Definitions: Probability

Philosophical Logic 2025/2026

1 Basic probabilistic notions

Definition 1.1 (Probability space). *A* probability space *is a triple* $(\Omega, \mathcal{F}, \Pr)$ *where:*

1. $\Omega \neq \emptyset$ *is a non-empty set, the* sample space*;*
2. $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ *is a collection of subsets of* Ω*, the* events*, such that:*
   (a) *if* $A \in \mathcal{F}$*, then* $\Omega \setminus A \in \mathcal{F}$ *(closed under complements);*
   (b) *if* $A, B \in \mathcal{F}$*, then* $A \cup B \in \mathcal{F}$ *(closed under finite unions);*
3. $\Pr : \mathcal{F} \to [0, 1]$ *is a function, the* probability measure*, such that:*
   (a) $\Pr(\Omega) = 1$*;*
   (b) *if* $A, B \in \mathcal{F}$ *and* $A \cap B = \emptyset$*, then* $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Definition 1.2 (Conditional probability). *Let* $(\Omega, \mathcal{F}, \Pr)$ *be a probability space and let* $A, B \in \mathcal{F}$ *with* $\Pr(B) > 0$*. The* conditional probability *of* A *given* B *is*

$$\Pr(A \mid B) := \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Definition 1.3 (Chain rule). *Let* $(\Omega, \mathcal{F}, \Pr)$ *be a probability space and let* $A_1, \dots, A_n \in \mathcal{F}$ *be events such that all the conditional probabilities below are defined. The* chain rule *(or* product rule*) states that*

$$\Pr(A_1 \cap \cdots \cap A_n) = \Pr(A_1)\ \Pr(A_2 \mid A_1)\ \Pr(A_3 \mid A_1 \cap A_2) \cdots \Pr(A_n \mid A_1 \cap \cdots \cap A_{n-1}).$$

For $n = 2$ *this reduces to*

$$\Pr(A \cap B) = \Pr(A)\ \Pr(B \mid A),$$

whenever $\Pr(A) > 0$*.*

2 Probabilities on a propositional language

Definition 2.1 (Probability function on a propositional language). *Let* $\mathcal{L}$ *be a propositional language (closed under* $\neg, \wedge, \vee$*), and let* $\models$ *denote classical consequence. A* probability function *on* $\mathcal{L}$ *is a function* $P : \mathcal{L} \to \mathbb{R}$ *such that, for all* $\varphi, \psi \in \mathcal{L}$*:*

1. $P(\varphi) \geq 0$;

2. if $\models \varphi$ (i.e. φ is a tautology), then $P(\varphi) = 1$;

3. if $\models \neg(\varphi \wedge \psi)$ (i.e. φ and ψ are mutually exclusive), then $P(\varphi \vee \psi) = P(\varphi) + P(\psi)$.

Definition 2.2 (Uncertainty). *Let P be a probability function on $\mathcal{L}$. For any sentence $\varphi \in \mathcal{L}$ the* uncertainty *(or* risk of error*) of φ relative to P is*

$$U_P(\varphi) := P(\neg\varphi) = 1 - P(\varphi).$$

3 Probabilistic consequence relations

Definition 3.1 (Probabilistic entailment $\models_P$). *Let Γ be a finite set of sentences of $\mathcal{L}$ and let $\varphi \in \mathcal{L}$. We say that Γ* probabilistically entails *φ and write*

$$\Gamma \models_P \varphi$$

iff for every probability function P on $\mathcal{L}$,

$$U_P(\varphi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma).$$

Definition 3.2 (Material conditional). *For sentences φ, ψ in the propositional language, the* material conditional *$\varphi \supset \psi$ is the abbreviation $\varphi \supset \psi := \neg\varphi \vee \psi$.*

Theorem 3.1 (Classical fragment (no conditionals)). *Let $\mathcal{L}_0$ be a propositional language* without *the connective $\rightarrow$, closed under $\neg, \wedge, \vee$, and let $\models_{CL}$ denote classical entailment on $\mathcal{L}_0$. Then for every finite $\Gamma \subseteq \mathcal{L}_0$ and every $\varphi \in \mathcal{L}_0$:*

$$\Gamma \models_{CL} \varphi \iff \Gamma \models_P \varphi,$$

4 Conditionals and probabilities

Theorem 4.1 (Conditional fragment and System **P**). *Let $\mathcal{L}_0$ be as above and let $\mathcal{L}^{\rightarrow}$ be the language obtained by adding to $\mathcal{L}_0$ a binary connective $\rightarrow$. Consider the fragment of $\mathcal{L}^{\rightarrow}$ consisting of* non-embedded *indicative conditionals of the form $\varphi \rightarrow \psi$ with $\varphi, \psi \in \mathcal{L}_0$, and let P range over probability functions on $\mathcal{L}^{\rightarrow}$ satisfying Adams' thesis*

$$P(\varphi \rightarrow \psi) = P(\psi \mid \varphi)$$

whenever $P(\varphi) > 0$.

Let Γ be a finite set of such non-embedded conditionals and let α be such a conditional. Then the consequence relation between conditionals induced by probabilistic entailment $\models_P$,

$$\Gamma \models_P \alpha,$$

coincides with System **P***: $\Gamma \models_P \alpha$ holds if and only if α is derivable from Γ using exactly the axioms and rules of System* **P**.