

# CONTINUOUS NONCROSSING PARTITIONS AND WEIGHTED CIRCULAR FACTORIZATIONS

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ABSTRACT. This article examines noncrossing partitions of the unit circle in the complex plane; we call these *continuous noncrossing partitions*. More precisely, we focus on the *degree- $d$*  continuous noncrossing partitions where unit complex numbers in the same block have identical  $d$ -th powers. We prove that the degree- $d$  continuous noncrossing partitions form a topological poset whose uncountable set of elements can be indexed by equivalence classes of objects we call *weighted linear factorizations* of factors of a  $d$ -cycle. Moreover, the maximal elements in this poset form a subspace homeomorphic to the dual Garside classifying space for the  $d$ -strand braid group.

The degree- $d$  continuous noncrossing partitions of the unit circle are a special case of a more general construction. For every choice of Coxeter element  $\mathbf{c}$  in any Coxeter group  $W$  we define a topological poset of equivalence classes of weighted linear factorizations of factors of  $\mathbf{c}$  in  $W$  whose elements we call *continuous  $\mathbf{c}$ -noncrossing partitions*. The maximal elements in this poset form a subspace homeomorphic to the one-vertex complex whose fundamental group is the corresponding dual Artin group.

## INTRODUCTION

Let  $\text{NC}(\mathbf{S})$  denote the set of all partitions of the unit circle  $\mathbf{S} \subset \mathbb{C}$  for which the convex hulls of the blocks are pairwise disjoint. This set is partially ordered by refinement, and we refer to its elements as *continuous noncrossing partitions*. The full continuous noncrossing partition poset  $\text{NC}(\mathbf{S})$  is quite complicated. For example, a geodesic lamination of a hyperbolic surface lifts to a lamination of the disk model of the hyperbolic plane, and this induces a continuous noncrossing partition of the circle whose blocks are the asymptotic ends of the leaves. In this article we focus on the simpler subposet  $\text{NC}_d(\mathbf{S})$  of *degree- $d$  continuous noncrossing partitions*, i.e. those where unit complex numbers in the same block have the same image under the map  $z \mapsto z^d$ . See Figure 1. Our first main theorem gives an algebraic construction of  $\text{NC}_d(\mathbf{S})$ .

**Theorem A** (Theorem 6.11). *The poset  $\text{NC}_d(\mathbf{S})$  of degree- $d$  continuous noncrossing partitions is isomorphic to the topological poset  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  defined using weighted linear factorizations of noncrossing permutations in  $\text{SYM}_d$ .*

The topological poset  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  is a special case of a general construction introduced here. Since the full definition is slightly complicated, we sketch the definition here and record the details in the body of the article. First, recall that several recent studies of affine and hyperbolic Artin groups have been based on constructions of the following form. Given a group  $G$  with a fixed conjugacy-closed generating set  $X$  and an element  $g \in \text{MON}(X) \subset G$ , the submonoid generated by

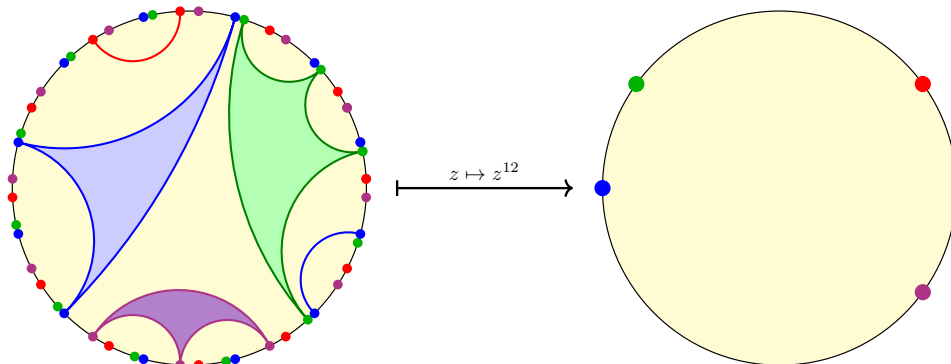


FIGURE 1. The degree-12 continuous noncrossing partition shown on the left has five non-trivial blocks, and these are sent under the map  $z \mapsto z^{12}$  to the four points shown on the right. The 48 points shown on the left are the preimages of these four points on the right. All other points in  $\mathbf{S}$  belong to singleton blocks.

$X$ , one can define a bounded graded poset  $P_g = [1, g]$  of reduced factorizations of  $g$  over  $X$ , a simplicial complex  $O_g = \Delta(P_g)$  which is the order complex of  $P_g$ , a one-vertex  $\Delta$ -complex  $K_g = q(O_g)$  which is a cellular quotient of  $O_g$ , and a group  $G_g = \pi_1(K_g)$  which is the fundamental group of the quotient. These are the *interval poset*  $P_g$ , its *order complex*  $O_g$ , the *interval complex*  $K_g$  and the *interval group*  $G_g$ , respectively. See [BM15, McC15, MS17, McC18, PS21, Hae22, DPS24].

When  $G$  is the symmetric group  $\text{SYM}_d$ ,  $X = T$  is the set of all transpositions in  $\text{SYM}_d$ , and  $\delta$  is the  $d$ -cycle  $(1\ 2\ \cdots\ d)$ , the poset  $P_\delta$  is the noncrossing partition lattice  $\text{NC}_d$  and the interval complex  $K_\delta$  is the *dual braid complex* with fundamental group  $\text{BRAID}_d$ . This space was introduced by Tom Brady [Bra01] and given a piecewise-Euclidean metric by Brady and the second author [BM10]. The dual braid complex is a classifying space for the  $d$ -strand braid group and it is conjectured to be locally  $\text{CAT}(0)$  with its “orthoscheme” metric. This claim has been verified for  $d \leq 7$  [BM10, HKS16, Jeo23]. A proof that these metric piecewise Euclidean complexes are nonpositively curved would resolve the long-standing conjecture that the braid group is a  $\text{CAT}(0)$  group. More generally, when  $G = W$  is a Coxeter group,  $X = T$  is the set of all reflections in  $W$ , and  $g = \mathbf{c}$  is a Coxeter element, (i.e. a product of the reflections in a simple system in some order), then  $P_g$  is the poset of  $\mathbf{c}$ -noncrossing partitions,  $K_g$  is the one-vertex complex for the dual Artin group and  $G_g$  is the dual Artin group  $\text{ART}^*(G, g)$ .

In this article, we introduce algebraic constructions that index the cells and the points of the order complex  $O_g$  and interval complex  $K_g$ . The cells are indexed by linear and circular factorizations. The points are indexed by weighted linear and weighted circular factorizations. The linear versions refer to the simplicial order complex  $O_g$ . In symbols,  $\text{CELLS}(O_g) = \text{CHAINS}(P_g) = \text{FACT}(G, g, \mathbf{I})$ , and  $\text{POINTS}(O_g) = \text{WCHAINS}(P_g) = \text{WFACT}(G, g, \mathbf{I})$ . The circular versions refer to the one-vertex interval complex  $K_g = q(O_g)$ . In symbols, we have  $\text{CELLS}(K_g) = q(\text{CELLS}(O_g)) = q(\text{CHAINS}(P_g)) = q(\text{FACT}(G, g, \mathbf{I})) = \text{FACT}(G, g, \mathbf{S})$ . See Tables 1 and 2. The rows in these tables vary the type of group under consideration, and in the final row we switch to additive notation.

	Cells	Points	TopPoset
General	$\text{FACT}(G, g, \mathbf{I})$	$\text{WFACT}(G, g, \mathbf{I})$	$\mathcal{F}(G, g, \mathbf{I})$
Coxeter	$\text{FACT}(W, \mathbf{c}, \mathbf{I})$	$\text{WFACT}(W, \mathbf{c}, \mathbf{I})$	$\mathcal{F}(W, \mathbf{c}, \mathbf{I})$
Symmetric	$\text{FACT}(\text{SYM}_d, \delta, \mathbf{I})$	$\text{WFACT}(\text{SYM}_d, \delta, \mathbf{I})$	$\mathcal{F}(\text{SYM}_d, \delta, \mathbf{I})$
Integer	$\text{COMP}(\mathbb{Z}, n, \mathbf{I})$	$\text{WCOMP}(\mathbb{Z}, n, \mathbf{I})$	$\mathcal{C}(\mathbb{Z}, n, \mathbf{I})$

TABLE 1. Linear factorizations of  $g$  describe the cells of the order complex  $O_g$  (first column). Weighted linear factorizations of  $g$  describe the points of  $O_g$  (second column). The disjoint union of order complexes  $\{O_h \mid h \leq g\}$  form a topological poset with a grading (third column).

	Cells	Points	TopPoset
General	$\text{FACT}(G, g, \mathbf{S})$	$\text{WFACT}(G, g, \mathbf{S})$	$\mathcal{F}(G, g, \mathbf{S})$
Coxeter	$\text{FACT}(W, \mathbf{c}, \mathbf{S})$	$\text{WFACT}(W, \mathbf{c}, \mathbf{S})$	$\mathcal{F}(W, \mathbf{c}, \mathbf{S})$
Symmetric	$\text{FACT}(\text{SYM}_d, \delta, \mathbf{S})$	$\text{WFACT}(\text{SYM}_d, \delta, \mathbf{S})$	$\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$
Integer	$\text{COMP}(\mathbb{Z}, n, \mathbf{S})$	$\text{WCOMP}(\mathbb{Z}, n, \mathbf{S})$	$\mathcal{C}(\mathbb{Z}, n, \mathbf{S})$

TABLE 2. Circular factorizations of  $g$  describe the cells of the interval complex  $K_g$  (first column). Weighted circular factorizations of  $g$  describe the points of  $K_g$  (second column). The circular topological poset (third column) is a quotient of the linear topological poset in the third column of Table 1.

Our third construction combines the weighted factorizations of elements below  $g$  into a topological poset. A *topological poset*, roughly speaking, is a poset where the elements have a topology and the order relation is a closed subspace (Definition 5.1). They were defined by Rade Živaljević in [Ž98] and inspired by the work of Vasiliev in [Vas91]. See also [Fol66, Bjo95, Ž16].

One of the motivating examples for the concept is the poset  $\mathcal{L}_n(\mathbb{R})$  of linear subspaces in  $\mathbb{R}^n$  partially ordered by inclusion. It has a rank function that sends each subspace to its dimension, and can be viewed topologically as the disjoint union of the Grassmannians  $\text{Gr}(i, \mathbb{R}^n)$  with  $i \in \{0, \dots, n\}$ . A more relevant example here is the poset of multisets of size at most  $n$  in an interval  $\mathbf{I}$ , also ordered by inclusion. The elements of rank  $k$  are the multisets of size  $k$  when counted with multiplicity, and these are in bijection with the points of a  $k$ -dimensional simplex, which gives them a natural topology. Figure 11 in Section 5 shows this topological graded poset for  $n = 3$ . It has a tetrahedron of points in rank 3, a triangle of points in rank 2, an interval of points in rank 1, and a single point in rank 0.

The elements of the linear topological poset  $\mathcal{F}(G, g, \mathbf{I})$  are precisely *weighted circular factorizations* of elements  $h$  in the interval  $P_g = [1, g]$ , and topologically it is a disjoint union of the order complexes  $\{O_h \mid 1 \leq h \leq g\}$ . The circular topological poset  $\mathcal{F}(G, g, \mathbf{S})$  is defined as a quotient of the linear topological poset  $\mathcal{F}(G, g, \mathbf{I})$ . See Definition 3.6 for a precise definition of the quotient map. Example 5.12 shows that the topology of  $\mathcal{F}(G, g, \mathbf{S})$  need not be a disjoint union of the interval complexes  $\{K_h \mid 1 \leq h \leq g\}$ . The components of  $\mathcal{F}(G, g, \mathbf{S})$  are instead cyclic covers of the

interval complexes  $K_h$ . The maximal elements of  $\mathcal{F}(G, g, \mathbf{S})$  form a copy of the interval complex  $K_g$  without taking a cyclic cover.

**Theorem B** (Theorem 5.13). *Let  $P_g$  be an interval poset in a marked group  $G$  with order complex  $O_g$  and interval complex  $K_g$ . In the circular topological poset  $\mathcal{F}(G, g, \mathbf{S})$ , the subspace of maximal elements is homeomorphic to  $K_g$ .*

When Theorem B is applied to a Coxeter group  $W$  with Coxeter element  $\mathbf{c}$ , the result is a topological poset whose maximal elements form the one-vertex complex for the dual Artin group  $\text{ART}^*(W, \mathbf{c})$ .

**Theorem C** (Corollary 5.14). *Let  $\mathbf{c} \in W$  be a Coxeter element in a Coxeter group generated by its reflections. The maximal elements of the circular topological poset  $\mathcal{F}(W, \mathbf{c}, \mathbf{S})$  form a subspace homeomorphic to the complex  $K_{\mathbf{c}}$  whose fundamental group is the dual Artin group  $\text{ART}^*(W, \mathbf{c})$ . In particular, the subspace of maximal elements in  $\text{NC}_d(\mathbf{S}) \cong \mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  is homeomorphic to the dual braid complex  $K_{\delta}$  whose fundamental group is the braid group  $\text{BRAID}_d$ .*

The ideals and filters in the poset  $\mathcal{F}(G, g, \mathbf{I})$  (and in its circular counterpart  $\mathcal{F}(G, g, \mathbf{S})$ ) are quite distinct. Let  $\mathbf{u} \in \mathcal{F}(G, g, \mathbf{I})$  be a weighted linear factorization of  $h \in P_g = [1, g]$  and let  $[\mathbf{u}]$  be its equivalence class in  $\mathcal{F}(G, g, \mathbf{S})$ . The lower set (ideal)  $\downarrow([\mathbf{u}])$  of all elements below  $[\mathbf{u}]$  in the partial order on  $\mathcal{F}(G, g, \mathbf{S})$  is isomorphic to a product of intervals  $[1, x_1] \times \cdots \times [1, x_k]$  for some  $x_1, \dots, x_k \in P_g$  (Remark 5.4). In particular, the lower set  $\downarrow([\mathbf{u}])$  is discrete when  $G$  is a discrete group and finite when  $G$  is a finite group, even though  $\mathcal{F}(G, g, \mathbf{S})$  itself is uncountable. The upper set  $\uparrow([\mathbf{u}])$  is more complicated: it is a circular topological poset for an element in  $[1, g]$ .

**Theorem D** (Corollary 5.15). *Let  $\mathbf{u}$  be a weighted linear factorization of  $h \in P_g$  with equivalence class  $[\mathbf{u}] \in \mathcal{F}(G, g, \mathbf{S})$ . Then the upper set  $\uparrow([\mathbf{u}])$  in  $\mathcal{F}(G, g, \mathbf{S})$  is isomorphic to  $\mathcal{F}(G, h', \mathbf{S})$  where  $h \cdot h' = g$ . Consequently, the maximal elements of  $\uparrow([\mathbf{u}])$  form a subspace which is isometric to the interval complex  $K_{h'}$  inside the interval complex  $K_g$ .*

In the specific case of degree- $d$  continuous noncrossing partitions  $\text{NC}_d(\mathbf{S})$ , there is a close connection with degree- $d$  polynomials that has already been noted in the literature. The preimage of the union of the real and imaginary coordinate axes under a degree- $d$  complex polynomial, for example, determines a degree- $d$  continuous noncrossing partition of  $\mathbf{S}$  in which each non-trivial block has size divisible by 4 [MSS07]. The blocks are determined by the points in the circle at infinity asymptotically approached by the ends of a connected component of the preimage. Additional constructions connecting continuous noncrossing partitions and complex polynomials can be found in our earlier articles [DM22, DM].

The theorems proved here allow us to tackle the following natural question: what is the relationship between the dual braid complex and other classifying spaces for the braid group? One classical example is the space  $\text{POLY}_d^{mc}(\mathbb{C}^*)$  of all monic complex polynomials with  $d$  roots which are centered at the origin and critical values in the punctured plane  $\mathbb{C}^*$ ; by observing that a polynomial has distinct roots if and only if it does not have 0 as a critical value, this space can be identified with the configuration space of  $d$  unordered points in the plane which are centered at the origin. Fox and Neuwirth showed in 1962 that  $\text{POLY}_d^{mc}(\mathbb{C}^*)$  is a classifying space for the  $d$ -strand braid group, which means that this space is homotopy equivalent

to the dual braid complex for abstract reasons. In this article we provide a more concrete connection.

**Theorem E** (Corollary 6.13). *The dual braid complex is a spine for  $\text{POLY}_d^{mc}(\mathbb{C}^*)$ . More specifically, the dual braid complex is homeomorphic to the subspace of polynomials with critical values on the unit circle, and there is a deformation retraction from  $\text{POLY}_d^{mc}(\mathbb{C}^*)$  to this subspace.*

The existence of the deformation retraction (without reference to a cell structure) was stated by W. Thurston in an unpublished manuscript from 2012, which was posthumously completed by Baik, Gao, Hubbard, Lei, Lindsey, and D. Thurston [TBY<sup>+</sup>20]. In this 2012 manuscript, Thurston introduced *degree- $d$ -invariant laminations*, which provide a useful tool for describing points in the subspace of polynomials with critical values on the unit circle. In this article, we use the preceding theorems to describe a characterization of points in the dual braid complex which aligns with Thurston’s degree- $d$ -invariant laminations, thus providing an explicit embedding of the dual braid complex into  $\text{POLY}_d^{mc}(\mathbb{C}^*)$ . To complete the proof of Theorem E, we invoke Thurston’s deformation retraction.

The connection between spaces of polynomials and the dual braid complex was pointed out to us by Daan Krammer in 2017, and his comment inspired much of the work in this article and other recent work by the authors [DM20, DM22, DM]. In [DM], we describe a bounded piecewise-Euclidean metric for  $\text{POLY}_d^{mc}(\mathbb{C}^*)$  and a finite cell structure for its metric completion such that the subspace  $\text{POLY}_d^{mc}(\mathbf{S})$  of polynomials with critical values on the unit circle inherits a metric cell structure which is isometric to the dual braid complex. In particular, [DM] contains an alternative direct proof of Theorem E which does not rely on Thurston’s argument.

A general version of Theorem E appears in work of David Bessis [Bes15, Section 10], where he uses a complex of spaces on the universal cover of  $\text{POLY}_d^{mc}(\mathbb{C}^*)$  together with tools from algebraic topology [Hat02, Section 4.G] to prove the result. In fact, his work proves a version of Theorem E that extends to all finite complex reflection groups, including all finite Coxeter groups. Concretely, Bessis proves that if  $\mathbf{c}$  is a Coxeter element for a finite Coxeter group  $W$ , then the interval complex  $K_{\mathbf{c}}$  is a spine for the corresponding quotient of the complexified hyperplane complement [Bes15]. The constructions introduced here might lead to an alternate algebraic proof of Bessis’ result.

**Structure of the article.** Sections 1 and 2 review the definitions and properties of interval posets, orthoschemes and interval complexes. Section 3 introduces linear and circular factorization posets and examines their combinatorial structure. Section 4 introduces weighted analogs of these factorizations and examines their topology and geometry. Section 5 defines graded topological posets of weighted factorizations and proves Theorems B, C, and D. Section 6 defines the poset  $\text{NC}_d(\mathbf{S})$  of degree- $d$ -invariant partitions of the circle and proves Theorems A and E.

## 1. POSETS

In this section we recall some background information for partially ordered sets and a special kind of metric simplex known as an orthoscheme. See [Sta12, Ch. 3] for a standard reference on posets and [Hat02, BM10] for background on simplices.

A partially ordered set  $P$  is a *lattice* if each pair of elements has a unique meet and a unique join. A poset is *bounded* if it has a unique maximum element and

a unique minimum element. A *chain* of length  $k$  in  $P$  is a collection of distinct elements  $x_0, \dots, x_k \in P$  such that  $x_0 < \dots < x_k$ , and a chain is *maximal* if it is not properly contained in another chain. We write  $\text{CHAINS}(P)$  for the poset of all chains in  $P$  under inclusion. We say that  $P$  is *graded* if there is a *rank function*  $\text{rk}: P \rightarrow \mathbb{N}$  such that for all  $x, y \in P$ , we have  $\text{rk}(x) < \text{rk}(y)$  whenever  $x < y$  and  $\text{rk}(x) + 1 = \text{rk}(y)$  whenever  $x < y$  and there is no  $z \in P$  with  $x < z < y$ . A finite bounded poset is graded if and only if all of its maximal chains have the same length, which we call the *height* of the poset. Given two elements  $x, y \in P$ , the set of all  $z \in P$  with  $x \leq z \leq y$  is the *interval*  $[x, y]$ . The set of all elements  $y \in P$  with  $x \leq y$  is called the *upper set* of  $x$  and is denoted  $\uparrow(x)$ . Similarly, the set of all  $z \in P$  with  $z \leq x$  is the *lower set* of  $x$ , denoted  $\downarrow(x)$ .

**Example 1.1** (Boolean lattice). The *Boolean lattice*  $\text{BOOL}(n)$  consists of all subsets for the  $n$ -element set  $\{1, \dots, n\}$ , partially ordered under inclusion, and it is indeed a lattice: given  $A, B \in \text{BOOL}(n)$ , the unique meet is  $A \cap B$  and the unique join is  $A \cup B$ . This poset is also graded, with rank function  $\text{rk}: \text{BOOL}(n) \rightarrow \mathbb{N}$  given by  $\text{rk}(A) = |A|$  for each  $A \subseteq \{1, \dots, n\}$ . Fixing an element  $A \in \text{BOOL}(n)$  with rank  $k$ , the lower set  $\downarrow(A)$  is isomorphic to the smaller Boolean lattice  $\text{BOOL}(k)$ , whereas the upper set  $\uparrow(A)$  is isomorphic to  $\text{BOOL}(n - k)$ . Finally, we write  $\text{BOOL}^*(n)$  to mean the subposet of nonempty elements in  $\text{BOOL}(n)$  and refer to this as the *truncated Boolean poset*.

There is a close connection between partially ordered sets and simplicial complexes. Each cell complex has an associated *face poset* and each poset has an associated *order complex*. We will make use of both operations, and we begin by describing the first.

**Definition 1.2** (Face posets). Let  $X$  be a simplicial complex. The *face poset*  $P(X)$  is defined to be the graded poset of all faces of  $X$  (including the empty face), ordered by inclusion.

For example, each face of an  $n$ -dimensional simplex can be specified precisely by a subset of the  $n + 1$  vertices, and the relation of incidence between two faces corresponds exactly to inclusion between the two subsets. In other words, the face poset for the  $n$ -simplex is isomorphic to the Boolean lattice  $\text{BOOL}(n + 1)$ .

**Definition 1.3** (Order complexes). Let  $P$  be a graded poset. The *order complex*  $\Delta(P)$  is the ordered simplicial complex with vertex set  $P$  and an ordered  $k$ -simplex on the vertices  $x_0, \dots, x_k$  whenever  $x_0 < \dots < x_k$  is a chain in  $P$ . By construction, there is a bijection between  $\text{CELLS}(\Delta(P))$  and  $\text{CHAINS}(P)$ . See Hatcher's book for background on ordered simplicial complexes and  $\Delta$ -complexes [Hat02].

The order complex of  $\text{BOOL}(n)$ , for example, is homeomorphic to the cell complex obtained by subdividing the cube  $[0, 1]^n \subset \mathbb{R}^n$  into  $n!$  top-dimensional simplices via the  $\binom{n}{2}$  hyperplanes with equations  $x_i = x_j$  where  $i \neq j$ . Each of the top-dimensional simplices is determined by a path of length  $n$  from  $(0, \dots, 0)$  to  $(1, \dots, 1)$  along the edges of the cube which gives an ordering of the vertices for each simplex. Moreover, we can promote this homeomorphism to an isometry with an appropriate choice of metric for the order complex.

**Definition 1.4** (Orthoschemes). The simplex spanned by points  $\mathbf{p}_0, \dots, \mathbf{p}_n$  in  $\mathbb{R}^n$  is called an  *$n$ -dimensional orthoscheme* if the set of  $n$  vectors  $\{\mathbf{p}_i - \mathbf{p}_{i-1} \mid i \in [n]\}$

element of $\text{BOOL}(3)$	face of the 2-orthoscheme
$\{1, 2, 3\}$	$0 \leq x_1 \leq x_2 \leq 1$
$\{1, 2\}$	$0 \leq x_1 \leq x_2 = 1$
$\{2, 3\}$	$0 = x_1 \leq x_2 < 1$
$\{1, 3\}$	$0 \leq x_1 = x_2 \leq 1$
$\{1\}$	$0 \leq x_1 = x_2 = 1$
$\{2\}$	$0 = x_1 \leq x_2 = 1$
$\{3\}$	$0 = x_1 = x_2 \leq 1$

TABLE 3. A bijection between subsets and (closed) faces.

is orthogonal. If those vectors are orthonormal, then the simplex is a *standard  $n$ -dimensional orthoscheme*, and it is isometric to the subset of points  $(x_1, \dots, x_n) \in \mathbb{R}^n$  subject to the inequalities  $0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1$ .

For each bounded graded poset  $P$ , we give the order complex  $\Delta(P)$  the *orthoscheme metric*, in which each maximal simplex is a standard orthoscheme with the order of its vertices determined by the order of the corresponding maximal chain in  $P$ . For more detail on this use of the orthoscheme metric, see [BM10, Sec. 5-6].

To conclude this section, we give a brief remark on the faces of orthoschemes.

**Remark 1.5.** Each face of a standard  $n$ -dimensional orthoscheme is itself an orthoscheme, but one which is not necessarily standard. Using the inequalities  $0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1$  to describe the standard  $n$ -dimensional orthoscheme, each nonempty  $A = \{t_0, \dots, t_k\}$  in  $\text{BOOL}(n+1)$  corresponds to the  $k$ -dimensional face determined by the equations

$$\begin{aligned} x_i &= 0 \text{ if } 1 \leq i \leq t_0; \\ x_i &= x_{i+1} \text{ if } t_j < i < i+1 \leq t_{j+1} \text{ for some } j \in \{0, 1, \dots, k-1\}; \\ x_i &= 1 \text{ if } t_k < i \leq n+1. \end{aligned}$$

In plain language, each element of  $\text{BOOL}(n+1)$  determines a face by changing some of the  $n+1$  inequalities into equalities. See Table 3 for an example when  $n=2$ . Furthermore, this face is isometric to the set of points  $(y_1, \dots, y_k) \in \mathbb{R}^k$  determined by the inequalities

$$0 \leq \frac{y_1}{\sqrt{t_1 - t_0}} \leq \frac{y_2}{\sqrt{t_2 - t_1}} \leq \dots \leq \frac{y_k}{\sqrt{t_k - t_{k-1}}} \leq 1.$$

Note in particular that two nonempty subsets  $\{t_0, \dots, t_k\}$  and  $\{s_0, \dots, s_k\}$  of  $[n+1]$  label isometric  $k$ -dimensional faces if  $t_i - t_{i-1} = s_i - s_{i-1}$  for all  $i \in \{1, 2, \dots, k\}$ .

## 2. COMPLEXES

In this section, we review the definition and basic properties of interval complexes and the dual braid complex. For the rest of the article  $G$  is a group and  $X$  is a fixed conjugacy-closed generating set for  $G$ .

**Definition 2.1** (Intervals and Order Complexes). Suppose  $g \in G$  can be written as a product of elements in  $X$  (i.e.  $g$  belongs to the *monoid* generated by  $X$ )

and let  $\ell(g)$  denote the length of a minimal factorization of  $g$  into elements of  $X$ . This induces a partial order on  $G$  by declaring that  $h \leq g$  if  $h \cdot h' = g$  and  $\ell(h) + \ell(h') = \ell(g)$ ; in other words,  $h \leq g$  if there is a minimal-length factorization of  $g$  into elements of  $X$  which has a minimal-length factorization of  $h$  as a left prefix. The *interval poset*  $P_g = [1, g]$  is the poset of all elements between the identity and  $g$  in this ordering. Observe that the order diagram or Hasse diagram of  $P_g$  is the directed graph obtained by taking the union of all geodesics from 1 to  $g$  in the right Cayley graph of  $G$  with respect to  $X$ . Since every geodesic has the same length, this makes the interval poset  $P_g$  into a bounded graded poset with height  $\ell(g)$ . The order complex of  $P_g$  is the ordered simplicial complex  $O_g = \Delta(P_g)$ .

The primary application of these general constructions is when  $G = W$  is a Coxeter group,  $X = T$  is the set of all reflections, and  $g = \mathbf{c}$  is a Coxeter element of  $W$ . Of particular interest is the “type A” case where  $G = \text{SYM}_d$ ,  $X = T$  is the set of all transpositions, and  $g = \delta$  is the  $d$ -cycle  $(1\ 2\ \cdots\ d)$ . Finally, the case where  $G$  is  $\mathbb{Z}$  is a fundamental building block of the general theory.

**Example 2.2.** Let  $G = \mathbb{Z}$  with generating set  $X = \{1\}$ . For each positive integer  $n$ ,  $\ell(n) = n$ , and the induced partial order is the usual one for  $\mathbb{Z}$ . The interval from 0 to  $n$  in this poset consists of a single chain with  $n + 1$  elements.

**Example 2.3.** Let  $G$  be the symmetric group  $\text{SYM}_d$  and let  $X = T$  be the set of all transpositions. Then for all  $g \in \text{SYM}_d$ , the length  $\ell(g)$  is called the *absolute reflection length* and the induced partial order is the *absolute order* on the symmetric group. If we define  $\delta$  to be the  $d$ -cycle  $(1\ 2\ \cdots\ d) \in \text{SYM}_d$ , then the interval  $[1, \delta]$  is the *lattice of noncrossing permutations* [McC06], [DM, Section 4]. For example, if  $\delta = (1\ 2\ 3) \in \text{SYM}_3$  and  $T = \{a, b, c\}$  where  $a = (1\ 2)$ ,  $b = (2\ 3)$  and  $c = (1\ 3)$ , then the interval  $[1, \delta]$  consists of five elements (see Figure 2). This structure produces the *dual presentation* for  $\text{BRAID}_3$ , defined by  $\langle a, b, c, \delta \mid ab = bc = ca = \delta \rangle$ . More generally, the dual presentation for the  $d$ -strand braid group  $\text{BRAID}_d$  has as its generating set the nontrivial elements of  $[1, \delta]$ , with relations consisting of all words which arise from closed loops in  $[1, \delta]$  which are based at the identity [Bra01, Bes03]. See also [McC15, MS17].

Because the generating set  $X$  is assumed to be closed under conjugation, the interval poset  $P_g = [1, g]$  also has a certain closure property.

**Lemma 2.4.** *If  $x_1, \dots, x_n \in G$  with  $x_1 \cdots x_n = g$  and  $\ell(x_1) + \cdots + \ell(x_n) = \ell(g)$ , then for any choice of integers  $1 \leq i_1 < \cdots < i_k \leq n$ , we have  $x_{i_1} \cdots x_{i_k} \in [1, g]$ .*

*Proof.* We can rewrite the factorization  $g = x_1 \cdots x_n$  by grouping terms to obtain  $g = w_0 x_{i_1} w_1 x_{i_2} w_2 \cdots w_{k-1} x_{i_k} w_k$ , where  $w_j$  is defined appropriately. Note that if the trivial  $w_j$  are removed, this is a merged factorization of  $g$  and

$$\ell(g) = \ell(w_0) + \ell(x_{i_1}) + \ell(w_1) + \cdots + \ell(w_{k-1}) + \ell(x_{i_k}) + \ell(w_k).$$

If we define  $z_j = x_{i_j} \cdots x_{i_k}$  for each  $j$ , then we can rearrange the first product to obtain

$$g = x_{i_1} \cdots x_{i_k} (z_1^{-1} w_0 z_1) (z_2^{-1} w_1 z_2) \cdots (z_k^{-1} w_{k-1} z_k) w_k.$$

By definition, we know that  $\ell(x_{i_1} \cdots x_{i_k}) \leq \ell(x_{i_1}) + \cdots + \ell(x_{i_k})$ , and since the generating set  $X$  is closed under conjugation, we have  $\ell(z_j^{-1} w_{j-1} z_j) = \ell(w_{j-1})$  for each  $j$ . Combining these with the equation above, we obtain

$$\ell(g) = \ell(x_{i_1} \cdots x_{i_k}) + \ell(w_0) + \cdots + \ell(w_n),$$

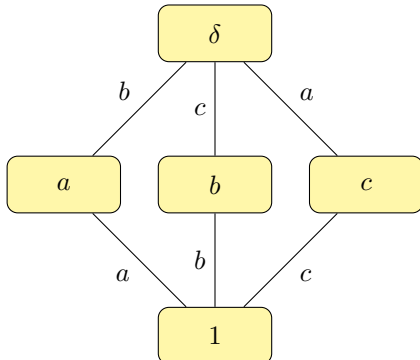


FIGURE 2. The interval poset  $P_\delta = [1, \delta]$  described in Example 2.3, depicted as a subgraph of the Cayley graph  $\text{Cay}(\text{SYM}_3, T)$ . Its order complex  $O_g$  is a triple of right-angled Euclidean triangles with a common hypotenuse.

from which it follows that  $x_{i_1} \cdots x_{i_k} \leq g$ .  $\square$

**Definition 2.5** (Dual braid complex). Let  $g \in G$ . The *interval complex*  $K_g$  associated to the interval  $P_g = [1, g]$  is the single-vertex  $\Delta$ -complex obtained by identifying faces in the order complex  $O_g = \Delta([1, g])$  as follows: the  $k$ -simplices labeled by chains  $x_0 < \cdots < x_k$  and  $y_0 < \cdots < y_k$  are identified if and only if  $x_{i-1}^{-1}x_i = y_{i-1}^{-1}y_i$  for all  $i \in \{1, \dots, k\}$ . Note that this identification is well-defined by the ordering given to each simplex. In particular, there is a well-defined surjective cellular map  $q: O_g \twoheadrightarrow K_g$  from the ordered simplicial complex  $O_g$  to the one-vertex  $\Delta$ -complex  $K_g$ . In the special case when  $G = \text{SYM}_d$  and  $X = T$  as outlined in Example 2.3, we write  $\delta = (1\ 2 \cdots d)$  and refer to the interval complex  $K_\delta$  as the (one vertex) *dual braid complex*<sup>1</sup>.

Brady introduced the dual braid complex in [Bra01] and showed that it is a classifying space for the  $d$ -strand braid group. Using the orthoscheme metric, it is conjectured that the dual braid complex is locally  $\text{CAT}(0)$  [BM10]. This has been shown when  $d \leq 7$  but remains open for higher values of  $d$  [BM10, HKS16, Jeo23].

**Remark 2.6.** If  $h \leq g$ , then the order complex  $O_h = \Delta([1, h])$  is isometric to a subcomplex of  $O_g = \Delta([1, g])$  and, by following the gluing described in Definition 2.5, we can see that the interval complex  $K_h$  is isometric to a subcomplex of  $K_g$ . When  $G = \text{SYM}_d$  and  $X = T$ , each permutation  $\gamma \in \text{SYM}_d$  can be written as a product of disjoint cycles  $\gamma = x_1 \cdots x_k$  and the interval  $[1, \gamma]$  is isomorphic to the product of intervals  $[1, x_1] \times \cdots \times [1, x_k]$ , so it follows that the interval complex  $K_\gamma$  is a subcomplex of  $K_\delta$  which is isometric to a product of smaller dual braid complexes.

### 3. CELLS AND FACTORIZATIONS

In this section we develop a convenient way of describing the cells in the order complex  $O_g$  and its quotient interval complex  $K_g = q(O_g)$ . To this end, we define

<sup>1</sup>In some articles, “dual braid complex” refers to the universal cover of  $K_g$ , rather than  $K_g$  itself. Here, the term refers to the one-vertex complex  $K_g$  since the universal cover is never needed.

two key posets for each  $g \in G$ :  $\text{FACT}(G, g, \mathbf{I})$ , the poset of linear factorizations of  $g$ , and  $\text{FACT}(G, g, \mathbf{S})$ , the poset of circular factorizations of  $g$ . As usual, let  $G$  be a group with conjugacy-closed generating set  $X$  and let  $P_g = [1, g]$  be the interval poset for an element  $g \in \text{MON}(X)$  (Definition 2.1). Let  $\ell: \text{MON}(X) \rightarrow \mathbb{Z}$  be the length function that takes each  $h \in \text{MON}(X)$  to its *length* as a word over  $X$ . Note that for  $h \in P_g$ ,  $\ell(h)$  is its rank.

**Definition 3.1** (Linear factorizations). A *linear factorization* of  $g \in \text{MON}(X)$  is a row vector  $\mathbf{x} = [x_L \ x_1 \ \cdots \ x_k \ x_R]$  with entries in  $\text{MON}(X)$  such that

- (1)  $x_i$  is nontrivial when  $i \in \{1, \dots, k\}$ ;
- (2)  $\ell(x_L) + \ell(x_1) + \cdots + \ell(x_k) + \ell(x_R) = \ell(g)$ ;
- (3)  $x_L \cdot x_1 \cdots x_k \cdot x_R = g$ .

Note that the entries here are elements of  $\text{MON}(X)$ , not necessarily elements in  $X$ . Also note that the left and right entries of  $\mathbf{x}$  ( $x_L$  and  $x_R$ ) are labeled differently from the others since they are treated differently. In particular, these elements are allowed to be trivial. For each  $i \in \{0, 1, \dots, k\}$ , a linear factorization

$$[x_L \ \cdots \ x_{i-1} \ x_i \ x_{i+1} \ x_{i+2} \ \cdots \ x_R]$$

of length  $k + 2$  can be *merged* at position  $i$  to obtain the linear factorization

$$[x_L \ \cdots \ x_{i-1} \ (x_i x_{i+1}) \ x_{i+2} \ \cdots \ x_R]$$

of length  $k + 1$  (where  $x_0$  and  $x_{k+1}$  are understood to mean  $x_L$  and  $x_R$  respectively). Let  $\text{FACT}(G, g, \mathbf{I})$  denote the set of all linear factorizations of  $g$ , equipped with the partial order  $\mathbf{x} \leq \mathbf{y}$  if  $\mathbf{x}$  can be obtained from  $\mathbf{y}$  by a sequence of merges.

By Lemma 2.4, we see that the elements of  $\text{SYM}_d$  which appear in linear factorizations of  $g$  are precisely those which belong to the interval  $[1, g]$ . We also note that linear factorizations are closely related to the “reduced products” in [McC] and the “block factorizations” in [Rip12], but with the slight variation that the first and last entries are permitted to be trivial.

**Example 3.2** ( $\text{FACT}(\text{SYM}_3, \delta, \mathbf{I})$ ). If  $G = \text{SYM}_3$  and  $X = \{a, b, c\}$  as in Example 2.3, then  $\text{FACT}(\text{SYM}_3, \delta, \mathbf{I})$  contains the 15 elements shown in Figure 3. There are 3 in rank 2, 7 in rank 1, and 5 in rank 0.

**Example 3.3** ( $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$ ). In the special case  $G = \mathbb{Z}$  and  $X = \{1\}$  we switch to additive notation and refer to the linear factorizations of  $n \in \mathbb{Z}$  as *linear compositions*. The set of all linear compositions is denoted  $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$ . The reason for this switch to additive notation is to make the map  $L$  from factorizations to compositions easier to define. See Definition 3.9. Unlike the more general case,  $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$  has a unique maximum element given by the row vector  $[0 \ 1 \ \cdots \ 1 \ 0]$  of length  $n + 2$ . See Figure 4 for an example when  $n = 2$ .

**Proposition 3.4.** *Let  $g \in G$ . Then  $\text{FACT}(G, g, \mathbf{I})$  is isomorphic to the poset of nonempty chains for  $[1, g]$ , ordered by inclusion.*

*Proof.* Let  $f$  be the function which sends the linear factorization  $[x_L \ x_1 \ \cdots \ x_k \ x_R]$  to the chain

$$x_L \leq x_L x_1 \leq x_L x_1 x_2 \leq \cdots \leq x_L x_1 \cdots x_k,$$

noting that  $x_0 x_1 \cdots x_i \in [1, g]$  for each  $i$ . Then  $f$  has an inverse which takes the chain  $y_0 < y_1 < \cdots < y_k$  to the linear factorization  $[y_0 \ y_0^{-1} y_1 \ \cdots \ y_{k-1}^{-1} y_k \ y_k^{-1} g]$ .

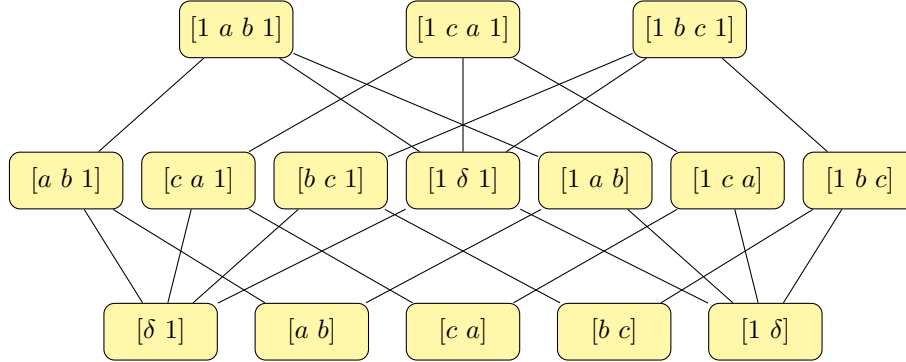


FIGURE 3. The poset  $\text{FACT}(\text{SYM}_3, \delta, \mathbf{I})$  of linear factorizations of the 3-cycle  $\delta = (1\ 2\ 3)$  with respect to the generators  $a = (1\ 2)$ ,  $b = (2\ 3)$ , and  $c = (1\ 3)$ .

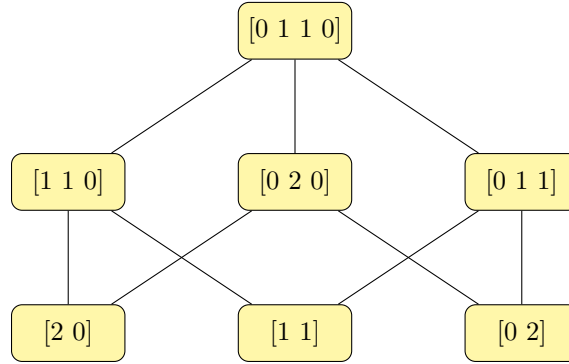


FIGURE 4. The poset  $\text{COMP}(\mathbb{Z}, 2, \mathbf{I})$  of linear compositions of 2.

Moreover, merging factorizations in  $\text{FACT}(G, g, \mathbf{I})$  corresponds exactly to taking subchains in  $[1, g]$ , so  $f$  is an order-preserving bijection with order-preserving inverse and thus the two posets are isomorphic.  $\square$

**Corollary 3.5.**  $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$  is isomorphic to  $\text{BOOL}^*(n+1)$ .

By identifying the ends of a linear factorization, we obtain a new object which we call a circular factorization.

**Definition 3.6** (Circular factorizations). Let  $g \in G$ . Define an equivalence relation on  $\text{FACT}(G, g, \mathbf{I})$  by declaring  $\mathbf{x} = [x_L\ x_1\ \cdots\ x_k\ x_R]$  and  $\mathbf{y} = [y_L\ y_1\ \cdots\ y_k\ y_R]$  to be equivalent if and only if  $x_i = y_i$  for all  $i \in \{1, \dots, k\}$ . We refer to the equivalence classes as *circular factorizations of  $g$* , each of which is represented by the unique element with 1 as its final entry. More concretely, the equivalence class of  $\mathbf{x}$  is denoted  $\bar{\mathbf{x}} = [gx_Rg^{-1}x_L \mid x_1\ \cdots\ x_k \mid 1]$ , where the vertical bars are indicators used to distinguish the first and last terms from the rest of the factorization. Let  $\text{FACT}(G, g, \mathbf{S})$  denote the set of all circular factorizations under the partial order  $\bar{\mathbf{x}} \leq \bar{\mathbf{y}}$  if  $\mathbf{x}' \leq \mathbf{y}'$  for some  $\mathbf{x}' \in \bar{\mathbf{x}}$  and  $\mathbf{y}' \in \bar{\mathbf{y}}$ . Define the

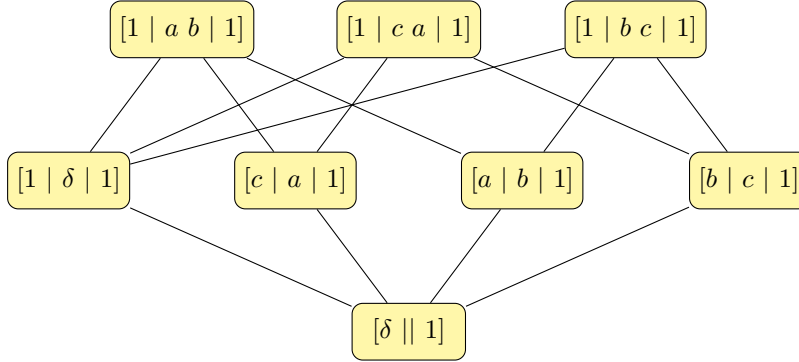


FIGURE 5. The poset  $\text{FACT}(\text{SYM}_3, \delta, \mathbf{S})$  of circular factorizations of the 3-cycle  $\delta = (1\ 2\ 3)$  with respect to the generators  $a = (1\ 2)$ ,  $b = (2\ 3)$ , and  $c = (1\ 3)$ .

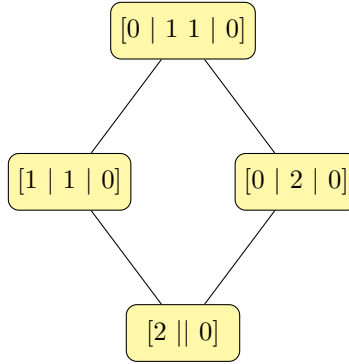


FIGURE 6. The poset  $\text{COMP}(\mathbb{Z}, 2, \mathbf{S})$  of circular compositions of 2.

order-preserving surjection  $q: \text{FACT}(G, g, \mathbf{I}) \rightarrow \text{FACT}(G, g, \mathbf{S})$  by sending each linear factorization to the equivalence class which contains it, i.e.  $q(\mathbf{x}) = \overline{\mathbf{x}}$ .

**Example 3.7** ( $\text{FACT}(\text{SYM}_3, \delta, \mathbf{S})$ ). When  $G = \text{SYM}_3$ ,  $X = T = \{a, b, c\}$ , and  $\delta = (1\ 2\ 3)$  as in Example 2.3, the poset  $\text{FACT}(\text{SYM}_3, \delta, \mathbf{S})$  has 8 elements. See Figure 5.

**Example 3.8** ( $\text{COMP}(\mathbb{Z}, n, \mathbf{S})$ ). When  $G = \mathbb{Z}$  and  $X = \{1\}$ , we denote the set  $\text{FACT}(\mathbb{Z}, g, \mathbf{S})$  by  $\text{COMP}(\mathbb{Z}, n, \mathbf{S})$  and refer to its elements as *circular compositions of  $n$* . When  $n = 2$ , for example, the poset  $\text{COMP}(\mathbb{Z}, 2, \mathbf{S})$  of circular compositions of 2 has four elements and is isomorphic to  $\text{BOOL}(2)$  — see Figure 6. More generally, it is important to note that while there is a rank-preserving bijection between  $\text{COMP}(\mathbb{Z}, n, \mathbf{S})$  and  $\text{BOOL}(n)$ , the two are not isomorphic when  $n > 2$  since there are relations in the former which do not appear in the latter. For example, each rank-one element of  $\text{BOOL}(3)$  lies beneath exactly two rank-two elements, but the circular composition  $[1\ 2\ 0]$  lies beneath  $[1\ 1\ 1\ 0]$ ,  $[0\ 1\ 2\ 0]$  and  $[0\ 2\ 1\ 0]$  in  $\text{COMP}(\mathbb{Z}, 3, \mathbf{S})$ .

$$\begin{array}{ccc}
\text{FACT}(G, g, \mathbf{I}) & \xrightarrow{L} & \text{COMP}(\mathbb{Z}, n, \mathbf{I}) \\
\downarrow q & & \downarrow q \\
\text{FACT}(G, g, \mathbf{S}) & \xrightarrow{\bar{L}} & \text{COMP}(\mathbb{Z}, n, \mathbf{S})
\end{array}$$

FIGURE 7. Our four key posets fit into a commutative diagram.

We close this section by providing combinatorial connections between the posets introduced in this section. See Figure 7 for the relevant commutative diagram.

**Definition 3.9** (Factorizations to compositions). Let  $g \in G$  with  $n = \ell(g)$  and define the function  $L: \text{FACT}(G, g, \mathbf{I}) \rightarrow \text{COMP}(\mathbb{Z}, n, \mathbf{I})$  by sending the factorization  $\mathbf{x} = [x_L \ x_1 \ \cdots \ x_k \ x_R]$  to the composition  $L(\mathbf{x}) = [\ell(x_L) \ \ell(x_1) \ \cdots \ \ell(x_k) \ \ell(x_R)]$ . Similarly, define  $\bar{L}: \text{FACT}(G, g, \mathbf{S}) \rightarrow \text{COMP}(\mathbb{Z}, n, \mathbf{S})$  by sending  $\bar{\mathbf{x}} = [x_L \mid x_1 \ \cdots \ x_k \mid 1]$  to  $\bar{L}(\bar{\mathbf{x}}) = [\ell(x_L) \mid \ell(x_1) \ \cdots \ \ell(x_k) \mid 0]$ . Note that if  $\mathbf{x}'$  is obtained from  $\mathbf{x}$  by performing a merge in position  $i$ , then we know  $\ell(x_{ix_{i+1}}) = \ell(x_i) + \ell(x_{i+1})$  by Lemma 2.4, and this means that  $L(\mathbf{x}')$  is obtained from  $L(\mathbf{x})$  by merging at position  $i$ . By similar reasoning, we know that if  $\bar{\mathbf{x}}' \leq \bar{\mathbf{x}}$ , then  $\bar{L}(\bar{\mathbf{x}}') \leq \bar{L}(\bar{\mathbf{x}})$ . Thus  $L$  and  $\bar{L}$  are both order-preserving functions.

**Lemma 3.10.** *Let  $\mathbf{x} \in \text{FACT}(G, g, \mathbf{I})$ . Then  $L$  restricts to an isomorphism from the lower set  $\downarrow(\mathbf{x})$  to the lower set  $\downarrow(L(\mathbf{x}))$ .*

*Proof.* As described in Definition 3.9, each element of the lower set  $\downarrow(\mathbf{x})$  is obtained from  $\mathbf{x}$  via a sequence of merges, and the same merges can be performed on  $L(\mathbf{x})$  to obtain an element of  $\downarrow(L(\mathbf{x}))$ . Two different sequences of merges produce the same element of  $\downarrow(\mathbf{x})$  if and only if the analogous sequence produces the same element of  $\downarrow(L(\mathbf{x}))$ , so  $L$  restricts to a bijection and thus an isomorphism.  $\square$

Note that Lemma 3.10 does not hold for  $\bar{L}$ , as it is possible to have elements  $\bar{\mathbf{x}} \in \text{FACT}(G, g, \mathbf{S})$  such that the restriction of  $\bar{L}$  to  $\downarrow(\bar{\mathbf{x}})$  is an order-preserving surjection that is not injective. Compare Figures 5 and 6 for an example.

**Proposition 3.11.** *The functions  $L$  and  $\bar{L}$  are surjective order-preserving maps and  $\bar{L}q = qL$ .*

*Proof.* By Definition 3.9, we know the two maps are order-preserving. Also, if  $\mathbf{x}$  is a maximal element of  $\text{FACT}(G, g, \mathbf{I})$ , then  $L(\mathbf{x}) = [0 \ 1 \ \cdots \ 1 \ 0]$ , the maximum element of  $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$ . By Lemma 3.10, we know that  $L$  restricts to an isomorphism from  $\downarrow(\mathbf{x})$  to  $\downarrow(L(\mathbf{x})) = \text{COMP}(\mathbb{Z}, n, \mathbf{I})$ , so the map  $L$  is surjective. The fact that  $\bar{L}q = qL$  follows directly from Definition 3.6 and Example 3.8. Since  $q$  and  $L$  are surjective,  $qL$  is surjective and  $\bar{L}q = qL$  is surjective. Thus  $\bar{L}$  must also be surjective.  $\square$

**Proposition 3.12.** *The poset  $\text{FACT}(G, g, \mathbf{I})$  is simplicial, i.e. each of its intervals is isomorphic to a Boolean lattice.*

*Proof.* For each  $\mathbf{x}, \mathbf{y} \in \text{FACT}(G, g, \mathbf{I})$  with  $\mathbf{x} \leq \mathbf{y}$ , choose a maximal element  $\mathbf{z}$  with  $\mathbf{x} \leq \mathbf{y} \leq \mathbf{z}$ . Then the interval  $[\mathbf{x}, \mathbf{y}]$  is contained within the lower set  $\downarrow(\mathbf{z})$ , which we know by Lemma 3.10 is isomorphic to  $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$ , which is itself a truncated

Boolean lattice (Corollary 3.5). Since every interval of a truncated Boolean lattice is isomorphic to a smaller Boolean lattice, the proof is complete.  $\square$

#### 4. POINTS AND WEIGHTED FACTORIZATIONS

In this section, we introduce a way of labeling individual points in the order complex  $O_g$  and the interval complex  $K_g$  by adding in weights. For each  $g \in G$ , we define  $\text{WFACT}(G, g, \mathbf{I})$ , the space of weighted linear factorizations of  $g$ , and  $\text{WFACT}(G, g, \mathbf{S})$ , the space of weighted circular factorizations of  $g$ . The points in each space are defined as weighted versions of poset elements from the previous section, or as decorated multisets in the interval  $\mathbf{I}$  or the circle  $\mathbf{S}$ .

**Definition 4.1** ( $G$ -multisets). Let  $S$  be a set and let  $G$  be a group. We define a  $G$ -multiset on  $S$  to be a function  $\mathbf{x}: S \rightarrow G$  such that  $\mathbf{x}(s)$  is the identity in  $G$  for all but finitely many  $s \in S$ . We denote the set of all  $G$ -multisets on  $S$  by  $\text{MULT}(G, S)$ .

We are interested in eight cases which arise from two choices:  $S$  is either the unit interval  $\mathbf{I}$  or the circle  $\mathbf{S}$ , and  $G$  is a general group, a Coxeter group  $W$ , the symmetric group  $\text{SYM}_d$ , or the integers  $\mathbb{Z}$ . See Tables 1 and 2 in the Introduction. The  $G$ -multisets arising from these cases are slightly unusual in that they are functions from uncountable sets to discrete groups which produce nontrivial output for only finitely many inputs, but the interpretation is natural: one should picture the set  $S$  with a finite number of special points labeled by non-trivial elements of  $G$ .

**Definition 4.2** ( $\text{WFACT}(G, g, \mathbf{I})$ ). Let  $g \in G$ . For each  $G$ -multiset  $\mathbf{u}: \mathbf{I} \rightarrow G$ , let  $0 = s_L < s_1 < \dots < s_k < s_R = 1$  be such that  $\mathbf{u}$  is nontrivial on the set  $\{s_1, \dots, s_k\}$  and trivial on its complement in  $(0, 1)$ . The  $G$ -multiset  $\mathbf{u}$  may or may not be trivial on 0 and 1. For each  $i$ , let  $x_i = \mathbf{u}(s_i)$  and define  $P(\mathbf{u}) = [x_L \ x_1 \ \dots \ x_k \ x_R]$ ; we say that  $\mathbf{u}$  is a *weighted linear factorization* of  $g$  if  $P(\mathbf{u})$  is a linear factorization of  $g$ . Note that  $P(\mathbf{u})$  necessarily has length at least 2. Denote the set of all weighted linear factorizations of  $g$  by  $\text{WFACT}(G, g, \mathbf{I})$  and observe that  $P$  is a surjective function  $P: \text{WFACT}(G, g, \mathbf{I}) \twoheadrightarrow \text{FACT}(G, g, \mathbf{I})$ . We will often use the convenient shorthand  $\mathbf{u} = 0^{x_L} s_1^{x_1} \dots s_k^{x_k} 1^{x_R}$  to denote elements of  $\text{WFACT}(G, g, \mathbf{I})$ .

**Example 4.3** ( $\text{WFACT}(\text{SYM}_3, \delta, \mathbf{I})$ ). If  $G = \text{SYM}_3$ ,  $T = \{a, b, c\}$  and  $\delta = (1 \ 2 \ 3)$  as in Example 2.3, then  $\text{WFACT}(\text{SYM}_3, \delta, \mathbf{I})$  is a 2-dimensional simplicial complex which consists of three triangles, all sharing a common edge. The 3 triangles, 7 edges, and 5 vertices of  $O_\delta = \text{WFACT}(\text{SYM}_3, \delta, \mathbf{I})$  shown in Figure 8 are labeled by the 15 elements of  $\text{FACT}(\text{SYM}_3, \delta, \mathbf{I})$  shown in Figure 3.

**Example 4.4** ( $\text{WCOMP}(\mathbb{Z}, n, \mathbf{I})$ ). When  $G = \mathbb{Z}$  and  $X = \{1\}$ , we denote the set  $\text{WFACT}(\mathbb{Z}, n, \mathbf{I})$  by  $\text{WCOMP}(\mathbb{Z}, n, \mathbf{I})$  and refer to its elements as *weighted linear compositions* of  $n$ . If we consider the action of  $\text{SYM}_n$  on the  $n$ -cube  $\mathbf{I}^n$  by permuting coordinates, then each element  $\mathbf{s} = 0^{a_L} s_1^{a_1} \dots s_k^{a_k} 1^{a_R}$  in  $\text{WCOMP}(\mathbb{Z}, n, \mathbf{I})$  can be viewed as a point on the quotient space  $\mathbf{I}^n / \text{SYM}_n$ , which is isometric to a standard  $n$ -dimensional orthoscheme. For each  $\mathbf{a} \in \text{COMP}(\mathbb{Z}, n, \mathbf{I})$ , the set of weighted linear compositions  $\mathbf{s}$  of  $n$  with  $P(\mathbf{s}) = \mathbf{a}$  forms an open face of the standard orthoscheme which we refer to as a (non-standard) *orthoscheme of shape  $\mathbf{a}$* . This recovers what we found in Corollary 3.5:  $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$  is the face poset for the standard  $n$ -dimensional orthoscheme. Moreover, Remark 1.5 tells us that elements

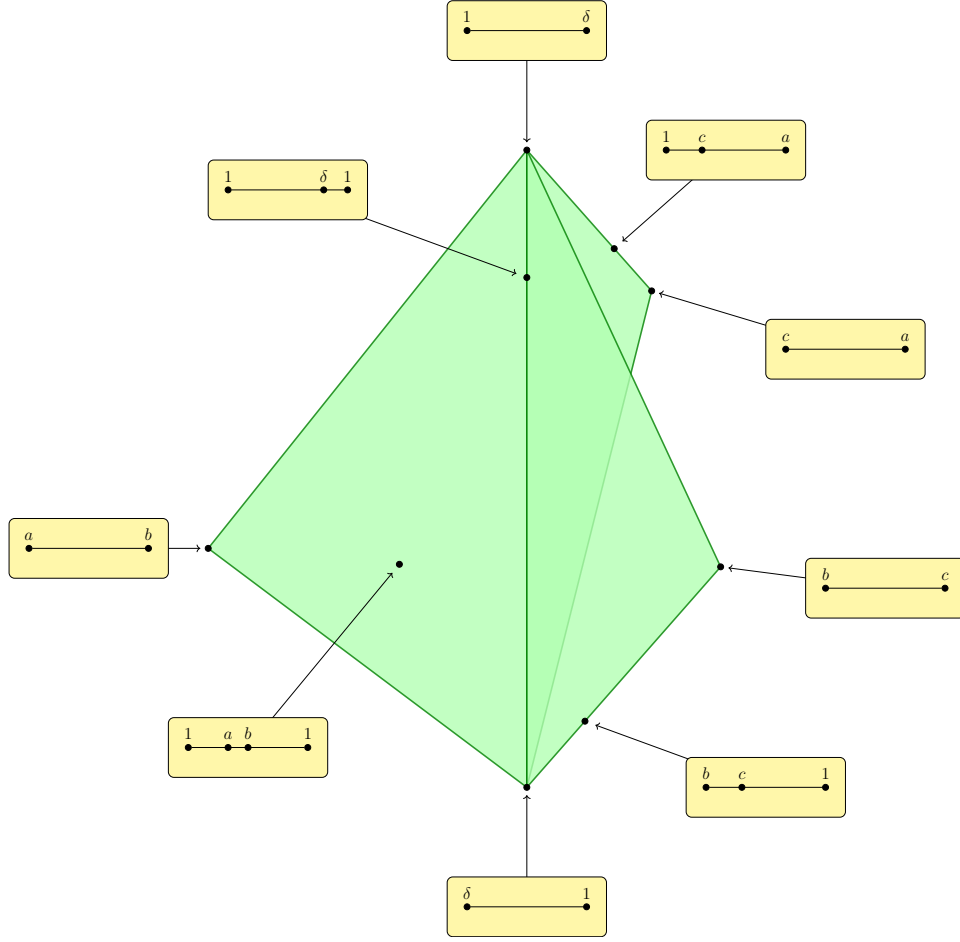


FIGURE 8. The order complex  $O_\delta = \Delta([1, \delta])$  has points labeled by elements of  $\text{WFACT}(\text{SYM}_3, \delta, \mathbf{I})$  and cells labeled by elements of  $\text{FACT}(\text{SYM}_3, \delta, \mathbf{I})$ .

$\mathbf{a} = [a_L \ a_1 \ \cdots \ a_k \ a_R]$  and  $\mathbf{b} = [b_L \ b_1 \ \cdots \ b_k \ b_R]$  in  $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$  label isometric faces of  $\text{WCOMP}(\mathbb{Z}, n, \mathbf{I})$  if  $a_i = b_i$  for all  $i \in \{1, \dots, k\}$ ; see Figure 9.

We can use the orthoscheme metric on  $\text{WCOMP}(\mathbb{Z}, n, \mathbf{I})$  and an extension of the map  $L$  to provide a piecewise-Euclidean metric for  $\text{WFACT}(G, g, \mathbf{I})$ .

**Definition 4.5** (Pullback metric). Let  $g \in G$  with  $\ell(g) = n$ . Abusing notation, we define the function  $L: \text{WFACT}(G, g, \mathbf{I}) \rightarrow \text{WCOMP}(\mathbb{Z}, n, \mathbf{I})$  by  $L(\mathbf{u}) = \ell \circ \mathbf{u}$  and observe that  $PL = LP$ , where the second occurrence of “ $L$ ” denotes the function from  $\text{FACT}(G, g, \mathbf{I})$  to  $\text{COMP}(\mathbb{Z}, n, \mathbf{I})$  given in Definition 3.9. For each  $\mathbf{x} \in \text{FACT}(G, g, \mathbf{I})$ , the set  $P^{-1}(\mathbf{x}) = \{\mathbf{u} \in \text{WFACT}(G, g, \mathbf{I}) \mid P(\mathbf{u}) = \mathbf{x}\}$  is sent bijectively via  $L$  to the set  $P^{-1}(L(\mathbf{x})) = \{\mathbf{s} \in \text{WCOMP}(\mathbb{Z}, n, \mathbf{I}) \mid P(\mathbf{s}) = L(\mathbf{x})\}$ , so we can pull back the metric and identify  $P^{-1}(\mathbf{x})$  with an open orthoscheme of shape  $L(\mathbf{x})$ . By Lemma 3.10, we know that the closure of this open orthoscheme is indeed the closed

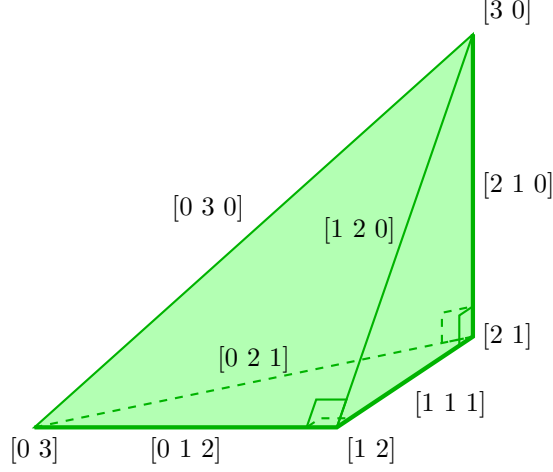


FIGURE 9. The space  $\text{WCOMP}(\mathbb{Z}, 3, \mathbf{I})$  is a 3-dimensional orthoscheme. Here we have labeled its vertices and open edges by the corresponding elements of  $\text{COMP}(\mathbb{Z}, 3, \mathbf{I})$ .

orthoscheme we would expect, so this endows  $\text{WFACT}(G, g, \mathbf{I})$  with the structure of a piecewise-Euclidean  $\Delta$ -complex with  $\text{FACT}(G, g, \mathbf{I})$  as its face poset.

The pullback metric defined in Definition 4.5 turns the simplicial complex shown in Figure 8 into 3 isosceles right triangles with a common hypotenuse. Note that the function  $L$  from  $\text{WFACT}(\text{SYM}_3, \delta, \mathbf{I})$  to  $\text{WCOMP}(\mathbb{Z}, 2, \mathbf{I})$  is a branched covering map with branch points along the shared hypotenuse.

**Proposition 4.6.** *The order complex  $O_g = \Delta([1, g])$  is isometric to the space  $\text{WFACT}(G, g, \mathbf{I})$  of weighted linear factorizations of  $g$ .*

*Proof.* This follows from Proposition 3.4 and Example 4.3. More explicitly, the differences between the consecutive real numbers  $0 = s_L < s_1 < \dots < s_k < s_R = 1$  are the barycentric coordinates of a point in the corresponding open simplex.  $\square$

Identifying the endpoints of  $\mathbf{I}$  to produce the circle  $\mathbf{S}$  yields a “circular” quotient of  $\text{WFACT}(G, g, \mathbf{I})$ .

**Definition 4.7** ( $\text{WFACT}(G, g, \mathbf{S})$ ). The equivalence relation given in Definition 3.6 transforms  $\text{FACT}(G, g, \mathbf{I})$  into  $\text{FACT}(G, g, \mathbf{S})$ , and this determines a quotient of the cell complex  $\text{WFACT}(G, g, \mathbf{I})$  by isometrically identifying faces. As described in Definition 1.4, each simplex in the order complex comes with an ordering of its vertices which determines the metric, and this information determines the gluing orientation. We refer to this quotient as the space of *weighted circular factorizations of  $g$* , denoted  $\text{WFACT}(G, g, \mathbf{S})$ . Each point  $\bar{\mathbf{u}}$  in this space can be viewed as a  $G$ -multiset  $\mathbf{S} \rightarrow G$  and uniquely represented as  $\bar{\mathbf{u}} = 0^{x_L} s_1^{x_1} \dots s_k^{x_k} 1^1$ , where  $q(\bar{\mathbf{u}})$  is the circular factorization  $[x_L \mid x_1 \cdots x_k \mid 1]$ . It follows that  $\text{FACT}(G, g, \mathbf{S})$  is the face poset for  $\text{WFACT}(G, g, \mathbf{S})$ .

**Example 4.8** ( $\text{WFACT}(\text{SYM}_3, \delta, \mathbf{S})$ ). If  $G = \text{SYM}_3$ ,  $T = \{a, b, c\}$  and  $\delta = (1\ 2\ 3)$  as in Example 4.3, then  $\text{WFACT}(\text{SYM}_3, \delta, \mathbf{S})$  is obtained from  $\text{WFACT}(\text{SYM}_3, \delta, \mathbf{I})$  by

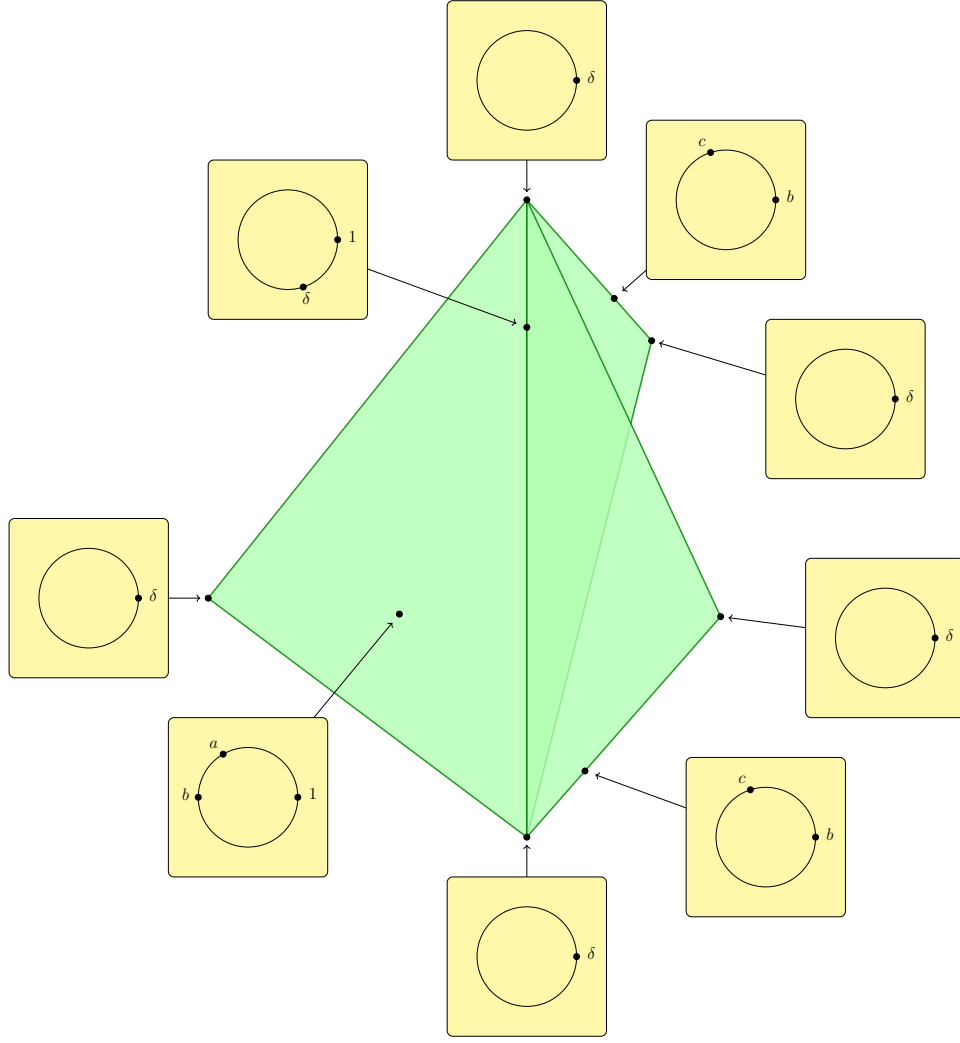


FIGURE 10. The interval complex  $K_\delta = \text{WFACT}(\text{SYM}_3, \delta, \mathbf{S})$  is obtained by gluing the points of the order complex  $O_\delta = \text{WFACT}(\text{SYM}_3, \delta, \mathbf{I})$  as described in Definition 4.7. The order complex is shown with its circular labels. From the circular labeling, one can see that all 5 vertices are identified and the short edges are identified in pairs.

identifying edges. The quotient  $K_g = \text{WFACT}(\text{SYM}_3, \delta, \mathbf{S})$  has 3 triangles, 4 edges, and 1 vertex, and these cells are labeled by the 8 elements of  $\text{FACT}(\text{SYM}_3, \delta, \mathbf{S})$  shown in Figure 5. For a depiction of the pointwise labels in  $K_g$ , see Figure 10.

**Example 4.9** ( $\text{WCOMP}(\mathbb{Z}, n, \mathbf{S})$ ). When  $G = \mathbb{Z}$  and  $X = \{1\}$ , we denote the set  $\text{WFACT}(\mathbb{Z}, n, \mathbf{S})$  by  $\text{WCOMP}(\mathbb{Z}, n, \mathbf{S})$  and refer to its elements as *weighted circular compositions of  $n$* . As discussed in Example 4.4,  $\text{WCOMP}(\mathbb{Z}, n, \mathbf{I})$  is isometric to a standard  $n$ -dimensional orthoscheme, so  $\text{WCOMP}(\mathbb{Z}, n, \mathbf{S})$  is obtained by identifying

faces of an orthoscheme according to the map  $q: \text{COMP}(\mathbb{Z}, n, \mathbf{I}) \rightarrow \text{COMP}(\mathbb{Z}, n, \mathbf{S})$  given in Definition 3.6. To give another way of viewing this identification, the inequalities  $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_1 + 1$  define a topological subspace of  $\mathbb{R}^n$  called a *column* which is isometric to the product of  $\mathbb{R}$  and an  $(n-1)$ -simplex (more specifically, a Coxeter simplex of type  $\tilde{A}_{n-1}$ . See [BM10, Section 8] and [DMW20, Section 8]). The infinite cyclic group generated by the isometry  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $T(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n, x_1 + 1)$  acts freely on the column and a fundamental domain for the action is a standard  $n$ -dimensional orthoscheme. The quotient by this free  $\mathbb{Z}$ -action is  $\text{WCOMP}(\mathbb{Z}, n, \mathbf{S})$ .

**Proposition 4.10.** *The interval complex  $K_g = q(O_g)$  for the interval  $P_g = [1, g]$  is isometric to the space  $\text{WFACT}(G, g, \mathbf{S})$  of weighted circular factorizations of  $g$ .*

*Proof.* The interval complex  $K_g$  for  $[1, g]$  is obtained from the order complex by identifying the faces labeled by chains  $x_0 < \cdots < x_k$  and  $y_0 < \cdots < y_k$  if and only if  $x_{i-1}^{-1}x_i = y_{i-1}^{-1}y_i$  for all  $i \in \{1, \dots, k\}$ . By Proposition 3.4, these faces are labeled by the linear factorizations

$$[x_L \ x_L^{-1}x_1 \ \cdots \ x_{k-1}^{-1}x_k \ x_k^{-1}g]$$

and

$$[y_L \ y_L^{-1}y_1 \ \cdots \ y_{k-1}^{-1}y_k \ y_k^{-1}g],$$

so the identification of faces in constructing the interval complex is identical to the equivalence relation established on  $\text{FACT}(G, g, \mathbf{I})$  when defining  $\text{FACT}(G, g, \mathbf{S})$ . Since this equivalence relation dictates the identification of faces in  $\text{WFACT}(G, g, \mathbf{I})$  when constructing  $\text{WFACT}(G, g, \mathbf{S})$  and  $\text{WFACT}(G, g, \mathbf{I})$  is isometric to the ordered simplicial complex  $O_g = \Delta([1, g])$  by Proposition 4.6, the proof is complete.  $\square$

## 5. TOPOLOGICAL POSETS

In this section we define two graded topological posets: a linear version  $\mathcal{F}(G, g, \mathbf{I})$ , which contains the weighted linear factorizations of all elements  $h$  in  $P_g = [1, g]$ , and a circular version  $\mathcal{F}(G, g, \mathbf{S})$ , which is a quotient of the linear version. We also prove Theorems B, C, and D. We begin by recalling the definition.

**Definition 5.1** (Topological posets). A *topological poset* is a poset  $\mathcal{P}$  with a Hausdorff topology  $\tau$  such that the order relation  $R := \{(p, q) \in \mathcal{P} \times \mathcal{P} \mid p \leq q\}$  is a closed subspace of  $\mathcal{P} \times \mathcal{P}$ . Morphisms are continuous poset maps.

This definition was given by Rade Živaljević [Ž98] inspired by the work of Vasiliev [Vas91]. The idea of combining posets and topology goes back to Folkman [Fol66] and Björner [Bjo95] has a survey of the area from the mid-1990s. For most natural examples, including the two given below, establishing the technical condition in Definition 5.1 is straightforward, and it is an exercise best left to the reader.

The linear topological poset  $\mathcal{F}(G, g, \mathbf{I})$  is easy to define and we establish its properties before moving on to the more complicated circular version  $\mathcal{F}(G, g, \mathbf{S})$ .

**Definition 5.2** ( $\mathcal{F}(G, g, \mathbf{I})$ ). For each  $g \in G$ , define  $\mathcal{F}(G, g, \mathbf{I})$  to be the disjoint union of topological spaces

$$\bigsqcup_{h \in [1, g]} \text{WFACT}(G, h, \mathbf{I}).$$

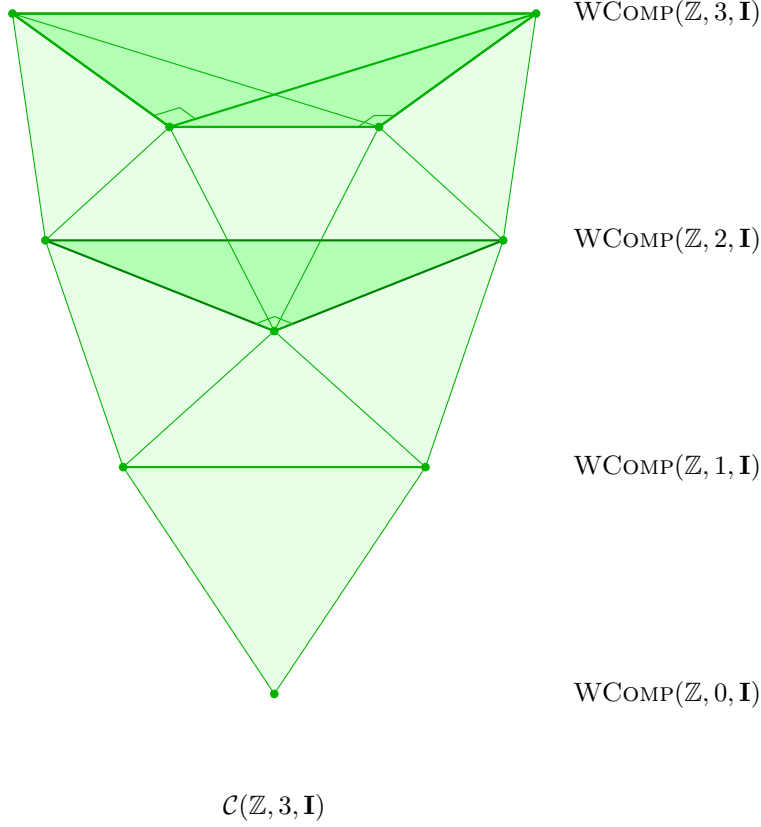


FIGURE 11. In the graded topological poset  $\mathcal{C}(\mathbb{Z}, 3, \mathbf{I})$ , the elements of rank  $k$  have the metric topology of the standard  $k$ -dimensional orthoscheme  $\text{WCOMP}(\mathbb{Z}, k, \mathbf{I})$ .

This space comes with a partial order: for  $\mathbf{u}, \mathbf{v} \in \mathcal{F}(G, g, \mathbf{I})$ , we say that  $\mathbf{v}$  is a *subfactorization* of  $\mathbf{u}$  and write  $\mathbf{v} \subseteq \mathbf{u}$  if  $\mathbf{v}(r) \leq \mathbf{u}(r)$  in  $G$  for all  $r \in \mathbf{I}$ . Define  $\rho: \mathcal{F}(G, g, \mathbf{I}) \rightarrow G$  by  $\rho(\mathbf{u}) = \prod_{r \in [0,1]} \mathbf{u}(r)$ , where the factors are arranged from left to right in increasing order of the real numbers  $r$ . Note that this uncountable product is well-defined since all but finitely many factors are trivial. The topological poset  $\mathcal{F}(G, g, \mathbf{I})$  is a graded poset of height  $\ell(g)$  with rank function  $\ell \circ \rho$ .

We illustrate this construction with  $G = \mathbb{Z}$  and  $g = n = 3$  since this is one of the few examples where one can sketch the full poset.

**Example 5.3.** When  $G = \mathbb{Z}$  and  $X = \{1\}$  we denote  $\mathcal{F}(\mathbb{Z}, n, \mathbf{I})$  by  $\mathcal{C}(\mathbb{Z}, n, \mathbf{I})$  and refer to subfactorizations as *subcompositions*. The elements of rank  $k$  in  $\mathcal{C}(\mathbb{Z}, n, \mathbf{I})$  are precisely those in  $\text{WCOMP}(\mathbb{Z}, k, \mathbf{I})$  by definition. Each point is labeled by an element of  $\text{WCOMP}(\mathbb{Z}, k, \mathbf{I})$ , i.e. a  $k$ -element multiset in the interval, and each cell is labeled by an element of  $\text{COMP}(\mathbb{Z}, k, \mathbf{I})$ , the list of multiplicities in their natural linear order. In particular, the elements in each rank form a standard metric orthoscheme. See Figure 11 for an illustration when  $n = 3$ . Note that there is a stark asymmetry in this ordering. For any element of rank  $k$  with  $0 < k < n$ ,

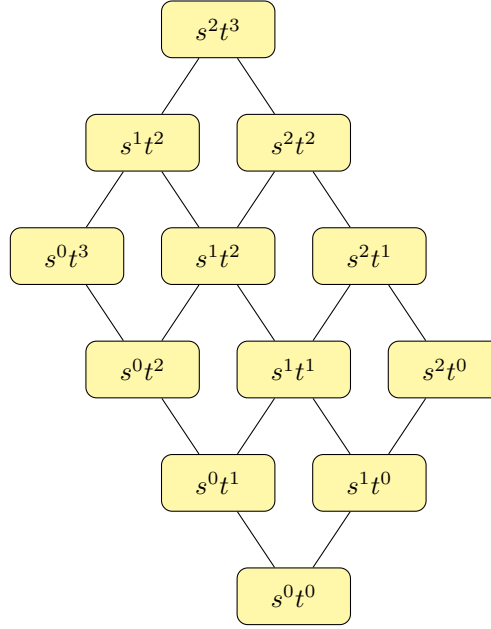


FIGURE 12. The lower set  $\downarrow(\mathbf{s})$  in  $\mathcal{C}(\mathbb{Z}, 5, \mathbf{I})$  for the 5-element multiset  $\mathbf{s} = 0^0s^2t^31^0$ . In the figure, the unused initial 0 and final 1 have been suppressed.

there is only a finite number of elements below it in rank  $k - 1$  (obtained by removing one of the elements of this  $k$ -element multiset), and there is a continuum of elements above it in rank  $k + 1$  (obtained by adding a random real number in  $\mathbf{I}$  to the multiset).

For each  $h \in P_g = [1, g]$ , there is an  $h'$  such that  $h \cdot h' = g$  and  $[1 \ h \ h' \ 1]$  is a linear factorization of  $g$ . The ideal or lower set  $\downarrow(h)$  of elements below  $h$  in  $P_g$  is isomorphic to the interval  $P_h$  and the filter or upper set  $\uparrow(h)$  of elements above  $h$  in  $P_g$  is isomorphic to  $P_{h'}$ . In the linear topological poset  $\mathcal{F}(G, g, \mathbf{I})$  (Definition 5.2), lower sets are also products of intervals inside  $P_g$ .

**Remark 5.4** (Lower sets). It is straightforward to compute the ideal or lower set of an element in  $\mathcal{F}(G, g, \mathbf{I})$ . For example, if  $\mathbf{u} \in \mathcal{F}(G, g, \mathbf{I})$  with  $\mathbf{u} = 0^{x_L} s_1^{x_1} \cdots s_k^{x_k} 1^{x_R}$ , then the lower set  $\downarrow(\mathbf{u})$  is isomorphic to the direct product  $P_{x_L} \times P_{x_1} \times \cdots \times P_{x_k} \times P_{x_R}$ . This is because each new exponent must be below the old exponent in the ordering of  $P_g$  and these choices are independent.

**Example 5.5.** If  $\mathbf{s} = 0^{a_L} s_1^{a_1} \cdots s_k^{a_k} 1^{a_R}$  is a multiset in  $\mathcal{C}(\mathbb{Z}, n, \mathbf{I})$ , then its lower set  $\downarrow(\mathbf{s})$  is isomorphic to the product of chain posets with lengths  $a_L, a_1, \dots, a_k, a_R$ . See Figure 12 for an illustration for the 5 element multiset  $\mathbf{s} = 0^0s^2t^31^0$  with  $0 < s < t < 1$ . Note that when  $\mathbf{s} = 0^1s_1^1 \cdots s_k^11^1$ , this means that the lower set  $\downarrow(\mathbf{s})$  is isomorphic to the Boolean lattice  $\text{BOOL}(k + 2)$ .

Since lower sets in  $\mathcal{F}(G, g, \mathbf{I})$  are products of subintervals in  $P_g$ , these are discrete when  $G$  is discrete and finite when  $G$  is finite. Upper sets in  $\mathcal{F}(G, g, \mathbf{I})$ , on the other

hand, are uncountable, but they have a familiar structure. To characterize upper sets in  $\mathcal{F}(G, g, \mathbf{I})$  we define operations on multisets and prove a technical lemma.

**Definition 5.6** (Multiset operations). Given  $G$ -multisets  $\mathbf{u}: \mathbf{I} \rightarrow G$  and  $\mathbf{v}: \mathbf{I} \rightarrow G$ , define the *product  $G$ -multiset*  $\mathbf{uv}$  by the formula  $(\mathbf{uv})(r) = \mathbf{u}(r)\mathbf{v}(r)$  for all  $r \in \mathbf{I}$  and the *inverse  $G$ -multiset*  $\mathbf{u}^{-1}$  by the formula  $\mathbf{u}^{-1}(r) = (\mathbf{u}(r))^{-1}$  for all  $r \in \mathbf{I}$ .

Note that if  $\mathbf{u}, \mathbf{v} \in \mathcal{F}(G, g, \mathbf{I})$  with  $\mathbf{v} \subseteq \mathbf{u}$ , then  $\mathbf{v}^{-1}\mathbf{u}$  is a  $G$ -multiset such that  $\mathbf{v}(\mathbf{v}^{-1}\mathbf{u}) = \mathbf{u}$ , and by Lemma 2.4,  $\mathbf{v}^{-1}\mathbf{u}$  is also an element of  $\mathcal{F}(G, g, \mathbf{I})$ .

**Lemma 5.7.** *Let  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{F}(G, g, \mathbf{I})$  with  $\mathbf{v} \subseteq \mathbf{u}$  and  $\mathbf{v} \subseteq \mathbf{w}$ . Then  $\mathbf{u} \subseteq \mathbf{w}$  if and only if  $\mathbf{v}^{-1}\mathbf{u} \subseteq \mathbf{v}^{-1}\mathbf{w}$ .*

*Proof.* Since  $\mathbf{v} \subseteq \mathbf{u}$  and  $\mathbf{v} \subseteq \mathbf{w}$ , we know that  $\ell(\mathbf{v}^{-1}(r)\mathbf{u}(r)) = \ell(\mathbf{u}(r)) - \ell(\mathbf{v}(r))$  and  $\ell(\mathbf{v}^{-1}(r)\mathbf{w}(r)) = \ell(\mathbf{w}(r)) - \ell(\mathbf{v}(r))$ . By definition,  $\mathbf{v}^{-1}\mathbf{u} \subseteq \mathbf{v}^{-1}\mathbf{w}$  if and only if  $\mathbf{v}^{-1}(r)\mathbf{u}(r) \leq \mathbf{v}^{-1}(r)\mathbf{w}(r)$  for all  $r \in \mathbf{I}$ , which is equivalent to saying that

$$\ell(\mathbf{v}^{-1}(r)\mathbf{u}(r)) + \ell(\mathbf{u}^{-1}(r)\mathbf{w}(r)) = \ell(\mathbf{v}^{-1}(r)\mathbf{w}(r)).$$

Plugging in, we find that  $\ell(\mathbf{u}(r)) + \ell(\mathbf{u}^{-1}(r)\mathbf{w}(r)) = \ell(\mathbf{w}(r))$ , which means that  $\mathbf{u}(r) \leq \mathbf{w}(r)$  for all  $r \in \mathbf{I}$ , i.e.  $\mathbf{u} \subseteq \mathbf{w}$ .  $\square$

We now use Lemma 5.7 to completely characterize upper sets in  $\mathcal{F}(G, g, \mathbf{I})$ .

**Theorem 5.8.** *Let  $\mathbf{v} \in \mathcal{F}(G, g, \mathbf{I})$  with  $\rho(\mathbf{v}) = h$  and let  $h'$  be the element in  $P_g$  with  $h \cdot h' = g$ . The upper set  $\uparrow(\mathbf{v})$  is isomorphic to the topological poset  $\mathcal{F}(G, h', \mathbf{I})$ . As a consequence, the maximal elements in  $\uparrow(\mathbf{v})$  form a subspace homeomorphic to the order complex  $O_{h'} = \Delta(P_{h'})$ .*

*Proof.* For the first claim, note that for each  $r \in I$ , the product  $\prod_{s>r} \mathbf{v}(s)$ , in which the factors are arranged left to right in increasing order of  $s$ , is a well-defined element of  $P_g$  since there are only finitely many  $s \in I$  such that  $\mathbf{v}(s)$  is nontrivial. Using this, we define  $\phi: \uparrow(\mathbf{v}) \rightarrow \mathcal{F}(G, h', \mathbf{I})$  by

$$\phi(\mathbf{u}) = \left( \prod_{s>r} \mathbf{v}(s) \right)^{-1} (\mathbf{v}^{-1}\mathbf{u})(r) \left( \prod_{s>r} \mathbf{v}(s) \right).$$

If  $\rho(\mathbf{u}) = h''$ , then we can see by definition that  $\rho(\phi(\mathbf{u})) = h^{-1}h''$  and therefore we do indeed have  $\phi(\mathbf{u}) \in \mathcal{F}(G, h', \mathbf{I}) = \mathcal{F}(G, h^{-1}g, \mathbf{I})$ . This function is a bijection with inverse  $\phi^{-1}: \mathcal{F}(G, h^{-1}g, \mathbf{I}) \rightarrow \uparrow(\mathbf{v})$  given by

$$\phi^{-1}(\mathbf{w}) = \mathbf{v}(r) \left( \prod_{s>r} \mathbf{v}(s) \right) \mathbf{w}(r) \left( \prod_{s>r} \mathbf{v}(s) \right)^{-1}.$$

By Lemma 5.7 and the fact that conjugation preserves the partial order on  $G$ , both  $\phi$  and  $\phi^{-1}$  are order-preserving maps. Therefore,  $\phi$  is a poset isomorphism. By Proposition 4.6, the maximal elements in  $\mathcal{F}(G, h', \mathbf{I})$  form a subspace homeomorphic to the order complex  $O_{h'} = \text{WFACT}(G, h', \mathbf{I})$ , proving the second claim.  $\square$

To describe this another way, we can write  $\mathbf{u} = \mathbf{v}(\mathbf{v}^{-1}\mathbf{u})$  and then deform  $\mathbf{u}$  by pushing the terms appearing from  $\mathbf{v}$  to the left end of the interval, conjugating the elements of  $\mathbf{v}^{-1}\mathbf{u}$  along the way to preserve the fact that the product is  $h''$ . The weighted factorization obtained by applying these conjugations to  $\mathbf{v}^{-1}\mathbf{u}$  is what we call  $\phi(\mathbf{u})$ , and the effect on the group elements is an example of a *Hurwitz move*. Viewing this as a continuous deformation makes clear that  $\phi$  is not just an isomorphism, but an isometry as well.

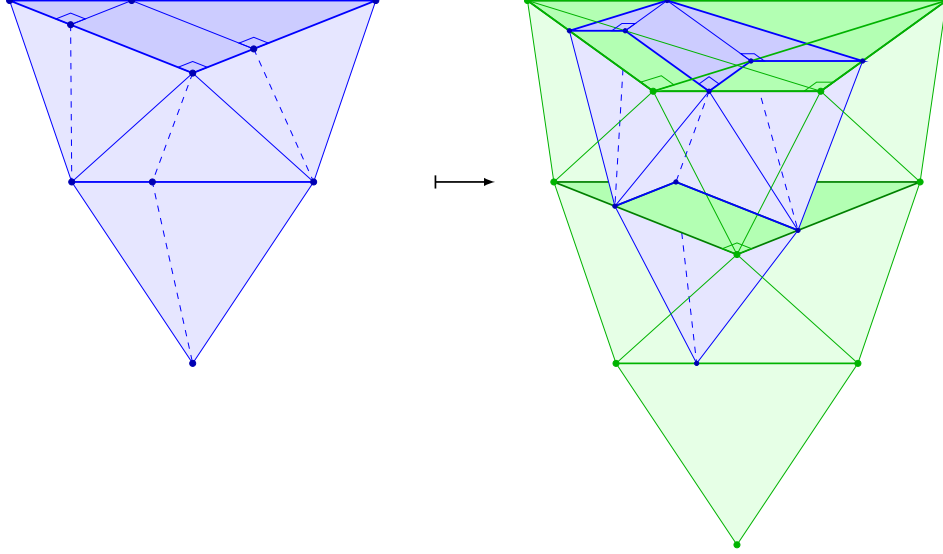


FIGURE 13. In  $\mathcal{C}(\mathbb{Z}, 3, \mathbf{I})$ , the upper set of a rank-1 element (i.e. a multiset of size 1 in the unit interval) is a copy of  $\mathcal{C}(\mathbb{Z}, 2, \mathbf{I})$ . The bending of the image is caused by the inequalities between the new multiset elements and the original multiset element.

**Example 5.9.** Let  $\mathbf{s} = [0^0 s^1 1^0]$  be a multiset of size 1 in the unit interval and an element of  $\mathcal{C}(\mathbb{Z}, 3, \mathbf{I})$ . The upper set  $\uparrow(\mathbf{s})$  is the collection of multisets of size at most 3 in  $\mathbf{I}$  which contains  $\mathbf{s}$  as one of its elements. By Theorem 5.8, this is a subposet isomorphic (as a topological poset) to  $\mathcal{C}(\mathbb{Z}, 2, \mathbf{I})$ . This is illustrated in Figure 13. The bending of the image under the embedding is caused by the inequalities between the one or two new multiset elements and the original multiset element  $\mathbf{s}$ .

We now turn our attention to the more complicated circular version.

**Definition 5.10** ( $\mathcal{F}(G, g, \mathbf{S})$ ). Define an equivalence relation on  $\mathcal{F}(G, g, \mathbf{I})$  by declaring  $\mathbf{u} = 0^{x_L} s_1^{x_1} \dots s_k^{x_k} 1^{x_R}$  and  $\mathbf{v} = 0^{y_L} s_1^{y_1} \dots s_k^{y_k} 1^{y_R}$  to be equivalent if  $x_i = y_i$  for all  $i \in \{1, \dots, k\}$  and  $h = g x_R g^{-1} x_L = g y_R g^{-1} y_L$ . We denote the set of all such equivalence classes  $[\mathbf{u}]$  by  $\mathcal{F}(G, g, \mathbf{S})$  and observe that it inherits the partial order by subfactorizations from  $\mathcal{F}(G, g, \mathbf{I})$ . In particular, each equivalence class  $[\mathbf{u}]$  can be uniquely represented by an element of the form  $\mathbf{w} = 0^{z_L} s_1^{z_1} \dots s_k^{z_k} 1^1$ , and applying  $\ell \circ \rho$  to these representatives provides a rank function for  $\mathcal{F}(G, g, \mathbf{S})$ . Finally, note that the quotient map  $q: \mathcal{F}(G, g, \mathbf{I}) \rightarrow \mathcal{F}(G, g, \mathbf{S})$  given by  $q(0^{x_L} s_1^{x_1} \dots s_k^{x_k} 1^{x_R}) = 0^h s_1^{x_1} \dots s_k^{x_k} 1^1$  is an order-preserving surjection.

**Remark 5.11.** Caution is required when working with  $\mathcal{F}(G, g, \mathbf{S})$ . It is quite possible that  $\mathbf{u}$  and  $\mathbf{v}$  belong to the same equivalence class even when  $\rho(\mathbf{u}) \neq \rho(\mathbf{v})$ . To give a small example, the weighted linear factorizations  $[1 \ a]$  and  $[g a g^{-1} \ 1]$  belong to the same equivalence class, but will have different values of  $\rho$  if  $a$  and  $g$  do not commute. In particular, the identified factorizations may belong to distinct order complexes  $O_{\rho(\mathbf{u})}$  and  $O_{\rho(\mathbf{v})}$ . This is why the rank function needed to be defined carefully. And this also shows that the clean definition of  $\mathcal{F}(G, g, \mathbf{I})$  as a disjoint

union of spaces  $O_h = \text{WFACT}(G, h, \mathbf{I})$  with  $h \in P_g$ , becomes more complicated in the circular version  $\mathcal{F}(G, g, \mathbf{S})$ . The elements in  $\mathcal{F}(G, g, \mathbf{S})$  cannot be partitioned into a disjoint union of the spaces  $K_h = \text{WFACT}(G, h, \mathbf{S})$ , one for each  $h$  in  $P_g$ .

An example of this gluing phenomenon already occurs in  $\text{SYM}_3$ .

**Example 5.12.** Let  $G = \text{SYM}_3$ ,  $\delta = (1\ 2\ 3)$  with  $\delta = ab = bc = ca$  as in Example 2.3. The maximal elements (of rank 2) in the linear topological poset  $\mathcal{F}(\text{SYM}_3, \delta, \mathbf{I})$  form a subspace homeomorphic to  $O_\delta$ , the three isosceles right triangles with a common hypotenuse shown in Figure 8. The rank 1 elements are three disjoint unit intervals,  $O_a = \text{WFACT}(\text{SYM}_3, a, \mathbf{I})$ ,  $O_b = \text{WFACT}(\text{SYM}_3, b, \mathbf{I})$  and  $O_c = \text{WFACT}(\text{SYM}_3, c, \mathbf{I})$ , and the unique rank 0 element is the complex  $O_1 = \text{WFACT}(\text{SYM}_3, 1, \mathbf{I})$ . Under the quotient identification defined above, the rank 2 elements  $\mathcal{F}(\text{SYM}_3, \delta, \mathbf{S})$  form the 1 vertex interval complex  $K_\delta$  schematically shown in Figure 10. But the three unit intervals in rank 1 are identified end-to-end to form a single circle of length 3, rather than three disjoint circles  $K_a \sqcup K_b \sqcup K_c$  each of length 1. This is because conjugating by  $\delta$  sends  $a$  to  $b$ ,  $b$  to  $c$  and  $c$  to  $a$ .

The gluing described in Example 5.12 is caused by the orbit of  $h \in P_g$  under the iterated conjugation by  $g$ . In general, if  $k$  is the smallest positive integer such that  $g^k$  commutes with  $h$ , then the  $k$  subspaces  $O_{h_i}$  with  $h_i = g^i h g^{-i}$  glue together to form a single cyclic  $k$ -sheeted cover of  $K_h$ . Note that when  $g$  and  $h$  commute, we get a copy of  $K_h$ . For example,  $\mathcal{C}(\mathbb{Z}, n, \mathbf{S})$  is a disjoint union of the spaces  $K_i$  for  $i \in \{0, 1, \dots, n\}$  since  $\mathbb{Z}$  is abelian. This also applies to the maximal elements of  $\mathcal{F}(G, g, \mathbf{S})$ , since  $g$  commutes with  $g$ , and the maximal elements form a subspace homeomorphic to the interval complex  $K_g = \text{WFACT}(G, g, \mathbf{S})$ . In particular, we obtain Theorem B as a rephrasing of Proposition 4.10.

**Theorem 5.13** (Theorem B). *Let  $P_g$  be an interval poset in a marked group  $G$  with order complex  $O_g$  and interval complex  $K_g$ . In the circular topological poset  $\mathcal{F}(G, g, \mathbf{S})$ , the subspace of maximal elements is homeomorphic to  $K_g$ .*

*Proof.* The only thing to check is that the quotient map defined in Definition 5.10 agrees with the quotient map defined in Definition 4.7 when restricted to maximal elements, and this is straightforward.  $\square$

As an immediate corollary we have Theorem C.

**Corollary 5.14** (Theorem C). *Let  $\mathbf{c}$  be a Coxeter element in a Coxeter group  $W$  generated by all reflections. The maximal elements of the circular topological poset  $\mathcal{F}(W, \mathbf{c}, \mathbf{S})$  form a subspace homeomorphic to the interval complex  $K_{\mathbf{c}}$  whose fundamental group is the dual Artin group  $\text{ART}^*(W, \mathbf{c})$ . In particular, the subspace of maximal elements in  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  is homeomorphic to the dual braid complex  $K_\delta$  whose fundamental group is the braid group  $\text{BRAID}_d$ .*

Once we prove Theorem A in Section 6, the topological poset  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  can be replaced with  $\text{NC}_d(\mathbf{S})$ . And Theorem D follows quickly from Theorem 5.8.

**Corollary 5.15** (Theorem D). *Let  $\mathbf{u}$  be a weighted linear factorization of  $h \in P_g$  with equivalence class  $[\mathbf{u}] \in \mathcal{F}(G, g, \mathbf{S})$ . Then the upper set  $\uparrow([\mathbf{u}])$  in  $\mathcal{F}(G, g, \mathbf{S})$  is isomorphic to  $\mathcal{F}(G, h', \mathbf{S})$  where  $h \cdot h' = g$ . Consequently, the maximal elements of  $\uparrow([\mathbf{u}])$  form a subspace which is isometric to the interval complex  $K_{h'}$  inside the interval complex  $K_g$ .*

*Proof.* By definition of the subfactorization order on  $\mathcal{F}(G, g, \mathbf{S})$  and the associated quotient map, we have  $\uparrow([\mathbf{v}]) = q(\uparrow(\mathbf{v}))$ , where  $\mathbf{v} \in \mathcal{F}(G, g, \mathbf{I})$  is any representative of the equivalence class  $[\mathbf{v}]$ . Thus, the first claim follows from Theorem 5.8. The second, following similar reasoning to the proof of Theorem 5.8, follows from Proposition 4.10.  $\square$

We conclude by discussing these results for our two running examples.

**Example 5.16.** Let  $\delta$  be the  $d$ -cycle  $(1 \cdots d) \in \text{SYM}_d$  and let  $[\mathbf{u}]$  be an equivalence class of a weighted linear factorization of  $\gamma \in P_\delta = [1, \delta]$ . Then the upper set  $\uparrow([\mathbf{u}])$  in  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  is isomorphic to  $\mathcal{F}(\text{SYM}_d, \gamma^{-1}\delta, \mathbf{S})$ . Moreover, the maximal elements of  $\uparrow([\mathbf{u}])$  form a subspace of the dual braid complex which is isometric to a product of dual braid complexes with smaller dimension.

**Example 5.17.** Let  $\mathbf{s} \in \mathcal{C}(\mathbb{Z}, n, \mathbf{I})$  with  $\rho(\mathbf{s}) = k$ . Then  $\uparrow(\mathbf{s})$  is isomorphic to  $\mathcal{C}(\mathbb{Z}, n - k, \mathbf{I})$ , and the maximal elements of  $\uparrow(\mathbf{s})$  form a subspace homeomorphic to an orthoscheme of dimension  $n - k$ . Similarly, the upper set  $\uparrow([\mathbf{s}])$  in  $\mathcal{C}(\mathbb{Z}, n, \mathbf{S})$  is homeomorphic to  $\mathcal{C}(\mathbb{Z}, n - k, \mathbf{S})$ , and the maximal elements of  $\uparrow([\mathbf{s}])$  form a subspace which is isometric to the quotient of the standard orthoscheme of dimension  $n - k$  described in Example 4.9.

## 6. CONTINUOUS NONCROSSING PARTITIONS

We now examine  $\mathcal{F}(G, g, \mathbf{S})$  in the special case when  $G = \text{SYM}_d$ ,  $X = T$ , and  $g = \delta = (1 \ 2 \ \cdots \ d)$  as described in Example 2.3. In particular, we introduce a new type of noncrossing partition and use this to prove Theorems A and E.

**Definition 6.1** (Noncrossing partitions). Let  $P$  be a subset of the complex plane and let  $\Pi(P)$  denote the lattice of all partitions of  $P$ , partially ordered by refinement. The elements of each partition are subsets of  $P$  called blocks, and we say that a partition is *noncrossing* if the convex hulls of its blocks are pairwise disjoint regions in  $\mathbb{C}$ . We define the *poset of noncrossing partitions for  $P$*  to be the subposet of noncrossing elements in  $\Pi(P)$ , denoted  $\text{NC}(P)$ .

When  $P$  is the vertex set for a convex  $n$ -gon,  $\text{NC}(P)$  is the *classical lattice of noncrossing partitions*  $\text{NC}(n)$ , originally defined by Kreweras in 1972 [Kre72]. The following theorem, proven 25 years later by Biane, illustrates our interest in  $\text{NC}(n)$ .

**Theorem 6.2** ([Bia97]). *Let  $\psi: \text{SYM}_d \rightarrow \Pi(d)$  be the function which sends each permutation to the partition formed by the orbits of its action on  $\{1, \dots, d\}$ . Then  $\psi$  restricts to an isomorphism from  $P_\delta = [1, \delta]$  to  $\text{NC}(d)$ .*

The decade following Biane's theorem produced a flurry of connections between the absolute order on the symmetric group and the combinatorics of noncrossing partitions—see the survey articles [BBG<sup>+</sup>19] and [McC06] for more background. One of these connections, involving a variation on noncrossing partitions due to Armstrong [Arm09], will be of use to us later in this section.

**Definition 6.3** (Shuffle partitions). Let  $\pi \in \Pi(dk)$ . Then  $\pi$  is a  *$k$ -shuffle partition* if  $a \equiv b \pmod{k}$  whenever  $a$  and  $b$  belong to the same block of  $\pi$ . When this is the case, note that for each  $j \in \{1, \dots, k\}$ , we may identify the set  $\{1, \dots, d\}$  with the equivalence class of  $j$  via the map  $m \mapsto (m - 1)k + j$  to obtain a partition  $\pi_j \in \Pi(d)$  which is induced by  $\pi$ . Then  $\pi$  is uniquely determined by the  $k$ -tuple  $(\pi_1, \dots, \pi_k)$ .

**Theorem 6.4** ([Arm09, Theorem 4.3.5]). *Let  $x_1, \dots, x_k \in [1, \delta]$ , let  $\pi_i = \psi(x_i)$  for each  $i$ , and define  $\pi$  to be the  $k$ -shuffle partition in  $\Pi(dk)$  which is determined by the  $k$ -tuple  $(\pi_1, \dots, \pi_k)$ . Then  $\pi$  is noncrossing if and only if  $\ell(x_1) + \dots + \ell(x_k) = \ell(x_1 \cdots x_k)$ .*

Some recent progress on the structure of  $\text{NC}(P)$  when  $P$  is finite (but not convex) can be found in [CDHM24], but seemingly little attention has been devoted to cases where  $P$  is infinite. Here, we are interested in the case where  $P$  is the unit circle  $\mathbf{S}$ . We refer to  $\text{NC}(\mathbf{S})$  as the *poset of continuous noncrossing partitions*, and we are particularly interested in a subposet of these which are compatible with a covering map for the circle.

**Definition 6.5** (Degree- $d$  partitions). Let  $f: \mathbf{S} \rightarrow \mathbf{S}$  be the standard degree- $d$  covering map  $f(z) = z^d$ . We say that a partition  $\pi \in \Pi(\mathbf{S})$  is a *degree- $d$  partition* if  $f(z) = f(w)$  whenever  $z$  and  $w$  belong to the same block of  $\pi$ . When this is the case, we may fix numbers  $0 = s_L < s_1 < \dots < s_k < 1$  such that  $z$  belongs to a nontrivial block of  $\pi$  only if  $f(z) = e^{2\pi i s_j}$  for some  $j$ , then identify each preimage  $f^{-1}(e^{2\pi i s_j})$  with the set  $\{1, \dots, d\}$  by reading off the preimages in increasing order of argument in  $[0, 2\pi)$ . If we let  $\pi_j$  be the partition in  $\Pi(d)$  determined by  $\pi$  in this way, then  $\pi$  can be uniquely denoted by the expression  $\pi = 0^{\pi_L} s_1^{\pi_1} \cdots s_k^{\pi_k}$ . It is clear from this construction that  $\pi$  is noncrossing if and only if the  $(k+1)$ -shuffle partition determined by  $(\pi_L, \pi_1, \dots, \pi_k)$  is noncrossing. Denote the subposet of all degree- $d$  partitions by  $\Pi_d(\mathbf{S})$  and the subposet of all noncrossing degree- $d$  partitions by  $\text{NC}_d(\mathbf{S})$ . Note that one may replace  $f$  with any covering map  $\mathbf{S} \rightarrow \mathbf{S}$  of degree  $d$ , and the resulting poset of “ $f$  noncrossing” partitions will be isomorphic to  $\text{NC}_d(\mathbf{S})$ .

**Example 6.6.** The degree-12 noncrossing partition in Figure 1 can be described by the shorthand  $\pi = 0^{\pi_L} 0.1^{\pi_1} 0.4^{\pi_2} 0.5^{\pi_3} 0.9^{\pi_4}$ , where

$$\begin{aligned} \pi_L &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}\}; \\ \pi_1 &= \{\{1\}, \{2\}, \{3\}, \{4, 5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}\}; \\ \pi_2 &= \{\{1, 2, 3, 11\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{12\}\}; \\ \pi_3 &= \{\{1\}, \{2\}, \{3, 6, 8\}, \{4\}, \{5\}, \{7\}, \{9\}, \{10\}, \{11, 12\}\}; \\ \pi_4 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8, 9, 10\}, \{11\}, \{12\}\}. \end{aligned}$$

See Figure 14 for an illustration (omitting the discrete partition  $\pi_L$ ). Note that when superimposed in the proper order, these partitions form a noncrossing hypertree on the vertex set  $\{1, \dots, 12\}$  as described in [McC].

It would have been reasonable to refer to the elements of  $\text{NC}_d(\mathbf{S})$  as “continuous shuffle partitions” considering their resemblance to Definition 6.3. Instead, we use the descriptor *degree- $d$  noncrossing partitions* to recognize their previous appearance in unpublished work of the late W. Thurston, which was later completed by Baik, Gao, Hubbard, Lei, Lindsey, and D. Thurston [TBY<sup>+</sup>20]. In the article, Thurston and his collaborators described a spine for the space of monic complex polynomials with  $d$  distinct roots, where each point is labeled by a “primitive major” of a “degree- $d$ -invariant” lamination of the disk. We have adopted the term “degree- $d$ ” but avoided the word “invariant” since these partitions are rarely invariant under a  $2\pi/d$  rotation.

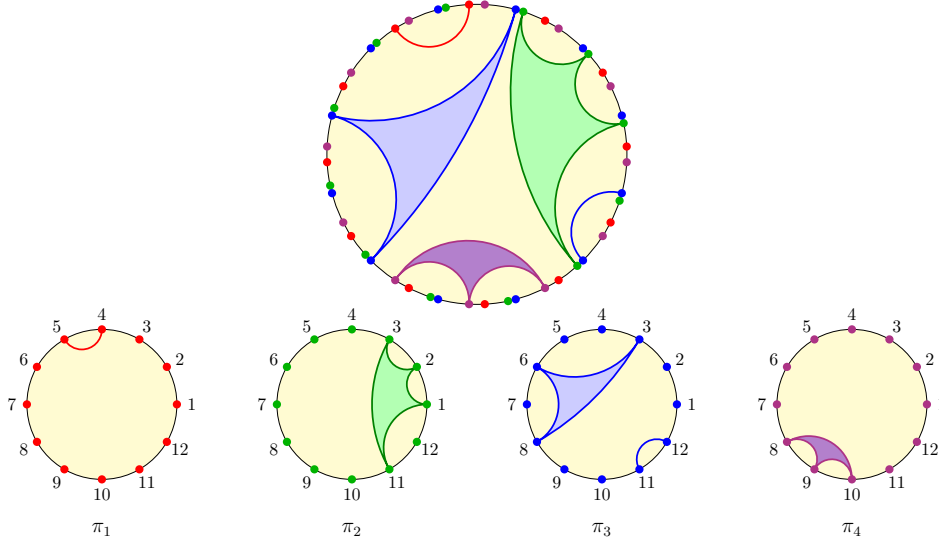


FIGURE 14. A degree-12-invariant noncrossing partition  $\pi$  together with its component partitions  $\pi_1, \pi_2, \pi_3, \pi_4 \in \text{NC}(d)$ , as described in Example 6.6.

The following definition and lemma rephrase a useful observation of Thurston's regarding the maximum number of non-singleton blocks in a degree- $d$  noncrossing partition.

**Definition 6.7** (Total criticality). Let  $\pi \in \Pi_d(\mathbf{S})$  and suppose that  $\pi$  has  $k$  non-singleton blocks, denoted  $A_1, \dots, A_k$ . The *total criticality* of the partition  $\pi$  is defined to be  $|A_1| + \dots + |A_k| - k$ . It is straightforward to see that the total criticality gives a rank function  $\text{rk}: \Pi_d(\mathbf{S}) \rightarrow \mathbb{N}$ , and this descends to a rank function for  $\text{NC}_d(\mathbf{S})$ .

The total criticality of a generic degree- $d$  partition can be arbitrarily large, but this is not the case for their noncrossing counterparts.

**Lemma 6.8** ([TBY<sup>+</sup>20, Proposition 2.1]). *If  $\pi \in \text{NC}_d(\mathbf{S})$ , then the total criticality of  $\pi$  is at most  $d - 1$ . Consequently,  $\text{NC}_d(\mathbf{S})$  is a graded poset of height  $d - 1$ .*

It follows from Lemma 6.8 that a degree- $d$  noncrossing partition has at most  $d - 1$  non-singleton blocks (in which case each has two elements). We can also give a clear characterization of the maximal elements in  $\text{NC}_d(\mathbf{S})$  as follows.

**Definition 6.9** (Complementary regions). Let  $\pi \in \text{NC}_d(\mathbf{S})$  and identify  $\mathbf{S}$  with the boundary of a disk. The *complementary regions* of  $\pi$  are the connected components of the disk after removing the convex hulls of the blocks of  $\pi$ .

**Lemma 6.10.** *The partition  $\pi \in \text{NC}_d(\mathbf{S})$  has  $\text{rk}(\pi) + 1$  complementary regions. Consequently, the maximal elements of  $\text{NC}_d(\mathbf{S})$  are those which have exactly  $d$  complementary regions.*

*Proof.* If  $\pi$  is a degree- $d$  noncrossing partition, then we can illustrate  $\pi$  in the disk and define a dual bipartite tree for  $\pi$  by placing a black vertex in each convex hull

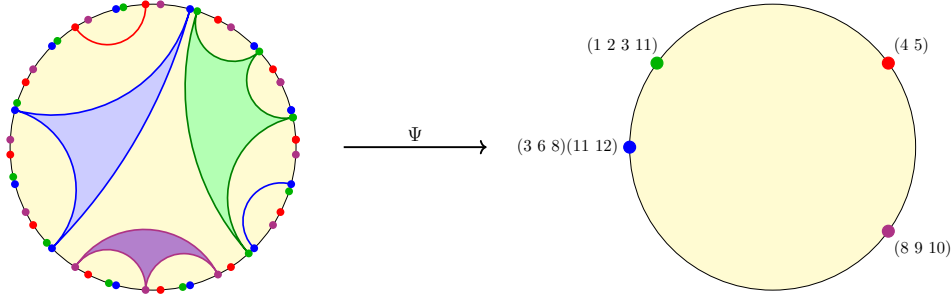


FIGURE 15. A continuous noncrossing partition in  $\text{NC}_{12}(\mathbf{S})$  and its corresponding weighted circular factorization in  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$

and a white vertex in each complementary region, then connecting a black vertex to a white vertex when the two corresponding regions are adjacent. If  $\pi$  has  $k$  non-singleton blocks, then the tree must have  $k$  black vertices and  $\text{rk}(\pi) + k = d - 1 + k$  edges. By the Euler characteristic, the number of vertices for a tree is always one more than the number of edges, so it follows that the number of white vertices, and thus the number of complementary regions, is  $\text{rk}(\pi) + 1$ . By Lemma 6.8, we may thus conclude that  $\pi$  is a maximal element of  $\text{NC}_d(\mathbf{S})$  if and only if it has  $d$  complementary regions.  $\square$

Given  $\pi \in \text{NC}_d(\mathbf{S})$ , we can draw the convex hulls of the blocks of  $\pi$  in the disk with unit circumference (rather than unit radius), then deformation retract each convex hull to a point. Under this transformation, the boundary of the disk becomes a metric graph known as a *cactus*, and in the special case where  $\pi$  is maximal, Lemma 6.10 tells us that this graph can be built by gluing together  $d$  circles, each of length  $1/d$ . These metric graphs make an appearance in [DM22], where we associate such a graph to each complex polynomial with distinct roots and critical values on the unit circle. In [DM22, Section 8], we also described a connection between continuous noncrossing partitions (then referred to as “real” noncrossing partitions) and a type of metric tree that we called a *banyan*. These connections between  $\text{NC}_d(\mathbf{S})$  and complex polynomials are of central importance in our ongoing paper series [DM].

We are now ready to prove Theorem A. Recall from Definition 4.1 that the set of all  $G$ -multisets from  $S$  to  $G$  is denoted by  $\text{MULT}(G, S)$ .

**Theorem 6.11** (Theorem A). *Define  $\Psi: \text{MULT}(\text{SYM}_d, \mathbf{S}) \rightarrow \Pi_d(\mathbf{S})$  by sending the  $\text{SYM}_d$ -multiset  $0^{x_L} s_1^{x_1} \cdots s_k^{x_k} 1^1$  to the partition  $0^{\psi(x_L)} s_1^{\psi(x_1)} \cdots s_k^{\psi(x_k)}$ . Then  $\Psi$  restricts to an isomorphism from  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  to  $\text{NC}_d(\mathbf{S})$ .*

*Proof.* Let  $\mathbf{x} = 0^{x_L} s_1^{x_1} \cdots s_k^{x_k} 1^1 \in \text{MULT}(\text{SYM}_d, \mathbf{S})$ , define  $\pi_j = \psi(x_j)$  for each  $j$  and consider the partition  $\Psi(\mathbf{x}) = 0^{\pi_L} s_1^{\pi_1} \cdots s_k^{\pi_k}$ . As outlined in Definition 6.5,  $\Psi(\mathbf{x})$  is noncrossing if and only if the shuffle partition in  $\Pi(dk + d)$  determined by  $(\pi_L, \dots, \pi_k)$  is noncrossing, and this is equivalent to having  $x_L, \dots, x_k \in [1, \delta]$  and  $\ell(x_L) + \cdots + \ell(x_k) = \ell(x_L \cdots x_k)$  by Theorem 6.4. By Lemma 2.4, this condition is satisfied by all elements of  $\mathcal{F}(\delta, \mathbf{S})$ , so  $\Psi$  restricts to a map  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S}) \rightarrow \text{NC}_d(\mathbf{S})$ . See Figure 15.

To see that  $\Psi$  is surjective, let  $\pi = 0^{\pi_L} s_1^{\pi_1} \cdots s_k^{\pi_k}$  be an element of  $\text{NC}_d(\mathbf{S})$  and note that  $k \leq d - 1$  by Lemma 6.8. By Theorem 6.2, we may define  $x_i = \psi^{-1}(\pi_i)$

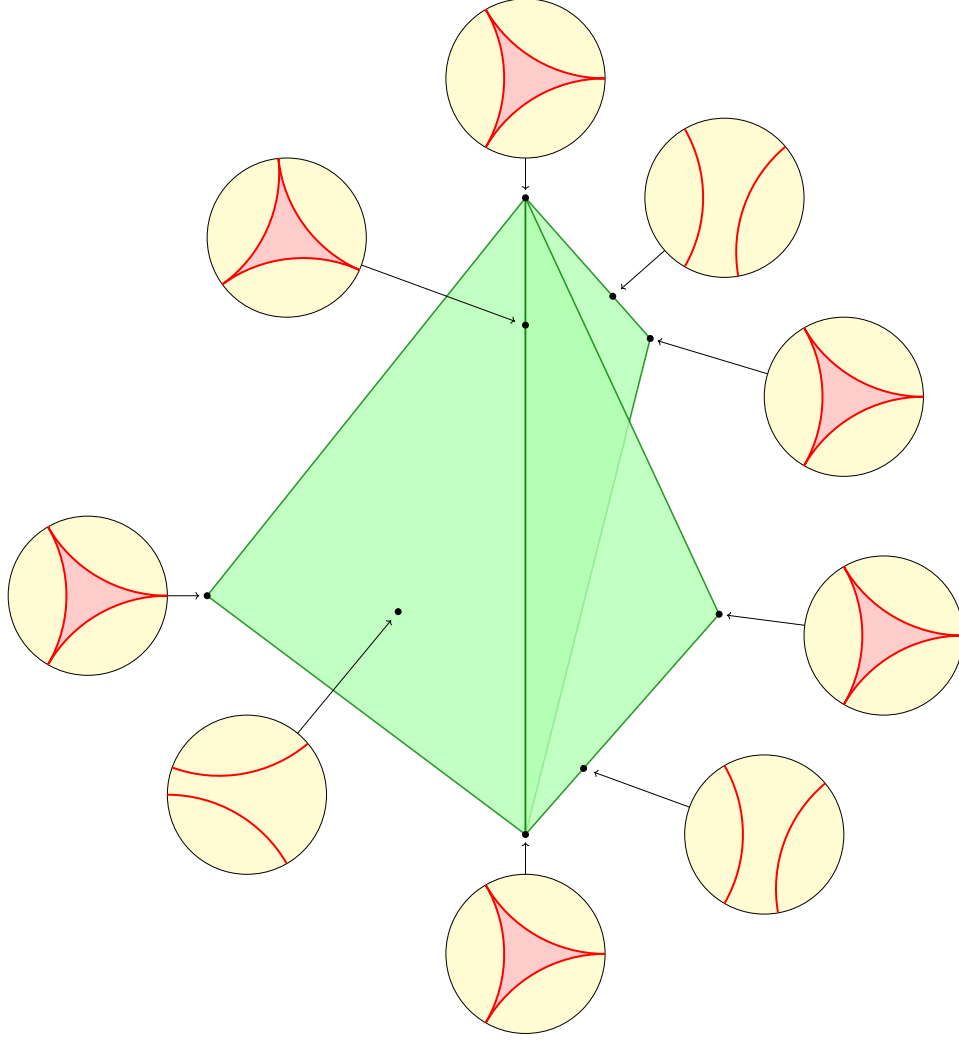


FIGURE 16. The isomorphism described in Theorem 6.11 allows us to label the points in the dual braid complex from Figure 10 by maximal elements of  $\text{NC}_3(\mathbf{S})$ . As before, points with the same label are glued together, which means that all five vertices are identified and the short edges are identified in pairs.

for each  $i$  and again apply Theorem 6.4 to see that  $0^{x_L} s_1^{x_1} \cdots s_k^{x_k} 1^1$  is an element of  $\mathcal{F}(\delta, \mathbf{S})$  which is sent to  $\pi$ . Injectivity of  $\Psi$  then follows from Theorem 6.2. And, by the definitions of the partial orders,  $\mathbf{x} \leq \mathbf{y}$  in  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  if and only if  $\Psi(\mathbf{x}) \leq \Psi(\mathbf{y})$  in  $\text{NC}_d(\mathbf{S})$ . Therefore,  $\Psi$  is an isomorphism and the proof is complete.  $\square$

As a consequence of Theorem 6.11, we can import the topology and cell structure from  $\mathcal{F}(\text{SYM}_d, \delta, \mathbf{S})$  to  $\text{NC}_d(\mathbf{S})$ —see Figure 16 for an example using the maximal elements of  $\text{NC}_3(\mathbf{S})$ . In [TBY<sup>+</sup>20], Thurston and his collaborators gave a topology

for the maximal elements of  $\text{NC}_d(\mathbf{S})$  using a slightly different metric: all edges of the dual braid complex would have equal length in their metric, whereas the lengths differ in the orthoscheme metric as described in Remark 1.5. Nevertheless, the two topologies are homeomorphic.

Let  $\text{POLY}_d^{mc}(U)$  denote the space of monic degree- $d$  complex polynomials for which the roots are centered at the origin and the critical values lie in the subspace  $U \subseteq \mathbb{C}$ . The following theorem from [TBY<sup>+</sup>20] uses the topology above to provide a spine for  $\text{POLY}_d^{mc}(\mathbb{C}^*)$ , the space of polynomials with distinct roots.

**Theorem 6.12** ([TBY<sup>+</sup>20, Theorem 9.2]). *The space of maximal elements in  $\text{NC}_d(\mathbf{S})$  is homeomorphic to  $\text{POLY}_d^{mc}(\mathbf{S})$ , and there is a deformation retraction from  $\text{POLY}_d^{mc}(\mathbb{C}^*)$  to  $\text{POLY}_d^{mc}(\mathbf{S})$ .*

**Corollary 6.13** (Theorem E). *The dual braid complex  $K_\delta$  is homeomorphic to the subspace  $\text{POLY}_d^{mc}(\mathbf{S})$  of polynomials with critical values on the unit circle, and there is a deformation retraction from  $\text{POLY}_d^{mc}(\mathbb{C}^*)$  to  $\text{POLY}_d^{mc}(\mathbf{S})$ .*

*Proof.* By Theorem 6.12,  $\text{POLY}_d^{mc}(\mathbb{C}^*)$  deformation retracts to  $\text{POLY}_d^{mc}(\mathbf{S})$ , which is homeomorphic to the maximal elements of  $\text{NC}_d(\mathbf{S})$ , which in turn is homeomorphic to the dual braid complex  $K_\delta$  by Theorems 5.13 and 6.11.  $\square$

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