  $v$ . If  $v \models^1 \Gamma$ , then for all  $v' \in \text{Prec}(v)$  we have  $v'(\Gamma) = 1$ . By classical consequence,  $v'(\varphi) = 1$  for all such  $v'$ , so  $v \models^1 \varphi$ .

# Supervaluations as sets of valuations

- ▶ For a three-valued  $v : P \rightarrow \{1, i, 0\}$ , let  $\text{Prec}(v)$  be the set of all classical *precisifications* of  $v$ .
- ▶ Alternatively, start from an arbitrary nonempty set  $V \subseteq \{0, 1\}^P$  of classical valuations and evaluate formulas pointwise over  $V$ .

Supertruth/superfalsity lift as:

$$V \models^1 \varphi \iff \forall v' \in V : v'(\varphi) = 1$$

$$V \models^0 \varphi \iff \forall v' \in V : v'(\varphi) = 0$$

This is equivalent to the original definition via  $v$ , taking  $V = \text{Prec}(v)$

# Pointed evaluations

Given a non-empty set of classical valuations  $V^1$ , define the pointed satisfaction relation  $V, v \models \varphi$  for  $v \in V$  by:

$V, v \models p$	iff	$v(p) = 1$
$V, v \models \neg\varphi$	iff	$V, v \not\models \varphi$
$V, v \models \varphi \wedge \psi$	iff	$V, v \models \varphi$ and $V, v \models \psi$
$V, v \models \varphi \vee \psi$	iff	$V, v \models \varphi$ or $V, v \models \psi$
$V, v \models \varphi \rightarrow \psi$	iff	$V, v \not\models \varphi$ or $V, v \models \psi$

## Definition (Supertruth)

Given a non-empty  $V$ , a formula  $\varphi$  is supertrue iff  $V, v \models \varphi$  for all  $v \in V$ . We write  $V \models^1 \varphi$ .

Likewise,  $\varphi$  is superfalse iff  $V \models^1 \neg\varphi$ . We write  $V \models^0 \varphi$ .

---

<sup>1</sup>Allowing empty  $V$  does not change the resulting logic (as  $\emptyset \models^1 \varphi$  for any  $\varphi$ ), but it matters for satisfiability.

# Local and Global

Likewise, global and local consequence can be recast in this way:

## Definition (Global consequence)

$\Gamma \models_g \varphi$  iff for all non-empty  $V$ , if  $V \models^1 \gamma$  for all  $\gamma \in \Gamma$ , then  $V \models^1 \varphi$ .

## Definition (Local consequence)

$\Gamma \models_l \varphi$  iff for all non-empty  $V$ , for all  $v \in V$ , if  $V, v \models \gamma$  for all  $\gamma \in \Gamma$ , then  $V, v \models \varphi$ .

# Bivalence vs. Law of Excluded Middle

- ▶ **Failure of bivalence:** There are non-empty  $V$  and  $p$  with neither  $V \models^1 p$  nor  $V \models^1 \neg p$ .
- ▶ **Validity of LEM:** Every classical tautology is supertrue. In particular for every non-empty  $V$ ,  $V \models^1 p \vee \neg p$ .

Let  $V = \{v_1, v_2\}$  with  $v_1(p) = 1$  and  $v_2(p) = 0$ . Then

$$V \not\models^1 p \quad \text{and} \quad V \not\models^1 \neg p$$

since  $p$  fails at  $v_2$  and  $\neg p$  fails at  $v_1$ .

# Modelling the Sorites

- Recall the descending Sorites sequence from  $p_1$  to  $p_N$ .

$$\begin{array}{c}
 p_1 \\
 p_1 \rightarrow p_2 \\
 \vdots \\
 p_{N-1} \rightarrow p_N \\
 \hline
 p_N
 \end{array}$$

- A faithful model  $V$  for a descending Sorites series satisfies:

- $V \models^1 p_1$
- $V \models^0 p_N$
- $\exists k (1 < k < N)$  with  $V \not\models^1 p_k$  and  $V \not\models^1 \neg p_k$
- $\forall v \in V \forall m \in \{1, \dots, N\} :$ 

$$\begin{cases}
 v(p_m) = 1 \Rightarrow \forall k (1 \leq k \leq m \Rightarrow v(p_k) = 1) \\
 v(p_m) = 0 \Rightarrow \forall k (m \leq k \leq N \Rightarrow v(p_k) = 0)
 \end{cases}$$

# An example

► Take  $N = 5$ . A faithful model is:

$V$	$p_1$	$p_2$	$p_3$	$p_4$	$p_5$
$v_1$	1	0	0	0	0
$v_2$	1	1	0	0	0
$v_3$	1	1	1	0	0
$v_4$	1	1	1	1	0

$$V \models^1 p_1$$

$$V \not\models^1 (p_1 \rightarrow p_2) \text{ and } V \not\models^0 (p_1 \rightarrow p_2)$$

$$V \not\models^1 (p_2 \rightarrow p_3) \text{ and } V \not\models^0 (p_2 \rightarrow p_3)$$

$$V \not\models^1 (p_3 \rightarrow p_4) \text{ and } V \not\models^0 (p_3 \rightarrow p_4)$$

$$V \not\models^1 (p_4 \rightarrow p_5) \text{ and } V \not\models^0 (p_4 \rightarrow p_5)$$

$$V \models^0 p_5$$



# An example (cut-off as a disjunction)

$V$	$p_1$	$p_2$	$p_3$	$p_4$	$p_5$
$v_1$	1	0	0	0	0
$v_2$	1	1	0	0	0
$v_3$	1	1	1	0	0
$v_4$	1	1	1	1	0

- ▶  $A : (p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_3) \wedge (p_3 \rightarrow p_4) \wedge (p_4 \rightarrow p_5)$
- ▶  $\neg A : (p_1 \wedge \neg p_2) \vee (p_2 \wedge \neg p_3) \vee (p_3 \wedge \neg p_4) \vee (p_4 \wedge \neg p_5)$
- ▶  $A$  is superfalse, and  $\neg A$  is supertrue . However, none of the disjuncts is supertrue (no specific cut-off point).

# Assessing the situation

**Supervaluationist answer to the Sorites:** not all conditional premises are supertrue (so the argument is blocked), *without committing to which step* (no conditional is superfalse).

- ▶ In first-order guise:  $\forall n (\varphi(n) \rightarrow \varphi(n+1))$  is superfalse (read: series of conjunctions), but  $\exists n (\varphi(n) \wedge \neg\varphi(n+1))$  is supertrue (read: series of disjunctions).
- ▶ Yet for each particular  $d$ ,  $\{\varphi(d) \wedge \neg\varphi(d+1)\}$  is not supertrue . (No singled-out cut-off.)

# The notion of truth

- ▶ Supervaluationism lifts truth from a single valuation to a set of valuations. This echoes two frameworks:
  1. **Modal logic:** formulas are evaluated relative to possible worlds. A formula is true in a model if it is true in **all worlds of the model**.
  2. **Team semantics:** formulas are evaluated w.r.t. a *team* (set of valuations). A formula is true in a model if it is true in **all valuations of the team**.
- ▶ Over the base language (without modal operators), both are equivalent to classical logic.
- ▶ But lifting truth to sets of valuations yields loss of bivalence.
- ▶ Adding modal operators (next) or defining different connectives yields logic whose consequence is different from classical propositional logic.

# Team Semantics

- In team semantics (Hodges 1997, Väänänen 2007), the satisfaction relation uses a *possibly empty* set  $V$  of valuations and is defined over sets (not pointed):

$$V \models p \quad \text{iff} \quad \forall v \in V : v(p) = 1$$

$$V \models \neg\varphi \quad \text{iff} \quad \forall v \in V : \{v\} \not\models \varphi$$

$$V \models \varphi \wedge \psi \quad \text{iff} \quad V \models \varphi \text{ and } V \models \psi$$

$$V \models \varphi \vee \psi \quad \text{iff} \quad \exists V', V'' (V' \cup V'' = V, V' \models \varphi, V'' \models \psi)$$

- Connection to supervaluationism (for the base language):

$$V \models \varphi \text{ iff } \forall v \in V : \{v\} \models \varphi \text{ (i.e. } v(\varphi) = 1)$$

- Note that this definition of disjunction (over possibly empty teams) gives classical disjunction:

$$V \models p \vee q \text{ iff } \forall v \in V : v(p) = 1 \text{ or } v(q) = 1$$

# Outline

1. Supervaluationism

2. Modalized Supervaluationism

3. Higher-order Vagueness

4. Truth-functionality

# Definitely operator

- ▶ Add a determinacy operator  $\Delta$  ("definitely"). Think of  $\Delta$  as a necessity operator where precisifications act as worlds.
- ▶ Intuitively,  $\Delta p$  is true at a precisification  $v'$  of  $v$  iff  $p$  holds at *all* precisifications of  $v$ .
- ▶ For simplicity, first take the accessibility relation to be universal:

$$V, v \models \Delta\varphi \quad \text{iff} \quad \forall v' \in V : V, v' \models \varphi$$

# Deduction Theorem

$$p \models_g \Delta p$$

Let  $V$  be any non-empty set of valuations and assume  $V \models^1 p$ .  
Then  $p$  holds at every  $v \in V$ , hence  $V \models^1 \Delta p$ .

$$\not\models_g p \rightarrow \Delta p$$

Take  $V = \{v_1, v_2\}$  with  $v_1(p) = 1$  and  $v_2(p) = 0$ . Then  $V, v_1 \models p$  but  $V, v_1 \not\models \Delta p$  (since not all  $v' \in V$  satisfy  $p$ ). Hence  $V, v_1 \not\models p \rightarrow \Delta p$ .  
Therefore  $p \rightarrow \Delta p$  is not globally valid.

**So the deduction theorem fails:**  $\Gamma, \varphi \models_g \psi \not\Rightarrow \Gamma \models_g \varphi \rightarrow \psi$

# Global vs. Local with $\Delta$

$$\varphi \models_g \Delta\varphi$$

$$\varphi \not\models_l \Delta\varphi$$

- *Global holds:* If  $V \models^1 p$  then  $v(p) = 1$  for all  $v \in V$ , so  $V, v \models \Delta p$  for all  $v$ . Hence  $V \models^1 \Delta p$ .
- *Local fails:* Take  $V = \{v_1, v_2\}$  with  $v_1(p) = 1$  and  $v_2(p) = 0$ . Then  $V, v_1 \models p$  but  $V, v_1 \not\models \Delta p$ .



# Semantics

- Formulas are evaluated not just wrt a set of valuations but a pair  $M = \langle V, R \rangle$  with  $V \neq \emptyset$  and  $R \subseteq V \times V$ . Each  $v \in V$  is a classical valuation  $v : P \rightarrow \{0, 1\}$ .

$M, v \models p$	iff	$v(p) = 1$
$M, v \models \neg\varphi$	iff	$M, v \not\models \varphi$
$M, v \models \varphi \wedge \psi$	iff	$M, v \models \varphi$ and $M, v \models \psi$
$M, v \models \varphi \vee \psi$	iff	$M, v \models \varphi$ or $M, v \models \psi$
$M, v \models \varphi \rightarrow \psi$	iff	$M, v \not\models \varphi$ or $M, v \models \psi$
$M, v \models \Delta\varphi$	iff	$\forall v' \in V (vRv' \Rightarrow M, v' \models \varphi)$

**Supertruth:**  $M \models^1 \varphi \iff \forall v \in V (M, v \models \varphi)$

**Global:**  $\Gamma \models_g \varphi \iff \forall M (M \models^1 \Gamma \Rightarrow M \models^1 \varphi)$

**Local (modal logic):**  $\Gamma \models_l \varphi \iff \forall M \forall v \in V (M, v \models \Gamma \Rightarrow M, v \models \varphi)$

# Outline

1. Supervaluationism

2. Modalized Supervaluationism

**3. Higher-order Vagueness**

4. Truth-functionality

# Higher-order Vagueness

- ▶ Define indeterminate:  $\nabla\varphi := \neg\Delta\varphi \wedge \neg\Delta\neg\varphi$ .
- ▶ With *universal* accessibility:

$\nabla\Delta p$  is not satisfiable.

## Frame constraints $\leftrightarrow$ modal axioms (for $\Delta$ )

- |                  |  |
|------------------|--|
| T (reflexive)    | $\Delta\varphi \rightarrow \varphi$  |
| 4 (transitive)   | $\Delta\varphi \rightarrow \Delta\Delta\varphi$  |
| B (symmetric)    | $\varphi \rightarrow \Delta\neg\Delta\neg\varphi$  |
| 5 (euclidean)    | $\neg\Delta\varphi \rightarrow \Delta\neg\Delta\varphi$                                      |
| S5 / equivalence | $T+4+5; T+4+B; T+B+5 \Rightarrow \Delta\Delta\varphi \vee \Delta\neg\Delta\varphi$ is valid. |

- ▶ **To allow higher-order vagueness:** we need to drop some axioms. Which ones to keep for a ‘Definitely’ operator?

# R is reflexive and transitive (S4-frames)

In reflexive and transitive frames, higher-order vagueness for  $\Delta$  is *possible*:  $\nabla \Delta p$  is satisfiable.

Let  $M = \langle V, R \rangle$  with  $V = \{a, b, c\}$  and

$$R = \{(x, x) \mid x \in V\} \cup \{(a, b), (a, c)\}$$

(This  $R$  is transitive: from  $aRb$  and  $bRb$  we get  $aRb$ ; similarly for  $c$ .)

Valuation:  $p(b) = 1$ ,  $p(c) = 0$  (value at  $a$  arbitrary).

- ▶  $R[b] = \{b\}$ , so  $M, b \models \Delta p$ .
- ▶  $R[c] = \{c\}$ , so  $M, c \not\models \Delta p$ .
- ▶ Therefore  $M, a \models \nabla \Delta p$ .

# Outline

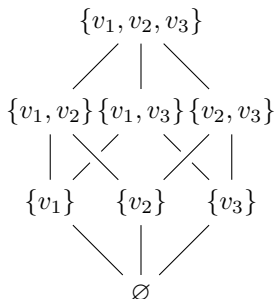
1. Supervaluationism
2. Modalized Supervaluationism
3. Higher-order Vagueness
4. Truth-functionality

# Supervaluations and truth-functionality

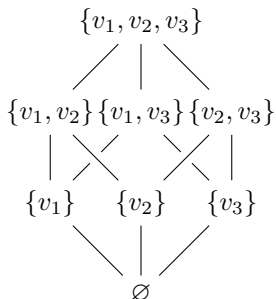
- ▶ **Truth-functionality:** the truth of a complex sentence is a function of the truth of its constituents.
- ▶ Supervaluationist theories are not truth-functional at the level of supertruth/superfalsity.
- ▶ For instance, the supertruth of  $\neg p$  is not determined solely by whether  $p$  is supertrue.

# An algebraic perspective

- ▶ Given  $V$ , the powerset  $\mathcal{P}(V)$  identifies formulas with their set of supporting valuations.
- ▶ Take  $V = \{v_1, v_2, v_3\}$  with  $v_1(p) = 1$ ,  $v_2(q) = 1$ , others 0. Then  $p$  corresponds to  $\{v_1\}$ , and  $p \vee q$  to  $\{v_1, v_2\}$ .



# Functionality via sets



Functionality is preserved *extensionally*:

$$f(p) = \{v \in V : v(p) = 1\}$$

$$f(\neg\varphi) = V \setminus f(\varphi)$$

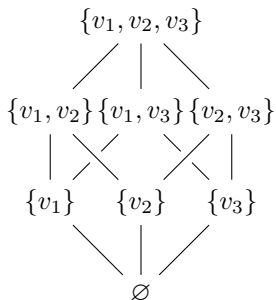
$$f(\varphi \vee \psi) = f(\varphi) \cup f(\psi)$$

$$f(\varphi \wedge \psi) = f(\varphi) \cap f(\psi)$$

$\varphi$  is supertrue in  $V$  iff  $f(\varphi) = V$ ; superfalse iff  $f(\varphi) = \emptyset$ .



# Supervaluations and degrees



- ▶ This representation suggests a degree-theoretic flavor.
- ▶ However, take  $V = \{v_1, v_2, v_3\}$  with  $v_1(p) = 1$ ,  $v_2(q) = 1$ , others 0.
- ▶ Then  $f(p) = \{v_1\}$  and  $f(\neg p) = \{v_2, v_3\}$ . But supervaluationism does *not* rank  $p$  as ‘less true’ than  $\neg p$ .

# Discussion

- ▶ **Disjunction** can be supertrue without any supertrue disjunct, undermining the intuitive idea that a true disjunction is always made true by one of its disjuncts.
- ▶ While theoremhood over the base language is classical, familiar *argument forms* needn't be globally supervalid with  $\Delta$  (e.g., conditional proof)
- ▶ Supertruth is not disquotational. If we identify ordinary truth with supertruth, the **T-schema is no longer valid** (though this might be an advantage for some).
- ▶ Higher-order vagueness is problematic with S5 axioms (plus a concern you have to address in your second assignment)
- ▶ Supervaluationists say “there is an  $n$  where the cutoff occurs” (since  $\exists n (\varphi(n) \wedge \neg\varphi(n+1))$  is supertrue) yet deny there *really* is a sharp cutoff. Some see this as talking *as if* sharp boundaries exist.

# Exercises

1. Show that (slide 25):

- (a)  $\Gamma \models_l \varphi \Rightarrow \Gamma \models_g \varphi$
- (b)  $\Gamma, \varphi \models_l \psi \Leftrightarrow \Gamma \models_l \varphi \rightarrow \psi$
- (c)  $\Gamma \models_g \varphi \rightarrow \psi \Rightarrow \Gamma, \varphi \models_g \psi$
- (d)  $\models_g \varphi \Leftrightarrow \models_l \varphi$
- (e)  $\varphi \models_g \psi \not\Rightarrow \neg\psi \models_g \neg\varphi$

(f) If  $R$  is not reflexive, then  $\not\models (\Delta\varphi \rightarrow \varphi)$

(g)  $\nabla\Delta p$  is satisfiable on reflexive and symmetric frames

2. On the set-theoretic preservation of functionality (slide 32), add the clauses for:

2.1 Universal  $\Delta$

2.2 General  $\Delta$