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Sol. 1 (a) Given : $u(x,y) = \cancel{\cos hx \cos}\ \cosh x \cos y$

To show : u is harmonic.

we know that $u(x,y)$ is harmonic iff

$u_{xx} + u_{yy} = 0$, where $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$

$u_{yy} = \dfrac{\partial^2 u}{\partial y^2}$

$\therefore u_{xx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}(\sinh x \times \cos y)$

$= \cosh x \cos y.$

and $u_{yy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}(\cosh x \cos y)\right)$

$= \dfrac{\partial}{\partial y}(-\cosh x \sin y) = -\cosh x \cos y$

$u_{xx} + u_{yy} = \cosh x \cos y + (-\cosh x \cos y) = 0$

$\therefore u(x,y)$ is harmonic.

(b) ~~That~~ Let v be the required harmonic conjugate of u.

$\therefore \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} \rightarrow$ eq(i)

From Cauchy riemann eq.

$\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \rightarrow$ eq(ii)

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from eq (ii)

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \cosh x \sin y$$

Integrating w.r.t. x

$$v = \sinh x \sin y + c(y) \quad \text{---eq (a)}$$

here, $c(y)$ is a function of y.

from eq (i)

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \sinh x \cos y$$

$$\Rightarrow \sinh x \cos y + c'(y) = \sinh(x) \cos(y) \quad \text{(from eq (a))}$$

$$c'(y) = 0$$

$\therefore$ $c(y)$ is constant let c.

$\therefore$ $\boxed{v = \sinh x \cos y + c}$ = harmonic conjugate of u.

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Sol. 2

(a) $\int_\gamma z\,dz$, where γ is the arc of line $y=x$ from 0 to $1+i$.

let $z = x+iy$

$\therefore$ ~~dy~~ $dz = dx + i\,dy$

lower limit = $(0,0)$ along $x=y$

upper limit = ~~(1+i)~~ $(1,1)$ along $x=y$.

$\therefore$ given integral is,

$$\int_{(0,0)}^{(1,1)} (x+iy)(dx+i\,dy) =$$

$$= \int_{(0,0)}^{(1,1)} (x\,dx - y\,dy) + i(y\,dx + x\,dy)$$

[$\because y = x$ so, $dy = dx$]

$$= \cancel{\int_{(0,0)}^{(1+i)} 2i(x\,dx) = i\frac{x^2}{2}}$$

$$= \int_0^1 2i(x\,dx) = 2i\left[\frac{x^2}{2}\right]_0^1 = i$$

$\therefore$ $\boxed{\int_0^{1+i} z\,dz = i}$, along $x=y$.

(b) $\int_\gamma z\,dz$, where γ is the arc of parabola $y=x^2$ from 0 to $1+i$.

Let $z = x+iy$

$dz = dx + i\,dy$

lower limit = $(0,0)$

upper limit = $(1,1)$ [$\because y = x^2$]

as $y = x^2$ $dy = 2x\,dx$

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$\therefore$ required integral $= \int_{(0,0)}^{(1,1)} (x+iy)(dx+idy)$

$= \int_0^1 (x+ix^2)(dx + i \times 2x dx)$

$= \int_0^1 (x+ix^2)(1+i2x)dx$

$= \int_0^1 (x + i2x^2 + ix^2 - 2x^3)dx$

$= \left[\frac{x^2}{2} + i\frac{x^3}{\cancel{3}} \times \cancel{3} - 2\times\frac{x^4}{4}\right]_0^1$

$= \frac{1}{2} + i - \frac{1}{2} = i$

$\therefore \boxed{\int_r z\,dz = i}$

Along, $y = x^2$
from 0 to $1+i$

Sol. 3

$$f(z) = \frac{z+2}{(z+1)^2 (z-2)}$$

for poles :-

$$(z+1)^2 (z-2) = 0$$

$\boxed{z = 2, -1}$ these are the required poles of the function

here, $z = -1$ has order 2

$z = 2$ has order 1

Now, we know that residue of a pole a :-

$$\text{Res}(f, a) = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-a)^k f(z) \right)$$

where k is the order of the pole

$$\therefore \text{Res}(f, 2) = \lim_{z \to 2} 1 \times \frac{(z-2)(z+2)}{(z+1)^2 (z-2)} = \frac{4}{9}$$

and

$$\text{Res}(f, -1) = \lim_{z \to -1} 1 \times \frac{d}{dz} \frac{(z+1)^2 (z+2)}{(z+1)^2 (z-2)}$$

$$= \lim_{z \to -1} \frac{(z-2) - (z+2)}{(z-2)^2} = \frac{-4}{9}$$

from residue theorem :

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{m} \text{res}(f, a_k)$$

where $f(z)$ has m ~~poles~~ finite poles.

and ~~ref~~ $\text{res}(f, a_k)$, is the residue of function at pole $a_k \in C$.

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Now, ~~given~~ we need to evaluate

$$\int_c \frac{(z+2)\,dz}{(z+1)^2(z-2)}$$, where c is a circle $|z|=3$, traversed in anti-clockwise direction.

$\therefore$ from residue theorem,

$$\int_c \frac{(z+2)\,dz}{(z+1)^2(-z-2)} = 2\pi i \,\text{Res}(f, a_k)$$

[$\because$ ~~Be~~ for circular reigon $|z|=3$
$z=-1, 2$ lie inside the reigon]

$$= 2\pi i\left(\frac{4}{9} - \frac{4}{9}\right) = 0$$

$$\boxed{\int_c \frac{(z+2)\,dz}{(z+1)^2(z-2)} = 0}$$, where c is a circle $|z|=3$

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4.
$$f(z) = \frac{1}{z(z-2)}$$

(a) Laurent Series for 0 < |z| < 2

we know that

$$\frac{1}{1-z} = \begin{cases} \sum_{n=0}^{\infty} z^n , & |z|<1 \rightarrow \text{eq(i)} \\ -\sum_{n=1}^{\infty} \frac{1}{z^n} , & |z|>1 \rightarrow \text{eq(ii)} \end{cases}$$

Let $g(z) = \frac{1}{z-2}$ so, $f(z) = \frac{g(z)}{z}$

∴ Laurent series for $g(z)$:

$$g(z) = \frac{1}{(z-2)} = \frac{-1}{2(1-\frac{z}{2})} = \frac{(-1)}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$\therefore g(z) = \frac{(-1)}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

So laurent series for $f(z)$

$$f(z) = \frac{g(z)}{z} = \frac{-1}{2z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$\text{or, } \boxed{f(z) = (-1)\sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}}}$$

(b) |z| > 2 from eq(ii) described above:-

$$\frac{1}{1-z} = -\sum_{n=1}^{\infty}\frac{1}{z^n} , \quad |z|>1$$

$$\therefore f(z) = \frac{1}{z(z-2)}$$

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Again let $g(z) = \frac{1}{z-2}$

$\therefore$ ~~...~~ $g(z) = \frac{(-1)}{2(1-(z/2))} = \frac{-1}{2} \times (-1) \sum_{n=1}^{\infty} \frac{1}{(\frac{z}{2})^n}, |z| > 2$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(z/2)^n}, \quad |z| > 2$$

Similarly, for $f(z) = \frac{g(z)}{z}$

So, laurent series for $f(z)$, when $|z| > 2$

$$\therefore f(z) = \frac{\left(\frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{1}{(z/2)^2}}{z}, \quad |z| > 2$$

$$\boxed{f(z) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{z^{n+1}}}, \quad |z| > 2$$

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Sol.5 given:

$$y'' + 5y' + 4y = 10e^{-3x} \quad -(i)$$

As it's a non homogenous equation so general equation can be written as,

$$y(x) = y_H(x) + y_{NH}(x) \quad -(ii)$$

where, $y_H(x)$ = homogenous part of $y(x)$

$y_{NH}(x)$ = Non homogenous part of $y(x)$

Now, let the roots of characteristic equation

$y'' + 5y' + 4y = 0$ be K_1, K_2 so the solution will be of the form

$$y_H(x) = Ae^{K_1 x} + Be^{K_2 x}$$

or, $K^2 + 5K + 4 = 0$

$(K+4)(K+1) = 0$

$K = -1, -4$

So, $\boxed{y_H(x) = Ae^{-x} + Be^{-4x}} \quad -(iii)$

And the non-homogenous part

let, $y_{NH}(x) = ce^{-3x}$

So, $y'_{NH}(x) = -3ce^{-3x} \rightarrow$ (a)

$y''_{NH}(x) = 9ce^{-3x} \rightarrow$ (b)

Substituting (a) and (b) in (i)

$$9ce^{-3x} + 5\times(-3c\times e^{-3x}) + 4\times ce^{-3x} = 10e^{-3x}$$

$9c - 15c + 4c = 10$

$-2c = 10$

$\boxed{c = -5}$

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Hence, $\boxed{y_{NH}(n) = -5e^{-3n}}$ - (iv)

from eq (ii), (iii), (iv), we get the generalized equation of $y(n)$ as,

$$\boxed{y(n) = Ae^{-n} + Be^{-4n} - 5e^{-3n}}$$

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Sol. 6

$y'' + 4y' + 4y = e^{-x}\cos x$

the characteristic homogenous eq. $y_H(x)$

$k^2 + 4k + 4 = 0$

$\Rightarrow (k+2)^2 = 0$

$\boxed{k = -2}$ repeated roots at $k = -2$

$\therefore y_H(x)$ will be of the form,

$\boxed{y_H(x) = Ae^{-2x} + Bxe^{-2x}}$ — (i)

i.e.

$y_H(x) = Ay_1(x) + By_2(x)$

So, $y_1(x) = e^{-2x}$, $y_2(x) = xe^{-2x}$

Now we can find wronskian as,

$$w(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$= y_1(x)y_2'(x) - y_2(x)y_1'(x)$

where, $y_1(x) = e^{-2x}$, $y_1'(x) = -2e^{-2x}$

$y_2(x) = xe^{-2x}$, $y_2'(x) = e^{-2x} - 2xe^{-2x}$

$\therefore w(y_1, y_2) =$

$= e^{-2x}(e^{-2x} - 2xe^{-2x}) - (xe^{-2x})(-2e^{-2x})$

$= e^{-4x} - 2xe^{-4x} + 2xe^{-4x}$

$\boxed{w(y_1, y_2) = e^{-4x}}$ — (ii)

Given: $y'' + 4y' + 4y = e^{-x}\cos x$

$$\therefore y_{NH}(x) = -y_1(x)\int \frac{y_2(x)f(x)\,dx}{w(y_1, y_2)} + y_2(x)\int \frac{y_1(x)f(x)\,dx}{w(y_1, y_2)}$$

where $f(x) = e^{-x}\cos x$

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$$y_{NH}(x) = -e^{-2x}\int \frac{xe^{-2x}e^{-x}\cos x}{e^{-4x}}\,dx + xe^{-2x}\int \frac{e^{-2x}\cdot e^{-x}\cos x\,dx}{e^{-4x}}$$

$$= -e^{-2x}\int \underbrace{xe^{x}\cos x\,dx}_{I} + xe^{-2x}\int e^{x}\cos x\,dx \quad -\text{(iii)}$$

$$I = \int \underset{I}{x}\,\underset{II}{e^{x}\cos x}\,dx \qquad \text{Applying Integration by parts}$$

$$I = x\int e^{x}\cos x\,dx - \int\left(\int e^{x}\cos x\,dx\right)dx$$

put I in (iii)

$$y_{NH}(x) = -e^{-2x}\times x\int e^{x}\cos x\,dx + e^{-2x}\int\left(\int e^{x}\cos x\,dx\right)dx$$
$$+\, xe^{-2x}\int e^{x}\cos x\,dx$$

$$y_{NH}(x) = e^{-2x}\int\left(\int e^{x}\cos x\,dx\right)dx = \int\left(\int e^{-x}\cos x\,dx\right)dx$$

Now, $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c$

[for standard results]

$$\therefore \int e^{-x}\cos x\,dx = \frac{e^{-x}}{2}(\sin x - \cos x) + c$$

So, $y_{NH}(x) = -\int\left(\int e^{-x}\cos x\,dx\right)dx = -\int \frac{e^{-x}}{2}(\sin x - \cos x)\,dx + \int c\,dx$

Also we know,

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$$

$$\therefore y_{NH}(x) = \frac{1}{2}\left[\frac{e^{-x}}{2}(-\sin x - \cos x) - \frac{e^{-x}}{2}(-\cos x + \sin x)\right] + cx$$

$$= -\frac{1}{2}\left[\frac{e^{-x}}{2}\times 2\sin x\right] + cx = +\frac{e^{-x}\sin x}{2} + cx$$

so general eq of $y(x)$ can be written as,

$$\boxed{y(x) = Ae^{-2x} + Bxe^{-2x} + \frac{e^{-x}}{2}\sin x + cx}$$