PROPERTY (T) FOR GROUPS GRADED BY ROOT SYSTEMS

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ABSTRACT. We introduce and study the class of groups graded by root systems. We prove that if Φ is an irreducible classical root system of rank ≥ 2 and G is a group graded by Φ, then under certain natural conditions on the grading, the union of the root subgroups is a Kazhdan subset of G. As the main application of this result we prove that for any reduced irreducible classical root system Φ of rank ≥ 2 and a finitely generated commutative ring R with 1, the Steinberg group $St_\Phi(R)$ and the elementary Chevalley group $\mathbb{E}_\Phi(R)$ have property (T).

CONTENTS

1. Introduction

1.1. **The main result.** In this paper by a *ring* we will always mean an associative ring with 1. In a recent work of the first two authors [EJ] it was shown that for any integer $n \geq 3$ and a finitely generated ring R, the elementary linear group $\mathrm{EL}_n(R)$ and the Steinberg group $\mathrm{St}_n(R)$ have Kazhdan's property (T) (in fact property (T) for $\mathrm{EL}_n(R)$ is a consequence of property (T) for $\mathrm{St}_n(R)$ since $\mathrm{EL}_n(R)$ is a quotient of $\mathrm{St}_n(R)$). In this paper we extend this result to elementary Chevalley groups and Steinberg groups corresponding to other classical root systems of rank ≥ 2 (see Theorem 1.1 below).

We will use the term *root system* in a very broad sense (see Section 4). By a *classical root system* we mean the root system of some semisimple algebraic group (such root systems are often called *crystallographic*).

If Φ is a reduced irreducible classical root system and R a commutative ring, denote by $\mathbb{G}_\Phi(R)$ the corresponding simply-connected Chevalley group over R and by $\mathbb{E}_\Phi(R)$ the *elementary subgroup* of $\mathbb{G}_\Phi(R)$, that is, the subgroup generated by the root subgroups with respect to the standard torus. For instance, if $\Phi = A_{n-1}$, then $\mathbb{G}_\Phi(R) = SL_n(R)$ and $\mathbb{E}_\Phi(R) = \mathrm{EL}_n(R)$. For brevity we will refer to $\mathbb{E}_\Phi(R)$ as an *elementary Chevalley group*. There is a natural epimorphism from the Steinberg group $\mathrm{St}_\Phi(R)$ onto $\mathbb{E}_\Phi(R)$.

Theorem 1.1. *Let Φ be a reduced irreducible classical root system of rank ≥ 2. Let R be a finitely generated ring, which is commutative if Φ is not of type A_n. Then the Steinberg group $\mathrm{St}_\Phi(R)$ and the elementary Chevalley group $\mathbb{E}_\Phi(R)$ have Kazhdan's property (T).*

Remark: There are many cases when $\mathbb{E}_\Phi(R) = \mathbb{G}_\Phi(R)$. For instance, this holds if $R = \mathbb{Z}[x_1, \ldots, x_k]$ or $R = F[x_1, \ldots, x_k]$, where F is a field, and Φ is of type A_n (see [Su]) or C_n (see [GMV]), with $n \geq 2$.

Steinberg groups and elementary Chevalley groups over rings are typical examples of *groups graded by root systems* which are introduced and studied in this paper. Our central result asserts that if G is any group graded by a (finite) root system Φ of rank ≥ 2, the grading satisfies certain non-degeneracy condition, and $\{X_\alpha\}_{\alpha \in \Phi}$ are the root subgroups, then $\cup X_\alpha$ is a Kazhdan subset of G (see Theorem 1.2 below). Theorem 1.1 follows primarily from this result and relative property (T) for the pair $(\mathrm{St}_2(R) \ltimes R^2, R^2)$ established by Shalom when R is commutative [Sh1] and by the third author for general R [Ka1]; however, additional considerations are needed in the case when Φ is not simply laced. Before discussing the proofs of these results, we briefly comment on the previous work on property (T) for Chevalley and Steinberg groups and the proof of the main theorem in [EJ].

1.2. **Property (T) for $\mathrm{EL}_n(R)$: summary of prior work.** By the 1967 foundational paper of Kazhdan [Kazh] and the subsequent work of Vaserstein [Va], the Chevalley groups $\mathbb{G}_\Phi(\mathbb{Z}) = \mathbb{E}_\Phi(\mathbb{Z})$ and $\mathbb{G}_\Phi(F[t]) = \mathbb{E}_\Phi(F[t])$, where F is a finite field, have property (T) for any reduced irreducible classical root system of rank ≥ 2. The question of whether the groups $\mathbb{E}_\Phi(R)$, with $rk(\Phi) \geq 2$ (in particular, the groups $\mathrm{EL}_n(R)$, $n \geq 3$) have property (T) for "larger" rings R remained completely open until the last few years. [1]

In 2005 Kassabov and Nikolov showed that the group $\mathrm{EL}_n(R)$, $n \geq 3$, has property (τ) (a certain weak form of property (T)) for any finitely generated commutative ring R, which gave an indication that these groups might also have property (T). This indication was partially confirmed by Shalom in 2006 who proved that the groups $\mathrm{EL}_n(R)$ have property (T) whenever R is commutative and $n \geq \mathrm{Kdim}(R) + 2$. In 2007 Vaserstein [Va] eliminated this restriction on the Krull dimension by showing that $\mathrm{EL}_n(R)$, $n \geq 3$, has property (T) for any finitely generated commutative ring R. Finally, in [EJ] the result was extended to arbitrary finitely generated (associative) rings, and the method of proof was very different from the one used by Shalom and Vaserstein. To explain the idea behind this method, we recall some standard terminology.

Let G be a discrete group and S a subset of G. Following the terminology in [BHV], we will say that S is a *Kazhdan subset* of G if every unitary representation of G containing almost S-invariant vectors must contain a G-invariant vector. By definition, G has property (T) if it has a finite Kazhdan subset; however, one can prove that G has (T) by finding an infinite Kazhdan subset K such that the pair (G, K) has relative property (T) (see Section 2 for details).

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[1] The situation in rank 1 is completely different. It is easy to see that $\mathrm{EL}_2(R)$ does not have (T) whenever R (possibly noncommutative) surjects onto $\mathbb{Z}$ or $F[t]$, with F a finite field. A more delicate argument shows that $\mathrm{EL}_2(R)$ does not have (T) for any infinite commutative ring R (with 1).

If $G = \mathrm{EL}_n(R)$, where $n \geq 3$ and R is a finitely generated ring, the afore-mentioned results of Shalom and Kassabov yield relative property (T) for the pair (G, X) where $X = \cup_{i \neq j} E_{ij}(R)$ is the union of root subgroups. Thus, establishing property (T) for G is reduced to showing that X is a Kazhdan subset. An easy way to prove the latter is to show that G is boundedly generated by X — this is the so-called bounded generation method of Shalom. However, G is known to be boundedly generated by X only in a few cases, namely, when R is a finite extension of $\mathbb{Z}$ or $F[t]$, with F a finite field. In [EJ], it was proved that X is a Kazhdan subset of $G = \mathrm{EL}_n(R)$ for any ring R using a different method, described in the next subsection.

1.3. **Almost orthogonality, codistance and a spectral criterion from** [EJ]. Suppose we are given a group G and a finite collection of subgroups $H_1, \ldots, H_k$ which generate G, and we want to know whether the union of these subgroups $X = \cup H_i$ is a Kazhdan subset of G. By definition, this will happen if and only if given a unitary representation V of G without (nonzero) invariant vectors, a unit vector $v \in V$ cannot be arbitrarily close to each of the subspaces V^{H_i} (where as usual V^K denotes the subspace of K-invariant vectors). In the simplest case $k = 2$ the latter property is equivalent to asserting that the angle between subspaces V^{H_1} and V^{H_2} must be bounded away from 0. For an arbitrary k, the closeness between the subspaces $V_1, \ldots, V_k$ of a Hilbert space V can be measured using the notion of a codistance introduced in [EJ]. We postpone the formal definition until Section 2; here we just say that the codistance between $\{V_i\}$, denoted by $codist(V_1, \ldots, V_k)$ is a real number in the interval $[1/k, 1]$, and in the above setting $\cup H_i$ is a Kazhdan subset of G if and only if $\sup\{codist(V^{H_1}, \ldots, V^{H_k})\} < 1$ where V runs over all representations of G with $V^G = \{1\}$.

Let H and K be subgroups of the same group. We define $\sphericalangle(H, K)$, the angle between H and K, to be the infimum of the set $\{\sphericalangle(V^H, V^K)\}$ where V runs over all representations of $\langle H, K \rangle$ without invariant vectors; we shall also say that H and K are ε-orthogonal if $\cos \sphericalangle(H, K) \leq \varepsilon$. The idea of using such angles to prove property (T) already appears in 1991 paper of Burger [Bu] and is probably implicit in earlier works on unitary representations. However, this idea has not been fully exploited until the paper of Dymara and Januszkiewicz [DJ] which shows that for a group G generated by k subgroups $H_1, \ldots, H_k$, property (T) can be established by controlling "local information", the angles between H_i and H_j. More precisely, in [DJ] it is proved that if H_i and H_j are ε-orthogonal for $i \neq j$ for sufficiently small ε, then $\cup H_i$ is a Kazhdan subset of G (so if in addition each H_i is finite, then G has property (T)). This "almost orthogonality method" was generalized in [EJ] using the notion of codistance. As a result, a new spectral criterion for property (T) was obtained, which is applicable to groups with a given graph of groups decomposition, as defined below.

Let G be a group and Γ a finite graph. A decomposition of G over Γ is a choice of a vertex subgroup $G_\nu \subseteq G$ for each vertex ν of Γ and an edge subgroup $G_e \subseteq G$ for each edge e of Γ such that

(i) G is generated by the vertex subgroups $\{G_\nu\}$
(ii) Each vertex group G_ν is generated by edge subgroups $\{G_e\}$, with e incident to ν
(iii) If an edge e connects ν and ν', then G_e is contained in $G_\nu \cap G_{\nu'}$.

The spectral criterion from [EJ] asserts that if a group G has a decomposition $(\{G_\nu\}, \{G_e\})$ over a graph Γ, and for each vertex ν of Γ the codistance between subgroups $\{G_e\}$, with e incident to ν, is sufficiently small with respect to the spectral gap of Γ, then the codistance between the vertex subgroups $\{G_\nu\}$ is less than 1 (and thus the union of vertex subgroups is a Kazhdan subset of G).

1.4. Groups graded by root systems and associated graphs of groups.

Property (T) for the groups $\mathbb{G}_\Phi(R)$ and $\mathrm{St}_\Phi(R)$ will be proved using certain generalization of the spectral criterion from [EJ]. First we shall describe the relevant graph decompositions of those groups, for simplicity concentrating on the case $G = \mathrm{EL}_n(R)$.

For each $n \geq 2$ consider the following graph denoted by $\Gamma(A_n)$. The vertices of $\Gamma(A_n)$ are labeled by the elements of the symmetric group $Sym(n+1)$, and two vertices σ and σ' are connected if any only if they are not opposite to each other in the Cayley graph of $Sym(n+1)$ with respect to the generating set $\{(12), (23), \ldots, (n, n+1)\}$.

Now let R be ring and $n \geq 3$. The group $G = \mathrm{EL}_n(R)$ has a natural decomposition over $\Gamma(A_{n-1})$ defined as follows. For each $\sigma \in Sym(n)$ the vertex group G_σ is defined to be the subgroup of G generated by $\{X_{ij} : \sigma(i) < \sigma(j)\}$ where $X_{ij} = \{e_{ij}(r) : r \in R\}$. Thus, vertex subgroups are precisely the maximal unipotent subgroups of G normalized by the diagonal subgroup; in particular, the vertex subgroup corresponding to the identity permutation is the upper-unitriangular subgroup of $\mathrm{EL}_n(R)$. If e is the edge connecting vertices σ and σ', we define the edge subgroup G_e to be the intersection $G_\sigma \cap G_{\sigma'}$ (note that this intersection is non-trivial precisely when σ and σ' are connected in $\Gamma(A_{n-1})$).

As already discussed in the last paragraph of § 1.2, property (T) for $G = \mathrm{EL}_n(R)$ is reduced to showing that $\cup X_{ij}$ is a Kazhdan subset of G. By the standard bounded generation argument [Sh1] it suffices to show that the larger subset $\cup_{\sigma \in Sym(n)} G_\sigma$ (the union of vertex subgroups in the above decomposition) is Kazhdan, and one might attempt to prove the latter by applying the spectral criterion from [EJ] to the decomposition of G over $\Gamma(A_{n-1})$ described above. This almost works. More precisely, the attempted application of this criterion yields a "boundary case", where equality holds in the place of the desired inequality. In order to resolve this problem, a slightly generalized version of the spectral criterion must be used. The precise conditions entering this generalized spectral criterion are rather technical (see Section 3), but these conditions are consequences of a very transparent property satisfied by $\mathrm{EL}_n(R)$, namely the fact that $\mathrm{EL}_n(R)$ is strongly graded by a root system of type A_{n-1} (which has rank ≥ 2) as defined below.

Let G be a group and Φ a classical root system. Suppose that G has a family of subgroups $\{X_\alpha\}_{\alpha \in \Phi}$ such that

$$(1.1) \qquad [X_\alpha, X_\beta] \subseteq \prod_{\gamma \in \Phi \cap (\mathbb{Z}_{>0}\alpha \oplus \mathbb{Z}_{>0}\beta)} X_\gamma$$

for any $\alpha, \beta \in \Phi$ such that $\alpha \neq -\lambda\beta$ with $\lambda \in \mathbb{R}_{>0}$. Then we will say that G is graded by Φ and $\{X_\alpha\}$ is a Φ-grading of G.

Clearly, for any root system Φ the Steinberg group $\mathrm{St}_\Phi(R)$ and the elementary Chevalley group $\mathbb{E}_\Phi(R)$ are graded by Φ (with $\{X_\alpha\}$ being the root subgroups). On the other hand, any group G graded by Φ has a natural graph decomposition. We already discussed how to do this for $\Phi = A_n$ (in which case the underlying graph

is $\Gamma(A_n)$ defined above). For an arbitrary Φ, the vertices of the underlying graph $\Gamma(\Phi)$ are labeled by the elements of W_Φ, the Coxeter group of type Φ, and given $w \in W_\Phi$, the vertex subgroup G_w is defined to be $\langle X_\alpha : w\alpha \in \Phi^+ \rangle$ where Φ^+ is the set of positive roots in Φ (with respect to some fixed choice of simple roots). The edges of $\Gamma(\Phi)$ and the edge subgroups of G are defined as in the case $\Phi = A_n$.

Once again, the above decomposition of G over $\Gamma(\Phi)$ corresponds to the boundary case of the spectral criterion from [EJ], and the generalized spectral criterion turns out to be applicable under the addition assumption that the grading of G by Φ is *strong*. Informally a grading is strong if the inclusion in (1.1) is not too far from being an equality (see Section 4 for precise definition). For instance, if Φ is a simply-laced system, a sufficient condition for a Φ-grading $\{X_\alpha\}$ to be strong is that $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ whenever $\alpha + \beta$ is a root.

We can now formulate the central result of this paper.

Theorem 1.2. *Let Φ be an irreducible classical root system of rank ≥ 2, and let G be a group which admits a strong Φ-grading $\{X_\alpha\}$. Then $\cup X_\alpha$ is a Kazhdan subset of G.*

Theorem 1.2 in the case $\Phi = A_n$ was already established in [EJ]; however, this was achieved by only considering the graph $\Gamma(A_2)$, called the *magic graph on six vertices* in [EJ]. This was possible thanks to an observation that every group strongly graded by A_{n-1}, $n \geq 3$. must also be strongly graded by A_2; for simplicity we illustrate the latter for $G = \mathrm{EL}_n(R)$. If $n = 3k$, the A_2 grading follows from the well-known isomorphism $\mathrm{EL}_{3k}(R) \cong \mathrm{EL}_3(\mathrm{Mat}_k(R))$, and for arbitrary n one should think of $n \times n$ matrices as "3×3 block-matrices with blocks of uneven size." This observation is a special case of the important concept of a *reduction of root systems* discussed in the next subsection (see Section 6 for full details).

The proof of Theorem 1.2 for arbitrary Φ follows the same general approach as in the case $\Phi = A_2$ done in [EJ], although some arguments which are straightforward for $\Phi = A_2$ require delicate considerations in the general case. Perhaps more importantly, the proof presented in this paper provides a conceptual explanation of certain parts of the argument from [EJ] and shows that there was really nothing "magic" about the graph $\Gamma(A_2)$.

1.5. **Further examples and applications.** So far we have discussed important, but very specific examples of groups graded by root systems – Chevalley and Steinberg groups. We shall now describe two general methods of constructing new groups graded by root systems. Thanks to Theorem 1.2 and suitable results on relative property (T), in this way we will also obtain new examples of Kazhdan groups.

The first method is simply an adaptation of the construction of twisted Chevalley groups to a slightly different setting. Suppose we are given a group G together with a grading $\{X_\alpha\}$ by a root system Φ and a finite group Q of automorphisms of G which permutes the root subgroups between themselves. Under some additional conditions we can use this data to construct a new group graded by a (different) root system. Without going into details, we shall mention that the new group, denoted by $\widehat{G^Q}$, surjects onto certain subgroup of G^Q, the group of Q-fixed points of G, and the new root system often coincides with the set of orbits under the induced action of Q on the original root system Φ. As a special case of this construction, we can take $G = \mathrm{St}_\Phi(R)$ for some ring R and let Q be the cyclic subgroup generated

by an automorphism of G of the form $d\sigma$ where σ is a ring automorphism of $\mathrm{St}_\Phi(R)$ and d is a diagram automorphism of $\mathrm{St}_\Phi(R)$ having the same order as σ.

In this way we shall obtain "Steinberg covers" of the usual twisted Chevalley groups over commutative rings of type ^2A_n, ^2D_n, $^3D_4$ and $^2E_6$. The Steinberg covers for the groups of type ^2A_n (which are unitary groups over rings with involution) can also be defined over non-commutative rings; moreover, the construction allows additional variations leading to a class of groups known as *hyperbolic unitary Steinberg groups* (see [HO], [Bak2]). Using this method one can also construct interesting families of groups which do not seem to have direct counterparts in the classical theory of algebraic groups; for instance, we will define Steinberg groups of type $^2F_4$ – these correspond to certain groups constructed by Tits [Ti] which, in turn, generalize twisted Chevalley groups of type $^2F_4$. We will show that among these Steinberg-type groups the ones graded by a root system of rank ≥ 2 have property (T) under some natural finiteness conditions on the data used to construct the twisted group.

The second method is based on the notion of a reduction of root systems defined below. This method does not directly produce new groups graded by root systems, but rather shows how given a group G graded by a root system Φ, one can construct a new grading of G by another root system of smaller rank.

If Φ and Ψ are two root systems, a reduction of Φ to Ψ is just a surjective map $\eta : \Phi \to \Psi \cup \{0\}$ which extends to a linear map between the real vector spaces spanned by Φ and Ψ, respectively. Now if G is a group with a Φ-grading X_α, for each $\beta \in \Psi$ we set $Y_\beta = \langle X_\alpha : \eta(\alpha) = \beta \rangle$. If the groups $\{Y_\beta\}_{\beta \in \Psi}$ happen to generate G (which is automatic, for instance, if η does not map any roots to 0), it is easy to see that $\{Y_\beta\}$ is a Ψ-grading of G. Furthermore, if the original Φ-grading was strong, then under some natural assumptions on the reduction η the new Ψ-grading will also be strong (reductions with this property will be called 2-good). We will show that every classical root system of rank ≥ 2 has a 2-good reduction to a classical root system of rank 2 (that is, A_2, B_2, BC_2 or G_2). [2] In this way we reduce the proof of Theorem 1.1 to Theorem 1.2 for classical root systems of rank 2. While Theorem 1.2 is not any easier to prove in this special case, using such reductions we obtain much better Kazhdan constants for the groups $\mathrm{St}_\Phi(R)$ and $\mathbb{E}_\Phi(R)$ than what we are able to deduce from direct application of Theorem 1.2.

So far our discussion was limited to groups graded by classical root systems, but our definition of Φ-grading makes sense for any finite subset Φ. In this paper by a *root system* we mean any finite subset of $\mathbb{R}^n$ symmetric about the origin, and Theorem 1.2 remains true for groups graded by any root system satisfying a minor technical condition (these will be called *regular root systems*). There are plenty of regular root systems, which are not classical, but there is no easy recipe for constructing interesting groups graded by them. What we know is that reductions can be used to obtain some exotic gradings on familiar groups – for instance, the groups $\mathrm{St}_n(R)$ and $\mathrm{EL}_n(R)$ naturally graded by A_{n-1} can also be strongly graded by I_n (the two-dimensional root system whose elements join the origin with the vertices of a regular n-gon). We believe that the analysis of this and other similar gradings can be used to construct new interesting groups, but we leave such an investigation for a later project.

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[2] The reduction of A_n to A_2 was already implicitly used in the proof of property (T) for $\mathrm{St}_{n+1}(R)$ in [EJ].

2. Preliminaries

In this section we shall define the notions of property (T), relative property (T), Kazhdan constants and Kazhdan ratios, orthogonality constants, angles and codistances between subspaces of Hilbert spaces, and recall some basic facts about them. We shall also state two new results on Kazhdan constants for nilpotent groups and group extensions, which will be established at the end of the paper (in Section 9).

All groups in this paper will be assumed discrete, and we shall consider their unitary representations on Hilbert spaces. By a *subspace* of a Hilbert space we shall mean a closed subspace unless indicated otherwise.

2.1. **Property** (T).

Definition. Let G be a group and S a subset of G.

  (a) Let V be a unitary representation of G. A nonzero vector $v \in V$ is called (S, ε)-invariant if
$$\|sv - v\| \leq \varepsilon \|v\| \text{ for any } s \in S.$$

  (b) Let V be a unitary representation of G without nonzero invariant vectors. The *Kazhdan constant* $\kappa(G, S, V)$ is the infimum of the set
$$\{\varepsilon > 0 : V \text{ contains an } (S, \varepsilon)\text{-invariant vector}\}.$$

  (c) The *Kazhdan constant* $\kappa(G, S)$ of G with respect to S is the infimum of the set $\{\kappa(G, S, V)\}$ where V runs over all representations of G without nonzero invariant vectors.

  (d) S is called a *Kazhdan subset of G* if $\kappa(G, S) > 0$.

  (e) A group G has *property* (T) if G has a finite Kazhdan subset, that is, if $\kappa(G, S) > 0$ for some finite subset S of G.

Remark: The Kazhdan constant $\kappa(G, S)$ may only be nonzero if S is a generating set for G (see, e.g. [BHV, Proposition 1.3.2]). Thus, a group with property (T) is automatically finitely generated. Furthermore, if G has property (T), then $\kappa(G, S) > 0$ for any finite generating set S of G, but the Kazhdan constant $\kappa(G, S)$ depends on S.

We note that if S is a "large" subset of G, positivity of the Kazhdan constant $\kappa(G, S)$ does not tell much about the group G. In particular, the following holds (see, e.g., [BHV, Proposition 1.1.5]):

Lemma 2.1. *For any group G we have $\kappa(G, G) \geq \sqrt{2}$.*

2.2. **Relative property** (T) **and Kazhdan ratios.** Relative property (T) has been originally defined for pairs (G, H) where H is a normal subgroup of G:

Definition. Let G be a group and H a normal subgroup of G. The pair (G, H) has *relative property* (T) if there exist a finite set S and $\varepsilon > 0$ such that if V is any unitary representation of G with an (S, ε)-invariant vector, then V has a (nonzero) H-invariant vector. The largest ε with this property (for a fixed set S) is called the *relative Kazhdan constant* of (G, H) with respect to S and denoted by $\kappa(G, H; S)$.

The generalization of the notion of relative property (T) to pairs (G, B), where B is an arbitrary subset of a group G, has been given by de Cornulier [Co] and can be defined as follows (see also a remark in [EJ, Section 2]):

Definition. Let G be a group and B a subset of G. The pair (G, B) has *relative property* (T) if for any $\varepsilon > 0$ there are a finite subset S of G and $\mu > 0$ such that if V is any unitary representation of G and $v \in V$ is (S, μ)-invariant, then v is (B, ε)-invariant.

In this more general setting it is not clear how to "quantify" the relative property (T) using a single real number. However, this is possible under the additional assumption that the dependence of μ on ε in the above definition may be expressed by a linear function. In this case we can define the notion of a Kazhdan ratio:

Definition. Let G be a group and B and S subsets of G. The *Kazhdan ratio* $\kappa_r(G, B; S)$ is the largest $\delta \in \mathbb{R}$ with the following property: if V is any unitary representation of G and $v \in V$ is $(S, \delta\varepsilon)$-invariant, then v is (B, ε)-invariant.

Somewhat surprisingly, there is a simple relationship between Kazhdan ratios and relative Kazhdan constants:

Observation 2.2. *Let G be a group, and let B and S be subsets of G. The following hold:*

(i) *If $\kappa_r(G, B; S) > 0$ and S is finite, then (G, B) has relative (T).*
(ii) *If B is a normal subgroup of G, then*

$$\sqrt{2}\kappa_r(G, B; S) \leq \kappa(G, B; S) \leq 2\kappa_r(G, B; S).$$

In particular, (G, B) has relative (T) if and only if $\kappa_r(G, B; S) > 0$ for some finite set S.
(iii) $\kappa(G, S) \geq \kappa(G, B)\kappa_r(G, B; S)$

Proof. (i) and (iii) follow immediately from the definition. The first inequality in (ii) holds by Lemma 2.1, and the second inequality in (ii) is a standard fact proved, for instance, in [Sh1, Corollary 2.3]). $\square$

2.3. **Using relative property** (T). A typical way to prove that a group G has property (T) is to find a subset K of such that

(a) K is a Kazhdan subset of G
(b) the pair (G, K) has relative property (T).

Clearly, (a) and (b) imply that G has property (T). Note that (a) is easy to establish when K is a large subset of G, while (b) is easy to establish when K is small, so to obtain (a) and (b) simultaneously one typically needs to pick K of intermediate size.

In all our examples, pairs with relative property (T) will be produced with the aid of the following fundamental result: if R is any finitely generated ring, then the pair $(\mathrm{EL}_2(R) \ltimes R^2, R^2)$ has relative property (T). This has been proved by Shalom [Sh1] for commutative R and by Kassabov [Ka1] in general. In fact, we shall use what formally is a stronger result, although its proof is identical to the one given in [Ka1]:

Theorem 2.3. *Let $R * R$ denote the free product of two copies of the additive group of R, and consider the semi-direct product $(R * R) \ltimes R^2$, where the first copy of R acts by upper-unitriangular matrices, that is, $r \in R$ acts as left multiplication by the matrix $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, and the second copy of R acts by lower-unitriangular matrices. Then the pair $((R * R) \ltimes R^2, R^2)$ has relative property (T).*

In Section 7 we shall state a slight generalization of this theorem along with an explicit bound for the relative Kazhdan constant.

Given a group G, once we have found some subsets S of G for which (G, S) has relative property (T), it is easy to produce many more subsets with the same property. First it is clear that if (G, B_i) has (T) for some finite collection of subsets $B_1, \ldots, B_k$, then $(G, \cup B_i)$ also has property (T). Here is a more interesting result of this kind, on which the bounded generation method is based.

Lemma 2.4 (Bounded generation principle). *Let G be a group, S a subset of G and $B_1, \ldots, B_k$ a finite collection of subsets of G. Let $B_1 \ldots B_k$ be the set of all elements of G representable as $b_1 \ldots b_k$ with $b_i \in B_i$.*

(a) *Suppose that (G, B_i) has relative (T) for each i. Then $(G, B_1 \ldots B_k)$ also has relative (T).*

(b) *Suppose in addition that $\kappa_r(G, B_i; S) > 0$ for each i. Then*

$$\kappa_r(G, B_1 \ldots B_k; S) \geq \frac{1}{\sum\limits_{i=1}^{k} \frac{1}{\kappa_r(G, B_i; S)}} \geq \frac{\min\{\kappa_r(G, B_i; S)\}_{i=1}^{k}}{k} > 0.$$

Proof. See [Sh1, Proof of Theorem 2.6]. □

For convenience, we introduce the following terminology:

Definition. Let G be a group and $B_1, \ldots, B_k$ a finite collection of subsets of G. Let H be another subset of G. We will say that H *lies in a bounded product of* $B_1, \ldots, B_k$ if there exists $N \in \mathbb{N}$ such that any element $h \in H$ can be written as $h = h_1 \ldots h_N$ with $h_i \in \cup_{j=1}^{k} B_j$ for all i.

By Lemma 2.4 if a group G has subsets $B_1, \ldots, B_k$ such that (G, B_i) has relative (T) for each i, then (G, H) has relative (T) for any subset H which lies in a bounded product of $B_1, \ldots, B_k$. This observation will be frequently used in the proof of property (T) for groups graded by non-simply laced root systems.

2.4. **Orthogonality constants, angles and codistances.** The notion of the orthogonality constant between two subspaces of a Hilbert space was introduced and successfully applied in [DJ] and also played a key role in [EJ]:

Definition. Let V_1 and V_2 be two (closed) subspaces of a Hilbert space V.

(i) The *orthogonality constant* $\mathrm{orth}(V_1, V_2)$ between V_1 and V_2 is defined by

$$\mathrm{orth}(V_1, V_2) = \sup\{|\langle v_1, v_2 \rangle| : \|v_i\| = 1, v_i \in V_i \text{ for } i = 1, 2\}$$

(ii) The quantity $\sphericalangle(V_1, V_2) = \arccos(\mathrm{orth}(V_1, V_2))$ will be called the *angle* between V_1 and V_2. Thus, $\sphericalangle(V_1, V_2)$ is the infimum of angles between a nonzero vector from V_1 and a nonzero vector from V_2.

These quantities are only of interest when the subspaces V_1 and V_2 intersect trivially. In general, it is more adequate to consider the corresponding quantities after factoring out the intersection. We call them the *reduced orthogonality constant and reduced angle.*

Definition. Let V_1 and V_2 be two subspaces in a Hilbert space V, and assume that neither of the subspaces V_1 and V_2 contains the other.

(i) The *reduced orthogonality constant* $\mathrm{orth}_{red}(V_1, V_2)$ between V_1 and V_2 is defined by

$$\mathrm{orth}_{red}(V_1, V_2) = \sup\{\langle v_1, v_2 \rangle \mid \|v_i\| = 1, v_i \in V_i, v_i \perp V_1 \cap V_2 \text{ for } i = 1, 2\}.$$

(ii) The quantity $\sphericalangle_{red}(V_1, V_2) = \arccos(\mathrm{orth}_{red}(V_1, V_2))$ will be called the *reduced angle between V_1 and V_2*. Thus, $\sphericalangle_{red}(V_1, V_2)$ is the infimum of the angles between nonzero vectors v_1 and v_2, where $v_i \in V_i$ and $v_i \perp V_1 \cap V_2$.

Remark: If V_1 and V_2 are two distinct planes in a three-dimensional space, then the reduced angle $\sphericalangle_{red}(V_1, V_2)$ coincides with the usual (geometric) angle between V_1 and V_2.

Reduced angles play a key role in Kassabov's paper [Ka2] (where they are called just 'angles'), but in the present paper the case of subspaces with trivial intersections will suffice. In fact, in the subsequent discussion we shall operate with orthogonality constants rather than angles.

The following simple result will be very important for our purposes.

Lemma 2.5. *Let V_1 and V_2 be two subspaces of a Hilbert space V. Then the reduced angle between the orthogonal complements $V_1^\perp$ and $V_2^\perp$ is equal to the reduced angle between V_1 and V_2. Equivalently,*

$$\mathrm{orth}_{red}(V_1^\perp, V_2^\perp) = \mathrm{orth}_{red}(V_1, V_2).$$

Proof. This result appears as [Ka2, Lemma 3.9] as well as [EJ, Lemma 2.4] (in a special case), but it has apparently been known to functional analysts for a long time (see [BGM] and references therein). □

The notion of codistance introduced in [EJ] generalizes orthogonality constants to the case of more than two subspaces.

Definition. *Let V be a Hilbert space, and let $\{U_i\}_{i=1}^n$ be subspaces of V. Consider the Hilbert space V^n and its subspaces $U_1 \times U_2 \times \ldots \times U_n$ and $\mathrm{diag}(V) = \{(v, v, \ldots, v) : v \in V\}$. The quantity*

$$\mathrm{codist}(\{U_i\}) = (\mathrm{orth}(U_1 \times \ldots \times U_n, \mathrm{diag}(V)))^2$$

will be called the codistance between the subspaces $\{U_i\}_{i=1}^n$. It is easy to see that

$$\mathrm{codist}(\{U_i\}) = \sup\left\{ \frac{\|u_1 + \cdots + u_n\|^2}{n(\|u_1\|^2 + \cdots + \|u_n\|^2)} : u_i \in U_i \right\}.$$

For any collection of n subspaces $\{U_i\}_{i=1}^n$ we have $\frac{1}{n} \le \mathrm{codist}(\{U_i\}) \le 1$, and $\mathrm{codist}(\{U_i\}) = \frac{1}{n}$ if and only if $\{U_i\}$ are pairwise orthogonal. In the case of two subspaces we have an obvious relation between $\mathrm{codist}(U, W)$ and $\mathrm{orth}(U, W)$:

$$\mathrm{codist}(U, W) = \frac{1 + \mathrm{orth}(U, W)}{2}.$$

Similarly one can define the *reduced codistance*, but we shall not use this notion (we refer the interested reader to [Ka2]).

We now define the orthogonality constants and codistances for subgroups of a given group.

Definition.

(a) Let H and K be subgroups of the same group and let $G = \langle H, K \rangle$, the group generated by H and K. We define $\mathrm{orth}(H, K)$ to be the supremum of the quantities $\mathrm{orth}(V^H, V^K)$ where V runs over all unitary representations of G without nonzero invariant vectors.

(b) Let $\{H_i\}_{i=1}^n$ be subgroups of the same group, and let $G = \langle H_1, \ldots, H_n \rangle$. The *codistance* between $\{H_i\}$, denoted by $\mathrm{codist}(\{H_i\})$, is defined to be the supremum of the quantities $\mathrm{codist}(V^{H_1}, \ldots, V^{H_n})$, where V runs over all unitary representations of G without nonzero invariant vectors.

The basic connection between codistance and property (T), already discussed in the introduction, is given by the following lemma (see [EJ, Lemma 3.1]):

Lemma 2.6 ([EJ])**.** *Let G be a group and $H_1, H_2, \ldots, H_n$ subgroups of G such that $G = \langle H_1, \ldots, H_n \rangle$. Let $\rho = \mathrm{codist}(\{H_i\})$, and suppose that $\rho < 1$. The following hold:*

(a) *$\kappa(G, \bigcup H_i) \geq \sqrt{2(1 - \rho)}$.*

(b) *Let S_i be a generating set of H_i, and let $\delta = \min\{\kappa(H_i, S_i)\}_{i=1}^n$. Then*

$$\kappa(G, \bigcup S_i) \geq \delta \sqrt{1 - \rho}.$$

(c) *Assume in addition that each pair (G, H_i) has relative property (T). Then G has property (T).*

2.5. Kazhdan constants for nilpotent groups and group extensions. We finish this section by formulating two theorems and one simple lemma which provide estimates for Kazhdan constants. These results are new (although they have been known before in some special cases [BHV, Ha, NPS]). The two theorems will be established in Section 9 of this paper, while the lemma will be proved right away.

The first theorem concerns relative Kazhdan constants in central extensions of groups:

Theorem 2.7. *Let G be a group, N a normal subgroup of G and Z a subgroup of $Z(G) \cap N$. Put $H = Z \cap [N, G]$. Assume that A, B and C are subsets of G satisfying the following conditions:*

(1) *A and N generate G,*

(2) *$\kappa(G/Z, N/Z; B) \geq \varepsilon$,*

(3) *$\kappa(G/H, Z/H; C) \geq \delta$.*

Then

$$\kappa(G, N; A \cup B \cup C) \geq \frac{1}{\sqrt{3}} \min\{\frac{12\varepsilon}{5\sqrt{72\varepsilon^2|A| + 25|B|}}, \delta\}.$$

In a typical application of this theorem the following additional conditions will be satisfied:

(a) The group G/N is finitely generated

(b) The group Z/H is finite

In this case (3) holds automatically, and (1) holds for some finite set A. Therefore, Theorem 2.7 implies that under the additional assumptions (a) and (b), relative property (T) for the pair $(G/Z, N/Z)$ implies relative property (T) for the pair (G, N).

The second theorem that we shall use gives a bound for the codistance between subgroups of a nilpotent group. It is not difficult to see that if G is an abelian

group generated by subgroups $X_1, \ldots, X_k$, then $\mathrm{codist}(X_1, \ldots, X_k) \leq 1 - \frac{1}{k}$ (and this bound is optimal). We shall prove a similar (likely far from optimal) bound in the case of countable nilpotent groups:

Theorem 2.8. *Let G be a countable nilpotent group of class c generated by subgroups $X_1, \ldots, X_k$. Then*

$$\mathrm{codist}(X_1, \ldots, X_k) \leq 1 - \frac{1}{4^{c-1}k}.$$

We finish with a technical lemma which yields certain supermultiplicativity property involving Kazhdan ratios. It can probably be applied in a variety of situations, but in this paper it will only be used to obtain a better Kazhdan constant for the Steinberg groups of type G_2:

Lemma 2.9. *Let G be a group, H a subgroup of G and N a normal subgroup of H. Suppose that there exists a subset Σ of G and real numbers $a, b > 0$ such that*

(1) $\kappa_r(G, N; \Sigma) \geq \frac{1}{a}$
(2) $\kappa(H/N, \overline{\Sigma \cap H}) \geq \frac{1}{b}$ *where $\overline{\Sigma \cap H}$ is the image of $\Sigma \cap H$ in H/N.*

Then $\kappa_r(G, H; \Sigma) \geq \frac{1}{\sqrt{2a^2 + 4b^2}}$.

Proof. Let V be a unitary representation of G and $v \in V$ such that

$$\|sv - v\| \leq \varepsilon \|v\| \text{ for any } s \in \Sigma.$$

Write $v = v_1 + v_2$, where $v_1 \in V^N$ and $v_2 \in (V^N)^\perp$. Since $\kappa_r(G, N; \Sigma) \geq \frac{1}{a}$, we obtain that for every $n \in N$

$$\|nv_2 - v_2\| = \|nv - v\| \leq a\varepsilon \|v\|.$$

On the other hand, by Lemma 2.1 there exists $n \in N$ such that $\|nv_2 - v_2\| \geq \sqrt{2}\|v_2\|$. Hence $\|v_2\| \leq \frac{a\varepsilon}{\sqrt{2}}\|v\|$.

Since N is normal in H, the subspaces V^N and $(V^N)^\perp$ are H-invariant. Hence for any $s \in \Sigma \cap H$ we have $\|sv_1 - v_1\| \leq \|sv - v\| \leq \varepsilon \|v\|$. By Observation 2.2(ii) we have

$$\kappa_r(H/N, H/N; S) \geq \frac{\kappa(H/N, H/N; S)}{2} = \frac{\kappa(H/N, S)}{2} \geq \frac{1}{2b}.$$

Thus, considering V^N as a representation of H/N, we obtain that $\|hv_1 - v_1\| \leq 2b\varepsilon\|v_1\| \leq 2b\varepsilon\|v\|$ for any $h \in H$. Hence for any $h \in H$,

$$\|hv - v\|^2 = \|hv_1 - v_1\|^2 + \|hv_2 - v_2\|^2 \leq 4b^2\varepsilon^2\|v\|^2 + 4\|v_2\|^2 \leq \varepsilon^2(2a^2 + 4b^2)\|v\|^2. \qquad \square$$

3. Generalized spectral criterion

3.1. **Graphs and Laplacians.** Let Γ be a finite graph without loops. We will denote the set of vertices of Γ by $\mathcal{V}(\Gamma)$ and the set of edges by $\mathcal{E}(\Gamma)$. For any edge $e = (x, y) \in \mathcal{E}(\Gamma)$, we let $\bar{e} = (y, x)$ be the inverse of e. We assume that if $e \in \mathcal{E}(\Gamma)$, then also $\bar{e} \in \mathcal{E}(\Gamma)$. If $e = (x, y)$, we let $e^- = x$ be the initial vertex of e and by $e^+ = y$ the terminal vertex of e.

Now assume that the graph Γ is connected, and fix a Hilbert space V. Let $\Omega^0(\Gamma, V)$ be the Hilbert space of functions $f : \mathcal{V}(\Gamma) \to V$ with the scalar product

$$\langle f, g \rangle = \sum_{y \in \mathcal{V}(\Gamma)} \langle f(y), g(y) \rangle$$

and let $\Omega^1(\Gamma, V)$ be the Hilbert space of functions $f : \mathcal{E}(\Gamma) \to V$ with the scalar product

$$\langle f, g \rangle = \frac{1}{2} \sum_{e \in \mathcal{E}(\Gamma)} \langle f(e), g(e) \rangle.$$

Define the linear operator

$$d : \Omega^0(\Gamma, V) \to \Omega^1(\Gamma, V) \text{ by } (df)(e) = f(e^+) - f(e^-).$$

We will refer to d as the *difference operator of* Γ.

The adjoint operator $d^* : \Omega^1(\Gamma, V) \to \Omega^0(\Gamma, V)$ is given by formula

$$(d^* f)(y) = \sum_{y = e^+} \frac{1}{2} \left(f(e) - f(\bar{e})) \right).$$

The symmetric operator $\Delta = d^* d : \Omega^0(\Gamma, V) \to \Omega^0(\Gamma, V)$ is called the *Laplacian of* Γ and is given by the formula

$$(\Delta f)(y) = \sum_{y = e^+} (f(y) - f(e^-)) = \sum_{y = e^+} df(e).$$

The smallest positive eigenvalue of Δ is commonly denoted by $\lambda_1(\Delta)$ and called the *spectral gap* of the graph Γ (clearly, it is independent of the choice of V).

3.2. Spectral criteria.

Definition. Let G be a group and Γ a finite graph without loops. A *graph of groups decomposition (or just a decompoisition) of* G *over* Γ is a choice of a vertex subgroup $G_\nu \subseteq G$ for every $\nu \in \mathcal{V}(\Gamma)$ and an edge subgroup $G_e \subseteq G$ for every $e \in \mathcal{E}(\Gamma)$ such that

(a) The vertex subgroups $\{G_\nu : \nu \in \mathcal{V}(\Gamma)\}$ generate G;
(b) $G_e = G_{\bar{e}}$ and $G_e \subseteq G_{e^+} \cap G_{e^-}$ for every $e \in \mathcal{E}(\Gamma)$.

We will say that the decomposition of G over Γ is *regular* if for each $\nu \in \mathcal{V}(\Gamma)$ the vertex group G_ν is generated by the edge subgroups $\{G_e : e^+ = \nu\}$

The following criterion is proved in [EJ]:

Theorem 3.1. *Let* Γ *be a connected k-regular graph and let G be a group with a given regular decomposition over* Γ. *For each $\nu \in \mathcal{V}(\Gamma)$ let p_ν be the codistance between the subgroups $\{G_e : e^+ = \nu\}$ of G_ν, and let $p = \max_\nu p_\nu$. Let Δ be the Laplacian of* Γ, *and assume that*

$$p < \frac{\lambda_1(\Delta)}{2k}.$$

Then $\cup_{\nu \in \mathcal{V}(\Gamma)} G_\nu$ *is a Kazhdan subset of G, and moreover*

$$\kappa(G, \cup G_\nu) \geq \sqrt{\frac{2(\lambda_1(\Delta) - 2pk)}{\lambda_1(\Delta)(1 - p)}}.$$

In [EJ] a slight modification of this criterion was applied to groups graded by root systems of type A_2 with their canonical graph of groups decompositions (as described in the introduction). In this case one has $p = \frac{\lambda_1(\Delta)}{2k}$, and thus Theorem 3.1 is not directly applicable; however this problem was bypassed in [EJ] using certain trick. We shall now describe a generalization of Theorem 3.1 which essentially formalizes that trick and allows us to handle the "boundary" case $p = \frac{\lambda_1(\Delta)}{2k}$.

First, we shall use extra data – in addition to a decomposition of the group G over the graph Γ, we choose a normal subgroup CG_ν of G_ν for each vertex ν of Γ, called the *core subgroup* of G_ν. We shall assume that if a representation of the vertex group G_ν does not have any CG_ν-invariant vectors, then the codistance between the fixed subspaces of the edge groups is bounded above by $\frac{\lambda_1(\Delta)}{2k} - \varepsilon$ for some $\varepsilon > 0$ (independent of the representation). In order to make use of this extra assumption, we need to know that there are sufficiently many representations of G_ν without CG_ν-invariant vectors (for instance, if $CG_\nu = \{1\}$, there are no such non-trivial representations), and thus we shall also require that the core subgroups CG_ν are not too small.

Let us now fix a group G, a regular decomposition of G over a graph Γ, and a normal subgroup CG_ν of G_ν for each $\nu \in \mathcal{V}(\Gamma)$.

Let V be a unitary representation of G. Let $\Omega^0(\Gamma, V)^{\{G_\nu\}}$ denote the subspace of $\Omega^0(\Gamma, V)$ consisting of all function $g : V(\Gamma) \to V$ such that $g(\nu) \in V^{G_\nu}$ for any $\nu \in \mathcal{V}(\Gamma)$. Similarly we define the subspace $\Omega^1(\Gamma, V)^{\{G_e\}}$ of $\Omega^1(\Gamma, V)$.

Define the projections ρ_1, ρ_2, ρ_3 on the space $\Omega^0(\Gamma, V)$ as follows: for a function $g \in \Omega^0(\Gamma, V)$ and $\nu \in \mathcal{V}(\Gamma)$ we set

$$\rho_1(g)(\nu) = \pi_{V^{G_\nu}}(g(\nu)) \qquad \rho_2(g)(\nu) = \pi_{(V^{G_\nu})^\perp \cap V^{CG_\nu}}(g(\nu))$$
$$\rho_3(g)(\nu) = \pi_{(V^{CG_\nu})^\perp}(g(\nu)),$$

that is, the values of $\rho_i(g)$ for $i = 1, 2$ and 3 at the vertex ν are the projections of the vector $g(\nu) \in V$ onto the subspaces V^{G_ν}, $(V^{G_\nu})^\perp \cap V^{CG_\nu}$ and $(V^{CG_\nu})^\perp$, respectively.

By construction ρ_1 is just the projection onto $\Omega^0(\Gamma, V)^{\{G_\nu\}}$, and we have

$$\|\rho_1(g)\|^2 + \|\rho_2(g)\|^2 + \|\rho_3(g)\|^2 = \|g\|^2 \text{ for every } g \in \Omega^0(\Gamma, V).$$

Similarly, we define the projections ρ_1, ρ_2, ρ_3 on the space $\Omega^1(\Gamma, V)$: for a function $g \in \Omega^1(\Gamma, V)$ and an edge $e \in \mathcal{E}(\Gamma)$ we set

$$\rho_1(g)(e) = \pi_{V^{G_{e^+}}}(g(e)) \qquad \rho_2(g)(e) = \pi_{(V^{G_{e^+}})^\perp \cap V^{CG_{e^+}}}(g(e))$$
$$\rho_3(g)(e) = \pi_{(V^{CG_{e^+}})^\perp}(g(e)),$$

Again we have

$$\|\rho_1(g)\|^2 + \|\rho_2(g)\|^2 + \|\rho_3(g)\|^2 = \|g\|^2 \text{ for every } g \in \Omega^1(\Gamma, V).$$

Claim 3.2. *The projections ρ_i, $i = 1, 2, 3$, preserve the subspace $\Omega^1(\Gamma, V)^{\{G_e\}}$.*

Proof. The projection ρ_1 preserve the space $\Omega^1(\Gamma, V)^{\{G_e\}}$ because G_{e^+} contains the group G_e. The other two projections preserve this space because CG_{e^+} is a normal subgroup of G_{e^+}. $\qquad\square$

Now we are ready to state the desired generalization of Theorem 3.1.

Theorem 3.3. *Let Γ be a connected k-regular graph. Let G be a group with a chosen regular decomposition over Γ, and choose a normal subgroup CG_ν of G_ν for each $\nu \in \mathcal{V}(\Gamma)$. Let $\bar{p} = \frac{\lambda_1(\Delta)}{2k}$, where Δ is the Laplacian of Γ. Suppose that*

(i) *For each vertex ν of Γ the codistance between the subgroups $\{G_e : e^+ = \nu\}$ of G_v is bounded above by $\bar{p}$.*

(ii) *There exists $\varepsilon > 0$ such that for any $\nu \in \mathcal{V}(\Gamma)$ and any unitary representation V of the vertex group G_ν without CG_ν invariant vectors, the codistance between the fixed subspaces of G_e, with $e^+ = \nu$, is bounded above by $\bar{p}(1 - \varepsilon)$;*

(iii) *There exist constants A, B such that for any unitary representation V of G and for any function $g \in \Omega^0(\Gamma, V)^{\{G_\nu\}}$ one has*

$$\|dg\|^2 \leq A\|\rho_1(dg)\|^2 + B\|\rho_3(dg)\|^2.$$

Then $\cup G_\nu$ is a Kazhdan subset of G and

$$\kappa(G, \cup G_\nu) \geq \sqrt{\frac{4\varepsilon k}{\varepsilon \lambda_1(\Delta)A + (2k - \lambda_1(\Delta))B}} > 0.$$

Remark: (a) Theorem 3.3 implies Theorem 3.1 as (ii) clearly holds with $\varepsilon = 1 - p/\bar{p} > 0$, and if we let the core subgroups CG_ν to be equal to the vertex groups, then the projection ρ_2 is trivial, and thus (iii) holds with $A = B = 1$.

(b) If ε is sufficiently large, one can show that the conclusion of Theorem 3.3 holds even if $\bar{p}$ is slightly larger than $\frac{\lambda_1(\Delta)}{2k}$, but we are unaware of any interesting applications of this fact.

(c) The informal assumption that the core subgroups are not "too small" discussed above is "hidden" in the condition (iii).

3.3. **Proof of Theorem 3.3.** The main step in the proof is to show that the image of $\Omega^0(\Gamma, V)^{\{G_\nu\}}$ under the the Laplacian Δ is sufficiently far from $(\Omega^0(\Gamma, V)^G)^\perp$:

Theorem 3.4. *Let A and B be as in Theorem 3.3. Then for any $g \in \Omega^0(\Gamma, V)^{\{G_\nu\}}$ we have*

$$\|\rho_1(\Delta g)\|^2 \geq \frac{\varepsilon}{B(1 - \bar{p}) + \varepsilon A\bar{p}}\|\Delta g\|^2.$$

Proof. Let g be an element of $\Omega^0(\Gamma, V)^{\{G_\nu\}}$. This implies that $dg \in \Omega^1(\Gamma, V)^{\{G_e\}}$ and therefore $\rho_i(dg) \in \Omega^1(\Gamma, V)^{\{G_e\}}$ for $i = 1, 2, 3$. We have

$$\rho_i(\Delta g)(\nu) = \sum_{e^+ = \nu} \rho_i(dg)(e).$$

For $i = 1$ we just use the triangle inequality:

$$\|\rho_1(\Delta g)(\nu)\|^2 = \left\|\sum_{e^+ = \nu} \rho_1(dg)(e)\right\|^2 \leq k \sum_{e^+ = \nu} \|\rho_1(dg)(e)\|^2.$$

Summing over all vertices we get

$$\|\rho_1(\Delta g)\|^2 \leq k \sum_{\nu \in \mathcal{V}(\Gamma)} \sum_{e^+ = \nu} \|\rho_1(dg)(e)\|^2 = 2k \|\rho_1(dg)\|^2.$$

If $i = 2$ and $i = 3$ the vectors $\rho_i(dg)(e)$ are in V^{G_e} and they are orthogonal to the spaces V^{G_ν} and V^{CG_ν}, respectively. Since $(V^{G_\nu})^\perp$ (resp. $(V^{CG_\nu})^\perp$) is a representation of G_ν without invariant (resp. CG_ν-invariant) vectors, we can use the bounds for codistances from (i) and (ii):

$$\|\rho_2(\Delta g)(\nu)\|^2 = \left\|\sum_{e^+ = \nu} \rho_2(dg)(e)\right\|^2 \leq k\bar{p} \sum_{e^+ = \nu} \|\rho_2(dg)(e)\|^2,$$

and

$$\|\rho_3(\Delta g)(\nu)\|^2 = \left\|\sum_{e^+ = \nu} \rho_3(dg)(e)\right\|^2 \leq k\bar{p}(1 - \varepsilon) \sum_{e^+ = \nu} \|\rho_3(dg)(e)\|^2.$$

Again, summing over all vertices ν yields

$$\|\rho_2(\Delta g)\|^2 \le 2k\bar{p}\,\|\rho_2(dg)\|^2$$

$$\|\rho_3(\Delta g)\|^2 \le 2k\bar{p}\,(1-\varepsilon)\,\|\rho_3(dg)\|^2.$$

Thus we have

$$\bar{p}\left(1-\frac{\varepsilon A}{B}\right)\|\rho_1(\Delta g)\|^2 + \|\rho_2(\Delta g)\|^2 + \|\rho_3(\Delta g)\|^2 \le$$

$$\le 2k\bar{p}\left(\left(1-\frac{\varepsilon A}{B}\right)\|\rho_1(dg)\|^2 + \|\rho_2(dg)\|^2 + (1-\varepsilon)\,\|\rho_3(dg)\|^2\right) =$$

$$= 2k\bar{p}\left(\|dg\|^2 - \frac{\varepsilon A}{B}\|\rho_1(dg)\|^2 - \varepsilon\|\rho_3(dg)\|^2\right).$$

Combining this inequality with the norm inequality from (iii) and the fact that by definition of Δ and $\lambda_1(\Delta)$ we have

$$\|dg\|^2 = \langle \Delta g, g\rangle \le \frac{1}{\lambda_1(\Delta)}\|\Delta g\|^2,$$

we get

$$\left(\bar{p}\left(1-\frac{\varepsilon A}{B}\right)-1\right)\|\rho_1(\Delta g)\|^2 + \|(\Delta g)\|^2 \le$$

$$\le 2k\bar{p}\left(1-\frac{\varepsilon}{B}\right)\|dg\|^2 \le \frac{2k\bar{p}}{\lambda_1(\Delta)}\left(1-\frac{\varepsilon}{B}\right)\|\Delta g\|^2 = \left(1-\frac{\varepsilon}{B}\right)\|\Delta g\|^2.$$

and so

$$\|\rho_1(\Delta g)\|^2 \ge \frac{\varepsilon}{B(1-\bar{p})+\varepsilon A\bar{p}}\|\Delta g\|^2. \quad \square$$

Proof of Theorem 3.3. The following argument is very similar to the one on p.325 of [EJ]. Let V be a unitary representation of G without invariant vectors. Let U denote the subspace of $\Omega^0(\Gamma, V)$ consisting of all constant functions, let $W = \Omega^0(\Gamma, V)^{\{G_\nu\}}$ and $V' = U + W$. Define a symmetric operator $\tilde{\Delta} : V' \to V'$ by $\tilde{\Delta} = \mathrm{proj}_{V'}\circ\Delta$. An easy computation shows that the kernel of $\tilde{\Delta}$ is U, and therefore its image is equal to $U^{\perp V'}$.

Let $P = \frac{\varepsilon}{B(1-\bar{p})+\varepsilon A\bar{p}}$. By Theorem 3.4 we have $\|\mathrm{proj}_W \Delta g\|^2 \ge P\|\Delta g\|^2$ for all $g \in W$. In fact, the same inequality holds for all $g \in V'$ since $U = \mathrm{Ker}\,\Delta$. Notice that when $g \in V'$ we also have $\|\mathrm{proj}_W \tilde{\Delta}g\|^2 = \|\mathrm{proj}_W \mathrm{proj}_{V'}\Delta g\|^2 = \|\mathrm{proj}_W \Delta g\|^2$ (since $W \subseteq V'$), and therefore

$$\|\mathrm{proj}_W \tilde{\Delta}g\|^2 = \|\mathrm{proj}_W \Delta g\|^2 \ge P\|\Delta g\|^2 \ge P\|\tilde{\Delta}g\|^2.$$

Since the image of $\tilde{\Delta}$ is equal to $U^{\perp V'}$, the obtained inequality can be restated in terms of codistance:

$$\mathrm{codist}(U^{\perp V'}, W^{\perp V'}) \le 1 - P.$$

By Lemma 2.5 this implies [3] that $\mathrm{codist}(U, W) \le 1 - P$. But by definition of the codistance and the definition of subspaces U and W this implies that the codistance

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[3] Here we are using that U and W have trivial intersection.

between the vertex subgroups $\{G_\nu\}$ of G is bounded above by $1 - P$. Finally, by Lemma 2.5 we have

$$\kappa(G, \cup G_\nu) \geq \sqrt{2(1 - \text{codist}(\{G_\nu\}))} \geq \sqrt{2P} = \sqrt{\frac{4\varepsilon k}{\varepsilon \lambda_1(\Delta)A + (2k - \lambda_1(\Delta))B}} > 0.$$

$\square$

4. Root systems

The definition of a root system used in this paper is much less restrictive than that of a classical root system. However, most constructions associated with root systems we shall consider are naturally motivated by the classical case.

Definition. Let E be a real vector space. A finite non-empty subset Φ of E is called a *root system in E* if

(a) Φ spans E;
(b) Φ does not contain 0;
(c) Φ is closed under inversion, that is, if $\alpha \in \Phi$ then $-\alpha \in \Phi$.

The dimension of E is called the *rank of Φ*.

Remark: Sometimes we shall refer to the pair (E, Φ) as a root system.

Definition. Let Φ be a root system in E.

(i) Φ is called *reduced* if any line in E contains at most two elements of Φ;
(ii) Φ is called *irreducible* if it cannot be represented as a disjoint union of two non-empty subsets, whose $\mathbb{R}$-spans have trivial intersection.
(iii) A subset Ψ of Φ is called a *subsystem of Φ* if $\Psi = \Phi \cap \mathbb{R}\Psi$, where $\mathbb{R}\Psi$ is the $\mathbb{R}$-span of Ψ.

The importance of the following definition will be explained later in this section.

Definition. A root system will be called *regular* if any root is contained in an irreducible subsystem of rank 2.

4.1. **Classical root systems.** In this subsection we define classical root systems and state some well-known facts about them.

Definition. A root system Φ in a space E will be called *classical* if E can be given the structure of a Euclidean space such that

(a) For any $\alpha, \beta \in \Phi$ we have $\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$;
(b) If $\alpha, \beta \in \Phi$, then $\alpha - \frac{2(\alpha, \beta)}{(\beta, \beta)}\beta \in \Phi$.

Any inner product on E satisfying (a) and (b) will be called *admissible*.

Fact 4.1. (a) *Every irreducible classical root system is isomorphic to one of the following:* $A_n, B_n(n \geq 2), C_n(n \geq 3), BC_n(n \geq 1), D_n(n \geq 4), E_6, E_7, E_8, F_4, G_2.$ *The only non-reduced systems on this list are those of type BC_n.*
(b) *Every irreducible classical root system of rank ≥ 2 is regular.*

If Φ is a classical irreducible root system in a space E, then an admissible inner product $(\cdot, \cdot)$ on E is uniquely defined up to rescaling. In particular, we can compare *lengths* of different roots in Φ without specifying the Euclidean structure. Furthermore, the following hold:
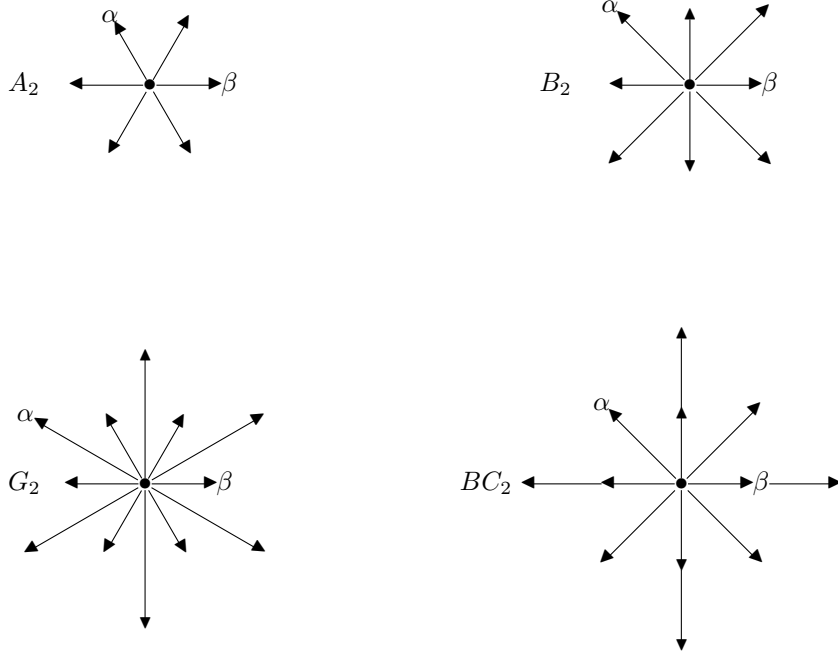
FIGURE 1. Classical irreducible root systems of rank 2.

(i) If $\Phi = A_n, D_n, E_6, E_7$ or E_8, all roots in Φ have the same length;

(ii) If $\Phi = B_n, C_n, F_4$ or G_2, there are two different root lengths in Φ;

(iii) If $\Phi = BC_n$, there are three different root lengths in Φ.

As usual, in case (ii), roots of smaller length will be called *short* and the remaining ones called *long*. In case (iii) roots of smallest length will be called *short*, roots of intermediate length called *long* and roots of largest length called *double*. The latter terminology is due to the fact that double roots in BC_n are precisely roots of the form 2α where α is also a root.

Definition. A subset Π of a classical root system Φ is called a *base* (or a *system of simple roots*) if every root in Φ is an integral linear combination of elements of Π with all coefficients positive or all coefficients negative. Thus every base Π of Φ determines a decomposition of Φ into two disjoint subsets $\Phi^+(\Pi)$ and $\Phi^-(\Pi) = -\Phi^+(\Pi)$, called the sets of *positive (resp. negative) roots with respect to* Π.

Example 4.2. Figure 1 illustrates each irreducible classical root system of rank 2 with a chosen base $\Pi = \{\alpha, \beta\}$.

If $\Phi = A_2$, then $\Phi^+(\Pi) = \{\alpha, \beta, \alpha + \beta\}$

If $\Phi = B_2$, then $\Phi^+(\Pi) = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$

If $\Phi = BC_2$, then $\Phi^+(\Pi) = \{\alpha, \beta, \alpha + \beta, 2\beta, \alpha + 2\beta, 2\alpha + 2\beta\}$

If $\Phi = G_2$, then $\Phi^+(\Pi) = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}$

Every classical root system Φ in a space E has a base; in fact, the number of (unordered) bases is equal to the order of the Weyl group of Φ. If Π is a base of Φ, it must be a basis of E. Observe that if $f : E \to \mathbb{R}$ is any functional which takes positive values on Π, then

$$\Phi^+(\Pi) = \{\alpha \in \Phi : f(\alpha) > 0\}.$$

Conversely, if $f : E \to \mathbb{R}$ is any functional which does not vanish on any of the roots in Φ, one can show that the set $\Phi_f = \{\alpha \in \Phi : f(\alpha) > 0\}$ coincides with $\Phi^+(\Pi)$ for some base Π. In fact, Π can be characterized as the elements $\alpha \in \Phi_f$ which are not representable as $\beta + \gamma$, with $\beta, \gamma \in \Phi_f$.

4.2. **General root systems.** In this subsection we extend the notions of a base and a set of positive roots from classical to arbitrary root systems. By the discussion at the end of the last subsection, if Φ is a classical root system, the sets of positive roots with respect to different bases of Φ are precisely the *Borel subsets* of Φ as defined below. The suitable generalization of the notion of a base, called the *boundary of a Borel set*, is less straightforward and will be given later. The terminology 'Borel subset' will be explained in §4.3.

Remark: If Φ is a reduced classical root system, the notions of boundary and base for Borel subsets of Φ coincide. However, if Φ is not reduced, the boundary of a Borel subset will be larger than its base.

Definition. Let Φ be a root system in a space E. Let $\mathfrak{F} = \mathfrak{F}(\Phi)$ denote the set of all linear functionals $f : E \to \mathbb{R}$ such that

(1) $f(\alpha) \neq 0$ for all $\alpha \in \Phi$;
(2) $f(\alpha) \neq f(\beta)$ for any distinct $\alpha, \beta \in \Phi$.

For $f \in \mathfrak{F}$, the set $\Phi_f = \{\alpha \in \Phi \mid f(\alpha) > 0\}$ is called the *Borel set of f*. The sets of this form will be called *Borel subsets of Φ*. We will say that two elements $f, f' \in \mathfrak{F}$ are equivalent and write $f \sim f'$ if $\Phi_f = \Phi_{f'}$.

Remark: Note that condition (2) implies condition (1) (if $f(\alpha) = 0$, then $f(-\alpha) = f(\alpha)$), but we will not use this fact.

Remark: Observe that for any $f \in \mathfrak{F}$ we can order the elements in Φ_f as follows:

$$\Phi_f = \{\alpha_{f,1}, \alpha_{f,2}, \dots \alpha_{f,k}\}$$

where $k = |\Phi_f| = |\Phi|/2$ and

$$f(\alpha_{f,1}) > f(\alpha_{f,2}) > \cdots > f(\alpha_{f,k}) > 0.$$

If f and g are equivalent functionals, their Borel sets Φ_f and Φ_g coincide, but the orderings on $\Phi_f = \Phi_g$ induced by f and g may be different. For instance, if $\Phi = A_2$ and $\{\alpha, \beta\}$ is a base of Φ, the functionals f and f' defined by $f(\alpha) = f'(\alpha) = 2$, $f(\beta) = 1$ and $f'(\beta) = 3$ define the same Borel set consisting of the roots α, β and $\alpha + \beta$, however the ordering induced by f and f' are different

$$f(\beta) < f(\alpha) < f(\alpha + \beta) \qquad f'(\alpha) < f'(\beta) < f'(\alpha + \beta).$$

Definition. Let Φ be a root system. Two Borel sets Φ_f and Φ_g will be called

- *opposite* if $\Phi_f \cap \Phi_g = \emptyset$ or, equivalently, $\Phi_g = \Phi_{-f}$;
- *co-maximal* if an inclusion $\Phi_h \supset \Phi_f \cap \Phi_g$ implies that $\Phi_h = \Phi_f$ or $\Phi_h = \Phi_g$;
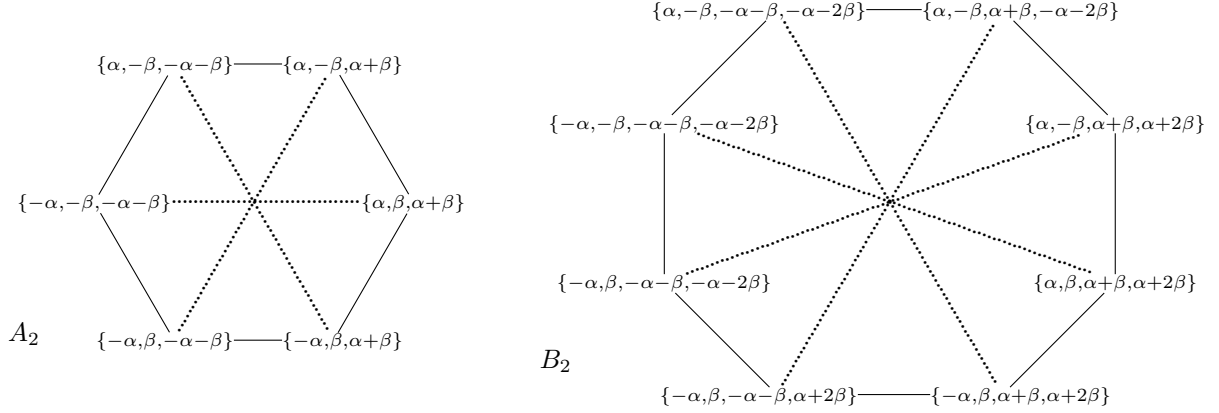
FIGURE 2. Borel Sets in root systems of type A_2 and B_2.

- *co-minimal* if Φ_f and Φ_{-g} are co-maximal.

Example 4.3. Figure 4.2 shows the Borel sets in root systems of type A_2 and B_2. Pairs of opposite Borel sets are connected with a dotted line and co-maximal ones are connected with a solid line.

Lemma 4.4. *Let Φ be a root system in a space E, and let Φ_f and Φ_g be distinct Borel sets. The following are equivalent:*

  (i) $\Phi_f \cap \Phi_{-g}$ *spans one-dimensional subspace*
  (ii) Φ_f *and Φ_g are co-maximal*
  (iii) *If $h \in \mathfrak{F}$ is such that $\Phi_h \supset \Phi_f \cap \Phi_g$, then $\Phi_h \cap \Phi_g = \Phi_f \cap \Phi_g$ or $\Phi_h \cap \Phi_f = \Phi_f \cap \Phi_g$.*

Proof. "(i)$\Rightarrow$ (ii)" Since $\Phi_f \cap \Phi_{-g}$ spans one-dimensional subspace, there exists a vector $v \in E$ such that

$$\Phi_f = (\Phi_f \cap \Phi_g) \cup (\Phi_f \cap \Phi_{-g}) \subset (\Phi_f \cap \Phi_g) \cup \mathbb{R}_{>0}v$$

and

$$\Phi_g = (\Phi_f \cap \Phi_g) \cup (\Phi_{-f} \cap \Phi_g) \subset (\Phi_f \cap \Phi_g) \cup \mathbb{R}_{<0}v.$$

Thus, if Φ_h contains $\Phi_f \cap \Phi_g$, then $\Phi_h = \Phi_f$ in the case $h(v) > 0$ or $\Phi_h = \Phi_g$ in the case $h(v) < 0$. Hence Φ_f and Φ_g are co-maximal.

"(ii)$\Rightarrow$ (iii)" is obvious.

"(iii)$\Rightarrow$ (i)" Let U be the subspace spanned by $\Phi_f \cap \Phi_{-g}$, and suppose that $\dim U > 1$. Since any sufficiently small perturbation of f does not change its equivalence class, we may assume that the the restrictions of f and g to U are linearly independent.

For any $x \in [0, 1]$ consider $h_x = xf + (1 - x)g$. Note that $h_0 = g$ is negative on $\Phi_f \cap \Phi_{-g}$ and $h_1 = f$ is positive on $\Phi_f \cap \Phi_{-g}$. Since the restrictions of f and g to U are linearly independent, by continuity there exists $x \in (0, 1)$ such that $h = h_x$ is positive on some but not all roots from $\Phi_f \cap \Phi_{-g}$. Thus there exist $\alpha, \beta \in \Phi$ such that $f(\alpha) > 0$, $g(\alpha) < 0$, $h(\alpha) > 0$ and $f(\beta) > 0$, $g(\beta) < 0$, $h(\beta) < 0$, so

$\Phi_f \cap \Phi_g \neq \Phi_f \cap \Phi_h$ and $\Phi_f \cap \Phi_g \neq \Phi_h \cap \Phi_g$. On the the hand, since $h = xf + (1-x)g$, it is clear that $\Phi_f \cap \Phi_g$ is contained in Φ_h. This contradicts (iii). $\qquad\square$

Part (a) of the next definition generalizes the notion of a base of a root system.

Definition. (a) *The boundary of a Borel set Φ_f is the set*

$$B_f = \bigcup_g (\Phi_f \setminus \Phi_g) = \bigcup_g (\Phi_f \cap \Phi_{-g}), \text{ where } \Phi_g \text{ and } \Phi_f \text{ are co-maximal.}$$

Equivalently,

$$B_f = \Phi_f \cap (\bigcup_g \Phi_g), \text{ where } \Phi_g \text{ and } \Phi_f \text{ are co-minimal.}$$

(b) *The core of a Borel set Φ_f is the set*

$$C_f = \Phi_f \setminus B_f = \bigcap_g (\Phi_f \cap \Phi_g), \text{ where } \Phi_g \text{ and } \Phi_f \text{ are co-maximal.}$$

If Φ is a classical reduced system and Φ_f is a Borel subset of Φ, it is easy to see that the boundary of Φ_f is precisely the base Π for which $\Phi^+(\Pi) = \Phi_f$. However for non-reduced systems this is not the case and the boundary also contains all roots which are positive multiples of the roots in the base.

Example 4.5. In each of the following examples we consider a classical rank 2 root system Φ, its base $\Pi = \{\alpha, \beta\}$ and the Borel set $\Phi^+(\Pi)$.

1. If $\Phi = A_2$, the core of the Borel set $\{\alpha, \beta, \alpha + \beta\}$ is $\{\alpha + \beta\}$ and the boundary is $\{\alpha, \beta\}$.
2. If $\Phi = B_2$, the core of the Borel set $\{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$ is $\{\alpha + \beta, \alpha + 2\beta\}$ and the boundary is again $\{\alpha, \beta\}$.
3. If $\Phi = BC_2$, the core of the Borel set $\{\alpha, \beta, 2\beta, \alpha + \beta, 2\alpha + 2\beta, \alpha + 2\beta\}$ is $\{\alpha + \beta, 2\alpha + 2\beta, \alpha + 2\beta\}$, while the boundary is $\{\alpha, \beta, 2\beta\}$.

Lemma 4.6. *Let Φ be a root system, $f \in \mathfrak{F} = \mathfrak{F}(\Phi)$, and let $\alpha, \beta \in \Phi_f$ be linearly independent. Then any root in Φ of the form $a\alpha + b\beta$, with $a, b > 0$, lies in C_f.*

Proof. Assume the contrary, in which case $a\alpha + b\beta \in B_f$. Thus, there exists $g \in \mathfrak{F}(\Phi)$ such that Φ_f and Φ_g are co-minimal and $a\alpha + b\beta \in \Phi_f \cap \Phi_g$. But then $g(\beta) > 0$ or $g(\alpha) > 0$, so $\Phi_f \cap \Phi_g$ contains linearly independent roots $a\alpha + b\beta$ and α or β. This contradicts Lemma 4.4. $\qquad\square$

Lemma 4.7. *Let Φ be a root system, Ψ a subsystem, $f \in \mathfrak{F} = \mathfrak{F}(\Phi)$ and f_0 the restriction of f to $\mathbb{R}\Psi$. Then the core C_{f_0} of Ψ_{f_0} is a subset of the core C_f of Φ_f.*

Proof. Let $v \in C_{f_0}$. We have to show that for any $g \in \mathfrak{F}$ such that Φ_f and Φ_g are co-maximal, $g(v) > 0$. Assume the contrary, that is, $g(v) < 0$. Then by Corollary 4.4, $\Phi_f \cap \Phi_{-g} \subset \mathbb{R}v$. Let g_0 be the restriction of g on Ψ. Then $\emptyset \neq \Psi_{f_0} \cap \Psi_{-g_0} \subseteq \Phi_f \cap \Phi_{-g} \subset \mathbb{R}v$. Again by Corollary 4.4, Ψ_{f_0} and Ψ_{-g_0} are co-minimal, so Ψ_{f_0} and Ψ_{g_0} are co-maximal. Since $v \in C_{f_0}$, we have $g(v) = g_0(v) > 0$, a contradiction. $\qquad\square$

Lemma 4.8. *Every root in an irreducible rank 2 system Φ is contained in the core of some Borel set.*

Proof. Let E be the vector space spanned by Φ, and take any $\alpha \in \Phi$. Since Φ is irreducible, there exist $\beta, \gamma \in \Phi$ such that α, β and γ are pairwise linearly independent. Replacing β by $-\beta$ and γ by $-\gamma$ if necessary, we can assume that $\alpha = b\beta + c\gamma$ with $b, c > 0$. If we now take any $f \in \mathfrak{F}(\Phi)$ such that $f(\beta) > 0$ and $f(\gamma) > 0$, then $\alpha \in C_f$ by Lemma 4.6. $\qquad\square$

Corollary 4.9. *Every root in a regular root system is contained in the core of some Borel set.*

Proof. This follows from Lemmas 4.7 and 4.8 and the fact that if Ψ is a subsystem of Φ, then any element of $\mathfrak{F}(\Psi)$ is the restriction of some element of $\mathfrak{F}(\Phi)$ to $\mathbb{R}\Psi$. $\qquad\square$

4.3. **Weyl graphs.** To each root system Φ we shall associate two graphs $\Gamma_l(\Phi)$ and $\Gamma_s(\Phi)$, called the *large Weyl graph* and the *small Weyl graph*, respectively. Both Weyl graphs $\Gamma_l = \Gamma_l(\Phi)$ and $\Gamma_s = \Gamma_s(\Phi)$ will have the same vertex set:

$$\mathcal{V}(\Gamma_l) = \mathcal{V}(\Gamma_s) = \mathfrak{F}(\Phi)/\sim \, .$$

Thus vertices of either graph are naturally labeled by Borel subsets of Φ: to each vertex $f \in \mathcal{V}(\Gamma_l) = \mathcal{V}(\Gamma_s)$ we associate the Borel set Φ_f.

- Two vertices f and g are connected in the large Weyl graph Γ_l if and only if their Borel sets are not opposite;
- Two vertices f and g are connected in the small Weyl graph Γ_s if and only if there exists functionals f' and g' such that $\Phi_f \subset C_{f'} \cup \Phi_g$ and $\Phi_g \subset C_{g'} \cup \Phi_f$.

To each (oriented) edge e in $\mathcal{E}(\Gamma_l)$ or $\mathcal{E}(\Gamma_s)$ we associate the set $\Phi_e = \Phi_{e+} \cap \Phi_{e-}$. Note that Φ_e is always non-empty by construction.

Remark: If Φ is an irreducible classical root system, both Weyl graphs of Φ are Cayley graphs of $W = W(\Phi)$, the Weyl group of Φ, but with respect to different generating sets. The large Weyl graph $\Gamma_l(\Phi)$ is the Cayley graph with respect to the set $W \setminus \{\alpha_{long}(\Phi)\}$ where $\alpha_{long}(\Phi)$ is the longest element of W relative to the (standard) Coxeter generating set S_Φ.

The generating set corresponding to the small Weyl graph $\Gamma_s(\Phi)$ is harder to describe. At this point we will just mention that it always contains the Coxeter generating set S_Φ, but it is equal to S_Φ only for systems of type A_2.

Example 4.10. Figure 4.3 shows the Weyl graphs in the root systems of type A_2 and B_2. The edges of the small Weyl graph are denoted by solid lines and ones in the large Weyl graph are either by solid or by dotted lines.

The structure of the large Weyl graph is very transparent.

Lemma 4.11. *Let Φ be a root system.*
  (a) *The large Weyl graph $\Gamma_l = \Gamma_l(\Phi)$ is a regular graph with N vertices and degree $N - 2$, where N is the number of distinct Borel sets of Φ.*
  (b) *The eigenvalues of the adjacency matrix of Γ_l are $N - 2$ with multiplicity 1, 0 with multiplicity $N/2$ and -2 with multiplicity $N/2 - 1$. Therefore the spectral gap of the Laplacian of Γ_l is equal to the degree of Γ_l.*

Proof. (a) is clear. (b) A constant function is an eigenvector with eigenvalue $N - 2$, any "antisymmetric" function (one with $F(x) = -F(\bar{x})$ where x and $\bar{x}$ are opposite vertices) has eigenvalue 0, and the space of antisymmetric functions has
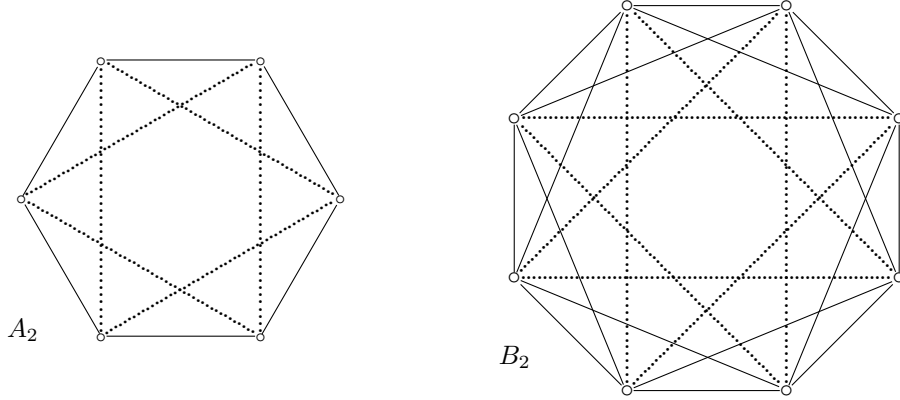
FIGURE 3. Weyl graphs corresponding to root systems of type A_2 and B_2.

dimension $N/2$. Finally, any "symmetric" function with sum 0 is an eigenfunction with eigenvalue -2, and the space of such functions has dimension $N/2 - 1$. $\square$

The role played by the small Weyl graph in this paper will be discussed at the end of this section. The key property we shall use is the following lemma:

Lemma 4.12. *Let Φ be a regular root system. Then the graph $\Gamma_s(\Phi)$ is connected.*

Proof. Let $f, g \in \mathfrak{F}$ be two functionals such that Φ_f and Φ_g are distinct. We prove that there exists a path in Γ_s from f to g by downward induction of $|\Phi_f \cap \Phi_g|$. If Φ_f and Φ_g are co-maximal, then f and g are connected (by an edge) in the small Weyl graph Γ_s by Lemma 4.4 and Corollary 4.9. If Φ_f and Φ_g are not co-maximal, then by Lemma 4.4 there exists h such that $\Phi_f \cap \Phi_g$ is properly contained in $\Phi_h \cap \Phi_f$ and $\Phi_h \cap \Phi_g$. By induction, there are paths that connects h with both f and g. Hence f and g are connected by a path in Γ_s. $\square$

Corollary 4.13. *Both large and small Weyl graphs of any irreducible classical root system of rank ≥ 2 are connected.*

We have computed the diameter of the small Weyl graph for some root systems, and in all these examples the diameter is at most 3. We believe that this is true in general.

Conjecture 4.14. *If Φ is an irreducible classical root system of rank ≥ 2, then the diameter of Γ_s is at most 3.*

4.4. **Groups graded by root systems.**

Definition. Let Φ be a root system and G a group. A Φ-*grading of G* (or just *grading of G*) is a collection of subgroups $\{X_\alpha\}_{\alpha \in \Phi}$ of G, called *root subgroups* such that

(i) G is generated by $\cup X_\alpha$;
(ii) For any $\alpha, \beta \in \Phi$, with $\alpha \notin \mathbb{R}_{<0}\beta$, we have

$$[X_\alpha, X_\beta] \subseteq \langle X_\gamma \mid \gamma = a\alpha + b\beta \in \Phi, \ a, b \geq 1 \rangle$$

If $\{X_\alpha\}_{\alpha\in\Phi}$ is a collection of subgroups satisfying (ii) but not necessarily (i), we will simply say that $\{X_\alpha\}_{\alpha\in\Phi}$ is a Φ-*grading* (without specifying the group).

Each grading of a group G by a root system Φ determines canonical graph of groups decompositions of G over the large and small Weyl graphs of Φ. The vertex and edge subgroups in these decompositions are defined as follows.

Definition. Let Φ be a root system, G a group and $\{X_\alpha\}_{\alpha\in\Phi}$ a Φ-grading of G. For each $f \in \mathcal{V}(\Gamma_l) = \mathcal{V}(\Gamma_s)$ we set

$$G_f = \langle X_\alpha \mid \alpha \in \Phi_f \rangle,$$

and for each $e \in \mathcal{E}(\Gamma_l) \supset \mathcal{E}(\Gamma_s)$ we set

$$G_e = \langle X_\alpha \mid \alpha \in \Phi_e \rangle.$$

We will call G_f the *Borel subgroup of G corresponding to f*.

Remark: We warn the reader that our use of the term 'Borel subgroup' is potentially misleading. Assume that Φ is classical, irreducible and reduced. Let F be a field and $G = \mathbb{E}_\Phi(F) = \mathbb{G}_\Phi(F)$ the corresponding simply-connected Chevalley group over F. Let $\{X_\alpha\}_{\alpha\in\Phi}$ be the root subgroups (relative to the standard torus H), so that $\{X_\alpha\}$ is a Φ-grading of G. Then Borel subgroups of G in our sense are smaller than Borel subgroups in the sense of Lie theory. In fact, Borel subgroups in our sense are precisely the unipotent radicals of those Borel subgroups in the sense of Lie theory which contain H. Equivalently, our Borel subgroups are maximal unipotent subgroups of G normalized by H.

Example 4.15. If G is a group graded by a root system of type A_2, Figure 4.4 shows the canonical decomposition of G over the large Weyl graph of A_2, called the "magic graph" in [EJ].

Definition. Let $\{X_\alpha\}$ be a Φ-grading of a group G. For each $f \in \mathfrak{F}(\Phi)$, the *core subgroup* G_{C_f} of G_f is the subgroup generated by the root subgroups in the core, that is,

$$G_{C_f} = \langle X_\alpha \mid \alpha \in C_f \rangle.$$

Lemma 4.16. *In the above notations, for each $f \in \mathfrak{F}$ the core subgroup G_{C_f} is a normal subgroup of G_f.*

Proof. This is an immediate consequence of Lemma 4.6. $\square$

Definition. Let Φ be a root system and $\{X_\alpha\}_{\alpha\in\Phi}$ a Φ-grading.
  (i) Take any functional $f \in \mathfrak{F}(\Phi)$ and any root $\gamma \in C_f$. We will say that the grading $\{X_\alpha\}$ is *strong at the pair* (γ, f) if

$$X_\gamma \subseteq \langle X_\beta \mid \beta \in \Phi_f \text{ and } \beta \notin \mathbb{R}\gamma \rangle.$$

  (ii) We will say that the grading $\{X_\alpha\}$ is *strong* if $\{X_\alpha\}$ is strong at every pair (γ, f) (with $\gamma \in C_f$).
  (iii) Given an integer k, we will say that the grading $\{X_\alpha\}_{\alpha\in\Phi}$ is *k-strong* if for any irreducible subsystem Ψ of rank k of Φ the grading $\{X_\alpha\}_{\alpha\in\Psi}$ is strong.
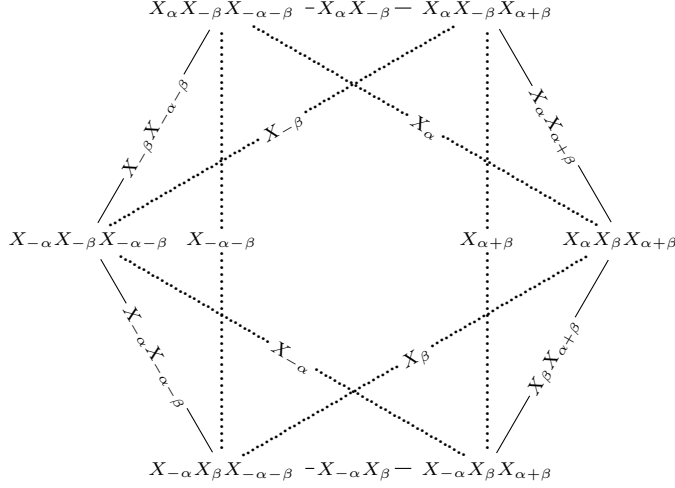
FIGURE 4. Weyl graph of groups for a groups graded by a root system of type A_2.

Remark: In Sections 7 and 8 we will need to verify that the natural gradings of certain Steinberg groups and twisted Steinberg groups are strong. In each of these examples the following property will be satisfied:

For any two functionals $f, f' \in \mathfrak{F}(\Phi)$ there exists an automorphism $w \in \mathrm{Aut}(G)$ which permutes the root subgroups $\{X_\alpha\}_{\alpha \in \Phi}$ between themselves, and the induced action of w on Φ sends the Borel subset Φ_f to the Borel subset $\Phi_{f'}$.

In the presence of this property, in order to prove that the grading $\{X_\alpha\}$ is strong it suffices to check that it is strong at (γ, f) where $f \in \mathfrak{F}(\Phi)$ is a fixed functional and γ runs over C_f. In each of our examples, we shall use a functional f such that Φ_f is the set of positive roots (with respect to a fixed system of simple roots). To simplify the terminology, we shall say that the grading is *strong at γ* if it is strong at (γ, f) for the f that we fixed.

Example 4.17. Let Φ be a root system of type A_2 and let $\{X_\gamma\}$. A sufficient condition for the grading to be strong is that $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ for any pairs of roots α and β such that $\alpha + \beta$ is also a root. This condition is also necessary under the additional assumption that every element in a Borel subgroup can be expressed uniquely as a product of elements in the 3 root subgroups (put in some fixed order).

Example 4.18. Let Φ be a root system of type B_2 and let $\{X_\gamma\}$ be a Φ-grading. Assume that there exists an abelian group R such that each of the root subgroups $\{X_\gamma\}$ is isomorphic to R; thus we can denote the elements of X_γ by $\{x_\gamma(r) : r \in R\}$ so that $x_\gamma(r + s) = x_\gamma(r)x_\gamma(s)$.

Now let $\{\alpha, \beta\}$ be a base of Φ, with α a long root. Let f be any functional such that $B_f = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$ (in which case $C_f = \{\alpha + \beta, \alpha + 2\beta\}$). By definition of grading there exist functions $p, q : R \times R \to R$ such that

$$[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(p(r, s)) x_{\alpha+2\beta}(q(r, s)) \text{ for all } r, s \in S$$

Then the grading is strong at the pair $(\alpha + \beta, f)$ (resp. $(\alpha + 2\beta, f)$) whenever the image of p (resp. q) generates R as a group.

If Φ is a non-reduced root system, it is sometimes useful to slightly modify a given Φ-grading using a simple operation called fattening:

Definition. Let Φ be a non-reduced root system and $\{X_\alpha\}_{\alpha \in \Phi}$ a Φ-grading of some group G. For each $\alpha \in \Phi$ we set $\widetilde{X}_\alpha = \langle X_{a\alpha} : a \geq 1 \rangle$. We will say that $\{\widetilde{X}_\alpha\}_{\alpha \in \Phi}$ is the fattening of the grading $\{X_\alpha\}_{\alpha \in \Phi}$.

It is easy to see that the fattening $\{\widetilde{X}_\alpha\}$ is also Φ-grading. Moreover, $\{\widetilde{X}_\alpha\}$ is strong whenever $\{X_\alpha\}$ is strong.

4.5. **A few words about the small Weyl graph.** We end this section explaining how the small Weyl graph and the notion of the core of a Borel set will be used in this paper. Unlike the large Weyl graph, which plays a central role in the proof of Theorem 5.1, the small Weyl graph is just a convenient technical tool.

As discussed in Section 3, given a group G graded by a regular root system Φ, Theorem 5.1 for G will be proved by applying the generalized spectral criterion (Theorem 3.3) to the canonical decomposition of G over the large Weyl graph Γ_l. The small Weyl graph Γ_s will be used to verify hypothesis (iii) in Theorem 3.3.

In fact, for many root systems we could use a different definition of the core of a Borel subset (leading to a different small Weyl graph and different core subgroups) which would still work for applications in Section 5. We could not make the cores any larger than we did (otherwise hypothesis (ii) in Theorem 3.3 would not hold), but we could often make then smaller – the only properties we need is that the small Weyl graph is connected (Lemma 4.12) and the core subgroups are normal in the ambient vertex groups.

For instance, if Φ is a simply-laced classical root system, we could let C_f consist of just one root, namely, the root of maximal height in the Borel set Φ_f (this definition coincides with ours for $\Phi = A_2$). In this case the small Weyl graph would become the Cayley graph of the Weyl group $W(\Phi)$ with respect to the standard Coxeter generating set. If Φ is a non-simply-laced classical root system, this seemingly more natural definition of the core does not work, although we could still make the core smaller except when $\Phi = B_2$ or BC_2.

5. Property (T) for groups graded by root systems

In this section we prove Theorem 1.2, in fact a slightly generalized version of it dealing with groups graded by regular (not necessarily classical) root systems.

Theorem 5.1. *Let Φ be a regular root system, and let G be a group which admits a strong Φ-grading $\{X_\alpha\}$. Then $\cup X_\alpha$ is a Kazhdan subset of G, and moreover the Kazhdan constant $\kappa(G, \cup X_\alpha)$ is bounded below by a constant κ_Φ which depends only on the root system Φ.*

Theorem 5.1 will be established by applying the generalized spectral criterion from Theorem 3.3 to the canonical decomposition of G over the large Weyl graph of Φ. Thus, we need to show that the hypotheses (i)-(iii) in Theorem 3.3 are satisfied in this setting.

For the rest of this section we fix a regular root system Φ and a group G with a strong Φ-grading $\{X_\alpha\}$. Let $\Gamma_l = \Gamma_l(\Phi)$ be the large Weyl graph of Φ. For each vertex of Γ_l we fix a functional $f \in \mathfrak{F}$ representing that vertex. The vertex itself will also be denoted by f, and the associated vertex subgroup will be denoted by G_f.

5.1. **Estimates for codistances.**

In this subsection we fix a vertex f of Γ_l and study representation theory of the vertex subgroup G_f of G. Throughout the subsection V will denote a *representation of G_f without G_f-invariant vectors*.

We shall first show that the codistance between the subspace $\{V^{G_e}\}$, where e runs over all vertices with $e^+ = f$, is bounded above by $1/2$. Then we will obtain a better bound for the codistance under the additional assumption that V has no CG_f-invariant vectors, where CG_f is the core subgroup of G_f.

Let E_f denote the set of all edges $e \in \mathcal{E}(\Gamma_l)$ with $e^+ = f$. Let $\{\alpha_{f,i}\}_{i=1}^{|\Phi|/2}$ be a (fixed) ordering of the roots in the Borel set Φ_f induced by f. For each $0 \le t \le |\Phi|/2$ we set
$$G_{f,t} = \langle X_{\alpha_{f,i}} : i \le t \rangle,$$
so that $\{1\} = G_{f,0} \subseteq G_{f,1} \subseteq \ldots \subseteq G_{f,|\Phi/2|} = G_f$. It is easy to see that
   (i) $G_{f,t}$ is a normal subgroup in G_f;
   (ii) $G_{f,t+1}/G_{f,t}$ is central in $G_f/G_{f,t}$,
so in particular the group G_f is nilpotent of class at most $|\Phi|/2|$.

Let $V_t = V^{G_{f,t}}$ and $V_t^\perp$ the orthogonal complement of V_t in V. Since $G_{f,t}$ is normal in G_f, both V_t and $V_t^\perp$ are G_f-invariant. Finally, let $V_{(t)} = V_{t-1} \cap V_t^\perp$. Since $V^{G_f} = \{0\}$ by assumption, we clearly have the decomposition
$$V = \bigoplus_t V_{(t)}.$$

Let $\pi_t : V \to V_{(t)}$ be the orthogonal projection. Thus for any $v \in V$ we have $v = \sum_{t=1}^{|\Phi|/2} \pi_t(v)$. Let
$$E_{f,t} = \{e \in E_f \mid \alpha_{f,t} \notin \Phi_e\}.$$

Claim 5.2. *For any t we have $|E_{f,t}| = |E_f|/2$.*

Proof. The neighbors of the vertex f in Γ_l can be grouped in pairs consisting of two opposite Borel sets. For any pair of opposite Borel subsets the root $\alpha_{f,t}$ lies in exactly one of them, which yields the claim. $\square$

Claim 5.3. *Let $v \in V_e = V^{G_e}$ for some $e \in E_f$. Then $\pi_t(v) = 0$ if $e \notin E_{f,t}$.*

Proof. Since each of the groups $G_{f,t}$ is normalized by G_e and v is G_e-invariant, its projection $\pi_t(v)$ is also G_e-invariant. On the other hand, by construction there are no $X_{\alpha_{f,t}}$-invariant vectors in $V_{(t)}$. This is impossible since by assumption $\alpha_{f,t} \in \Phi_e$, and thus $X_{\alpha_{f,t}} \subset G_e$. $\square$

Combining the last two claims, we deduce that the projection π_t is trivial when restricted to at least half of the subspaces V^{G_e}.

Corollary 5.4. *Let $v_e \in V_e = V^{G_e}$ for all $e \in E_f$. Then for any t at most $|E_f|/2$ of the vectors $\{\pi_t(v_e) : e \in E_f\}$ are nonzero.*

After these preparations we can obtain our first bound for the codistance:

Theorem 5.5. *(a) Let $v_e \in V^{G_e}$ for all $e \in E_f$. Then*

$$\left\| \sum v_e \right\|^2 \le \frac{|E_f|}{2} \sum \|v_e\|^2.$$

(b) The codistance between the subspaces $\{V^{G_e} : e \in E_f\}$ is bounded above by $1/2$.

Proof. (a) Using the decomposition of V as the direct sum of $V_{(t)}$ we obtain

$$\left\| \sum_{e \in E_f} v_e \right\|^2 = \left\| \sum_t \sum_{e \in E_f} \pi_t(v_e) \right\|^2 = \sum_t \left\| \sum_{e \in E_f} \pi_t(v_e) \right\|^2$$

Using Corollary 5.4 we get

$$\sum_t \left\| \sum_{e \in E_f} \pi_t(v_e) \right\|^2 \le \sum_t \frac{|E_f|}{2} \sum_{e \in E_f} \|\pi_t(v_e)\|^2 =$$

$$= \frac{|E_f|}{2} \sum_{e \in E_f} \sum_t \|\pi_t(v_e)\|^2 = \frac{|E_f|}{2} \sum_{e \in E_f} \|v_e\|^2.$$

(b) follows from (a) and the definition of codistance. $\qquad \square$

Next we establish a better bound for the codistance between $\{V^{G_e} : e \in E_f\}$ assuming that there are no vectors invariant under the core subgroup CG_f (see Theorem 5.9 below).

Claim 5.6. *The set*

$$\bigcup \{\Phi_e : \ e \in E_{f,t}\}$$

contains all roots from Φ_f, which are not multiples of $\alpha_{f,t}$.

Proof. Let $\beta \in \Phi_f$, and assume that β is not a multiple of $\alpha_{f,t}$. Then there exists another functional $f' \in \mathfrak{F}$ such that $\beta \in \Phi_{f'}$ but $\alpha_{f,t} \notin \Phi_{f'}$. Hence Φ_f and $\Phi_{f'}$ are connected by an edge $e \in E_f$ (because $\Phi_f \ne \Phi_{f'}$ and $\Phi_f \cap \Phi_{f'} \ne \emptyset$). Then $\beta \in \Phi_e$ but $\alpha_{f,t} \notin \Phi_e$. Thus by definition $e \in E_{f,t}$ and β lies in the set defined above. $\quad \square$

Claim 5.7. *Let G_f^t denote the subgroup generated by $\{G_e : e \in E_{f,t}\}$ Then G_f^t contains the core subgroup CG_f.*

Proof. By Claim 5.6 the group G_f^t contains the root subgroup X_β for each $\beta \in \Phi_f$ which is not a multiple of $\alpha_{f,t}$.

If the root $\alpha_{f,t}$ lies on the boundary of Φ_f, then the set $\Phi_f \setminus \mathbb{R}\alpha_{f,t}$ contains the core $C\Phi_f$, and thus G_f^t contains the core subgroup CG_f. If the root $\alpha_{f,t}$ lies in the core $C\Phi_f$, the inclusion $CG_f \subseteq G_f^t$ follows from the assumption that the grading is strong. $\qquad \square$

Claim 5.7 implies the following:

Corollary 5.8. *The intersection $\bigcap_{e \in E_{f,t}} V^{G_e}$ is inside V^{CG_f}.*

Theorem 5.9. *Let N be the number of Borel subsets in Φ and set*

$$\varepsilon_\Phi = \frac{8}{(N-2)4^{|\Phi|/2}}.$$

Let V be a representation of G_f without CG_f-invariant vectors. The following hold:

(a) *Let $v_e \in V^{G_e}$ for all $e \in E_f$. Then*

$$\left\| \sum v_e \right\|^2 \le \frac{|E_f|}{2}\left(1 - \varepsilon_\Phi\right) \sum \|v_e\|^2.$$

(b) *The codistance between the subspaces $\{V^{G_e} : e \in E_f\}$ is bounded above by $\left(1 - \varepsilon_\Phi\right)/2$.*

Proof. (a) Recall that by Lemma 4.11(a), $|E_f| = N - 2$, and by Claim 5.2, $|E_{f,t}| = |E_f|/2 = (N-2)/2$. Since $V^{CG_f} = \{0\}$, by Claim 5.7 for any t the group G_f^t acts on V without invariant vectors. The group G_f^t is nilpotent of class $\le |\Phi|/2$ and is generated by G_e with $e \in E_{f,t}$. By Theorem 2.8, given vectors $v_{e,t} \in V^{G_e}$ for all $e \in E_{f,t}$ we have

$$\left\| \sum v_{e,t} \right\|^2 \le \left(1 - \varepsilon_\Phi\right)|E_{f,t}| \sum \|v_{e,t}\|^2.$$

These inequalities combined with the decomposition $V = \bigoplus V_{(t)}$ yield

$$\left\| \sum_{e \in E_f} v_e \right\|^2 = \sum_t \left\| \sum_{e \in E_f} \pi_t(v_e) \right\|^2 \le \sum_t \left(1 - \varepsilon_\Phi\right)|E_{f,t}| \sum_{e \in E_{f,t}} \|\pi_t(v_e)\|^2 =$$

$$\frac{|E_f|}{2}\left(1 - \varepsilon_\Phi\right) \sum_{e \in E_f} \sum_t \|\pi_t(v_e)\|^2 = \frac{|E_f|}{2}\left(1 - \varepsilon_\Phi\right) \sum_{e \in E_f} \|v_e\|^2.$$

(b) is just a reformulation of (a) in terms of codistance. $\square$

5.2. **Norm estimates.** In this subsection we establish the "norm inequality" (Theorem 5.15) which is needed to verify hypothesis (iii) in Theorem 3.3. This inequality will be proved by considering both the small and the large Weyl graphs. We note that this is the only part of the paper where the small Weyl graph is used. In this subsection we assume that V is a *representation of the whole group G*.

Recall that the large Weyl graph Γ_l and the small Weyl graph Γ_s have the same sets of vertices. Also recall that $\Omega^0(\Gamma_l, V) = \Omega^0(\Gamma_s, V)$ is the set of all functions from $\mathcal{V}(\Gamma_l) = \mathcal{V}(\Gamma_s)$ to V and $\Omega^0(\Gamma_l, V)^{\{G_\nu\}}$ the set of all functions $g \in \Omega^0(\Gamma_l, V)$ such that $g(f) \in V^{G_f}$ for each vertex f. Denote by d_l and d_s the difference operators of Γ_l and Γ_s, respectively.

Lemma 5.10. *Let $g \in \Omega^0(\Gamma_l, V)^{\{G_\nu\}} = \Omega^0(\Gamma_s, V)^{\{G_\nu\}}$. If Φ is a regular root system, then*

(a) $\|d_s g\|^2 \le \|d_l g\|^2$
(b) $\|d_l g\|^2 \le C_\Phi \|d_s g\|^2$,

where the constant C_Φ depends only on the root system.

Proof. (a) is clear since Γ_s is a subgraph of Γ_l and (b) holds since Γ_s is connected. Indeed, for each edge e in $\mathcal{E}(\Gamma_l)$ we can find a path in Γ_s connecting the endpoints and write $g(e^+) - g(e^-) = \sum_i (g(e_i^+) - g(e_i^-))$. Use the triangle inequality we get

$$\|g(e^+) - g(e^-)\|^2 \le k \sum_i \|g(e_i^+) - g(e_i^-)\|^2$$

where k is the length of the path. $\qquad\square$

Example 5.11. The constant C_Φ can be easily computed for "small" root systems, e.g., one can take $C_{A_2} = 5$, $C_{B_2} = C_{BC_2} = 3$ and $C_{G_2} = 2$. It is unclear how the constant C_Φ depends on the rank of the root system.

For the rest of this subsection, for a subgroup H of G we denote by

$$\pi_H : V \to V^H \qquad \text{and} \qquad \pi_{H^\perp} : V \to (V^H)^\perp$$

the projections onto V^H and its orthogonal complement $(V^H)^\perp$, respectively.

For an edge e of Γ_l we let $GR_e = \langle X_\alpha : \alpha \in \Phi_{e^+} \setminus \Phi_e \rangle$

Claim 5.12. *For any edge e of Γ_l and any $v \in V^{G_e}$ we have $\pi_{G_{e^+}}(v) = \pi_{GR_e}(v)$.*

Proof. Let $k = |\Phi_e|$ and let $\{\beta_j\}_{1 \leq j \leq k}$ be the roots in Φ_e ordered so that

$$f(\beta_1) > f(\beta_2) > \ldots > f(\beta_k).$$

For $1 \leq i \leq k$ let H_i be the subgroup generated by GR_e and $\{X_{\beta_j}\}_{1 \leq j \leq i}$. By construction $H_0 = GR_e$, $H_k = G_{e^+}$ and each H_i is normalized by $X_{\beta_{i+1}}$.

By assumption, the vector v is $X_{\beta_{i+1}}$-invariant for each $0 \leq i \leq k-1$. Hence $\pi_{H_i}(v)$ is also $X_{\beta_{i+1}}$-invariant, so $\pi_{H_i}(v) = \pi_{H_{i+1}}(v)$. Combining these equalities for all i we get $\pi_{G_{e^+}}(v) = \pi_{H_k}(v) = \pi_{H_0}(v) = \pi_{GR_e}(v)$. $\qquad\square$

Now recall that $\Omega^1(\Gamma_l, V)$ (resp. $\Omega^1(\Gamma_s, V)$) is the space of all functions from $\mathcal{E}(\Gamma_l)$ (resp. $\mathcal{E}(\Gamma_s)$) to V. Notice that these two spaces are different unlike $\Omega^0(\Gamma_l, V) = \Omega^0(\Gamma_s, V)$.

As in Section 3 we have projections ρ_1, ρ_2, ρ_3 defined on each of those spaces with $\|\rho_1(g)\|^2 + \|\rho_2(g)\|^2 + \|\rho_3(g)\|^2 = \|g\|^2$ for all g. Note that in our new notations, for any $g \in \Omega^1(\Gamma_l, V)$ (resp. $g \in \Omega^1(\Gamma_s, V)$) and $e \in \mathcal{E}(\Gamma_l)$ (resp. $e \in \mathcal{E}(\Gamma_s)$) we have

$$\rho_1(g)(e) = \pi_{G_{e^+}}(g(e)) \qquad\qquad \rho_3(g)(e) = \pi_{CG_{e^+}^\perp}(g(e)).$$

The following is the main result of this subsection:

Theorem 5.13. *For any $g \in \Omega^0(\Gamma_l, V)^{\{G_\nu\}}$ we have*

$$\|d_s g\|^2 \leq \|\rho_1(d_s g)\|^2 + D_\Phi \|\rho_3(d_l g)\|^2,$$

where the constant D_Φ depends only on the root system.

Remark: Notice that the second term on the right hand side involves the differential of the large Borel graph, while the other two terms involve the differential of the small Borel graph.

Proof. Let e be an edge in the small Borel graph Γ_s. Since $g(e^+) = \pi_{G_{e^+}}(g(e^+)) = \pi_{GR_e}(g(e^+))$ and $g(e^-) \in V^{G_e}$, by Claim 5.12 we have

$$\|g(e^+) - g(e^-)\|^2 = \|\pi_{GR_e}(g(e^+) - g(e^-))\|^2 + \|\pi_{GR_e^\perp}(g(e^+) - g(e^-))\|^2 =$$
$$\|\pi_{G_{e^+}}(g(e^+) - g(e^-))\|^2 + \|\pi_{GR_e^\perp}(g(e^-))\|^2.$$

By the definition of the small Borel graph, we can find another vertex f' such that

$$\Phi_{e^+} \setminus \Phi_{e^-} = \Phi_{e^+} \setminus \Phi_e \subset C_{f'}.$$

This implies that $GR_e \subseteq CG_{f'} \subset G_{f'}$, and so $\pi_{GR_e^\perp}(g(f')) = 0$. Therefore

$$\|g(e^+) - g(e^-)\|^2 = \|\pi_{G_{e^+}}(g(e^+) - g(e^-))\|^2 + \|\pi_{GR_e^\perp}(g(f') - g(e^-))\|^2.$$

Since clearly $f' \neq e^-$, there is an edge e' in Γ_l connecting f' and e^-, with $(e')^+ = f'$ and $(e')^- = e^-$. The inclusion $GR_e \subseteq CG_{f'}$ implies that

$$\|\pi_{GR_e^\perp}\big(g(f') - g(e^-)\big)\|^2 \leq \|\pi_{CG_{f'}^\perp}\big(g(f') - g(e^-)\big)\|^2 =$$
$$\|\rho_3(d_l g)(e')\|^2 \leq \|\rho_3(d_l g)\|^2.$$

Summing over all edges e of Γ_s we get

$$\|d_s g\|^2 = \|\rho_1(d_s g)\|^2 + \sum_{e \in \mathcal{E}(\Gamma_s)} \|\pi_{GR_e^\perp}\big(g(f') - g(e^-)\big)\|^2 \leq$$
$$\|\rho_1(d_s g)\|^2 + \frac{|\mathcal{E}(\Gamma_s)|}{2}\|\rho_3(d_l g)\|^2. \quad \square$$

Example 5.14. Carefully doing the above estimates in the case of some "small" root systems gives that once can take $D_\Phi = 1$ if Φ is of type A_2, B_2, BC_2 or G_2.

Combining Lemma 5.10(b), Theorem 5.13 and the obvious inequality $\|\rho_1(d_s g)\| \leq \|\rho_1(d_l g)\|$, we obtain the desired inequality, which verifies hypothesis (iii) of Theorem 3.3 in our setting. Its statement only involves the large Weyl graph:

Corollary 5.15. *Let g be a function in $\Omega^0(\Gamma_l, V)^{\{G_\nu\}}$. Then*

$$\|d_l g\|^2 \leq A_\Phi \|\rho_1(d_l g)\|^2 + B_\Phi \|\rho_3(d_l g)\|^2,$$

where A_Φ and B_Φ are constants which depend on the root system Φ.

Example 5.16. In the case $\Phi = A_2$, the above estimates show that one can take $A_{A_2} = 5$ and $B_{A_2} = 5$. These bounds are not optimal — in [EJ] it is shown that one can use $A_{A_2} = 3$ and $B_{A_2} = 5$.

5.3. **Proof of Theorem 5.1.**

Proof of Theorem 5.1. As explained at the beginning of this section, we shall apply Theorem 3.3 to the canonical decomposition of G over the large Weyl graph $\Gamma = \Gamma_l \Phi$. Let us check that inequalities (i)-(iii) are satisfied.

By Lemma 4.11 the spectral gap of the Laplacian $\lambda_1(\Delta)$ is equal to the degree of Γ. Hence in the notations of Theorem 3.3 we have $\bar{p} = 1/2$. Thus, (i) holds by Theorem 5.5 and (ii) holds by Theorem 5.9. Finally, (iii) holds by Corollary 5.15.

Since all parameters in these inequalities depend only on Φ, Theorem 3.3 yields that $\kappa(G, \cup G_f) \geq \mathcal{K}_\Phi > 0$, with $\mathcal{K}_\Phi$ depending only on Φ. Finally, to obtain the same conclusion with G_f's replaced by root subgroups X_α's, we only need to observe that each G_f lies in a bounded product of root subgroups, where the number of factors does not exceed $|\Phi|/2$. $\square$

6. REDUCTIONS OF ROOT SYSTEMS

6.1. **Reductions.**

Definition. Let Φ be a root system in a space $V = \mathbb{R}\Phi$. A *reduction of Φ* is a surjective linear map $\eta : V \to V'$ where V' is another non-trivial real vector space. The set $\Phi' = \eta(\Phi) \setminus \{0\}$ is called the *induced root system*. We will also say that η is a *reduction of Φ to Φ'* and symbolically write $\eta : \Phi \to \Phi'$.

Lemma 6.1. *Let Φ be a root system, η a reduction of Φ, and Φ' the induced root system. Let $\{X_\alpha\}_{\alpha \in \Phi}$ be a Φ-grading. For any $\alpha' \in \Phi'$ put*

$$Y_{\alpha'} = \langle X_\alpha \mid \eta(\alpha) = \alpha' \rangle.$$

Then $\{Y_{\alpha'}\}_{\alpha' \in \Phi'}$ is a Φ'-grading, which will be called the coarsened grading.

Proof. This is a direct consequence of the following fact: if $A = \langle S_1 \rangle$ and $B = \langle S_2 \rangle$ are two subgroups of the same group, then $[A, B]$ is contained in the subgroup generated by all possible commutators in $S_1 \cup S_2$ of length at least 2 with at least one entry from S_1 and S_2. $\square$

A reduction $\eta : \Phi \to \Phi'$ enables us to replace a grading of a given group G by the "large" root system Φ by the coarsened grading by the "small" root system Φ' which may be easier to analyze. Since we are mostly interested in strong gradings, we would like to have a sufficient condition on η which ensures that the coarsened Φ'-grading is strong under some natural assumption on the initial Φ-grading.

Definition. Let $k \geq 2$ be an integer. A reduction η of Φ to Φ' is called *k-good* if

    (a) for any $\gamma \in \ker\eta \cap \Phi$, there exists an irreducible regular subsystem Ψ of rank k such that $\gamma \in \Psi$ and $\ker\eta \cap \Psi \subseteq \mathbb{R}\gamma$;

    (b) for any $f \in \mathfrak{F}(\Phi')$, $\gamma' \in C_f$ and $\gamma \in \Phi$ with $\eta(\gamma) = \gamma'$, there exists an irreducible subsystem Ψ of rank k of Φ and $g \in \mathfrak{F}(\Psi)$ such that $\gamma \in C_g$, $\eta(\Psi_g) \subseteq \Phi'_f$ and $\Psi \cap \eta^{-1}(\mathbb{R}\gamma') \subset \mathbb{R}\gamma$.

Lemma 6.2. *Let Φ be a root system, let η be a k-good reduction of Φ, and $\Phi' = \eta(\Phi) \setminus \{0\}$ the induced root system. Let $\{X_\alpha\}_{\alpha \in \Phi}$ be a k-strong grading of a group G. Then the coarsened grading $\{Y_{\alpha'}\}_{\alpha' \in \Phi'}$ is a strong grading of G.*

Proof. First let us show that G is generated by $\{Y_{\alpha'}\}$. Since the subgroups $\{X_\alpha\}_{\alpha \in \Phi}$ generate G, it is enough to show that every X_γ lies in the subgroup generated by $\{Y_{\alpha'}\}$. This is clear if $\eta(\gamma) \neq 0$. Assume now that $\eta(\gamma) = 0$. Since the reduction η is k-good, Φ has an irreducible regular subsystem Ψ of rank k such that $\gamma \in \Psi$ and $\ker\eta \cap \Psi \subseteq \mathbb{R}\gamma$. Since Ψ is regular, $\gamma \in C_f$ for some $f \in \mathfrak{F}(\Psi)$ by Corollary 4.9. Since the grading $\{X_\alpha\}_{\alpha \in \Phi}$ is k-strong, X_γ lies in the group generated by $\{X_\alpha\}_{\alpha \in \Psi_f \setminus \mathbb{R}\gamma} \subseteq \{X_\alpha\}_{\alpha \in \Phi \setminus \ker\eta}$, and so X_γ lies in the subgroup generated by $\{Y_{\alpha'}\}$.

The fact that $\{Y_{\alpha'}\}$ is a strong grading of G follows directly from part (b) of the definition of a k-good reduction and the assumption that the grading $\{X_\alpha\}_{\alpha \in \Phi}$ is k-strong. $\square$

Corollary 6.3. *Let Φ be a system such that any root lies in an irreducible subsystem of rank k. If a Φ-grading of a group G is k-strong, then it is also strong.*

Proof. The identity map $\mathbb{R}\Phi \to \mathbb{R}\Phi$ is clearly a reduction. It is k-good if and only if any root of Φ lies in an irreducible subsystem of rank k.

$\square$

6.2. **Examples of good reductions.** In this subsection we present several examples of 2-good reductions that will be used in this paper. In particular, we will establish the following result:

Proposition 6.4. *Every irreducible classical root system of rank > 2 admits a 2-good reduction to an irreducible classical root system of rank 2.*

The following elementary fact can be proved by routine case-by-case verification.

Claim 6.5. *Let Φ be a classical irreducible root system of rank ≥ 2. For $\alpha \in \Phi$ let $N_\Phi(\alpha)$ be the set of all $\beta \in \Phi$ such that $\mathrm{span}\{\alpha, \beta\} \cap \Phi$ is an irreducible rank 2 system. Then for any $\alpha \in \Phi$, the set $\{\alpha\} \cup N_\Phi(\alpha)$ spans $\mathbb{R}\Phi$.*

Now let $\eta : \Phi \to \Phi'$ be a reduction of root systems, where Φ is classical irreducible of rank ≥ 2. Claim 6.5 implies that η always satisfies condition (a) in the definition of a 2-good reduction. In order to speed up verification of condition (b) in the examples below we shall use symmetries of the "large" root system Φ which project to symmetries of the "small" root system Φ' under η. Formally, we shall use the following observation:

Suppose that a group Q acts linearly on Φ (hence also on $\mathbb{R}\Phi$), preserving the subspace $\ker \eta$. Thus we have an induced action of Q on Φ' given by

$$q\,\eta(\alpha) = \eta(q\,\alpha) \text{ for all } \alpha \in \Phi, q \in Q.$$

Then to prove that η satisfies condition (b) in the definition of a k-good reduction it suffices to check that condition for one representatives from each Q-orbit in $\{(f', \gamma) : f \in \mathfrak{F}(\Phi')/\sim, \eta(\gamma) \in C'_f\}$.

In the following examples $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$.

Reduction 6.6. The system A_n (or rather its canonical realization) is defined to be the set of vectors of $\mathbb{R}^{n+1}$ of length $\sqrt{2}$ with integer coordinates that sum to 0 (note that A_n spans a proper subspace of $\mathbb{R}^{n+1}$). Thus

$$A_n = \{e_i - e_j : 1 \leq i, j \leq n+1, i \neq j\}.$$

Fix $1 \leq i < j < n+1$. Then the map $\eta : \mathbb{R}^{n+1} \to \mathbb{R}^3$ defined by

$$\eta(x_1, \ldots, x_{n+1}) = (x_1 + \ldots + x_i, x_{i+1} + \ldots + x_j, x_{j+1} + \ldots + x_{n+1}),$$

is a reduction of A_n to A_2. It is clear that each of these reductions is 2-good.

Reduction 6.7. The system B_n consists of all integer vectors in $\mathbb{R}^n$ of length 1 or $\sqrt{2}$. Thus

$$B_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm e_i : 1 \leq i \leq n\}.$$

A natural reduction of B_n to B_2 is given by the map $\eta \colon \mathbb{R}^n \to \mathbb{R}^2$ defined by

$$\eta(x_1, \ldots, x_n) = (x_1, x_2).$$

Let us show that this reduction is 2-good. Let Q be the dihedral group of order 8, acting naturally on the first two coordinates of $\mathbb{R}^n$. This action preserves $\ker \eta$, and the induced Q-action on $\{(f, \gamma') : f \in \mathfrak{F}(B_2), \gamma' \in C_f\}$ has two orbits. The following table shows how to verify condition (b) from the definition of a good reduction for one representative in each orbit (using the notations from that definition). We do not specify the functionals f and g themselves; instead we list the bases (or boundaries) of the corresponding Borel sets Φ'_f and Ψ_g.

γ'	γ	base of Φ'_f	base of Ψ_g
(1,0)	$e_1 + xe_i$ $(i \geq 3)$	(1,-1), (0,1)	$e_1 - e_2, e_2 + xe_i$
(1,1)	$e_1 + e_2$	(1,-1),(0,1)	$e_1 - e_2, e_2 + xe_i$ $(i \geq 3)$

Reduction 6.8. The system D_n consists of all integer vectors in $\mathbb{R}^n$ of length $\sqrt{2}$. Thus

$$D_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}.$$

A natural reduction of $D_n(n \geq 3)$ to B_2 is given by the map $\eta \colon \mathbb{R}^n \to \mathbb{R}^2$ defined by

$$\eta(x_1, \ldots, x_n) = (x_1, x_2).$$

This reduction is 2-good – the proof is similar to the case of B_n.

Reduction 6.9. The system C_n consists of all integer vectors in $\mathbb{R}^n$ of length $\sqrt{2}$ together with all vectors of the form $2e$, where e is an integer vector of length 1. Thus

$$C_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm 2e_i : 1 \leq i \leq n\}.$$

A natural reduction of $C_n(n \geq 3)$ to BC_2 is given by the map $\eta \colon \mathbb{R}^n \to \mathbb{R}^2$ defined by

$$\eta(x_1, \ldots, x_n) = (x_1, x_2).$$

Let us show that this reduction is 2-good. We use the same action of the dihedral group of order 8 as in the example $B_n \to B_2$, but this time there are three orbits in $\{(f, \gamma') : f \in \mathfrak{F}(BC_2), \gamma' \in C_f\}$. The following table covers all the cases.

γ'	γ	base of Φ'_f	base of Ψ_g
(1,0)	$e_1 + xe_i$ $(i \geq 3)$	(1,-1), (0,1)	$e_1 - e_2, e_2 + xe_i$
(1,1)	$e_1 + e_2$	(1,-1),(0,1)	$e_1 - e_2, e_2 + e_i$ $(i \geq 3)$
(2,0)	$2e_1$	(1,-1), (0,1)	$e_1 - e_2, e_2$

Reduction 6.10. The root system C_n also admits a natural reduction to B_2.

Fix $1 \leq i < n$. The map $\eta_i \colon \mathbb{R}^n \to \mathbb{R}^2$ given by

$$\eta_i(x_1, \ldots, x_n) = (x_1 + \ldots + x_i, x_{i+1} + \ldots + x_n)$$

is a reduction of C_n to C_2. Composing η_i with some isomorphism $C_2 \to B_2$, we obtain an explicit reduction of C_n to B_2.

For instance, in the case $i = n - 1$ we obtain the following reduction η from C_n to B_2:

$$\eta(x_1, \ldots, x_n) = \left(\frac{x_1 + \ldots + x_{n-1} - x_n}{2}, \frac{x_1 + \ldots + x_{n-1} + x_n}{2} \right).$$

Let us show that it is 2-good. This time we take $Q = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, acting on C_n via the maps $\varphi_{\varepsilon_1, \varepsilon_2}$, with $\varepsilon_1, \varepsilon_2 = \pm 1$, defined by

$$\varphi_{\varepsilon_1, \varepsilon_2}(x_i) = \varepsilon_1 x_i \text{ for } 1 \leq i \leq n - 1, \quad \varphi_{\varepsilon_1, \varepsilon_2}(x_n) = \varepsilon_2 x_n.$$

There are four Q-orbits in $\{(f, \gamma') : f \in \mathfrak{F}(B_2), \gamma' \in C_f\}$. All the cases are described in the following table:

γ'	γ	base of Φ'_f	base of Ψ_g
(1,0)	$e_i - e_n$ $(i \geq n - 1)$	(1,-1), (0,1)	$-2e_n, e_i + e_n$
(1,1)	$2e_i$	(1,-1),(0, 1)	$-2e_n, e_i + e_n$
(1,1)	$e_i + e_j$ $(1 \leq i < j \leq n - 1)$	(1,-1),(0, 1)	$e_j - e_n, e_i + e_n$
(0,1)	$e_i + e_n$ $(i \geq n - 1)$	(1,1),(-1,0)	$2e_i, -e_i + e_n$
(-1,1)	$2e_n$ $(i \geq n - 1)$	(1,1),(-1,0)	$2e_i, -e_i + e_n$

Reduction 6.11. The system BC_n is the union of B_n and C_n (in their standard realizations). Thus

$$BC_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm e_i, \pm 2e_i : 1 \leq i \leq n\}.$$

Once again, the map $\eta : \mathbb{R}^n \to \mathbb{R}^2$ given by

$$\eta(x_1, \ldots, x_n) = (x_1, x_2)$$

is a reduction of BC_n to BC_2. To show that this reduction is 2-good we use the same action of the dihedral group of order 8 as in the reductions $B_n \to B_2$ and $C_n \to BC_2$. There are three Q-orbits in $\{(f, \gamma') : f \in \mathfrak{F}(BC_2), \gamma' \in C_f\}$, whose representatives are listed in the following table.

γ'	γ	base of Φ'_f	base of Ψ_g
(1,0)	$e_1 + xe_i \ (i \geq 3)$	(1,-1), (0,1)	$e_1 - e_2, e_2 + xe_i$
(1,1)	$e_1 + e_2$	(1,-1),(0,1)	$e_1 - e_2, e_2 + xe_i \ (i \geq 3)$
(2,0)	$2e_1$	(1,-1), (0,1)	$e_1 - e_2, e_2$

Reduction 6.12. The system G_2 consists of 12 vectors of lengths $\sqrt{2}$ and $\sqrt{6}$ of $\mathbb{R}^3$ with integer coordinates that sum to 0. Thus

$$G_2 = \{e_i - e_j : 1 \leq i, j \leq 3, i \neq j\} \cup \{\pm(2e_i - e_j - e_k) : 1 \leq i, j, k \leq 3, i \neq j \neq k \neq i\}$$

The system F_4 consists of vectors v of length 1 or $\sqrt{2}$ in $\mathbb{R}^4$ such that the coordinates of $2v$ are all integers and are either all even or all odd. Thus

$$F_4 = \{\pm e_i : 1 \leq i \leq 4\} \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\} \cup \{\pm e_i \pm e_j : 1 \leq i \leq j \leq 4\}.$$

A reduction of F_4 to G_2 is given by the map $\eta : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$\eta(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_2 - x_3, x_3 - x_1).$$

Let us show that this reduction is 2-good. This time we use an action of $S_3 \times \mathbb{Z}/2\mathbb{Z}$ where S_3 permutes the first three coordinates, and the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ sends (x_1, x_2, x_3, x_4) to $(-x_1, -x_2, -x_3, x_4)$. This reduces all the calculations to the following cases.

γ'	γ	base of Φ'_f	base of Ψ_g
(0,1,-1)	$(1, 1, 0, 0)$	(1,-1,0), (-1,2,-1)	$(1, 0, 1, 0), (0, 1, -1, 0)$
(0,1,-1)	$(0, 0, -1, x)$	(1,-1,0), (-1,2,-1)	$(0, -1, 0, x), (0, 1, -1, 0)$
(0,1,-1)	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, x)$	(1,-1,0), (-1,2,-1)	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, x), (0, 1, -1, 0)$
(1,0,-1)	$(0, -1, -1, 0)$	(1,-1,0), (-1,2,-1)	$(0, -1, 0, 0), (0, 1, -1, 0)$
(1,0,-1)	$(1, 0, 0, x)$	(1,-1,0), (-1,2,-1)	$(1, 0, 1, 0), (0, 0, -1, x)$
(1,0,-1)	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, x)$	(1, -1,0), (-1,2,-1)	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, x), (0, 0, -1, 0)$
(1,1,-2)	$(1, 0, -1, 0)$	(1, -1,0), (-1,2,-1)	$(1, -1, 0, 0), (0, 1, -1, 0)$
(2,-1,-1)	$(1, -1, 0, 0)$	(1, -1,0), (-1,2,-1)	$(0, -1, 0, 1), (1, 0, 0, -1)$

Reduction 6.13. The root system E_8 consists of the vectors of length $\sqrt{2}$ in $\mathbb{Z}^8$ and $(\mathbb{Z} + \frac{1}{2})^8$ such that the sum of all coordinates is an even number. The system E_7 is the intersection of E_8 with the hyperplane of vectors orthogonal to $(0, 0, 0, 0, 0, 0, 1, -1)$ in E_8 and the system E_6 is the intersection of E_7 with the hyperplane of vectors orthogonal to $(0, 0, 0, 0, 1, -1, 0, 0)$. The map $\eta : \mathbb{R}^8 \to \mathbb{R}^3$

$$\eta(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_1 - x_2, x_2 - x_3, -x_3 - x_1).$$

is a reduction of E_8 to G_2, and the restriction of η to $\mathbb{R}^7$ (resp. $\mathbb{R}^6$) is a reduction of E_7 (resp. E_6) to G_2.

Each of these reductions is 2-good, and the proof is similar to the case $F_4 \to G_2$.

Reduction 6.14. Let $n \geq 3$. Then the map $\eta : \mathbb{R}^n \to \mathbb{R}^3$ defined by

$$\eta(x_1, \ldots, x_n) = (x_1, x_2, x_3)$$

is a 3-good reduction of BC_n to BC_3. The proof is analogous to the previous examples.

7. Steinberg groups over commutative rings

In this section we prove property (T) for Steinberg groups of rank ≥ 2 over finitely generated commutative rings and estimate asymptotic behavior of Kazhdan constants.

7.1. Graded covers. Let Γ be a finite graph and G a group with a chosen decomposition $(\{G_\nu\}_{\nu \in \mathcal{V}(\Gamma)}, \{G_e\}_{e \in \mathcal{E}(\Gamma)})$ over Γ. If H is another group with a decomposition $(\{H_\nu, \{H_e\}\})$ over Γ, we will say that the decomposition $(\{H_\nu\}, \{H_e\})$ is isomorphic to $(\{G_\nu\}, \{G_e\})$ if there are isomorphisms $\iota_\nu : G_\nu \cong H_\nu$ for each $\nu \in \mathcal{V}(\Gamma)$ and $\iota_e : G_e \cong H_e$ for each $e \in \mathcal{E}(\Gamma)$ such that $\iota_{e^+|G_e} = \iota_e$ and $\iota_{\bar{e}} = \iota_e$.

Among all groups which admit a decomposition over the graph Γ isomorphic to $(\{G_\nu\}, \{G_e\})$ there is the "largest" one, which surjects onto any other group with this property. This group will be called the *cover of G corresponding to* $(\{G_\nu\}_{\nu \in \mathcal{V}(\Gamma)}, \{G_e\}_{e \in \mathcal{E}(\Gamma)})$ and can be defined as the free product of the vertex subgroups $\{G_\nu\}_{\nu \in \mathcal{V}(\Gamma)}$ amalgamated along the edge subgroups $\{G_e\}_{e \in \mathcal{E}(\Gamma)}$. We will be particularly interested in the special case of this construction dealing with decompositions associated to gradings by root systems.

Definition. Let G be a group, Φ a root system and $\{X_\alpha\}_{\alpha \in \Phi}$ a Φ-grading of G. Let $\Gamma_l \Phi$ be the large Weyl graph of Φ, and consider the canonical decomposition of G over $\Gamma_l \Phi$. The cover of G corresponding to this decomposition will be called the *graded cover of G with respect to the grading* $\{X_\alpha\}$.

Graded covers may be also defined using the generators and relations. Assume that $G_f = \langle \cup_{\alpha \in \Phi_f} X_\alpha | R_f \rangle$ for each $f \in \mathcal{F}(\Phi)$. Then the graded cover of G with respect to $\{X_\alpha\}_{\alpha \in \Phi}$ is isomorphic to

$$\langle \cup_{\alpha \in \Phi} X_\alpha | \cup_{f \in \mathcal{F}(\Phi)} R_f \rangle.$$

Observe that if $\pi : G \to G'$ is an epimorphism, and $\{X_\alpha\}$ is a Φ-grading of G, then $\{\pi(X_\alpha)\}$ is a Φ-grading of G'. If in addition π is injective on all the Borel subgroups of G, then the graded covers of G and G' coincide.

Here is a simple observation about automorphisms of graded covers recorded here for later use.

Definition. If G is a group and $\{X_\alpha\}$ is a grading of G, an automorphism $\pi \in \mathrm{Aut}(G)$ will be called *graded* (with respect to $\{X_\alpha\}$) if π permutes the root subgroups $\{X_\alpha\}$ between themselves. The group of all graded automorphisms of G will be denoted by $\mathrm{Aut}_{gr}(G)$.

Lemma 7.1. *Let G be a group, $\{X_\alpha\}$ a grading of G and $\widetilde{G}$ the graded cover of G with respect to $\{X_\alpha\}$. Then each graded automorphism of G naturally lifts to a graded automorphism of $\widetilde{G}$, and the obtained map $\mathrm{Aut}_{gr}(G) \to \mathrm{Aut}_{gr}(\widetilde{G})$ is a monomorphism.*

7.2. **Steinberg groups over commutative rings.** In this subsection we sketch the definition of Steinberg groups and show that the natural grading of these groups is strong. Our description of Steinberg groups follows Steinberg's lecture notes on Chevalley groups [St] and Carter's book [Ca].

Let Φ be a reduced irreducible classical root system and $\mathcal{L}$ a simple complex Lie algebra corresponding to Φ. Let $\mathcal{H}$ be Cartan subalgebra of $\mathcal{L}$. Then $\mathcal{H}$ is abelian and we have the following decomposition of $\mathcal{L}$:

$$\mathcal{L} = \mathcal{H} \oplus (\oplus_{\alpha \in \Phi} \mathcal{L}_\alpha),$$

where $\mathcal{L}_\alpha = \{l \in \mathcal{L} : [h,l]_L = \alpha(h)l \text{ for all } h \in \mathcal{H}\}$ (as usual we consider Φ as a subset of $\mathcal{H}^*$). It is known that each $\mathcal{L}_\alpha$ is one dimensional. Let $l = \dim \mathcal{H}$, and fix a a system of simple roots $\{\alpha_1, \ldots, \alpha_l\}$.

For any $\alpha, \beta \in \Phi$ we put $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$, where $(\cdot, \cdot)$ is an admissible scalar product on Φ (since $(\cdot, \cdot)$ is unique up to scalar multiples, the pairing $\langle \cdot, \cdot \rangle$ is well defined). For any $\alpha \in \Phi$ let $h_\alpha \in \mathcal{H}$ be the unique element such that

$$(7.1) \qquad \beta(h_\alpha) = \langle \beta, \alpha \rangle.$$

Note that $h_{\alpha_1}, \ldots, h_{\alpha_l}$ is a basis of $\mathcal{H}$.

Proposition 7.2. *There exist nonzero elements $x_\alpha \in \mathcal{L}_\alpha$ for $\alpha \in \Phi$ such that*

$$(7.2) \qquad [x_\alpha, x_{-\alpha}]_L = h_\alpha,$$

$$(7.3) \qquad [x_\alpha, x_\beta]_L = \begin{cases} \pm(r+1)x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \\ 0 & \text{if } \alpha+\beta \notin \Phi \end{cases},$$

where $\text{length}(\beta) \geq \text{length}(\alpha)$ *and* $r = \max\{s \in \mathbb{Z} : \beta - s\alpha \in \Phi\}$.

Any basis $\{h_{\alpha_i}, x_\alpha : 1 \leq i \leq l, \alpha \in \Phi\}$ of $\mathcal{L}$ with this property is called a *Chevalley basis*. It is unique up to sign changes and automorphisms of $\mathcal{L}$. From now on we fix such a basis $\mathcal{B}$. Denote by $\mathcal{L}_\mathbb{Z}$ the subset of $\mathcal{L}$ of all linear combinations of the elements of $\mathcal{B}$ with integer coefficients. By the properties of a Chevalley basis $\mathcal{L}_\mathbb{Z}$ is closed under Lie brackets. For any commutative ring R put $\mathcal{L}_R = R \otimes_\mathbb{Z} \mathcal{L}_\mathbb{Z}$. The Lie bracket of $\mathcal{L}_\mathbb{Z}$ is naturally extended to a Lie bracket of $\mathcal{L}_R$.

Proposition 7.3. *Let $S = \mathbb{Q}[t,s]$ and $T = \mathbb{Z}[t,s]$ be the polynomial rings in two variables over $\mathbb{Q}$ and $\mathbb{Z}$, respectively. Then for every $\alpha \neq -\beta \in \Phi$,*

(1) *the derivation $\text{ad}(tx_\alpha) = t\,\text{ad}(x_\alpha)$ of $\mathcal{L}_S$ is nilpotent;*
(2) *$x_\alpha(t) = \exp(t\,\text{ad}(x_\alpha)) = \sum_{i=0}^\infty \frac{t^i}{i!} \text{ad}(x_\alpha)^i$ is a well defined automorphism of $\mathcal{L}_S$ which preserves $\mathcal{L}_T$,*
(3) *there are $c_{ij}(\alpha,\beta) \in \mathbb{Z}$ such that the following equality holds:*

$$(7.4) \qquad [x_\alpha(t), x_\beta(s)] = \prod_{i,j} x_{i\alpha+j\beta}(c_{ij}(\alpha,\beta)t^i s^j),$$

where the product on the right is taken over all roots $i\alpha + j\beta \in \Phi$, with $i, j \in \mathbb{N}$, arranged in some fixed order.

Remark: If we use a different Chevalley basis, the constants $c_{ij}(\alpha,\beta)$ may change the sign. If we change the order in the product in (7.4), the constants $c_{ij}(\alpha,\beta)$ may change, although $c_{11}(\alpha,\beta)$ will not change.

Let $A_\alpha(t)$ be the matrix representing $x_\alpha(t)$ with respect to $\mathcal{B}$ (note that $\mathcal{L}_S$ is a free S-module). By Proposition 7.3(2), its coefficients are in T. Consider a

commutative ring R and let $r \in R$. We denote by $x_\alpha(r)$ the automorphism of $\mathcal{L}_R$ represented by $A_\alpha(r)$ (the matrix obtained from $A_\alpha(t)$ by replacing t by r) with respect to $\mathcal{B}$. We denote by $X_\alpha = X_\alpha(R)$ the set $\{x_\alpha(r) : r \in R\}$. This is a subgroup of $\mathrm{Aut}(\mathcal{L}_R)$ isomorphic to $(R, +)$. By Proposition 7.3(3), $\{X_\alpha\}_{\alpha \in \Phi}$ is a Φ-grading.

Definition. Let Φ be a reduced irreducible classical root system and R a commutative ring.

(a) The subgroup of $\mathrm{Aut}(\mathcal{L}_R)$ generated by $\cup_\alpha X_\alpha$ is called the *adjoint elementary Chevalley group over R corresponding to* Φ and will be denoted by $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$.

(b) The *Steinberg group* $\mathrm{St}_\Phi(R)$ is the graded cover of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ with respect to the grading $\{X_\alpha\}_{\alpha \in \Phi}$.

Remark: Elementary Chevalley groups of simply-connected type (and other non-adjoint types) can be constructed in a similar way, except that the adjoint representation of the Lie algebra $\mathcal{L}_R$ should be replaced by a different representation. The graded cover for each such group is isomorphic to $\mathrm{St}_\Phi(R)$.

The Steinberg group $\mathrm{St}_\Phi(R)$ can also be defined as the group generated by the elements $\{x_\alpha(r) : \alpha \in \Phi, r \in R\}$ subject to the following relations for every $\alpha \neq -\beta \in \Phi$ and $t, u \in R$:

$$
\begin{aligned}
x_\alpha(t) x_\alpha(u) &= x_\alpha(t + u) \\
[x_\alpha(t), x_\beta(u)] &= \prod_{i,j \in \mathbb{N}, i\alpha + j\beta \in \Phi} x_{i\alpha + j\beta}(c_{ij}(\alpha, \beta) t^i u^j).
\end{aligned}
$$

Note that while the second definition of Steinberg groups has an advantage of being explicit, the first one shows that the isomorphism class of $\mathrm{St}_\Phi(R)$ does not depend on the choice of Chevalley basis.

Remark: Note that according to our definition the Steinberg group $\mathrm{St}_{A_1}(R)$ is the free product of two copies of $(R, +)$. This definition does not coincide with the usual definition in the literature, but it is convenient for the purposes of this paper.

Remark: If we do not assume that R is commutative, then $\mathcal{L}_R$ does not have a natural structure of a Lie algebra over R, and so the above construction of $\mathrm{St}_\Phi(R)$ is not valid. However in the case $\Phi = A_n$, we can still define the Steinberg group as the graded cover of $\mathrm{EL}_{n+1}(R)$. When $\Phi \neq A_n$ we are not aware of any natural way to define the Steinberg group $\mathrm{St}_\Phi(R)$ when R is noncommutative.

The following proposition will be used frequently in the rest of the paper. It shows that some "natural" subgroups of Steinberg groups are quotients of Steinberg groups.

Definition. Let Φ be a root system. A subset Ψ of Φ is called a **weak subsystem** if $\Phi \cap (\sum_{\gamma \in \Psi} \mathbb{Z}\gamma) = \Psi$.

Proposition 7.4. *Let Φ be a reduced irreducible classical root system and Ψ an irreducible weak subsystem. Then Ψ is classical and the subgroup H of $\mathrm{St}_\Phi(R)$ generated by $\{X_\gamma : \gamma \in \Psi\}$ is a quotient of $\mathrm{St}_\Psi(R)$.*

Proof. Note that Ψ is a root system. Moreover, Ψ is classical, since if $(\cdot, \cdot)$ is an admissible scalar product on Φ, then $(\cdot, \cdot)$ restricted to Ψ is an admissible scalar product on Ψ.

Let $\mathcal{L}$ be a simple complex Lie algebra corresponding to Φ and $\{h_{\alpha_i}, x_\alpha : 1 \leq i \leq l, \alpha \in \Phi\}$ a Chevalley basis of $\mathcal{L}$. We claim that the Lie subalgebra $\mathcal{L}^\Psi$ generated by $\{x_\alpha : \alpha \in \Psi\}$ is a simple complex Lie algebra corresponding to the root system Ψ, and moreover $\{x_\alpha : \alpha \in \Psi\}$ is a part of a Chevalley basis of $\mathcal{L}^\Psi$. By Proposition 7.2, to prove this it suffices to check that

(i) the pairing $\langle \cdot, \cdot \rangle_\Psi$ on Ψ is obtained from the pairing $\langle \cdot, \cdot \rangle_\Phi$ on Φ by restriction;

(ii) If $\alpha, \beta \in \Psi$, then the value of r in relation (7.3) does not change if Φ is replaced by Ψ.

Assertion (i) is clear since the scalar product on Ψ is obtained from the scalar product on Φ by restriction. Assertion (ii) holds since Ψ is a weak subsystem and the value of r in (7.3) depends only on the structure of the $\mathbb{Z}$-lattice generated by α and β.

Thus, the values of the coefficients $c_{ij}(\alpha, \beta)$, with $\alpha, \beta \in \Psi$, do not depend on whether we consider α, β as roots of Φ or Ψ. It follows that the defining relations of $\mathrm{St}_\Psi(R)$ hold in H, so H is a quotient of $\mathrm{St}_\Psi(R)$. $\qquad\square$

Remark: In most cases a weak subsystem of a classical root system is also a subsystem, but not always. For instance, the long roots of G_2 form a weak subsystem of type A_2, but they do not form a subsystem. Note that the short roots of G_2 do not even form a weak subsystem.

Next we explicitly describe some relations in the Steinberg groups corresponding to root systems of rank 2.

Proposition 7.5. *There exists a Chevalley basis such that*

(A) *if $\Phi = A_2 = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)$, then*

$$[x_\alpha(t), x_\beta(t)] = x_{\alpha+\beta}(tu), \quad [x_{-\alpha}(t), x_{\alpha+\beta}(u)] = x_\beta(tu).$$

(B) *if $\Phi = B_2 = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\alpha + 2\beta)$, then*

$$[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(tu)x_{\alpha+2\beta}(tu^2), \quad [x_{-\alpha}(t), x_{\alpha+\beta}(u)] = x_\beta(tu)x_{\alpha+2\beta}(-tu^2),$$

$$[x_{\alpha+\beta}(t), x_\beta(u)] = x_{\alpha+2\beta}(2tu).$$

(G) *if $\Phi = G_2 = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\alpha + 2\beta), \pm(\alpha + 3\beta), \pm(2\alpha + 3\beta)\}$, then*

$$[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(tu)x_{\alpha+2\beta}(tu^2)x_{\alpha+3\beta}(tu^3), \quad [x_\alpha(t), x_{\alpha+3\beta}(u)] = x_{2\alpha+3\beta}(tu),$$

$$[x_{-\alpha}(t), x_{\alpha+\beta}(u)] = x_\beta(tu)x_{\alpha+2\beta}(-tu^2)x_{2\alpha+3\beta}(-tu^3),$$

$$[x_{-\alpha}(t), x_{2\alpha+3\beta}(u)] = x_{\alpha+3\beta}(tu),$$

$$[x_{\alpha+\beta}(t), x_\beta(u)] = x_{\alpha+2\beta}(2tu)x_{\alpha+3\beta}(3tu^2)x_{2\alpha+3\beta}(3t^2u).$$

Proof. We start with some Chevalley basis. Changing x_γ by $-x_\gamma$ for some $\gamma \in \Phi$ we may obtain a new Chevalley basis such that that

(1) if $\Phi = A_2$, then

$$[x_\alpha, x_\beta]_L = x_{\alpha+\beta}, \quad [x_{-\alpha}, x_{\alpha+\beta}]_L = x_\beta.$$

(2) if $\Phi = B_2$, then

$$[x_\alpha, x_\beta]_L = x_{\alpha+\beta}, \quad [x_{\alpha+\beta}, x_\beta]_L = 2x_{\alpha+2\beta}, \quad [x_{-\alpha}, x_{\alpha+\beta}]_L = x_\beta.$$

(3) if $\Phi = G_2$, then

$$[x_\alpha, x_\beta]_L = x_{\alpha+\beta}, \quad [x_{\alpha+\beta}, x_\beta]_L = 2x_{\alpha+2\beta}, \quad [x_{\alpha+2\beta}, x_\beta]_L = 3x_{\alpha+3\beta},$$
$$[x_\alpha, x_{\alpha+3\beta}]_L = x_{2\alpha+3\beta}, \quad [x_{\alpha+\beta}, x_{\alpha+2\beta}]_L = -2x_{2\alpha+3\beta},$$
$$[x_{-\alpha}, x_{\alpha+\beta}]_L = x_\beta, \quad [x_{-\alpha}, x_{2\alpha+3\beta}]_L = x_{\alpha+3\beta}$$

Now the result follows from straightforward calculations $\square$

Proposition 7.6. *Let Φ be a reduced irreducible classical root system of rank $l \geq 2$ and R a commutative ring, let $G = \mathrm{St}_\Phi(R)$ and $\{X_\alpha : \alpha \in \Phi\}$ the root subgroups of G. Then $\{X_\alpha : \alpha \in \Phi\}$ is a k-strong grading of G for any $2 \leq k \leq l$, and in particular, it is strong.*

Proof. Let Ψ be an irreducible subsystem of Φ. Then by Proposition 7.4, Ψ is classical, and the subgroup H generated by $\{X_\alpha : \alpha \in \Psi\}$ is a quotient of the Steinberg group $\mathrm{St}_\Psi(R)$. Thus, the grading $\{X_\alpha : \alpha \in \Psi\}$ of H is strong if the natural Ψ-grading of $\mathrm{St}_\Psi(R)$ is strong. Since Ψ is regular, using Corollary 6.3 with $k = 2$, we deduce that it is enough to prove Proposition 7.6 when $l = k = 2$. In this case the result easily follows from Proposition 7.5. We illustrate this for $\Phi = G_2$.

Consider a functional f and let $\{\alpha, \beta\}$ be a base on which f takes positive values. Then the core C_f is equal to $\{\alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}$. Using the relations described in Proposition 7.5 for the case $\Phi = G_2$, we have

$$X_{\alpha+\beta} \quad \subseteq \quad [X_\alpha, X_\beta] X_{\alpha+2\beta} X_{\alpha+3\beta},$$

$$X_{\alpha+2\beta} \quad \subseteq \quad [X_\alpha, X_\beta] X_{\alpha+\beta} X_{\alpha+3\beta},$$

$$X_{\alpha+3\beta} \quad \subseteq \quad [X_\alpha, X_\beta] X_{\alpha+\beta} X_{\alpha+2\beta},$$

$$X_{2\alpha+3\beta} \quad \subseteq \quad [X_\alpha, X_{\alpha+3\beta}].$$

Hence the grading $\{X_\alpha\}_{\alpha \in G_2}$ is strong. $\square$

7.3. Standard sets of generators of Steinberg groups.

Let R be a commutative ring generated by $T = \{t_0 = 1, t_1, \ldots, t_d\}$. We denote by T^* the set

$$\{t_{i_1} \cdots t_{i_k} : 0 \leq i_1 < \ldots < i_k \leq d\}.$$

In the following proposition we describe a set of generators of $\mathrm{St}_\Phi(R)$ that we will call *standard*.

Proposition 7.7. *Let Φ be a reduced irreducible classical root system of rank at least 2 and R a commutative ring generated by $T = \{t_0 = 1, t_1, \ldots, t_d\}$. Let $\Sigma = \Sigma_\Phi(T)$ be the following set:*

(1) *if $\Phi = A_n, B_n (n \geq 3), D_n, E_6, E_7, E_8, F_4$,*

$$\Sigma = \{x_\alpha(t) : \ \alpha \in \Phi, \ t \in T\},$$

(2) *if $\Phi = B_2, C_n$,*

$$\Sigma = \left\{ \begin{array}{ll} x_\alpha(t), \ t \in T & \alpha \in \Phi \text{ is a short root} \\ x_\alpha(t), \ t \in T^* & \alpha \in \Phi \text{ is a long root} \end{array} \right\},$$

(3) *if $\Phi = G_2$,*

$$\Sigma = \left\{ \begin{array}{ll} x_\alpha(t), \ t \in T & \alpha \in \Phi \text{ is a long root} \\ x_\alpha(t), \ t \in T^* & \alpha \in \Phi \text{ is a short root} \end{array} \right\}.$$

Then Σ generates $\mathrm{St}_\Phi(R)$.

Proof. First we consider the case $\Phi = A_n, B_n (n \geq 3), D_n, E_6, E_7, E_8, F_4$. We prove by induction on k that for any $\gamma \in \Phi$ and any monomial m in variables from T of degree k, the element $x_\gamma(m)$ lies in the subgroup generated by Σ. This statement clearly implies the proposition.

The base of induction is clear. Assume that the statement is true for monomials of degree $\leq k$. Let m be a monomial of degree $k+1$.

If $\Phi = A_n, D_n, E_6, E_7, E_8, F_4$, then we can find a subsystem Ψ of Φ isomorphic to A_2 which contains γ. We write $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1, \gamma_2 \in \Psi$, and $m = m_1 m_2$, where m_1, m_2 are monomials of degree $\leq k$. Then $x_\gamma(m) = [x_{\gamma_1}(m_1), x_{\gamma_2}(m_2)]^{\pm 1}$ and we can apply the inductive hypothesis.

If $\Phi = B_n, n \geq 3$, then any long root lies in an irreducible subsystem isomorphic to A_2, whence the statement holds when γ is a long root. Assume γ is a short root. Then there are a long root α and a short root β such that $\gamma = \alpha + \beta$. Note that α and β generate a subsystem of type B_2. Without loss of generality we may assume that the relations of Proposition 7.5 hold. Then we obtain that

$$x_\gamma(m) = x_{\alpha+\beta}(m) = [x_\alpha(m), x_\beta(1)]x_{\alpha+2\beta}(-m).$$

Since the roots α and $\alpha + 2\beta$ are long, by induction $x_\gamma(m)$ lies in the subgroup generated by Σ.

In the case $\Phi = B_2$ the proposition is an easy consequence of the following lemma (we do not need the second part of this lemma now; it will be used later).

Lemma 7.8. *Consider the semidirect product* $\mathrm{St}_{A_1}(R) \ltimes N$, *where N is the subgroup of* $\mathrm{St}_{B_2}(R)$ *generated by* $X_\beta, X_{\alpha+\beta}, X_{\alpha+2\beta}$ *and we view* $\mathrm{St}_{A_1}(R)$ *as the free product of* X_α *and* $X_{-\alpha}$ *with corresponding action on N. Let*

$$S_1 = \{x_\alpha(t), x_{-\alpha}(t) : \ t \in T^* \cup T^2\}; \quad S_2 = \{x_\beta(t), x_{\alpha+\beta}(t) : \ t \in T\} \ and$$
$$S_3 = \{x_{\alpha+2\beta}(t) : \ t \in T^*\}.$$

Let G be the subgroup generated by the set $S = S_1 \cup S_2 \cup S_3$. Then the following hold:

(1) *G contains N;*
(2) *$X_{\alpha+2\beta}/([N,G] \cap X_{\alpha+2\beta})$ is of exponent 2 and generated by S_3.*

Proof. Without loss of generality we may assume that the relations from Proposition 7.5 hold in $\mathrm{St}_{B_2}(R)$.

We prove by induction on k that for any $\gamma \in \{\alpha+\beta, \alpha+2\beta, \beta\}$ and any monomial m in variables from T of degree k, the element $x_\gamma(m)$ lies in the subgroup generated by S. This clearly implies the first statement.

The base of induction is clear. Assume that the statement holds for all monomials of degree $\leq k$. Let m be a monomial of degree $k+1$.

Case 1: $\gamma = \alpha + 2\beta$. If $m \in T^*$, then $x_{\alpha+2\beta}(m) \in S_3$. If $m \notin T^*$, we can write $m = m_1 m_2^2$ with $m_1 \in T^*$ and $m_2 \neq 1$, and we obtain that

$$x_{\alpha+2\beta}(m_1 m_2^2) = [x_\alpha(m_1), x_\beta(m_2)]x_{\alpha+\beta}(-m_1 m_2).$$

Thus, by induction, $x_{\alpha+2\beta}(m) \in G$.

Case 2: $\gamma = \alpha + \beta$. If $m \in T^*$, then

$$x_{\alpha+\beta}(m) = [x_\alpha(m), x_\beta(1)]x_{\alpha+2\beta}(-m)$$

and we are done. If $m \notin T^*$, we write $m = t^2 m_1$, where $t \in T \setminus \{1\}$. Then we obtain that

$$x_{\alpha+\beta}(m) = [x_\alpha(t^2), x_\beta(m_1)][x_\alpha(1), x_\beta(-tm_1)]x_{\alpha+\beta}(tm_1) \in G.$$

Case 3: $\gamma = \beta$. This case is analogous to Case 2.

The proof of the second part is an easy exercise based on Proposition 7.5. $\square$

We now go back to the proof of Proposition 7.7. In the case $\Phi = B_2$ the result follows from Lemma 7.8(a) since for any $t \in T$ and long root $\gamma \in B_2$, the element $x_\gamma(t^2)$ can be expressed as a product of elements from Σ. For instance, in the case $\gamma = \alpha + 2\beta$ we have

$$(7.5) \qquad x_{\alpha+2\beta}(t^2) = [x_\alpha(1), x_\beta(t)]x_{\alpha+\beta}(-t).$$

If $\Phi = C_n$, then any root lies in a subsystem of type B_2, and so the result follows from the previous case.

Finally, consider the case $\Phi = G_2$. The proof that the long root subgroups lie in the subgroup generated by Σ is as in the case of A_2, because the long roots of G_2 form a weak subsystem of type A_2 and we can use Proposition 7.4. Arguing similarly to the case of B_2 we obtain that the short root subgroups also lie in the subgroup generated by Σ. $\square$

7.4. Property (T) for the Steinberg groups.

In this subsection we establish property (T) for the Steinberg groups (of rank ≥ 2) over finitely generated rings and obtain asymptotic lower bounds for the Kazhdan constants. It will be convenient to use the following notation.

If κ is some quantity depending on $\mathbb{N}$-valued parameters $n_1, \ldots, n_r$ (and possibly some other parameters) and $f : \mathbb{N}^r \to \mathbb{R}_{>0}$ is a function, we will write

$$\kappa \succcurlyeq f(n_1, \ldots, n_r)$$

if there exists an absolute constant $C > 0$ such that $\kappa \geq Cf(n_1, \ldots, n_r)$ for all $n_1, \ldots, n_r \in \mathbb{N}$.

Let Φ be a reduced irreducible classical root system of rank ≥ 2 and R a finitely generated ring (which is commutative if Φ is not of type A). By Proposition 7.6 the standard grading $\{X_\alpha\}$ of $\mathrm{St}_\Phi(R)$ is strong, so to prove property (T) it suffices to check relative property (T) for each of the pairs $(\mathrm{St}_\Phi(R), X_\alpha)$. However, in order to obtain a good bound for the Kazhdan constant of $\mathrm{St}_\Phi(R)$ with respect to a finite generating set of the form $\Sigma_\Phi(T)$ (as defined in Proposition 7.7), we need to proceed slightly differently.

We shall use a good reduction of Φ to a root system of rank 2 (of type A_2, B_2, BC_2 or G_2) described in § 6.2. Proposition 7.6 implies that the coarsened grading $\{Y_\beta\}$ of $\mathrm{St}_\Phi(R)$ is also strong, so Theorem 5.1 can be applied to this grading, this time yielding a much better Kazhdan constant.

To complete the proof of property (T) for $\mathrm{St}_\Phi(R)$ we still need to establish relative property (T), this time for the pairs $(\mathrm{St}_\Phi(R), Y_\beta)$. Qualitatively, this is not any harder than proving relative (T) for $(\mathrm{St}_\Phi(R), X_\alpha)$; however, we also need to explicitly estimate the corresponding Kazhdan ratios (which will affect the eventual bound for the Kazhdan constant of $\mathrm{St}_\Phi(R)$ with respect to a finite generating set).

Terminology: For brevity, in the sequel instead of saying *relative property* (T) *for the pair* (G, H) we will often say *relative property* (T) *for* H if G is clear from the context.

Our main tool for proving relative property (T) is the following result of Kassabov [Ka1] which generalizes Theorem 2.3:

Proposition 7.9 (Kassabov). *Let $n \geq 2$ and R a ring generated by $T = \{1 = t_0, t_1, \ldots, t_d\}$. Let $\{\alpha_1, \cdots, \alpha_n\}$ be a system of simple roots of A_n. Consider the semidirect product $G = \mathrm{St}_{A_{n-1}}(R) \ltimes N$, where $N \cong R^n$ is the subgroups of $\mathrm{St}_{A_n}(R)$ generated by $X_{\alpha_1}, X_{\alpha_1+\alpha_2}, \ldots, X_{\alpha_1+\cdots+\alpha_n}$ and the action of $\mathrm{St}_{A_{n-1}}(R)$ on N comes from the action of the subgroup $\langle X_\gamma : \gamma \in \mathbb{R}\alpha_2 + \ldots + \mathbb{R}\alpha_n \rangle$ of $\mathrm{St}_{A_n}(R)$ on N (we will refer to this action as the standard action of $\mathrm{St}_{A_{n-1}}(R)$ on R^n). Let $S = \Sigma_{A_{n-1}}(T) \cup (\Sigma_{A_n}(T) \cap N)$ and let $G' = \langle S, N \rangle$. Then*

$$\kappa_r(G', N; S) \succcurlyeq \frac{1}{\sqrt{d+n}}.$$

Remark: Note that if $n \geq 3$ we have $G' = G$.

Combining Proposition 7.9 and Theorem 2.7, we obtain the following result which immediately implies relative property (T) for the root subgroups of $\mathrm{St}_{B_2}(R)$.

Corollary 7.10. *Let $\{\alpha, \beta\}$ be a system of simple roots of B_2, with α a long root. Consider the semidirect product $\mathrm{St}_{A_1}(R) \ltimes N$, where N is the subgroup generated by $X_\beta, X_{\alpha+\beta}, X_{\alpha+2\beta}$ and we view $\mathrm{St}_{A_1}(R)$ as the free product of X_α and $X_{-\alpha}$ with the corresponding action on N. Let*

$$S = \{x_\alpha(t_1), x_{-\alpha}(t_1), x_\beta(t_2), x_{\alpha+\beta}(t_2), x_{\alpha+2\beta}(t_3) : t_1 \in T^* \cup T^2, \ t_2 \in T, t_3 \in T^*\}.$$

Then

$$\kappa(\mathrm{St}_{A_1}(R) \ltimes N, N; S) \succcurlyeq \frac{1}{2^{d/2}}.$$

Proof. Let $A = \{x_\alpha(t), x_{-\alpha}(t) : t \in T^* \cup T^2\}$, $B = \{x_\alpha(t), x_{-\alpha}(t), x_\beta(t), x_{\alpha+\beta}(t) : t \in T\}$ and let $C = \{x_{\alpha+2\beta}(t) : t \in T^*,\}$. Let G be the subgroup of $\mathrm{St}_{A_1}(R) \ltimes N$ generated by $S = A \cup B \cup C$.

By Lemma 7.8(1), N is a subgroup of G. Let $Z = X_{\alpha+2\beta}$. The relations for $\mathrm{St}_{B_2}(R)$, described in Propositions 7.5, imply that $(\mathrm{St}_{A_1}(R) \ltimes N)/Z \cong \mathrm{St}_{A_1}(R) \ltimes (N/Z)$ is isomorphic to $\mathrm{St}_{A_1}(R) \ltimes R^2$ (with the standard action of $\mathrm{St}_{A_1}(R)$ on R^2), and the image of G/Z under this isomorphism contains the subgroup denoted by G' in Proposition 7.9. Hence $\kappa(G/Z, N/Z; B) \succcurlyeq \frac{1}{\sqrt{d}}$.

Now let $H = Z \cap [N, G]$. By Lemma 7.8(2), Z/H is an elementary abelian 2-group generated by C, whence $\kappa(G/H, Z/H; C) \succcurlyeq \frac{1}{\sqrt{|C|}} \succcurlyeq \frac{1}{2^{d/2}}$. Since $Z \subseteq Z(G) \cap N$ and AN generates G/N, by Theorem 2.7 we have $\kappa(G, N; S) \succcurlyeq \frac{1}{2^{d/2}}$. Since G is a subgroup of $\mathrm{St}_{A_1}(R) \ltimes N$, we are done. $\square$

Before turning to the case-by-case verification of relative property (T) (which is the main part of the proof of Theorem 7.11 below), we briefly summarize how this will be done for different root systems. Let Φ be a reduced irreducible classical root system with $rk(\Phi) \geq 2$.

(1) If Φ is simply-laced, relative property (T) for the root subgroups of $\mathrm{St}_\Phi(R)$ will follow almost immediately from Proposition 7.9 (with the aid of Proposition 7.4).

(2) If Φ is non-simply-laced and $\Phi \neq B_2$, relative property (T) for some of the root subgroups (either short ones or long ones) will again follow from Proposition 7.9. To prove relative property (T) for the remaining root subgroups, we will show that each of them is contained in a bounded product of a finite set and root subgroups for which relative (T) has already been established.

(3) Finally, the most difficult case $\Phi = B_2$ has already been established in Corollary 7.10.

Explicit bounds for the Kazhdan constants and Kazhdan ratios will follow form Observation 2.2 and Lemma 2.4; these results will be often used without further mention.

We are now ready to prove the main result of this section.

Theorem 7.11. *Let Φ be a reduced irreducible classical root system of rank at least 2 and R a ring (which is commutative if Φ is not of type A) generated by a finite set $T = \{1 = t_0, t_1, \ldots, t_d\}$. Let Σ be the corresponding standard generating set of $\mathrm{St}_\Phi(R)$. Then*

$$\kappa(\mathrm{St}_\Phi(R), \Sigma) \succcurlyeq \mathcal{K}(\Phi, d)$$

where

$$\mathcal{K}(\Phi, d) = \begin{cases} \frac{1}{\sqrt{n+d}} & if \ \Phi = A_n, B_n(n \geq 3), D_n \\ \frac{1}{\sqrt{d}} & if \ \Phi = E_6, E_7, E_8, F_4 \\ \frac{1}{2^{d/2}} & if \ \Phi = B_2, G_2 \\ \frac{1}{\sqrt{n+2^d}} & if \ \Phi = C_n \end{cases}$$

Proof. Let $G = \mathrm{St}_\Phi(R)$. We will show using case-by-case analysis that there exists a root system Φ' of type A_2, B_2, BC_2 or G_2 and a strong Φ'-grading $\{Y_\beta\}$ of G such that

$$\kappa_r(G, \cup Y_\beta; \Sigma) \succcurlyeq \mathcal{K}(\Phi, d).$$

This will imply the assertion of the theorem since $\kappa(G, \Sigma) \geq \kappa(G, \cup Y_\beta) \kappa_r(G, \cup Y_\beta; \Sigma)$ and $\kappa(G, \cup Y_\beta) \geq C$ for some absolute constant $C > 0$ by Theorem 5.1.

Case $\Phi = A_n$. This case is covered by [EJ]; however, we include the argument for completeness.

We use the reduction of A_n to A_2 given by the map

$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, x_2, \sum_{i \geq 3} x_i).$$

The coarsened grading is strong by Reduction 6.6, Lemma 6.2 and Proposition 7.6. The new root subgroups are

$$Y_{(-1,1,0)} = X_{e_2-e_1}, \quad Y_{(1,0,-1)} = \prod_{i=3}^{n+1} X_{e_1-e_i}, \quad Y_{(-1,0,1)} = \prod_{i=3}^{n+1} X_{e_i-e_1},$$

$$Y_{(1,-1,0)} = X_{e_1-e_2}, \quad Y_{(0,1,-1)} = \prod_{i=3}^{n+1} X_{e_2-e_i}, \quad \text{and} \quad Y_{(0,-1,1)} = \prod_{i=3}^{n+1} X_{e_i-e_2}.$$

We shall prove that $\kappa_r(G, Y_{(1,0,-1)}; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}$; the other cases are similar. The roots $\{e_i - e_j : 2 \leq i \neq j \leq n+1\}$ form a subsystem of type A_{n-1}. Thus, if H is

the subgroup generated by $\{X_{e_i - e_j} : 2 \leq i \neq j \leq n+1\}$ and

$$E = \prod_{i=2}^{n+1} X_{e_1 - e_i} \supset Y_{(1,0,-1)},$$

then by Proposition 7.4 there is a natural epimorphism $\mathrm{St}_{A_{n-1}}(R) \ltimes R^n \to H \ltimes E$. Hence using Proposition 7.9 we have

$$\kappa_r(\mathrm{St}_{A_n}(R), Y_{(1,0,-1)}; \Sigma) \geq \kappa_r(H \ltimes E, E; \Sigma) \geq \kappa_r(\mathrm{St}_{A_{n-1}}(R) \ltimes R^n, R^n; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}.$$

Case $\Phi = B_n$, $n \geq 3$. We use the reduction of B_n to B_2 given by the map

$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, x_2).$$

The coarsened grading is strong by Reduction 6.7, Lemma 6.2 and Proposition 7.6. The new root subgroups are

$$Y_{(\pm 1, 0)} = X_{\pm e_1} \prod_{i=3}^{n} X_{\pm e_1 - e_i} \prod_{i=3}^{n} X_{\pm e_1 + e_i}, \quad Y_{(\pm 1, \pm 1)} = X_{\pm e_1 \pm e_2},$$

$$Y_{(0, \pm 1)} = X_{\pm e_2} \prod_{i=3}^{n} X_{\pm e_2 - e_i} \prod X_{\pm e_2 + e_i}.$$

We shall prove that $\kappa_r(G, Y_{(1,0)}; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}$; the other cases are similar. Since $\{e_i - e_j : 1 \leq i \neq j \leq n\}$ form a subsystem of type A_{n-1}, so arguing as in the previous case, we obtain that $\kappa_r(\mathrm{St}_{B_n}(R), \prod_{i=3}^{n} X_{e_1 - e_i}; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}$. The same argument applies to $\kappa_r(\mathrm{St}_{B_n}(R), \prod_{i=3}^{n} X_{e_1 + e_i}; \Sigma)$. It remains to show that $\kappa_r(\mathrm{St}_{B_n}(R), X_{e_1}; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$.

By the same argument as above, $\kappa_r(G, X_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$ for any long root γ. From Proposition 7.5 it follows that

$$X_{e_1} \subseteq [x_{e_2}(1), X_{e_1 - e_2}] X_{e_1 + e_2} \subseteq \Sigma X_{e_1 - e_2} \Sigma X_{e_1 - e_2} X_{e_1 + e_2}.$$

Therefore $\kappa_r(G, X_{e_1}; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$ by Lemma 2.4.

Case $\Phi = D_n$, $n \geq 3$. We use the reduction of D_n to B_2 given by the map

$$(x_1, \ldots, x_n) \mapsto (x_1, x_2).$$

The coarsened grading is strong by Reduction 6.8, Lemma 6.2 and Proposition 7.6. The new root subgroups are

$$Y_{(\pm 1, 0)} = \prod_{i=3}^{n} X_{\pm e_1 - e_i} \prod_{i=3}^{n} X_{\pm e_1 + e_i}, \quad Y_{(\pm 1, \pm 1)} = X_{\pm e_1 \pm e_2},$$

$$Y_{(0, \pm 1)} = \prod_{i=3}^{n} X_{\pm e_2 - e_i} \prod_{i=3}^{n} X_{\pm e_2 + e_i}.$$

The proof of relative property (T) is the same as in the case of B_n.

Case $\Phi = F_4$, E_6, E_7 *or* E_8. In this case we do not have to do any reduction to a root system of bounded rank, since the rank is already bounded. The grading is strong by Proposition 7.6. However, if one wants to obtain explicit estimates for the Kazhdan constants, one can use Reductions 6.12 and 6.13 (again the coarsened gradings are strong from Lemma 6.2 and Proposition 7.6).

In order to prove that $\kappa_r(\mathrm{St}_\Phi(R), X_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$ for any root $\gamma \in \Phi$, we simply observe that γ lies in a subsystem of type A_2, and the same argument as in the case A_2 can be applied.

Case $\Phi = B_2$. The grading is strong by Proposition 7.6. We only have to show that $\kappa_r(\mathrm{St}_{B_2}(R), \cup_\gamma X_\gamma; \Sigma) \succcurlyeq \frac{1}{2^{d/2}}$, which is a direct consequence of (7.5) and Corollary 7.10.

Case $\Phi = C_n$, $n \geq 3$. We use the reduction of C_n to BC_2 given by the map

$$(x_1, \ldots, x_n) \mapsto (x_1, x_2).$$

The coarsened grading is strong by Reduction 6.9, Lemma 6.2 and Proposition 7.6. The new root subgroups are

$$Y_{(\pm 1, 0)} = \langle \prod_{i=3}^n X_{\pm e_1 - e_i}, \quad \prod_{i=3}^n X_{\pm e_1 + e_i} \rangle, \quad Y_{(\pm 1, \pm 1)} = X_{\pm e_1 \pm e_2}, \quad Y_{(\pm 2, 0)} = X_{\pm 2e_1},$$

$$Y_{(0, \pm 1)} = \langle \prod_{i=3}^n X_{\pm e_2 - e_i}, \prod_{i=3}^n X_{\pm e_2 + e_i} \rangle \quad \text{and} \quad Y_{(0, \pm 2)} = X_{\pm 2e_2}.$$

Note that $Y_{(1,0)} \neq AB$ for $A = \prod_{i=3}^n X_{\pm e_1 - e_i}$ and $B = \prod_{i=3}^n X_{\pm e_1 + e_i}$; however, it is easy to see that $Y_{(1,0)} = ABABAB$. Similar factorization exists for other short root subgroups.

Arguing as in the case $\Phi = B_n, n \geq 3$, we conclude that $\kappa_r(\mathrm{St}_\Phi(R), Y_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}$ when γ is a short or a long root. If γ is a double root, then γ lies in a weak subsystem of type B_2. Thus, from Corollary 7.10 and Proposition 7.4 we obtain that $\kappa_r(\mathrm{St}_\Phi(R), Y_\gamma; \Sigma) \succcurlyeq \frac{1}{2^{d/2}}$. Hence $\kappa_r(\mathrm{St}_\Phi(R), \cup Y_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+2^d}}$.

Case $\Phi = G_2$. The grading is strong by Proposition 7.6. If γ is a long root, then $\kappa_r(\mathrm{St}_{G_2}(R), X_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$, because the long roots form a weak subsystem of type A_2.

Now, we will show that if γ is a short root, then $\kappa_r(\mathrm{St}_{G_2}(R), X_\gamma; \Sigma) \succcurlyeq \frac{1}{2^{d/2}}$. Without loss of generality we may assume that the relations from Proposition 7.5 for $\mathrm{St}_{G_2}(R)$ hold and $\gamma = \alpha + 2\beta$.

Calculating $[x_\alpha(r), x_\beta(2)][x_\alpha(2r), x_\beta(1)]^{-1}$ we see that

$$X_\gamma(2R) = \{x_\gamma(2r) : \ r \in R\} \subseteq X_\alpha^{x_\beta(2)} X_\alpha X_\alpha^{x_\beta(1)} X_{\alpha+3\beta} X_{2\alpha+3\beta},$$

so $X_\gamma(2R)$ lies inside a bounded product of long root subgroups and 2 fixed elements of Σ.

Similarly, calculating $[x_\alpha(r), x_\beta(t)][x_\alpha(tr), x_\beta(1)]^{-1}$ for any $1 \neq t \in T$ we obtain that $X_\gamma((t^2 - t)R) = \{x_\gamma((t^2 - t)r) : \ r \in R\}$ lies inside a bounded product of long root subgroups and 2 fixed elements of Σ.

Let $I = 2R + \sum_{t \in T}(t^2 - t)R$. Since $|T| = d + 1$, using our previous observations and Lemma 2.4(b), we conclude that

$$\kappa_r(\mathrm{St}_{G_2}(R), X_\gamma(I); \Sigma) \succcurlyeq \frac{1}{d\sqrt{d}}.$$

The group $X_\gamma(R)/X_\gamma(I)$ is an elementary abelian 2-group generated by $S = \{x_\gamma(t) : \ t \in T^*\}$. Since $|S| = 2^{d+1}$, we have

$$\kappa(X_\gamma(R)/X_\gamma(I), S) \succcurlyeq \frac{1}{2^{d/2}}.$$

By Lemma 2.9, these two inequalities imply that $\kappa_r(\mathrm{St}_{G_2}(R), X_\gamma(R); \Sigma) \succcurlyeq \frac{1}{2^{d/2}}$.

$\square$

8. Twisted Steinberg groups

8.1. **Constructing twisted groups.** In this subsection we introduce a general method for constructing new groups graded by root systems from old ones using the machinery of twists. The method generalizes the construction of twisted Chevalley groups [St].

Let Φ be a root system, G a group and $\{X_\alpha\}_{\alpha \in \Phi}$ a Φ-grading of G. Let $Q \subset \mathrm{Aut}(G)$ be a group of automorphisms of G such that

  (i) Each element of Q is a graded automorphism (as defined in § 7.1), so that there is an induced action of Q on Φ.
  (ii) Q acts linearly on Φ, that is, if $\sum \lambda_i \alpha_i = 0$ for some $\lambda_i \in \mathbb{R}$ and $\alpha_i \in \Phi$, then $\sum \lambda_i q(\alpha_i) = 0$ for any $q \in Q$.

Remark: In all our applications Q will be a finite group (in fact, usually a cyclic group).

Let $V = \mathbb{R}\Phi$ be the $\mathbb{R}$-vector space spanned by Φ. Suppose we are given another $\mathbb{R}$-vector space W and a reduction $\eta : V \to W$ such that

(8.1) $\eta(q\alpha) = \eta(\alpha)$ for any $\alpha \in \Phi$ and $q \in Q$.

Let $\Psi = \eta(\Phi) \setminus \{0\}$ be the induced root system and let $\{Y_\beta\}_{\beta \in \Psi}$ be the coarsened grading, that is, $Y_\beta = \langle X_\alpha : \eta(\alpha) = \beta \rangle$. Finally, let $Z_\beta = Y_\beta^Q$ be the set of Q-fixed points in Y_β, and assume that the following additional condition holds:

  (iii) $\{Z_\alpha\}_{\alpha \in \Psi}$ is a Ψ-grading.

Then we define the **twisted group** $\widehat{G^Q}$ to be the graded cover of $\langle Z_\alpha : \alpha \in \Psi \rangle$ with respect to the Ψ-grading $\{Z_\alpha\}_{\alpha \in \Psi}$.

Here is a slightly technical but easy-to-use criterion which ensures that condition (iii) holds. This criterion will be applicable in all of our examples.

Proposition 8.1. *Let Ψ' be the set of all roots in Ψ which are not representable as $a\gamma$ for some other $\gamma \in \Psi$ and $a > 1$ (in particular, $\Psi' = \Psi$ if Ψ is reduced). Assume that*

  (a) *For any $\gamma, \delta \in \Psi$ such that $\delta = a\gamma$ with $a \geq 1$, we have $Y_\delta \subseteq Y_\gamma$.*
  (b) *For any Borel subset B of Ψ, any element of $\langle Y_\gamma \rangle_{\gamma \in B}$ can be **uniquely** written as $\prod_{i=1}^k y_{\gamma_i}$ where $\gamma_1, \ldots, \gamma_k$ are the roots in $B \cap \Psi'$ taken in some fixed order and $y_{\gamma_i} \in Y_{\gamma_i}$ for all i.*

Then $\{Z_\alpha\}_{\alpha \in \Psi}$ is a Ψ-grading.

Proof. Let $\alpha, \beta \in \Psi$ such that $\beta \notin \mathbb{R}_{<0}\alpha$, in which case there exists a Borel subset B containing both α and β. Let $\Omega = \{\gamma \in \Psi : \gamma = a\alpha + b\beta \text{ with } \alpha, \beta \geq 1\}$, and let $\gamma_1, \ldots, \gamma_k$ be the roots in $B \cap \Psi'$ (as in condition (b)). Let I be the set of all $i \in \{1, \ldots, k\}$ such that $\mathbb{R}_{\geq 1}\gamma_i \cap \Omega \neq \emptyset$. For each $i \in I$ let $\delta_i = a_i\gamma_i$ such that $\delta_i \in \Omega$ and $a_i \in \mathbb{R}_{\geq 1}$ is smallest possible.

Now take any $z \in Z_\alpha$ and $w \in Z_\beta$. Since $\{Y_\gamma\}_{\gamma \in \Psi}$ is a Ψ-grading, using condition (a) it is easy to show that $[z, w] = \prod_{i \in I} y_{\delta_i}$ where $y_{\delta_i} \in Y_{\delta_i}$ for each i. Since z and w are fixed by Q, for any $q \in Q$ we have $[z, w] = \prod_{i \in I} q(y_{\delta_i})$.

Thus, we have obtained two factorizations for $[z, w]$, and since $Y_{\delta_i} \subseteq Y_{\gamma_i}$, they both satisfy the requirement in (b). Therefore, (b) implies that $q(y_{\delta_i}) = y_{\delta_i}$ for each i. Hence $y_{\delta_i} \in Y_{\delta_i}^Q = Z_{\delta_i}$ for each $i \in I$, and so $[z, w] \in \langle Z_\gamma : \gamma \in \Omega \rangle$. $\qquad \square$

If Q is finite, one always has a natural choice for the pair (W, η) satisfying (8.1). Indeed, by condition (ii) the action of Q on Φ extends to a linear action of Q on V. Then we can take $W = V^Q$, the subspace of Q-invariant vectors and $\eta : V \to W$ the natural projection, that is,

$$(8.2) \qquad \eta(v) = \frac{1}{|Q|} \sum_{q \in Q} qv.$$

In fact, in all our examples the pair (W, η) will be of this form up to isomorphism, but it will be more convenient to define W and η first and then check (8.1) rather than realize W as the subspace of Q-invariant vectors in V.

Remark: If Q and Q' are conjugate in the group $\mathrm{Aut}_{gr}(G)$ of graded automorphisms of G, then the corresponding twisted groups $\widehat{G^Q}$ and $\widehat{G^{Q'}}$ are easily seen to be isomorphic.

We will discuss in detail six families of twisted groups. The first five families will all be of the following form (while the construction of the sixth family will involve minor modifications). Let Φ be classical, reduced and irreducible of rank ≥ 2. We take $G = St_\Phi(R)$ for some ring R, which is commutative if Φ is not of type A. The acting group Q will be a finite (usually cyclic) subgroup of $\mathrm{Aut}(G)$ whose elements are compositions of diagram, ring and diagonal automorphisms (as defined in § 8.2).

The latter restriction on Q implies that it naturally acts on the corresponding adjoint elementary Chevalley group $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$. Let $\mathbb{E}_\Phi^{\mathrm{ad}}(R)^Q$ be the subgroup of Q-fixed points of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$, and let $\widetilde{\mathbb{E}_\Phi^{\mathrm{ad}}(R)^Q}$ be the group generated by the intersections of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)^Q$ with the root subgroups of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$.

Thus, if $\widehat{St_\Phi(R)^Q}$ is the twisted group obtained via the above procedure, there is a natural epimorphism from $\widehat{St_\Phi(R)^Q}$ onto $\widetilde{\mathbb{E}_\Phi^{\mathrm{ad}}(R)^Q}$. We will refer to $\widehat{St_\Phi(R)^Q}$ as a *twisted Steinberg group*, and to $\widetilde{\mathbb{E}_\Phi^{\mathrm{ad}}(R)^Q}$ as a *twisted Chevalley group*, and we will also say that $\widehat{St_\Phi(R)^Q}$ is a *Steinberg cover* of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)^Q$.

We note that the term 'twisted Chevalley group' usually has a more restricted meaning – instead of all possible finite groups of automorphisms Q as above, one only considers those which are used in (canonical) realizations of finite simple groups of twisted Lie type. In this section we will mostly deal with the Steinberg covers for these types of twisted Chevalley groups. The obtained Steinberg groups are summarized below and will be studied in Examples 1-5.

1. Groups $St_{C_n}^\omega(R, *)$ where R is a ring, $*$ is an involution on R and ω is a central unit of R satisfying $\omega^* = \omega^{-1}$. These groups are Steinberg covers for *hyperbolic unitary groups* (see [HO]). The special case $\omega = 1$ corresponds to twisted Chevalley groups of type ${}^2A_{2n-1}$ (unitary groups in even dimension).

2. Groups $St_{BC_n}(R, *)$ where R is a ring and $*$ is an involution on R. These groups are Steinberg covers for twisted Chevalley groups of type ${}^2A_{2n}$ (unitary groups in odd dimension).

3. Groups $St_{B_n}(R, \sigma)$ where R is a commutative ring and σ is an involution on R. These groups are Steinberg covers for twisted Chevalley groups of type ^2D_n.

4. Groups $St_{G_2}(R, \sigma)$ where R is a commutative ring and σ an automorphism of R of order 3. These groups are Steinberg covers for twisted Chevalley groups of type $^3D_4$.

5. Groups $St_{F_4}(R, \sigma)$ where R is a commutative ring and σ an involution of R. These groups are Steinberg covers for twisted Chevalley groups of type $^2E_6$.

Remark: Our notations for the twisted Steinberg groups are chosen in such a way that the subscript indicates the root system by which this twisted Steinberg group is naturally graded.

In Example 3 we shall define a more general family of twisted groups (which will include the groups $St_{C_n}(R, \sigma)$ as a special case) using an observation that the (classical) Steinberg group $St_{D_n}(R)$ arises as the twisted Steinberg group $St_{C_n}^{-1}(R, *)$ (from Example 1) in the special case when $*$ is trivial (and R is commutative).

In the last example (Example 6) we construct certain groups $St_{{}^2F_4}(R, *)$, where R is a commutative ring of characteristic 2 and $* : R \to R$ is an injective homomorphism such that $(r^*)^* = r^2$. These groups are graded by a root system in $\mathbb{R}^2$ with 24 roots and can be defined as graded covers of certain "algebraic-like" groups constructed by Tits [Ti]. We note that the standard definition of twisted Chevalley groups of type $^2F_4$ is only valid when R is a perfect field, in which case they coincide with Tits' groups. Unlike Examples 1-5, in the construction of the groups $St_{{}^2F_4}(R, *)$, the initial group G will not be the entire Steinberg group $St_{F_4}(R)$, but certain subgroup of it. The general twisting procedure will also be slightly modified in this example, as we will have to apply the *fattening* operation (defined in § 4.4) to the coarsened grading $\{Y_\beta\}$.

8.2. **Graded automorphisms of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ and $St_\Phi(R)$.** In this section we describe some natural families of graded automorphisms of (non-twisted) adjoint elementary Chevalley groups and Steinberg groups. Each automorphism will be defined via its action on the root subgroups, and we will need to justify that it can be extended to the entire group.

In the case of elementary Chevalley groups and Steinberg groups over commutative rings the following observation will provide the justification:

Observation 8.2. *Let Φ be a reduced irreducible classical root system, R a commutative ring and $\mathcal{L}_R$ the R-Lie algebra of type Φ, as defined in Section 7. Let $f \in \mathrm{Aut}(\mathcal{L}_R)$ be an automorphism which permutes the root subspaces of $\mathcal{L}_R$. Then $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$, considered as a subgroup of $\mathrm{Aut}(\mathcal{L}_R)$, is normalized by f, and moreover, the conjugation by f permutes the root subgroups $\{X_\alpha(R)\}$ of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$. Thus f naturally induces a graded automorphism of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ and hence also induces a graded automorphism of $St_\Phi(R)$ by Lemma 7.1.*

If $G = St_{A_m}(R)$, with R noncommutative, the existence of the automorphism of G with a given action on the root subgroups is easy to establish using the standard presentation of $St_{A_m}(R)$ recalled below.

As usual, we realize A_m as the subset $\{e_i - e_j : 1 \le i \ne j \le m+1\}$ of $\mathbb{R}^{m+1}$. The group $St_{A_m}(R)$ has generators $\{x_{e_i - e_j}(r) : 1 \le i \ne j \le m+1, r \in R\}$ and

relations

$$x_{e_i-e_j}(r+s) = x_{e_i-e_j}(r)x_{e_i-e_j}(s) \text{ and}$$

$$[x_{e_i-e_j}(r), x_{e_k-e_l}(s)] = \begin{cases} x_{e_i-e_l}(rs) & \text{if } j = k \text{ and } i \neq l \\ x_{e_k-e_j}(-sr) & \text{if } j \neq k \text{ and } i = l \\ 0 & \text{if } j \neq k \text{ and } i \neq l \end{cases}$$

In each of the following examples we fix a ring R and a reduced irreducible classical root system Φ, and G will denote one of the groups $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ or $St_\Phi(R)$, unless additional restrictions are imposed.

Type I: *ring automorphisms.* Let σ be an automorphism of the ring R. Then we can define the automorphism φ_σ of G by

$$\varphi_\sigma(x_\alpha(r)) = x_\alpha(\sigma(r)), \ \alpha \in \Phi, \ r \in R.$$

If R is commutative, φ_σ is well defined since it is induced (as in Observation 8.2) by the automorphism of $\mathcal{L}_R$ which sends $r \otimes l$ (where $l \in \mathcal{L}_\mathbb{Z}$ and $r \in R$) to $\sigma(r) \otimes l$ (we assume that automorphisms act from the right).

If R is arbitrary and $G = St_{A_n}(R)$, the automorphism φ_σ is well defined since it clearly respects the defining relations of G (the same argument applied to automorphisms of $St_{A_n}(R)$ of other types described below).

Type II: *diagonal automorphisms.* Let $Z(R)^\times$ be the group of invertible elements of $Z(R)$, let $\mathbb{Z}\Phi$ denote the $\mathbb{Z}$-span of Φ, and let $\mu : \mathbb{Z}\Phi \to Z(R)^\times$ be a homomorphism. Then we can define the automorphism χ_μ of G given by

$$\chi_\mu(x_\alpha(r)) = x_\alpha(\mu(\alpha)r), \ \alpha \in \Phi, \ r \in R.$$

If R is commutative, χ_μ is well defined since it is induced by the automorphism of $\mathcal{L}_R$ which fixes h_α and sends x_α to $\mu(\alpha)x_\alpha$ for any $\alpha \in \Phi$.

Type III: *root system automorphisms (commutative case).* In this example we assume that R is commutative. Let $V = \mathbb{R}\Phi$ be the $\mathbb{R}$-span of Φ, and let π be an automorphism of V which stabilizes Φ (equivalently, we can start with an automorphism of Φ and uniquely extend it to an automorphism of V). Then there are constants $\gamma_\alpha = \pm 1 (\alpha \in \Phi)$ such that the map

$$\lambda_\pi(x_\alpha(r)) = x_{\pi(\alpha)}(\gamma_\alpha r), \ \alpha \in \Phi, \ r \in R,$$

can be extended to an automorphism of G.

The existence of an automorphism of $\mathcal{L}_R$ which induces λ_π is a consequence of the Isomorphism Theorem for simple Lie algebras ([Ca, Theorem 3.5.2], see also [Ca, Proposition 12.2.3]).

Note that there is no canonical choice for the constants γ_α (except when $\Phi = A_n$), so in our notation λ_π is only unique up to a diagonal automorphism (which acts as multiplication by ± 1 on each root subgroup).

Type III': *mixed automorphisms of $St_{A_m}(R)$.* In this example we assume that $G = St_{A_m}(R)$ and R is arbitrary. Let V be the $\mathbb{R}$-span of A_m. It is well known that every automorphism of A_m has the form

$$a(\pi, \delta) : e_i - e_j \mapsto (-1)^\delta(e_{\pi(i)} - e_{\pi(j)})$$

for some permutation $\pi \in \Sigma_{m+1}$ and $\delta = 0, 1$. In particular, $\mathrm{Aut}(A_m)$ has order $2(m+1)!$ (if $m \geq 2$), and it is easy to see that the automorphisms with $\delta = 0$ are precisely the elements of the Weyl group of A_m.

If R is commutative, we have already associated an automorphism of G of type III to each element of $\mathrm{Aut}(A_m)$. The type III automorphism of G corresponding to

$a(\pi, 0) \in \mathrm{Aut}(A_m)$ can be defined even if R is not commutative. It will be denoted by λ_π^+ and is given by

$$\lambda_\pi^+(x_{e_i - e_j}(r)) = x_{e_{\pi(i)} - e_{\pi(j)}}(r), \ \alpha \in \Phi, \ r \in R.$$

Similarly, if R is commutative, we will denote by λ_π^- the type III automorphism of G corresponding to $a(\pi, 1) \in \mathrm{Aut}(A_m)$. It is given by

$$\lambda_\pi^-(x_{e_i - e_j}(r)) = x_{e_{-\pi(i)} + e_{\pi(j)}}(-r), \ \alpha \in \Phi, \ r \in R.$$

The formula for λ_π^- will not define an automorphism of G if R is noncommutative. However, if we are given an anti-automorphism $*$ of R, for each $\pi \in \Sigma_{m+1}$ we can define an automorphism $\lambda_{\pi,*}^-$ of G by setting

$$\lambda_{\pi,*}^-(x_{e_i - e_j}(r)) = x_{e_{-\pi(i)} + e_{\pi(j)}}(-r^*), \ \alpha \in \Phi, \ r \in R.$$

These automorphisms will be called *mixed*.

Note that if R is commutative, then $\lambda_{\pi,*}^-$ is just the composition of λ_π^- and the ring automorphism φ_*.

The collection of twisted groups that can be constructed using these four types of automorphisms and their compositions is clearly too large for case-by-case analysis and is beyond the scope of this paper. We shall concentrate on automorphisms which yield natural analogues of twisted Chevalley groups listed at the end of § 8.1.

Among all root system automorphisms of particular importance are *diagram automorphisms* – the ones induced by an automorphism of the Dynkin diagram of Φ. For instance, in the case $\Phi = A_m$, there is unique diagram automorphism (for a given choice of simple roots) – in the above notations it is the automorphism λ_π^- where $\pi \in \Sigma_{m+1}$ is given by $\pi(i) = m + 2 - i$. Each of the twisted Chevalley groups of type $^k\Phi$, where $\Phi = A_n, D_n$ or E_6 and $k = 2$, or $\Phi = D_4$ and $k = 3$, is obtained from $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ using the twisting by the composition of a diagram automorphism and a ring automorphism of the same order k.

8.3. Unitary Steinberg groups over non-commutative rings with involution.
In this subsection we shall define (twisted) Steinberg groups corresponding to (quasi-split) unitary groups and, in the case of even dimension, their generalizations, called hyperbolic unitary Steinberg groups. We shall establish property (T) for most of those groups. To simplify the exposition, we will not provide explicit estimates for the Kazhdan constants, although in most cases reasonably good estimates can be obtained by adapting the arguments from Section 7.

Throughout this subsection we fix a ring R, and let $* : R \to R$ be an involution, that is, an anti-automorphism of order 2.

As we already stated, in the classical setting unitary groups are obtained from Chevalley groups of type A_m via twisting by the order 2 automorphism

$$Dyn_* = \lambda_{\pi,*}^- \text{ where } \pi \text{ is the permutation } i \mapsto m + 2 - i.$$

In even dimension (that is, if m is odd), there is an interesting generalization of this construction, where instead of Dyn_* one uses the composition of Dyn_* with a suitable diagonal automorphism of order 2.

To each $\omega \in Z(R)^\times$ we can associate a homomorphism from $\mathbb{Z}A_m$ to $Z(R)^\times$ also denoted by ω and given by

$$\omega(e_i - e_j) = \begin{cases} 1 & \text{if } i,j \leq (m+1)/2 \text{ or } i,j > (m+1)/2 \\ \omega & \text{if } i \leq (m+1)/2 < j \\ \omega^{-1} & \text{if } j \leq (m+1)/2 < i \end{cases}$$

Note that the homomorphism $1 : \mathbb{Z}A_m \to Z(R)^\times$ is the trivial homomorphism.

For each such ω we define the automorphism q_ω of $St_{A_m}(R)$ given by

(8.3)
$$q_\omega = Dyn_* \chi_\omega.$$

(recall that χ_ω is a diagonal automorphism, defined in § 8.2).

Now let

$$U(R) = \{r \in R^\times : rr^* = 1\} \quad \text{and} \quad U(Z(R)) = U(R) \cap Z(R).$$

It is easy to see that if m is odd and $\omega \in U(Z(R))$, then q_ω has order 2.

The groups obtained from Chevalley groups of type A_m via twisting by q_ω (with m odd and $\omega \in U(Z(R))$) are called *hyperbolic unitary groups*. These groups have been originally defined by Bak [Bak1] and are discussed in detail in the book by Hahn and O'Meara [HO] (see also [Bak2]).

Remark: It is easy to show that if χ is any diagonal automorphism of order 2 of the Chevalley group $SL_{m+1}(R)$, then χ is graded conjugate to q_ω for some ω as above if m is odd, and graded conjugate to Dyn_* if m is even. This yields a simple characterization of hyperbolic unitary groups among all twisted Chevalley groups.

Before turning to Example 1, we introduce some additional terminology from [HO] (we note that our notations are different from [HO]).

Definition. Let $\omega \in U(Z(R))$. Put

$$Sym_{-\omega}(R) = \{r \in R : r^*\omega = -r\} \quad \text{and} \quad Sym_{-\omega}^{min}(R) = \{r - r^*\omega : r \in R\}.$$

A *form parameter* of the triple $(R, *, \omega)$ is a subgroup I of $(R, +)$ such that

(i) $Sym_{-\omega}^{min}(R) \subseteq I \subseteq Sym_{-\omega}(R)$
(ii) For any $u \in I$ and $s \in R$ we have $s^*us \in I$.

The following simplified terminology will be used in the case $\omega = \pm 1$.

- The set $Sym_1(R)$ will be denoted by $Sym(R)$, and its elements will be called symmetric.
- The set $Sym_{-1}(R)$ will be denoted by $Asym(R)$, and its elements will be called antisymmetric

For a subset A of $Sym_{-\omega}(R)$ we let $\langle A \rangle_{-\omega}$ be the form parameter generated by A, that is,

$$\langle A \rangle_{-\omega} = \{x \in Sym_{-\omega}(R) : \; x = \sum_{i=1}^{k} s_i a_i s_i^* + (r - r^*\omega) \text{ with } a_i \in A, s_i, r \in R\}.$$

Remark: If $A = \{a_1, \ldots, a_m\}$ is finite, then any element $x \in \langle A \rangle_{-\omega}$ has an expansion in the form

$$x = \sum_{i=1}^{m} s_i a_i s_i^* + (r - \omega r^*),$$

that is, with only one term sas^* for each $a \in A$. This is because $uau^* + vav^* = (u+v)a(u+v)^* + (r - r^*\omega)$ for $r = -uav^*$.

Example 1: *Hyperbolic unitary Steinberg groups.* Let $\Phi = A_{2n-1}$ and $G = St_\Phi(R)$. Fix $\omega \in U(Z(R))$, let $q = q_\omega \in \mathrm{Aut}(G)$ and $Q = \langle q \rangle$.

The twisted group $\widehat{G^Q}$ constructed in this example will be denoted by $St^\omega_{C_n}(R, *)$. This group is graded by the root system C_n and corresponds to the group of transformations preserving the sesquilinear form

$$f(u, v) = \sum_{i=1}^{n} u_i v_{\bar{i}}^* + \omega u_{\bar{i}} v_i^* \text{ on } R^{2n} \text{ where } \bar{i} = 2n + 1 - i.$$

Remark: This form is ω-hermitian, that is, $f(v, u) = \omega(f(u, v)^*)$. For more

information on groups fixing this form see [HO, Chapter 5.3].

We shall use the standard realization for both A_{2n-1} and C_n, and to avoid confusion we shall denote the roots of A_{2n-1} by $e_i - e_j$, with $1 \leq i \neq j \leq 2n$, and the roots of C_n by $\pm\varepsilon_i \pm \varepsilon_j$ and $\pm 2\varepsilon_i$, with $1 \leq i \neq j \leq n$.

The action of q on the root subgroups of G is given by

$$q(x_{e_i - e_j}(r)) = \begin{cases} x_{e_{\bar{j}} - e_{\bar{i}}}(-r^*) & \text{if } i, j \leq n \text{ or } i, j > n \\ x_{e_{\bar{j}} - e_{\bar{i}}}(-\omega r^*) & \text{if } i \leq n < j \\ x_{e_{\bar{j}} - e_{\bar{i}}}(-\omega^* r^*) & \text{if } j \leq n < i \end{cases}$$

Define $\eta : \bigoplus_{i=1}^{2n} \mathbb{R}e_i \to \bigoplus_{i=1}^{n} \mathbb{R}\varepsilon_i$ by $\eta(e_i) = \varepsilon_i$ if $i \leq n$ and $\eta(e_i) = -\varepsilon_{\bar{i}}$ if $i > n$. It is straightforward to check that η is q-invariant. Then

$$\eta(e_i - e_j) = \begin{cases} \varepsilon_i - \varepsilon_j & \text{if } i, j \leq n \\ \varepsilon_i + \varepsilon_{\bar{j}} & \text{if } i \leq n, \; j > n \\ -\varepsilon_{\bar{i}} - \varepsilon_j & \text{if } i > n, \; j \leq n \\ \varepsilon_{\bar{j}} - \varepsilon_{\bar{i}} & \text{if } i, j > n \end{cases}$$

so the root system $\Psi = \eta(\Phi) \setminus \{0\}$ is indeed of type C_n (with standard realization).

Let $\{Y_\gamma\}_{\gamma \in \Psi}$ denote the coarsened Ψ-grading of G. If $\gamma \in \Psi$ is a short root, the corresponding root subgroup Y_γ consists of elements $\{y_\gamma(r, s) : r, s \in R\}$ where

$$y_{\varepsilon_i - \varepsilon_j}(r, s) \quad = \quad x_{e_i - e_j}(r) x_{e_{\bar{j}} - e_{\bar{i}}}(s)$$

$$y_{\pm(\varepsilon_i + \varepsilon_j)}(r, s) \quad = \quad x_{\pm(e_i - e_{\bar{j}})}(r) x_{\pm(e_j - e_{\bar{i}})}(s)$$

If $\gamma \in \Psi$ is a long root, the corresponding root subgroup Y_γ consists of elements $\{y_\gamma(r) : r \in R\}$ where

$$y_{2\varepsilon_i}(r) = x_{e_i - e_{\bar{i}}}(r).$$

Computing q-invariants and letting $Z_\gamma = Y_\gamma^q$, we get

$$Z_{\varepsilon_i - \varepsilon_j} \quad = \quad \{z_{\varepsilon_i - \varepsilon_j}(r) = x_{e_i - e_j}(r) x_{e_{\bar{j}} - e_{\bar{i}}}(-r^*) : \ r \in R\} \ \text{ for } i < j$$

$$Z_{\varepsilon_i - \varepsilon_j} \quad = \quad \{z_{\varepsilon_i - \varepsilon_j}(r) = x_{e_i - e_j}(-r^*) x_{e_{\bar{j}} - e_{\bar{i}}}(r) : \ r \in R\} \ \text{ for } i > j$$

$$Z_{\varepsilon_i + \varepsilon_j} \quad = \quad \{z_{\varepsilon_i + \varepsilon_j}(r) = x_{e_i - e_{\bar{j}}}(r) x_{e_j - e_{\bar{i}}}(-\omega r^*) : \ r \in R\} \ \text{ for } i < j$$

$$Z_{-\varepsilon_i - \varepsilon_j} \quad = \quad \{z_{-\varepsilon_i - \varepsilon_j}(r) = x_{-e_i + e_{\bar{j}}}(-r^*) x_{-e_j + e_{\bar{i}}}(\omega^* r) : \ r \in R\} \ \text{ for } i < j$$

$$Z_{2\varepsilon_i} \quad = \quad \{z_{2\varepsilon_i}(r) = x_{e_i - e_{\bar{i}}}(r) : \ r \in Sym_{-\omega}(R)\}$$

$$Z_{-2\varepsilon_i} \quad = \quad \{z_{-2\varepsilon_i}(r) = x_{-e_i + e_{\bar{i}}}(-r^*) : \ r \in Sym_{-\omega}(R)\}$$

Note that

$\qquad Z_\gamma \cong (R, +)$ if γ is a short root and
$\qquad Z_\gamma \cong (Sym_{-\omega}(R), +)$ if γ is a long root

It is easy to see that the hypothesis of Proposition 8.1 holds in this example. Hence $\{Z_\gamma\}_{\gamma \in \Psi}$ is a Ψ-grading, and we can form the graded cover $\widehat{G^Q}$.

Thus, by definition $\widehat{G^Q} = \langle Z \mid E \rangle$ where $Z = \sqcup_{\gamma \in \Psi} Z_\gamma$ and E is the set of commutation relations (inside G) expressing the elements of $[Z_\gamma, Z_\delta]$ in terms of $\{Z_{a\gamma + b\delta} : a, b \geq 1\}$ (where $\delta \notin \mathbb{R}_{<0}\gamma$). These relations are obtained by straightforward calculation.

Below we list the non-trivial commutation relations between the positive root subgroups (omitting the relations where the commutator is equal to 1).

$\quad (E1) \qquad [z_{\varepsilon_i - \varepsilon_j}(r), z_{\varepsilon_j - \varepsilon_k}(s)] = z_{\varepsilon_i - \varepsilon_k}(rs) \ \text{ for } i < j < k$

$\quad (E2) \qquad [z_{\varepsilon_i - \varepsilon_j}(r), z_{\varepsilon_i + \varepsilon_j}(s)] = z_{2\varepsilon_i}(sr^* - \omega rs^*) \ \text{ for } i < j$

$\quad (E3) \qquad [z_{2\varepsilon_j}(r), z_{\varepsilon_i - \varepsilon_j}(s)] = z_{\varepsilon_i + \varepsilon_j}(-sr) z_{2\varepsilon_i}(srs^*) \ \text{ for } i < j$

$\quad (E4) \qquad [z_{\varepsilon_i - \varepsilon_j}(r), z_{\varepsilon_j + \varepsilon_k}(s)] = \begin{cases} z_{\varepsilon_i + \varepsilon_k}(rs) & \text{for } i < j < k \\ z_{\varepsilon_i + \varepsilon_k}(sr^*) & \text{for } k < i < j \\ z_{\varepsilon_i + \varepsilon_k}(-\omega rs^*) & \text{for } i < k < j \end{cases}$

The remaining relations (involving negative root subgroups) are analogous. We list just one of these relations since it will be explicitly used later in the paper.

$\quad (E5) \qquad [z_{-2\varepsilon_i}(r), z_{\varepsilon_i + \varepsilon_j}(s)] = \begin{cases} z_{\varepsilon_j - \varepsilon_i}(-r^*s) z_{2\varepsilon_j}(s^*rs) & \text{for } i < j \\ z_{\varepsilon_j - \varepsilon_i}(sr^*) z_{2\varepsilon_j}(srs^*) & \text{for } i > j \end{cases}$

The group $\widehat{G^Q}$ we just constructed will be denoted by $St_{C_n}^\omega(R, *)$.

Variations of $St_{C_n}^\omega(R, *)$ involving form parameters. The defining relations show that $St_{C_n}^\omega(R, *)$ admits a natural family of subgroups also graded by C_n, obtained by decreasing long root subgroups.

Let J be a form parameter of $(R, *, \omega)$. Given $\gamma \in C_n$, let

$$Z_{J, \gamma} = \begin{cases} Z_\gamma & \text{if } \gamma \text{ is a short root} \\ \{z_\gamma(r) : r \in J\} & \text{if } \gamma \text{ is a long root.} \end{cases}$$

The defining relations of $St^{\omega}_{C_n}(R,*)$ imply that $\{Z_{J,\gamma}\}_{\gamma \in C_n}$ is a grading. Define $\overline{St}^{\omega}_{C_n}(R,*,J)$ to be the subgroup of $St^{\omega}_{C_n}(R,*)$ generated by $Z_J := \cup Z_{J,\gamma}$, and let $St^{\omega}_{C_n}(R,*,J)$ be the graded cover of $\overline{St}^{\omega}_{C_n}(R,*,J)$. It is not hard to show that $\overline{St}^{\omega}_{C_n}(R,*,J)$ has the presentation $\langle Z_J | E_J \rangle$ where $E_J \subseteq E$ is set of those commutation relations of $\overline{St}^{\omega}_{C_n}(R,*)$ which only involve generators from Z_J.

Here are two important observations. The first one is that non-twisted Steinberg groups of type C_n and D_n are special cases of the groups $\{St^{\omega}_{C_n}(R,*,J)\}$. The second observation describes some natural isomorphisms between these groups.

Observation 8.3. *Assume that the ring R is commutative and the involution $*$ is trivial. The following hold:*

(1) *The group $St^{-1}_{C_n}(R,*)$ coincides with $St_{C_n}(R)$, the usual (non-twisted) Steinberg group of type C_n.*

(2) *$J = \{0\}$ is a possible form parameter of $(R,*,1)$, and the group $St^1_{C_n}(R,*,\{0\})$ coincides with $St_{D_n}(R)$, the usual Steinberg group of type D_n. This happens because the long root subgroups in the C_n-grading on $St^1_{C_n}(R,*,\{0\})$ are trivial, and we can "remove" those roots to obtain a D_n-grading.*

Observation 8.4. *Let $\omega \in U(Z(R))$, and let $\omega' = \omega \mu^{-1} \mu^*$ for some $\mu \in Z(R)$. Then the automorphisms q_ω and $q_{\omega'}$ are graded-conjugate and so $St^{\omega}_{C_n}(R,*)$ and $St^{\omega'}_{C_n}(R,*)$ are isomorphic.*

Remark: An explicit isomorphism is constructed as follows. If $Z_\gamma = \{z_\gamma(r)\}$ are the root subgroups of $St^{\omega}_{C_n}(R)$ and $Z'_\gamma = \{z'_\gamma(r)\}$ are the root subgroups of $St^{\omega'}_{C_n}(R)$, then the map φ defined on root subgroups as

$$\varphi(z_\gamma(r)) = \begin{cases} z'_\gamma(r) & \text{if } \gamma = \varepsilon_i - \varepsilon_j \\ z'_\gamma(\mu^* r) & \text{if } \gamma = \varepsilon_i + \varepsilon_j \\ z'_\gamma(\mu^{-1} r) & \text{if } \gamma = -\varepsilon_i - \varepsilon_j \end{cases}$$

is an isomorphism.

We now turn to the proof of property (T) for hyperbolic unitary Steinberg groups.

Lemma 8.5. *Let R be a ring with involution $*$, let $\omega \in U(Z(R))$, and let J be a form parameter of $(R,*,\omega)$.*

(a) *If $n \geq 3$, the C_n-grading on $St^{\omega}_{C_n}(R,*,J)$ is strong.*

(b) *Assume that the left ideal of R generated by J equals R. Then the C_n-grading on $St^{\omega}_{C_n}(R,*,J)$ is 2-strong (in particular, the grading is strong if $n = 2$).*

Proof. (a) The grading is strong with respect to long root subgroups by relations (E3) with $s = 1$. If $n \geq 3$, the grading is strong with respect to short root subgroups by relations (E1).

(b) If $n = 2$, the grading is strong with respect to short root subgroups by relations (E3). The same argument shows that the grading is 2-strong for any $n \geq 2$. $\square$

Proposition 8.6. *Let R be a ring with involution $*$, $\omega \in U(Z(R))$ and J a form parameter of $(R,*,\omega)$. Assume that J is finitely generated as a form parameter. The following hold:*

(a) *The group $H = St^{\omega}_{C_n}(R,*,J)$ has property (T) for any $n \geq 3$.*

(b) *Assume in addition that $\omega = -1$, $1_R \in J$ (so, in particular, the left ideal of R generated by J equals R), and R is a finitely generated right module over its subring generated by a finite set of elements from J. Then the group $St_{C_2}^{-1}(R, *, J)$ has property (T).*

Proof. Lemma 8.5 ensures that the C_n-grading is strong, so we only need to check relative property (T) for root subgroups.

(a) Relations (E1) ensure that any short root subgroup Z_γ can be put inside a group which is a quotient of $St_{A_2}(R) = St_3(R)$ (by Proposition 7.4) and hence the pair (H, Z_γ) has relative property (T). To prove relative (T) for long root subgroups we realize each of them as a subset of a bounded product of short root subgroups and some finite set. Without loss of generality, we will establish the required factorization for the long root $\gamma = 2\varepsilon_1$.

Let T be a finite set which generates J as a form parameter of $(R, *, \omega)$. By the remark following the definition of a form parameter, any $r \in J$ can be written as $r = \sum_{t \in T} s_t t s_t^* + (u - \omega u^*)$ for some $s_t, u \in R$. Relations (E2) and (E3) yield the following identity:

$$z_{2\varepsilon_1}(r) = \prod_{t \in T} [z_{2\varepsilon_2}(t), z_{\varepsilon_1 - \varepsilon_2}(s_t)] z_{\varepsilon_1 + \varepsilon_2}\left(\sum_{t \in T} s_t t\right) [z_{\varepsilon_1 - \varepsilon_2}(1), z_{\varepsilon_1 + \varepsilon_2}(u)]$$

It follows that

$$Z_{2\varepsilon_1} \subseteq \prod_{t \in T} Z_{\varepsilon_1 - \varepsilon_2}^{z_{2\varepsilon_2}(t)} Z_{\varepsilon_1 - \varepsilon_2} Z_{\varepsilon_1 + \varepsilon_2} Z_{\varepsilon_1 - \varepsilon_2} Z_{\varepsilon_1 + \varepsilon_2} Z_{\varepsilon_1 - \varepsilon_2} Z_{\varepsilon_1 + \varepsilon_2}.$$

The set $\{z_{2\varepsilon_2}(t) : t \in T\}$ of conjugating elements is finite, so we obtained the desired factorization.

(b) Relative property (T) in this case will be established in Proposition 8.12 in § 8.6. $\square$

Example 2: *Unitary Steinberg groups in odd dimension.* Let $\Phi = A_{2n}$ and $G = St_\Phi(R)$. Let $q = Dyn_* \in \operatorname{Aut}(G)$ and $Q = \langle q \rangle$.

The twisted group $\widehat{G^Q}$ constructed in this example will be denoted by $St_{BC_n}(R, *)$ and graded by the root system BC_n. It corresponds to the group of transformations preserving the Hermitian form

$$f(u, v) = u_{n+1} v_{n+1}^* + \sum_{i=1}^n (u_i v_{\bar{i}}^* + u_{\bar{i}} v_i^*) \text{ on } R^{2n+1} \text{ where } \bar{i} = 2n + 2 - i.$$

The action of $q = Dyn_*$ on the root subgroups of G is given by

$$q : x_{e_i - e_j}(r) \mapsto x_{e_{\bar{j}} - e_{\bar{i}}}(-r^*).$$

Define $\eta : \bigoplus_{i=1}^{2n+1} \mathbb{R} e_i \to \bigoplus_{i=1}^n \mathbb{R} \varepsilon_i$ by $\eta(e_i) = \varepsilon_i$ if $i \leq n$ and $\eta(e_i) = -\varepsilon_{\bar{i}}$ if $i \geq n+2$ and $\eta(e_{n+1}) = 0$. Similarly to Example 1, we check that η is q-invariant and the root system $\Psi = \eta(\Phi) \setminus \{0\}$ is indeed of type BC_n.

If $\gamma \in \Psi$ is a long root, the corresponding root subgroup Y_γ consists of elements $\{y_\gamma(r, s) : r, s \in R\}$ where

$$y_{\varepsilon_i - \varepsilon_j}(r, s) = x_{e_i - e_j}(r) x_{e_{\bar{j}} - e_{\bar{i}}}(s)$$
$$y_{\pm(\varepsilon_i + \varepsilon_j)}(r, s) = x_{\pm(e_i - e_{\bar{j}})}(r) x_{\pm(e_j - e_{\bar{i}})}(s).$$

If $\gamma \in \Psi$ is a short root, the root subgroup Y_γ consists of elements $\{y_\gamma((r,s,t)) : r, s, t \in R\}$ where

$$y_{\pm\varepsilon_i}((r,s,t)) = x_{\pm(e_i - e_{n+1})}(r) x_{\pm(e_{n+1} - e_{\bar{\imath}})}(s) x_{\pm(e_i - e_{\bar{\imath}})}(t).$$

Note that the groups $Y_{\pm\varepsilon_i}$ are not abelian, and multiplication in them is determined by

$$y_{\varepsilon_i}((r_1, s_1, t_1)) y_{\varepsilon_i}((r_2, s_2, t_2)) \quad = \quad y_{\varepsilon_i}((r_1 + r_2, s_1 + s_2, t_1 + t_2 - r_2 s_1))$$

$$y_{-\varepsilon_i}((r_1, s_1, t_1)) y_{-\varepsilon_i}((r_2, s_2, t_2)) \quad = \quad y_{-\varepsilon_i}((r_1 + r_2, s_1 + s_2, t_1 + t_2 + s_1 r_2))$$

Finally, the double root subgroup $Y_{\pm 2\varepsilon_i}$ is the subgroup of $Y_{\pm\varepsilon_i}$ consisting of all elements of the form $y_{\pm\varepsilon_i}((0,0,t))$ where $t \in R$.

Now, calculating $Z_\alpha = Y_\alpha^{\langle q \rangle}$ we obtain that

$$Z_{\varepsilon_i - \varepsilon_j} \quad = \quad \{z_{\varepsilon_i - \varepsilon_j}(r) = x_{e_i - e_j}(r) x_{e_{\bar{\jmath}} - e_{\bar{\imath}}}(-r^*) : \ r \in R\} \ \text{ for } i < j,$$

$$Z_{\varepsilon_i - \varepsilon_j} \quad = \quad \{z_{\varepsilon_i - \varepsilon_j}(r) = x_{e_i - e_j}(-r^*) x_{e_{\bar{\jmath}} - e_{\bar{\imath}}}(r) : \ r \in R\} \ \text{ for } i > j,$$

$$Z_{\varepsilon_i + \varepsilon_j} \quad = \quad \{z_{\varepsilon_i + \varepsilon_j}(r) = x_{e_i - e_{\bar{\jmath}}}(r) x_{e_j - e_{\bar{\imath}}}(-r^*) : \ r \in R\} \ \text{ for } i < j,$$

$$Z_{-(\varepsilon_i + \varepsilon_j)} \quad = \quad \{z_{-(\varepsilon_i + \varepsilon_j)}(r) = x_{-e_i + e_{\bar{\jmath}}}(-r^*) x_{-e_j + e_{\bar{\imath}}}(r) : \ r \in R\} \ \text{ for } i < j,$$

$$Z_{\varepsilon_i} \quad = \quad \{z_{\varepsilon_i}(r,t) = x_{e_i - e_{n+1}}(r) x_{e_{n+1} - e_{\bar{\imath}}}(-r^*) x_{e_i - e_{\bar{\imath}}}(t) : \ r, t \in R, \ rr^* = t + t^*\},$$

$$Z_{-\varepsilon_i} \quad = \quad \{z_{-\varepsilon_i}(r,t) = x_{-e_i + e_{n+1}}(-r^*) x_{-e_{n+1} + e_{\bar{\imath}}}(r) x_{-e_i + e_{\bar{\imath}}}(-t^*) : \ r, t \in R, \ rr^* = t + t^*\},$$

$$Z_{\pm 2\varepsilon_i} \quad = \quad \{z_{\pm\varepsilon_i}(0,t) \in Z_{\pm\varepsilon_i}\}.$$

Clearly,

$$Z_\gamma \cong (R, +) \text{ if } \gamma \text{ is a long root and}$$
$$Z_\gamma \cong (Asym(R), +) \text{ if } \gamma \text{ is a double root.}$$

When $\gamma \in \Psi$ is a short root, the group Z_γ may not be abelian. We also have a natural injection $Z_\gamma / Z_{2\gamma} \to (R, +)$ which need not be an isomorphism.

Applying Proposition 8.1 we obtain that $\{Z_\gamma\}_{\gamma \in \Psi}$ is a grading. The corresponding graded cover $\widehat{G^Q}$ will be denoted by $St_{BC_n}(R, *)$.

Below we list the non-trivial commutation relations between the positive root subgroups (again the the remaining relations are similar).

$(E1) \qquad [z_{\varepsilon_i - \varepsilon_j}(r), z_{\varepsilon_j - \varepsilon_k}(s)] = z_{\varepsilon_i - \varepsilon_k}(rs) \ \text{ for } i < j < k$

$(E2) \qquad [z_{\varepsilon_i - \varepsilon_j}(r), z_{\varepsilon_i + \varepsilon_j}(s)] = z_{\varepsilon_i}((0, sr^* - rs^*)) \ \text{ for } i < j$

$(E3) \qquad [z_{\varepsilon_j}((r,t)), z_{\varepsilon_i - \varepsilon_j}(s)] = z_{\varepsilon_i}((-sr, sts^*)) z_{\varepsilon_i + \varepsilon_j}(-st) \ \text{ for } i < j$

$(E4) \qquad [z_{\varepsilon_i}((r,t)), z_{\varepsilon_j}((s,q))] = z_{\varepsilon_i + \varepsilon_j}(-rs^*) \ \text{ for } i < j$

$(E5) \qquad [z_{\varepsilon_i - \varepsilon_j}(r), z_{\varepsilon_j + \varepsilon_k}(s)] = \begin{cases} z_{\varepsilon_i + \varepsilon_k}(rs) & \text{for } i < j < k \\ z_{\varepsilon_i + \varepsilon_k}(sr^*) & \text{for } k < i < j \\ z_{\varepsilon_i + \varepsilon_k}(-rs^*) & \text{for } i < k < j \end{cases}$

As in Example 1 we can construct a family of generalizations of $St_{BC_n}(R, *)$, this time by decreasing the short root subgroups. Let I be a left ideal of R. For

each $\gamma \in BC_n$ we put

$$Z_{I,\gamma} = \begin{cases} Z_\gamma & \text{if } \gamma \text{ is a long or a double root} \\ \{z_\alpha(r,t) \in Z_\alpha : r \in I\} & \text{if } \gamma \text{ is a short root.} \end{cases}$$

We define $St_{BC_n}(R,*,I)$ to be the graded cover of the subgroup of $St_{BC_n}(R,*)$ generated by $\cup_{\gamma \in BC_n} Z_{I,\gamma}$.

Observation 8.7. *The group* $St_{BC_n}(R,*,\{0\})$ *is isomorphic to* $St^1_{C_n}(R,*)$.

Proposition 8.8. *Assume that* $\{r \in I : \exists t \in R, rr^* = t + t^*\}$ *is finitely generated as a left ideal. The following hold:*

(a) *The group* $St_{BC_n}(R,*,I)$ *has property* (T) *for any* $n \geq 3$.
(b) *Assume in addition that there exists an antisymmetric element* $\mu \in Z(R)$, *and* R *is generated (as a ring) by a finite set of elements from* $Sym(R)$. *Then the group* $St_{BC_2}(R,*,I)$ *has property* (T).

Proof. We shall prove (a) and (b) simultaneously. The fact that the grading is strong in both cases is verified as in Lemma 8.5, so we only need to check relative property (T). Observe that the set Ψ of long and double roots in BC_n is a weak subsystem of type C_n, so the corresponding root subgroups generate a quotient of $St^1_{C_n}(R,*)$ (this is proved similarly to Proposition 7.4). Hence relative property (T) for long and double root subgroups follows directly from Proposition 8.6(a) if $n \geq 3$. Note that the existence of an antisymmetric element $\mu \in Z(R)$ implies that $St^1_{C_2}(R,*) \cong St^{-1}_{C_2}(R,*)$ by Observation 8.4. Thus, relative property (T) in the case $n = 2$ follows from Proposition 8.6(b).

Finally, we claim that every short root subgroup lies in a bounded product of fixed conjugates of long and double root subgroups. This follows easily from relations (E3) and the fact that $\{r \in I : \exists t \in R, rr^* = t + t^*\}$ is finitely generated as a left ideal. $\square$

8.4. **Twisted groups of types** ^2D_n **and** $^{2,2}A_{2n-1}$. Recall that the next family on our agenda were the Steinberg covers of the twisted Chevalley groups of type ^2D_n ($n \geq 3$). These groups can be constructed using our general twisting procedure by taking $G = St_{D_n}(R)$, where R is a commutative ring endowed with involution σ and $Q \subseteq \text{Aut}(G)$ the subgroup of order 2 generated by Dyn_σ, the composition of the ring automorphism φ_σ and the Dynkin involution of D_n. However, we shall present a more general construction, making use of Observation 8.3(2).

Recall that the Steinberg group $St_{D_n}(R)$, for R commutative, was realized as the group $St^1_{C_n}(R,*,\{0\})$ where $*$ is the trivial involution. It turns our that if we start with any ring R (not necessarily commutative) endowed with an involution $*$ and an automorphism σ of order ≤ 2 which commutes with $*$, then the analogous twisting on $St^1_{C_n}(R,*)$ can be constructed.

Example 3: *Steinberg groups* $St^1_{BC_n}(R,*,\sigma)$. Let R be a ring endowed with an involution $*$ and an automorphism σ of order ≤ 2 which commutes with $*$. The fixed subring of σ will be denoted by R^σ. In this example we will construct the group $\text{St}^1_{BC_n}(R,*,\sigma)$ graded by the root system BC_n.

Let $\Phi = C_{n+1}$ and $G = St^1_{C_{n+1}}(R,*)$, the group constructed in Example 1 with $\omega = 1$. Denote the roots of Φ by $\pm\varepsilon_i \pm \varepsilon_j$ and $\pm 2\varepsilon_i$, and let $\{Z_\gamma\}_{\gamma \in \Phi}$ be the grading of G constructed in Example 1.

Let ρ be the automorphism of $\oplus_{i=1}^{n+1} \mathbb{R}\varepsilon_i$ given by

$$\rho(\varepsilon_i) = \varepsilon_i \text{ for } 1 \leq i \leq n \text{ and } \rho(\varepsilon_{n+1}) = -\varepsilon_{n+1}.$$

Clearly ρ stabilizes Φ. We claim that there exists an automorphism $q = q_\sigma \in \text{Aut}(G)$ of order 2 such that

$$(8.4) \qquad q(z_\gamma(r)) = z_{\rho(\gamma)}(\pm\sigma(r)) \text{ for all } \gamma \in \Phi, r \in R.$$

(for some choice of signs). Unlike Examples 1 and 2, we cannot prove the existence of such q by referring to general results from § 8.2. One (rather tedious) way to prove this is first to define q as an automorphism of the free product $*_{\gamma \in \Phi} Z_\gamma$ (using (8.4)), and then show that for a suitable choice of signs in (8.4), q respects the defining relations of G established in Example 1 and hence induces an automorphism of G. However, we will also give a conceptual argument for the existence of q at the end of this example.

Now let $\eta : \oplus_{i=1}^{n+1} \mathbb{R}\varepsilon_i \to \oplus_{i=1}^n \mathbb{R}\alpha_i$ be the reduction given by $\eta(\varepsilon_i) = \alpha_i$ for $i \leq n$ and $\eta(\varepsilon_{n+1}) = 0$. It is clear that η is q-invariant and the induced root system $\Psi = \eta(\Phi) \setminus \{0\} = \{\pm\alpha_i \pm \alpha_j\} \cup \{\pm\alpha_i\} \cup \{\pm2\alpha_i\}$ is of type BC_n.

Let $\{W_\alpha\}_{\alpha \in \Phi}$ be the q-invariants of the Φ-grading of G. By Proposition 8.1 $\{W_\alpha\}$ is a grading, and thus we can form the graded cover $\widehat{G^{\langle q \rangle}}$ which will be denoted by $\text{St}^1_{BC_n}(R, *, \sigma)$.

An easy calculation shows that

$$W_{\pm\alpha_i \pm \alpha_j} = \{w_{\pm\alpha_i \pm \alpha_j}(r) = z_{\pm\varepsilon_i \pm \varepsilon_j}(r) : r \in R^\sigma\}$$

$$W_{\pm\alpha_i} = \{w_{\pm\alpha_i}(r,t) = z_{\pm(\varepsilon_i-\varepsilon_n)}(r)z_{\pm(\varepsilon_i+\varepsilon_n)}(\sigma(r))z_{\pm2\varepsilon_i}(t) : t \in Asym(R), \ t - r\sigma(r^*) \in R^\sigma\}$$

$$W_{\pm2\alpha_i} = \{w_{\pm2\alpha_i}(t) = z_{\pm2\varepsilon_i}(t) : t \in Asym(R^\sigma)\} = \{w_{\pm\alpha_i}(0,t) : t \in Asym(R^\sigma)\}$$

The commutation relations between the positive root subgroups of the grading $\{W_\alpha\}$ are as follows.

(E1) $[w_{\alpha_i-\alpha_j}(r), w_{\alpha_j-\alpha_k}(s)] = w_{\alpha_i-\alpha_k}(rs)$ for $i < j < k$

(E2) $[w_{\alpha_i-\alpha_j}(r), w_{\alpha_i+\alpha_j}(s)] = w_{\alpha_i}(0, sr^* - rs^*)$ for $i < j$

(E3) $[w_{\alpha_j}(r,t), w_{\alpha_i-\alpha_j}(s)] = w_{\alpha_i}(-sr, sts^*)w_{\alpha_i+\alpha_j}(s(r\sigma(r^*) - t))$ for $i < j$

(E4) $[w_{\alpha_i}(r,t), w_{\alpha_j}(s,q)] = w_{\alpha_i+\alpha_j}(-r\sigma(s^*) - \sigma(r)s^*)$ for $i < j$

(E5) $[w_{\alpha_i-\alpha_j}(r), w_{\alpha_j+\alpha_k}(s)] = \begin{cases} w_{\alpha_i+\alpha_k}(rs) & \text{for } i < j < k \\ w_{\alpha_i+\alpha_k}(sr^*) & \text{for } k < i < j \\ w_{\alpha_i+\alpha_k}(-rs^*) & \text{for } i < k < j \end{cases}$

Variations of the groups $St_{BC_n}(R, *, \sigma)$. Let $I \subseteq R$ be a left R^σ-submodule and $J \subseteq Asym(R^\sigma)$ a form parameter of $(R^\sigma, *, 1)$. Then we can define the group $St^1_{BC_n}(R, *, \sigma, I, J)$ by decreasing the short and double root subgroups. For a root $\alpha \in BC_n$ we put

$$W_{I,J,\alpha} = \begin{cases} W_\alpha & \text{if } \alpha \text{ is a long root} \\ \{w_\alpha(r,t) \in W_\alpha : r \in I\} & \text{if } \alpha \text{ is a short root} \\ \{w_\alpha(t) \in W_\alpha : t \in J\} & \text{if } \alpha \text{ is a double root} \end{cases}$$

We define $St^1_{BC_n}(R, *, \sigma, I, J)$ to be the graded cover of the subgroup of $St^1_{BC_n}(R, *, \sigma)$ generated by $\cup_{\alpha \in BC_n} W_{I,J,\alpha}$.

Now assume that R is commutative and the involution $*$ is trivial. Then $J = \{0\}$ is a valid form parameter of $(R^\sigma, *, 1)$, and the double root subgroups of

$St^1_{BC_n}(R, *, \sigma, I, \{0\})$ are trivial. Hence we obtain a group graded by a system of type B_n. This group will be denoted by $St_{B_n}(R, \sigma, I)$.

We let $St_{B_n}(R, \sigma) = St_{B_n}(R, \sigma, R) = St^1_{BC_n}(R, \mathrm{id}, \sigma, R, \{0\})$ This is the Steinberg cover for the twisted Chevalley group of type $^2D_{n+1}$ over R, which we discussed at the beginning of this example.

We now state a sufficient condition for the groups $St^1_{BC_n}(R, *, \sigma, I, J)$ to have property (T).

Proposition 8.9. *Assume that*

   (i) *R^σ is finitely generated as a ring*
  (ii) *$\{r \in I : \exists t \in Asym(R), t - r\sigma(r^*) \in R^\sigma\}$ is finitely generated as an R^σ-module*
 (iii) *J is finitely generated as a form parameter of $(R^\sigma, *, 1)$.*

Then the group $St^1_{BC_n}(R, *, \sigma, I, J)$ has property (T) for any $n \geq 3$.

Proof. The proof is analogous to that of Proposition 8.8(a). $\qquad\square$

Another definition of the groups $St^1_{BC_n}(R, *, \sigma)$. There is a less intuitive, but in some sense more convenient, way to construct the groups $St^1_{BC_n}(R, *, \sigma)$. The construction we described uses the twist by q_σ on the group $St^1_{C_{n+1}}(R, *)$ which, in turn, was itself constructed using the twist by Dyn_* on $St_{A_{2n+1}}(R)$. It is easy to see that $St^1_{BC_n}(R, *, \sigma)$ can also be obtained directly from $St_{A_{2n+1}}(R)$ as follows.

Let π' be the permutation $(n+1, n+2)$ and τ the automorphism of $St_{A_{2n+1}}(R)$ defined by

$$\tau(x_{e_i - e_j}(r)) = x_{e_{\pi'(i)} - e_{\pi'(j)}}(\sigma(r)).$$

Note that τ commutes with Dyn_*, and let Q be the group generated by τ and Dyn_* (so that $Q \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). Then $St^1_{BC_n}(R, *, \sigma)$ can be obtained from $St_{A_{2n+1}}(R)$ using the twist by Q. One advantage of this approach is that the existence of the automorphism q_σ defined above follows automatically, without case-by-case verification.

Summary of Examples 1-3. For the reader's convenience below we list all the twisted Steinberg groups constructed in Examples 1-3, including the key special cases and relations between them. In all examples, $n \geq 2$ is an integer, R is a ring and $*$ is an involution on R.

1. The groups $St^\omega_{C_n}(R, *, J)$ where ω is an element of $U(Z(R))$ and J is a form parameter of $(R, *, \omega)$.
Special cases:

   (i) $St^\omega_{C_n}(R, *) = St^\omega_{C_n}(R, *, R)$;
  (ii) $St_{C_n}(R) = St^{-1}_{C_n}(R, \mathrm{id})$ where R is commutative;
 (iii) $St_{D_n}(R) = St^1_{C_n}(R, \mathrm{id}, \{0\})$ where R is commutative.

2. The groups $St_{BC_n}(R, *, I)$ where I is a left ideal of R.
Special cases:

   (i) $St^1_{C_n}(R, *) = St_{BC_n}(R, *, \{0\})$.

3. The groups $St^1_{BC_n}(R, *, \sigma, I, J)$ where σ is an automorphism of order ≤ 2 commuting with $*$, $I \subseteq R$ is a left R^σ-submodule and $J \subseteq Asym(R)$ is a form parameter of $(R^\sigma, *, 1)$.
Special cases:

(i) $St_{B_n}(R, \sigma, I) = St^1_{BC_n}(R, id, \sigma, I, \{0\})$ where R is commutative;

(ii) $St_{B_n}(R, \sigma) = St_{B_n}(R, \sigma, R)$;

(iii) $St_{B_n}(R) = St_{B_n}(R, id)$.

8.5. **Further twisted examples.** In this subsection we prove property (T) for twisted Steinberg groups of type $^3D_4$ and $^2E_6$. In all examples R is a finitely generated commutative ring and $\sigma : R \to R$ a finite order automorphism of R.

Example 4: *Steinberg groups of type* $^3D_4$. The group in this example will be denoted by $St_{G_2}(R, \sigma)$ and is graded by the root system G_2. It is the Steinberg cover for the twisted Chevalley group of type $^3D_4$ over R.

We use the standard realization of D_n in $\mathbb{R}^n$: $D_n = \{\pm e_i \pm e_j : 1 \leq i \neq j \leq n\}$. The commutation relations in $St_{D_n}(R)$ are as follows:

$$[x_{e_i-e_j}(r), x_{e_j-e_k}(s)] = x_{e_i-e_k}(rs)$$

$$[x_{e_i-e_j}(r), x_{e_j+e_k}(s)] = \begin{cases} x_{e_i+e_k}(rs) & \text{if } i,j < k \text{ or } i,j > k \\ x_{e_i+e_k}(-rs) & \text{if } j > k > i \text{ or } i > k > j \end{cases}$$

$$[x_{e_i-e_j}(r), x_{-e_i-e_k}(s)] = \begin{cases} x_{-e_j-e_k}(-rs) & \text{if } i,j < k \text{ or } i,j > k \\ x_{-e_j-e_k}(rs) & \text{if } j > k > i \text{ or } i > k > j \end{cases}$$

$$[x_{e_k+e_i}(r), x_{-e_j-e_k}(s)] = \begin{cases} x_{e_i-e_j}(rs) & \text{if } i,j < k \text{ or } i,j > k \\ x_{e_i-e_j}(-rs) & \text{if } j > k > i \text{ or } i > k > j \end{cases}$$

We realize G_2 as the set of vectors $\pm(\varepsilon_i - \varepsilon_j)$ and $\pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k)$ where $i, j, k \in \{1, 2, 3\}$ are distinct. We let

$$\alpha = 2\varepsilon_2 - 2\varepsilon_1 - \varepsilon_3 \quad \text{and} \quad \beta = \varepsilon_1 - \varepsilon_2$$

and take $\{\alpha, \beta\}$ as our system of simple roots.

Let $\Phi = D_4$ (with standard realization) and $G = St_\Phi(R)$. Let $\sigma : R \to R$ be an automorphism of order 3 and π the isometry of $\mathbb{R}^4$ represented by the following matrix with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

Then it is clear that π stabilizes D_4. Let $q = \varphi_\sigma \lambda_\pi \in \text{Aut}(G)$ (as defined in § 8.2). With a suitable choice of signs in the definition of λ_π, we can assume that q maps $x_{\pm(e_2-e_3)}(r)$ to $x_{\pm(e_2-e_3)}(\sigma(r))$, $x_{\pm(e_1-e_2)}(r)$ to $x_{\pm(e_3-e_4)}(\sigma(r))$, $x_{\pm(e_3-e_4)}(r)$ to $x_{\pm(e_3+e_4)}(\sigma(r))$ and $x_{\pm(e_3+e_4)}(r)$ to $x_{\pm(e_1-e_2)}(\sigma(r))$. Then q is an automorphism of order 3.

Define $\eta : \bigoplus_{i=1}^4 \mathbb{R}e_i \to \bigoplus_{i=1}^3 \mathbb{R}\varepsilon_i$ by $\eta(e_1) = \varepsilon_1 - \varepsilon_3$, $\eta(e_2) = \varepsilon_2 - \varepsilon_3$, $\eta(e_3) = \varepsilon_1 - \varepsilon_2$ and $\eta(e_4) = 0$. It is easy to see that the root system $\Psi = \eta(\Phi)$ is of type G_2. Furthermore,

$$\eta^{-1}(\beta) = \{e_1 - e_2, e_3 - e_4, e_3 + e_4\} \qquad \eta^{-1}(\alpha) = \{e_2 - e_3\},$$

$$\eta^{-1}(\alpha + \beta) = \{e_1 - e_3, e_2 - e_4, e_2 + e_4\} \qquad \eta^{-1}(\alpha + 3\beta) = \{e_1 + e_3\},$$

$$\eta^{-1}(\alpha + 2\beta) = \{e_1 - e_4, e_1 + e_4, e_2 + e_3\} \qquad \eta^{-1}(2\alpha + 3\beta) = \{e_1 + e_2\} \qquad .$$

If $\gamma \in \Psi$ is a short root, the corresponding root subgroup Y_γ consists of elements

$$\{y_\gamma(r, s, t) = x_{\gamma_1}(r)x_{\gamma_2}(s)x_{\gamma_3}(t) : \eta^{-1}(\gamma) = \{\gamma_1, \gamma_2, \gamma_3\}, \ r, s, t \in R\}.$$

If $\gamma \in \Psi$ is a long root, then

$$Y_\gamma = \{y_\gamma(r) = x_{\gamma_1}(r) : \ \eta(\gamma_1) = \gamma, \ r \in R\}.$$

If $\gamma \in \Psi$ is a short root, the corresponding root subgroup $Z_\gamma = Y_\gamma^q$ is isomorphic to $(R, +)$, and if $\gamma \in \Psi$ is a long root, then $Z_\gamma \cong (R^\sigma, +)$. Positive root subgroups can be explicitly described as follows (we define $z_\gamma(r)$ in such way that the relations in the non-twisted case coincide with relation from Proposition 7.5):

$$Z_\beta = \{z_\beta(r) = x_{e_1-e_2}(r)x_{e_3-e_4}(\sigma(r))x_{e_3+e_4}(\sigma^2(r)) : \ r \in R\},$$

$$Z_{\alpha+\beta} = \{z_{\alpha+\beta}(r) = x_{e_1-e_3}(-r)x_{e_2-e_4}(\sigma(r))x_{e_2+e_4}(\sigma^2(r)) : \ r \in R\},$$

$$Z_{\alpha+2\beta} = \{z_{\alpha+2\beta}(r) = x_{e_1-e_4}(-r)x_{e_2+e_3}(-\sigma(r))x_{e_1+e_4}(-\sigma^2(r)) : \ r \in R\},$$

$$Z_\alpha = \{z_\alpha(r) = x_{e_2-e_3}(r) : r \in R^\sigma\}, \quad Z_{\alpha+3\beta} = \{z_{\alpha+3\beta}(r) = x_{e_1+e_3}(r) : \ r \in R^\sigma\},$$

$$Z_{2\alpha+3\beta} = \{z_{2\alpha+3\beta}(r) = x_{e_1+e_2}(r) : \ r \in R^\sigma\}.$$

Below we list the commutation relations between positive root subgroups which will be used in the sequel:

(E1) $\quad [z_\alpha(t), z_\beta(u)] = z_{\alpha+\beta}(tu) \cdot z_{2\alpha+\beta}(tu\sigma(u)) \cdot z_{3\alpha+\beta}(tu\sigma(u)\sigma^2(u))$

(E2) $\quad [z_\alpha(t), z_{\alpha+3\beta}(u)] = z_{2\alpha+3\beta}(tu)$

(E3) $\quad [z_{\alpha+\beta}(t), z_\beta(u)] = z_{\alpha+2\beta}(t\sigma(u) + u\sigma(t)) \cdot z_{\alpha+3\beta}(u\sigma(u)\sigma^2(t) + u\sigma(t)\sigma^2(u) + t\sigma(u)\sigma^2(u))$
$z_{2\alpha+3\beta}(t\sigma(t)\sigma^2(u) + t\sigma(u)\sigma^2(t) + u\sigma(t)\sigma^2(t))$

Proposition 8.10. *The group $G = St_{G_2}(R, \sigma)$ has property (T) provided*

(i) *R^σ is finitely generated as a ring*

(ii) *R is a finitely generated module over R^σ*

Proof. As usual, we need to check two things

(a) The Ψ-grading of G is strong at each root subgroup

(b) The pair (G, Z_γ) has relative (T) for each $\gamma \in \Psi$

Relation (E2) implies the condition (a) for the root subgroup $Z_{2\alpha+3\beta}$. Condition (a) for the root subgroups $Z_{\alpha+3\beta}$ and $Z_{\alpha+\beta}$ follow from relation (E1) as we can take $u = 1$ and let t be an arbitrary element of R^σ in the case of $Z_{\alpha+3\beta}$ and take $t = 1$ and let u be an arbitrary element of R in the case of $Z_{\alpha+\beta}$. In the non-twisted case there is no problem with $Z_{2\alpha+\beta}$ either as we can take $r = 1$ and arbitrary $s \in R$ (in general we cannot do this as s must come from R^σ).

To check the required property for the subgroup $Z_{2\alpha+\beta}$ in the general (twisted) case we need to show that elements of the form $u\sigma(u)t$ with $u \in R, t \in R^\sigma$ span R. Indeed, denote this span by M. Then M contains all elements of R^σ, in particular all elements of the form $u + \sigma(u) + \sigma^2(u)$. It also contains all elements of the form $(u+1)(\sigma(u+1)) - u\sigma(u) - 1 = u + \sigma(u)$. Since $u = u + \sigma(u) + \sigma^2(u) - (\sigma(u) + \sigma(\sigma(u)))$, we are done with (a).

We now prove (b). The subgroup of G generated by long root subgroups is isomorphic to a quotient of $St_3(R^\sigma)$ (and R^σ is finitely generated), so condition (b) for long root subgroups holds by Theorem 7.11. It remains to check (b) for short root subgroups. We shall show that any short root subgroup lies in a bounded product of long root subgroups and finite sets. By symmetry, it is enough to establish this property for $Z_{\alpha+2\beta}$. For any set S we put $Z_{\alpha+2\beta}(S) = \{z_{\alpha+2\beta}(s) : s \in S\}$.

Put $A = \{u\sigma(u) : u \in R\}$. In the proof of (a) we showed that A generates R as an R^σ-module. Thus by our assumption there is a finite subset $U \subseteq A$ which generates R as an R^σ-module. Let S be a finite generating set of R^σ.

Now fix $s \in S$ and $u \in U$, and let $t \in R^\sigma$ be arbitrary. Similarly to the case of non-twisted G_2, if we calculate the quantity $[z_\alpha(t), z_\beta(su)][z_\alpha(ts), z_\beta(u)]^{-1}$ using relation (E1), we obtain that $\{z_{\alpha+2\beta}(t(s^2 - s)u\sigma(u)) : t \in R^\sigma\}$ lies in a bounded product of long root subgroups and fixed elements of short root subgroups. Similarly, this property holds for the set $\{z_{\alpha+2\beta}(2tu\sigma(u)) : t \in R^\sigma\}$ and hence also for the set $Z_{\alpha+2\beta}(IU)$ where $I = 2R^\sigma + \sum_{s \in S}(s^2 - s)R^\sigma$ and $IU = \{\sum_{u \in U} r_u u : r_u \in I\}$.

As we have already seen (in the case of non-twisted G_2), I is a finite index ideal of R^σ whence IU has finite index in $R^\sigma U = R$. Hence $Z_{\alpha+2\beta}$ can be written as a product of $Z_{\alpha+2\beta}(IU)$ and some finite set. This finishes the proof of (b). $\qquad\square$

Example 5: *Steinberg groups of type $^2E_6$/twisted Steinberg group of type F_4.* The group in this example will be denoted by $St_{F_4}(R, \sigma)$ and is graded by the root system F_4. We will only sketch the details of the construction.

Let $\Phi = E_6$ and $G = St_\Phi(R)$. Let $\{\alpha_1, \ldots, \alpha_6\}$ be a system of simple roots of Φ ordered as shown below:
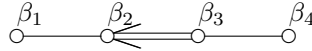
$$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5$$
$$\underset{\alpha_6}{\bullet}$$
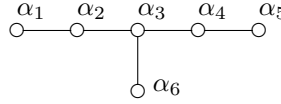
Let $\sigma : R \to R$ be an automorphism of order 2, let π be the automorphism of Φ given by $\pi(\alpha_i) = \alpha_{6-i}$ for $i = 1, 2, 4, 5$ and $\pi(\alpha_i) = \alpha_i$ for $i = 3, 6$, and let $q = \lambda_\pi \varphi_\sigma \in \text{Aut}(G)$. With a suitable choice of signs in the definition of λ_π, we can assume that q is an automorphism of G of order 2 and is given by

$$q(x_{\pm\alpha_i}(r)) = \begin{cases} x_{\pm\alpha_{6-i}}(-\sigma(r)) & \text{for } i = 1, 2, 4, 5 \\ x_{\pm\alpha_i}(-\sigma(r)) & \text{for } i = 3, 6. \end{cases}$$

Let V be the $\mathbb{R}$-span of Φ, and consider the induced action of q on V (so that $q(\alpha_i) = \pi(\alpha_i)$ for each i). Let $W = V^q$ be the subspace of q-invariants and define $\eta : V \to W$ by (8.2), that is,

$$\eta(v) = \frac{v + qv}{2}.$$

It is easy to see that $\Psi = \eta(\Phi)$ is a root system of type F_4 with base $\beta_1, \beta_2, \beta_3, \beta_4$ where $\beta_1 = \frac{\alpha_1 + \alpha_5}{2}, \beta_1 = \frac{\alpha_2 + \alpha_4}{2}, \beta_3 = \alpha_3$ and $\beta_4 = \alpha_6$:

$$\beta_1 \quad\quad \beta_2 \quad\quad \beta_3 \quad\quad \beta_4$$

As in Example 4, if $\gamma \in \Psi$ is a short root, the root subgroup Z_γ is isomorphic to $(R, +)$, and if $\gamma \in \Psi$ is a long root, then $Z_\gamma \cong (R^\sigma, +)$.

Proposition 8.11. *The group $St_{F_4}(R, \sigma)$ has property (T) provided R^σ is a finitely generated ring.*

Proof. The proof is identical to the case of classical (non-twisted) F_4. $\qquad\square$

8.6. **Proof of relative property (T) for type C_2.** In this subsection we prove relative property (T) for the pairs (G, Z_γ) where G is a twisted Steinberg group of the form $St_{C_2}^{-1}(R, *, J)$ for a suitable triple $(R, *, J)$ and Z_γ is one of its root subgroups. The main ingredient in the proof is Theorem 2.7.

Proposition 8.12. *Let R be a ring with involution $*$ and J a form parameter of $(R, *, -1)$ containing 1_R. Assume that*

(1) *There is a finite subset $T_+ = \{t_1, \ldots, t_d\}$ of J and $a_1, \ldots, a_l \in R$ such that $R = \sum_{i=1}^l a_i R_0$, where R_0 denotes the ring generated by T_+.*

(2) *J is generated as a form parameter by a finite set $U = \{u_1, \ldots, u_D\}$.*

Let

$$W_1 = \{w \in R : w \text{ is a monomial in } T_+ \text{ of degree } \leq d\}$$

and

$$W_2 = \{w + w^* \in R : w \text{ is a monomial in } T_+ \text{ of degree } \leq 2d + 1.\}$$

Let

$$W_{short} = \bigcup_{i=1}^l \{a_i w : w \in W_1 \cup \{1\}\}$$

and

$$W_{long} = W_2 \cup \bigcup_{i=1}^l \{a_i w a_i^* : w \in W_2\} \cup U \cup T_+ \cup \{t^2 : t \in T_+\}.$$

For a short root $\gamma \in C_2$ set $S_\gamma = \{z_\gamma(r) : r \in W_{short}\}$, for a long root $\gamma \in C_2$ set $S_\gamma = \{z_\gamma(r) : r \in W_{long}\}$, and let $S = \cup_{\gamma \in C_2} S_\gamma$. Then for every $\gamma \in C_2$ we have

$$\kappa_r(St_{C_2}^{-1}(R, *, J); Z_\gamma, S) > 0.$$

Proof. Let α, β be a base for C_2 with α a long root. As established in Example 1, we have the following relations between positive root subgroups:

(8.5) $$[z_\beta(r), z_{\alpha+\beta}(s)] = z_{\alpha+2\beta}(rs^* + sr^*)$$

(8.6) $$[z_\alpha(r), z_\beta(s)] = z_{\alpha+\beta}(-sr)z_{\alpha+2\beta}(srs^*)$$

(8.7) $$[z_{-\alpha}(r), z_{\alpha+\beta}(s)] = z_\beta(sr)z_{\alpha+2\beta}(srs^*).$$

(in the relations (8.7) we use the fact that long root subgroups only contain symmetric elements).

Let $N = \langle Z_{\alpha+\beta}, Z_\beta, Z_{\alpha+2\beta}\rangle$, $S^+ = \cup\{S_\gamma\}_{\gamma \in \pm\alpha, \alpha+\beta, \beta, \alpha+2\beta}$, $G = \langle S^+\rangle$, $Z = Z_{\alpha+2\beta}$ and $H = Z \cap [N, G]$. We claim that Proposition 8.12 follows from Lemma 8.13 below and Theorem 2.7.

Lemma 8.13. *The following hold:*

(a) *N is contained in G;*

(b) *Z/H is a group of exponent 2 generated by (the image of) $S_{\alpha+2\beta}$.*

Indeed, let E be the subgroup of the Steinberg group $St_2(R_0)$ generated by $\{x_{12}(r), x_{21}(r) : r \in W_{long}\}$. Relations (8.6) and (8.7) and Lemma 8.13(a) imply that $(G/Z, N/Z)$ as a pair is a quotient of $(E \ltimes (\oplus_{i=1}^l R_0^2), \oplus_{i=1}^l R_0^2)$, that is, there exists an epimorphism $\pi : E \ltimes (\oplus_{i=1}^l R_0^2) \to G/Z$ such that $\pi(\oplus_{i=1}^l R_0^2) = N/Z$, $\pi(x_{12}(r)) = x_\alpha(-r)$ and $\pi(x_{21}(r)) = x_{-\alpha}(r)$ for $r \in W_{long}$. Since W_{long} contains $T_+ \cup \{1\}$ and T_+ generates R_0 as a ring, the pair $(G/Z, N/Z)$ has relative (T) by Proposition 7.9 and Lemma 2.4(a).

This result and Lemma 8.13(b) imply that the hypotheses of Theorem 2.7 hold if we put $A = B = S^+$ and $C = S_{\alpha+2\beta}$. Applying this theorem we deduce that $\kappa(G, N; S^+) > 0$, so in particular $\kappa_r(St_{C_2}^{-1}(R, *, J); Z_\gamma, S) > 0$ for $\gamma \in \{\beta, \alpha+\beta, \alpha+$

$2\beta\}$. By symmetry we obtain the same result for all $\gamma \in C_2$ (applying the above argument to different bases of C_2).

Before proving Lemma 8.13, we establish another auxiliary result, from which Lemma 8.13 will follow quite easily.

Lemma 8.14. *For any $r \in R$ the following hold:*

- (i) $z_{\alpha+\beta}(r) \in G$ *and* $z_\beta(r) \in G$
- (ii) $z_{\alpha+\beta}(r) \in [N,G]\langle S_{\alpha+2\beta}\rangle$

Proof. Note that it suffices to prove both statements when r is of the form $r = a_i w$, where w is a monomial in T^+. Let us prove that both (i) and (ii) hold for such r by induction on $m = length(w)$.

If $m \leq d$, then $z_{\alpha+\beta}(r), z_\beta(r) \in S^+ \subset G$ by definition of S^+. Also by (8.5) we have $z_{\alpha+\beta}(r) = z_{\alpha+2\beta}(rr^*)[z_\alpha(1), z_\beta(r)]^{-1}$, so both (i) and (ii) hold.

Now fix $m > d$, and assume that for any monomial $w' \in T_+$ of length less than m both (i) and (ii) hold for $r = a_i w'$.

Claim 8.15. *Let q be some tail of w with $2 \leq length(q) \leq d+1$ so that $w = pq$ for some p. Then (i) and (ii) hold for $r = a_i pq = a_i w$ if and only if (i) and (ii) hold for $r = a_i pq^*$.*

Remark: Note that if $q = t_{i_1} \ldots t_{i_s}$, then $q^* = t_{i_s} \ldots t_{i_1}$ is the monomial obtained from q by reversing the order of letters.

Proof. Consider the element $v = p(q + q^*)$. Then

$$[z_\alpha(q + q^*), z_\beta(a_i p)] = z_{\alpha+\beta}(-a_i v)z_{\alpha+2\beta}(a_i p(q + q^*)p^* a_i^*).$$

Notice that $p(q + q^*)p^* = u + u^*$ for $u = pqp^*$. Furthermore, $length(u) \leq 2m - 2$, so we can write $u = w_1 w_2^*$ where w_1 and w_2 are monomials of length $< m$. Then

$$z_{\alpha+2\beta}(a_i(u + u^*)a_i^*) = z_{\alpha+2\beta}(a_i w_1(a_i w_2)^* + a_i w_2(a_i w_1)^*)$$
$$= [z_\beta(a_i w_1), z_{\alpha+\beta}(a_i w_2)] \in G \cap [N,G] \quad \text{by induction.}$$

Since $z_\alpha(q + q^*) \in S^+$ and $z_\beta(a_i p) \in G$ by induction, we get

$z_{\alpha+\beta}(a_i v) = z_{\alpha+\beta}(a_i(pq + pq^*)) = z_{\alpha+2\beta}(a_i(u+u^*)a_i^*)[z_\alpha(q+q^*), z_\beta(a_i p)]^{-1} \in G \cap [N,G].$

Hence $z_{\alpha+\beta}(a_i pq) \in G \iff z_{\alpha+\beta}(a_i pq^*) \in G$ and $z_{\alpha+\beta}(a_i pq) \in [N,G]\langle S_{\alpha+2\beta}\rangle \iff z_{\alpha+\beta}(a_i pq^*) \in [N,G]\langle S_{\alpha+2\beta}\rangle$. A similar argument shows that $z_\beta(a_i pq) \in G \iff z_\beta(a_i pq^*) \in G$. □

Thus, in order to prove that $z_{\alpha+\beta}(a_i w), z_\beta(a_i w) \in G$ we are allowed to replace w by another word obtained by reversing some tail of w of length $\leq d+1$, and this operation can be applied several times. The corresponding permutations clearly generate the full symmetric group on $d+1$ letters, and since T_+ has d elements, we can assume that w has a repeated letter at the end: $w = pt^2$ where $t \in T_+$. But then we have

$$[z_\alpha(t^2), z_\beta(a_i p)] = z_{\alpha+\beta}(-a_i w)z_{\alpha+2\beta}(a_i pt^2 p^* a_i^*)$$

and

$$[z_\alpha(1), z_\beta(a_i pt)] = z_{\alpha+\beta}(-a_i pt)z_{\alpha+2\beta}(a_i pt^2 p^* a_i^*).$$

Since by induction $z_\beta(a_i pt) \in G$, $z_\beta(a_i p) \in G$, and $z_{\alpha+\beta}(a_i pt) \in G \cap [N,G]\langle S_{\alpha+2\beta}\rangle$, we conclude that $z_{\alpha+\beta}(a_i w) \in G \cap [N,G]\langle S_{\alpha+2\beta}\rangle$ as well. A similar argument shows that $z_\beta(a_i w) \in G$. □

Proof of Lemma 8.13. By Lemma 8.14, G contains the root subgroups Z_β and $Z_{\alpha+\beta}$. Relations (8.5) and (8.6) and the assumption that J is generated by U as a form parameter easily imply that $Z = Z_{\alpha+2\beta} \subseteq G$ and $Z \subseteq [N, G]Z_{\alpha+\beta}$. The first inclusion completes the proof of Lemma 8.13(a). The second inclusion combined with Lemma 8.14(ii) shows that $Z \subseteq [N, G]\langle S_{\alpha+2\beta} \rangle$, which proves the second assertion of Lemma 8.13(b). Finally, from (8.6) we get that $z_{\alpha+2\beta}(2r) = [z_\beta(r), z_{\alpha+\beta}(1)]$ for every $r \in J$, which proves the first assertion of Lemma 8.13(b). $\qquad\square$

$\hfill\square$

8.7. Twisted groups of type $^2F_4$.

Let R be a commutative ring of characteristic 2 and $* : R \to R$ an injective homomorphism such that $(r^*)^* = r^2$ for any $r \in R$. We will use a standard realization of the root system F_4 inside $\mathbb{R}^4$:

$$F_4 = \{\pm e_i, \pm e_i \pm e_j, \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) : 1 \le i \ne j \le 4\}.$$

Let $G = St_{F_4}(R)$. Consider another root system, normalizing the roots from F_4:

$$\overline{F_4} = \{\bar{v} = \tfrac{v}{|v|} : v \in F_4\} = \{\pm e_i, \tfrac{1}{\sqrt{2}}(\pm e_i \pm e_j), \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) : 1 \le i \ne j \le 4\}.$$

If $\{X_\gamma\}_{\gamma \in F_4}$ denotes the standard grading of G, then since R has a characteristic 2, $\{X_{\bar{\gamma}} = X_\gamma\}_{\bar{\gamma} \in \overline{F_4}}$ is an $\overline{F_4}$-grading. We consider an $\overline{F_4}$-subgrading defined in the following way. For any $\gamma \in F_4$ we put

$$\tilde{X}_{\bar{\gamma}} = \{\tilde{x}_{\bar{\gamma}}(r) : r \in R\} \text{ where } \tilde{x}_{\bar{\gamma}}(r) = \begin{cases} x_\gamma(r) & \text{if } \gamma \text{ is a short root} \\ x_\gamma(r^*) & \text{if } \gamma \text{ is a long root} \end{cases}$$

The commutation relations between the elements of the root subgroups of this grading are the following. Let $\alpha, \beta \in \overline{F_4}$. Then we have

$$[\tilde{x}_\alpha(r), \tilde{x}_\beta(s)] = \begin{cases} 1 & \text{if the angle between } \alpha \text{ and } \beta \text{ is } \frac{\pi}{4} \text{ or } \frac{\pi}{2} \\ \tilde{x}_{\alpha+\beta}(rs) & \text{if the angle between } \alpha \text{ and } \beta \text{ is } \frac{2\pi}{3} \\ \tilde{x}_{\sqrt{2}\alpha+\beta}(r^*s)\tilde{x}_{\alpha+\sqrt{2}\beta}(rs^*) & \text{if the angle between } \alpha \text{ and } \beta \text{ is } \frac{3\pi}{4} \end{cases}.$$

We denote by $\tilde{G}$ the graded cover of the group generated by $\{\tilde{X}_\alpha\}_{\alpha \in \overline{F_4}}$. These very symmetric relations permit construct graded automorphisms of $\tilde{G}$ from isometries of the root system. Let ρ be an isometry of $\mathbb{R}^4$ which preserves $\overline{F_4}$. Then we can define an automorphism of $\tilde{G}$, denoted by the same symbol ρ:

$$\rho(\tilde{x}_\alpha(r)) = \tilde{x}_{\rho(\alpha)}(r).$$

Let q be the isometry represented by the following matrix with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

and τ be the isometry represented by the following matrix with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then q has order 2 and τ has order 8. Moreover they commute.

Example 6: *Twisted groups of type $^2F_4$.* The group in this example is denoted by $St_{^2F_4}(R)$ and is obtained from $\tilde{G}$ using the twist by the automorphism q. Define

$$\eta : \bigoplus_{i=1}^4 \mathbb{R}e_i \to \bigoplus_{i=1}^2 \mathbb{R}\varepsilon_i$$

by $\eta(e_1) = (1 + \sqrt{2})\varepsilon_1$, $\eta(e_2) = \varepsilon_1$, $\eta(e_3) = (1 + \sqrt{2})\varepsilon_2$ and $\eta(e_4) = \varepsilon_2$. It is easy to see that the root system $\Psi = \eta(\overline{F_4})$ is as in the diagram below.
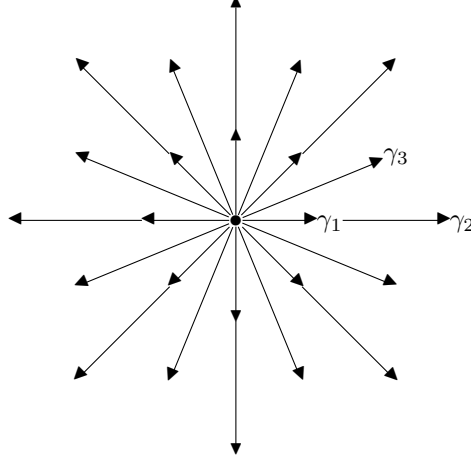
We see that there are three types of roots. We shall call them short, long and double by analogy with BC_2 even though this time the double roots are $(\sqrt{2} + 1)$ times longer than the short ones.

The short roots are

$$\{\pm\varepsilon_i, \frac{1}{\sqrt{2}}(\pm\varepsilon_i \pm \varepsilon_j): \ 1 \leq i \neq j \leq 2\},$$

the double roots are

$$\{\pm(\sqrt{2} + 1)\varepsilon_i, \frac{\sqrt{2}+1}{\sqrt{2}}(\pm\varepsilon_i \pm \varepsilon_j): \ 1 \leq i \neq j \leq 2\}$$

and the long roots are

$$\{\frac{1}{\sqrt{2}}((1 + \sqrt{2})\varepsilon_i + \varepsilon_j): \ 1 \leq i \neq j \leq 2\}.$$

In this example we denote by $\{Y_\alpha\}_{\alpha \in \Psi}$ not the coarsened grading on $\tilde{G}$, but its fattening (see § 4.5). If $\alpha \in \Psi$ is a long or double root, the corresponding root subgroup Y_α consists of elements

$$\{y_\alpha(r, s) = \tilde{x}_{\alpha_1}(r)\tilde{x}_{\alpha_2}(s): \ \eta^{-1}(\alpha) = \{\alpha_1, \alpha_2\}, \ r, s, t \in R\}.$$

If $\alpha \in \Psi$ is a short root, then Y_α consists of elements

$$y_\alpha(r, s, t, u) = \tilde{x}_{\alpha_1}(r)\tilde{x}_{\alpha_2}(s)\tilde{x}_{(1+\sqrt{2})\alpha_1}(t)\tilde{x}_{(1+\sqrt{2})\alpha_2}(u),$$

where $\eta^{-1}(\alpha) = \{\alpha_1, \alpha_2\}$ and $r, s, t, u \in R$ (the short root subgroups are the ones being fattened).

Since τ commutes with q, it permutes the root subgroups $\{Y_\alpha\}$. Moreover this action has 3 orbits corresponding to short, double and long roots. Thus, we will

describe $Z_\alpha = Y_\alpha^{\langle q \rangle}$ for one root of each type: $\gamma_1 = \varepsilon_1$, $\gamma_2 = (1 + \sqrt{2})\varepsilon_1$ and $\gamma_3 = \frac{1}{\sqrt{2}}((1 + \sqrt{2})\varepsilon_1 + \varepsilon_2)$.

$$Z_{\gamma_1} = \{z_{\gamma_1}(r, s) = \tilde{x}_{\frac{e_1 - e_2}{\sqrt{2}}}(r)\tilde{x}_{e_2}(r)\tilde{x}_{e_1}(r^*r + s)\tilde{x}_{\frac{e_1 + e_2}{\sqrt{2}}}(s) : r, s \in R\},$$

$$Z_{\gamma_2} = \{z_{\gamma_2}(r) = z_{\gamma_1}(0, r) = \tilde{x}_{e_1}(r)\tilde{x}_{\frac{e_1 + e_2}{\sqrt{2}}}(r) : r \in R\},$$

$$Z_{\gamma_3} = \{z_{\gamma_3}(r) = \tilde{x}_{\frac{e_1 + e_4}{\sqrt{2}}}(r)\tilde{x}_{\frac{1}{2}(e_1 + e_2 + e_3 - e_4)}(r) : r \in R\}.$$

If $\gamma \in \Psi$ is a long or double root, the corresponding root subgroup Z_γ is isomorphic to $(R, +)$, and if $\gamma \in \Psi$ is a short root, then Z_γ is nilpotent of class 2.

To simplify the notation we denote $\tau^i(\gamma_k)$ by $\gamma_{k,i}$ and $z_{\gamma_{k,i}}(r)$ by $z_{k,i}(r)$. Note that τ acts on the root system Ψ as a counterclockwise rotation by $\frac{\pi}{4}$.

We list the relevant commutation relations between the elements of root subgroups. All the commutation relations may be found in the Tits paper [Ti] (observe that our notation is slightly different).

($E1$) $[z_{1,i}(r, s), z_{1,i+1}(t, u)] = z_{3,i}(rt)$

($E2$) $[z_{1,i}(r, s), z_{3,i+1}(t)] = z_{1,i+1}(0, rt) = z_{2,i+1}(rt)$

($E3$) $[z_{1,0}(r, s), z_{3,3}(t)] \equiv z_{1,3}(tr, 0)z_{1,2}(t^*s, 0)z_{1,1}(tr^*r + ts, 0) \pmod{\prod_{i=1}^{3} Z_{\gamma_{3,i-1}}Z_{\gamma_{2,i}}}$

($E4$) $[z_{3,i+4}(r), z_{2,i+2}(s)] = z_{2,i+3}(rs)$

($E5$) $[z_{3,i}(r), z_{2,i+3}(s)] = z_{2,i+2}(rs)$

($E6$) $[z_{2,i}(r), z_{2,i+3}(s)] \equiv z_{3,i+1}(rs) \pmod{Z_{\gamma_{2,i+1}}Z_{\gamma_{2,i+2}}}$

Proposition 8.16. *Let R be a finitely generated ring. Then the group $St_{2F_4}(R)$ has property (T).*

Proof. As usual, we need to check two things:

    (a) The Ψ-grading of $St_{2F_4}(R)$ is strong at each root subgroup

    (b) The pair $(St_{2F_4}(R), Z_\gamma)$ has relative (T) for each $\gamma \in \Psi$

We check the condition (a) for the Borel subgroup corresponding to the Borel set with boundary $\{\gamma_{1,0}, \gamma_{2,0}, \gamma_{3,3}\}$. Relation (E1) implies condition (a) for the long root subgroups (that is, the root subgroups $Z_{\gamma_{3,i}}$). From relations (E2) and (E3) we obtain condition (a) for the short root subgroups. This also implies condition (a) for the double root subgroups since they are contained in the short root subgroups.

Now let us prove (b). Relations (E4) and (E5) imply that the pair

$$(\langle Z_{\gamma_{3,i}}, Z_{\gamma_{3,i+4}}, Z_{\gamma_{2,i+2}}, Z_{\gamma_{2,i+3}}\rangle, Z_{\gamma_{2,i+2}}Z_{\gamma_{2,i+3}})$$

is a quotient of $(St_2(R) \ltimes R^2, R^2)$. This yields relative property (T) for the double root subgroups. It follows from relation (E6) that any long root subgroup lies in a bounded product of double root subgroups, so relative (T) also holds for the long root subgroups.

It remains to prove relative property (T) for the short root subgroups. By symmetry, it suffices to treat the subgroup Z_{γ_1}. Fix $s \in R$, and consider relation (E3) with $r = 0$ and t arbitrary. It implies that the set

$$P_0 = \{z_{1,2}(t^*s, 0)z_{1,1}(ts, 0) : t \in R\}$$

lies in a bounded product of the double and long root subgroups. The same holds for each of the sets $P_1 = \{z_{1,2}(t^*, 0)z_{1,1}(t, 0) : t \in R\}$ (setting $s = 1$ in P_0),

$P_2 = \{z_{1,2}(t^*s^*, 0)z_{1,1}(ts, 0) : t \in R\}$ (replacing t by ts in P_1) and $P_3 = \{z_{1,2}(t^*(s + s^*), 0)z_{1,1}(t(s + s^*), 0) : t \in R\}$ (replacing s by $s + s^*$ in P_0). Considering the product $P_0P_2P_3$ and using relation (E1) with $i = 1$, we conclude that the set $P = \{z_{1,1}(t(s + s^*), 0) : t \in R\}$ lies in a bounded product of the double and long root subgroups (for any fixed s).

Let I be the ideal of R generated by $\{s + s^* : s \in R\}$. Since R is finitely generated, it is Noetherian, and so I is generated by a finite subset of $\{s + s^* : s \in R\}$. Hence, by what we just proved the set $\{z_{1,1}(r, 0) : r \in I\}$ also lies in a bounded product of the double and long root subgroups. On the other hand, I has finite index in R since the quotient ring R/I is generated by idempotent elements (as $(r^*)^* = r^2$). Thus, the set $\{z_{1,1}(r, 0) : r \in R\}$ lies in a bounded product of the double and long root subgroups and fixed elements of G. Since the short root subgroup Z_{γ_1} is a product of $\{z_{1,1}(r, 0) : r \in R\}$ and Z_{γ_2}, we have proved relative property (T) for Z_{γ_1}. □

9. Estimating relative Kazhdan constants

The goal of this section is to prove Theorems 2.7 and 2.8 from Section 2. For convenience we shall use the following terminology and notations:

- A unitary representation V of a group G will be referred to as a *G-space*
- If U is a subspace of V, by P_U we denote the operator of orthogonal projection onto U. For any nonzero $v \in V$ we set $P_v = P_{\mathbb{C}v}$.

9.1. Hilbert-Schmidt scalar product.
Consider the space $HS(V)$ of Hilbert-Schmidt operators on V, i.e., linear operators $A : V \to V$ such that $\sum_i \|A(e_i)\|^2$ is finite where $\{e_i\}$ is an orthonormal basis of V. The space $HS(V)$ is endowed with the *Hilbert-Schmidt scalar product* given by

$$\langle A, B \rangle = \sum_i \langle A(e_i), B(e_i) \rangle.$$

By a standard argument this definition does not depend on the choice of $\{e_i\}$. The associated norm on $HS(V)$ will be called the *Hilbert-Schmidt norm.*

If V is a unitary representation of a group G then $HS(V)$ is also a unitary representation of G – the action of an element $g \in A$ on an operator $A \in HS(V)$ is defined by $(gA)(v) = gA(g^{-1}v)$. If the element g acts by a scalar on V, for instance if g is in the center of G and V is an irreducible representation, then g acts trivially on $HS(V)$.

For any unit vector $v \in V$ the projection $P_v : V \to V$ is an element in $HS(V)$ of norm 1. The map $v \to P_v$ does not preserve the scalar product. However, we have the following explicit formula for $\langle P_u, P_v \rangle$.

Lemma 9.1. *If u and v are unit vectors in a Hilbert space V, then*

$$\langle P_u, P_v \rangle = |\langle u, v \rangle|^2 \quad \text{and therefore} \quad \|P_u - P_v\| \leq \sqrt{2}\|u - v\|$$

Proof. Consider an orthonormal basis $\{e_i\}$ such that $e_1 = u$ and the subspace spanned by $\{e_1, e_2\}$ coincides with the subspace spanned by $\{u, v\}$. Then

$$\langle P_u, P_v \rangle = \sum_i \langle P_u(e_i), P_v(e_i) \rangle = \langle u, \langle u, v \rangle v \rangle = |\langle u, v \rangle|^2.$$

Therefore,

$$\|P_u - P_v\|^2 = 2(1 - |\langle u, v \rangle|^2) = (1 + |\langle u, v \rangle|)\|u - v\|^2 \leq 2\|u - v\|^2. \quad □$$

One can define a non-linear, norm preserving, map $\iota : V \to HS(V)$ by

$$\iota(v) = \|v\| P_v$$

The following lemma imposes a restriction on the change of codistance between vectors under the map ι.

Lemma 9.2. *Let $v_1, \ldots, v_k$ be vectors in V. Then*

$$2 \operatorname{codist}(v_1, v_2, \ldots, v_k) - 1 \le \operatorname{codist}(\iota(v_1), \iota(v_2), \ldots, \iota(v_k))$$

where $\operatorname{codist}(u_1, \ldots, u_k)$ denotes the ratio $\dfrac{\|\sum u_i\|^2}{k \sum \|u_i\|^2}$.

Proof. The inequality $\cos^2 \varphi \ge 2 \cos \varphi - 1$ implies that

$$\langle \iota(v), \iota(w) \rangle = \|v\|\|w\| \left| \left\langle \frac{v}{\|v\|}, \frac{w}{\|w\|} \right\rangle \right|^2 \ge 2 \langle v, w \rangle - \|v\|\|w\|$$

Therefore

$$\left\| \sum \iota(v_i) \right\|^2 = \sum_{i,j} \langle \iota(v_i), \iota(v_j) \rangle \ge$$

$$\sum_{i,j} (2 \langle v_i, v_j \rangle - \|v_i\|\|v_j\|) = 2 \left\| \sum v_i \right\|^2 - \left(\sum \|v_i\| \right)^2,$$

which translates into the stated inequality between codistances. $\qquad\square$

Let $\{(U_i, \langle \ , \ \rangle_i)\}_{i \in I}$ be a family of Hilbert spaces. Recall that the Hilbert direct sum of U_i's denoted by $\oplus_{i \in I} U_i$ is the Hilbert space consisting of all families $(u_i)_{i \in I}$ with $u_i \in U_i$ such that $\sum_i \langle u_i, u_i \rangle_i < \infty$ with inner product

$$\langle (u_i), (w_i) \rangle = \sum_i \langle u_i, w_i \rangle_i.$$

Let V be a unitary representation of G and let N be a subgroup of G. Denote by $(\hat{N})_f$ the set of equivalence classes of irreducible finite dimensional representations of N. Let $\pi \in (\hat{N})_f$. Denote by $V(\pi)$ the N-subspace of V generated by all irreducible N-subspaces of V isomorphic to π. Applying Zorn's Lemma we obtain that $V(\pi)$ is isomorphic to a Hilbert direct sum of N-spaces isomorphic to π. We may also decompose V as a Hilbert direct sum $V = V_\infty \oplus (\oplus_{\pi \in (\hat{N})_f} V(\pi))$, where V_∞ is the orthogonal complement of $\oplus_{\pi \in (\hat{N})_f} V(\pi)$ in V.

Lemma 9.3. *Let $v \in V$ be a unit vector and $v = v_\infty + \sum_{\pi \in (\hat{N})_f} v_\pi$ the decomposition of v such that $v_\infty \in V_\infty$ and $v_\pi \in V(\pi)$. Then*

$$\|P_{HS(V)^N}(P_v)\|^2 \le \sum_{\pi \in (\hat{N})_f} \frac{\|v_\pi\|^4}{\dim \pi}$$

(where the norm on the left-hand side is the Hilbert-Schmidt norm). Moreover, if V is an irreducible N-space, then $\|P_{HS(V)^N}(P_v)\|^2 = \frac{1}{\dim V}$.

Proof. Let $T \in HS(V)^N$. Then T preserves the decomposition $V = V_\infty \oplus (\oplus_{\pi \in (\hat{N})_f} V(\pi))$. Moreover, by Proposition A.1.12 of [BHV] T sends V_∞ to zero. Hence we have a decomposition

$$HS(V)^N = \oplus_{\pi \in (\hat{N})_f} HS(V)^{N,\pi}$$

where $HS(V)^{N,\pi}$ is the subspace of operators from $HS(V)^N$ which act trivially on the orthogonal complement of $V(\pi)$. Thus, we may write $T = \sum_{\pi \in (\hat{N})_f} T_\pi$ where $T_\pi \in HS(V)^{N,\pi}$, and T_{π_1} and T_{π_2} are orthogonal for non-isomorphic π_1 and π_2. Note also that

$$\|P_{HS(V)^N}(P_v)\|^2 = \sum_{\pi \in (\hat{N})_f} \|P_{HS(V)^{N,\pi}}(P_v)\|^2. \qquad (***)$$

Now fix $\pi \in (\hat{N})_f$, and decompose $V(\pi)$ as a Hilbert direct sum $\oplus_{i \in I} U_i$ of (pairwise orthogonal) N-spaces $\{U_i\}$ each of which is isomorphic to π. Note that

$$HS(V)^{N,\pi} = \oplus_{i,j \in I} HS(V)^{N,\pi}_{i,j}$$

where $HS(V)^{N,\pi}_{i,j}$ is the subspace of operators from $HS(V)^{N,\pi}$ which map U_i onto U_j and acts trivially on the orthogonal complement of U_i. A standard application of Schur's lemma shows that each subspace $HS(V)^{N,\pi}_{i,j}$ is one-dimensional. Thus, if for each $i, j \in I$ we choose an element $T_{i,j} \in HS(V)^{N,\pi}_{i,j}$ with $\|T_{i,j}\| = 1$, then $\{T_{i,j}\}$ form an orthonormal basis of $HS(V)^{N,\pi}$. Therefore,

$$\|P_{HS(V)^{N,\pi}}(P_v)\|^2 = \sum_{i,j} |\langle P_v, T_{i,j} \rangle|^2 \qquad (!!!)$$

Decompose v_π as $v_\pi = \sum_{i \in I} u_i$, where $u_i \in U_i$. Since there exists an orthonormal basis of V containing v, we have

$$\langle P_v, T_{i,j} \rangle = \langle v, T_{i,j}(v) \rangle = \langle u_j, T_{i,j}(u_i) \rangle.$$

Next note that $T^*_{i,j} T_{i,j}$ is an element of $HS(V)^{N,\pi}_{i,i}$ and thus by an earlier remark must act as multiplication by some scalar λ_i on U_i. Moreover, $\lambda_i = \frac{1}{\dim \pi}$ because if $f_1, \ldots, f_k$ is an orthonormal basis for U_i, then

$$\lambda_i \dim \pi = \sum_{l=1}^{k} \langle T^*_{i,j} T_{i,j} f_l, f_l \rangle = \langle T_{i,j} f_l, T_{i,j} f_l \rangle = \|T_{i,j}\|^2 = 1.$$

Hence $\|T_{i,j} u_i\|^2 = |\langle T^*_{i,j} T_{i,j} u_i, u_i \rangle| = \frac{\|u_i\|^2}{\dim \pi}$, whence $|\langle P_v, T_{i,j} \rangle|^2 \leq \frac{\|u_i\|^2 \|u_j\|^2}{\dim \pi}$, and (!!!) yields

$$\|P_{HS(V)^{N,\pi}}(P_v)\|^2 \leq \sum_{i,j \in I} \frac{\|u_i\|^2 \|u_j\|^2}{\dim \pi} = \frac{\|v_\pi\|^4}{\dim \pi}.$$

Combining this result with (***) we deduce the first assertion of the lemma.

Now we prove the second assertion. Assume that V is an irreducible N-space. As above, if V is infinite dimensional, then $HS(V)^N = 0$, so $\|P_{HS(V)^N}(P_v)\| = 0$. If V is finite-dimensional, then $HS(V)^N = HS(V)^G$ is one-dimensional consisting of scalar operators. The operator of multiplication by λ has Hilbert-Schmidt norm $|\lambda|\sqrt{\dim \pi}$, so we can assume that $T_{1,1}$ acts as multiplication by $\frac{1}{\sqrt{\dim \pi}}$. Therefore,

$$\|P_{HS(V)^N}(P_v)\|^2 = |\langle P_v, T_{1,1} \rangle|^2 = |\langle v, T_{1,1} v \rangle|^2 = \frac{1}{\dim \pi}. \quad \square$$

9.2. Relative property (*T*) for group extensions. We start with a simple result which reduces verification of relative property (*T*) to the case of irreducible representations.

Lemma 9.4. *Let G be a group, N a normal subgroup of G and S a finite subset of G. Assume that there exists a set of positive numbers $\{\varepsilon_s : s \in S\}$ such that for any irreducible G-space U without nonzero N-invariant vectors and any $0 \neq u \in U$ there exists $s \in S$ with $\|su - u\| \geq \varepsilon_s \|u\|$. Then*

$$\kappa(G, N; S) \geq \frac{1}{\sqrt{\sum_{s \in S} \frac{1}{\varepsilon_s^2}}}.$$

Proof. Let V be a G-space without nonzero N-invariant vectors. Write V as a direct integral $\int_Z^\oplus V(z) d\mu(z)$ of irreducible G-spaces over some measure space Z. Take any $0 \neq v \in V$, and write it as $v = \int_Z v(z)$ with $v(z) \in V(z)$ for all z. For every $s \in S$ we put

$$Z_s = \{z \in Z : \|sv(z) - v(z)\| \geq \varepsilon_s \|v(z)\|\}.$$

Since $\cup_{s \in S} Z_s = Z$, we have

$$\sum_{s \in S} \int_{Z_s} \|v(z)\|^2 d\mu(z) \geq \int_Z \|v(z)\|^2 d\mu(z) = \|v\|^2,$$

and therefore there exists $g \in S$ such that

$$\int_{Z_g} \|v(z)\|^2 d\mu(z) \geq \frac{\|v\|^2}{\varepsilon_g^2 \sum_{s \in S} \frac{1}{\varepsilon_s^2}}.$$

Thus,

$$\|gv - v\|^2 \geq \int_{Z_g} \|gv(z) - v(z)\|^2 d\mu(z) > \int_{Z_g} \varepsilon_g^2 \|v(z)\|^2 d\mu(z) \geq \frac{\|v\|^2}{\sum_{s \in S} \frac{1}{\varepsilon_s^2}}. \quad \square$$

We are now ready to prove Theorem 2.7 whose statement (in fact, an extended version of it) is recalled below.

Theorem 9.5. *Let G be a group, N a normal subgroup of G and $Z \subseteq Z(G) \cap N$. Put $H = Z \cap [N, G]$. Assume that the subsets A, B and C of G satisfy the following conditions*

(1) *A and N generate G,*
(2) *$\kappa(G/Z, N/Z; B) \geq \varepsilon$,*
(3) *$\kappa(G/H, Z/H; C) \geq \delta$.*

Then the following hold:

(a)

$$\kappa(G, H; A \cup B) \geq \frac{12\varepsilon}{5\sqrt{72\varepsilon^2 |A| + 25|B|}}.$$

(b)

$$\kappa(G, N; A \cup B \cup C) \geq \frac{1}{\sqrt{3}} \min\{\frac{12\varepsilon}{5\sqrt{72\varepsilon^2 |A| + 25|B|}}, \delta\}.$$

Proof. (a) Using Lemma 9.4 we are reduced to proving the following claim:

Claim. *Let V be a non-trivial irreducible G-space without nonzero H-invariant vectors. Then there is no unit vector $v \in V$ such that*

$$\|sv - v\| \leq \frac{\sqrt{2}}{5} \text{ for any } s \in A \text{ and } \|sv - v\| \leq \frac{12\varepsilon}{25} \text{ for any } s \in B.$$

Let us assume the contrary, and let $v \in V$ be a unit vector satisfying the above conditions. First we shall show that

$$(9.1) \qquad \|P_{HS(V)^N}(P_v)\|^2 \leq \frac{337}{625}$$

Case 1: V has an N-eigenvector. In this case V is spanned by N-eigenvectors, and thus we may write $v = \sum_i v_i$, where v_i are N-eigenvectors corresponding to distinct characters.

Assume that $\|v_j\| > \frac{4}{5}$ for some j. Since N is normal in G, any $g \in G$ sends the vector v_j to some eigenvector for N. Consider the subgroup

$$K = \{g \in G : g^{-1}ngv_j = nv_j \text{ for any } n \in N\}$$

consisting of elements fixing the character corresponding to v_j. Note that v_j is $[K, N]$-invariant. Since V has no nonzero H-invariant vectors and $H \subseteq [G, N]$, K is a proper subgroup of G. Thus, since $N \subseteq K$, there should exist $s \in A$ which is not in K. In particular $\langle sv_j, v_j \rangle = 0$ as sv_j and v_j are both N-eigenvectors corresponding to distinct characters. Hence

$$\|sv - v\|^2 = \|sv_j - (v - v_j)\|^2 + \|s(v - v_j) - v_j\|^2 > \frac{2}{25}.$$

But this contradicts to the choice of v.

Hence $\|v_i\| \leq \frac{4}{5}$ for all i, and Lemma 9.3 easily implies that, $\|P_{HS(V)^N}(P_v)\|^2 \leq (4/5)^4 + (3/5)^4 = \frac{337}{625}$ (where the equality is achieved if after reindexing $\|v_1\| = 4/5$, $\|v_2\| = 3/5$ and $v_i = 0$ for $i \neq 1, 2$).

Case 2: V has no N-eigenvectors. Then we get directly from Lemma 9.3 that $\|P_{HS(V)^N}(P_v)\|^2 \leq \frac{1}{2} < \frac{337}{625}$.

Thus, we have established (9.1) in both cases. Let $Q = P_{(HS(V)^N)^\perp}(P_v)$. Then $\|Q\| \geq \sqrt{1 - \frac{337}{625}} = \frac{12\sqrt{2}}{25}$, so Lemma 9.1 yields

$$(9.2) \qquad \|sQ - Q\| = \|sP_v - P_v\| = \|P_{sv} - P_v\| \leq \sqrt{2}\|sv - v\| \leq \frac{12\sqrt{2}\varepsilon}{25} \leq \varepsilon\|Q\|$$

for every $s \in B$.

Since V is an irreducible G-space, the elements of Z act as scalars on V, so Z acts trivially on $HS(V)$. Thus, $(HS(V)^N)^\perp$ is a G/Z-space without nonzero N/Z-invariant vectors, so (9.2) violates the assumption $\kappa(G/Z, N/Z; B) \geq \varepsilon$. This contradiction proves the claim and hence also part (a).

(b) Let V be a G-space without non-trivial N-invariant vectors and $0 \neq v \in V$. Let U be the orthogonal complement of V^Z in V and W the orthogonal complement of U^H in U. Then $V = V^Z \oplus U^H \oplus W$, so the projection of v onto at least one of the three subspaces V^Z, U^H and W has norm at least $\frac{\|v\|}{\sqrt{3}}$.

Case 1: $\|P_{V^Z}(v)\| \geq \frac{\|v\|}{\sqrt{3}}$. Then by condition (2) there exists $s \in B$ such that

$$\|sv - v\| \geq \|sP_{V^Z}(v) - P_{V^Z}(v)\| > \frac{\varepsilon\|v\|}{\sqrt{3}} > \frac{12\varepsilon}{5\sqrt{72\varepsilon^2|A| + 25|B|}} \frac{\|v\|}{\sqrt{3}}.$$

Case 2: $\|P_{U^H}(v)\| \geq \frac{\|v\|}{\sqrt{3}}$. Since U^H is a representation of G/H without nonzero Z/H-invariant vectors, by condition (3) there exists $s \in C$ such that

$$\|sv - v\| \geq \|sP_{U^H}(v) - P_{U^H}(v)\| > \frac{\delta\|v\|}{\sqrt{3}}.$$

Case 3: $\|P_W(v)\| \geq \frac{\|v\|}{\sqrt{3}}$. In this case we can apply part (a) to deduce that there exists $s \in A \cup B$ such that

$$\|sv - v\| \geq \|sP_{V^H}(v) - P_{V^H}(v)\| > \frac{12\varepsilon}{5\sqrt{72\varepsilon^2|A| + 25|B|}} \frac{\|v\|}{\sqrt{3}}.$$

$\square$

Remark: Theorem 9.5 generalizes a similar result due to Serre in the case $G = N$ (see, e.g., [BHV, Theorem 1.7.11] or [Ha, Theorem 1.8]). The case of a pair of subgroups (G, N) is also considered in [NPS, Lemma 1.1].

9.3. Codistance bounds in nilpotent groups. Let G be a nilpotent group generated by k subgroups $X_1, \ldots, X_k$. In this subsection we prove Theorem 2.8 which gives a bound for the codistance $codist(\{X_i\})$. The case when $k = 2$ and G is of nilpotency class 2 was considered in Section 4 of [EJ]. Here we strengthen and generalize those results.

We will use the following auxiliary result.

Lemma 9.6. *Let (Z, μ) be a measure space and $z \to V(z)$ a measurable field of Hilbert spaces over Z. Let $A(z)$ and $B(z)$ be subspaces of $V(z)$. Put $A = \int_Z^{\oplus} A(z)$ and $B = \int_Z^{\oplus} B(z)$. Then for any measurable subset Z_1 of Z such that $\mu(Z \setminus Z_1) = 0$,*

$$\mathrm{orth}(A, B) \leq \sup_{z \in Z_1} \mathrm{orth}(A(z), B(z)).$$

Proof. Let $a = a(z) \in A(z)$ and $b = b(z) \in B(z)$ be two vectors. Then

$$
\begin{aligned}
|\langle a, b \rangle| &= \int_Z |\langle a(z), b(z) \rangle| \, d\mu(z) = \int_{Z_1} |\langle a(z), b(z) \rangle| \, d\mu(z) \\
&\leq \int_{Z_1} \mathrm{orth}(A(z), B(z)) \|a(z)\| \|b(z)\| d\mu(z) \\
&\leq \sup_{z \in Z_1} \mathrm{orth}(A(z), B(z)) \|a\| \|b\|.
\end{aligned}
$$

$\square$

Corollary 9.7. *Let G be a countable group generated by subgroups $X_1, \ldots, X_k$. Then $\mathrm{codist}(X_1, \ldots, X_k)$ is equal to the supremum of the quantities $\mathrm{codist}(V^{X_1}, \ldots, V^{X_k})$, where V runs over all non-trivial irreducible unitary representations of G.*

Proof. Let V be a unitary representations of G without G-invariant vectors. By [BHV, Theorem F.5.3], $V \cong \int_Z V(z) d\mu(z)$ for some measurable field of irreducible G-spaces $V(z)$ over a standard Borel space Z. Put

$$Z_0 = \{z \in Z : V(z) \text{ is a trivial } G\text{-space}\}.$$

Then, Z_0 is a measurable subspace of Z, and since V does not have non-trivial G-invariant vectors, $\mu(Z_0) = 0$. Thus, by Lemma 9.6,

$$
\begin{aligned}
\mathrm{codist}(V^{X_1}, \ldots, V^{X_k}) &= (\mathrm{orth}(V^{X_1} \times \cdots \times V^{X_k}, \mathrm{diag}\, V))^2 \\
&\leq \sup_{z \in Z \setminus Z_0} (\mathrm{orth}(V(z)^{X_1} \times \cdots \times V(z)^{X_k}, \mathrm{diag}\, V(z)))^2 \\
&= \sup_{z \in Z \setminus Z_0} \mathrm{codist}(V(z)^{X_1}, \ldots, V(z)^{X_k}).
\end{aligned}
$$

$\square$

We are now ready to prove the main result of this subsection.

Theorem 9.8. *Let G be a countable group generated by subgroups $X_1, \ldots, X_k$. Let H be a subgroup of $Z(G)$, and let m be the minimal dimension of an irreducible representation of G which is not trivial on H. Denote by $\bar{X}_i$ the image of X_i in G/H, and let $\varepsilon = 1 - \mathrm{codist}(\bar{X}_1, \ldots, \bar{X}_k)$. Then $\mathrm{codist}(X_1, \ldots, X_k) \leq 1 - \frac{(m-1)\varepsilon}{2m}$.*

Proof. By Corollary 9.7 we only have to consider non-trivial irreducible G-spaces. Let V be a non-trivial irreducible G-space, and let $n = \dim V \in \mathbb{N} \cup \infty$. If H acts trivially on V, there is nothing to prove since $\varepsilon > \frac{(m-1)\varepsilon}{2m}$. Thus, we can assume that H acts non-trivially, so $n \geq m$.

Now take any vectors $v_i \in V^{X_i}$ $(i = 1, \ldots, k)$. It is sufficient to show that $\mathrm{codist}(v_1, \ldots, v_k) \leq 1 - \frac{(n-1)\varepsilon}{2n}$. Note that Lemma 9.3 implies that

$$
\|P_{HS(V)^G}(\iota(v_i))\|^2 = \frac{1}{n}\|v_i\|^2 = \frac{1}{n}\|\iota(v_i)\|^2
$$

and so

$$
\|P_{HS(V)^G}(\sum_{i=1}^{k} \iota(v_i))\|^2 \leq k \sum_{i=1}^{k} \|P_{HS(V)^G}(\iota(v_i))\|^2 \leq \frac{k}{n} \sum_{i=1}^{k} \|\iota(v_i)\|^2
$$

On the other hand, since $Z(G)$ acts trivially on $HS(V)$, the action factors through G/H. This means that $(HS(V)^G)^\perp$ is a G/H-space without invariant vectors and by the definition of codistance

$$
\begin{aligned}
\|P_{(HS(V)^G)^\perp}(\sum_{i=1}^{k} \iota(v_i))\|^2 &\leq k\, \mathrm{codist}(\bar{X}_1, \ldots, \bar{X}_k) \sum_{i=1}^{k} \|P_{(HS(V)^G)^\perp}(\iota(v_i))\|^2 \\
&= \frac{k(n-1)}{n} \mathrm{codist}(\bar{X}_1, \ldots, \bar{X}_k) \sum_{i=1}^{k} \|\iota(v_i)\|^2.
\end{aligned}
$$

Combining these inequalities gives that $\left\|\sum_{i=1}^{k} \iota(v_i)\right\|^2$ is bounded above by

$$
k \left(\frac{1}{n} + \mathrm{codist}(\bar{X}_1, \ldots, \bar{X}_k) \frac{n-1}{n} \right) \sum_{i=1}^{k} \|\iota(v_i)\|^2
$$

Therefore

$$
\mathrm{codist}(\iota(v_1), \ldots, \iota(v_k)) \leq \left(\frac{1}{n} + \mathrm{codist}(\bar{X}_1, \ldots, \bar{X}_k) \frac{n-1}{n} \right),
$$

which combined with Lemma 9.2 gives the following inequality equivalent to the one in the statement of the theorem:

$$2 \operatorname{codist}(v_1, \ldots, v_k) - 1 \le \left(\frac{1}{n} + \operatorname{codist}(\bar{X}_1, \ldots, \bar{X}_k) \frac{n-1}{n} \right). \quad \square$$

We are now ready to prove Theorem 2.8 (whose statement is recalled below).

Theorem 9.9. *Let G be a countable nilpotent group of class c generated by subgroups $X_1, \ldots, X_k$. Then*

$$\operatorname{codist}(X_1, \ldots, X_k) \le 1 - \frac{1}{4^{c-1}k}.$$

Proof. We prove the theorem by induction on the nilpotency class c. The induction step follows from Theorem 9.8. To establish the base case $c = 1$, in which case G is abelian, we use a separate induction on k. The case $k = 2$ holds by [EJ, Lemma 3.4]. We now do the induction step on k. Let V be a G-space without G-invariant vectors, and let $v_i \in V^{X_i}$. Then using the induction hypothesis in the fourth line, we obtain

$$
\begin{aligned}
\| \sum_{i=1}^{k} v_i \|^2 &= \| \sum_{i=1}^{k-1} P_{(V^{X_k})^\perp}(v_i) \|^2 + \| \sum_{i=1}^{k} P_{V^{X_k}}(v_i) \|^2 \\[2mm]
&\le (k-1) \sum_{i=1}^{k-1} \| P_{(V^{X_k})^\perp}(v_i) \|^2 + \| (k-2) \frac{\sum_{i=1}^{k-1} P_{V^{X_k}}(v_i)}{k-2} + v_k \|^2 \\[2mm]
&\le (k-1) \sum_{i=1}^{k-1} \| P_{(V^{X_k})^\perp}(v_i) \|^2 + \frac{k-1}{k-2} \| \sum_{i=1}^{k-1} P_{V^{X_k}}(v_i) \|^2 + (k-1) \| v_k \|^2 \\[2mm]
&\le (k-1) \sum_{i=1}^{k-1} \| P_{V^{(X_k)^\perp}}(v_i) \|^2 + (k-1) \sum_{i=1}^{k-1} \| P_{V^{X_k}}(v_i) \|^2 + (k-1) \| v_k \|^2 \\[2mm]
&= (k-1) \sum_{i=1}^{k} \| v_i \|^2. \quad \square
\end{aligned}
$$

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