Homework #9. Due Tuesday, November 23rd

Reading:

1. For this homework assignment: online class notes (Lectures 18-21) and Steinberg, parts of 4.2-4.4.

2. Plan for upcoming classes. Thu, Nov 18: permutation representations (online Lecture 22, Chapter 7 in Steinberg). Tue, Nov 23: TBA.

Problems:

For problems (or their parts) marked with a *, a hint is given later in the assignment. Do not to look at the hint(s) until you seriously tried to solve the problem without it.

1. Let p be a prime and $G = \mathrm{Heis}(\mathbb{Z}_p)$, the Heisenberg group over $\mathbb{Z}_p$ defined in HW#7.2

(a) Determine the number of conjugacy classes of G and their sizes. As in HW#8.6, you can work directly with matrices or with their expressions in terms of the generators x, y, z introduced in HW#7.2.

(b) Let $\omega \neq 1$ be a p^{th} root of unity, that is, $\omega = e^{\frac{2\pi k i}{p}}$ with $1 \leq k \leq p-1$. Let V be a p-dimensional complex vector space with basis $e_{[0]}, e_{[1]}, \ldots, e_{[p-1]}$ where we think of indices as elements of $\mathbb{Z}_p$. Prove that there exists a representation (ρ_ω, V) of G such that

- $\rho_\omega(z)e_{[k]} = \omega e_{[k]}$ for each k (that is, $\rho_\omega(z)$ is just the scalar multiplication by ω),
- $\rho_\omega(y)e_{[k]} = e_{[k+1]}$ for each k (that is, $\rho_\omega(y)$ cyclically permutes the basis vectors) and finally
- $\rho_\omega(x)e_{[k]} = \omega^k e_{[k]}$ for each k.

(c) Prove that every representation in (b) is irreducible (do not do this directly from definition) and every irreducible complex representation of G is either one-dimensional or equivalent to (ρ_ω, V) for some ω.

2. Let (ρ, V) be a representation of a group G. Recall that the dual representation (ρ^*, V^*) is defined by $\rho^*(g)(f) = f \circ \rho(g)^{-1}$ for all $f \in V^*$. Prove parts (1) and (2) of Claim 21.1 from class:

(1) (ρ^*, V^*) is indeed a representation

(2) If β is any basis of V and β^* is the dual basis of V^*, then $[\rho^*(g)]_{\beta^*} = ([\rho(g)]_\beta^{-1})^T$

Note: Part (2) has almost nothing to do with representation theory. Recall from Lecture 9 that given an operator $A \in \mathrm{End}(V)$, its adjoint $A^* \in \mathrm{End}(V^*)$ is defined by $A^*(f) = f \circ A$ (recall that this notion of adjoint is related to but slightly different from adjoints in complex inner product spaces). What you need to prove is Claim 9.2 from online notes which asserts that $[A^*]_{\beta^*} = ([A]_\beta)^T$ (make sure to explain how (2) follows from this).

3.* Let $G = S_n$ for some $n \geq 2$ and χ an irreducible complex character of G. Prove that χ is real-valued, that is, $\chi(g) \in \mathbb{R}$ for all $g \in G$.

4. Give an example of two representations V and W of the same group which are not equivalent, but have the same character. Recall that by Corollary 22.2 from class this cannot happen if G is finite and representations are complex.

5. Let (ρ, V) and (ρ', V) be complex representations of a finite group G (the vector space V is the same for both representations). Suppose that $\rho'(g)$ is conjugate to $\rho(g)$ in $\mathrm{GL}(V)$ for every $g \in G$. Prove that the representations (ρ, V) and (ρ', V) are equivalent. **Note:** The result is not automatic since the matrix which conjugates $\rho'(g)$ to $\rho(g)$ may depend on g.

Hint for 3: Use our discussion in Lecture 21 (=online Lecture 18) to find a general condition on a finite group G which guarantees that all of its complex characters are real-valued and then show that this condition holds for S_n.