Math 4452, Spring 2024. Midterm #2

due Tuesday, April 16th, by 11:59pm on Canvas

Directions: Provide complete arguments (do not skip steps). State clearly any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given.

Rules: You are not allowed to discuss midterm problems with each other. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use class notes (your own notes as well as notes posted on collab), previous homework assignments, our main textbook "Coding theory: a first course" and lectures notes by J. Hall and Y. Lindell. The use of other books or other online resources is prohibited.

1. (9 pts) In each part determine if a code with given parameters exists. Make sure to prove your answer. You can use any result stated in our main textbook, regardless of whether we proved it in class or not.

(a) binary $[12, 11, 2]$-linear code
(b) binary $[12, 6, 5]$-linear code
(c) binary $[12, 5, 6]$-linear code
(d) ternary $[12, 6, 6]$-linear code

2. (9 pts) Recall that $A_q(n, d)$ is the largest size of a q-ary code of length n and distance d. Prove that if $d < n$, then

$$A_q(n, d) \leq q \cdot A_q(n - 1, d).$$

Give a detailed argument. **Hint:** Let X be a set and $K \in \mathbb{N}$. One way to prove that $|X| \leq qK$ is to represent X as a disjoint union of q subsets $X_1, \ldots, X_q$ such that $|X_i| \leq K$ for all i.

3. (8 pts) Prove that there are no perfect codes of even distance. **Hint:** Remember that we gave 3 equivalent definitions of a perfect code. It is important to pick the right condition to approach this problem.

4. (12 pts) In HW#6 we determined the distribution of weights in the extended binary Golay code G_{24} assuming without proof that G_{24}

has exactly 759 words of weight 8. The goal of this problem is to prove the latter statement.

Parts (a)-(d) below deal with G_{23}, not G_{24}. For each $k \in \mathbb{N}$ let us denote by $n_k(G_{23})$ the number of words of weight k in G_{23}.

(a) According to Problem 5(b) in HW#6 the following holds: For every $w \in \mathbb{F}_2^{23}$ with $wt(w) = 4$ there exists $c \in G_{23}$ such that $wt(c) = 7$ and $d(w, c) = 3$. Explain why such c must be unique.

(b) Use (a) to prove that

$$\binom{7}{4} \cdot n_7(G_{23}) = \binom{23}{4}.$$

Deduce that $n_7(G_{23}) = 253$. **Hint:** (a) defines a natural map from the set of all words of length 4 to the set of words of length 7 in G_{23}.

(c) Now prove that for every $w \in \mathbb{F}_2^{23}$ with $wt(w) = 5$ there exists unique $c \in G_{23}$ such that either $wt(c) = 7$ and $d(w, c) = 2$ or $wt(c) = 8$ and $d(w, c) = 3$.

(d) Use (c) to find a relation of the form $An_7(G_{23}) + Bn_8(G_{23}) = C$ where A, B, C are some (explicit) binomial coefficients. Use this relation to prove that $n_8(G_{23}) = 506$.

(e) Now use (b) and (d) to prove that G_{24} has exactly 759 words of weight 8.

5. (10 pts) Problem 7.22. Make sure to give a detailed argument. **Hint:** Some of the homework problems are very relevant.

6. (12 pts) Prove that there are exactly 4 cyclic codes of length 8 and dimension 2 over $\mathbb{F}_3$. For each such code explicitly compute its generator polynomial and generator matrix in the standard form. Make sure to justify all the steps. **Hint:** You need the result of one of the homework problems to start on this problem.

Bonus. (5 pts) Determine (with proof) the largest number of pairwise inequivalent codes among the 4 cyclic codes from Problem 6.