ON FINITENESS PROPERTIES OF THE JOHNSON FILTRATIONS

MIKHAIL ERSHOV AND SUE HE

Abstract. Let Γ denote either the automorphism group of the free group of rank $n \geq 4$ or the mapping class group of an orientable surface of genus $n \geq 5$ with at most 1 boundary component, and let G be either the subgroup of IA-automorphisms or the Torelli subgroup of Γ, respectively. We prove that if $N = 2$ or $N > 2$ and $6N - 5 \leq n$, then any subgroup of G containing $\gamma_N G$, the N^{th} term of the lower central series of G, is finitely generated (in particular, the result applies to the Johnson kernel in the mapping class group case). We also prove that if $12(N-1) \leq n$ and K is any subgroup of G containing $\gamma_N G$, then $G/[K, K]$ is nilpotent. Finally, we prove that if H is any finite index subgroup of Γ containing $\gamma_N G$, with $12(N-1) \leq n$, then H has finite abelianization.

1. Introduction

Let F_n denote the free group of rank n, and let $IA_n \subset \mathrm{Aut}\,(F_n)$ denote the subgroup of automorphisms of F_n which act as identity on the abelianization; equivalently, IA_n is the kernel of the natural map $\mathrm{Aut}\,(F_n) \to \mathrm{Aut}\,(F_n^{ab}) \cong GL_n(\mathbb{Z})$. More generally, for each $k \in \mathbb{N}$ define $IA_n(k)$ to be the kernel of the natural map $\mathrm{Aut}\,(F_n) \to \mathrm{Aut}\,(F_n/\gamma_{k+1}F_n)$. The filtration $IA_n = IA_n(1) \supset IA_n(2) \supset \ldots$ was first introduced and studied by Andreadakis in [An]. It is easy to see that $\gamma_k IA_n \subseteq IA_n(k)$ for each k, and in [Ba] it was shown that $\gamma_2 IA_n = IA_n(2)$.

The filtration $\{IA_n(k)\}_{k\in\mathbb{N}}$ is often referred to as the *Johnson filtration* and owes its name to the corresponding filtration in mapping class groups whose study was initiated by Johnson [Jo4]. Let Σ_g^1 be an orientable surface of genus $g \geq 2$ with 1 boundary component and $\mathrm{Mod}\,_g^1$ its mapping class group. The fundamental group $\pi = \pi(\Sigma_g^1)$ is free of rank $2g$, and for each $k \in \mathbb{N}$ there is a natural homomorphism $\mathrm{Mod}\,_g^1 \to \mathrm{Aut}\,(\pi/\gamma_{k+1}\pi)$. Denote the kernel of this homomorphism by $\mathcal{I}_g^1(k)$. The subgroups $\mathcal{I}_g^1 = \mathcal{I}_g^1(1)$ and $\mathcal{J}_g^1 = \mathcal{I}_g^1(2)$ are well known as the *Torelli subgroup* and the *Johnson kernel*, respectively, and the filtration $\{\mathcal{I}_g^1(k)\}_{k\in\mathbb{N}}$ is called the *Johnson filtration* of $\mathrm{Mod}\,_g^1$. Again one has $\gamma_k\mathcal{I}_g^1 \subseteq \mathcal{I}_g^1(k)$ for each k, but this time the inclusion is known to be strict already for $k = 2$. The mapping class group $\mathrm{Mod}\,_g$ of a closed orientable surface of genus g is a quotient of $\mathrm{Mod}\,_g^1$, and the Johnson filtration $\{\mathcal{I}_g(k)\}_{k\in\mathbb{N}}$ of $\mathrm{Mod}\,_g$ is defined to be the image of $\{\mathcal{I}_g^1(k)\}_{k\in\mathbb{N}}$ in $\mathrm{Mod}\,_g^1$.

A basic open question about Johnson filtrations is which of their terms are finitely generated (it is easy to see that if some term is not finitely generated, then so are all the subsequent terms in the same filtration). A complete answer was known for the first terms: already in 1930s Magnus [Ma] proved that $IA_n = IA_n(1)$ is finitely generated for all $n \geq 2$, and in 1980 Johnson [Jo5] proved that the Torelli group $\mathcal{I}_g^1$ (and hence also its quotient $\mathcal{I}_g$) is finitely generated for $g \geq 3$, while $\mathcal{I}_2$ is infinitely generated by [MM] (hence

The work of the first author was partially supported by the NSF grant DMS-1201452.

the same is true for $\mathcal{I}_2(k)$ and $\mathcal{I}_2^1(k)$ for all k). Our first theorem settles in the positive the finite generation problem for arbitrary terms in sufficiently large rank:

Theorem 1.1. *Let $N \geq 2$ be an integer, and let G be one of the following groups:*

(a) *$G = IA_n$ where $n \geq 4$ if $N = 2$ and $n \geq 6N - 5$ if $N > 2$.*
(b) *$G = \mathcal{I}_g^1$ where $g \geq 5$ if $N = 2$ and $g \geq 6N - 5$ if $N > 2$.*

Then $\gamma_N G$ is finitely generated.

Since G in Theorem 1.1 is itself finitely generated, it follows that any subgroup of G containing $\gamma_N G$ (in particular, the N^{th} term of the Johnson filtration) is finitely generated. In particular, Theorem 1.1 answers (in the positive) the well known open question about finite generation of the Johnson kernel $\mathcal{I}_g^1(2)$ and its quotient $\mathcal{I}_g(2)$ for $g \geq 5$.

Remark. After this paper was completed, the authors learned that Church and Putman [CP2] independently proved a result almost identical to Theorem 1.1. The paper [CP2] was completed after an earlier version of this paper was posted on arXiv where Theorem 1.1 was proved for $N = 2$, with the extra restriction $g \geq 12$ in part (b).

For some time it was generally expected that the second and higher terms of the Johnson filtrations are never finitely generated and moreover have infinite first rational homology. The latter was refuted by the recent breakthrough results of Dimca and Papadima [DP] and Papadima and Suciu [PS2] who established that $H_1(\mathcal{I}_g(2), \mathbb{Q})$ for $g \geq 4$ and $H_1(IA_n(2), \mathbb{Q})$ for $n \geq 5$ are finite-dimensional. In addition, in [DHP] it was proved that $H_1(\mathcal{I}_g(2), \mathbb{Q})$ is a unipotent $\mathcal{I}_g/\mathcal{I}_g(2)$-module in the sense that it is annihilated by some power of the augmentation ideal of $\mathbb{Q}[\mathcal{I}_g/\mathcal{I}_g(2)]$. Our second theorem generalizes this result from rational to integral homology and also applies to higher terms of the Johnson filtrations:

Theorem 1.2. *Let $N \geq 2$ be an integer, and let G be one of the following groups.*

(a) *$G = IA_n$ where $n \geq 4$ if $N = 2$ and $n \geq 12(N - 1)$ if $N > 2$.*
(b) *$G = \mathcal{I}_g^1$ where $g \geq 12(N - 1)$.*

Let K be any subgroup of G containing $\gamma_N G$. Then $G/[K, K]$ is nilpotent or, equivalently, $K/[K, K] \cong H_1(K, \mathbb{Z})$ considered as a G/K-module is annihilated by some power of the augmentation ideal of $\mathbb{Z}[G/K]$.

Remark. The equivalence of the above conditions will be explained in the proof of Theorem 1.2.

Both Theorems 1.1 and 1.2 will be deduced from certain properties of representations of G that we discuss below. Let G be an arbitrary group. By a *character* of G we will mean a homomorphism $\chi : G \to \mathbb{R}$, and let $\operatorname{Hom}(G, \mathbb{R})$ be the set of all characters of G. In [BNS], Bieri, Neumann and Strebel introduced certain subset $\Sigma(G)$ of $\operatorname{Hom}(G, \mathbb{R}) \setminus \{0\}$, now known as the *BNS invariant* which, in the case when G is finitely generated, completely determines which subgroups of G containing $[G, G]$ are finitely generated: [1]

Theorem 1.3 ([BNS])**.** *Let G be a finitely generated group, and let N be a subgroup of G containing $[G, G]$. Then N is finitely generated if and only if $\Sigma(G)$ contains every non-trivial character of G which vanishes on N.*

[1] Usually, $\Sigma(G)$ is defined as a set of equivalence classes of characters where two characters are equivalent if they are positive scalar multiples of each other.

In particular, $[G,G]$ is finitely generated if and only if $\Sigma(G) = \mathrm{Hom}\,(G,\mathbb{R}) \setminus \{0\}$. More generally, $\gamma_N G$ is finitely generated for some N if and only if for every $1 \le k < N$ the set $\Sigma(\gamma_{k-1}G)$ contains every nonzero character of $\gamma_{k-1}G$ which vanishes on $\gamma_k G$ (this follows from Theorem 1.3 by induction). The set $\Sigma(G)$ admits many different characterizations. In order to prove Theorem 1.1 we will use the characterization in terms of actions on real trees due to Brown [Br].

Note that characters of a group G are in natural bijection with one-dimensional real representations of G whose image lies in $\mathbb{R}_{>0}$. In order to prove Theorem 1.2 we consider irreducible representations of G which vanish on $\gamma_N G$ (with G and N as in Theorem 1.2), although, somewhat surprisingly, this time we will deal with representations over finite fields, in fact fields of prime order. Theorem 1.2 will be obtained as a direct combination of Theorems 1.4 and 1.5 below.

Definition. Let K be a normal subgroup of a group G and F a field. We will say that the triple (G,K,F) is *nice* if the following holds: let V be a non-trivial finite-dimensional irreducible representation of G over F such that K acts trivially on V. Then

$$H^1(G,V) = 0.$$

Theorem 1.4. *Let G be a finitely generated group, and let K be a normal subgroup of G such that $Q = G/K$ is nilpotent. Assume that (G,K,F) is nice for any finite field F of prime order. Then the quotient $G/[K,K]$ is nilpotent, so in particular the abelianization $K^{ab} = K/[K,K]$ is finitely generated.*

Theorem 1.5. *Let G and N be as in Theorem 1.2. Then $(G,\gamma_N G,F)$ is nice for any field F.*

We now briefly comment on how Theorems 1.1 and 1.5 will be proved. For simplicity, below we will restrict to the case $N = 2$ in Theorem 1.1 (allowing arbitrary N in Theorem 1.5). Both results easily follow from the existence of a *ρ-centralizing generating* (ρ-CG) sequence in G where in Theorem 1.1 ρ is a non-trivial character of G and in Theorem 1.5 ρ is a non-trivial irreducible representation of G (over an arbitrary field) with $\mathrm{Ker}\,\rho \supseteq \gamma_N G$. The precise definition of a ρ-CG sequence will be given in § 3. Here we will just mention that such a sequence exists (for a given ρ) whenever G has finitely many pairwise commuting elements $t_1,\ldots,t_k$ whose centralizers generate G and such that $\rho(t_i)$ is non-trivial and central (in $\rho(G)$) for each i. Note that by our hypotheses we will always deal with representations ρ such that $\rho(G)$ is nilpotent (and non-trivial) and hence always contains non-trivial central elements.

The group IA_n has a very simple generating set constructed by Magnus already in 1930s (see § 5 for its definition). A generating set for $\mathcal{I}_g^1$ which is in many ways analogous to Magnus' generating set for IA_n was constructed in a recent paper of Church and Putman [CP1], which made use of an earlier work of Putman [Pu2] and the original work of Johnson [Jo5] (see § 8.1). We will refer to these generating sets as *standard*. We will show that if ρ is a "generic" representation of G which vanishes on $\gamma_N G$ (with G and N as above), then the existence of a ρ-CG sequence in G easily follows from the basic relations between the standard generators. For a general ρ, we will use the observation that G is a normal subgroup of Γ (with $\Gamma = \mathrm{Aut}^+(F_n)$, the subgroup of "orientation-preserving" automorphisms or $\Gamma = \mathrm{Mod}_g^1$), and precomposing ρ by the conjugation by a fixed element of Γ does not affect the existence of a ρ-CG sequence.

From this point on the proofs of Theorems 1.1 and 1.5 will proceed slightly differently. To prove Theorem 1.1 for $N = 2$ (which amounts to proving the equality $\Sigma(G) = \mathrm{Hom}\,(G, \mathbb{R}) \setminus \{0\}$) we will first show that $\Sigma(G)$ contains the complement of the union of finitely many proper linear subspaces of $\mathrm{Hom}\,(G, \mathbb{R})$ (this complement will be one of our classes of generic characters). We will complete the proof by an elementary algebraic geometry argument using Γ-invariance of $\Sigma(G)$ and the fact that the action of Γ on $\mathrm{Hom}\,(G, \mathbb{R})$ (which factors through the action of Γ/G) extends to a rational representation of a connected real algebraic group containing Γ/G as a Zariski-dense subgroup.

To prove Theorem 1.5 we will use a different class of generic representations (see Lemma 4.1) and show that an arbitrary ρ (with $\mathrm{Ker}\,\rho \supseteq \gamma_N G$) can be conjugated to a generic one. In the case $G = IA_n$ and $N = 2$ this can be done quite explicitly (see § 7). In the other cases an explicit calculation may be possible, but would be cumbersome, so we take a more conceptual approach. First it is not hard to see that the relevant information about the conjugation action of Γ on G is captured by the induced action of Γ/G on $L(G) = \bigoplus_{n=1}^{\infty} \gamma_n G/\gamma_{n+1} G$, the Lie algebra of G with respect to the lower central series. Second, in both cases $(\Gamma, G) = (\mathrm{Aut}^+(F_n), IA_n)$ and $(\mathrm{Mod}_n^1, \mathcal{I}_n^1)$, the quotient Γ/G contains a natural copy of $SL_n(\mathbb{Z})$ which therefore acts on $L(G)$. In § 5 we will establish a sufficient condition on a group G (assuming $SL_n(\mathbb{Z})$ acts on $L(G)$ as above) which guarantees that G has a ρ-CG sequence for any non-trivial representation ρ of G with image of sufficiently small nilpotency class. One of the hypotheses in Theorem 5.3 is that the abelianization $G^{ab} = G/[G, G]$ is a *regular* $SL_n(\mathbb{Z})$*-module*, another technical notion which will also be introduced in § 5. While the proof of Theorem 5.3 will deal with the action of $SL_n(\mathbb{Z})$ on the entire Lie algebra $L(G)$, its hypotheses only involve the action on G^{ab}, the degree 1 component of $L(G)$. This is very important since while the structure of G^{ab} as an $SL_n(\mathbb{Z})$-module is completely understood and easy to describe (for $G = IA_n$ or $\mathcal{I}_g^1$), this is not the case with higher degree components of $L(G)$.

At the end of the paper we will show that Theorem 1.5 also yields new results on the abelianization of finite index subgroups in $\mathrm{Aut}\,(F_n)$, IA_n, Mod_g^1 and $\mathcal{I}_g^1$:

Theorem 1.6. *Let G, N and K be as in Theorem 1.2, and let $\Gamma = \mathrm{Aut}\,(F_n)$ if $G = IA_n$ and $\Gamma = \mathrm{Mod}_g^1$ if $G = \mathcal{I}_g^1$. The following hold:*

1. *If H is a finite index subgroup of G which contains K, then the restriction map $H^1(G, \mathbb{C}) \to H^1(H, \mathbb{C})$ is an isomorphism*
2. *If H is a finite index subgroup of Γ which contains K, then H has finite abelianization.*

In the case $\Gamma = \mathrm{Mod}_g^1$, $G = \mathcal{I}_g^1$ and $K = \mathcal{I}_g^1(2)$ both assertions of Theorem 1.6 have been previously proved by Putman (see [Pu3, Thm B] and [Pu1, Thm B]) for all $g \geq 3$ by a different method. It is interesting to note that while we will use Theorem 1.5 to prove both Theorem 1.2 and Theorem 1.6, an opposite implication was used in [DHP] where [Pu3, Thm B] was one of the tools in the proofs of special cases of Theorem 1.2 and Theorem 1.5.

Finally, let us comment on the restrictions on n, g and N in the statements of Theorems 1.1, 1.2, 1.5 and 1.6. If $n = 2$, the assertions of all four theorems are easily seen to be false already for $N = 2$ since IA_2 is a free group of rank 2. As shown in [GL, Prop 6.2] (see also [BV]), $\mathrm{Aut}\,(F_3)$ contains a finite index subgroup H such that $[IA_3 : H \cap IA_3] = 2$ (so $H \supset [IA_3, IA_3]$) and H has infinite abelianization. This result combined with the proof of Theorem 1.6 implies that Theorem 1.5 and both parts of Theorem 1.6 are false for $n = 3$ and $N = 2$. We expect that Theorems 1.1 and 1.2 do not hold in this case as well, but as far as we know these questions have not been settled. The restrictions $g \geq 5$ in Theorem 1.1 and $g \geq 12$ in Theorem 1.2 for $G = \mathcal{I}_g^1$ and $N = 2$ are probably the drawbacks of our methods, and we expect that in both cases the results remains true for all $g \geq 4$ (and possibly even for $g = 3$). It is possible that some (or even all) of Theorems 1.2, 1.5 and 1.6 remain true for arbitrary N, but the proof of such a result would almost certainly require new ideas.

Organization. The paper is organized as follows. In § 2 we will prove Theorem 1.4. In § 3 we will introduce the notion of a ρ-CG sequence and reduce Theorems 1.1 and 1.5 to the problem of existence of ρ-CG sequences for suitable ρ (see Proposition 3.4). In the first part of § 4 we will introduce the notion of a good $\mathbf{n}$-group where $\mathbf{n} = \{1, 2, \ldots, n\}$, which captures the key properties of basic commutation relations in $\mathrm{Aut}\,(F_n)$ and Mod_g^1. We will then show that in a good $\mathbf{n}$-group a ρ-CG sequence exists for any generic representation ρ with image of sufficiently small nilpotency class (see Lemma 4.1).

In the second part of § 4 we will introduce the notion of a good twin $\mathbf{n}$-group and establish our main criterion for finite generation of (some) terms of the lower central series in such groups (Theorem 4.5). In § 5 we will introduce the notion of a regular $SL_n(\mathbb{Z})$-module, discuss basic examples and properties of such modules and state Theorem 5.3, which will then be proved in § 6.

At the end of § 5 we will show that the hypotheses of Theorem 4.5 and Theorem 5.3 hold for $G = IA_n$. The corresponding verification for $G = \mathcal{I}_g^1$ will be made in § 8. Once these verifications are completed, we will immediately deduce Theorem 1.1 and Theorem 1.5 from Theorem 4.5 and Theorem 5.3, respectively, except when $G = IA_n$, $N = 2$ and $4 \leq n < 7$ in Theorem 1.1 or $G = IA_n$, $N = 2$ and $4 \leq n < 12$ in Theorem 1.5. The proof of those theorems in the remaining cases will be given in § 7. Finally, in § 9 we will prove Theorem 1.6 and also give a summary of previously known results of the same kind.

Acknowledgments. We are extremely grateful to Andrei Rapinchuk who encouraged us to take on this project and suggested the general approach to the proof of Theorem 1.6. We are also indebted to Andrei Jaikin who proposed a tremendous simplification of our original proof of Theorem 1.4. Finally, we would like to thank Andrew Putman and Zezhou Zhang for helpful comments on earlier versions of this paper.

Notation. When considering cohomology of a group G with coefficients in some module, we will sometimes use the notation $H^k(G, M)$ where M is the underlying space of the module and sometimes $H^k(G, \rho)$ when ρ is the action of G on M. We hope that this inconsistency will not cause a confusion.

2. The abelianization of the kernel of a homomorphism to a nilpotent group

In this short section we prove Theorem 1.4 using the following results of Roseblade and Robinson. A major part of the argument below (including the use of Theorem 2.2) was suggested to us by Andrei Jaikin who substantially simplified our original proof. The latter was inspired by the proof of [PS2, Thm 3.6].

Theorem 2.1 (Roseblade)**.** *Let Q be a virtually polycyclic group. Then*

(a) *Every simple $\mathbb{Z}[Q]$-module is finite and thus is an $\mathbb{F}_p[Q]$-module for some prime p.*
(b) *Assume now that Q is nilpotent, let Ω be the augmentation ideal of $\mathbb{Z}[Q]$ and V a finitely generated $\mathbb{Z}[Q]$-module. If $\Omega M = 0$ for every simple quotient M of V, then $\Omega^N V = 0$ for some $N \in \mathbb{N}$.*

Parts (a) and (b) of Theorem 2.1 are special cases of [Ro, Cor A] and [Ro, Thm B], respectively.

Theorem 2.2 (Robinson [Rb])**.** *Let Q be a nilpotent group, and let M be a Q-module. Assume that either $H_0(Q, M) = 0$ and M is Noetherian or $H^0(Q, M) = 0$ and M is Artinian. Then $H^i(Q, M) = 0$ and $H_i(Q, M) = 0$ for all $i > 0$.*

Proof of Theorem 1.4. First we claim that K^{ab} is finitely generated as a $\mathbb{Z}[Q]$-module. We know that K contains $\gamma_n G$ for some n, and it suffices to prove that $(\gamma_n G)^{ab}$ is finitely generated as a $\mathbb{Z}[G]$-module. Let X be a finite generating set for G. Then $(\gamma_n G)^{ab}$ is generated as an abelian group by left-normed commutators in X of length at least n. But if c is such a commutator and $x \in X$, then $(c, x) = c^{-1}x^{-1}cx$ is equal to $x^{-1}.c - c$ in $(\gamma_n G)^{ab}$, so by straightforward induction $(\gamma_n G)^{ab}$ is generated by left-normed commutators in X of length exactly n as a $\mathbb{Z}[G]$-module.

Next we explain why the two assertions in the statement of Theorem 1.4 are equivalent. Let Ω denote the augmentation ideal of $\mathbb{Z}[Q]$, and suppose that $\Omega^N K^{ab} = \{0\}$ for some $N \in \mathbb{N}$. Then $[K, \underbrace{G, \ldots, G}_{N \text{ times}}] \subseteq [K, K]$, so if $K \supseteq \gamma_n G$, then $[K, K] \supseteq \gamma_{n+N} G$ and $G/[K, K]$ is nilpotent. The reverse implication is analogous. We proceed with proving that $\Omega^N K^{ab} = \{0\}$ for some $N \in \mathbb{N}$ by contradiction.

Let $V = K^{ab}$, and assume that $\Omega^N K^{ab} \neq \{0\}$ for any $N \in \mathbb{N}$. By Theorem 2.1(b) there is a simple Q-module M which is a quotient of V such that $\Omega M \neq 0$, so Q acts on M non-trivially. Moreover, by Theorem 2.1(a), M is a finite $F[Q]$-module for some finite field F of prime order, so $H^1(G, M) = 0$ by the hypotheses of Theorem 1.4.

Since K acts trivially on V and hence on M, we have a natural isomorphism

$$H^1(K, M) \cong Hom(V, M)$$

Under this isomorphism $H^1(K, M)^Q$, the subspace of Q-invariant elements of $H^1(K, M)$, maps to $Hom_Q(V, M)$, the subspace of Q-module homomorphisms from V to M. Since M is a quotient of V as Q-module, we have $Hom_Q(V, M) \neq 0$ and hence $H^1(K, M)^Q \neq 0$.

On the other hand, we have the inflation-restriction sequence

$$0 \to H^1(Q, M) \to H^1(G, M) \to H^1(K, M)^Q \to H^2(Q, M) \to H^2(G, M).$$

Since M is finite, simple and non-trivial, it is both Artinian and Noetherian, and both $H^0(Q, M)$ and $H_0(Q, M)$ are trivial. Thus, using either condition in Theorem 2.2, we conclude that $H^1(Q, M) = H^2(Q, M) = 0$, so $H^1(G, M) \cong H^1(K, M)^Q \neq 0$, a contradiction. □

3. Centralizing generating sequences

Definition. Let G be a group, let $g_1, g_2, \ldots, g_k$ be a finite sequence of elements of G, and for each $1 \le i \le k$ let $G_i = \langle g_j : 1 \le j \le i \rangle$ be the subgroup generated by the first i elements of the sequence.

(a) We will say that the sequence $\{g_n\}$ is *centralizing* if for each $2 \le i \le k$ there exists $j < i$ such that $[g_i, g_j] \in G_{i-1}$.

(b) Now let ρ be a homomorphism from G to another group, and let $C_\rho(G)$ be the set of all $g \in G$ such that $\rho(g)$ is a non-trivial central element of $\rho(G)$. We will say that $\{g_n\}$ is *ρ-centralizing* if $g_1 \in C_\rho(G)$ and for each $2 \le i \le k$ there exists $j < i$ such that $g_j \in C_\rho(G)$ and $[g_i, g_j] \in G_{i-1}$.

Definition. A sequence $g_1, g_2, \ldots, g_k$ in a group G will be called *centralizer generating* if the union of the centralizers $\cup C(g_i)$ generates G.

For brevity we will say that a sequence $g_1, \ldots, g_k$ is

(i) *ρ-CG* (for some ρ) if it is ρ-centralizing and the set $\{g_1, \ldots, g_k\}$ generates G

(i) *CCG* if it is both centralizing and centralizer generating.

The following observation is obvious.

Lemma 3.1. *Let G be a finitely generated group, and let $g_1, \ldots, g_k$ be a CCG sequence in G. Then $g_1, \ldots, g_k$ can be extended to a sequence which is ρ-CG for any ρ such that $g_i \in C_\rho(G)$ for all i.*

In this section we will show that

(i) if ρ is an irreducible representation of G over some field, then the existence of a ρ-CG sequence implies that $H^1(G, \rho) = 0$;

(ii) if $\rho : G \to \mathbb{R}$ is a non-trivial character, the existence of a ρ-CG sequence implies that ρ lies in $\Sigma(G)$, the BNS invariant of G.

We will start with briefly recalling Brown's characterization of $\Sigma(G)$ in terms of actions on $\mathbb{R}$-trees [Br] and establishing a very simple property of group 1-cocycles.

We will need only very basic properties of group actions on $\mathbb{R}$-trees (see, e.g. [AB] for the proofs). Let X be an $\mathbb{R}$-tree with metric d, and suppose that a group G acts on X by isometries; denote the action by α. For each $g \in G$ define $l(g) = \inf_{x \in X} d(\alpha(g)x, x)$ (it is known that the infimum is always attained), and let $\mathcal{A}_g = \{x \in X : d(\alpha(g)x, x) = l(g)\}$. An element $g \in G$ is called *elliptic* (with respect to α) if $l(g) = 0$, in which case $\mathcal{A}_g$ is the fixed point set of g. If $l(g) > 0$, then g is called *hyperbolic*. In this case $\mathcal{A}_g$ is a line, called *the axis of* g, and g acts on $\mathcal{A}_g$ as a translation by $l(g)$; moreover, $\mathcal{A}_g$ is the unique line invariant under the action of g.

An action α is called

- *abelian* if there exists a character $\chi : G \to \mathbb{R}$ such that $l(g) = |\chi(g)|$ for all $g \in G$. If such χ exists, it is unique up to sign, and we will say that α is associated with χ.
- *non-trivial* if it has no (global) fixed points and no (global) invariant line.

The following characterization of the BNS invariant $\Sigma(G)$ was obtained by Brown [Br]:

Theorem 3.2. *Let G be a finitely generated group, and let $\chi : G \to \mathbb{R}$ be a non-trivial character. Then $\chi \in \Sigma(G)$ if and only if G has no non-trivial abelian actions associated* *with χ. Thus, by Theorem 1.3, $[G,G]$ is finitely generated if and only if G has no non-trivial abelian actions.*

Now let (ρ, V) be a non-trivial irreducible representation of a group G over a field F. Recall that $H^1(G,\rho) = Z^1(G,\rho)/B^1(G,\rho)$ where

$$Z^1(G,\rho) = \{f : G \to V : f(xy) = f(x) + \rho(x)f(y) \text{ for all } x,y \in G\}$$

$$B^1(G,\rho) = \{f : G \to V : \text{ there exists } v \in V \text{ s.t. } f(x) = \rho(x)v - v \text{ for all } x \in G\}.$$

Lemma 3.3. *Let G, ρ and V be as above, and define $C_\rho(G)$ as at the beginning of this section. The following hold:*

(i) *There exists a coboundary $b \in B^1(G,\rho)$ with $f(g) = b(g)$*

(ii) *Suppose that $f(g) = 0$. Then $f(x) = 0$ for every $x \in G$ such that $f(gx) = f(xg)$; in particular, $f(x) = 0$ for every $x \in G$ which commutes with g.*

Proof. (i) Since $g \in C_\rho(G)$ and ρ is irreducible, by Schur's Lemma the operator $\rho(g) - 1$ is invertible. Thus, if $v = (\rho(g) - 1)^{-1} f(g)$, then the map $b(x) = \rho(x)v - v$ is a coboundary with the required property.

(ii) We have $f(xg) = f(x) + \rho(x)f(g) = f(x)$ and $f(gx) = f(g) + \rho(g)f(x) = \rho(g)f(x)$, so $(\rho(g) - 1)f(x) = 0$ and hence $f(x) = 0$. □

Proposition 3.4. *Let G be a finitely generated group. The following hold:*

(a) *If (ρ, V) is an irreducible representation of G over some field and G admits a ρ-CG sequence, then $H^1(G,\rho) = 0$*

(b) *Let $\rho : G \to \mathbb{R}$ be a non-trivial character. If G admits a ρ-CG sequence, then $\rho \in \Sigma(G)$. In particular, if G admits a ρ-CG sequence for every non-trivial character ρ, then $[G,G]$ is finitely generated.*

Proof. In both parts of the proof we let $g_1, \ldots, g_k$ be a ρ-CG sequence and $G_i = \langle g_1, \ldots, g_i \rangle$ for $1 \le i \le k$ (so that $G_k = G$).

(a) Take any $f \in Z^1(G,\rho)$. By Lemma 3.3(i), after modifying f by a coboundary, we can assume that $f(g_1) = 0$. Then $f = 0$ on G_1, and we will now prove that $f = 0$ on G_i for all $1 \le i \le k$ by induction on i.

Take $i \ge 2$, and assume that $f = 0$ on G_{i-1}. By hypothesis there exists $j < i$ and $r \in G_{i-1}$ such that $g_j \in C_\rho(G)$ and $g_i g_j = g_j g_i r$. Then $f(g_i g_j) = f(g_j g_i) + \rho(g_j g_i) f(r) = f(g_j g_i)$, whence $f(g_i) = 0$ by Lemma 3.3(ii). Since $G_i = \langle G_{i-1}, g_i \rangle$, we have $f(G_i) = 0$ as desired.

(b) The following argument is similar to the proof of the main theorem in [OK]. By Theorem 3.2 we need to show that if (α, X) is an abelian action of G on an $\mathbb{R}$-tree X associated to ρ, then α is trivial. By assumption g_1 is hyperbolic (with respect to α). We claim that its axis $\mathcal{A} = \mathcal{A}_{g_1}$ is invariant under the entire group G (which will finish the proof). We will prove that $\mathcal{A}$ is invariant under G_i for all $1 \le i \le k$ by induction on i.

Take $i \ge 2$, and assume that $\mathcal{A}$ is invariant under G_{i-1} (in particular, $\mathcal{A}$ is the axis for any hyperbolic element in G_{i-1} by the uniqueness of the invariant line of a hyperbolic element). By hypothesis there exists $j < i$ and $r \in G_{i-1}$ such that g_j is hyperbolic and $g_i^{-1} g_j g_i = g_j r$. The element $g_i^{-1} g_j g_i$ is also hyperbolic (being a conjugate of g_j), and its axis is $\alpha(g_i^{-1})(\mathcal{A})$. On the other hand $g_i^{-1} g_j g_i = g_j r \in G_{i-1}$, so $\mathcal{A}$ is the axis of $g_i^{-1} g_j g_i$. Thus, $\alpha(g_i^{-1})(\mathcal{A}) = \mathcal{A}$, so $\mathcal{A}$ is g_i-invariant and hence G_i-invariant. □

4. $\mathbf{n}$-groups and twin $\mathbf{n}$-groups

For the rest of the paper, given $n \in \mathbb{N}$, denote by $\mathbf{n}$ the set $\{1, 2, \ldots, n\}$. In the first part of this section we will introduce the notion of a good $\mathbf{n}$-group and explain how one can easily construct ρ-CG sequences (for suitable ρ) in such groups. In the second part we will define twin $\mathbf{n}$-groups and establish a sufficient condition on a finitely generated good twin $\mathbf{n}$-group G which guarantees that $\gamma_k G$ is finitely generated for sufficiently small k (see Theorem 4.5).

4.1. **n-groups.**

Definition. Let $n \in \mathbb{N}$. An $\mathbf{n}$-*group* is a group G endowed with a collection of subgroups $\{G_I\}_{I \subseteq \mathbf{n}}$ such that

(i) $G_{\mathbf{n}} = G$

(ii) $G_I \subseteq G_J$ whenever $I \subseteq J$

Given $d \in \mathbb{N}$, we will say that G is *generated in degree* d if $G = \langle G_I : |I| = d \rangle$.

We will say that G is a *very good* $\mathbf{n}$-group if G_I and G_J commute (elementwise) whenever I and J are disjoint.

A simple example of a very good $\mathbf{n}$-group is $G = SL_n(\mathbb{Z})$ with $\{G_I\}$ defined as follows. Let $e_1, \ldots, e_n$ be the standard basic of $\mathbb{Z}^n$. Given $I \subseteq \mathbf{n}$, let $\mathbb{Z}^I = \oplus_{i \in I} \mathbb{Z} e_i$, and let

$$G_I = \{g \in G : g(\mathbb{Z}^I) \subseteq \mathbb{Z}^I \text{ and } g(e_j) = e_j \text{ for all } j \notin I\}. \tag{4.1}$$

Clearly, $SL_n(\mathbb{Z})$ is generated in degree 2. As we will explain at the end of the next section, $\mathrm{Aut}\,(F_n)$ and $\mathrm{Aut}^+(F_n)$ are also very good $\mathbf{n}$-groups generated in degree 2 while IA_n is a very good $\mathbf{n}$-group generated in degree 3. As we will show in § 8, the Torelli group $\mathcal{I}_g^1$ also has a (non-trivial) structure of a $\mathbf{g}$-group generated in degree 3, but that group is not *very good*. Fortunately, the Torelli group is still *good* according to the definition below, which will be sufficient for our purposes.

Definition. Let $n \in \mathbb{N}$, and let I and J be disjoint subsets of $\mathbf{n}$. We will say that I and J are *crossed* if there exist $i_1, i_2 \in I$ and $j_1, j_2 \in J$ such that $i_1 < j_1 < i_2 < j_2$ or $j_1 < i_1 < j_2 < i_2$. Otherwise I and J will be called *uncrossed*.

Note that if I consists of consecutive integers, then I and J are uncrossed for any J disjoint from I.

Definition. An $\mathbf{n}$-group G will be called *good* if G_I and G_J commute for any I and J which are disjoint and uncrossed.

The following lemma can be (very) informally thought of as saying that if ρ is a "generic" homomorphism from a good $\mathbf{n}$-group G to a group of sufficiently small nilpotency class, then G admits a ρ-CG sequence. This lemma will be a key tool in the proof of Theorem 1.5.

Lemma 4.1. *Let G be a good $\mathbf{n}$-group generated in degree d, and let ρ be a homomorphism from G to another group such that*

(*) *there exists disjoint subsets $I_1, \ldots, I_{d+1}$ of $\mathbf{n}$ each of which consists of consecutive integers and elements $g_k \in I_k$ such that each $\rho(g_k) \in \rho(G)$ is central and non-trivial.*

Then G has a ρ-CG sequence.

Proof. By assumption, the elements $g_1, \ldots, g_{d+1}$ commute with each other. Now take any $I \subseteq \mathbf{n}$ with $|I| = d$. There exists $1 \leq k \leq d+1$ such that $I \cap I_k = \emptyset$. Then every element of G_I commutes with g_k. Since G is generated in degree d, the sequence $g_1, \ldots, g_{d+1}$ is CCG and hence by Lemma 3.1 G has a ρ-CG sequence. $\square$

4.2. **Twin n-groups.**

Definition. Let G be a group, $n \in \mathbb{N}$, and suppose that for each $I \subseteq \mathbf{n}$ we have chosen two subgroups G_I, G'_I of G such that $(G, \{G_I\})$ and $(G, \{G'_I\})$ are both $\mathbf{n}$-groups. Then we will say that G is a *twin* $\mathbf{n}$*-group* (with respect to $\{G_I\}, \{G'_I\}$). We will say that

(i) the twin group $\mathbf{n}$-group G is *good* if both $(G, \{G_I\})$ and $(G, \{G'_I\})$ are good $\mathbf{n}$-groups and in addition G_I commutes with G'_J for any disjoint I and J.
(ii) the twin group $\mathbf{n}$-group G is generated in degree d if both $(G, \{G_I\})$ and $(G, \{G'_I\})$ are generated in degree d.

If $(G, \{G_I\})$ is a very good $\mathbf{n}$-group, then G clearly becomes a good twin $\mathbf{n}$-group if we set $G'_I = G_I$. In § 8 we will show that the mapping class group Mod_g^1 (which is a good, but not a very good $\mathbf{g}$-group) is also a good twin $\mathbf{g}$-group.

If H is a subgroup of a twin $\mathbf{n}$-group G, then H is also a twin $\mathbf{n}$-group with respect to the subgroups $\{H_I = H \cap G_I, H'_I = H \cap G'_I\}$, and of course, H is good whenever G is good. If H is normal in G, we will say that H is *normally generated in degree* d if each of the sets $\bigcup_{|I|=d} H_I$ and $\bigcup_{|I|=d} H'_I$ generates H as a normal subgroup of G.

Our next goal is to produce a large class of CCG sequences in good twin $\mathbf{n}$-groups (under suitable conditions). This will be done using Lemma 4.2 below (see the remark following the lemma).

Let H be a subgroup of a group Γ. Denote by $\mathcal{C}_\Gamma(H)$ the graph whose vertices are conjugates of H and where $H^x = x^{-1}Hx$ and $H^y = y^{-1}Hy$ are connected by an edge if and only if they commute (elementwise). We would like to find a sufficient condition for $\mathcal{C}_\Gamma(H)$ to be connected. Clearly, Γ has a natural right action on $\mathcal{C}_\Gamma(H)$, so if S is any generating set for Γ and for every $s \in S$ there is a path in $\mathcal{C}_\Gamma(H)$ connecting H and H^s, then $\mathcal{C}_\Gamma(H)$ is connected.

Lemma 4.2. *Let $e \in \mathbb{N}$ and let Γ be a good twin $\mathbf{n}$-group satisfying the following two conditions:*

(i) *Γ has a generating set S contained in $\bigcup_{|J|=e} \Gamma_J \cap \Gamma'_J$*
(ii) *given any $J_1, J_2 \subseteq \mathbf{n}$ with $|J_1| = |J_2|$, there exists $a \in \Gamma$ such that $(\Gamma_{J_1})^a = \Gamma'_{J_2}$.*

Let H be a subgroup of Γ_I for some I. The following hold:

(a) *Assume that $n \geq 2|I| + e - 1$. Then $\mathcal{C}_\Gamma(H)$ is connected.*
(b) *Let K be a normal subgroup of Γ containing H, and assume that K is normally generated in degree m (as a subgroup of Γ) with $n \geq m + |I|$. Then the centralizers of the vertices of $\mathcal{C}_\Gamma(H)$ generate K.*

Remark. *Suppose that H and K satisfy the hypotheses of (a) and (b) above and K is finitely generated. Then there is a finite connected subgraph X of $\mathcal{C}_\Gamma(H)$ such that the centralizers of the vertices of X generate K. Let $H^{a_1}, \ldots, H^{a_m}$ be the vertices of X ordered* *in such a way that for all $2 \leq i \leq m$, the subgroup H^{a_i} is connected by an edge to H^{a_j} for some $j < i$. Then for any $x_1, \ldots, x_m \in H$ the sequence $x_1^{a_1}, \ldots, x_m^{a_m}$ is CCG in K.*

Proof. (a) Let S be as in (i) above, and take any $s \in S$; by assumption $s \in \Gamma_J \cap \Gamma'_J$ for some J with $|J| = e$. As explained right before Lemma 4.2, we only need to show that H is connected (by a path) to H^s. If I and J are disjoint, then s commutes with every element of H (since $H \subseteq \Gamma_I$ and $s \in \Gamma'_J$), so $H^s = H$. If I and J are not disjoint, then $|I \cup J| \leq |I| + e - 1$ and $H^s \subseteq \Gamma_{I \cup J}$. Since $n \geq 2|I| + e - 1$, we can find a subset I' with $|I'| = |I|$ such that I' is disjoint from $I \cup J$. By (ii), there exists $a \in \Gamma$ such that $(\Gamma_I)^a = \Gamma'_{I'}$. Then $H^a \subseteq \Gamma'_{I'}$ commutes with both H and H^s, so there is a path (of length 2) in $\mathcal{C}_\Gamma(H)$ connecting H and H^s.

(b) Given $J \subseteq \mathbf{n}$ with $|J| = m$, choose I' disjoint from J with $|I'| = |I|$ and $a \in \Gamma$ such that $(\Gamma_I)^a = \Gamma'_{I'}$. Then the centralizer of $H^a \subseteq \Gamma'_{I'}$ in K contains $K \cap \Gamma_J$. Since K is normally generated in degree m and the subgroup generated by the centralizers of the vertices of $\mathcal{C}_\Gamma(H)$ is clearly normal in Γ, this subgroup must be the entire K. □

Lemma 4.3. *Let G be a twin $\mathbf{n}$-group generated in degree d. Then $\gamma_k G$ is normally generated in degree kd for every $k \in \mathbb{N}$.*

Proof. The result is a direct consequence of the following simple fact: if S is any generating set of (any group) G, then the set $\{[s_1, \ldots, s_k] : s_i \in S\}$ of left-normed commutators of length k in elements of S generates $\gamma_k G$ as a normal subgroup of G. □

The following lemma is probably well known but we include a proof for completeness.

Lemma 4.4. *Let G be a normal subgroup of a group Γ such that $G^{ab} \otimes \mathbb{R} \cong H_1(G, \mathbb{R})$ is finite-dimensional. Suppose $Q = \Gamma/G$ is a Zariski dense subgroup in a real affine algebraic group L and there exists a rational representation of L on $G^{ab} \otimes \mathbb{R}$ which extends the action of Q on G^{ab}. Then for any $k \in \mathbb{N}$ there exists a rational representation of L on $(\gamma_k G/\gamma_{k+1} G) \otimes \mathbb{R}$ which extends the conjugation action of Q on $\gamma_k G/\gamma_{k+1} G$.*

Proof. Fix $k \in \mathbb{N}$, and for brevity set $A = G/[G, G]$, $B = \gamma_k G/\gamma_{k+1} G$, $A_\mathbb{R} = A \otimes \mathbb{R}$ and $B_\mathbb{R} = B \otimes \mathbb{R}$. Note that there is a surjective Q-equivariant map $\pi : A^{\otimes k} \to B$ given by

$$\pi(g_1[G, G] \otimes \ldots \otimes g_k[G, G]) = [g_1, \ldots, g_k]\gamma_{k+1} G$$

which naturally extends to a (still surjective) Q-equivariant map $\pi_\mathbb{R} : A_\mathbb{R}^{\otimes k} \to B_\mathbb{R}$. On the other hand, the rational representation of L on $A_\mathbb{R}$ yields a representation on $A_\mathbb{R}^{\otimes k}$ which is also rational and extends the action of Q on $A^{\otimes k}$. Let $C_\mathbb{R} = \operatorname{Ker} \pi_\mathbb{R}$. Since $C_\mathbb{R}$ is Zariski closed and Q-invariant and Q is Zariski dense in L, we conclude that $C_\mathbb{R}$ is L-invariant. Thus, we obtain a rational representation of L on $B_\mathbb{R} \cong A_\mathbb{R}^{\otimes k}/C_\mathbb{R}$ which by construction extends the Q-action. □

Remark. *Let $B_\mathbb{R}$ be as in the above proof. The obtained rational representation of L on $B_\mathbb{R}$ yields the dual representation of L on $B_\mathbb{R}^*$ (which is still rational). The underlying space of $B_\mathbb{R}^*$ can be naturally identified with* $\operatorname{Hom}(\gamma_k G/\gamma_{k+1} G, \mathbb{R})$*, and the representation of L on $B_\mathbb{R}^*$ extends the action of $Q = \Gamma/G$ on* $\operatorname{Hom}(\gamma_k G/\gamma_{k+1} G, \mathbb{R})$ *given by*

$$((aG).\chi)(g\gamma_{k+1} G) = \chi((a^{-1} g a)\gamma_{k+1} G) \tag{4.2}$$

We are now ready to state and prove the main result of this subsection, which will be used to prove Theorem 1.1.

Theorem 4.5. *Let Γ be a twin $\mathbf{n}$-group and G a normal subgroup which is finitely generated (as a group). Assume that conditions (i) and (ii) of Lemma 4.2 hold and also the hypotheses of Lemma 4.4 hold with L connected. Then the following hold:*

(a) *Assume that G is generated in degree d (as a twin $\mathbf{n}$-group) and $n \geq 2(k-1)d+e-1$ (with e coming from condition (i) of Lemma 4.2). Then $\gamma_k G$ is finitely generated.*
(b) *Assume that G is normally generated in degree f (as a subgroup of Γ). If $n \geq 2f+e-1$, then $[G,G]$ is finitely generated.*

Proof. (a) We argue by induction on k, with the case $k=1$ being vacuous. Suppose now that $k \geq 2$ with $n \geq (2k-1)d$ and we already proved that $\gamma_{k-1}G$ is finitely generated. By Theorem 1.3, to prove that $\gamma_k G$ is finitely generated we need to show that

$$\mathrm{Hom}\,(\gamma_{k-1}G/\gamma_k G, \mathbb{R})) \setminus \{0\} \subseteq \Sigma(\gamma_{k-1}G)$$

where we identify $\mathrm{Hom}\,(\gamma_{k-1}G/\gamma_k G, \mathbb{R}))$ with $\{\chi \in \mathrm{Hom}\,(\gamma_{k-1}G), \mathbb{R}) : \chi(\gamma_k G) = 0\}$. Assume this is not case, and choose a nonzero character $\psi \in \mathrm{Hom}\,(\gamma_{k-1}G/\gamma_k G, \mathbb{R})$ which does not lie in $\Sigma(\gamma_{k-1}G)$. By Lemma 4.3, the set $A = \bigcup\limits_{|I|=(k-1)d} (\gamma_{k-1}G \cap G_I)$ generates $\gamma_{k-1}G$ as a normal subgroup of G, whence $\gamma_{k-1}G = \langle A \rangle \gamma_k G$. Since ψ vanishes on $\gamma_k G$, there must exist I with $|I| = (k-1)d$ and $x \in \gamma_{k-1}G \cap G_I$ such that $\psi(x) \neq 0$.

Let $D = \mathrm{Hom}\,(\gamma_{k-1}G/\gamma_k G, \mathbb{R}) \setminus \Sigma(\gamma_{k-1}G)$. It is clear that D is Γ/G-invariant under the action given by (4.2). Let C be an irreducible component of D (in Zariski topology) which contains ψ. Since D has finitely many irreducible components, C must be S-invariant for some finite index subgroup S of Γ/G.

Now let $H = \gamma_{k-1}G \cap G_I$ (with I chosen above). Then $2|I|+e-1 = 2(k-1)d-1 \leq n$, so the hypothesis of Lemma 4.2(a) holds. Since $\gamma_{k-1}G$ is normally generated in degree $m = (k-1)d$ (as a subgroup of G and hence as a subgroup of Γ), the hypothesis of Lemma 4.2(b) also holds with $K = \gamma_{k-1}G$. Thus, the graph $\mathcal{C}_\Gamma(H)$ is connected and the centralizers of its vertices generate $\gamma_{k-1}G$, so by the remark after Lemma 4.2 there exists a sequence $x_1, \ldots, x_m$ which is CCG for $\gamma_{k-1}G$ such that each x_i is a Γ-conjugate of x chosen above. For each $1 \leq i \leq m$ let

$$U_i = \{\chi \in \mathrm{Hom}\,(\gamma_{k-1}G/\gamma_k G, \mathbb{R}) : \chi(x_i) = 0\}.$$

By Lemma 3.1, $\gamma_{k-1}G$ has a ρ-CG sequence for any $\rho \in \mathrm{Hom}\,(\gamma_{k-1}G, \mathbb{R}) \setminus \cup U_i$, so $\mathrm{Hom}\,(\gamma_{k-1}G/\gamma_k G, \mathbb{R}) \setminus \cup U_i \subseteq \Sigma(\gamma_{k-1}G)$ by Proposition 3.4. Thus $C \subseteq \cup_{i=1}^m U_i$, and by irreducibility $C \subseteq U_i$ for some i. Since C is S-invariant, it is contained in the S-invariant linear subspace $U = \bigcap\limits_{s \in S} sU_i$. Since S has finite index in Γ/G which, in turn, is Zariski dense in L and L is connected, S must also be Zariski dense in L. As in the proof of Lemma 4.4, since U is Zariski closed and S-invariant, it must be L-invariant and hence Γ/G-invariant under the action (4.2). Since $\psi \in U$, for every $a \in \Gamma$ the character ψ_a given by $\psi_a(g) = \psi(a^{-1}ga)$ lies in U, and therefore ψ vanishes at every Γ-conjugate of x_i. But by construction, one of these conjugates is x itself, a contradiction.

(b) The proof of (b) is an easy variation of the proof of (a) in the case $k=2$. Assume that there exists a nonzero character $\psi \in \mathrm{Hom}\,(G, \mathbb{R}) \setminus \Sigma(G)$. Since G is normally generated in Γ in degree f, there exists $I \subseteq \mathbf{n}$ with $|I| = f$, $x \in G_I$ and $a \in \Gamma$ such that $\psi(a^{-1}xa) \neq 0$. The character ψ_a defined as above also does not lie in $\Sigma(G)$ and satisfies $\psi_a(x) \neq 0$. The rest of the proof is identical to that of (a), with ψ replaced by ψ_a. □

5. Regular $SL_n(\mathbb{Z})$-modules

In this section we introduce another slightly technical concept, that of a regular $SL_n(\mathbb{Z})$-module. Informally speaking, a regular $SL_n(\mathbb{Z})$-module is a graded (in a somewhat unconventional sense) $SL_n(\mathbb{Z})$-module whose structure is compatible with the canonical $\mathbf{n}$-group structure on $SL_n(\mathbb{Z})$ defined in the previous section.

Definition. Let V be an abelian group (written additively) and $n \in \mathbb{N}$. An $\mathbf{n}$-*grading of* V is a collection of additive subgroups $\{V_I : I \subseteq \mathbf{n}\}$ of V such that

$$\sum_{I \subseteq \mathbf{n}} V_I = V$$

(note that the sum is not required to be direct). The *degree* of V (with respect to the grading $\{V_I\}$), denoted by $\deg(V)$, is the smallest integer k such that $V_I = 0$ whenever $|I| > k$ (this automatically implies that $\sum_{|I| \le k} V_I = V$). If no such k exists, we set $\deg(V) = \infty$.

As usual, $E_{ij} \in SL_n(\mathbb{Z})$ will denote the matrix which has 1's on the diagonal and at the position (i, j) and 0 everywhere else. By $F_{ij} \in SL_n(\mathbb{Z})$ we will denote the matrix obtained from the identity matrix by swapping i^{th} and j^{th} rows and then multiplying the j^{th} row by -1. Note that $F_{ij} = E_{ij} E_{ji}^{-1} E_{ij}$.

Definition. Let $\{G_I : I \subseteq \mathbf{n}\}$ be the subgroups of $SL_n(\mathbb{Z})$ defined by (4.1). Let V be an $SL_n(\mathbb{Z})$-module endowed with an $\mathbf{n}$-grading $\{V_I\}$. We will say that V is *regular* if the following properties hold:

(1) G_J acts trivially on V_I if $I \cap J = \emptyset$
(2) If $I \subseteq \mathbf{n}$, $i \in I$, $j \notin I$, then for any $g \in \{E_{ij}^{\pm 1}, E_{ji}^{\pm 1}\}$ and $v \in V_I$ we have
$$gv - v \in V_{I \setminus \{i\} \cup \{j\}} + V_{I \cup \{j\}}$$
(3) If $I \subseteq \mathbf{n}$, $i, j \in \mathbf{n}$ with $i \neq j$, then $F_{ij} V_I = V_{(i,j)I}$ where $(i, j)I$ is the image of I under the transposition (i, j). In particular, if $i \in I$ and $j \notin I$, then $F_{ij} V_I = V_{I \setminus \{i\} \cup \{j\}}$.

Condition (2) in the above definition is tailored specifically for the purposes of this paper. Our notion of a regular $SL_n(\mathbb{Z})$-module certainly has some formal similarities with the notion of an FI-module from [CEF], but it is not clear if there are deep connections between the two notions.

The following result provides our starting examples of regular $SL_n(\mathbb{Z})$-modules.

Lemma 5.1. *Let R be a commutative ring with 1, $n \ge 2$, let R^n be a free R-module of rank n with basis $e_1, \ldots, e_n$, and let $(R^n)^* = Hom_R(R^n, R)$ be the dual module, with dual basis $e_1^*, \ldots, e_n^*$. Consider R^n and $(R^n)^*$ as $SL_n(\mathbb{Z})$-modules with standard actions. Define $(R^n)_{\{i\}} = Re_i$, $(R^n)^*_{\{i\}} = Re_i^*$ for $1 \le i \le n$ and $(R^n)_I = 0$, $(R^n)^*_I = 0$ for $|I| \neq 1$. Then with respect to these gradings R^n and $(R^n)^*$ are regular of degree 1.*

Proof. The only non-obvious part is condition (ii) in the definition of a regular $SL_n(\mathbb{Z})$-module. Condition (ii) is vacuous if $|I| \neq 1$, so assume that $I = \{i\}$ and $j \neq i$, in which case $I \setminus \{i\} \cup \{j\} = \{j\}$ and $I \cup \{j\} = \{i, j\}$.

We have $E_{ij}^{\pm 1}(re_i) = re_i$, $E_{ji}^{\pm 1}(re_i) - re_i = \pm re_j \in (R^n)_{\{j\}}$, $E_{ji}^{\pm 1}(re_i^*) = re_i^*$, $E_{ij}^{\pm 1}(re_i^*) - re_i^* = \mp re_j^* \in (R^n)^*_{\{j\}}$, so (ii) holds (note that in this case terms from $V_{I\cup\{j\}}$ do not arise). □

Lemma 5.2. *The following hold:*

(a) *Let V and W be regular $SL_n(\mathbb{Z})$-modules.*
   (i) *Define $(V\oplus W)_I = V_I \oplus W_I$. Then $V\oplus W$ is regular and* $\deg(V\oplus W) = \max\{\deg(V), \deg(W)\}$.
   (ii) *Define $(V\otimes_{\mathbb{Z}} W)_I = \sum\limits_{I=I_1\cup I_2} V_{I_1}\otimes W_{I_2}$. Then $V\otimes_{\mathbb{Z}} W$ is regular and* $\deg(V\otimes_{\mathbb{Z}} W) \le \deg(V) + \deg(W)$

(b) *Let V be a regular $SL_n(\mathbb{Z})$-module and U a submodule. For $I\subseteq \mathbf{n}$ set $U_I = U\cap V_I$ and $(V/U)_I = V_I + U$. Then V/U is always regular with* $\deg(V/U)\le \deg(V)$, *and U is regular with* $\deg(U)\le\deg(V)$ *provided $U = \sum\limits_{I\subseteq \mathbf{n}} U_I$.*

(c) *Suppose that $SL_n(\mathbb{Z})$ acts on an $\mathbb{N}$-graded Lie ring $L = \oplus_{m=1}^{\infty} L(m)$ by graded automorphisms. Suppose that L is generated in degree 1 (as a Lie ring) and that $L(1)$ is a regular $SL_n(\mathbb{Z})$-module of degree d. For each $m>1$ and $I\subseteq\mathbb{N}$ define*

$$L(m)_I = \sum_{I=I_1\cup I_2\cup\ldots\cup I_m} [L(1)_{I_1}, L(1)_{I_2}, \ldots, L(1)_{I_m}] \qquad (***)$$

Then $L(m)$ is a regular $SL_n(\mathbb{Z})$-module of degree at most md.

Proof. Parts (a)(i) and (b) are straightforward.

(a)(ii) As in the proof of Lemma 5.1, we only need to check condition (ii) in the definition of a regular module. So take $I\subseteq\mathbf{n}$, $i\in I$ and $j\notin I$. It is enough to check the condition for v of the from $v = x\otimes y$ where $x\in V_{I_1}$, $y\in W_{I_2}$ and $I_1\cup I_2 = I$. We will consider the case when i belongs to both I_1 and I_2 (the case when i only lies in one of those sets is analogous). Thus, $I_1 = K_1\cup\{i\}$ and $I_2 = K_2\cup\{i\}$ with $i\notin K_1, K_2$.

Let $g\in\{E_{ij}^{\pm 1}, E_{ji}^{\pm 1}\}$. Since V and W are regular, we have $gx = x + x_1 + x_2$ and $gy = y + y_1 + y_2$ where $x_1\in V_{K_1\cup\{j\}}$, $x_2 = V_{K_1\cup\{i,j\}}$ $y_1\in W_{K_2\cup\{j\}}$, $x_2 = W_{K_2\cup\{i,j\}}$. Then $x_1\otimes y_1\in (V\otimes W)_{K_1\cup K_2\cup\{j\}} = (V\otimes W)_{I\setminus\{i\}\cup\{j\}}$ and each of the 7 terms $x\otimes y_1$, $x\otimes y_2$, $x_1\otimes y$, $x_1\otimes y_2$, $x_2\otimes y$, $x_2\otimes y_1$ and $x_2\otimes y_2$ lies in $(V\otimes W)_{K_1\cup K_2\cup\{i,j\}} = (V\otimes W)_{I\cup\{j\}}$, so condition (ii) holds.

(c) Consider the map $\varphi : L(1)^{\otimes m}\to L(m)$ given by $\varphi(v_1\otimes\ldots\otimes v_m) = [v_1,\ldots,v_m]$ (where the commutator on the right-hand side is left-normed). Since $SL_n(\mathbb{Z})$ acts on L by graded automorphisms, φ is a homomorphism of $SL_n(\mathbb{Z})$-modules, and since L is generated in degree 1, φ is surjective. Thus, $L(m)$ is a quotient of $L(1)^{\otimes m}$ as an $SL_n(\mathbb{Z})$-module, and the grading (***) coincides with the quotient grading defined in (b). Thus, (c) follows from (a)(ii) and (b). □

Given a group G, let $L(G) = \oplus_{i=1}^{\infty}\gamma_i G/\gamma_{i+1}G$. The graded abelian group $L(G)$ has a natural structure of a graded Lie ring with the bracket on homogeneous elements defined by

$$[g\gamma_{i+1}G, h\gamma_{j+1}G] = [g,h]\gamma_{i+j+1}G \text{ for all } g\in\gamma_i G \text{ and } h\in\gamma_j G,$$

where $[g,h] = g^{-1}h^{-1}gh$. The bracket operation is well defined since $[\gamma_i G, \gamma_j G]\subseteq\gamma_{i+j}G$. It is clear from the definition that $L(G)$ is generated in degree 1 as a Lie ring.

Theorem 5.3. *Let G be a finitely generated group and $L = L(G) = \oplus_{i=1}^{\infty}\gamma_i G/\gamma_{i+1}G$. Suppose we are given another group Γ which contains G as a normal subgroup and a homomorphism $\varphi : SL_n(\mathbb{Z}) \to \Gamma/G$. Define the action of $SL_n(\mathbb{Z})$ on L by automorphisms by*

$$x.(g\gamma_{i+1}G) = \left(\varphi(x)g\varphi(x)^{-1}\right)\gamma_{i+1}G \text{ for all } x \in SL_n(\mathbb{Z}),\ i \in \mathbb{N} \text{ and } g \in \gamma_i G. \tag{5.1}$$

Let $d \in \mathbb{N}$, and suppose that

(i) *G has the structure of a good $\mathbf{n}$-group generated in degree d.*
(ii) *$L(1) = G/[G,G] = G^{ab}$ has the structure of a regular $SL_n(\mathbb{Z})$-module of degree at most d.*
(iii) *For every $I \subseteq \mathbf{n}$, with $|I| \geq d$, the image of G_I in $G^{ab} = L(1)$ contains $\sum\limits_{J\subseteq I} L(1)_J$*

Let $N = \left[\frac{n}{d(d+1)}\right] + 1$. If ρ is any non-trivial homomorphism from G to another group with $\mathrm{Ker}\,(\rho) \supseteq \gamma_N G$*, then G has a ρ-CG sequence.*

As mentioned in the introduction, Theorem 1.1 and Theorem 1.5 in the majority of cases will follow from Theorem 4.5 and Theorem 5.3, respectively, for $G = IA_n$ and $G = \mathcal{I}_g^1$, once we verify the hypotheses of those theorems (with suitable values of the parameters involved). Checking the relevant conditions for $G = IA_n$ is easy and will be done at the end of this section. The corresponding verification for the Torelli groups requires more work and will be given in § 8. The "exceptional" cases $G = IA_n$, $N = 2$ with $4 \leq n < 7$ (resp. $4 \leq n < 12$) of Theorem 1.1 (resp. Theorem 1.5) will be treated separately in § 7.

5.1. **Verification of hypotheses of Theorems 4.5 and 5.3 for $G = IA_n$.** In this subsection we will verify the hypotheses of Theorem 4.5(a) and Theorem 5.3 for $G = IA_n$. In both theorems we take $\Gamma = \mathrm{Aut}^+(F_n)$ (see the definition below) and $d = 3$, and in Theorem 4.5 we let $L = SL_n(\mathbb{R})$ and $e = 2$. We will not be using Theorem 4.5(b) (even though we could) since it would not yield any strengthening of Theorem 1.1 compared to what we obtain from Theorem 7.1.

Define the maps $F_i \in \mathrm{Aut}\,(F_n)$ for $1 \leq i \leq n$ and $R_{ij}, L_{ij} \in \mathrm{Aut}\,(F_n)$ for $1 \leq i \neq j \leq n$ by

$$F_i : \begin{cases} x_i \mapsto x_i^{-1} \\ x_k \mapsto x_k \text{ for } k \neq i \end{cases}, \quad R_{ij} : \begin{cases} x_j \mapsto x_j x_i \\ x_k \mapsto x_k \text{ for } k \neq j \end{cases}, \quad L_{ij} : \begin{cases} x_j \mapsto x_i x_j \\ x_k \mapsto x_k \text{ for } k \neq j \end{cases}$$

It is well known that $\mathrm{Aut}\,(F_n)$ is generated by these maps. The subgroup $\mathrm{Aut}^+(F_n)$ generated by R_{ij} and L_{ij} has index 2 in $\mathrm{Aut}\,(F_n)$ and is equal to the preimage of $SL_n(\mathbb{Z})$ under the natural projection $\pi : \mathrm{Aut}\,(F_n) \to GL_n(\mathbb{Z})$. Note that $\pi(R_{ij}) = \pi(L_{ij}) = E_{ij}$.

Define the structure of an $\mathbf{n}$-group on $\Gamma = \mathrm{Aut}^+(F_n)$ as follows: for each $I \subseteq \mathbf{n}$ let Γ_I be the subgroup consisting of automorphisms which leave the subgroup $\langle x_i : i \in I\rangle$ invariant and fix x_j for every $j \notin I$. Clearly, the obtained $\mathbf{n}$-group is very good, so we automatically get a good twin $\mathbf{n}$-group structure by setting $\Gamma'_I = \Gamma_I$.

Since Γ_I contains R_{ij}, L_{ij} for $i, j \in I$, we conclude that Γ is generated in degree 2; moreover condition (i) of Lemma 4.2 holds for $e = 2$. For each permutation $\sigma \in S_n$ let $F_\sigma \in \mathrm{Aut}^+(F_n)$ be the automorphism which sends x_i to $x_{\sigma(i)}$ for each $i > 1$ and x_1 to $x_{\sigma(1)}^{\pm 1}$ (with the sign chosen so that $F_\sigma \in \mathrm{Aut}^+(F_n)$). It is clear that $\Gamma_I^{F_\sigma} = \Gamma_{\sigma^{-1}(I)}$, and therefore condition (ii) of Lemma 4.2 also holds.

We will consider $G = IA_n$ with the twin $\mathbf{n}$-group structure induced from Γ (that is, $G_I = G'_I = G \cap \Gamma_I$). In [Ma], Magnus proved that IA_n is generated by the automorphisms K_{ij}, $i \neq j$ and K_{ijk}, i, j, k distinct, given by

$$K_{ij} : \begin{cases} x_i \mapsto x_j^{-1} x_i x_j \\ x_k \mapsto x_k \text{ for } k \neq i \end{cases}, \qquad K_{ijk} : \begin{cases} x_i \mapsto x_i[x_j, x_k] \\ x_l \mapsto x_l \text{ for } l \neq i \end{cases}$$

Thus, IA_n is generated in degree 3. Since $K_{ijk}^{-1} = K_{ikj}$, it is enough to consider K_{ijk} with $j < k$. The elements K_{ij} and K_{ijk} with $j < k$ will be referred to as the *standard generators* of IA_n.

Next we describe IA_n^{ab} as a $GL_n(\mathbb{Z})$-module (where we identify $GL_n(\mathbb{Z})$ with $\mathrm{Aut}\,(F_n)/IA_n$, as explained at the beginning of this subsection). Recall that IA_n and $IA_n(2)$ are defined as the kernels of the natural maps $\mathrm{Aut}\,(F_n) \to \mathrm{Aut}\,(F_n/\gamma_2 F_n)$ and $\mathrm{Aut}\,(F_n) \to \mathrm{Aut}\,(F_n/\gamma_3 F_n)$, respectively. Define a map $\psi : IA_n \to Hom(F_n/\gamma_2 F_n, \gamma_2 F_n/\gamma_3 F_n)$ by

$$(\psi(g))(x + \gamma_2 F_n) = x^{-1} g(x) + \gamma_3 F_n \quad \text{ for all } g \in IA_n, x \in F_n.$$

It is easy to see that ψ is a well defined homomorphism, $\mathrm{Ker}\,\psi = IA_n(2)$, and the elements $\{\psi(K_{ij}), \psi(K_{ijk})\}$ span $Hom(F_n/\gamma_2 F_n, \gamma_2 F_n/\gamma_3 F_n)$, so ψ is surjective. Since $IA_n(2) = [IA_n, IA_n]$ by [Ba, Lemma 5], we deduce an isomorphism of abelian groups:

$$IA_n^{ab} \cong Hom(F_n/\gamma_2 F_n, \gamma_2 F_n/\gamma_3 F_n)$$

Now let $V = \mathbb{Z}^n$. Then $F_n/\gamma_2 F_n \cong V$ and $\gamma_2 F_n/\gamma_3 F_n \cong V \wedge V$ as abelian groups. Thus,

$$IA_n^{ab} \cong Hom(V, V \wedge V) \cong V^* \otimes (V \wedge V) \qquad (***)$$

as abelian groups, and it is straightforward to check that this is actually an isomorphism of $GL_n(\mathbb{Z})$-modules (see [Kaw, Thm 6.1] for a self-contained proof of this isomorphism). The isomorphism (***) and Lemmas 5.1 and 5.2 imply that IA_n^{ab} is a regular $SL_n(\mathbb{Z})$-module generated in degree 3, so condition (ii) in Theorem 5.3 holds.

If $e_1, \ldots, e_n$ is the standard basis of V and $e_1^*, \ldots, e_n^*$ is the corresponding dual basis in V^*, then the grading on $W = V^* \otimes (V \wedge V)$ coming from Proposition 5.1 and Lemma 5.2 is given by

$$W_I = \sum_{I = \{i,j,k\}} \mathbb{Z} e_i^* \otimes (e_j \wedge e_k) \text{ where we do not require that } i, j, k \text{ are distinct.}$$

It is easy to check that under the isomorphism (***), K_{ij} maps to $e_i^* \otimes (e_i \wedge e_j)$ and K_{ijk} maps to $e_i^* \otimes (e_j \wedge e_k)$. This shows that condition (iii) in Theorem 5.3 also holds.

Finally, it is clear that the above representation of $SL_n(\mathbb{Z})$ on W extends to a rational (in fact, polynomial) representation of $SL_n(\mathbb{R})$ on $W \otimes \mathbb{R}$ which completes the verification of hypotheses of Theorem 4.5.

6. Proof of Theorem 5.3

Throughout this section we fix the notations introduced in Theorem 5.3. Without loss of generality we can assume that the homomorphism $\varphi : SL_n(\mathbb{Z}) \to \Gamma/G$ is surjective since if not, we can replace Γ by $\varphi^{-1}(\Gamma)$.

Let m be the largest integer such that $\rho(\gamma_m G)$ is non-trivial, and let

$$H = \mathrm{Ker}\,\rho \cap \gamma_m G.$$

Thus by assumptions $1 \le m \le N-1$, whence $n \ge md(d+1)$, and $\gamma_m G \backslash H \subseteq C_\rho(G)$, where as before $C_\rho(G)$ is the set of all elements of G such that $\rho(g)$ is central and non-trivial.

Now let $L(1) = G/[G,G]$ and $L(m) = \gamma_m G/\gamma_{m+1}G$. Given $I \subseteq \mathbf{n}$, define $L(m)_I$ as in Lemma 5.2(c). Then $\deg(L(m)) \le md$, that is,

$$L(m) = \sum_{|I| \le md} L(m)_I \quad \text{and} \quad L(m)_I = 0 \text{ if } |I| > md \tag{6.1}$$

Claim 6.1. *If $|I| \ge d$, then the image of $\gamma_m G_I$ in $L(m)$ contains $L(m)_J$ for every $J \subseteq I$.*

Proof. Fix $J \subseteq I$. By definition $L(m)_J$ is spanned by elements of the form $[x_1, \ldots, x_m]$ where $x_k \in L(1)_{J_k}$ for some $J_k \subseteq J$. Since $|I| \ge d$, hypothesis (iii) of Theorem 5.3 implies that there exist elements $\{g_k \in G_I\}_{k=1}^d$ such that $x_k = g_k[G,G]$. Then $[x_1, \ldots, x_m]$ is the image in $L(m)$ of the element $[g_1, \ldots, g_m]$ which lies in $\gamma_m G_I$. $\square$

Let U be any abelian group which contains an isomorphic copy of every finitely generated abelian group, and let

$$\Omega = Hom_{\mathbb{Z}}(L(m), U)$$

We will define the action of $SL_n(\mathbb{Z})$ on Ω in the usual way:

$$(g\lambda)(x) = \lambda(g^{-1}x) \text{ for all } g \in SL_n(\mathbb{Z}), \lambda \in \Omega \text{ and } x \in L(m).$$

Since G is a finitely generated group, $L(m)$ is a finitely generated abelian group, whence every subgroup of $L(m)$ is the kernel of some element of Ω. Choose $\lambda_H \in \Omega$ such that $\operatorname{Ker}\lambda_H = H/\gamma_{m+1}G$ (recall that $H = \gamma_m G \cap \operatorname{Ker}\rho$).

Definition. Let $\lambda \in \Omega$. Define *supp*(λ), the *support of* λ, to be the set of all subsets $I \subseteq \mathbf{n}$ such that λ does not vanish on $L(m)_I$, that is, $L(m)_I \not\subseteq \operatorname{Ker}\lambda$.

Note that while λ_H may not be uniquely determined, its support does not depend on the choice. Let us now see how the support of λ_H changes if we precompose ρ by the conjugation by an element of Γ. Given $a \in \Gamma$, define ρ_a by $\rho_a(x) = \rho(a^{-1}xa)$, and choose $g \in SL_n(\mathbb{Z})$ such that $\varphi(g) = aG$. Then

$$\begin{aligned}(\operatorname{Ker}\rho_a \cap \gamma_m G)/\gamma_{m+1}G &= a(\operatorname{Ker}\rho \cap \gamma_m G)a^{-1}/\gamma_{m+1}G \\ &= g\left((\operatorname{Ker}\rho \cap \gamma_m G)/\gamma_{m+1}G\right) = g(H/\gamma_{m+1}G) = g\operatorname{Ker}\lambda_H = \operatorname{Ker}(g\lambda_H).\end{aligned}$$

Claim 6.2. *Let $M = d(N-1)$, and for $1 \le k \le d+1$ define*

$$I_k = \{(k-1)M+1, (k-1)M+2, \ldots, kM\}$$

(recall that $(d+1)M \le n$). Suppose there exists $g \in SL_n(\mathbb{Z})$ such that supp$(g\lambda_H)$ contains $d+1$ subsets $J_1, \ldots, J_{d+1}$ with $J_k \subseteq I_k$. Then Theorem 5.3 holds for ρ.

Proof. Let $a \in \Gamma$ be such that $\varphi(g) = aG$. By Claim 6.1, for each $1 \le k \le d+1$ the image of $\gamma_m G_{I_k}$ in $L(m)$ is not contained in $\operatorname{Ker}(g\lambda_H)$, and by the above computation $\operatorname{Ker}(g\lambda_H) = (\operatorname{Ker}\rho_a \cap \gamma_m G)/\gamma_{m+1}G$. Thus, there exist elements $g_k \in \gamma_m G_{I_k} \setminus \operatorname{Ker}\rho_a \subseteq C_{\rho_a}(G) \cap G_{I_k}$ for $1 \le k \le d+1$. By Lemma 4.1, G has a ρ_a-CG sequence, say $x_1, \ldots, x_l$, whence it also has a ρ-CG sequence, namely $a^{-1}x_1 a, \ldots, a^{-1}x_l a$. $\square$

To prove Theorem 5.3 it remains to show that there exists $g \in SL_n(\mathbb{Z})$ satisfying the hypotheses of Claim 6.2. This easily follows from Proposition 6.3 below.

Definition. Let $0 \neq \lambda \in \Omega$ and $s \in \mathbb{N}$. If $A_1, \ldots, A_s$ are elements of $supp(\lambda)$, define

$$D(A_1, \ldots, A_s) = \sum_{i=1}^{s} |A_i| - \sum_{i \neq j} |A_i \cap A_j|$$

The maximum possible value of $D(A_1, \ldots, A_s)$ will be denoted by $D_s(\lambda)$, and any s-tuple in $supp(\lambda)$ on which this maximum is achieved will be called *maximally disjoint for* λ.

Proposition 6.3. *Let $0 \neq \lambda$ and s be as above and let $\{A_1, \ldots, A_s\} \subseteq supp(\lambda)$ be maximally disjoint. Then one of the following holds:*

(i) *$A_1, \ldots, A_s$ are disjoint*
(ii) $\cup A_i = \mathbf{n}$
(iii) *There exists $g \in SL_n(\mathbb{Z})$ such that $D_s(g\lambda) > D_s(\lambda)$*

Before proving Proposition 6.3, we shall deduce Theorem 5.3 from it. Indeed, let $s = d+1$, and assume Proposition 6.3 holds. Then case (ii) above cannot occur unless (i) also holds. Indeed, if $A_1, \ldots, A_{d+1} \in \mathrm{supp}\,(\lambda)$, then $|A_k| \leq md$ by (6.1), so $|\cup_{i=1}^{d+1} A_i| \leq (d+1) \cdot md \leq n$ with equality only possible if A_i are disjoint. If we now choose $g_0 \in SL_n(\mathbb{Z})$ such that $D_{d+1}(g_0\lambda_H)$ is maximal possible, then applying Proposition 6.3 to $\lambda = g_0\lambda_H$ we must be in case (i).

Let $A_1, \ldots, A_{d+1}$ be disjoint subsets in $supp(g_0\lambda_H)$. Since $|A_k| \leq md \leq M$, there exists a permutation $\sigma \in S_n$ such that $\sigma(A_k) \subseteq I_k$ for each k where I_k are as in Claim 6.2. Condition (3) in the definition of a regular module implies that $supp(F_{ij}\mu) = (i,j)supp(\mu)$ for all $\mu \in \Omega$. Thus if we write σ as a product of transpositions $\sigma = \prod(i_t, j_t)$ and let $g = \prod F_{i_t j_t}$, then $\mathrm{supp}\,(gg_0\lambda_H)$ contains $\sigma(A_k)$ for each k, as desired.

Proof of Proposition 6.3. Assume that none of the conditions (i)-(iii) holds. Without loss of generality we can assume that $|A_1 \cap A_2| \neq \emptyset$, and choose $i \in A_1 \cap A_2$ and $j \notin \cup_{i=1}^{s} A_i$. Since the s-tuple $(A_1, \ldots, A_s)$ is maximally disjoint, $supp(\lambda)$ does not contain $A_k \setminus \{i\} \cup \{j\}$ and $A_k \cup \{j\}$ for any k.

Now take any $g \in \{E_{ij}^{\pm 1}, E_{ji}^{\pm 1}\}$. We claim that $supp(g\lambda)$ contains A_k for any k. Indeed, by assumption $A_k \in \mathrm{supp}\,(\lambda)$, so $\lambda(x) \neq 0$ for some $x \in L(m)_{A_k}$. By conditions (1) and (2) in the definition of a regular module, $g^{-1}x - x \in L(m)_{A_k \setminus \{i\} \cup \{j\}} + L(m)_{A_k \cup \{j\}} \subseteq \mathrm{Ker}\,\lambda$. Hence $(g\lambda)(x) = \lambda(g^{-1}x) = \lambda(x) \neq 0$, so $A_k \in \mathrm{supp}\,(g\lambda)$.

Since $supp(g\lambda)$ contains $A_1, \ldots, A_s$ and $(A_1, \ldots, A_s)$ is also maximally disjoint for λ, we must have $D_s(g\lambda) \geq D_s(\lambda)$. On the other hand, $D_s(g\lambda) \leq D_s(\lambda)$ by our hypothesis, so $D_s(g\lambda) = D_s(\lambda)$ and $(A_1, \ldots, A_s)$ is also maximally disjoint for $g\lambda$.

Applying the same argument to $E_{ij}\lambda$, we conclude that $(A_1, \ldots, A_s)$ is also maximally disjoint for $E_{ji}^{-1}E_{ij}\lambda$ and likewise maximally disjoint for $E_{ij}E_{ji}^{-1}E_{ij}\lambda$. In particular, $A_1 \setminus \{i\} \cup \{j\} \notin supp(E_{ij}E_{ji}^{-1}E_{ij}\lambda)$. On the other hand, $E_{ij}E_{ji}^{-1}E_{ij} = F_{ij}$, and condition (3) in the definition of a regular module implies that $supp(F_{ij}\lambda)$ contains $(i,j)A_1 = A_1 \setminus \{i\} \cup \{j\}$, a contradiction. $\square$

7. Proof of Theorems 1.1 and 1.5 for IA_n with $N = 2$

In this section we will prove Theorem 1.1 for $G = IA_n$, $N = 2$, $4 \leq n < 7$ and Theorem 1.5(a) for $G = IA_n$, $N = 2$, $4 \leq n < 12$, the cases which do not follow from

Theorem 4.5 and Theorem 5.3, respectively.[2] By Proposition 3.4, we are reduced to proving the following result:

Theorem 7.1. *Let $G = IA_n$ for some $n \geq 4$, and let ρ be a non-trivial homomorphism from G to another group with $\operatorname{Ker} \rho \supseteq [G, G]$. Then G has a ρ-CG sequence.*

The general method of proof of Theorem 7.1 will be similar to that of Theorem 5.3, but we will make use of specific relations in IA_n and properties of the action of $SL_n(\mathbb{Z})$ on IA_n^{ab} which are not captured by the notions of a good $\mathbf{n}$-group and a regular $SL_n(\mathbb{Z})$-module, respectively.

We start with the list of relations in IA_n that will be used in the proof.

Lemma 7.2. *Let $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{n}$, and assume that $\{a_i\}$ are distinct and $\{b_i\}$ are distinct. The following relations hold:*

(R1) $[K_{a_1a_2}, K_{b_1b_2}] = 1$ *if* $a_1 \neq b_1, b_2$ *and* $b_1 \neq a_1, a_2$
(R2) $[K_{a_1a_2}, K_{b_1b_2b_3}] = 1$ *if* $a_1 \neq b_1, b_2, b_3$ *and* $b_1 \neq a_1, a_2$
(R3) $[K_{a_1a_2a_3}, K_{b_1b_2b_3}] = 1$ *if* $a_1 \neq b_1, b_2, b_3$ *and* $b_1 \neq a_1, a_2, a_3$
(R4) $[K_{bcd}, K_{ab}] = [K_{ad}, K_{ac}]$ *if* $a, b, c, d \in \mathbf{n}$ *are distinct*

For the rest of the section, G, n, ρ and V be as in Theorem 7.1, and define Ω as in § 6 with $m = 1$. By discussion at the end of § 5,

$$G^{ab} = L(1) \cong \bigoplus_{j<k} \mathbb{Z} e_i^* \otimes (e_j \wedge e_k) \oplus \bigoplus_{i \neq j} \mathbb{Z} e_i^* \otimes (e_i \wedge e_j)$$

under the map which sends K_{ijk} to $e_i^* \otimes (e_j \wedge e_k)$ and K_{ij} to $e_i^* \otimes (e_i \wedge e_j)$. Given $\lambda \in \Omega$, define

$$c_{ijk}(\lambda) = \lambda(e_i^* \otimes (e_j \wedge e_k)).$$

The following two observations can be verified by direct computation. Observation 7.3 shows that when we act by an elementary matrix E_{ij} on an arbitrary $\lambda \in \Omega$, at most 4 of the coefficients c_{xxy} will change.

Observation 7.3. *Let $i, j, a \in \mathbf{n}$ be distinct and $\lambda \in \Omega$. Then*

$$c_{aaj}(E_{ij}\lambda) = c_{aaj}(\lambda) - c_{aai}(\lambda)$$
$$c_{iij}(E_{ij}\lambda) = c_{iij}(\lambda) - c_{jji}(\lambda)$$
$$c_{jja}(E_{ij}\lambda) = c_{jja}(\lambda) + c_{jai}(\lambda)$$
$$c_{iia}(E_{ij}\lambda) = c_{iia}(\lambda) - c_{jai}(\lambda)$$

and $c_{xxy}(E_{ij}\lambda) = c_{xxy}(\lambda)$ for all $(x, y) \neq (a, j), (i, j), (j, a), (i, a)$.

Observation 7.4. *Let $i, j \in \mathbf{n}$ with $i \neq j$ and let σ be the transposition (i, j). Then for any $\lambda \in \Omega$ and $x, y, z \in \mathbf{n}$ we have $c_{xyz}(F_{ij}\lambda) = \pm c_{\sigma(x)\sigma(y)\sigma(z)}(\lambda)$.*

In the proof of the following lemma we will repeatedly use Observation 7.3 without an explicit reference.

Lemma 7.5. *Assume that $n \geq 4$. Then for any nonzero $\lambda \in \Omega$ there exists $g \in SL_n(\mathbb{Z})$ such that $c_{112}(g\lambda), c_{221}(g\lambda), c_{334}(g\lambda)$ and $c_{443}(g\lambda)$ are all nonzero.*

[2] The restrictions $n < 7$ and $n < 12$ will not be used in the proof.

Proof. Step 1: There exists $g_1 \in SL_n(\mathbb{Z})$ such that $c_{112}(g_1\lambda) \neq 0$.

First of all, by Observation 7.4 it suffices to find distinct $a, b \in \mathbf{n}$ such that $c_{aab}(g_1\lambda) \neq 0$.

If $c_{xxy}(\lambda) \neq 0$ for some $x \neq y$, we are done. Suppose now that $c_{xxy}(\lambda) = 0$ for all $x \neq y$. Then there must exist distinct distinct a, b, c with $c_{abc}(\lambda) \neq 0$, in which case

$$c_{aab}(E_{ca}\lambda) = c_{aab}(\lambda) + c_{abc}(\lambda) = 0 + c_{abc}(\lambda) \neq 0.$$

Step 2: There exists $g_2 \in SL_n(\mathbb{Z})$ such that $c_{112}(g_2\lambda), c_{113}(g_2\lambda) \neq 0$.

By Step 1, we can assume that $c_{112}(\lambda) \neq 0$. If $c_{113}(\lambda) \neq 0$, we are done. And if $c_{113}(\lambda) = 0$, then $c_{113}(E_{23}\lambda) = c_{113}(\lambda) - c_{112}(\lambda) = 0 - c_{112}(\lambda) \neq 0$ and $c_{112}(E_{23}\lambda) = c_{112}(\lambda) \neq 0$

Step 3: There exists $g_3 \in SL_n(\mathbb{Z})$ such that either $c_{112}(g_3\lambda), c_{113}(g_3\lambda), c_{221}(g_3\lambda) \neq 0$ or $c_{112}(g_3\lambda), c_{334}(g_3\lambda) \neq 0$

By Step 2, we can assume that $c_{112}(\lambda), c_{113}(\lambda) \neq 0$. If $c_{224}(\lambda) \neq 0$ or $c_{221}(\lambda) \neq 0$, we are done (using Observation 7.4 in the former case), so assume that $c_{224}(\lambda) = c_{221}(\lambda) = 0$. We consider 3 cases.

Case 1: $c_{223}(\lambda) \neq 0$. Then

$$c_{224}(E_{34}\lambda) = c_{224}(\lambda) - c_{223}(\lambda) = -c_{223}(\lambda) \neq 0$$

$$c_{113}(E_{34}\lambda) = c_{113}(\lambda) \neq 0$$

Hence $c_{113}(E_{34}\lambda), c_{224}(E_{34}\lambda) \neq 0$.

Case 2: $c_{223}(\lambda) = 0$ and $c_{132}(\lambda) = 0$. Then

$$c_{112}(E_{21}\lambda) = c_{112}(\lambda) \neq 0$$

$$c_{221}(E_{21}\lambda) = c_{221}(\lambda) - c_{112}(\lambda) = -c_{112}(\lambda) \neq 0$$

$$c_{113}(E_{21}\lambda) = c_{113}(\lambda) + c_{132}(\lambda) = c_{113}(\lambda) \neq 0.$$

Case 3: $c_{223}(\lambda) = 0$ and $c_{132}(\lambda) \neq 0$.

Again we have $c_{112}(E_{21}\lambda) \neq 0$ and $c_{221}(E_{21}\lambda) \neq 0$, and this time

$$c_{223}(E_{21}\lambda) = c_{223}(\lambda) - c_{132}(\lambda) = -c_{132}(\lambda) \neq 0.$$

Hence we are done by Observation 7.4.

Step 4: There exists $g_4 \in SL_n(\mathbb{Z})$ such that $c_{112}(g_4\lambda), c_{334}(g_4\lambda) \neq 0$.

By Step 3, we can assume that $c_{112}(\lambda) \neq 0$, $c_{221}(\lambda) \neq 0$ and $c_{113}(\lambda) \neq 0$. If $c_{443}(\lambda) \neq 0$ or $c_{224}(\lambda) \neq 0$, we are done (by Observation 7.4).

So assume that $c_{443}(\lambda) = c_{224}(\lambda) = 0$. Then

$$c_{224}(E_{14}\lambda) = c_{224}(\lambda) - c_{221}(\lambda) = -c_{221}(\lambda) \neq 0$$

Case 1: $c_{431}(\lambda) = 0$. Then $c_{113}(E_{14}\lambda) = c_{113}(\lambda) - c_{431}(\lambda) \neq 0$, so $c_{113}(E_{14}\lambda) \neq 0$ and $c_{224}(E_{14}\lambda) \neq 0$.

Case 2: $c_{431}(\lambda) \neq 0$. Then $c_{443}(E_{14}\lambda) = c_{443}(\lambda) + c_{431}(\lambda) = c_{431}(\lambda) \neq 0$ and $c_{221}(E_{14}\lambda) = c_{221}(\lambda) \neq 0$, so $c_{443}(E_{14}\lambda) \neq 0$ and $c_{221}(E_{14}\lambda) \neq 0$.

In both cases we are done by Observation 7.4.

Final Step. By Step 4, we can assume that $c_{112}(\lambda), c_{334}(\lambda) \neq 0$. Set $\alpha = 0$ if $c_{221}(\lambda) \neq 0$ and $\alpha = 1$ if $c_{221}(\lambda) = 0$ and $\beta = 0$ if $c_{443}(\lambda) \neq 0$ and $\beta = 1$ if $c_{443}(\lambda) = 0$. Then $g = E_{21}^{\alpha} E_{43}^{\beta}$ satisfies the assertion of Lemma 7.5. $\square$

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. Let $H = \operatorname{Ker} \rho$. By Lemma 7.5, after precomposing ρ with conjugation by a suitable element of $g \in \operatorname{Aut}^+(F_n)$ (which has the effect of replacing H by gHg^{-1} and λ_H by $g\lambda_H$), we can assume that H has empty intersection with the set $S = \{K_{12}, K_{21}, K_{34}, K_{43}\}$. Using relations (R1)-(R3) of Lemma 7.2, it is easy to show that the only standard generators of IA_n which do not commute with an element of S are $K_{134}, K_{234}, K_{312}, K_{412}$. However, by relations (R4) each of these 4 elements commutes with an element of S modulo the subgroup generated by all K_{xy}. Thus, if we order the standard generators of IA_n so that $K_{12}, K_{34}, K_{21}, K_{43}$ come first (in this order) and $K_{134}, K_{234}, K_{312}, K_{412}$ come last, we obtain a ρ-CG sequence. $\square$

8. Verifying hypotheses of Theorems 4.5 and 5.3 for the Torelli groups.

Let $g \geq 3$, and let $\Sigma = \Sigma_g^1$ be an orientable surface of genus g with 1 boundary component. The mapping class group $\operatorname{Mod}_g^1 = \operatorname{Mod}(\Sigma_g^1)$ is defined as the subgroup of orientation preserving homeomorphisms of Σ_g^1 which fix $\partial\Sigma_g^1$ pointwise modulo the isotopies which fix $\partial\Sigma_g^1$ pointwise.

Choose a base point p_0 on the boundary $\partial\Sigma$. Since Σ has a bouquet of $2g$ circles as a deformation retract, the fundamental group $\pi = \pi_1(\Sigma, p_0)$ is free of rank $2g$; moreover, for a suitable choice of a free generating set $\{\alpha_i, \beta_i\}_{i=1}^g$ of π, the boundary $\partial\Sigma$ represents $\Pi = \prod_{i=1}^g [\alpha_i, \beta_i]$. Thus, there is a natural homomorphism from Mod_g^1 to the subgroup of automorphisms of π which fix Π. It is well known that this map is an isomorphism.

Thus, we can identify Mod_g^1 with a subgroup of $\operatorname{Aut}(F_{2g})$, and using this identification we can define the Johnson filtration $\{\mathcal{I}_g^1(k)\}_{k=1}^{\infty}$ by $\mathcal{I}_g^1(k) = \operatorname{Mod}_g^1 \cap IA_{2g}(k)$. The subgroups $\mathcal{I}_g^1 = \mathcal{I}_g^1(1)$ and $\mathcal{J}_g^1 = \mathcal{I}_g^1(2)$ are known as the *Torelli subgroup* and the *Johnson kernel*, respectively. Note that $\mathcal{I}_g^1$ can also be defined as the set of all elements of Mod_g^1 which act trivially on the integral homology group $H_1(\Sigma_g^1)$.

For the rest of this section we set $\mathcal{M} = \operatorname{Mod}_g^1$, $\mathcal{I} = \mathcal{I}_g^1 = \mathcal{I}_g^1(1)$ and $\mathcal{J} = \mathcal{J}_g^1 = \mathcal{I}_g^1(2)$. Our goal is to verify the hypotheses of Theorem 4.5 and Theorem 5.3 for $\Gamma = \mathcal{M}$, $G = \mathcal{I}$ and $n = g$. In both cases we take $d = 3$, and in Theorem 4.5 we let $e = f = 2$ and $L = Sp_{2g}(\mathbb{R})$. The homomorphism $\varphi : SL_g(\mathbb{Z}) \to \mathcal{M}/\mathcal{I}$ in Theorem 5.3 will be defined in § 8.6 (assuming the canonical isomorphism $\mathcal{M}/\mathcal{I} \cong Sp(V)$ defined in § 8.2).

8.1. **The twin g-group structure on $\mathcal{M}$.** We start by recalling the construction from [CP1, §4.1]. It will be convenient to think of Σ_g^1 as a (closed) disk with g handles attached; call the handles $H_1, \ldots, H_g$. Choose disjoint subsurfaces $X_1, \ldots, X_g$, each homeomorphic to Σ_1^1, such that X_i contains H_i, and let $Y = \Sigma \setminus \cup_{i=1}^g \operatorname{Int}(X_i)$. Fix a point $p \in Int(Y)$, and for each $1 \leq i \leq g$ choose simple paths δ_i and δ_i' in Y which connect p to a point on ∂X_i such that the following hold (see Figure 1):

(i) the paths δ_i are disjoint apart from p;
(ii) the paths δ_i' are disjoint apart from p;

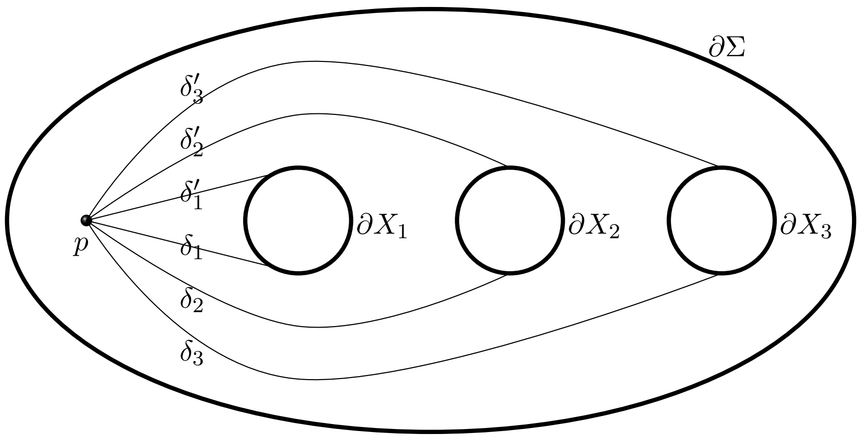

FIGURE 1.

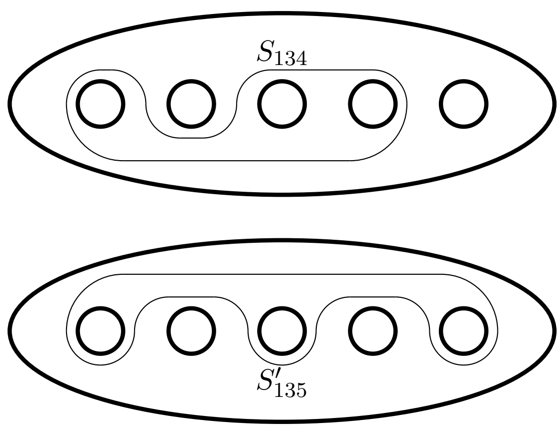

FIGURE 2. The surfaces S_{134} and S'_{135} perturbed by an isotopy (only parts of the surfaces inside Y are shown)

(iii) if C is a small circle around p, then the intersection points $C\cap\delta_1,\dots,C\cap\delta_g$ (resp. $C\cap\delta'_1,\dots,C\cap\delta'_g$) appear in clockwise (resp. counterclockwise) order.

Now for each $I\subseteq\mathbf{g}$ define S_I (resp. S'_I) to be a closed regular neighborhood of $\cup_{i\in I}(\delta_i\cup X_i)$ (resp. $\cup_{i\in I}(\delta'_i\cup X_i)$). Then S_I and S'_I are well defined up to isotopy (see Figure 2).

The following properties (1)-(5) are straightforward. Given two surfaces R and S, we will say that certain relation between R and S holds up to isotopy if there exist surfaces isotopic to R and S, respectively, satisfying the stated relation:

(1) $S_{\mathbf{g}}$ and $S'_{\mathbf{g}}$ are isotopic to Σ;
(2) if $I\subseteq J$, then up to isotopy, S_I is contained in S_J and S'_I is contained in S'_J;
(3) if I and J are disjoint and uncrossed, then up to isotopy, S_I and S_J are disjoint and S'_I and S'_J are disjoint;
(4) if I and J are any disjoint subsets, then S_I and S'_J are disjoint up to isotopy.
(5) if I consists of consecutive integers, then S_I and S'_I are isotopic.

Now define $\mathcal{M}_I$ (resp. $\mathcal{M}'_I$) to be the subgroup of $\mathcal{M}$ consisting of mapping classes which have a representative supported on S_I (resp. S'_I), and let $\mathcal{I}_I = \mathcal{M}_I \cap \mathcal{I}$ and $\mathcal{I}'_I = \mathcal{M}'_I \cap \mathcal{I}$. Properties (1)-(4) above clearly imply that $\mathcal{M}$ is a good twin $\mathbf{g}$-group (here we use the obvious fact that homeomorphisms supported on disjoint surfaces commute). A deep result of Church and Putman [CP1, Proposition 4.5] based on earlier work of Putman [Pu2] asserts that $\mathcal{I} = \langle \mathcal{I}_I : |I| = 3 \rangle = \langle \mathcal{I}'_I : |I| = 3 \rangle$, so $\mathcal{I}$ is generated in degree 3 as a twin $\mathbf{g}$-group.

8.2. $H_1(\Sigma)$ **as an** $Sp_{2g}(\mathbb{Z})$**-module.** Let $V = H_1(\Sigma)$. Then V is a free abelian group of rank $2g$ endowed with the canonical symplectic form

$$([\alpha], [\beta]) \mapsto [\alpha] \cdot [\beta]$$

where $[\alpha] \cdot [\beta]$ is the algebraic intersection number between the closed curves α and β on Σ. Clearly the action of $\mathcal{M}/\mathcal{I}$ on V preserves this form, so there is a canonical group homomorphism $\mathcal{M}/\mathcal{I} \to Sp(V)$ where $Sp(V)$ is the group of automorphisms of V preserving the above form. It is well known that this homomorphism is an isomorphism, so from now on we will identify $\mathcal{M}/\mathcal{I}$ with $Sp(V)$.

Out next goal is to describe the abelianization $\mathcal{I}^{ab}$ as an $Sp(V)$-module. First we will introduce two quotients of $\mathcal{I}^{ab}$, the largest torsion-free quotient and the largest quotient of exponent 2. The corresponding homomorphisms defined on $\mathcal{I}$ are called the *Johnson homomorphism* and the *Birman-Craig-Johnson homomorphism* and will be denoted by τ and σ, respectively.

8.3. **Johnson homomorphism.** The symplectic form introduced above yields a canonical isomorphism of $Sp(V)$-modules $V^* \cong V$. By the same logic as in the case of automorphisms of free groups, there exists a homomorphism of $Sp(V)$-modules

$$\tau : \mathcal{I} \to V \otimes (V \wedge V)$$

with $\mathrm{Ker}\,(\tau) = \mathcal{J}$, the Johnson kernel. It is called the *Johnson homomorphism.* Unlike the case of automorphisms of free groups, τ is not surjective, and Johnson [Jo3] showed that its image is spanned by elements of the form $u \otimes (v \wedge w) + v \otimes (w \wedge u) + w \otimes (u \wedge v)$. The latter subspace is clearly $Sp(V)$-isomorphic to $\wedge^3 V$, so $\mathcal{I}/\mathcal{J} \cong \wedge^3 V$ as $Sp(V)$-modules.

We will use an explicit formula for the values of τ on a suitable generating set for $\mathcal{I}$, which is discussed below. For each simple closed curve γ on Σ denote by $T_\gamma \in \mathcal{M} = \mathrm{Mod}\,(\Sigma)$ the Dehn twist about γ.

Definition. Let (γ, δ) be a pair of disjoint non-separating simple closed curves on Σ. It is called a *bounding pair of genus k* if

(i) γ and δ are homologous to each other and non-homologous to zero.
(ii) the union $\gamma \cup \delta$ separates Σ; moreover, if $\Sigma_{\gamma,\delta}$ is the connected component of $\Sigma \setminus (\gamma \cup \delta)$ which does not contain $\partial\Sigma$, then $\Sigma_{\gamma,\delta}$ has genus k.

If (γ, δ) is a bounding pair of genus k, then the map $T_\gamma T_\delta^{-1}$ always lies in $\mathcal{I}$ and is called a *k-BP map.*

Johnson [Jo1] proved that $\mathcal{I}$ is generated by 1-BP maps, so it suffices to know the values τ on such elements. Let (γ, δ) be a bounding pair of genus 1, and assume that γ is oriented in such a way that $\Sigma_{\gamma,\delta}$ is on its left. Let $c \in V = H_1(\Sigma)$ be the homology class of γ, and choose any $a, b \in H_1(\Sigma_{\gamma,\delta}) \subset H_1(\Sigma)$ such that $a \cdot b = 1$. Then

$$\tau(T_\gamma T_\delta^{-1}) = a \wedge b \wedge c. \tag{8.1}$$

For the justification of (8.1) and the related formula (8.2) below see [Jo6, § 2] and references therein.

We already stated that $\mathcal{I}$ is generated in degree 3. At this point we are ready to explain why the remaining hypotheses of Theorem 4.5 hold. The mapping class group $\mathcal{M}$ has a generating set consisting of Dehn twists each of which lies in $\mathcal{M}_{\{i,i+1\}}$ for some $1 \leq i \leq g-1$ (see [Jo5, Thm 1]). By property (5) in § 8.1 we have $\mathcal{M}_I = \mathcal{M}'_I$ if I consists of consecutive integers and thus condition (i) of Lemma 4.2 holds with $e = 2$. If $I, J \subseteq \mathbf{g}$ with $|I| = |J|$, there exists $f \in \mathcal{M}$ with $f(S_I) = S'_J$ (see, e.g., the proof of [CP1, Lemma 4.1]), and therefore $\mathcal{M}_I$ and $\mathcal{M}'_J$ are conjugate in $\mathcal{M}$. Hence condition (ii) of Lemma 4.2 also holds.

By the change of coordinates principle (see, e.g. [FM, § 1.3]), all 1-BP maps are conjugate in $\mathcal{M}$, and it is clear that there exist 1-BP maps which lie in $\mathcal{I}_I$ (or $\mathcal{I}'_I$) with $|I| = 2$. Since $\mathcal{I}$ is generated by 1-BP maps, $\mathcal{I}$ is normally generated in degree $f = 2$ as a subgroup of $\mathcal{M}$.

Finally, since $\mathcal{I}/\mathcal{J}$ is the largest torsion-free abelian quotient of $\mathcal{I}$, we have a natural isomorphism $\mathcal{I}^{ab} \otimes \mathbb{R} \cong (\mathcal{I}/\mathcal{J}) \otimes \mathbb{R}$. Since $\mathcal{I}/\mathcal{J} \cong \wedge^3 V$ as $Sp(V)$-modules, the action of $\mathcal{M}/\mathcal{I} \cong Sp(V)$ on $\mathcal{I}/\mathcal{J}$ clearly extends to a polynomial representation of $Sp_{2g}(\mathbb{R})$ on $\mathcal{I}^{ab} \otimes \mathbb{R}$.

In order to check conditions (ii) and (iii) of Theorem 5.3 we need to understand the entire abelianization $\mathcal{I}^{ab}$ as an $Sp(V)$-module. This will be done in the remaining three subsections.

8.4. **Birman-Craig-Johnson homomorphism.** Now let $V_{\mathbb{F}_2} = V \otimes \mathbb{F}_2 \cong H_1(\Sigma, \mathbb{F}_2)$. Let B be the ring of polynomials over $\mathbb{F}_2$ in formal variables $X = \{\overline{v} : v \in H_{\mathbb{F}_2} \setminus \{0\}\}$ subject to relations

(R1) $\overline{v + w} = \overline{v} + \overline{w} + v \cdot w$ for all $\overline{v}, \overline{w} \in X$
(R2) $\overline{v}^2 = \overline{v}$ for all $\overline{v} \in X$

The group $Sp(V)$ has a natural action on B by ring automorphisms such that $g(\overline{v}) = \overline{gv}$ for all $\overline{v} \in X$. Let B_n be the subspace of B consisting of elements representable by a polynomial of degree at most n. Then each B_n is an $Sp(V)$-submodule, and it is easy to see that $B_n/B_{n-1} \cong \wedge^n V_{\mathbb{F}_2}$ for each $n \geq 1$ (as $Sp(V)$-modules).

In [Jo2], Johnson constructed a surjective homomorphism $\sigma : \mathcal{I} \to B_3$ which induces an $Sp(V)$-module homomorphism $\mathcal{I}^{ab} \to B_3$. Following [BF], we will refer to σ as the *Birman-Craig-Johnson (BCJ) homomorphism.* We will not discuss the conceptual definition of σ in terms of the Rochlin invariant and instead give an explicit formula for σ on 1-BP maps:

$$\sigma(T_\gamma T_\delta^{-1}) = \overline{a}\overline{b}(\overline{c} + 1). \tag{8.2}$$

where a, b and c are defined as in (8.1) except that this time they are mod 2 homology classes.

8.5. **The full abelianization.** Let $\alpha : \wedge^3 V \to \wedge^3 V_{\mathbb{F}_2}$ be the natural reduction map, and let $\beta : B_3 \to \wedge^3 V_{\mathbb{F}_2}$ be the unique linear map such that $\beta(B_2) = 0$ and $\beta(\overline{u}\,\overline{v}\,\overline{w}) = u \wedge v \wedge w$

for all $u, v, w \in V_{\mathbb{F}_2}$. Clearly α and β are both $Sp(V)$-module homomorphisms. Let

$$W = \{(u, v) \in \wedge^3 V \oplus B_3 : \alpha(u) = \beta(v)\} \tag{8.3}$$

Theorem 8.1. *The map* $(\tau, \sigma) : \mathcal{I}^{ab} \to \wedge^3 V \times B_3$ *given by* $g + [G, G] \mapsto (\tau(g), \sigma(g))$ *is injective and* $\mathrm{Im}\,((\tau, \sigma)) = W$.

Proof. The fact that $\mathrm{Im}\,(\tau, \sigma) \subseteq W$ holds by [Jo3, Theorem 4] and also follows immediately from (8.1) and (8.2). Once this is established, the opposite inclusion follows from $\sigma(\mathrm{Ker}\,\tau) = \mathrm{Ker}\,\beta$, and the latter holds by [Jo2, Lemma 4]. Finally, injectivity of (τ, σ) is proved in [Jo6]. □

Now let $I \subseteq \mathbf{g}$ be any subset with $|I| \geq 3$, and let S_I be the corresponding subsurface of Σ introduced in § 8.1. Since S_I is itself a closed orientable surface of genus ≥ 3 with one boundary component, we can repeat the entire construction described in this section starting with S_I instead of Σ, so in particular we can define the modules $V(I) = H_1(S_I)$ and $B_3(I)$ and the homomorphisms $\tau_I : \mathcal{I}_I \to \wedge^3 V(I)$ and $\sigma_I : \mathcal{I}_I \to B_3(I)$ (recall that $\mathcal{I}_I$ was defined as the subgroup of $\mathcal{I}$ consisting of mapping classes supported on S_I, but this group is canonically isomorphic to the Torelli subgroup of $\mathrm{Mod}\,(S_I)$).

Proposition 8.2. *The following diagrams are commutative*

$$\begin{tikzcd} (\mathcal{I}_I)^{ab} \arrow[r, "\tau_I"] \arrow[d] & \wedge^3 V(I) \arrow[d] \\ \mathcal{I}^{ab} \arrow[r, "\tau"] & \wedge^3 V \end{tikzcd} \qquad \begin{tikzcd} (\mathcal{I}_I)^{ab} \arrow[r, "\sigma_I"] \arrow[d] & B_3(I) \arrow[d] \\ \mathcal{I}^{ab} \arrow[r, "\sigma"] & B_3 \end{tikzcd} \tag{8.4}$$

where the vertical maps are induced by the natural inclusions $\mathcal{I}_I \to \mathcal{I}$ *and* $H_1(S_I) \to H_1(\Sigma)$.

Proof. For both diagrams, it is enough to check commutativity for the values on the generators $T_\gamma T_\delta^{-1}$. This follows immediately from (8.1) and (8.2) since if γ and δ are curves on S_I satisfying (i)-(iii), replacing S_I by Σ will not change the surface $\Sigma_{\gamma,\delta}$ or the homology classes a, b and c. □

8.6. $\mathcal{I}^{ab}$ **as a regular** $SL_g(\mathbb{Z})$**-module.** For each $1 \leq i \leq g$ choose any basis $\{a_i, b_i\}$ for $H_1(S_i) \subset V$ s.t. $a_i \cdot b_i = 1$. Then $V = \bigoplus_{i=1}^{g} (\mathbb{Z}a_i \oplus \mathbb{Z}b_i)$, and $\{a_i, b_i\}_{i=1}^{g}$ is a symplectic basis for V, that is, $a_i \cdot a_j = b_i \cdot b_j = 0$ for all i, j, and $a_i \cdot b_j = \delta_{ij}$. Now define $\varphi : SL_g(\mathbb{Z}) \to Sp(V)$ by

$$\varphi(A) = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix},$$

where the above matrix is with respect to the ordered basis $(a_1, \ldots, a_g, b_1, \ldots, b_g)$. This yields $SL_g(\mathbb{Z})$-module structures on $\wedge^3 V$, B_3 and hence on W defined by (8.3). We also obtain an action of $SL_g(\mathbb{Z})$ on $L(\mathcal{I})$ by Lie algebra automorphisms via (5.1) using φ above (recall that $Sp(V)$ is canonically isomorphic to $\mathcal{M}/\mathcal{I}$), and it is straightforward to check that $(\tau, \sigma) : \mathcal{I}^{ab} \to W$ from Theorem 8.1 is an isomorphism of $SL_g(\mathbb{Z})$-modules.

We could proceed working directly with W, but things can be simplified further with the following observations. Let $\pi : B_3 \to \oplus_{i=0}^{3} \wedge^i V_{\mathbb{F}_2}$ be the unique linear map such that $\pi(1) = 1$, $\pi(\overline{x}) = x$, $\pi(\overline{x}\,\overline{y}) = x \wedge y$ and $\pi(\overline{x}\,\overline{y}\,\overline{z}) = x \wedge y \wedge z$ where x, y, z are distinct elements of the basis $\{a_i, b_i\}_{t=1}^{g}$. Clearly, π is bijective, and it is straightforward to check that π is an isomorphism of $SL_g(\mathbb{Z})$-modules [3] (but not an isomorphism of $Sp(V)$-modules!)

Now define $\pi' : \wedge^3 V \oplus B_3 \to \wedge^3 V \oplus \bigoplus_{i=0}^{3} \wedge^i V_{\mathbb{F}_2}$ by $\pi'(u,v) = (u, \pi(v) - \alpha(u))$. Clearly, π' is an isomorphism of $SL_g(\mathbb{Z})$-modules, and it is easy to show that $\pi'(W) = \wedge^3 V \oplus \bigoplus_{i=0}^{2} \wedge^i V_{\mathbb{F}_2}$. Thus we have the following isomorphism of $SL_g(\mathbb{Z})$-modules:

$$\lambda = \pi' \circ (\tau, \sigma) : \mathcal{I}^{ab} \to \wedge^3 V \oplus \left(\bigoplus_{i=0}^{2} \wedge^i V_{\mathbb{F}_2} \right). \qquad (***)$$

It is easy to see that V is a direct sum of the natural $SL_g(\mathbb{Z})$-module $\mathbb{Z}^g$ and its dual. Using Lemmas 5.1 and 5.2 and the isomorphism λ, we endow $\mathcal{I}^{ab}$ with the structure of a regular $SL_g(\mathbb{Z})$-module generated in degree 3, so condition (ii) in Theorem 5.3 holds. Here is an explicit description of the obtained grading on $\mathcal{I}^{ab}$. The symbol e_t stand for a_t or b_t.

(1) If $I = \{i < j < k\}$, then $(\mathcal{I}^{ab})_I = \lambda^{-1}(\oplus \mathbb{Z} e_i \wedge e_j \wedge e_k)$.
(2) If $I = \{i < j\}$, then $(\mathcal{I}^{ab})_I = \lambda^{-1}((\oplus \mathbb{Z} e_i \wedge a_j \wedge b_j) \oplus (\oplus \mathbb{Z} e_j \wedge a_i \wedge b_i) \oplus (\oplus \mathbb{F}_2 e_i \wedge e_j))$.
(3) If $I = \{i\}$, then $(\mathcal{I}^{ab})_I = \lambda^{-1}(\mathbb{F}_2 a_i \oplus \mathbb{F}_2 b_i)$
(4) If $I = \emptyset$, then $(\mathcal{I}^{ab})_I = \lambda^{-1}(\mathbb{F}_2)$

It remains to check condition (iii). In the argument below the reader should not confuse $(\mathcal{I}^{ab})_I$, the I-component of $\mathcal{I}^{ab}$ as defined above, with $(\mathcal{I}_I)^{ab}$, the abelianization of the group $\mathcal{I}_I$. Let $I \subseteq \mathbf{g}$, with $|I| \geq 3$, and let $\iota_I : (\mathcal{I}_I)^{ab} \to \mathcal{I}^{ab}$ be the natural map. We need to show that $\iota_I((\mathcal{I}_I)^{ab})$ contains $\sum_{J \subseteq I} (\mathcal{I}^{ab})_J$ or, equivalently, that $\lambda \circ \iota_I((\mathcal{I}_I)^{ab})$ contains $\sum_{J \subseteq I} \lambda((\mathcal{I}^{ab})_J)$. We claim that the latter two sets are both equal to $Z_I := \wedge^3 V(I) \oplus \left(\bigoplus_{i=0}^{2} \wedge^i V(I)_{\mathbb{F}_2} \right)$ (where we identify $V(I) = H_1(S_I)$ with its image in $V = H_1(\Sigma)$). Indeed, $\sum_{J \subseteq I} \lambda((\mathcal{I}^{ab})_J) = Z_I$ directly from (1)-(3) above, while the equality $\lambda \circ \iota_I((\mathcal{I}_I)^{ab}) = Z_I$ follows easily from Proposition 8.2.

9. Abelianization of finite index subgroups in $\mathrm{Aut}\,(F_n)$ and $\mathrm{Mod}\,(\Sigma_g^1)$

A group G is said to have *property (FAb)* if every finite index subgroup of G has finite abelianization. Clearly $\mathrm{Aut}\,(F_2)$ does not have (FAb) since it projects onto $GL_2(\mathbb{Z})$, which is a virtually free group. The group $\mathrm{Aut}\,(F_3)$ also does not have (FAb) – this was proved by completely different methods in [Mc2] and [GL]. The question whether $\mathrm{Aut}\,(F_n)$ has (FAb) for $n \geq 4$ is wide open. It was previously known that all subgroups of $\mathrm{Aut}\,(F_n)$ containing IA_n, with $n \geq 3$, have finite abelianization – this has been proved independently in [Bh] and [BV]; for another result of this kind see [Ki, Thm 5.3.2].

In the case of mapping class groups, it was proved in [Mc1] that $\mathrm{Mod}\,^1_2$ does not have (FAb), and it is a well known conjecture of Ivanov that $\mathrm{Mod}\,^1_g$ has (FAb) for $g \geq 3$. In the

[3]This is true since $SL_g(\mathbb{Z})$ preserves both $V_A = \oplus \mathbb{Z} a_i$ and $V_B = \oplus \mathbb{Z} b_i$ and the intersection form vanishes on both V_A and V_B

case $g \geq 3$, in [Ha] it was proved that H^{ab} is finite for every finite index subgroup of Mod_g^1 containing the Torelli subgroup (see also [Mc1] for a short elementary proof). In [Pu1] this result was extended to all subgroups containing a large portion of the Johnson kernel (in suitable sense); in particular, to all subgroups containing the Johnson kernel itself. [4]

In this section we will prove Theorem 1.6 restated below. In particular we will establish finiteness of abelianization for all finite index subgroups of $\mathrm{Aut}(F_n)$ and Mod_n^1 which contain the N^{th} term of the Johnson filtration, provided $n \geq 12(N-1)$.

Theorem 1.7. *Let G, N and K be as in Theorem 1.2, and let $\Gamma = \mathrm{Aut}(F_n)$ if $G = IA_n$ and $\Gamma = \mathrm{Mod}_g^1$ if $G = \mathcal{I}_g^1$. The following hold:*

1. *If H is a finite index subgroup of G which contains K, then the restriction map $H^1(G, \mathbb{C}) \to H^1(H, \mathbb{C})$ is an isomorphism*
2. *If H is a finite index subgroup of Γ which contains K, then H has finite abelianization.*

Theorem 1.6 is a direct consequence of Theorem 1.5 and the following general result.

Theorem 9.1. *Let Γ be a group and G and K normal subgroups of Γ with $K \subseteq G$. Assume that $(G, K, \mathbb{C})$ is nice. The following hold:*

1. *Let H be a subgroup of Γ which contains K, is normalized by G and such that $[GH : H] < \infty$. Then the restriction map $H^1(GH, \mathbb{C}) \to H^1(H, \mathbb{C})$ is an isomorphism.*
2. *If H is any finite index subgroup of G containing K, then the restriction map $H^1(G, \mathbb{C}) \to H^1(H, \mathbb{C})$ is an isomorphism.*
3. *Assume in addition that Γ is finitely generated and M^{ab} is finite for every finite index subgroup M of Γ which contains G. Then H^{ab} is finite for every finite index subgroup H of Γ which contains K.*

The extra hypothesis in part (3) holds for $(\Gamma, G) = (\mathrm{Aut}(F_n), IA_n)$ with $n \geq 3$ or $(\Gamma, G) = (\mathrm{Mod}_g^1, \mathcal{I}_g^1)$ for $g \geq 2$ by the results from [Bh], [BV] and [Ha] mentioned at the beginning of this section.

Proof of Theorem 9.1. (1) By Shapiro's Lemma $H^1(H, \mathbb{C}) \cong H^1(GH, Coind_H^{GH}(\mathbb{C}))$ where $Coind_H^{GH}(\mathbb{C})$ is the coinduced module. By assumption, H is a normal finite index subgroup of GH, so we have the following isomorphisms of GH-modules:

$$Coind_H^{GH}(\mathbb{C})) \cong \mathbb{C}[Q] \cong \bigoplus_{V \in Irr(Q)} (dim V)V$$

where $Q = GH/H$ and $Irr(Q)$ is the set of equivalence classes of irreducible complex representations of Q. Hence

$$H^1(H, \mathbb{C}) \cong H^1(GH, \mathbb{C}) \oplus \bigoplus_{V \in Irr(Q) \setminus V_0} H^1(GH, V)^{\dim(V)}$$

where V_0 is the trivial representation of Q. Moreover, the inclusion $H^1(GH, \mathbb{C}) \to H^1(H, \mathbb{C})$ coming from the above isomorphism is the restriction map.

[4]The results of [Ha] and [Pu1] apply to mapping class groups of surfaces with an arbitrary number of punctures and boundary components

Thus, we only need to show that $H^1(GH, V) = 0$ for every non-trivial irreducible representation V of Q. Take any such representation V. The exact sequence of groups $1 \to G \to GH \to GH/G \to 1$ yields the following inflation-restriction sequence:

$$0 \to H^1(GH/G, V^G) \to H^1(GH, V) \to H^1(G, V)^{GH}. \qquad (***)$$

Since V is irreducible and non-trivial, we have $V^G = 0$. Since $(G, K, \mathbb{C})$ is nice and K acts trivially on V, we have $H^1(G, V) = 0$. Thus (***) implies that $H^1(GH, V) = 0$, as desired.

(2) If H is normal in G, the result follows directly from (1). In general, since H has finite index in G and K is normal in G, there exists a finite index normal subgroup H' of G with $K \subseteq H' \subseteq H$. Then by (1) the composite restriction map $H^1(G, \mathbb{C}) \to H^1(H, \mathbb{C}) \to H^1(H', \mathbb{C})$ is an isomorphism, whence $H^1(H, \mathbb{C}) \to H^1(H', \mathbb{C})$ is surjective. Injectivity of $H^1(H, \mathbb{C}) \to H^1(H', \mathbb{C})$ is automatic since H' has finite index in H.

(3) Again the result follows from (1) if H is normal in Γ. Indeed, GH has finite index in Γ, so $(GH)^{ab}$ is finite, whence $H^1(GH, \mathbb{C}) = 0$ and thus $H^1(H, \mathbb{C}) = 0$ by (1). Since H has finite index in Γ and Γ is finitely generated, H is also finitely generated, so $H^1(H, \mathbb{C}) = 0$ forces H^{ab} to be finite. In general, we can find a finite index subgroup H' of H which is normal in Γ, and H^{ab} is finite whenever $(H')^{ab}$ is finite. □

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University of Virginia
E-mail address: `ershov@virginia.edu`

University of Virginia
E-mail address: `suzy.he@gmail.com`