Solutions to Homework #10

1. Problem 3(b)(c)(d) from Section 6.2.

Solution: (b) Since 2 and 4 both lie in the domain of f and $f(2) = f(4)$, the function f is not injective, hence not bijective, hence has no inverse.

(d) This function is also not injective (and hence has no inverse). For instance, $f(\{1\}) = f(\{2\}) = \{1\}$.

(c) We first show that f is bijective.

We start with injectivity. Let $A_1 = \mathbb{N}$ and $A_2 = \mathbb{Z}_{\geq 0}$. Take any $a, b \in \mathbb{Z}$ such that $f(a) = f(b)$. It is clear from the definition that $f(x)$ is even if $x \in A_1$ and $f(x)$ is odd if $x \in A_2$. Thus, $f(a) = f(b)$ implies that a and b both lie in A_1 or both lie in A_2. We consider two cases accordingly:

Case 1: $a, b \in A_1$. Then $f(a) = f(b)$ implies $2a = 2b$, and dividing by 2, we get $a = b$.

Case 2: $a, b \in A_2$. Then $f(a) = f(b)$ implies $-2a + 1 = -2b + 1$, and subtracting 1 from both sides and dividing by -2, we also get $a = b$.

Thus f is injective.

Next we check that f is surjective. Take any $y \in \mathbb{N} =$ codomain of f. We need to show that $y = f(a)$ for some $a \in A$.

If y is even, we can write $y = 2a$ for some $a \in \mathbb{Z}$. Also since $y > 0$, we have $a > 0$, so $a \in A_1$ and hence $y = f(a)$. If y is odd, we can write $y = 2b + 1$ for some $b \in \mathbb{Z}$. Again since $y > 0$, we have $2b + 1 \geq 1$, so $b \geq 0$. If we now let $a = -b$, then $a \in A_2$ and $y = -2a + 1$, so $y = f(a)$. Thus, f is surjective.

Thus, we proved that f is bijective and hence has an inverse. By definition, the inverse $f^{-1} : \mathbb{N} \to \mathbb{Z}$ is given by the formula $f^{-1}(y) = a$ where $a \in \mathbb{Z}$ is the unique number such that $f(a) = y$. According to our computation in the proof of surjectivity, if y is even, then a is the solution of the equation $2a = y$, that is, $a = \frac{y}{2}$, and if y is odd, then a is the solution of the equation $-2a + 1 = y$, that is, $a = \frac{1-y}{2}$. Thus, f^{-1} is given by the formula

$$f^{-1}(y) = \begin{cases} \frac{y}{2} & \text{if } y \text{ is even} \\ \frac{1-y}{2} & \text{if } y \text{ is odd.} \end{cases}$$

Note that a different (slightly more conceptual) way to compute inverses of piecewise functions was described at the end of Lecture 25.

2. Problem 5(b)(d) from Section 6.2.

Solution: (b) $g^{-1}(\{-2,-1\}) = \{x \in \mathbb{R} \mid \lceil x \rceil = -2 \text{ or } \lceil x \rceil = -1\} = \{x \in \mathbb{R} \mid \lceil x \rceil = -2\} \cup \{x \in \mathbb{R} \mid \lceil x \rceil = -1\} = (-3,-2] \cup (-2,-1] = (-3,1]$.

(d) $g(\{k\pi \mid k \in \{2,3,6,8,10\}\}) = \{g(2\pi), g(3\pi), g(6\pi), g(8\pi), g(10\pi)\} = \{\lceil 2\pi \rceil, \lceil 3\pi \rceil, \lceil 6\pi \rceil, \lceil 8\pi \rceil, \lceil 10\pi \rceil\} = \{7, 10, 19, 26, 32\}$.

3. Problem 6(a)(b) from Section 6.2.

Solution: (a) We have $h(2) = 2$, $h(4) = 2+2 = 4$, $h(6) = 2+3 = 5$, $h(8) = 2+2+2 = 6$, $h(10) = 2+5 = 7$, so $h(\{2,4,6,8,10\}) = \{2,4,5,6,7\}$.

(b) To compute this set we need to find all possible ways to write $2,4,6,8$ and 10 as a sum of primes (possibly with repetitions, a single prime is allowed and the order does not matter). The number 2 is itself prime and cannot be written as a sum of two or more primes. The only way to write 4 is $2+2$; 6 can be written as $2+2+2$ or $3+3$; 8 can be written as $2+2+2+2$, $2+3+3$ or $3+5$; finally 10 can be written $2+2+2+2+2$, $2+2+3+3$ or $2+3+5$, $3+7$ or $5+5$. Thus,

$h^{-1}(\{2,4,6,8,10\}) = \{2, 2^2, 2^3, 3^2, 2^4, 2\cdot 3^2, 3\cdot 5, 2^5, 2^2\cdot 3^2, 2\cdot 3\cdot 5, 3\cdot 7, 5^2\} = \{2,4,8,9,16,18,15,32,36,30,21,25\} = \{2,4,8,9,15,16,18,21,25,30,32,36\}$.

4. Problem 7 from Section 6.2. Note that $f^{-1}(b)$ in the statement of the problem denotes $f^{-1}(\{b\})$.

Recall that by definition $f : A \to B$ is

- injective if $\forall b \in B$ there is at most one $a \in A$ such that $f(a) = b$
- surjective if $\forall b \in B$ there is at least one $a \in A$ such that $f(a) = b$

Reformulating this definition using preimages, we obtain the following statements:

$f : A \to B$ is injective if $\forall b \in B$ the set $f^{-1}(\{b\})$ has at most one element.

$f : A \to B$ is surjective if $\forall b \in B$ the set $f^{-1}(\{b\})$ has at least one element (equivalently, $f^{-1}(\{b\}) \neq \emptyset$).

5. Problem 9 from Section 6.2: Let $f : A \to B$ be a function. If $C \subseteq D \subseteq A$, then $f(C) \subseteq f(D)$. Does the converse hold?

Solution: We need to show the implication $y \in f(C) \Rightarrow y \in f(D)$. So take any $y \in f(C)$. By definition this means $y = f(c)$ for some $c \in C$. Since $C \subseteq D$, we have $c \in D$ and hence $y = f(c) \in f(D)$.

The converse fails in general; in fact, it fails for any non-injective function. Indeed, suppose that f is not injective, so there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ but $f(a_1) = f(a_2)$. Let $C = \{a_1\}$ and $D = \{a_2\}$. Then $C \not\subseteq D$, but $f(C) = f(D) = \{f(a_1)\}$, so in particular, $f(C) \subseteq f(D)$.

However, one can show that the converse holds for any injective function f (see Problem 1 on the list of practice problems for the final).

6. Problem 11 from Section 6.2. Let $f : A \to B$ be a function and C and D subsets of A. Then

(a) $f(C \setminus D) \supseteq f(C) \setminus f(D)$
(b) $f(C \setminus D) = f(C) \setminus f(D) \iff f$ is injective

Solution: (a) As in 5, we need to prove the implication $y \in f(C) \setminus f(D) \Rightarrow y \in f(C \setminus D)$. So take any $y \in f(C) \setminus f(D)$. This means $y \in f(C)$ and $y \notin f(D)$. Since $y \in f(C)$, we must have $y = f(c)$ for some $c \in C$.

Claim: $c \notin D$. *Proof of the claim:* By contradiction. Suppose $c \in D$. Then $y = f(c) \in f(D)$, which contradicts our assumption above.

Thus, $c \in C$ and $c \notin D$, so $c \in C \setminus D$. Hence $y = f(c) \in f(C \setminus D)$, as desired.

(b) "$\Rightarrow$" We will prove the contrapositive: if f is not injective, then there exist C and D such that $f(C \setminus D) \neq f(C) \setminus f(D)$. We use the same counterexample as in 5. Since f is not injective, there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ but $f(a_1) = f(a_2)$. Let $C = \{a_1\}$ and $D = \{a_2\}$. Then $C \setminus D = C$, so $f(C \setminus D) = \{f(a_1)\}$. On the other hand, $f(C) \setminus f(D) = \{f(a_1)\} \setminus \{f(a_1)\} = \emptyset$ and hence $f(C \setminus D) \neq f(C) \setminus f(D)$.

"$\Leftarrow$" Will be posted later (see Problem 2 on the practice problems list)

7. Problem 12 from Section 6.2. Let $f : A \to B$ be a function. If $C \subseteq D \subseteq B$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

Solution: We need to prove the implication $x \in f^{-1}(C) \Rightarrow x \in f^{-1}(D)$. So suppose $x \in f^{-1}(C)$. By definition this means that $f(x) \in C$. Since $C \subseteq D$, we have $f(x) \in D$ and hence $x \in f^{-1}(D)$.

The converse fails for any non-surjective function. Let $f : A \to B$ be non-surjective, and let $C = B$ (the codomain of f) and $D = f(A)$ (the image of f). Then $f^{-1}(C) = f^{-1}(D) = A$ (for any function, the preimage of the codomain and the preimage of the image is the whole domain since for any element a of the domain must we must have $f(a) \in f(A)$ and $f(a) \in B$ by the definition of the image and the definition of a function, respectively). In particular, $f^{-1}(C) \subseteq f^{-1}(D)$. On the other hand, since f is not surjective, $f(A)$ is a proper subset of B, so $C \not\subseteq D$.

In analogy with Problem 5, one can show that the converse holds for any surjective function f.

8. Problem 13 from Section 6.2. Let $f : A \to B$ be a function and C and D subsets of B. Then $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$.

Solution: Instead of checking inclusions in both directions, we will prove directly the two sided implication $x \in f^{-1}(C \setminus D) \iff x \in f^{-1}(C) \setminus f^{-1}(D)$.

So let $x \in A$ (we can start with this assumption since both $f^{-1}(C \setminus D)$ and $f^{-1}(C) \setminus f^{-1}(D)$ are clearly subsets of A). Then

$x \in f^{-1}(C \setminus D) \iff f(x) \in C \setminus D \iff f(x) \in C$ and $f(x) \notin D \iff x \in f^{-1}(C)$ and $x \notin f^{-1}(D) \iff x \in f^{-1}(C) \setminus f^{-1}(D)$.

9. Problem 17 from Section 6.2.

Answer: $g^{-1}(0)$ is the set of all natural numbers one of whose digits is 0.

$g^{-1}(1)$ is the set of all natural numbers all of whose digits are 1.

To simplify the answer in all the remaining cases, we introduce an additional notation. Given $n \in \mathbb{N}$ which has at least one digit different from 1, let $u(n)$ be the number obtained from n by removing all the occurrences of 1 (e.g. $u(21311) = 23$).

If $n = 2, 3, 5$ and 7, then $g^{-1}(n)$ is the set of all natural numbers with $u(n) = n$.

$g^{-1}(4)$ is the set of all natural numbers with $u(n) = 4$ or $u(n) = 22$

$g^{-1}(6)$ is the set of all natural numbers with $u(n) = 6, 23$ or 32.

$g^{-1}(8)$ is the set of all natural numbers with $u(n) = 8, 24, 42$ or 222.

$g^{-1}(9)$ is the set of all natural numbers with $u(n) = 9$ or 33.

$g^{-1}(10)$ is the set of all natural numbers with $u(n) = 25$ or 52.

$g^{-1}(11)$ is the empty set. In general, $g^{-1}(n) = \emptyset \iff n$ has a prime divisor larger than 7.

$g^{-1}(12)$ is the set of all natural numbers with $u(n) = 26, 62, 34, 43, 223, 232$ or 322.