Math 3000. Solutions to the First Midterm.

1. (6 pts) Let A, B and C be subsets of a set U. Use truth tables to prove that

$$(A \setminus B) \cup (B \setminus C) \cup (C \setminus A) = (A \cup B \cup C) \setminus (A \cap B \cap C).$$

Solution: As usual, we compute the truth tables and verify that the columns corresponding to the statements $x \in (A \setminus B) \cup (B \setminus C) \cup (C \setminus A)$ and $x \in (A \cup B \cup C) \setminus (A \cap B \cap C)$ are identical:

$x \in A$	$x \in B$	$x \in C$	$x \in A \setminus B$	$x \in B \setminus C$	$x \in C \setminus A$	$x \in (A \setminus B) \cup (B \setminus C) \cup (C \setminus A)$
T	T	T	F	F	F	F
T	T	F	F	T	F	T
T	F	T	T	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	T	F	F	T	F	T
F	F	T	F	F	T	T
F	F	F	F	F	F	F

$x \in A$	$x \in B$	$x \in C$	$x \in A \cup B \cup C$	$x \in A \cap B \cap C$	$x \in (A \cup B \cup C) \setminus (A \cap B \cap C)$
T	T	T	T	T	F
T	T	F	T	F	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	F	F	F

2. (6 pts) Use mathematical induction to prove that for any integer $n \geq 2$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1} \qquad \text{statement } S_n$$

Solution: We prove that S_n is true for any $n \geq 2$ by induction on n.

Base case ($n = 2$): LHS of $S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3}$; RHS of $S_2 = \frac{2}{2+1} = \frac{2}{3}$. Thus, S_2 is true.

Induction step $S_n \Rightarrow S_{n+1}$ (where $n \geq 2$). Fix $n \geq 2$, and assume that S_n is true, that is, $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$. Then

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(n+1) \cdot (n+2)}$$
$$= \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} \right) + \frac{1}{(n+1) \cdot (n+2)}$$
$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

where the second equality holds by the induction hypothesis.

Thus, S_{n+1} is true as well.

3. (6 pts) Let $a = 382$ and $b = 26$. Use the Euclidean algorithm to compute $gcd(a, b)$ and find $u, v \in \mathbb{Z}$ such that $au + bv = gcd(a, b)$.

Solution: First we compute $gcd(a, b)$:
$382 = 26 \cdot 14 + 18$; $26 = 18 \cdot 1 + 8$; $18 = 8 \cdot 2 + 2$. Since 2 divides 8, it follows that $gcd(382, 26) = gcd(26, 18) = gcd(18, 8) = gcd(8, 2) = 2$.

Now we express 2 as a linear combination of 26 and 382 using the above equalities:
$2 = 18-8\cdot 2 = 18-(26-18)\cdot 2 = 18\cdot 3-26\cdot 2 = (382-26\cdot 14)\cdot 3-26\cdot 2 = 382 \cdot 3 - 26 \cdot 44 = 382 \cdot 3 + 26 \cdot (-44)$. Thus, $u = 3$ and $v = -44$ have the required properties.

4. (6 pts) Let $a, b \in \mathbb{Z}$. Prove that if $4 \mid (a^2 + b^2)$, then a and b are both even. Make sure not to skip steps in your argument.

Solution: We prove this by contrapositive. Since "a and b are both even" is another way of saying "a is even and b is even", the constrapositive of the implication we are asked to prove is

"Let $a, b \in \mathbb{Z}$. If a is odd OR b is odd, then $4 \nmid (a^2 + b^2)$."

The hypothesis "a is odd OR b is odd" means that either a and b are both odd or exactly one of the numbers a and b is odd (while the other is even). Accordingly we consider two cases:

Case 1: a and b are both odd. By definition there exist $k, l \in \mathbb{Z}$ such that $a = 2k + 1$ and $b = 2l + 1$. Then $a^2 + b^2 = (2k + 1)^2 + (2l + 1)^2 = 4k^2 + 4k + 1 + 4l^2 + 4l + 1 = 4(k^2 + k + l^2 + l) + 2$. Since $k^2 + k + l^2 + l \in \mathbb{Z}$ and $0 \leq 2 < 4$, it follows that $a^2 + b^2$ has remainder 2 when divided by 4. By uniqueness of the remainder, it follows that 4 does not divide $a^2 + b^2$ (if 4 divided $a^2 + b^2$, the remainder would have been equal to 0).

Alternatively, we can finish the proof by contradiction. Assume, by way of contradiction, that $4 \mid (a^2+b^2)$, so $a^2+b^2 = 4r$ for some $r \in \mathbb{Z}$. Combining this with the above equality $a^2 + b^2 = 4(k^2 + k + l^2 + l) + 2$, we get $4r = 4(k^2 + k + l^2 + l) + 2$ which implies $4(r - (k^2 + k + l^2 + l)) = 2$ and hence $r - (k^2 + k + l^2 + l) = \frac{1}{2}$. This is clearly impossible since LHS of the last equation is an integer while RHS is not.

Case 2: exactly one of the numbers a and b is odd. By symmetry, it suffices to consider the case a is even, b is odd. We can proceed exactly as in case 1, but this time a simpler argument exists. Indeed, in class and Section 2.1 of the book it was proved that the product of two even integers is even, the product of two odd integers is odd, and the sum of an even integer and an odd integer is odd. Thus, if a is even and b is odd, then $a^2 = a \cdot a$ is even, $b^2 = b \cdot b$ is odd and hence $a^2 + b^2$ is odd. In particular, $a^2 + b^2$ is not divisible by 4.

5. Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}_{>0}$ be the set of all positive real numbers. Consider the following statement

$$\forall x \in \mathbb{R}_{>0}\ \exists y \in \mathbb{R} \text{ s.t. } ((y^2 = x^2) \wedge (y \neq x)). \qquad (***)$$

(a) Determine whether the statement (***) is true or false and prove your answer (please be detailed)

(b) Write down the negation of the statement (***) without using the symbols $\sim$, $\not\forall$ and $\nexists$

Solution: (a) The statement is true. Indeed, take any $x \in \mathbb{R}_{>0}$, and define $y = -x$. Then $y \in \mathbb{R}$, $y \neq x$ (the only $t \in \mathbb{R}$ for which $-t = t$ is $t = 0$, and we assume that $x > 0$), and finally $y^2 = (-x)^2 = x^2$.

(b) Using the formal negation rules, we obtain the following statement:

$$\exists x \in \mathbb{R}_{>0} \text{ s.t. } \forall y \in \mathbb{R}\ ((y^2 \neq x^2) \vee (y = x)).$$

6. (a) Let $a, b, c \in \mathbb{Z}$. Prove the following properties **directly from definition of divisibility.** Do not use properties of divisibility proved in class, book or homework.

(i) if $c \mid a$, then $c \mid ad$ for any $d \in \mathbb{Z}$

(ii) if $c \mid a$ and $c \mid (a + b)$, then $c \mid b$.

(b) Let $p, n, m \in \mathbb{Z}$ where p is prime. Assume that $p \mid mn$ and $p \mid (m^2 + n^2)$. Prove that $p \mid m$ and $p \mid n$.

Solution: (a)(i) If $c \mid a$, then $a = ck$ for some $k \in \mathbb{Z}$, so $ad = ckd = c \cdot (kd)$. Since $kd \in \mathbb{Z}$, we have $c \mid ad$ by definition.

(ii) We are given that $a = ck$ and $a + b = cl$ for some $k, l \in \mathbb{Z}$. Therefore, $b = (a + b) - a = cl - ck = c(l - k)$. Since $l - k \in \mathbb{Z}$, we get $c \mid b$.

(b) Since $p \mid mn$, by Euclid's lemma, we have $p \mid m$ or $p \mid n$. Since the assumption in the problem is completely symmetric with respect to m and n, WOLOG we can assume that $p \mid m$.

Since $p \mid m$, applying part (a)(i) with $c = p$ and $a = d = m$, we get $p \mid m^2$. Since we also know that $p \mid (m^2 + n^2)$, part (a)(ii) (with $c = p$, $a = m^2$ and $b = n^2$) yields $p \mid n^2$. Thus, $p \mid n \cdot n$, and by Euclid's lemma we get $p \mid n$.

Thus, we proved that $p \mid m$ and $p \mid n$.