

ON FINITENESS PROPERTIES OF THE JOHNSON FILTRATIONS

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ABSTRACT. Let Γ denote either the automorphism group of the free group of rank $n \geq 4$ or the mapping class group of an orientable surface of genus $n \geq 12$ with 1 boundary component, and let G be either the subgroup of IA -automorphisms or the Torelli subgroup of Γ , respectively. We prove that any subgroup of G containing $[G, G]$ (in particular, the Johnson kernel in the mapping class group case) is finitely generated. We also prove that if $N \leq 1 + \frac{n}{12}$ and K is any subgroup of G containing $\gamma_N G$, the N^{th} term of the lower central series of G (for instance, if K is the N^{th} term of the Johnson filtration of G), then the abelianization $K/[K, K]$ is finitely generated. Finally, we prove that if H is any finite index subgroup of Γ containing $\gamma_N G$, then H has finite abelianization.

1. INTRODUCTION

Let F_n denote the free group of rank n , and let $IA_n \subset \text{Aut}(F_n)$ denote the subgroup of automorphisms of F_n which act as identity on the abelianization; equivalently, IA_n is the kernel of the natural map $\text{Aut}(F_n) \rightarrow \text{Aut}(F_n^{ab}) \cong GL_n(\mathbb{Z})$. More generally, for each $k \in \mathbb{N}$ define $IA_n(k)$ to be the kernel of the natural map $\text{Aut}(F_n) \rightarrow \text{Aut}(F_n/\gamma_{k+1}F_n)$. The filtration $IA_n = IA_n(1) \supset IA_n(2) \supset \dots$ was first introduced and studied by Andreadakis in [An]. It is easy to see that $\gamma_k IA_n \subseteq IA_n(k)$ for each k , and in [Ba] it was shown that $\gamma_2 IA_n = IA_n(2)$.

The filtration $\{IA_n(k)\}_{k \in \mathbb{N}}$ is often referred to as the *Johnson filtration* and owes its name to the corresponding filtration in mapping class groups whose study was initiated by Johnson [Jo4]. Let Σ_g^1 be an orientable surface of genus $g \geq 2$ with 1 boundary component and Mod_g^1 its mapping class group. The fundamental group $\pi = \pi(\Sigma_g^1)$ is free of rank $2g$, and for each $k \in \mathbb{N}$ there is a natural homomorphism $\text{Mod}_g^1 \rightarrow \text{Aut}(\pi/\gamma_{k+1}\pi)$. Denote the kernel of this homomorphism by $\mathcal{I}_g^1(k)$. The subgroups $\mathcal{I}_g^1 = \mathcal{I}_g^1(1)$ and $\mathcal{J}_g^1 = \mathcal{I}_g^1(2)$ are well known as the *Torelli subgroup* and the *Johnson kernel*, respectively, and the filtration $\{\mathcal{I}_g^1(k)\}_{k \in \mathbb{N}}$ is called the *Johnson filtration* of Mod_g^1 . Again one has $\gamma_k \mathcal{I}_g^1 \subseteq \mathcal{I}_g^1(k)$ for each k , but this time the inclusion is known to be strict already for $k = 2$.

The mapping class group Mod_g of a closed orientable surface of genus g is a quotient of Mod_g^1 , and the Johnson filtration $\{\mathcal{I}_g(k)\}_{k \in \mathbb{N}}$ of Mod_g is defined to be the image of $\{\mathcal{I}_g^1(k)\}_{k \in \mathbb{N}}$ in Mod_g^1 .

A basic open question about Johnson filtrations is which of their terms are finitely generated (it is easy to see that if some term is not finitely generated, then so are all the subsequent terms in the same filtration). A complete answer is known for the first terms: already in 1930s Magnus [Ma] proved that $IA_n = IA_n(1)$ is finitely generated for all $n \geq 2$, and in 1980 Johnson [Jo5] proved that the Torelli group \mathcal{I}_g^1 (and hence also its

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quotient \mathcal{I}_g) is finitely generated for $g \geq 3$, while \mathcal{I}_2 is infinitely generated by [MM] (hence the same is true for $\mathcal{I}_2(k)$ and $\mathcal{I}_2^1(k)$ for all k). Our first theorem settles in the positive the finite generation problem for the second terms in sufficiently large rank:

Theorem 1.1. *Let G be either IA_n for $n \geq 4$ or \mathcal{I}_g^1 for $g \geq 12$. Then $[G, G]$ is finitely generated.*

Since G in Theorem 1.1 is itself finitely generated, it follows that any subgroup of G containing $[G, G]$ is finitely generated; in particular, the Johnson kernel $\mathcal{I}_g^1(2)$ and its quotient $\mathcal{I}_g(2)$ are finitely generated for $g \geq 12$.

Theorem 1.1 generalizes recent breakthrough results of Dimca and Papadima [DP] and Papadima and Suciu [PS2] who established finite dimensionality of the first rational homology $H_1(\mathcal{I}_g(2), \mathbb{Q})$ for $g \geq 4$ and $H_1(IA_n(2), \mathbb{Q})$ for $n \geq 5$, respectively.

The finite generation question remains open for the third and higher terms of the Johnson filtrations, but we will prove that at least several terms (beyond the second one) have finitely generated first *integral* homology:

Theorem 1.2. *Let $N \geq 2$ be an integer, and let (G, K) be one of the following pairs of groups.*

- (a) $G = IA_n$ and $K = \gamma_N IA_n$ where $n \geq 4$ if $N = 2$ and $n \geq 12(N - 1)$ if $N > 2$.
- (b) $G = \mathcal{I}_g^1$ and $K = \gamma_N \mathcal{I}_g^1$ where $g \geq 12(N - 1)$.

Then the quotient $G/[K, K]$ is nilpotent. In particular, for any subgroup $G \supseteq L \supseteq K$, the abelianization $L/[L, L] \cong H_1(L, \mathbb{Z})$ is finitely generated.

Thus, each of the following groups has finitely generated abelianization:

$IA_n(2)$ for $n \geq 4$, $IA_n(N)$ for $n \geq 12(N - 1)$, and $\mathcal{I}_g^1(N)$ and $\mathcal{I}_g(N)$ for $g \geq 12(N - 1)$.

Remark. The assertion that $G/[K, K]$ is nilpotent is equivalent to saying that $K/[K, K] \cong H_1(K, \mathbb{Z})$ considered as a G/K -module is unipotent in the sense that it is annihilated by some power of the augmentation ideal of $\mathbb{Z}[G/K]$. In the case $G = \mathcal{I}_g$ and $K = \mathcal{I}_g(2)$, $g \geq 4$, it has already been proved in [DHP] that $H_1(K, \mathbb{Q})$ is a unipotent G/K -module.

Both Theorems 1.1 and 1.2 will be deduced from certain properties of representations of G that we discuss below. Let G be an arbitrary group. By a *character* of G we will mean a homomorphism $\chi : G \rightarrow \mathbb{R}$. Two characters χ and χ' will be considered equivalent if they are positive scalar multiples of each other. The equivalence class of a character χ will be denoted by $[\chi]$, and the set of equivalence classes of nonzero characters will be denoted by $S(G)$. In [BNS], Bieri, Neumann and Strebel introduced certain subset $\Sigma(G)$ of $S(G)$, now known as the *BNS invariant* which, in the case when G is finitely generated, completely determines which subgroups of G containing $[G, G]$ are finitely generated:

Theorem 1.3 ([BNS]). *Let G be a finitely generated group, and let N be a subgroup of G containing $[G, G]$. Then N is finitely generated if and only if $\Sigma(G)$ contains $[\chi]$ for every non-trivial χ which vanishes on N . In particular, $[G, G]$ is finitely generated if and only if $\Sigma(G) = S(G)$.*

Thus, Theorem 1.1 is a direct consequence of Theorem 1.3 and the following result:

Theorem 1.4. *Let G be as in Theorem 1.1. Then $\Sigma(G) = S(G)$.*

The set $\Sigma(G)$ admits many different characterizations. In order to prove Theorem 1.4 we will use the characterization in terms of actions on real trees due to Brown [Br].

Note that characters of a group G are in natural bijection with one-dimensional real representations of G whose image lies in $\mathbb{R}_{>0}$. In order to prove Theorem 1.2 we consider irreducible representations of G which vanish on K (with G and K as in Theorem 1.2), although, somewhat surprisingly, this time we will deal with representations over finite fields, in fact fields of prime order. Theorem 1.2 will be obtained as a direct combination of Theorems 1.5 and 1.6 below, just like Theorem 1.1 follows from Theorems 1.3 and 1.4.

Definition. Let K be a normal subgroup of a group G and F a field. We will say that the triple (G, K, F) is *nice* if the following holds: let V be a non-trivial finite-dimensional irreducible representation of G over F such that K acts trivially on V . Then

$$H^1(G, V) = 0.$$

Theorem 1.5. *Let G be a finitely generated group, and let K be a normal subgroup of G such that $Q = G/K$ is nilpotent. Assume that (G, K, F) is nice for any finite field F of prime order. Then the quotient $G/[K, K]$ is nilpotent, so in particular the abelianization $K^{ab} = K/[K, K]$ is finitely generated.*

Theorem 1.6. *Let (G, K) be as in Theorem 1.2. Then (G, K, F) is nice for any field F .*

We now briefly comment on how Theorems 1.4 and 1.6 will be proved. Both results easily follow from the existence of a ρ -centralizing generating set in G where ρ is a non-trivial irreducible representation of G (over an arbitrary field) with $\text{Ker } \rho \supseteq K$ (in the case of Theorem 1.4 we set $K = [G, G]$). The precise definition of a ρ -centralizing set will be given in § 3. An important special case of a ρ -centralizing set is a set S which contains a subset T of pairwise commuting elements such that every element of S commutes with at least one element of T and such that $\rho(t)$ is a non-trivial central element of $\rho(G)$ for each $t \in T$. Note that by our hypotheses we will always deal with representations ρ such that $\rho(G)$ is nilpotent (and non-trivial) and hence always contains non-trivial central elements.

The group IA_n has a very simple generating set constructed by Magnus already in 1930s (see § 4 for its definition). A generating set for \mathcal{I}_g^1 which is in many ways analogous to Magnus' generating set for IA_n was constructed in a recent paper of Church and Putman [CP], which made use of an earlier work of Putman [Pu2] and the original work of Johnson [Jo5] (see § 7.1). We will refer to these generating sets as *standard*. We will show that if ρ is a “sufficiently random” representation of G which vanishes on K (with G and K as above), then the existence of a ρ -centralizing generating set in G easily follows from the basic relations between the standard generators; in this case a ρ -centralizing generating set we construct will be an overset of the standard generating set. In the general case we will use the fact that G is a normal subgroup of Γ (with $\Gamma = IA_n$ or \mathcal{M}_g^1), and precomposing ρ by the conjugation by a fixed element of Γ does not affect the existence of a ρ -centralizing generating set.

Thus, we are reduced to the problem of showing that an arbitrary ρ (satisfying the above restrictions) can be conjugated to a sufficiently random one. In the case $G = IA_n$ and $K = [IA_n, IA_n]$ this can be done fairly explicitly (see § 6). In the general case an explicit calculation may be possible, but would be cumbersome, so we take a more conceptual approach. First it is not hard to see that the relevant information about the

conjugation action of Γ on G is completely captured by the induced action of Γ/G on $L(G) = \bigoplus_{n=1}^{\infty} \gamma_n G / \gamma_{n+1} G$, the Lie algebra of G with respect to the lower central series.

Second, in both cases $(\Gamma, G) = (\text{Aut}(F_n), IA_n)$ and $(\mathcal{M}_n^1, \mathcal{I}_n^1)$, the quotient Γ/G contains a natural copy of $SL_n(\mathbb{Z})$. In § 4 we will state our main criterion for the existence of a ρ -centralizing generating set in a group G (see Theorem 4.3). One of the hypotheses in Theorem 4.3 is that G is a normal subgroup of a group Γ such that Γ/G contains a copy of $SL_n(\mathbb{Z})$ for suitable n and such that the abelianization $G^{ab} = G/[G, G]$ is a *regular* $SL_n(\mathbb{Z})$ -module, another technical notion which will also be introduced in § 4. Note that while the proof of Theorem 4.3 will deal with the action of $SL_n(\mathbb{Z})$ on the entire Lie algebra $L(G)$, its hypotheses only involve the action on G^{ab} , the degree 1 component of $L(G)$. This is very important since while the structure of G^{ab} as an $SL_n(\mathbb{Z})$ -module is completely understood and easy to describe (for $G = IA_n$ or \mathcal{I}_g^1), this is not the case with higher degree components of $L(G)$.

At the end of the paper we will show that Theorem 1.6 also yields new results on the abelianization of finite index subgroups in $\text{Aut}(F_n)$, IA_n , Mod_g^1 and \mathcal{I}_g^1 :

Theorem 1.7. *Let (G, K) be as in Theorem 1.2, and let $\Gamma = \text{Aut}(F_n)$ if $G = IA_n$ and $\Gamma = \text{Mod}_g^1$ if $G = \mathcal{I}_g^1$. The following hold:*

- (1) *If H is a finite index subgroup of G which contains K , then the restriction map $H^1(G, \mathbb{C}) \rightarrow H^1(H, \mathbb{C})$ is an isomorphism*
- (2) *If H is a finite index subgroup of Γ which contains K , then H has finite abelianization.*

In the case $\Gamma = \text{Mod}_g^1$, $G = \mathcal{I}_g^1$ and $K = \mathcal{I}_g^1(2)$ both assertions of Theorem 1.7 have been previously proved by Putman (see [Pu3, Thm B] and [Pu1, Thm B]) by a different method. It is interesting to note that while we will use Theorem 1.6 to prove both Theorem 1.2 and Theorem 1.7, an opposite implication was used in [DHP] where [Pu3, Thm B] was one of the tools in the proofs of special cases of Theorem 1.2 and Theorem 1.6.

Finally, let us comment on the restrictions on n, g and N in the statements of Theorems 1.1, 1.2, 1.6 and 1.7 (where we set $N = 2$ in Theorem 1.1). If $n = 2$, the assertions of all four theorems are easily seen to be false already for $N = 2$ since IA_2 is a free group of rank 2. As shown in [BV], $\text{Aut}(F_3)$ contains a finite index subgroup H such that $[IA_3 : H \cap IA_3] = 2$ (so $H \supset [IA_3, IA_3]$) and H has infinite abelianization. This result combined with the proof of Theorem 1.7 implies that Theorem 1.6 and both parts of Theorem 1.7 are false for $n = 3$ and $N = 2$. We expect that Theorems 1.1 and 1.2 do not hold in this case as well, but as far as we know these questions have not been settled. The restriction $g \geq 12$ in the mapping class group case is likely the drawback of our method, and we expect that the result remains true for all $g \geq 4$ (and possibly even for $g = 3$). It may be possible to slightly improve the bound $g \geq 12$ using the same method and more delicate calculations, but it is unlikely that one can push it all the way to $g = 4$. More generally, in the cases $n \geq 4$ and $g \geq 3$ it is possible that some (or even all) of the above theorems remain true for arbitrary N , but the proof of such a result would almost certainly require new ideas.

Organization. The paper is organized as follows. In § 2 we will prove Theorem 1.5. In § 3 we will introduce the notion of a ρ -centralizing generating set and reduce Theorems 1.4

and 1.6 to the problem of existence of ρ -centralizing generating sets for suitable ρ (see Proposition 3.4). In § 4 we will introduce the notion of a regular $SL_n(\mathbb{Z})$ -module, discuss basic examples and properties of such modules and state Theorem 4.3, which gives a criterion for the existence of a ρ -centralizing generating set. At the end of § 4 we will show that the hypotheses of Theorem 4.3 are satisfied for $G = IA_n$ with $n \geq 12(N - 1)$. The corresponding verification for $G = \mathcal{I}_g^1$ and $g \geq 12(N - 1)$ will be made in § 7. Theorem 4.3 will be proved in § 5. The results mentioned in the last three sentences imply Theorem 1.4 except when $G = IA_n$, $4 \leq n < 12$ and Theorem 1.6 except when $G = IA_n$, $N = 2$, $4 \leq n < 12$. The proof of those theorems in the remaining cases will be given in § 6. Finally, in § 8 we will prove Theorem 1.7 and also give a summary of previously known results of the same kind.

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Notation. When considering cohomology of a group G with coefficients in some module, we will sometimes use the notation $H^k(G, M)$ where M is the underlying space of the module and sometimes $H^k(G, \rho)$ when ρ is the action of G on M . We hope that this inconsistency will not cause a confusion.

2. THE ABELIANIZATION OF THE KERNEL OF A HOMOMORPHISM TO A NILPOTENT GROUP

In this short section we prove Theorem 1.5 using the following results of Roseblade and Robinson. A major part of the argument below (including the use of Theorem 2.2) was suggested to us by Andrei Jaikin who substantially simplified our original proof. The latter was inspired by the proof of [PS2, Thm 3.6].

Theorem 2.1 (Roseblade). *Let Q be a virtually polycyclic group. Then*

- (a) *Every simple $\mathbb{Z}[Q]$ -module is finite and thus is an $\mathbb{F}_p[Q]$ -module for some prime p .*
- (b) *Assume now that Q is nilpotent, let Ω be the augmentation ideal of $\mathbb{Z}[Q]$ and V a finitely generated $\mathbb{Z}[Q]$ -module. If $\Omega M = 0$ for every simple quotient M of V , then $\Omega^N V = 0$ for some $N \in \mathbb{N}$.*

Parts (a) and (b) of Theorem 2.1 are special cases of [Ro, Cor A] and [Ro, Thm B], respectively.

Theorem 2.2 (Robinson [Rb]). *Let Q be a nilpotent group, and let M be a Q -module. Assume that either $H_0(Q, M) = 0$ and M is Noetherian or $H^0(Q, M) = 0$ and M is Artinian. Then $H^i(Q, M) = 0$ and $H_i(Q, M) = 0$ for all $i > 0$.*

Proof of Theorem 1.5. First we claim that K^{ab} is finitely generated as a $\mathbb{Z}[Q]$ -module. We know that K contains $\gamma_n G$ for some n , and it suffices to prove that $(\gamma_n G)^{ab}$ is finitely generated as a $\mathbb{Z}[G]$ -module. Let X be a finite generating set for G . Then $(\gamma_n G)^{ab}$ is generated as an abelian group by left-normed commutators in X of length at least n . But if c is such a commutator and $x \in X$, then $(c, x) = c^{-1}x^{-1}cx$ is equal to $x^{-1}.c - c$ in

$(\gamma_n G)^{ab}$, so by straightforward induction $(\gamma_n G)^{ab}$ is generated by left-normed commutators in X of length exactly n as a $\mathbb{Z}[G]$ -module.

Now let Ω denote the augmentation ideal of $\mathbb{Z}[Q]$. To prove the theorem it suffices to show that $\Omega^N K^{ab} = \{0\}$ for some $N \in \mathbb{N}$ (in fact, the two statements are equivalent). Indeed, the equality $\Omega^N K^{ab} = \{0\}$ means that $[K, \underbrace{G, \dots, G}_{N \text{ times}}] \subseteq [K, K]$, so if $K \supseteq \gamma_n G$,

then $[K, K] \supseteq \gamma_{n+N} G$ and $G/[K, K]$ is nilpotent.

Let $V = K^{ab}$, and assume that $\Omega^N K^{ab} \neq \{0\}$ for any $N \in \mathbb{N}$. By Theorem 2.1(b) there is a simple Q -module M which is a quotient of V such that $\Omega M \neq 0$, so Q acts on M non-trivially. Moreover, by Theorem 2.1(a), M is a finite $F[Q]$ -module for some finite field F of prime order, so $H^1(G, M) = 0$ by the hypotheses of Theorem 1.5.

Since K acts trivially on V and hence on M , we have a natural isomorphism

$$H^1(K, M) \cong \text{Hom}(V, M)$$

Under this isomorphism $H^1(K, M)^Q$, the subspace of Q -invariant elements of $H^1(K, M)$, maps to $\text{Hom}_Q(V, M)$, the subspace of Q -module homomorphisms from V to M . Since M is a quotient of V as Q -module, we have $\text{Hom}_Q(V, M) \neq 0$ and hence $H^1(K, M)^Q \neq 0$.

On the other hand, we have the inflation-restriction sequence

$$0 \rightarrow H^1(Q, M) \rightarrow H^1(G, M) \rightarrow H^1(K, M)^Q \rightarrow H^2(Q, M) \rightarrow H^2(G, M).$$

Since M is finite and simple, it is both Artinian and Noetherian, and both $H^0(Q, M)$ and $H_0(Q, M)$ are trivial. Thus, using either condition in Theorem 2.2, we conclude that $H^1(Q, M) = H^2(Q, M) = 0$, so $H^1(G, M) \cong H^1(K, M)^Q \neq 0$, a contradiction. \square

3. CENTRALIZING GENERATING SETS

Let G be a finitely generated group. In this section we will introduce the technical notion of a ρ -centralizing set in G where ρ is an irreducible representation of G over some field F . We will show that

- (i) the existence of a ρ -centralizing generating set implies that $H^1(G, \rho) = 0$;
- (ii) in the case $F = \mathbb{R}$ and $\rho = e^\chi$ for some non-trivial character $\chi : G \rightarrow \mathbb{R}$, the existence of a ρ -centralizing generating set implies that $[\chi]$ lies in $\Sigma(G)$, the BNS invariant of G .

We will start with briefly recalling Brown's characterization of $\Sigma(G)$ in terms of actions on \mathbb{R} -trees [Br] and establishing a very simple property of group 1-cocycles.

We will need only very basic properties of group actions on \mathbb{R} -trees (see, e.g. [AB] for the proofs). Let X be an \mathbb{R} -tree with metric d , and suppose that a group G acts on X by isometries; denote the action by α . For each $g \in G$ define $l(g) = \inf_{x \in X} d(\alpha(g)x, x)$ (it is known that the infimum is always attained), and let $\mathcal{A}_g = \{x \in X : d(\alpha(g)x, x) = l(g)\}$. An element $g \in G$ is called *elliptic* (with respect to α) if $l(g) = 0$, in which case \mathcal{A}_g is the fixed point set of g . If $l(g) > 0$, then g is called *hyperbolic*. In this case \mathcal{A}_g is a line, called *the axis of g* , and g acts on \mathcal{A}_g as a translation by $l(g)$; moreover, \mathcal{A}_g is the unique line invariant under the action of g .

An action α is called

- *abelian* if there exists a character $\chi : G \rightarrow \mathbb{R}$ such that $l(g) = |\chi(g)|$ for all $g \in G$. If such χ exists, it is unique up to sign, and we will say that α is associated with χ .
- *non-trivial* if it has no (global) fixed points and no (global) invariant line.

The following characterization of the BNS invariant $\Sigma(G)$ was obtained by Brown [Br]:

Theorem 3.1. *Let G be a finitely generated group, and let $\chi : G \rightarrow \mathbb{R}$ be a non-trivial character. Then $[\chi] \in \Sigma(G)$ if and only if G has no non-trivial abelian actions associated with χ . Thus, by Theorem 1.3, $[G, G]$ is finitely generated if and only if G has no non-trivial abelian actions.*

Now let (ρ, V) be a non-trivial irreducible representation of a group G over a field F . Recall that $H^1(G, \rho) = Z^1(G, \rho)/B^1(G, \rho)$ where

$$Z^1(G, \rho) = \{f : G \rightarrow V : f(xy) = f(x) + \rho(x)f(y) \text{ for all } x, y \in G\}$$

$$B^1(G, \rho) = \{f : G \rightarrow V : \text{there exists } v \in V \text{ s.t. } f(x) = \rho(x)v - v \text{ for all } x \in G\}.$$

Lemma 3.2. *Let G, ρ and V be as above, and let $C_\rho(G)$ be the set of all $g \in G$ such that $\rho(g)$ is a non-trivial central element of $\rho(G)$. Let $f \in Z^1(G, \rho)$ be a cocycle, and let $g \in C_\rho(G)$. The following hold:*

- (i) *There exists a coboundary $b \in B^1(G, \rho)$ with $f(g) = b(g)$*
- (ii) *Suppose that $f(g) = 0$. Then $f(x) = 0$ for every $x \in G$ such that $f(gx) = f(xg)$; in particular, $f(x) = 0$ for every $x \in G$ which commutes with g .*

Proof. (i) Since $g \in C_\rho(G)$, by Schur's Lemma the operator $\rho(g) - 1$ is invertible. Thus, if $v = (\rho(g) - 1)^{-1}f(g)$, then the map $b(x) = \rho(x)v - v$ is a coboundary with the required property.

(ii) We have $f(xg) = f(x) + \rho(x)f(g) = f(x)$ and $f(gx) = f(g) + \rho(g)f(x) = \rho(g)f(x)$, so $(\rho(g) - 1)f(x) = 0$ and hence $f(x) = 0$. \square

Definition. Let G, ρ and $C_\rho(G)$ be as above. Let $S = \{g_1, g_2, \dots, g_k\}$ be a finite ordered subset of G , and for each $1 \leq i \leq k$ let $G_i = \langle g_1, \dots, g_i \rangle$. We will say that S is a ρ -centralizing set if $g_1 \in C_\rho(G)$ and for each $2 \leq i \leq k$ there exists $j < i$ such that $g_j \in C_\rho(G)$ and $[g_i, g_j] \in G_{i-1}$.

The following straightforward observation describes a particularly simple instance where G has a ρ -centralizing generating set:

Lemma 3.3. *Let G, ρ and V be as above. Suppose that $C_\rho(G)$ contains a subset T such that elements of T commute with each other and the union of the centralizers of elements of T generates G . Then G admits a ρ -centralizing generating set.*

Proposition 3.4. *Let G be a finitely generated group. The following hold:*

- (a) *If (ρ, V) is an irreducible representation of G over some field and G admits a ρ -centralizing generating set, then $H^1(G, \rho) = 0$*
- (b) *Let $\chi : G \rightarrow \mathbb{R}$ be a non-trivial character and $\rho = e^\chi$ the corresponding 1-dimensional representation. If G admits a ρ -centralizing generating set, then $[\chi] \in \Sigma(G)$. In particular, if G admits a π -centralizing generating set for every non-trivial 1-dimensional \mathbb{R} -representation π , then $[G, G]$ is finitely generated.*

Proof. In both parts of the proof we let $S = \{g_1, \dots, g_k\}$ be a ρ -centralizing generating set of G and $G_i = \langle g_1, \dots, g_i \rangle$ for $1 \leq i \leq k$ (so that $G_k = G$).

(a) Take any $f \in Z^1(G, \rho)$. By Lemma 3.2(i), after modifying f by a coboundary, we can assume that $f(g_1) = 0$. Then $f = 0$ on G_1 , and we will now prove that $f = 0$ on G_i for all $1 \leq i \leq k$ by induction on i .

Take $i \geq 2$, and assume that $f = 0$ on G_{i-1} . By hypothesis there exists $j < i$ and $r \in G_{i-1}$ such that $g_j \in C_\rho(G)$ and $g_i g_j = g_j g_i r$. Then $f(g_i g_j) = f(g_j g_i) + \rho(g_j g_i) f(r) = f(g_j g_i)$, whence $f(g_i) = 0$ by Lemma 3.2(ii). Since $G_i = \langle G_{i-1}, g_i \rangle$, we have $f(G_i) = 0$ as desired.

(b) The following argument is similar to the proof of the main theorem in [OK]. By Theorem 3.1 we need to show that if (α, X) is an abelian action of G on an \mathbb{R} -tree X associated to χ , then α is trivial. By assumption g_1 is hyperbolic (with respect to α). We claim that its axis $\mathcal{A} = \mathcal{A}_{g_1}$ is invariant under the entire group G (which will finish the proof). We will prove that \mathcal{A} is invariant under G_i for all $1 \leq i \leq k$ by induction on i .

Take $i \geq 2$, and assume that \mathcal{A} is invariant under G_{i-1} (in particular, \mathcal{A} is the axis for any hyperbolic element in G_{i-1} by the uniqueness of the invariant line of a hyperbolic element). By hypothesis there exists $j < i$ and $r \in G_{i-1}$ such that g_j is hyperbolic and $g_i^{-1} g_j g_i = g_j r$. The element $g_i^{-1} g_j g_i$ is also hyperbolic (being a conjugate of g_j), and its axis is $\alpha(g_i^{-1})(\mathcal{A})$. On the other hand $g_i^{-1} g_j g_i = g_j r \in G_{i-1}$, so \mathcal{A} is the axis of $g_i^{-1} g_j g_i$. Thus, $\alpha(g_i^{-1})(\mathcal{A}) = \mathcal{A}$, so \mathcal{A} is g_i -invariant and hence G_i -invariant. \square

4. \mathbf{n} -GROUPS AND REGULAR $SL_n(\mathbb{Z})$ -MODULES

For the rest of the paper, given $n \in \mathbb{N}$, denote by \mathbf{n} the set $\{1, 2, \dots, n\}$.

Definition. Let $n \in \mathbb{N}$. An \mathbf{n} -group is a group G endowed with a collection of subgroups $\{G_I\}_{I \subseteq \mathbf{n}}$ such that

- (i) $G_{\mathbf{n}} = G$
- (ii) $G_I \subseteq G_J$ whenever $I \subseteq J$

We will say that G is a *good* \mathbf{n} -group if the following extra condition holds:

- (iii) G_I and G_J commute (elementwise) whenever $I \cap J = \emptyset$ and I consists of consecutive integers, that is, $I = \{i, i+1, \dots, i+|I|-1\}$ for some $i \in \mathbf{n}$.

Given $d \in \mathbb{N}$, we will say that G is *generated in degree d* if $G = \langle G_I : |I| = d \rangle$.

The basic example of a good \mathbf{n} -group that we will use in this paper is $G = SL_n(\mathbb{Z})$ with $\{G_I\}$ defined as follows. Let e_1, \dots, e_n be the standard basis of \mathbb{Z}^n . Given $I \subseteq \mathbf{n}$, let $\mathbb{Z}^I = \oplus_{i \in I} \mathbb{Z} e_i$, and let

$$(4.1) \quad G_I = \{g \in G : g(\mathbb{Z}^I) \subseteq \mathbb{Z}^I \text{ and } g(e_j) = e_j \text{ for all } j \notin I\}.$$

Clearly, $SL_n(\mathbb{Z})$ is generated in degree 2, and G_I commutes with G_J for any disjoint subsets I and J (in particular, condition (iii) holds). As we will explain at the end of this section, IA_n has a natural structure of a good \mathbf{n} -group generated in degree 3 and also satisfying a stronger form of condition (iii). The unnatural restriction that I consists of consecutive integers in (iii) is imposed so that the Torelli group \mathcal{T}_g^1 satisfies the definition of a good \mathbf{g} -group for $g \geq 3$ – this easily follows from the results of [CP] and will be explained in § 7.

Next we introduce another slightly technical concept, that of a regular $SL_n(\mathbb{Z})$ -module. Informally speaking, a regular $SL_n(\mathbb{Z})$ -module is a graded (in a somewhat unconventional sense) $SL_n(\mathbb{Z})$ -module which satisfies some key properties of an R -form of a finite-dimensional $SL_n(\mathbb{C})$ -module (where R is some commutative ring).

Definition. Let V be an abelian group (written additively). A *grading* of V is a collection of additive subgroups $\{V_I : I \subseteq \mathbf{n}\}$ of V such that

$$\sum_{I \subseteq \mathbf{n}} V_I = V$$

(note that the sum is not required to be direct). The *degree* of V (with respect to the grading $\{V_I\}$), denoted by $\deg(V)$, is the smallest integer k such that $V_I = 0$ whenever $|I| > k$ (this automatically implies that $\sum_{|I| \leq k} V_I = V$). If no such k exists, we set $\deg(V) = \infty$.

As usual, $E_{ij} \in SL_n(\mathbb{Z})$ will denote the matrix which has 1's on the diagonal and at the position (i, j) and 0 everywhere else. By $F_{ij} \in SL_n(\mathbb{Z})$ we will denote the matrix obtained from the identity matrix by swapping i^{th} and j^{th} rows and then multiplying the j^{th} row by -1 . Note that $F_{ij} = E_{ij}E_{ji}^{-1}E_{ij}$.

Definition. Let $\{G_I : I \subseteq \mathbf{n}\}$ be the subgroups of $SL_n(\mathbb{Z})$ defined by (4.1). Let V be an $SL_n(\mathbb{Z})$ -module endowed with a grading $\{V_I\}$. We will say that V is *regular* if the following properties hold:

- (1) G_J acts trivially on V_I if $I \cap J = \emptyset$
- (2) If $I \subseteq \mathbf{n}$, $i \in I$, $j \notin I$, then for any $g \in \{E_{ij}^{\pm 1}, E_{ji}^{\pm 1}\}$ and $v \in V_I$ we have

$$gv - v \in V_{I \setminus \{i\} \cup \{j\}} + V_{I \cup \{j\}}$$

- (3) If $I \subseteq \mathbf{n}$, $i, j \in \mathbf{n}$ with $i \neq j$, then $F_{ij}V_I = V_{(i,j)I}$ where $(i,j)I$ is the image of I under the transposition (i, j) . In particular, if $i \in I$ and $j \notin I$, then $F_{ij}V_I = V_{I \setminus \{i\} \cup \{j\}}$.

Condition (2) in the above definition is tailored specifically for the purposes of this paper. Our notion of a regular $SL_n(\mathbb{Z})$ -module certainly has some formal similarities with the notion of an FI -module from [CEF], but it is not clear if there are deep connections between the two notions.

The following result provides our starting examples of regular $SL_n(\mathbb{Z})$ -modules.

Lemma 4.1. *Let R be a commutative ring with 1, $n \geq 2$, let R^n be a free R -module of rank n with basis e_1, \dots, e_n , and let $(R^n)^* = \text{Hom}_R(R^n, R)$ be the dual module, with dual basis e_1^*, \dots, e_n^* . Consider R^n and $(R^n)^*$ as $SL_n(\mathbb{Z})$ -modules with standard actions. Define $(R^n)_{\{i\}} = Re_i$, $(R^n)_{\{i\}}^* = Re_i^*$ for $1 \leq i \leq n$ and $(R^n)_I = 0$, $(R^n)_I^* = 0$ for $|I| \neq 1$. Then with respect to these gradings R^n and $(R^n)^*$ are regular of degree 1.*

Proof. The only non-obvious part is condition (ii) in the definition of a regular $SL_n(\mathbb{Z})$ -module. Condition (ii) is vacuous if $|I| \neq 1$, so assume that $I = \{i\}$ and $j \neq i$, in which case $I \setminus \{i\} \cup \{j\} = \{j\}$ and $I \cup \{j\} = \{i, j\}$.

We have $E_{ij}^{\pm 1}(re_i) = re_i$, $E_{ji}^{\pm 1}(re_i) - re_i = \pm re_j \in (R^n)_{\{j\}}$, $E_{ji}^{\pm 1}(re_i^*) = re_i^*$, $E_{ij}^{\pm 1}(re_i^*) - re_i^* = \mp re_j^* \in (R^n)_{\{j\}}^*$, so (ii) holds (note that in this case terms from $V_{I \cup \{j\}}$ do not arise). \square

Lemma 4.2. *The following hold:*

- (a) *Let V and W be regular $SL_n(\mathbb{Z})$ -modules.*
 - (i) *Define $(V \oplus W)_I = V_I \oplus W_I$. Then $V \oplus W$ is regular and $\deg(V \oplus W) = \max\{\deg(V), \deg(W)\}$.*
 - (ii) *Define $(V \otimes_{\mathbb{Z}} W)_I = \sum_{I=I_1 \cup I_2} V_{I_1} \otimes W_{I_2}$. Then $V \otimes_{\mathbb{Z}} W$ is regular and $\deg(V \otimes_{\mathbb{Z}} W) \leq \deg(V) + \deg(W)$.*
- (b) *Let V be a regular $SL_n(\mathbb{Z})$ -module and U a submodule. For $I \subseteq \mathbf{n}$ set $U_I = U \cap V_I$ and $(V/U)_I = V_I + U$. Then V/U is always regular with $\deg(V/U) \leq \deg(V)$, and U is regular with $\deg(U) \leq \deg(V)$ provided $U = \sum_{I \subseteq \mathbf{n}} U_I$.*
- (c) *Suppose that $SL_n(\mathbb{Z})$ acts on an \mathbb{N} -graded Lie ring $L = \bigoplus_{m=1}^{\infty} L(m)$ by graded automorphisms. Suppose that L is generated in degree 1 (as a Lie ring) and that $L(1)$ is a regular $SL_n(\mathbb{Z})$ -module of degree d . For each $m > 1$ and $I \subseteq \mathbb{N}$ define*

$$L(m)_I = \sum_{I=I_1 \cup I_2 \cup \dots \cup I_m} [L(1)_{I_1}, L(1)_{I_2}, \dots, L(1)_{I_m}] \quad (**)$$

Then $L(m)$ is a regular $SL_n(\mathbb{Z})$ -module of degree at most md .

Proof. Parts (a)(i) and (b) are straightforward.

(a)(ii) As in the proof of Lemma 4.1, we only need to check condition (ii) in the definition of a regular module. So take $I \subseteq \mathbf{n}$, $i \in I$ and $j \notin I$. It is enough to check the condition for v of the form $v = x \otimes y$ where $x \in V_{I_1}$, $y \in W_{I_2}$ and $I_1 \cup I_2 = I$. We will consider the case when i belongs to both I_1 and I_2 (the case when i only lies in one of those sets is analogous). Thus, $I_1 = K_1 \cup \{i\}$ and $I_2 = K_2 \cup \{i\}$ with $i \notin K_1, K_2$.

Let $g \in \{E_{ij}^{\pm 1}, E_{ji}^{\pm 1}\}$. Since V and W are regular, we have $gx = x + x_1 + x_2$ and $gy = y + y_1 + y_2$ where $x_1 \in V_{K_1 \cup \{j\}}$, $x_2 = V_{K_1 \cup \{i,j\}}$, $y_1 \in W_{K_2 \cup \{j\}}$, $y_2 = W_{K_2 \cup \{i,j\}}$. Then $x_1 \otimes y_1 \in (V \otimes W)_{K_1 \cup K_2 \cup \{j\}} = (V \otimes W)_{I \setminus \{i\} \cup \{j\}}$ and each of the 7 terms $x \otimes y_1$, $x \otimes y_2$, $x_1 \otimes y$, $x_1 \otimes y_2$, $x_2 \otimes y$, $x_2 \otimes y_1$ and $x_2 \otimes y_2$ lies in $(V \otimes W)_{K_1 \cup K_2 \cup \{i,j\}} = (V \otimes W)_{I \cup \{j\}}$, so condition (ii) holds.

(c) Consider the map $\varphi : L(1)^{\otimes m} \rightarrow L(m)$ given by $\varphi(v_1 \otimes \dots \otimes v_m) = [v_1, \dots, v_m]$ (where the commutator on the right-hand side is left-normed). Since $SL_n(\mathbb{Z})$ acts on L by graded automorphisms, φ is a homomorphism of $SL_n(\mathbb{Z})$ -modules, and since L is generated in degree 1, φ is surjective. Thus, $L(m)$ is a quotient of $L(1)^{\otimes m}$ as an $SL_n(\mathbb{Z})$ -module, and the grading (**) coincides with the quotient grading defined in (b). Thus, (c) follows from (a)(ii) and (b). \square

Given a group G , let $L(G) = \bigoplus_{i=1}^{\infty} \gamma_i G / \gamma_{i+1} G$. The graded abelian group $L(G)$ has a natural structure of a graded Lie ring with the bracket on homogeneous elements defined by

$$[g\gamma_{i+1}G, h\gamma_{j+1}G] = [g, h]\gamma_{i+j+1}G \text{ for all } g \in \gamma_i G \text{ and } h \in \gamma_j G,$$

where $[g, h] = g^{-1}h^{-1}gh$. The bracket operation is well defined since $[\gamma_i G, \gamma_j G] \subseteq \gamma_{i+j} G$. It is clear from the definition that $L(G)$ is generated in degree 1 as a Lie ring.

Theorem 4.3. *Let G be a finitely generated group and $L = L(G) = \bigoplus_{i=1}^{\infty} \gamma_i G / \gamma_{i+1} G$. Suppose we are given another group Γ which contains G as a normal subgroup and a*

homomorphism $\varphi : SL_n(\mathbb{Z}) \rightarrow \Gamma/G$. Define the action of $SL_n(\mathbb{Z})$ on L by automorphisms by

$$(4.2) \quad x \cdot (g + \gamma_{i+1}G) = \varphi(x)g\varphi(x)^{-1} + \gamma_{i+1}G \text{ for all } x \in SL_n(\mathbb{Z}), \quad i \in \mathbb{N} \text{ and } g \in \gamma_i G.$$

Let $d \in \mathbb{N}$, let $N = \left\lceil \frac{n}{d(d+1)} \right\rceil + 1$, and suppose that

- (i) G has the structure of a good \mathbf{n} -group generated in degree d .
- (ii) $L(1) = G/[G, G] = G^{ab}$ has the structure of a regular $SL_n(\mathbb{Z})$ -module of degree at most d .
- (iii) For every $I \subseteq \mathbf{n}$, with $|I| \geq d$, the image of G_I in $G^{ab} = L(1)$ contains $\sum_{J \subseteq I} L(1)_J$

Then G has a ρ -centralizing generating set for any non-trivial irreducible representation ρ (over an arbitrary field F) such that $\text{Ker}(\rho) \supseteq \gamma_N G$.

In the remainder of this section we will show how Theorem 4.3 can be applied to $G = IA_n$. The corresponding verification for the Torelli groups requires more work and will be given in § 7.

4.1. Hypotheses of Theorem 4.3 hold for $G = IA(n)$ with $n \geq 3$. In [Ma], Magnus proved that IA_n is generated by the automorphisms K_{ij} , $i \neq j$ and K_{ijk} , i, j, k distinct, given by

$$K_{ij} : \begin{cases} x_i \mapsto x_j^{-1} x_i x_j \\ x_k \mapsto x_k \text{ for } k \neq i \end{cases}, \quad K_{ijk} : \begin{cases} x_i \mapsto x_i [x_j, x_k] \\ x_l \mapsto x_l \text{ for } l \neq i \end{cases}$$

Since $K_{ijk}^{-1} = K_{ikj}$, it is enough to consider K_{ijk} with $j < k$. The elements K_{ij} and K_{ijk} with $j < k$ will be referred to as the *standard generators* of IA_n .

We claim that hypotheses of Theorem 4.3 are satisfied with $\Gamma = \text{Aut}(F_n)$, $G = IA_n$, $\varphi : SL_n(\mathbb{Z}) \rightarrow \text{Aut}(F_n)/IA_n \cong GL_n(\mathbb{Z})$ the standard embedding and $d = 3$.

Define the structure of an \mathbf{n} -group on IA_n as follows: for each $I \subseteq \mathbf{n}$ let $(IA_n)_I = \langle K_{ij}, K_{ijk} : i, j, k \in I \rangle$. All properties in the definition of a good \mathbf{n} -group are clear, and IA_n is generated in degree 3, so condition (i) in the statement of Theorem 4.3 holds.

Next we describe IA_n^{ab} as a $GL_n(\mathbb{Z})$ -module. Recall that IA_n and $IA_n(2)$ are defined as the kernels of the natural maps $\text{Aut}(F_n) \rightarrow \text{Aut}(F_n/\gamma_2 F_n)$ and $\text{Aut}(F_n) \rightarrow \text{Aut}(F_n/\gamma_3 F_n)$, respectively. Define a map $\psi : IA_n \rightarrow \text{Hom}(F_n/\gamma_2 F_n, \gamma_2 F_n/\gamma_3 F_n)$ by

$$(\psi(g))(x + \gamma_2 F_n) = x^{-1}g(x) + \gamma_3 F_n \quad \text{for all } g \in IA(n), x \in F_n.$$

It is easy to see that ψ is a well defined homomorphism, $\text{Ker} \psi = IA_n(2)$, and the elements $\{\psi(K_{ij}), \psi(K_{ijk})\}$ span $\text{Hom}(F_n/\gamma_2 F_n, \gamma_2 F_n/\gamma_3 F_n)$, so ψ is surjective. Since $IA_n(2) = [IA_n, IA_n]$ by [Ba, Lemma 5], we deduce an isomorphism of abelian groups:

$$IA_n^{ab} \cong \text{Hom}(F_n/\gamma_2 F_n, \gamma_2 F_n/\gamma_3 F_n)$$

Now let $V = \mathbb{Z}^n$. Then $F_n/\gamma_2 F_n \cong V$ and $\gamma_2 F_n/\gamma_3 F_n \cong V \wedge V$ as abelian groups. Thus,

$$IA_n^{ab} \cong \text{Hom}(V, V \wedge V) \cong V^* \otimes (V \wedge V) \quad (***)$$

as abelian groups, and it is straightforward to check that this is actually an isomorphism of $GL_n(\mathbb{Z})$ -modules (see [Kaw, Thm 6.1] for a self-contained proof of this isomorphism). The isomorphism (***) and Lemmas 4.1 and 4.2 imply that IA_n^{ab} is a regular $SL_n(\mathbb{Z})$ -module generated in degree 3, so condition (ii) in Theorem 4.3 holds.

If e_1, \dots, e_n is the standard basis of V and e_1^*, \dots, e_n^* is the corresponding dual basis in V^* , then the grading on $W = V^* \otimes (V \wedge V)$ coming from Proposition 4.1 and Lemma 4.2 is given by

$$W_I = \sum_{I=\{i,j,k\}} \mathbb{Z}e_i^* \otimes (e_j \wedge e_k) \text{ where we do not require that } i, j, k \text{ are distinct.}$$

Finally, it is easy to check that under the isomorphism (**), K_{ij} maps to $e_i^* \otimes (e_i \wedge e_j)$ and K_{ijk} maps to $e_i^* \otimes (e_j \wedge e_k)$. This shows that condition (iii) in Theorem 4.3 also holds.

5. PROOF OF THEOREM 4.3

Throughout this section we fix the notations introduced in Theorem 4.3. Without loss of generality we can assume that the homomorphism $\varphi : SL_n(\mathbb{Z}) \rightarrow \Gamma/G$ is surjective since if not, we can replace Γ by $\varphi^{-1}(\Gamma)$.

Let (ρ, V) be a non-trivial irreducible representation of G over a field F with $\gamma_N G \subseteq \text{Ker } \rho$. Recall that $C_\rho(G)$ denotes the set of all elements of G such that $\rho(g)$ is central and non-trivial.

Let m be the largest integer such that $\gamma_m G$ acts non-trivially on V , and let

$$H = \text{Ker } \rho \cap \gamma_m G.$$

Thus by assumptions $1 \leq m \leq N-1$, whence $n \geq md(d+1)$, and $\gamma_m G \setminus H \subseteq C_\rho(G)$.

Now let $L(1) = G/[G, G]$ and $L(m) = \gamma_m G / \gamma_{m+1} G$. Given $I \subseteq \mathbf{n}$, define $L(m)_I$ as in Lemma 4.2(c). Then $\deg(L(m)) \leq md$, that is,

$$(5.1) \quad L(m) = \sum_{|I| \leq md} L(m)_I \quad \text{and} \quad L(m)_I = 0 \text{ if } |I| > md$$

Claim 5.1. *If $|I| \geq d$, then the image of $\gamma_m G_I$ in $L(m)$ contains $L(m)_J$ for every $J \subseteq I$.*

Proof. Fix $J \subseteq I$. By definition $L(m)_J$ is spanned by elements of the form $[x_1, \dots, x_m]$ where $x_k \in L(1)_{J_k}$ for some $J_k \subseteq J$. Since $|I| \geq d$, hypothesis (iii) of Theorem 4.3 implies that there exist elements $\{g_k \in G_I\}_{k=1}^d$ such that $x_k = g_k[G, G]$. Then $[x_1, \dots, x_m]$ is the image in $L(m)$ of the element $[g_1, \dots, g_m]$ which lies in $\gamma_m G_I$. \square

Let U be any abelian group which contains an isomorphic copy of every finitely generated abelian group, and let

$$\Omega = \text{Hom}_{\mathbb{Z}}(L(m), U)$$

We will define the action of $SL_n(\mathbb{Z})$ on Ω in the usual way:

$$(g\lambda)(x) = \lambda(g^{-1}x) \text{ for all } g \in SL_n(\mathbb{Z}), \lambda \in \Omega \text{ and } x \in L(m).$$

Since G is a finitely generated group, $L(m)$ is a finitely generated abelian group, whence every subgroup of $L(m)$ is the kernel of some element of Ω . Choose $\lambda_H \in \Omega$ such that $\text{Ker } \lambda_H = H/\gamma_{m+1} G$ (recall that $H = \gamma_m G \cap \text{Ker } \rho$).

Definition. Let $\lambda \in \Omega$. Define $\text{supp}(\lambda)$, the *support* of λ , to be the set of all subsets $I \subseteq \mathbf{n}$ such that λ does not vanish on $L(m)_I$, that is, $L(m)_I \not\subseteq \text{Ker } \lambda$.

Proposition 5.2. *For $1 \leq k \leq d+1$ let $I_k = \{(k-1)md+1, (k-1)md+2, \dots, kmd\}$ (note that $I_k \subseteq \mathbf{n}$ since $n \geq md(d+1)$ by assumption). Suppose that $\text{supp}(\lambda_H)$ contains $d+1$ subsets J_1, \dots, J_{d+1} such that $J_k \subseteq I_k$ for each k . Then G has a ρ -centralizing generating set.*

Remark. *Representations ρ satisfying the hypotheses of Proposition 5.2 are the “sufficiently random” ones we have referred to in the introduction.*

Proof. For each k we have $J_k \subseteq I_k$ and $|I_k| = md \geq d$. By Claim 5.1, the image of $\gamma_m G_{I_k}$ in $L(m)$ is not contained in $\text{Ker } \lambda_H = H/\gamma_{m+1}G$, so there exist elements $g_k \in \gamma_m G_{I_k} \setminus H \subseteq C_\rho(G) \cap G_{I_k}$ for $1 \leq k \leq d+1$. Since I_1, \dots, I_{d+1} are disjoint and consist of consecutive integers, the elements g_k commute with each other by condition (iii) in the definition of a good \mathbf{n} -group.

Now take any $I \subseteq \mathbf{n}$ with $|I| = d$. There exists $1 \leq k \leq d+1$ such that $I \cap I_k = \emptyset$. Then every element of G_I commutes with g_k . Since G is generated in degree d , applying Lemma 3.3 with $T = \{g_1, \dots, g_k\}$, we conclude that G has a ρ -centralizing generating set. \square

To treat the general case, we observe that if we precompose $\rho : G \rightarrow GL(V)$ with a conjugation by an element of Γ , existence (or non-existence) of a ρ -centralizing generating set will not be affected. Suppose the new representation ρ' is given by $\rho'(x) = \rho(a^{-1}xa)$ with $a \in \Gamma$, and choose $g \in SL_n(\mathbb{Z})$ such that $\varphi(g) = aG$. Then

$$\begin{aligned} (\text{Ker } \rho' \cap \gamma_m G) / \gamma_{m+1} G &= a(\text{Ker } \rho \cap \gamma_m G) a^{-1} / \gamma_{m+1} G \\ &= g((\text{Ker } \rho \cap \gamma_m G) / \gamma_{m+1} G) = g(H / \gamma_{m+1} G) = g \text{Ker } \lambda_H = \text{Ker } (g\lambda_H). \end{aligned}$$

Thus to prove Theorem 4.3 it suffices to show that there exists $g \in SL_n(\mathbb{Z})$ such that $\text{supp}(g\lambda_H)$ contains $d+1$ subsets J_1, \dots, J_{d+1} with $J_k \subseteq I_k$ (with I_k from Proposition 5.2). This easily follows from Proposition 5.3 below.

Definition. Let $0 \neq \lambda \in \Omega$ and $s \in \mathbb{N}$. If A_1, \dots, A_s are elements of $\text{supp}(\lambda)$, define

$$D(A_1, \dots, A_s) = \sum_{i=1}^s |A_i| - \sum_{i \neq j} |A_i \cap A_j|$$

The maximum possible value of $D(A_1, \dots, A_s)$ will be denoted by $D_s(\lambda)$, and any s -tuple in $\text{supp}(\lambda)$ on which this maximum is achieved will be called *maximally disjoint* for λ .

Proposition 5.3. *Let $0 \neq \lambda$ and s be as above and let $\{A_1, \dots, A_s\} \subseteq \text{supp}(\lambda)$ be maximally disjoint. Then one of the following holds:*

- (i) A_1, \dots, A_s are disjoint
- (ii) $\cup A_i = \mathbf{n}$
- (iii) There exists $g \in SL_n(\mathbb{Z})$ such that $D_s(g\lambda) > D_s(\lambda)$

First we explain why Proposition 5.3 finishes the proof of Theorem 4.3. Indeed, let $s = d+1$. Then case (ii) above cannot occur unless (i) also holds. Indeed, if $A_1, \dots, A_{d+1} \in \text{supp}(\lambda)$, then $|A_k| \leq md$ by (5.1), so $|\cup_{i=1}^{d+1} A_i| \leq (d+1) \cdot md \leq n$ with equality only possible if A_i are disjoint. If we now choose $g_0 \in SL_n(\mathbb{Z})$ such that $D_{d+1}(g_0\lambda_H)$ is maximal possible, then applying Proposition 5.3 to $\lambda = g_0\lambda_H$ we must be in case (i).

Let A_1, \dots, A_{d+1} be disjoint subsets in $\text{supp}(g_0\lambda_H)$. Since $|A_k| \leq md$, there exists a permutation $\sigma \in S_n$ such that $\sigma(A_k) \subseteq I_k$ for each k where I_k are as in Proposition 5.2. Condition (3) in the definition of a regular module implies that $\text{supp}(F_{ij}\mu) = (i, j)\text{supp}(\mu)$ for all $\mu \in \Omega$. Thus if we write σ as a product of transpositions $\sigma = \prod (i_t, j_t)$ and let $g = \prod F_{i_t j_t}$, then $\text{supp}(gg_0\lambda_H)$ contains $\sigma(A_k)$ for each k , as desired.

Proof of Proposition 5.3. Assume that none of the conditions (i)-(iii) holds. Without loss of generality we can assume that $|A_1 \cap A_2| \neq \emptyset$, and choose $i \in A_1 \cap A_2$ and $j \notin \cup_{i=1}^s A_i$. Since the s -tuple (A_1, \dots, A_s) is maximally disjoint, $\text{supp}(\lambda)$ does not contain $A_k \setminus \{i\} \cup \{j\}$ and $A_k \cup \{j\}$ for any k .

Now take any $g \in \{E_{ij}^{\pm 1}, E_{ji}^{\pm 1}\}$. We claim that $\text{supp}(g\lambda)$ contains A_k for any k . Indeed, by assumption $A_k \in \text{supp}(\lambda)$, so $\lambda(x) \neq 0$ for some $x \in L(m)_{A_k}$. By conditions (1) and (2) in the definition of a regular module, $g^{-1}x - x \in L(m)_{A_k \setminus \{i\} \cup \{j\}} + L(m)_{A_k \cup \{j\}} \subseteq \text{Ker } \lambda$. Hence $(g\lambda)(x) = \lambda(g^{-1}x) = \lambda(x) \neq 0$, so $A_k \in \text{supp}(g\lambda)$.

Since $\text{supp}(g\lambda)$ contains A_1, \dots, A_s and (A_1, \dots, A_s) is also maximally disjoint for λ , we must have $D_s(g\lambda) \geq D_s(\lambda)$. On the other hand, $D_s(g\lambda) \leq D_s(\lambda)$ by our hypothesis, so $D_s(g\lambda) = D_s(\lambda)$ and (A_1, \dots, A_s) is also maximally disjoint for $g\lambda$.

Applying the same argument to $E_{ij}\lambda$, we conclude that (A_1, \dots, A_s) is also maximally disjoint for $E_{ji}^{-1}E_{ij}\lambda$ and likewise maximally disjoint for $E_{ij}E_{ji}^{-1}E_{ij}\lambda$. In particular, $A_1 \setminus \{i\} \cup \{j\} \notin \text{supp}(E_{ij}E_{ji}^{-1}E_{ij}\lambda)$. On the other hand, $E_{ij}E_{ji}^{-1}E_{ij} = F_{ij}$, and condition (3) in the definition of a regular module implies that $\text{supp}(F_{ij}\lambda)$ contains $(i, j)A_1 = A_1 \setminus \{i\} \cup \{j\}$, a contradiction. \square

6. PROOF OF THE MAIN THEOREMS FOR IA_n WITH $N = 2$

In this section we will prove Theorem 1.4 for $G = IA_n$, $4 \leq n < 12$ and Theorem 1.6(a) in the case $N = 2$, $4 \leq n < 12$ (these are the only cases which do not follow from Theorem 4.3).¹ By Proposition 3.4, we are reduced to proving the following result:

Theorem 6.1. *Let $G = IA_n$ for some $n \geq 4$, and let (ρ, V) be a non-trivial irreducible representation of G with $\text{Ker } \rho \supseteq [G, G]$. Then G has a ρ -centralizing generating set.*

The general method of proof of Theorem 6.1 will be similar to that of Theorem 4.3, but we will make use of specific relations in IA_n and properties of the action of $SL_n(\mathbb{Z})$ on IA_n^{ab} which are not captured by the notions of a good \mathbf{n} -group and a regular $SL_n(\mathbb{Z})$ -module, respectively.

We start with the list of relations in IA_n that will be used in the proof.

Lemma 6.2. *Let $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{n}$, and assume that $\{a_i\}$ are distinct and $\{b_i\}$ are distinct. The following relations hold:*

- (R1) $[K_{a_1 a_2}, K_{b_1 b_2}] = 1$ if $a_1 \neq b_1, b_2$ and $b_1 \neq a_1, a_2$
- (R2) $[K_{a_1 a_2}, K_{b_1 b_2 b_3}] = 1$ if $a_1 \neq b_1, b_2, b_3$ and $b_1 \neq a_1, a_2$
- (R3) $[K_{a_1 a_2 a_3}, K_{b_1 b_2 b_3}] = 1$ if $a_1 \neq b_1, b_2, b_3$ and $b_1 \neq a_1, a_2, a_3$
- (R4) $[K_{bcd}, K_{ab}] = [K_{ad}, K_{ac}]$ if $a, b, c, d \in \mathbf{n}$ are distinct

¹The restriction $n \leq 12$ will not be used in the proof.

For the rest of the section, G, n, ρ and V be as in Theorem 6.1, and define Ω as in § 5 with $m = 1$. By discussion at the end of § 4,

$$G^{ab} = L(1) \cong \bigoplus_{j < k} \mathbb{Z}e_i^* \otimes (e_j \wedge e_k) \oplus \bigoplus_{i \neq j} \mathbb{Z}e_i^* \otimes (e_i \wedge e_j)$$

under the map which sends K_{ijk} to $e_i^* \otimes (e_j \wedge e_k)$ and K_{ij} to $e_i^* \otimes (e_i \wedge e_j)$. Given $\lambda \in \Omega$, define

$$c_{ijk}(\lambda) = \lambda(e_i^* \otimes (e_j \wedge e_k)).$$

The following two observations can be verified by direct computation. Observation 6.3 shows that when we act by an elementary matrix E_{ij} on an arbitrary $\lambda \in \Omega$, at most 4 of the coefficients c_{xy} will change.

Observation 6.3. *Let $i, j, a \in \mathbf{n}$ be distinct and $\lambda \in \Omega$. Then*

$$\begin{aligned} c_{aaj}(E_{ij}\lambda) &= c_{aaj}(\lambda) - c_{aai}(\lambda) \\ c_{iij}(E_{ij}\lambda) &= c_{iij}(\lambda) - c_{jji}(\lambda) \\ c_{jja}(E_{ij}\lambda) &= c_{jja}(\lambda) + c_{jai}(\lambda) \\ c_{iaa}(E_{ij}\lambda) &= c_{iaa}(\lambda) - c_{jai}(\lambda) \end{aligned}$$

and $c_{xxy}(E_{ij}\lambda) = c_{xxy}(\lambda)$ for all $(x, y) \neq (a, j), (i, j), (j, a), (i, a)$.

Observation 6.4. *Let $i, j \in \mathbf{n}$ with $i \neq j$ and let σ be the transposition (i, j) . Then for any $\lambda \in \Omega$ and $x, y, z \in \mathbf{n}$ we have $c_{xyz}(F_{ij}\lambda) = \pm c_{\sigma(x)\sigma(y)\sigma(z)}(\lambda)$.*

In the proof of the following lemma we will repeatedly use Observation 6.3 without an explicit reference.

Lemma 6.5. *Assume that $n \geq 4$. Then for any nonzero $\lambda \in \Omega$ there exists $g \in SL_n(\mathbb{Z})$ such that $c_{112}(g\lambda), c_{221}(g\lambda), c_{334}(g\lambda)$ and $c_{443}(g\lambda)$ are all nonzero.*

Proof. Step 1: There exists $g_1 \in SL_n(\mathbb{Z})$ such that $c_{112}(g_1\lambda) \neq 0$.

First of all, by Observation 6.4 it suffices to find distinct $a, b \in \mathbf{n}$ such that $c_{aab}(g_1\lambda) \neq 0$.

If $c_{xxy}(\lambda) \neq 0$ for some $x \neq y$, we are done. Suppose now that $c_{xxy}(\lambda) = 0$ for all $x \neq y$. Then there must exist distinct a, b, c with $c_{abc}(\lambda) \neq 0$, in which case

$$c_{aab}(E_{ca}\lambda) = c_{aab}(\lambda) + c_{abc}(\lambda) = 0 + c_{abc}(\lambda) \neq 0.$$

Step 2: There exists $g_2 \in SL_n(\mathbb{Z})$ such that $c_{112}(g_2\lambda), c_{113}(g_2\lambda) \neq 0$.

By Step 1, we can assume that $c_{112}(\lambda) \neq 0$. If $c_{113}(\lambda) \neq 0$, we are done. And if $c_{113}(\lambda) = 0$, then $c_{113}(E_{23}\lambda) = c_{113}(\lambda) - c_{112}(\lambda) = 0 - c_{112}(\lambda) \neq 0$ and $c_{112}(E_{23}\lambda) = c_{112}(\lambda) \neq 0$

Step 3: There exists $g_3 \in SL_n(\mathbb{Z})$ such that either $c_{112}(g_3\lambda), c_{113}(g_3\lambda), c_{221}(g_3\lambda) \neq 0$ or $c_{112}(g_3\lambda), c_{334}(g_3\lambda) \neq 0$

By Step 2, we can assume that $c_{112}(\lambda), c_{113}(\lambda) \neq 0$. If $c_{224}(\lambda) \neq 0$ or $c_{221}(\lambda) \neq 0$, we are done (using Observation 6.4 in the former case), so assume that $c_{224}(\lambda) = c_{221}(\lambda) = 0$. We consider 3 cases.

Case 1: $c_{223}(\lambda) \neq 0$. Then

$$c_{224}(E_{34}\lambda) = c_{224}(\lambda) - c_{223}(\lambda) = -c_{223}(\lambda) \neq 0$$

$$c_{113}(E_{34}\lambda) = c_{113}(\lambda) \neq 0$$

Hence $c_{113}(E_{34}\lambda), c_{224}(E_{34}\lambda) \neq 0$.

Case 2: $c_{223}(\lambda) = 0$ and $c_{132}(\lambda) = 0$. Then

$$c_{112}(E_{21}\lambda) = c_{112}(\lambda) \neq 0$$

$$c_{221}(E_{21}\lambda) = c_{221}(\lambda) - c_{112}(\lambda) = -c_{112}(\lambda) \neq 0$$

$$c_{113}(E_{21}\lambda) = c_{113}(\lambda) + c_{132}(\lambda) = c_{113}(\lambda) \neq 0.$$

Case 3: $c_{223}(\lambda) = 0$ and $c_{132}(\lambda) \neq 0$.

Again we have $c_{112}(E_{21}\lambda) \neq 0$ and $c_{221}(E_{21}\lambda) \neq 0$, and this time

$$c_{223}(E_{21}\lambda) = c_{223}(\lambda) - c_{132}(\lambda) = -c_{132}(\lambda) \neq 0.$$

Hence we are done by Observation 6.4.

Step 4: There exists $g_4 \in SL_n(\mathbb{Z})$ such that $c_{112}(g_4\lambda), c_{334}(g_4\lambda) \neq 0$.

By Step 3, we can assume that $c_{112}(\lambda) \neq 0$, $c_{221}(\lambda) \neq 0$ and $c_{113}(\lambda) \neq 0$. If $c_{443}(\lambda) \neq 0$ or $c_{224}(\lambda) \neq 0$, we are done (by Observation 6.4).

So assume that $c_{443}(\lambda) = c_{224}(\lambda) = 0$. Then

$$c_{224}(E_{14}\lambda) = c_{224}(\lambda) - c_{221}(\lambda) = -c_{221}(\lambda) \neq 0$$

Case 1: $c_{431}(\lambda) = 0$. Then $c_{113}(E_{14}\lambda) = c_{113}(\lambda) - c_{431}(\lambda) \neq 0$, so $c_{113}(E_{14}\lambda) \neq 0$ and $c_{224}(E_{14}\lambda) \neq 0$.

Case 2: $c_{431}(\lambda) \neq 0$. Then $c_{443}(E_{14}\lambda) = c_{443}(\lambda) + c_{431}(\lambda) = c_{431}(\lambda) \neq 0$ and $c_{221}(E_{14}\lambda) = c_{221}(\lambda) \neq 0$, so $c_{443}(E_{14}\lambda) \neq 0$ and $c_{221}(E_{14}\lambda) \neq 0$.

In both cases we are done by Observation 6.4.

Final Step. By Step 4, we can assume that $c_{112}(\lambda), c_{334}(\lambda) \neq 0$. Set $\alpha = 0$ if $c_{221}(\lambda) \neq 0$ and $\alpha = 1$ if $c_{221}(\lambda) = 0$ and $\beta = 0$ if $c_{443}(\lambda) \neq 0$ and $\beta = 1$ if $c_{443}(\lambda) = 0$. Then $g = E_{21}^\alpha E_{43}^\beta$ satisfies the assertion of Lemma 6.5. \square

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let $H = \text{Ker } \rho$. By Lemma 6.5, after precomposing ρ with conjugation by a suitable element of $g \in \text{Aut}(F_n)$ (which has the effect of replacing H by gHg^{-1} and λ_H by $g\lambda_H$), we can assume that H has empty intersection with the set $S = \{K_{12}, K_{21}, K_{34}, K_{43}\}$. Using relations (R1)-(R3) of Lemma 6.2, it is easy to show that the only standard generators of IA_n which do not commute with an element of S are $K_{134}, K_{234}, K_{312}, K_{412}$. However, by relations (R4) each of these 4 elements commutes with an element of S modulo the subgroup generated by all K_{xy} . Thus, if we order the standard generators of IA_n so that $K_{12}, K_{34}, K_{21}, K_{43}$ come first (in this order) and $K_{134}, K_{234}, K_{312}, K_{412}$ come last, we obtain a ρ -centralizing generating set. \square

7. VERIFYING HYPOTHESES OF THEOREM 4.3 FOR THE TORELLI SUBGROUPS.

Let $g \geq 3$, and let $\Sigma = \Sigma_g^1$ be an orientable surface of genus g with 1 boundary component. The mapping class group $\text{Mod}_g^1 = \text{Mod}(\Sigma_g^1)$ is defined as the subgroup of orientation preserving homeomorphisms of Σ_g^1 which fix $\partial\Sigma_g^1$ pointwise modulo the isotopies which fix $\partial\Sigma_g^1$ pointwise.

Choose a base point p_0 on the boundary $\partial\Sigma$. Since Σ has a bouquet of $2g$ circles as a deformation retract, the fundamental group $\pi = \pi_1(\Sigma, p_0)$ is free of rank $2g$; moreover, for a suitable choice of a free generating set $\{\alpha_i, \beta_i\}_{i=1}^g$ of π , the boundary $\partial\Sigma$ represents $\Pi = \prod_{i=1}^g [\alpha_i, \beta_i]$. Thus, there is a natural homomorphism from Mod_g^1 to the subgroup of automorphisms of π which fix Π . It is well known that this map is an isomorphism.

Thus, we can identify Mod_g^1 with a subgroup of $\text{Aut}(F_{2g})$, and using this identification we can define the Johnson filtration $\{\mathcal{I}_g^1(k)\}_{k=1}^\infty$ by $\mathcal{I}_g^1(k) = \text{Mod}_g^1 \cap IA_{2g}(k)$. The subgroups $\mathcal{I}_g^1 = \mathcal{I}_g^1(1)$ and $\mathcal{J}_g^1 = \mathcal{I}_g^1(2)$ are known as the *Torelli subgroup* and the *Johnson kernel*, respectively. Note that \mathcal{I}_g^1 can also be defined as the set of all elements of Mod_g^1 which act trivially on the integral homology group $H_1(\Sigma_g^1)$.

For the rest of this section we set $\mathcal{M} = \text{Mod}_g^1$, $\mathcal{I} = \mathcal{I}_g^1 = \mathcal{I}_g^1(1)$ and $\mathcal{J} = \mathcal{J}_g^1 = \mathcal{I}_g^1(2)$. Our goal is to show that the hypotheses of Theorem 4.3 hold for $\Gamma = \mathcal{M}$, $G = \mathcal{I}$, $d = 3$ and $n = g$. The homomorphism $\varphi : SL_g(\mathbb{Z}) \rightarrow \mathcal{M}/\mathcal{I}$ will be defined in § 7.6 (assuming the canonical isomorphism $\mathcal{M}/\mathcal{I} \cong Sp(V)$ defined in § 7.2).

7.1. The \mathbf{g} -group structure on \mathcal{I} . We can think of Σ_g^1 as a (closed) disk with g handles attached; call the handles H_1, \dots, H_g . Choose disjoint subsurfaces S_1, \dots, S_g , each homeomorphic to Σ_1^1 , such that S_i contains H_i . As shown in [CP, §4.1], for each $I \subseteq \mathbf{g}$, one can construct a subsurface S_I homeomorphic to $\Sigma_{|I|}^1$ such that the following properties hold:

- (1) $S_{\{i\}} = S_i$ for each $i \in \mathbf{n}$ and $S_{\mathbf{n}} = \Sigma$
- (2) $S_I \subseteq S_J$ if $I \subseteq J$
- (3) (see Figure 2 in [CP, §4.1]) Let $I, J \subseteq \mathbf{g}$ be disjoint subsets satisfying one of the additional properties:
 - (i) $i < j$ for all $i \in I$ and $j \in J$ (or vice versa)
 - (ii) There exist $j_1, j_2 \in J$ with $j_1 < j_2$ such that the interval $\{j_1 + 1, \dots, j_2 - 1\}$ contains I and does not contain any elements of J .

Then there exist disjoint subsurfaces S'_I and S'_J isotopic to S_I and S_J , respectively.

Note that if I and J are disjoint and I consists of consecutive integers, then (i) or (ii) above must hold.

Now define \mathcal{I}_I to be the subgroup of \mathcal{I} consisting of mapping classes which have a representative supported on S_I . Properties (1)-(3) above imply that $\mathcal{I} = \mathcal{I}_g^1$ is a good \mathbf{g} -group (here we use the obvious fact that homeomorphisms supported on disjoint surfaces commute), and by [CP, Proposition 4.5], this \mathbf{g} -group is generated in degree 3.

7.2. $H_1(\Sigma)$ as an $Sp_{2g}(\mathbb{Z})$ -module. Let $V = H_1(\Sigma)$. Then V is a free abelian group of rank $2g$ endowed with the canonical symplectic form

$$([\alpha], [\beta]) \mapsto [\alpha] \cdot [\beta]$$

where $[\alpha] \cdot [\beta]$ is the algebraic intersection number between the closed curves α and β on Σ . Clearly the action of \mathcal{M}/\mathcal{I} on V preserves this form, so there is a canonical group homomorphism $\mathcal{M}/\mathcal{I} \rightarrow Sp(V)$ where $Sp(V)$ is the group of automorphisms of V preserving the above form. It is well known that this homomorphism is an isomorphism, so from now on we will identify \mathcal{M}/\mathcal{I} with $Sp(V)$.

Our next goal is to describe the abelianization \mathcal{I}^{ab} as an $Sp(V)$ -module. First we will introduce two quotients of \mathcal{I}^{ab} , the largest torsion-free quotient and the largest quotient of exponent 2. The corresponding homomorphisms defined on \mathcal{I} are called the *Johnson homomorphism* and the *Birman-Craig-Johnson homomorphism* and will be denoted by τ and σ , respectively.

7.3. Johnson homomorphism. The symplectic form introduced above yields a canonical isomorphism of $Sp(V)$ -modules $V^* \cong V$. By the same logic as in the case of automorphisms of free groups, there exists a homomorphism of $Sp(V)$ -modules

$$\tau : \mathcal{I} \rightarrow V \otimes (V \wedge V)$$

with $\text{Ker}(\tau) = \mathcal{J}$, the Johnson kernel. It is called the *Johnson homomorphism*. Unlike the case of automorphisms of free groups, τ is not surjective, and Johnson [Jo3] showed that its image is spanned by elements of the form $u \otimes (v \wedge w) + v \otimes (w \wedge u) + w \otimes (u \wedge v)$. The latter subspace is clearly $Sp(V)$ -isomorphic to $\wedge^3 V$, so $\mathcal{I}/\mathcal{J} \cong \wedge^3 V$ as $Sp(V)$ -modules.

We will use an explicit formula for the values of τ on a suitable generating set for \mathcal{I} , which is discussed below. For each simple closed curve γ on Σ denote by $T_\gamma \in \mathcal{M} = \text{Mod}(\Sigma)$ the Dehn twist about γ . Now let (γ, δ) be a pair of disjoint non-separating simple closed curves on Σ such that

- (i) γ and δ are homologous to each other and non-homologous to zero.
- (ii) the union $\gamma \cup \delta$ separates Σ ; moreover, if $\Sigma_{\gamma, \delta}$ is the connected component of $\Sigma \setminus (\gamma \cup \delta)$ which does not contain $\partial\Sigma$, then $\Sigma_{\gamma, \delta}$ has genus 1.
- (iii) γ is oriented in such a way that $\Sigma_{\gamma, \delta}$ is on its left.

Johnson [Jo1] proved that the Torelli group \mathcal{I} is generated by elements of the form $T_\gamma T_\delta^{-1}$, with (γ, δ) as above, so it suffices to know the values τ on such elements.

Given (γ, δ) as above, let $c \in V = H_1(\Sigma)$ be the homology class of γ , and choose any $a, b \in H_1(\Sigma_{\gamma, \delta}) \subset H_1(\Sigma)$ such that $a \cdot b = 1$. Then

$$(7.1) \quad \tau(T_\gamma T_\delta^{-1}) = a \wedge b \wedge c.$$

For the justification of (7.1) and the related formula (7.2) below see [Jo6, § 2] and references therein.

7.4. Birman-Craig-Johnson homomorphism. Now let $V_{\mathbb{F}_2} = V \otimes \mathbb{F}_2 \cong H_1(\Sigma, \mathbb{F}_2)$. Let B be the ring of polynomials over \mathbb{F}_2 in formal variables $X = \{\bar{v} : v \in H_{\mathbb{F}_2} \setminus \{0\}\}$ subject to relations

- (R1) $\overline{v + w} = \bar{v} + \bar{w} + v \cdot w$ for all $\bar{v}, \bar{w} \in X$
- (R2) $\bar{v}^2 = \bar{v}$ for all $\bar{v} \in X$

The group $Sp(V)$ has a natural action on B by ring automorphisms such that $g(\bar{v}) = \overline{gv}$ for all $\bar{v} \in X$. Let B_n be the subspace of B consisting of elements representable by a polynomial of degree at most n . Then each B_n is an $Sp(V)$ -submodule, and it is easy to see that $B_n/B_{n-1} \cong \wedge^n V_{\mathbb{F}_2}$ for each $n \geq 1$ (as $Sp(V)$ -modules).

In [Jo2], Johnson constructed a surjective homomorphism $\sigma : \mathcal{I} \rightarrow B_3$ which induces an $Sp(V)$ -module homomorphism $\mathcal{I}^{ab} \rightarrow B_3$. Following [BF], we will refer to σ as the *Birman-Craig-Johnson (BCJ) homomorphism*. We will not discuss the conceptual definition of σ in terms of the Rochlin invariant and instead give an explicit formula for σ on elements $T_\gamma T_\delta^{-1}$ with (γ, δ) satisfying (i)-(iii) above:

$$(7.2) \quad \sigma(T_\gamma T_\delta^{-1}) = \bar{a}\bar{b}(\bar{c} + 1).$$

where a, b and c are defined as in (7.1) except that this time they are mod 2 homology classes.

7.5. The full abelianization. Let $\alpha : \wedge^3 V \rightarrow \wedge^3 V_{\mathbb{F}_2}$ be the natural reduction map, and let $\beta : B_3 \rightarrow \wedge^3 V_{\mathbb{F}_2}$ be the unique linear map such that $\beta(B_2) = 0$ and $\beta(\bar{u} \bar{v} \bar{w}) = u \wedge v \wedge w$ for all $u, v, w \in V_{\mathbb{F}_2}$. Clearly α and β are both $Sp(V)$ -module homomorphisms. Let

$$(7.3) \quad W = \{(u, v) \in \wedge^3 V \oplus B_3 : \alpha(u) = \beta(v)\}$$

Theorem 7.1. *The map $(\tau, \sigma) : \mathcal{I}^{ab} \rightarrow \wedge^3 V \times B_3$ given by $g + [G, G] \mapsto (\tau(g), \sigma(g))$ is injective and $\text{Im}((\tau, \sigma)) = W$.*

Proof. The fact that $\text{Im}(\tau, \sigma) \subseteq W$ holds by [Jo3, Theorem 4] and also follows immediately from (7.1) and (7.2). Once this is established, the opposite inclusion follows from $\sigma(\text{Ker } \tau) = \text{Ker } \beta$, and the latter holds by [Jo2, Lemma 4]. Finally, injectivity of (τ, σ) is proved in [Jo6]. \square

Now let $I \subseteq \mathfrak{g}$ be any subset with $|I| \geq 3$, and let S_I be the corresponding subsurface of Σ introduced in § 7.1. Since S_I is itself a closed orientable surface of genus ≥ 3 with one boundary component, we can repeat the entire construction described in this section starting with S_I instead of Σ , so in particular we can define the modules $V(I) = H_1(S_I)$ and $B_3(I)$ and the homomorphisms $\tau_I : \mathcal{I}_I \rightarrow \wedge^3 V(I)$ and $\sigma_I : \mathcal{I}_I \rightarrow B_3(I)$ (recall that \mathcal{I}_I was defined as the subgroup of \mathcal{I} consisting of mapping classes supported on S_I , but this group is canonically isomorphic to the Torelli subgroup of $\text{Mod}(S_I)$).

Proposition 7.2. *The following diagrams are commutative*

$$(7.4) \quad \begin{array}{ccc} (\mathcal{I}_I)^{ab} & \xrightarrow{\tau_I} & \wedge^3 V(I) \\ \downarrow & & \downarrow \\ \mathcal{I}^{ab} & \xrightarrow{\tau} & \wedge^3 V \end{array} \quad \begin{array}{ccc} (\mathcal{I}_I)^{ab} & \xrightarrow{\sigma_I} & B_3(I) \\ \downarrow & & \downarrow \\ \mathcal{I}^{ab} & \xrightarrow{\sigma} & B_3 \end{array}$$

where the vertical maps are induced by the natural inclusions $\mathcal{I}_I \rightarrow \mathcal{I}$ and $H_1(S_I) \rightarrow H_1(\Sigma)$.

Proof. For both diagrams, it is enough to check commutativity for the values on the generators $T_\gamma T_\delta^{-1}$. This follows immediately from (7.1) and (7.2) since if γ and δ are curves on S_I satisfying (i)-(iii), replacing S_I by Σ will not change the surface $\Sigma_{\gamma, \delta}$ or the homology classes a, b and c . \square

7.6. \mathcal{I}^{ab} as a regular $SL_g(\mathbb{Z})$ -module. For each $1 \leq i \leq g$ choose any basis $\{a_i, b_i\}$ for $H_1(S_i) \subset V$ s.t. $a_i \cdot b_i = 1$. Then $V = \bigoplus_{i=1}^g (\mathbb{Z}a_i \oplus \mathbb{Z}b_i)$, and $\{a_i, b_i\}_{i=1}^g$ is a symplectic basis for V , that is, $a_i \cdot a_j = b_i \cdot b_j = 0$ for all i, j , and $a_i \cdot b_j = \delta_{ij}$. Now define $\varphi : SL_g(\mathbb{Z}) \rightarrow Sp(V)$ by

$$\varphi(A) = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix},$$

where the above matrix is with respect to the ordered basis $(a_1, \dots, a_g, b_1, \dots, b_g)$. This yields $SL_g(\mathbb{Z})$ -module structures on $\wedge^3 V$, B_3 and hence on W defined by (7.3). We also obtain an action of $SL_g(\mathbb{Z})$ on $L(\mathcal{I})$ by Lie algebra automorphisms via (4.2) using φ above (recall that $Sp(V)$ is canonically isomorphic to \mathcal{M}/\mathcal{I}), and it is straightforward to check that $(\tau, \sigma) : \mathcal{I}^{ab} \rightarrow W$ from Theorem 7.1 is an isomorphism of $SL_g(\mathbb{Z})$ -modules.

We could proceed working directly with W , but things can be simplified further with the following observations. Let $\pi : B_3 \rightarrow \bigoplus_{i=0}^3 \wedge^i V_{\mathbb{F}_2}$ be the unique linear map such that $\pi(1) = 1$, $\pi(\bar{x}) = x$, $\pi(\bar{x}\bar{y}) = x \wedge y$ and $\pi(\bar{x}\bar{y}\bar{z}) = x \wedge y \wedge z$ where x, y, z are distinct elements of the basis $\{a_i, b_i\}_{i=1}^g$. Clearly, π is bijective, and it is straightforward to check that π is an isomorphism of $SL_g(\mathbb{Z})$ -modules² (but not an isomorphism of $Sp(V)$ -modules!)

Now define $\pi' : \wedge^3 V \oplus B_3 \rightarrow \wedge^3 V \oplus \bigoplus_{i=0}^3 \wedge^i V_{\mathbb{F}_2}$ by $\pi'(u, v) = (u, \pi(v) - \alpha(u))$. Clearly, π' is

an isomorphism of $SL_g(\mathbb{Z})$ -modules, and it is easy to show that $\pi'(W) = \wedge^3 V \oplus \bigoplus_{i=0}^2 \wedge^i V_{\mathbb{F}_2}$.

Thus we have the following isomorphism of $SL_g(\mathbb{Z})$ -modules:

$$\lambda = \pi' \circ (\tau, \sigma) : \mathcal{I}^{ab} \rightarrow \wedge^3 V \oplus \left(\bigoplus_{i=0}^2 \wedge^i V_{\mathbb{F}_2} \right). \quad (***)$$

It is easy to see that V is a direct sum of the natural $SL_g(\mathbb{Z})$ -module \mathbb{Z}^g and its dual. Using Lemmas 4.1 and 4.2 and the isomorphism λ , we endow \mathcal{I}^{ab} with the structure of a regular $SL_g(\mathbb{Z})$ -module generated in degree 3, so condition (ii) in Theorem 4.3 holds. Here is an explicit description of the obtained grading on \mathcal{I}^{ab} . The symbol e_t stand for a_t or b_t .

- (1) If $I = \{i < j < k\}$, then $(\mathcal{I}^{ab})_I = \lambda^{-1}(\oplus \mathbb{Z}e_i \wedge e_j \wedge e_k)$.
- (2) If $I = \{i < j\}$, then $(\mathcal{I}^{ab})_I = \lambda^{-1}((\oplus \mathbb{Z}e_i \wedge a_j \wedge b_j) \oplus (\oplus \mathbb{Z}e_j \wedge a_i \wedge b_i) \oplus (\oplus \mathbb{F}_2 e_i \wedge e_j))$.
- (3) If $I = \{i\}$, then $(\mathcal{I}^{ab})_I = \lambda^{-1}(\mathbb{F}_2 a_i \oplus \mathbb{F}_2 b_i)$
- (4) If $I = \emptyset$, then $(\mathcal{I}^{ab})_I = \lambda^{-1}(\mathbb{F}_2)$

It remains to check condition (iii). In the argument below the reader should not confuse $(\mathcal{I}^{ab})_I$, the I -component of \mathcal{I}^{ab} as defined above, with $(\mathcal{I}_I)^{ab}$, the abelianization of the group \mathcal{I}_I . Let $I \subseteq \mathbf{g}$, with $|I| \geq 3$, and let $\iota_I : (\mathcal{I}_I)^{ab} \rightarrow \mathcal{I}^{ab}$ be the natural map. We need to show that $\iota_I((\mathcal{I}_I)^{ab})$ contains $\sum_{J \subseteq I} (\mathcal{I}^{ab})_J$ or, equivalently, that $\lambda \circ \iota_I((\mathcal{I}_I)^{ab})$ contains $\sum_{J \subseteq I} \lambda((\mathcal{I}^{ab})_J)$. We claim that the latter two sets are both equal to $Z_I := \wedge^3 V(I) \oplus$

²This is true since $SL_g(\mathbb{Z})$ preserves both $V_A = \oplus \mathbb{Z}a_i$ and $V_B = \oplus \mathbb{Z}b_i$ and the intersection form vanishes on both V_A and V_B

$\left(\bigoplus_{i=0}^2 \wedge^i V(I)_{\mathbb{F}_2}\right)$ (where we identify $V(I) = H_1(S_I)$ with its image in $V = H_1(\Sigma)$). Indeed, $\sum_{J \subseteq I} \lambda((\mathcal{I}^{ab})_J) = Z_I$ directly from (1)-(3) above, while the equality $\lambda \circ \iota_I((\mathcal{I}_I)^{ab}) = Z_I$ follows easily from Proposition 7.2.

8. ABELIANIZATION OF FINITE INDEX SUBGROUPS IN $\text{Aut}(F_n)$ AND $\text{Mod}(\Sigma_g^1)$

A group G is said to have *property (FAb)* if every finite index subgroup of G has finite abelianization. Clearly $\text{Aut}(F_2)$ does not have (FAb) since it projects onto $GL_2(\mathbb{Z})$, which is a virtually free group. The group $\text{Aut}(F_3)$ also does not have (FAb) – this was proved by completely different methods in [Mc2] and [GL]. The question whether $\text{Aut}(F_n)$ has (FAb) for $n \geq 4$ is wide open. To the best of our knowledge, the only (sufficiently general) class of finite index subgroups of $\text{Aut}(F_n)$ for which abelianization was (previously) known to be finite are subgroups containing IA_n – this has been proved for all $n \geq 3$ independently in [Bh] and [BV]. Slightly more was known in the case of mapping class groups. In [Mc1], it was proved that Mod_2^1 does not have (FAb), and it is a well known conjecture of Ivanov that Mod_g^1 has (FAb) for $g \geq 3$. In the case $g \geq 3$, in [Ha] it was proved that H^{ab} is finite for every finite index subgroup of Mod_g^1 containing the Torelli subgroup (see also [Mc1] for a short elementary proof). In [Pu1] this result was extended to all subgroups containing a large portion of the Johnson kernel (in suitable sense); in particular, to all subgroups containing the Johnson kernel itself.³

In this section we will prove Theorem 1.7 restated below. In particular we will establish finiteness of abelianization for all finite index subgroups of $\text{Aut}(F_n)$ and Mod_n^1 which contain the N^{th} term of the Johnson filtration, provided $n \geq 12(N-1)$.

Theorem 1.7. *Let (G, K) be as in Theorem 1.2, and let $\Gamma = \text{Aut}(F_n)$ if $G = IA_n$ and $\Gamma = \text{Mod}_g^1$ if $G = \mathcal{I}_g^1$. The following hold:*

- (1) *If H is a finite index subgroup of G which contains K , then the restriction map $H^1(G, \mathbb{C}) \rightarrow H^1(H, \mathbb{C})$ is an isomorphism*
- (2) *If H is a finite index subgroup of Γ which contains K , then H has finite abelianization.*

Theorem 1.7 is a direct consequence of Theorem 1.6 and the following general result.

Theorem 8.1. *Let Γ be a group and G and K normal subgroups of Γ with $K \subseteq G$. Assume that (G, K, \mathbb{C}) is nice. The following hold:*

- (1) *Let H be a subgroup of Γ which contains K , is normalized by G and such that $[GH : H] < \infty$. Then the restriction map $H^1(GH, \mathbb{C}) \rightarrow H^1(H, \mathbb{C})$ is an isomorphism.*
- (2) *If H is any finite index subgroup of G containing K , then the restriction map $H^1(G, \mathbb{C}) \rightarrow H^1(H, \mathbb{C})$ is an isomorphism.*
- (3) *Assume in addition that Γ is finitely generated and M^{ab} is finite for every finite index subgroup M of Γ which contains G . Then H^{ab} is finite for every finite index subgroup H of Γ which contains K .*

³The results of [Ha] and [Pu1] apply to mapping class groups of surfaces with an arbitrary number of punctures and boundary components

The extra hypothesis in part (3) holds for $(\Gamma, G) = (\text{Aut}(F_n), IA_n)$ with $n \geq 3$ or $(\Gamma, G) = (\text{Mod}_g^1, \mathcal{I}_g^1)$ for $g \geq 2$ by the results from [Bh], [BV] and [Ha] mentioned at the beginning of this section.

Proof of Theorem 8.1. (1) By Shapiro's Lemma $H^1(H, \mathbb{C}) \cong H^1(GH, \text{Coind}_H^{GH}(\mathbb{C}))$ where $\text{Coind}_H^{GH}(\mathbb{C})$ is the coinduced module. By assumption, H is a normal finite index subgroup of GH , so we have the following isomorphisms of GH -modules:

$$\text{Coind}_H^{GH}(\mathbb{C}) \cong \mathbb{C}[Q] \cong \bigoplus_{V \in \text{Irr}(Q)} (\dim V)V$$

where $Q = GH/H$ and $\text{Irr}(Q)$ is the set of equivalence classes of irreducible complex representations of Q . Hence

$$H^1(H, \mathbb{C}) \cong H^1(GH, \mathbb{C}) \oplus \bigoplus_{V \in \text{Irr}(Q) \setminus V_0} H^1(GH, V)^{\dim(V)}$$

where V_0 is the trivial representation of Q . Moreover, the inclusion $H^1(GH, \mathbb{C}) \rightarrow H^1(H, \mathbb{C})$ coming from the above isomorphism is the restriction map.

Thus, we only need to show that $H^1(GH, V) = 0$ for every non-trivial irreducible representation V of Q . Take any such representation V . The exact sequence of groups $1 \rightarrow G \rightarrow GH \rightarrow GH/G \rightarrow 1$ yields the following inflation-restriction sequence:

$$0 \rightarrow H^1(GH/G, V^G) \rightarrow H^1(GH, V) \rightarrow H^1(G, V)^{GH}. \quad (***)$$

Since V is irreducible and non-trivial, we have $V^G = 0$. Since (G, K, \mathbb{C}) is nice and K acts trivially on V , we have $H^1(G, V) = 0$. Thus (***) implies that $H^1(GH, V) = 0$, as desired.

(2) If H is normal in G , the result follows directly from (1). In general, since H has finite index in G and K is normal in G , there exists a finite index normal subgroup H' of G with $K \subseteq H' \subseteq H$. Then by (1) the composite restriction map $H^1(G, \mathbb{C}) \rightarrow H^1(H, \mathbb{C}) \rightarrow H^1(H', \mathbb{C})$ is an isomorphism, whence $H^1(H, \mathbb{C}) \rightarrow H^1(H', \mathbb{C})$ is surjective. Injectivity of $H^1(H, \mathbb{C}) \rightarrow H^1(H', \mathbb{C})$ is automatic since H' has finite index in H .

(3) Again the result follows from (1) if H is normal in Γ . Indeed, GH has finite index in Γ , so $(GH)^{ab}$ is finite, whence $H^1(GH, \mathbb{C}) = 0$ and thus $H^1(H, \mathbb{C}) = 0$ by (1). Since H has finite index in Γ and Γ is finitely generated, H is also finitely generated, so $H^1(H, \mathbb{C}) = 0$ forces H^{ab} to be finite. In general, we can find a finite index subgroup H' of H which is normal in Γ , and H^{ab} is finite whenever $(H')^{ab}$ is finite. \square

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