Algebra-I, Fall 2018. Midterm #2. Due Friday, Nov 16th, by 1pm

Scoring system: Exam consists of **5** problems, each worth 9 points. If s_1, s_2, s_3, s_4, s_5 are your individual scores in decreasing order, your total is $s_1 + s_2 + s_3 + s_4 + \frac{2}{3}s_5$. Thus, the maximal possible total is 42, but the score of 40 counts as 100%.

Directions: Each problem is worth 10 points, all 4 problems will be counted. Provide complete arguments (do not skip steps). State clearly any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given.

Rules: You are not allowed to discuss midterm problems with each other. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use your class notes, previous homework assignments and the book by Dummit and Foote. The use of other books is allowed, but not encouraged. If you happen to run across a problem very similar or identical to one on the midterm which is solved in another book, do not consult that solution. The use of any online resources (except the class webpage) is absolutely prohibited.

1. Let R be a ring with 1 and I an ideal (not necessarily commutative). Let us say that an ideal I of R is *good* if every element of the set $1 + I = \{1 + x : x \in I\}$ is invertible in R.

(a) Prove that if I is an ideal such that every $a \in I$ is nilpotent, then I is a good

(b) Prove that if I is a good ideal, then $1 + I$ is a subgroup of $R^\times$

(c) Prove that if I and J are good ideals and $IJ = JI$, then IJ is also a good ideal and $[1 + I, 1 + J] \subseteq 1 + IJ$

(d) Deduce that if I is an ideal such that $I^n = 0$ for some n, then the group $1 + I$ is nilpotent.

2. Let G be a group. A subgroup H of G will be called *essential* if $H \cap K \neq \{1\}$ for every non-trivial subgroup K of G.

(a) Let p be a prime and $k \geq 2$. Prove that the group $\mathbb{Z}/p^k\mathbb{Z}$ has a proper essential subgroup.

(b) Assume that H_1 is an essential subgroup of G_1 and H_2 is an essential subgroup of G_2. Prove that $H_1 \times H_2$ is an essential subgroup of $G_1 \times G_2$.

(c) Let G be a finite abelian group. Prove that G does not have a proper essential subgroup if and only if G is a direct product of groups of prime order.

3. In all parts of this problem G is a non-abelian group. A *maximal abelian* subgroup of G is a maximal element of the set of all abelian subgroups of G ordered by inclusion.

(a) Prove that the union of all maximal abelian subgroups of G is equal to G and the intersection of all maximal abelian subgroups of G is $Z(G)$.

(b) Prove that G has at least 3 maximal abelian subgroups.

(c) Give an example of G which has exactly 3 maximal abelian subgroups.

4.

(a) Let R be a commutative ring with 1, and let $R[x]$ be the ring of polynomials over R in one variable x. Prove that the ideal (x) is maximal if and only if R is a field.

(b) (not related to 4(a)). Let R be a commutative ring with 1. Let Ω be the set of all ideals I of R such that every element of I is 0 or a zero divisor. Let P be a maximal element of Ω (such P always exists, but you are not asked to prove it). Prove that P is prime.

Hint: It is a well-known fact that for any non-nilpotent element $f \in R$ there exists a prime ideal P of R such that $f \notin P$. While this result is not directly related to Problem 4(b), its proof (see e.g. DF, Proposition 12 on p.674) may help you with 4(b).

5. Find the number of solutions to the equation $x^2 = x$ in $\mathbb{Z}_{4004}$ (note: $1001 = 7 \cdot 11 \cdot 13$). **Hint:** if you are not sure how to start, review HW#9.