Definitions: Non-monotonic Logic

Philosophical Logic 2025/2026

Throughout, let L be a propositional language based on a set of atoms Prop, and let $\models$ denote classical consequence on L. A valuation on a non-empty set of worlds W is a function $V : W \times \mathsf{Prop} \to \{0, 1\}$, and classical truth at worlds in a model, $\mathcal{M}, w \models \alpha$, is defined as usual.

1 System C: Cumulative Consequence

Definition 1.1 (System C and cumulative consequence relation)**.** *A (single-conclusion) non-monotonic consequence relation* $\mid\!\sim$ *on* L *is* cumulative *if it satisfies the following principles for all formulas* $\varphi, \psi, \chi \in L$*:*

1. ***Reflexivity****:* $\varphi \mid\!\sim \varphi$.
2. ***Left logical equivalence****: if* $\varphi \models \psi$ *and* $\psi \models \varphi$*, and* $\varphi \mid\!\sim \chi$*, then* $\psi \mid\!\sim \chi$.
3. ***Right weakening****: if* $\varphi \models \psi$ *and* $\chi \mid\!\sim \varphi$*, then* $\chi \mid\!\sim \psi$.
4. ***Cut****: if* $\varphi \wedge \psi \mid\!\sim \chi$ *and* $\varphi \mid\!\sim \psi$*, then* $\varphi \mid\!\sim \chi$.
5. ***Cautious monotonicity****: if* $\varphi \mid\!\sim \psi$ *and* $\varphi \mid\!\sim \chi$*, then* $\varphi \wedge \psi \mid\!\sim \chi$.

2 Cumulative Models

Definition 2.1 (Cumulative model)**.** *Let* W *be a non-empty set of worlds. A* cumulative model *is a quadruple*

$$\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$$

such that:

1. S *is a non-empty set of states;*
2. $\ell : S \to \mathcal{P}(W) \setminus \{\emptyset\}$ *labels each state with a non-empty set of worlds;*
3. $\prec$ *is a binary relation on* S*;*
4. $V : W \times \mathsf{Prop} \to \{0, 1\}$ *is a classical valuation on* W*;*
5. *for every formula* $\alpha \in L$*, the set* $[\![\alpha]\!]^{\mathcal{M}_C} = \{s \in S \mid \mathcal{M}_C, s \models \alpha\}$ *is smooth with respect to* $\prec$.

Definition 2.2 (State satisfaction)**.** *Let $\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$ be a cumulative model. For a state $s \in S$ and a formula $\alpha \in L$ we define:*

$$\mathcal{M}_C, s \models \alpha \quad \Longleftrightarrow \quad \forall w \in \ell(s)\ (\mathcal{M}_C, w \models \alpha)$$

Definition 2.3 (Truth set of a formula at states)**.** *Let $\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$ be a cumulative model. The* truth set *of a formula $\alpha \in L$ at the level of states is $[\![\alpha]\!]^{\mathcal{M}_C} = \{s \in S \mid \mathcal{M}_C, s \models \alpha\}$.*

Definition 2.4 (Smoothness (for states))**.** *Let $(S, \prec)$ be a set of states equipped with a binary relation $\prec$. A subset $A \subseteq S$ is* smooth *(with respect to $\prec$) iff for every $s \in A$:*

- *either s is $\prec$-minimal in A, i.e. there is no $s' \in A$ such that $s' \prec s$;*
- *or there exists a $\prec$-minimal $s' \in A$ with $s' \prec s$.*

Definition 2.5 (Cumulative consequence in a model)**.** *Let $\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$ be a cumulative model. The associated* cumulative consequence relation *$\mid\!\sim_{\mathcal{M}_C}$ is defined by:*

$$\alpha \mid\!\sim_{\mathcal{M}_C} \beta \quad \Longleftrightarrow \quad \text{for every } \prec\text{-minimal } s \in [\![\alpha]\!]^{\mathcal{M}_C},\ \ \mathcal{M}_C, s \models \beta.$$

2.1 Soundness and completeness of system C

Definition 2.6 (Satisfaction in a cumulative model)**.** *Let K be a set of conditionals of the form $\alpha \mid\!\sim \beta$, and $\mathcal{M}_C = \langle S, \ell, \prec, V \rangle$ a cumulative model. We say that $\mathcal{M}_C$* satisfies *K iff for every $\alpha \mid\!\sim \beta \in K$ we have*

$$\alpha \mid\!\sim_{\mathcal{M}_C} \beta.$$

Definition 2.7 (Cumulative entailment)**.** *Let K be a set of conditionals. For formulas $\alpha, \beta \in L$ we write*

$$K \models_{\mathrm{C}} \alpha \mid\!\sim \beta$$

iff for every cumulative model $\mathcal{M}_C$, if $\mathcal{M}_C$ satisfies K, then

$$\alpha \mid\!\sim_{\mathcal{M}_C} \beta.$$

In the special case $K = \emptyset$ we say that $\alpha \mid\!\sim \beta$ is valid in all cumulative models *and write $\models_{\mathrm{C}} \alpha \mid\!\sim \beta$.*

Definition 2.8 (Derivability in system **C**)**.** *Let K be a set of conditionals. We write*

$$K \vdash_{\mathrm{C}} \alpha \mid\!\sim \beta$$

iff $\alpha \mid\!\sim \beta$ can be obtained from premises in K by a finite number of applications of the rules of system **C***. We write $\vdash_{\mathrm{C}} \alpha \mid\!\sim \beta$ for $\emptyset \vdash_{\mathrm{C}} \alpha \mid\!\sim \beta$.*

Theorem 2.1 (Soundness of system **C**)**.** *If $K \vdash_{\mathrm{C}} \alpha \mid\!\sim \beta$, then $K \models_{\mathrm{C}} \alpha \mid\!\sim \beta$. In particular, every rule of system* **C** *preserves validity in cumulative models.*

Theorem 2.2 (Completeness / representation for system **C**)**.** *Let $\mid\!\sim$ be a non-monotonic consequence relation on L. Then the following are equivalent:*

1. $\mid\!\sim$ *is cumulative (i.e. it satisfies all rules of system* **C***).*
2. *There exists a cumulative model* $\mathcal{M}_C$ *such that for all formulas* $\alpha, \beta \in L$,

$$\alpha \mid\!\sim \beta \quad \iff \quad \alpha \mid\!\sim_{\mathcal{M}_C} \beta$$

As a consequence, for every set K *of conditionals and all* $\alpha, \beta \in L$:

$$K \vdash_{\mathrm{C}} \alpha \mid\!\sim \beta \quad \iff \quad K \vDash_{\mathrm{C}} \alpha \mid\!\sim \beta$$

3 Preferential Models and System P

Definition 3.1 (Preferential model). *A* preferential model *is a triple*

$$\mathcal{M}_P = \langle W, \prec, V \rangle$$

such that:

1. W *is a non-empty set of worlds;*
2. $\prec$ *is a strict partial order on* W *(irreflexive and transitive);*
3. $V : W \times \mathsf{Prop} \to \{0, 1\}$ *is a classical valuation on* W;
4. *for every formula* $\alpha \in L$, *the truth set*

$$[\![\alpha]\!]^{\mathcal{M}_P} = \{w \in W \mid \mathcal{M}_P, w \vDash \alpha\}$$

is smooth with respect to $\prec$.

Definition 3.2 (Smoothness (for worlds)). *Let* $(W, \prec)$ *be as above. A subset* $A \subseteq W$ *is* smooth *(with respect to* $\prec$*) iff for every* $w \in A$:

- *either* w *is* $\prec$*-minimal in* A,
- *or there exists a* $\prec$*-minimal* $w' \in A$ *with* $w' \prec w$.

Definition 3.3 (Preferential consequence). *Let* $\mathcal{M}_P = \langle W, \prec, V \rangle$ *be a preferential model. The associated* preferential consequence relation $\mid\!\sim_{\mathcal{M}_P}$ *is defined by:*

$$\alpha \mid\!\sim_{\mathcal{M}_P} \beta \quad \iff \quad \textit{for every } \prec\textit{-minimal } w \in [\![\alpha]\!]^{\mathcal{M}_P}, \;\; \mathcal{M}_P, w \vDash \beta.$$

Thus from α *we defeasibly infer* β *iff every most normal* α*-world satisfies* β.

Definition 3.4 (System **P**). *A non-monotonic consequence relation* $\mid\!\sim$ *on* L *satisfies* system **P** *(preferential consequence) iff:*

- *it is cumulative in the sense of Definition 1.1, and*
- *in addition it satisfies the rule:*

  (Or) *If* $\varphi \mid\!\sim \chi$ *and* $\psi \mid\!\sim \chi$, *then* $\varphi \vee \psi \mid\!\sim \chi$.

3.1 Knowledge bases and preferential entailment

Definition 3.5 (Default knowledge base)**.** *A* (default) knowledge base *is a set K of conditional assertions of the form*

$$\alpha \mid\sim \beta$$

where $\alpha, \beta \in L$.

Definition 3.6 (Satisfaction of a knowledge base)**.** *Let K be a default knowledge base and $\mathcal{M}_P = \langle W, \prec, V \rangle$ a preferential model. We say that $\mathcal{M}_P$* satisfies *K iff for every conditional $\alpha \mid\sim \beta \in K$ we have*

$$\alpha \mid\sim_{\mathcal{M}_P} \beta.$$

Definition 3.7 (Preferential entailment)**.** *Let K be a default knowledge base. For formulas $\alpha, \beta \in L$ we write*

$$K \models_{\text{pref}} \alpha \mid\sim \beta$$

iff for every preferential model $\mathcal{M}_P$, if $\mathcal{M}_P$ satisfies K, then

$$\alpha \mid\sim_{\mathcal{M}_P} \beta.$$

3.2 Soundness and completeness of system P

Definition 3.8 (Derivability in system **P**)**.** *Let K be a default knowledge base. We write*

$$K \vdash_{\mathbf{P}} \alpha \mid\sim \beta$$

iff $\alpha \mid\sim \beta$ can be obtained from premises in K by a finite number of applications of the rules of system **P** *(i.e. the rules of system* **C** *together with rule **(Or)**) We write $\vdash_{\mathbf{P}} \alpha \mid\sim \beta$ for $\emptyset \vdash_{\mathbf{P}} \alpha \mid\sim \beta$.*

Theorem 3.1 (Soundness of system **P**)**.** *If $K \vdash_{\mathbf{P}} \alpha \mid\sim \beta$, then $K \models_{\text{pref}} \alpha \mid\sim \beta$. Equivalently, every rule of system* **P** *is valid in all preferential models.*

Theorem 3.2 (Completeness / representation for system **P**)**.** *Let $\mid\sim$ be a non-monotonic consequence relation on L. Then the following are equivalent:*

1. *$\mid\sim$ satisfies all rules of system* **P***.*
2. *There exists a preferential model $\mathcal{M}_P$ such that for all formulas $\alpha, \beta \in L$,*

$$\alpha \mid\sim \beta \quad \Longleftrightarrow \quad \alpha \mid\sim_{\mathcal{M}_P} \beta$$

In particular, for every default knowledge base K and formulas $\alpha, \beta \in L$:

$$K \vdash_{\mathbf{P}} \alpha \mid\sim \beta \quad \Longleftrightarrow \quad K \models_{\text{pref}} \alpha \mid\sim \beta$$