Math 8851. Homework #5. To be completed by Thu, Mar 2

1. Let G be a group. Prove that the following are equivalent:

(a) G arises as an HNN-extension, that is, there exists a group H, isomorphic subgroups A and B of H and an isomorphism $\varphi : A \to B$ such that $G \cong \langle H, t \mid t^{-1}at = \varphi(a) \text{ for all } a \in A\rangle$.

(b) There exists an epimorphism $\pi : G \to \mathbb{Z}$.

Hint: For (b)$\Rightarrow$(a) use the fact that $\mathbb{Z}$ is a free group (of rank 1).

2. Prove that if G is an HNN extension, then G acts on the associated tree T without edge inversions. **Note:** This is related to Problem 1. I am not sure you can use the result of Problem 1, but solution to 1 should definitely help.

3. Let $A_1, \ldots, A_k$ be a finite collection of groups, and consider the natural epimorphism π from the free product $*_{i=1}^{k} A_i = A_1 * A_2 * \ldots * A_k$ to the direct product $\prod_{i=1}^{k} A_i$. The group $C = \operatorname{Ker} \pi$ is called the *Cartesian subgroup* of $*_{i=1}^{k} A_i$. The Cartesian subgroup C is always free – this is an immediate consequence of the Kurosh Subgroup Theorem which we will discuss next week (see the statement at the end of the problem). Note that if $A_1, \ldots, A_k$ are finite, then $G = *_{i=1}^{k} A_i$ is finitely generated and C has finite index in G and hence C is also finitely generated (by the Schreier formula). The goal of this problem is to compute the rank of C in this special case.

The rank can be computed using the main theorem of Bass-Serre theory (which we will also discuss next week), but in this exercise we take a different approach based on the notion of rational Euler characterstic which is introduced below.

Let Ω be the smallest class of groups such that

(i) Ω contains the trivial group $\{1\}$ and $\mathbb{Z}$

(ii) Ω is closed under finite direct products

(iii) Ω is closed under finite free products

(iv) Ω is closed under taking finite index subgroups and finite index supergroups

For each group $G \in \Omega$ one can uniquely define the **rational Euler characteristic** $\chi(G) \in \mathbb{Q}$ such that the following properties hold:

(a) $\chi(\{1\}) = 1$ and $\chi(\mathbb{Z}) = 0$

(b) $\chi(G * H) = \chi(G) + \chi(H) - 1$ for any $G, H \in \Omega$

(c) $\chi(G \times H) = \chi(G)\chi(H)$ for any $G, H \in \Omega$

(d) If $G \in \Omega$ and H is a subgroup of index n in G, then $\chi(H) = n\chi(G)$.

The basic idea is that if G is a group which has a finite CW-complex X as its classifying space, then one should have $\chi(G) = \chi(X)$, but then the definition of Euler characteristic has to be extended to a larger class of groups. An explanation of how this can be done is given in the following paper:

C.T.C. Wall, *Rational Euler characteristics.* Proc. Cambridge Philos. Soc. 57 1961 182–184.

Now the actual problem begins. Let $A_1, \ldots, A_k$ be finite groups, set $n_i = |A_i|$, and let F_r be a free group of rank r. Let $G = A_1 * A_2 * \ldots * A_k * F_r$, and let C be the kernel of the natural epimorphism $G \to A_1 \times A_2 \times \ldots \times A_k$ which sends F_r to $\{1\}$ (if $r = 0$, then $G = A_1 * A_2 * \ldots * A_k$ and C is exactly the Cartesian subgroup).

(a) Prove that C is free (using the Kurosh Subgroup Theorem, see the statement below).

(b) Use Euler characteristic to prove that

$$rk(C) = \prod_{i=1}^{k} n_i(r + k - 1 - \sum_{i=1}^{k} \frac{1}{n_i}) + 1.$$

Theorem (Kurosh Subgroup Theorem)**.** *Let $G = *_{\alpha \in I} G_\alpha$ be the free product of some family of groups $\{G_\alpha : \alpha \in I\}$ (here I may be infinite). Then any subgroup of G can be decomposed as a free product $F * (*_{\alpha \in I} H_\alpha)$ where F is free and each H_α is conjugate (in G) to a subgroup of G_α.*

4. Let G be a finitely generated group.

(a) Prove that for any finite group H there are only finitely many homomorphisms from G to H. **Hint:** A homomorphism from G is completely determined by its values on generators.

(b) Prove that for any $n \in \mathbb{N}$ there are only finitely many normal subgroups of index n in G. Then deduce that G has only finitely many subgroups of index n.