

# GROUPS OF POSITIVE WEIGHTED DEFICIENCY AND THEIR APPLICATIONS

MIKHAIL ERSHOV AND ANDREI JAIKIN-ZAPIRAIN

**ABSTRACT.** In this paper we introduce the concept of weighted deficiency for abstract and pro- $p$  groups and study groups of positive weighted deficiency which generalize Golod-Shafarevich groups. In order to study weighted deficiency we introduce weighted versions of the notions of rank for groups and index for subgroups and establish weighted analogues of several classical results in combinatorial group theory, including the Schreier index formula.

Two main applications of groups of positive weighted deficiency are given. First we construct infinite finitely generated residually finite  $p$ -torsion groups in which every finitely generated subgroup is either finite or of finite index – these groups can be thought of as residually finite analogues of Tarski monsters. Second we develop a new method for constructing just-infinite groups (abstract or pro- $p$ ) with prescribed properties; in particular, we show that graded group algebras of just-infinite groups can have exponential growth. We also prove that every group of positive weighted deficiency has a hereditarily just-infinite quotient. This disproves a conjecture of Boston on the structure of quotients of certain Galois groups and solves Problem 15.18 from Kurovka notebook.

## 1. INTRODUCTION

**1.1. What this paper is about.** Let  $\Gamma$  be a finitely presented group. Recall that the *deficiency* of  $\Gamma$  denoted by  $def(\Gamma)$  is the maximum value of the quantity  $|X| - |R|$  where  $(X, R)$  runs over all presentations of  $\Gamma$  by generators and relations. Note that  $def(\Gamma)$  cannot equal  $\infty$  as  $|X| - |R|$  does not exceed  $d(\Gamma)$ , the number of generators of  $\Gamma$ . One of the best known results involving deficiency is a theorem of Baumslag and Pride asserting that a group of deficiency  $\geq 2$  has a finite index subgroup which homomorphically maps onto a non-abelian free group. While the conclusion of this theorem is very strong, the class of groups of deficiency  $\geq 2$  to which it applies is rather limited. In this paper we shall consider groups of *positive weighted deficiency* (*PWD*), a much broader class of groups which nevertheless exhibit a number of “largeness” properties.

Groups of positive weighted deficiency generalize another class of groups, first introduced by Golod and Shafarevich in 1964 as a tool for solving two outstanding problems: the general Burnside problem in group theory and the Hilbert class field problem in number theory. In the last two decades there has been a lot of interest in studying the general structure of Golod-Shafarevich groups, in particular, establishing similarities between Golod-Shafarevich groups and free groups. A recent work in this area [EJ] dealt with what were called *generalized Golod-Shafarevich groups* – these are groups of positive weighted deficiency in our terminology. The reason that a larger class of groups was considered in

---

*Date:* July 11th, 2010.

2000 *Mathematics Subject Classification.* Primary 20F05, 20F50, Secondary 20E18, 20E07, 20F69, 17B50.

*Key words and phrases.* Golod-Shafarevich, deficiency, Tarski monsters, just-infinite.

The first author is supported by the NSF grant DMS-0901703.

The second author is supported by Spanish Ministry of Science and Innovation, grant MTM2008-06680.

[EJ] was not an attempt to seek the most general results, but the fact that the class of PWD groups is more natural than that of Golod-Shafarevich groups and is easier to work with.

The purpose of this paper is three-fold. First, we shall give a strong motivation for the concept of weighted deficiency and thus provide a new point of view on (generalized) Golod-Shafarevich groups. Second, we obtain new largeness results about PWD groups, building on the techniques introduced in [EJ]. Third, we provide two important applications of PWD groups:

- (i) construction of residually finite analogues of Tarski monsters (see §1.4)
- (ii) a new construction of hereditarily just-infinite abstract and pro- $p$  groups (see §1.5).

**1.2. Weighted deficiency.** We shall define weighted deficiency in two categories of groups – countable abstract groups and countably based pro- $p$  groups, but here for simplicity we limit our discussion to finitely generated groups. First we introduce a class of functions on pro- $p$  groups with values in the interval  $[0, 1)$  which we call *valuations* (see § 2.2) – informally speaking these are “multiplicative valuations” (in the usual sense) with some additional properties. Now let  $X$  be a finite set and  $F = F(X)$  the free pro- $p$  group on  $X$ . Given any function  $W_0 : X \rightarrow (0, 1)$ , one can extend it to a valuation  $W$  on  $F$  such that the value of  $W$  on each element of  $F$  is largest possible. Any function  $W : F \rightarrow [0, 1)$  obtained in such a way will be called a *weight function on  $F$  with respect to  $X$*  (see § 2.4 for a formal definition).

If  $W$  is a weight function on  $F(X)$  with respect to  $X$ , for any subset  $S$  of  $F(X)$  we define  $W(S) = \sum_{s \in S} W(s) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ . Given a pro- $p$  group  $G$ , we define the *weighted deficiency*<sup>1</sup> of  $G$  denoted by  $wdef(G)$  to be the maximum value of the quantity

$$W(X) - W(R) - 1$$

where  $(X, R)$  runs over all pro- $p$  presentations of  $G$  and  $W$  runs over weight functions on  $F(X)$  with respect to  $X$ .

If  $\Gamma$  is an abstract group and  $p$  is a prime, one could define the *weighted  $p$ -deficiency* of  $\Gamma$  in the same way (using abstract presentations instead of pro- $p$  presentations), but it will be more convenient to define the weighted  $p$ -deficiency of  $\Gamma$  as the weighted deficiency of its pro- $p$  completion  $\Gamma_{\hat{p}}$  (the small difference between these two definitions will be discussed in Section 6). Finally, the *weighted deficiency* of  $\Gamma$  is the supremum of its weighted  $p$ -deficiencies over all primes  $p$ .

Sometimes it is useful to think about weighted deficiency in a slightly different way. If  $(X, R)$  is a presentation of a pro- $p$  group  $G$  and  $F$  is the free pro- $p$  group  $X$ , then any weight function on  $F$  with respect to  $X$  induces certain valuation on  $G$ ; conversely, we will show that any valuation on  $G$  comes from some weight function. Therefore, given a pro- $p$  group  $G$  and a valuation  $W$  on  $G$ , we can consider the *deficiency of  $G$  with respect to  $W$* , denoted by  $def_W(G)$  – it is defined in the same way as  $wdef(G)$ , except that instead of all weight functions we only consider the ones which induce  $W$ . Thus,  $wdef(G)$  can also be defined as the supremum of the set  $\{def_W(G)\}$  where  $W$  runs over all valuations on  $G$ . In this language  $G$  is a Golod-Shafarevich group if and only if  $def_W(G) > 0$  for some *uniform* valuation  $W$  of  $G$  (see § 2.4 for the definition).

This definition does not reveal much about the structure of groups of positive weighted deficiency (PWD). To begin with, it is not clear at all that PWD groups cannot be finite

<sup>1</sup>Our notion of weighted deficiency is somewhat similar to the notion of  $p$ -deficiency introduced in a recent paper of Schlage-Puchta [SP].

(or even trivial). The fact that PWD groups are always infinite is a consequence of the *Golod-Shafarevich inequality* first proved in a slightly weaker form in [GSh] (see [Ko] for a full version). It is also easy to see that there exist torsion PWD groups. Combining these two results, Golod [Go] produced the first examples of infinite finitely generated torsion groups. In this paper we shall give a different and arguably more elementary proof of the fact that PWD groups must be infinite (see § 4.1).

Many important families of abstract and pro- $p$  groups have positive weighted deficiency. First of all, it is clear that any group of deficiency  $\geq 2$  has PWD. It takes a simple computation to show that a group  $G$  (abstract or pro- $p$ ) has positive weighted deficiency whenever  $\text{def}(G) \geq 0$  and  $d(G/[G, G]G^p) \geq 5$  where  $d(\cdot)$  stands for the minimal of generators. The latter property holds, possibly after passing to a finite index subgroup, for

- (i) fundamental groups of hyperbolic 3-manifolds
- (ii) many groups of the form  $\text{Gal}(K_{p,S}/K)$  where  $K$  is a number field,  $S$  is a finite set of primes of  $K$  and  $K_{p,S}$  is the maximal  $p$ -extension of  $K$  unramified outside of  $S$ .

In addition, the class of groups of positive weighted deficiency is closed under two natural operations:

- (a) If  $G$  is a pro- $p$  PWD group, then any open subgroup of  $G$  also has PWD.
- (b) If  $G$  is PWD (abstract or pro- $p$ ), then  $G \times H$  has PWD for any other group  $H$ .

We note that the smaller class of Golod-Shafarevich groups contains the families (i) and (ii), but is not closed under operations (a) and (b).

**1.3. Quotients of PWD groups.** Unlike groups of deficiency  $\geq 2$ , PWD groups need not be finitely presented and do not necessarily satisfy the conclusion of the Baumslag-Pride theorem, but they do have various largeness properties, as was already evident from the original work of Golod and Shafarevich. A common type of statement about PWD groups is the existence of an infinite quotient with a prescribed property. Here are two typical results of this kind:

- (1) Every abstract PWD group has an infinite torsion quotient which still has PWD [Wil2]
- (2) Every abstract PWD group has an infinite quotient with Kazhdan's property (T) [EJ]

One of the key results in this paper provides a natural addition to this list. It shows that given an abstract PWD group  $\Gamma$  and a finitely generated subgroup  $\Lambda$  of  $\Gamma$ , one has a surprising amount of control on the image of  $\Lambda$  in a suitable quotient of  $\Gamma$ :

**Theorem 1.1.** *Let  $\Gamma$  be an abstract PWD group and  $\Lambda$  a finitely generated subgroup of  $\Gamma$ . Then  $\Gamma$  has an infinite quotient  $\Gamma'$  such that*

- (i) *the image of  $\Lambda$  in  $\Gamma'$  is either finite or of finite index;*
- (ii) *some finite index subgroup of  $\Gamma'$  has PWD.*

An intriguing part of the proof of Theorem 1.1 is a very simple way to decide which of the two possibilities in (i) holds (it does not imply that the other possibility does not hold). Let  $G = \Gamma_{\widehat{p}}$  be the pro- $p$  completion of  $\Gamma$ , let  $L$  be the closure of (the image of)  $\Lambda$  in  $G$ , and let  $W$  be a valuation on  $G$  such that  $\text{def}_W(G) > 0$ . Then the quantity we need to look at is the  $W$ -index of  $L$  in  $G$ , denoted by  $[G : L]_W$ . The  $W$ -index  $[G : L]_W$  may take values in  $\mathbb{R}_{\geq 1} \cup \{\infty\}$  and is finite whenever the usual index  $[G : L]$  is finite (note that  $[G : L] \leq [\Gamma : \Lambda]$ ). The converse, however, is very far from being true – for instance, if  $\Lambda$  is normal in  $\Gamma$  and the quotient  $\Gamma/\Lambda$  is of subexponential growth, then  $[G : L]_W$  is finite (for any  $W$ ).

Going back to Theorem 1.1, we will show that if  $[G : L]_W < \infty$ , the image of  $\Lambda$  in  $\Gamma'$  can be made of finite index, and if  $[G : L]_W = \infty$ , the image of  $\Lambda$  can be made finite. In addition to playing a key role in the proof of Theorem 1.1,  $W$ -index has many similarities with the usual index, of which probably the most interesting one is the weighted version of the Schreier index formula (see Theorem 3.12). We believe that this notion of  $W$ -index deserves further study.

Let us mention another new result about quotients of PWD groups. It will be used in the proof of one of our results about just-infinite groups (see Theorem 1.8 below), but is probably of independent interest:

**Theorem 1.2.** *Every abstract PWD group has a quotient with LERF which still has PWD.*

Property LERF plays a prominent role in many recent works in geometric group theory and 3-manifold topology, and there are very few groups for which LERF has been established. While Theorem 1.2 does not prove LERF for any “familiar” groups, it provides what we believe is one of the most general constructions of groups with LERF.

Using Theorem 1.2 we will also obtain a slight improvement of the main result of [EJ]:

**Theorem 1.3.** *Every abstract PWD group has an infinite residually finite quotient with property (T).*

We now turn to the discussion of applications of PWD groups discovered in this paper.

**1.4. Constructing residually finite monsters.** In 1980 Ol’shanskii [Ol] proved the following remarkable theorem:

**Theorem 1.4** (Ol’shanskii). *For every sufficiently large prime  $p$  there exist an infinite group  $\Gamma$  in which every proper subgroup is cyclic of order  $p$ .*

Groups constructed in this theorem along with their torsion-free counterparts were the first examples of Tarski monsters, and many other groups with extremely unusual finiteness properties have been constructed since then. Most of the “monster” constructions produce groups which are neither finitely presented nor residually finite, and there is significant interest in building monsters satisfying one of these properties (or showing that this is impossible). A major progress in this direction was obtained a few years ago by Ol’shanskii and Sapir [OS] who constructed non-amenable finitely presented torsion-by-cyclic groups.

If  $\Gamma$  is a residually finite group, it clearly cannot satisfy the conclusion of Theorem 1.4, and the best approximation one can hope for is that  $\Gamma$  is a globally zero-one group as defined below.

**Definition.** Let  $\Gamma$  be a finitely generated abstract group which is not virtually cyclic. We will say that

- (a)  $\Gamma$  is a *globally zero-one group* if every subgroup of  $\Gamma$  is either finite or of finite index.
- (b)  $\Gamma$  is a *locally zero-one group* if every finitely generated subgroup of  $\Gamma$  is either finite or of finite index.

The existence of residually finite globally zero-one groups remains a major open problem (see § 8.1 for a brief discussion). In this paper we construct the first examples of residually finite locally zero-one groups:

**Theorem 1.5.** *Every abstract group of positive weighted deficiency has a residually finite quotient which is locally zero-one.*

Theorem 1.5 will be proved using repeated applications of Theorem 1.1 (in fact, a slightly stronger version of it).

We note that other examples of residually finite torsion groups with “unusual” properties were constructed by Ol’shanskii and Osin [OO].

**1.5. Applications to just-infinite groups.** An abstract (resp. profinite) group is called *just-infinite* if it is infinite and all its proper quotients<sup>2</sup> are finite. Being just-infinite is as close to being simple as a profinite or residually finite group can be – this is one of the reasons why just-infinite groups are an interesting class to study. A classification theorem due to Wilson [Wil1] implies that any abstract just-infinite group  $\Gamma$  satisfies one of the following mutually exclusive conditions:

- (i)  $\Gamma$  is virtually simple
- (ii)  $\Gamma$  is *hereditarily just-infinite*, which means that  $\Gamma$  is residually finite and every finite index subgroup of  $\Gamma$  is also just-infinite
- (iii)  $\Gamma$  has a finite index subgroup isomorphic to a direct power  $\Lambda^n$  where  $\Lambda$  is another just-infinite group and  $n \geq 2$

We note that the most interesting groups in class (iii) are branch groups. There is an analogous statement in the profinite case (see [Gr2]) except that this time there are only two “possibilities”, as infinite profinite groups cannot be simple.

An easy application of Zorn lemma shows that any infinite finitely generated abstract or virtually pro- $p$  group has a just-infinite quotient [Gr2, Proposition 3], but it seems that not much was known regarding what type of just-infinite groups one may obtain as quotients of a given class of groups. We are able to address this issue in this paper. Let  $\Gamma$  be an abstract PWD group and  $\Omega$  its residually finite locally zero-one quotient from Theorem 1.5. Note that any infinite quotient of  $\Omega$  must also be a locally zero-one group; moreover, we will show that any infinite quotient of  $\Omega$  has infinite profinite completion. It follows that just-infinite quotients of  $\Omega$  cannot be of type (i) or (iii), so any just-infinite quotient of  $\Omega$  must be hereditarily just-infinite. Thus Theorem 1.5 has the following consequence:

**Corollary 1.6.** *Every abstract group of positive weighted deficiency has a hereditarily just-infinite torsion quotient.*

It was not even known whether finitely generated hereditarily just-infinite torsion groups exist (this question was posed as Problem 15.18 in Kurovka notebook [Kou]). We note that examples of just-infinite torsion groups of other types were known – e.g., the groups in [Ol] are simple torsion groups, while the first Grigorchuk’s group [Gr1] is a just-infinite branch torsion group.

Our next result is a pro- $p$  analogue of Corollary 1.6 which will require more work:

**Theorem 1.7.** *Every pro- $p$  group of positive weighted deficiency has a hereditarily just-infinite quotient.*

Theorem 1.7 disproves the following number-theoretic conjecture of Boston. Let  $p$  be a fixed prime,  $K$  a number field,  $S$  a finite set of primes of  $K$  none of which lies above  $p$  and  $K_{p,S}$  the maximal  $p$ -extension of  $K$  unramified outside of  $S$ . Boston conjectured that whenever the Galois group  $\text{Gal}(K_{p,S}/K)$  is infinite, all of its just-infinite quotients must be branch groups (see, e.g. [Bo, § 2]). As we already mentioned, many groups of the form  $\text{Gal}(K_{p,S}/K)$  have positive weighted deficiency (see, e.g., [Ko]), and thus by Theorem 1.7 Boston’s conjecture is false for each of those groups.

<sup>2</sup>By a quotient of a profinite group we will always mean a quotient by a *closed* normal subgroup

We also obtain several results showing that hereditarily just-infinite groups may be rather large. For instance, we will prove the following:

**Theorem 1.8.** *For every prime  $p$  there exists a hereditarily just-infinite abstract group  $\Delta$  such that if  $\omega$  is the augmentation ideal of the group algebra  $\mathbb{F}_p[\Delta]$ , then the graded algebra  $gr\mathbb{F}_p[\Delta] = \bigoplus_{n \geq 0} \omega^n / \omega^{n+1}$  has exponential growth.*

The corresponding result for pro- $p$  groups also holds, and this time we will establish the pro- $p$  version first and then deduce the result about abstract groups (Theorem 1.8).

We note that another result showing that hereditarily just-infinite profinite groups can be large (in a very different sense) was recently obtained by Wilson [Wil4, Theorem A].

We finish with a peculiar application involving Kazhdan's property (T). In [Er] it was shown that groups of positive weighted deficiency can have property (T). Hence Theorem 1.5 implies the existence of residually finite locally zero-one groups with property (T). Note that an infinitely generated subgroup of a locally zero-one group must be a union of finite groups and hence amenable. Thus, we deduce the following:

**Corollary 1.9.** *There exist infinite residually finite groups in which all finitely generated subgroups have (T) and all infinitely generated subgroups are amenable.*

**1.6. Organization.** In Section 2 we define weight functions and valuations on pro- $p$  groups and introduce some standard tools that will be used in this paper. In Section 3 we establish basic properties of weight functions and valuations, including the weighted Schreier formula. In Section 4 we discuss weighted deficiency of pro- $p$  groups; the case of abstract groups is deferred until Section 6. Section 5 contains the proof of the pro- $p$  version of Theorem 1.1 which is a key step in the proof of Theorem 1.5. In Section 7 we introduce property LERF and some of its variations and prove Theorem 1.2 and Theorem 1.3. In Section 8 we establish the main applications of PWD groups; in particular, we prove Theorem 1.5, Corollary 1.6 and Theorem 1.7. Theorem 1.1 will be established in the course of the proof of Theorem 1.5. Finally, Section 9 is devoted to the proof of Theorem 1.8.

**1.7. Acknowledgments.** We would like to thank Dan Segal for suggesting an idea that helped initiate this project and Efim Zelmanov for proposing Theorem 1.7 (as a conjecture).

## 2. PRELIMINARIES

In this section we introduce some of the key notions that will be used in this paper, including valuations on pro- $p$  groups and profinite  $\mathbb{F}_p$ -algebras and weight functions on free pro- $p$  groups. We also recall several (mostly standard) results about graded associative and restricted Lie algebras and fix basic terminology (some of which is not standard).

**Some conventions.** In this paper the word 'countable' will mean finite or countably infinite. All abstract groups that we consider will be assumed countable and all pro- $p$  groups will be assumed countably based (hence also countably generated). We are mostly interested in finitely generated groups (abstract or pro- $p$ ), but the larger classes of countable abstract groups (resp. countably based pro- $p$  groups) are more convenient to work with since they are closed not only under taking homomorphic images but also under taking subgroups (closed in the pro- $p$  case).

**2.1. Algebras of noncommutative powers series and free pro- $p$  groups.** In this short subsection we define free pro- $p$  groups and relate them to algebras of noncommutative powers series. For more details and proofs of the results referred to as ‘well known’ the reader may consult [RZ] or [Wil3].

If  $X$  is a countable set, by  $F(X)$  we shall denote the free pro- $p$  group on  $X$  (where a prime  $p$  is fixed in advance). If  $X$  is finite,  $F(X)$  can be defined as the pro- $p$  completion of the free abstract group on  $X$  and is characterized by the usual universal property of a free object in the category of pro- $p$  groups.

If  $X = \{x_1, x_2, \dots\}$  is countably infinite, the definition is slightly different. In this case  $F(X)$  is a pro- $p$  group containing  $X$  with the following property: if  $G$  is any pro- $p$  group and  $f : X \rightarrow G$  is a set map such that the sequence  $\{f(x_i)\}$  converges to 1 in  $G$ , then  $f$  uniquely extends to a continuous homomorphism  $f_* : F(X) \rightarrow G$ . One can show that  $F(X)$  is isomorphic to the inverse limit of free pro- $p$  groups on  $\{x_1, \dots, x_n\}$ . Note that  $F(X)$  is much smaller than the pro- $p$  completion of the free abstract group on  $X$ , the latter not even being countably based.

It is well known that if  $X$  is any countable set, then any closed subgroup of the free pro- $p$  group  $F(X)$  is isomorphic to  $F(Y)$  for a countable set  $Y$ .

Free pro- $p$  groups are very convenient to study using their embedding into (multiplicative groups of) algebras of power series over  $\mathbb{F}_p$ . If  $Q = \{q_1, \dots, q_n\}$  is a finite set, then  $\mathbb{F}_p\langle\langle Q \rangle\rangle = \mathbb{F}_p\langle\langle q_1, \dots, q_n \rangle\rangle$  denotes the  $\mathbb{F}_p$ -algebra of noncommutative power series in  $Q$  over  $\mathbb{F}_p$ . If  $Q = \{q_1, q_2, \dots\}$  is countably infinite, then  $\mathbb{F}_p\langle\langle Q \rangle\rangle$  can be defined as the inverse limit of  $\{\mathbb{F}_p\langle\langle q_1, \dots, q_n \rangle\rangle\}$ .

If  $Q$  is a countable set and  $A = \mathbb{F}_p\langle\langle Q \rangle\rangle$ , then the closed subgroup of  $A^*$  generated by  $X = \{1 + q : q \in Q\}$  is free on  $X$ . Moreover, the natural map from the completed group algebra  $\mathbb{F}_p[[F(X)]]$  to  $A$  that extends the inclusion map  $X \rightarrow A$  is an isomorphism.

## 2.2. Valuations.

**Definition.** Let  $B$  be an associative profinite  $\mathbb{F}_p$ -algebra with 1. A continuous function  $w : B \rightarrow [0, 1]$  is called a *valuation* if

- (i)  $w(f) = 0$  if and only if  $f = 0$ ;
- (ii)  $w(1) = 1$
- (iii)  $w(f + g) \leq \max\{w(f), w(g)\}$  for any  $f, g \in B$ ;
- (iv)  $w(fg) \leq w(f)w(g)$  for any  $f, g \in B$ .

**Remark:** Valuations in our sense are sometimes called *multiplicative valuations*. The usual (additive) valuations are logarithms of multiplicative valuations.

For each  $\alpha \in (0, 1]$  we set

$$B_\alpha = \{b \in B : w(b) \leq \alpha\} \text{ and } B_{<\alpha} = \{b \in B : w(b) < \alpha\}.$$

Since  $[0, \alpha)$  is open in  $[0, 1)$  and  $w$  is continuous,  $B_{<\alpha}$  is an open ideal of  $B$  (hence has finite index in  $B$ ). Thus, each valuation automatically satisfies an additional condition:

- (v) For each  $\alpha > 0$  the set  $\text{Im } w \cap (\alpha, 1]$  is finite, where  $\text{Im } w$  is the image of  $w$ .

It also follows that  $B_{<\alpha} = B_\beta$  for some  $\beta$ .

Next we define valuations on pro- $p$  groups.

**Definition.** Let  $G$  be a pro- $p$  group. A continuous function  $W : G \rightarrow [0, 1)$  is called a *valuation* if

- (i)  $W(g) = 0$  if and only if  $g = 1$ ;

- (ii)  $W(fg) \leq \max\{W(f), W(g)\}$  for any  $f, g \in G$ ;
- (iii)  $W([f, g]) \leq W(f)W(g)$  for any  $f, g \in G$ ;
- (iv)  $W(g^p) \leq W(g)^p$  for any  $g \in G$ .

Each valuation  $W$  on a pro- $p$  group  $G$  satisfies the following additional conditions:

- (v) For each  $\alpha > 0$  the set  $\text{Im } W \cap (\alpha, 1]$  is finite;
- (vi)  $W(g^{-1}) = W(g)$  for each  $g \in G$ .

Condition (v) is established as in the case of algebras, and (vi) follows from (ii), continuity of  $W$  and the fact that  $g^{-1}$  lies in the closure of the set  $\{g^n : n \in \mathbb{N}\}$ .

The basic connection between valuations on pro- $p$  groups and profinite  $\mathbb{F}_p$ -algebras is very simple. If  $B$  is any profinite  $\mathbb{F}_p$ -algebra and  $G$  is a closed subgroup of  $B^*$ , then for any valuation  $w$  on  $B$  we get the corresponding valuation  $W$  on  $G$  by setting  $W(g) = w(g - 1)$ .

**Definition.** Let  $G$  be a pro- $p$  group, let  $w$  be a valuation on its completed group algebra  $\mathbb{F}_p[[G]]$ , and define the valuation  $W$  on  $G$  by  $W(g) = w(g - 1)$ . We will say that the valuations  $w$  and  $W$  are *compatible*.

Again let  $G$  be a pro- $p$  group and  $W$  is a valuation on  $G$ . For each  $\alpha \in (0, 1]$  we set

$$(2.1) \quad G_\alpha = \{g \in G : W(g) \leq \alpha\} \text{ and } G_{<\alpha} = \{g \in G : W(g) < \alpha\}.$$

Similarly to the algebra case, each of the subgroups  $G_\alpha$  and  $G_{<\alpha}$  is open and normal in  $G$  and  $G_{<\alpha} = G_\beta$  for some  $\beta$ . Since clearly  $\cap G_\alpha = \{1\}$ , we deduce that  $\{G_\alpha\}$  is a base of neighborhoods of 1 in  $G$ , so in particular  $G$  must be countably based.

If  $H$  is a closed subgroup of  $G$  then the restriction of  $W$  to  $H$  is a valuation of  $H$ . If  $N$  is a closed normal subgroup of  $G$  then  $W$  induces a valuation on  $G/N$  (also denoted by  $W$ ) – it is defined by setting

$$W(xN) = \min\{W(y) : y \in xN\}.$$

Note that in the notations (2.1) we have

$$H_\alpha = H \cap G_\alpha \quad \text{and} \quad (G/N)_\alpha = (G_\alpha N)/N.$$

If  $S$  is a countable subset of  $G$ , we define

$$W(S) = \sum_{s \in S} W(s).$$

Finally, we introduce the notion of a pseudo-valuation that will be occasionally used.

**Definition.** Let  $G$  be a pro- $p$  group. A continuous function  $W : G \rightarrow [0, 1)$  is called a *pseudo-valuation* if it satisfies conditions (ii), (iii) and (iv) in the definition of a valuation (but not necessarily condition (i)).

Thus, valuations are precisely pseudo-valuations which take positive values on non-identity elements.

**2.3. Graded algebras associated to valuations.** It is common in combinatorial group theory to consider Lie, restricted Lie and associative algebras graded by natural numbers  $\mathbb{N}$  or integers  $\mathbb{Z}$ . We will naturally encounter gradings by more general abelian semigroups.

**Definition.** Let  $\Omega$  be an abelian semigroup (written multiplicatively) and  $F$  a field.

(a) Let  $A$  be an associative  $F$ -algebra. A collection of  $F$ -subspaces  $\{A_\alpha\}_{\alpha \in \Omega}$  of  $A$  will be called an  $\Omega$ -grading of  $A$  if

$$(i) \quad A = \bigoplus_{\alpha \in \Omega} A_\alpha$$



- (ii)  $A_\alpha A_\beta \subseteq A_{\alpha\beta}$  for each  $\alpha, \beta \in \Omega$
- (b) Assume that  $\text{char} F = p$  and let  $L$  be a restricted Lie  $F$ -algebra. A collection of  $F$ -subspaces  $\{L_\alpha\}_{\alpha \in \Omega}$  of  $L$  will be called an  $\Omega$ -grading of  $L$  if
  - (i)  $L = \bigoplus_{\alpha \in \Omega} L_\alpha$
  - (ii)  $[L_\alpha, L_\beta] \subseteq L_{\alpha\beta}$  for each  $\alpha, \beta \in \Omega$
  - (iii)  $L_\alpha^p \subseteq L_{\alpha^p}$  for each  $\alpha \in \Omega$

**Remark:** If  $L$  is a restricted Lie algebra graded by  $\Omega$ , its (restricted) universal enveloping algebra  $\mathcal{U}(L)$  admits a natural grading by  $\Omega \sqcup \{1\}$  where 1 is the identity element adjoined to  $\Omega$ .

We are now ready to define graded associative (resp. restricted Lie) algebras corresponding to valuations on profinite  $\mathbb{F}_p$ -algebras (resp. pro- $p$  groups). These algebras will be graded by subsemigroups of the multiplicative group  $((0, 1), \cdot)$ .

**Definition.** (a) Let  $B$  be an associative profinite  $\mathbb{F}_p$ -algebra,  $w$  a valuation on  $B$  and  $\Omega = \langle \text{Im } w \rangle$ , the subsemigroup of  $((0, 1), \cdot)$  generated by  $\text{Im } w$ . The corresponding  $\Omega$ -graded (associative) algebra  $gr_w(B)$  is defined by

$$gr_w(B) = \bigoplus_{\alpha \in \Omega} B_\alpha / B_{<\alpha}$$

with componentwise addition and multiplication given by

$$(b + B_{<\alpha}) \cdot (c + B_{<\beta}) = bc + B_{<\alpha\beta}$$

For each  $b \in B$  we let  $\bar{b} = b + B_{<w(b)}$  be the image of  $b$  in  $gr_w(B)$ .

(b) Let  $G$  be a pro- $p$  group,  $W$  a valuation on  $G$  and  $\Omega = \langle \text{Im } W \rangle$ . The corresponding  $\Omega$ -graded restricted Lie algebra  $L_W(G)$  is defined by

$$L_W(G) = \bigoplus_{\alpha \in \Omega} G_\alpha / G_{<\alpha}$$

with componentwise addition, Lie bracket given by

$$[gG_{<\alpha}, hG_{<\beta}] = [g, h]G_{<\alpha\beta}$$

and  $p$ -power operation given by

$$(gG_{<\alpha})^p = g^p G_{<\alpha^p}.$$

For each  $g \in G$  we let  $\text{LT}(g) = gG_{<W(g)}$  be the image of  $g$  in  $L_W(G)$ .

In the examples of this type the grading set  $\Omega$  is often finitely generated, and in any case  $\Omega \cap (\alpha, 1]$  is finite for any  $\alpha > 0$ , as observed above. This property makes it possible to give inductive arguments for such gradings. Note that the familiar  $\mathbb{N}$ -gradings naturally appear in this setting in the special case when  $\Omega$  is the set of positive powers of a fixed real number  $\alpha \in (0, 1)$ .

We shall use the following elementary result:

**Lemma 2.1.** *Let  $G$  be a pro- $p$  group,  $w$  a valuation on  $\mathbb{F}_p[[G]]$ , and let  $W$  be the valuation on  $G$  compatible with  $w$ . Then the linear map  $L_W(G) \rightarrow gr_w(\mathbb{F}_p[[G]])$  given by*

$$gG_{<W(g)} \mapsto (g - 1) + \mathbb{F}_p[[G]]_{<W(g)} \text{ for each } g \in G$$

*is a monomorphism of restricted Lie algebras, and thus we have a natural homomorphism of graded associative algebras  $\mathcal{U}_W(G) \rightarrow gr_w(\mathbb{F}_p[[G]])$  where  $\mathcal{U}_W(G)$  is the (restricted) universal enveloping algebra of  $L_W(G)$ .*

**2.4. Weight functions.** The algebras of power series  $\mathbb{F}_p\langle\langle Q \rangle\rangle$  and free pro- $p$  groups admit a class of valuations with particularly nice properties. These valuations are called weight functions.

**Definition.** Let  $Q = \{q_1, q_2, \dots\}$  and  $A = \mathbb{F}_p\langle\langle Q \rangle\rangle$ . A continuous function  $w : A \rightarrow [0, 1]$  is called a *weight function with respect to  $Q$*  if

- (a)  $w$  is a valuation on  $A$
- (b)  $w(fg) = w(f)w(g)$  for any  $f, g \in A$
- (c) If  $f = \sum_{\alpha} \lambda_{\alpha} m_{\alpha}$  where each  $\lambda_{\alpha} \in \mathbb{F}_p$ , each  $m_{\alpha}$  is of the form  $q_{i_1} \dots q_{i_l}$  and all  $m_{\alpha}$  are distinct, then  $w(f) = \max\{w(m_{\alpha}) : \lambda_{\alpha} \neq 0\}$ .

If  $w$  is a weight function on  $A = \mathbb{F}_p\langle\langle Q \rangle\rangle$  (with respect to  $Q$ ), then clearly  $w$  is determined by its values on  $Q$ . Conversely, if  $w_0 : Q \rightarrow (0, 1)$  is any function such that  $\lim_{i \rightarrow \infty} w(q_i) = 0$  if  $Q$  is infinite, then  $w_0$  can be (uniquely) extended to a weight function  $w$  on  $A$ ; it is also easy to see that if  $w'$  is any valuation on  $A$  such that  $w'|_Q = w_0$ , then  $w'(f) \leq w(f)$  for any  $f \in A$ . This provides a simple characterization of weight functions among all valuations.

Now let  $F$  be a free pro- $p$  group. Recall from § 2.1 that if  $X$  is a free generating set of  $F$ , there is a canonical isomorphism  $\mathbb{F}_p[[F]] \cong \mathbb{F}_p\langle\langle Q \rangle\rangle$  where  $Q = \{x - 1 : x \in X\}$ . We define weight functions on free pro- $p$  groups using this isomorphism.

**Definition.** Let  $F$  be a free pro- $p$  group and  $W$  a valuation on  $F$ .

- (a) Let  $X = \{x_1, x_2, \dots\}$  be a free generating set of  $F$ . We will say that  $W$  is a *weight function with respect to  $X$*  if  $W$  is compatible with a weight function on  $\mathbb{F}_p[[F]]$  with respect to  $\{x - 1 : x \in X\}$ .
- (b) We will call  $W$  a *weight function* if  $W$  is a weight function with respect to some free generating set  $X$ . Any set  $X$  with this property will be called  *$W$ -free*.

While the definition of a weight function with respect to  $X$  substantially depends on  $X$ , we will see that for any weight function  $W$  there are a lot of  $W$ -free generating sets. We will often need to establish that a given function  $W$  is a weight function without specifying a  $W$ -free generating set, and the above definition is not well suited for this purpose. Instead we shall obtain a “coordinate-free” characterization of weight functions in terms of associated restricted Lie algebras.

If  $F$  is a finitely generated pro- $p$  group, there is a natural family of weight functions on  $F$  which we call *uniform*.

**Definition.**

- (a) Let  $F$  be a finitely generated free pro- $p$  group,  $W$  a weight function on  $F$ . We will say that  $W$  is *uniform* if there is a  $W$ -free generating set  $X$  of  $F$  such that  $W$  restricted to  $X$  is constant.
- (b) Let  $G$  be a finitely generated pro- $p$  group and  $W$  a valuation on  $G$ . We will say that  $W$  is *uniform* if there exists a finitely generated free pro- $p$  group  $F$ , an epimorphism  $\pi : F \rightarrow G$  and a uniform weight function  $\widehat{W}$  on  $F$  which induces  $W$  under  $\pi$ .

The following result is straightforward.

**Proposition 2.2.** *Let  $G$  be a finitely generated pro- $p$  group and  $W$  a valuation on  $G$ .*

- (a) *If  $W$  is uniform, then the associated filtration  $\{G_{\alpha}\}$  coincides with the Zassenhaus series of  $G$  (apart from repetitions). In particular, each  $G_{\alpha}$  is characteristic in  $G$ .*

- (b) Assume that  $G$  is free and  $W$  is a weight function. Then the converse of (a) holds, that is,  $\{G_\alpha\}$  coincides with the Zassenhaus series of  $G$  only if  $W$  is uniform.

From results of Section 3 it will be clear that a weight function  $W$  on a finitely generated free pro- $p$  group  $F$  is uniform if and only if every free generating set of  $F$  is  $W$ -free.

**2.5. Free restricted Lie algebras.** In this subsection we collect some facts about free restricted Lie algebras that will be used in the paper. Most of these facts can be found in [Ba]. We start with a classical result of E. Witt [Wt], for a recent exposition of which we recommend [BKS] (see also [Ba, Theorem 8, p.68]):

**Theorem 2.3** (Witt). Every subalgebra of a free restricted Lie algebra is free.

**Remark:** If  $L$  a restricted Lie algebra, by a subalgebra (resp. ideal) of  $L$  we shall mean a subalgebra (resp. ideal) closed under the  $p$ -power operation. By the universal enveloping algebra of  $L$  we shall mean its restricted universal enveloping algebra.

Free restricted Lie algebras can be explicitly realized inside free associative algebras.

**Proposition 2.4.** Let  $Q$  be a set and  $F$  a field of characteristic  $p$ . Let  $A = F\langle Q \rangle$  be the free associative  $F$ -algebra on  $Q$  (that is, the algebra of noncommutative polynomials), and let  $L$  be the restricted Lie subalgebra of  $A$  generated by  $Q$ . Then  $L$  is a free restricted  $F$ -Lie algebra on  $Q$ . Moreover, if  $\mathcal{U}(L)$  is the universal enveloping algebra of  $L$ , then the natural map  $\mathcal{U}(L) \rightarrow A$  induced by the embedding  $L \rightarrow A$  is an isomorphism.

*Proof.* This is a standard result which follows, e.g., from [Ba, Proposition 14, p.66] and [Ba, Corollary, p.52].  $\square$

The next result provides a simple way to verify when a given generating set of a free restricted Lie algebra is a free generating set.

**Lemma 2.5.** Let  $L$  be a free restricted Lie algebra and  $S$  a generating set of  $L$ . Then  $S$  is a free generating set of  $L$  if and only if the elements of  $S$  are linearly independent mod  $[L, L] + L^p$ .

*Proof.* The forward direction is immediate by Proposition 2.4. Conversely, assume that  $S$  is linearly independent mod  $[L, L] + L^p$ , and let  $S_0$  be a finite subset of  $S$ . By Theorem 2.3 the subalgebra  $L_0 = \langle S_0 \rangle$  is free. If  $X$  is a free generating set of  $L_0$ , then  $|X| = \dim L_0 / ([L_0, L_0] + L_0^p) \geq \dim L_0 / (([L, L] + L^p) \cap L_0) \geq |S_0|$ . Thus,  $|X| = |S_0|$  and  $S_0$  is also a free generating set of  $L_0$  since finitely generated free restricted Lie algebras are hopfian (that a finitely generated free restricted Lie algebra is hopfian follows easily from the fact that it is residually finite-dimensional). Hence  $S$  freely generates  $L$ .  $\square$

Our last result concerns possible gradings on free restricted Lie algebras. If  $L$  is a free restricted Lie algebra and  $\Omega$  is an abelian semigroup, one can obtain a class of  $\Omega$ -gradings on  $L$  as follows. Choose a free generating set  $Q$  of  $L$  and an arbitrary function  $D : Q \rightarrow \Omega$ ; then there is unique  $\Omega$ -grading  $\{L_\alpha\}$  of  $L$  such that  $q \in L_{D(q)}$  for each  $q \in Q$ . We shall now show that if  $\Omega$  is a subsemigroup of  $((0, 1), \cdot)$  such that  $\Omega \cap (\alpha, 1)$  is finite for any  $\alpha > 0$ , then all  $\Omega$ -gradings of  $L$  are obtained in such a way.

**Proposition 2.6.** Let  $\Omega$  be a subsemigroup of  $((0, 1), \cdot)$  such that  $\Omega \cap (\alpha, 1)$  is finite for any  $\alpha > 0$ . Let  $L$  be a free restricted Lie algebra. Then for any  $\Omega$ -grading on  $L$  there exists a free generating set of  $L$  consisting of homogeneous elements.

*Proof.* Let  $L = \bigoplus_{\alpha \in \Omega} L_\alpha$  be an  $\Omega$ -grading of  $L$ . Since  $[L, L] + L^p$  is a graded subspace of  $L$ , we can find its graded complement  $U$  in  $L$ . Let  $S$  be a graded basis of  $U$ . We will show by (downward) induction on  $\alpha \in \Omega$  that  $L_\alpha \leq \langle S \rangle$ .

If  $\alpha = \max \Omega$ , then  $L_\alpha \cap ([L, L] + L^p) = \{0\}$ , whence  $L_\alpha \subseteq U$ . Now take any  $\alpha \in (0, 1)$ , and assume that  $L_\beta \subseteq \langle S \rangle$  for any  $\beta > \alpha$ . Take any  $l \in L_\alpha$ . Then we can write  $l$  as  $l = u + k^p + \sum_i [l_i, m_i]$ , where  $u \in U$ ,  $k \in L_{\alpha^{1/p}}$ ,  $l_i \in L_{\alpha_i}$ ,  $m_i \in L_{\beta_i}$  and  $\alpha_i \beta_i = \alpha$ . Hence by the induction hypothesis  $l \in \langle S \rangle$ .

Thus  $S$  generates  $L$ , and so by Lemma 2.5  $S$  is a free generating set.  $\square$

### 3. SOME PROPERTIES OF VALUATIONS AND WEIGHT FUNCTIONS

In this section we establish basic properties of valuations and weight functions on pro- $p$  groups. Some of these properties have already been proved in [EJ] (usually under additional restrictions), but we have chosen to give independent proofs even when a precise citation to [EJ] was possible. This enables us to make this section self-contained and give simpler and more transparent proofs for these results.

In § 3.1 we obtain several characterizations of weight functions among all valuations on free pro- $p$  groups. Given a valuation  $W$  on a pro- $p$  group  $G$ , in § 3.2 we define the  $W$ -rank of  $G$  and the notion of a  $W$ -optimal generating set and establish several characterizations of  $W$ -optimal generating sets. Finally, in § 3.3 we define the notion of  $W$ -index for every closed subgroup of  $G$  and prove the weighted analogue of the Schreier formula relating  $W$ -rank and  $W$ -index.

**3.1. Characterizations of weight functions.** We begin by recalling the definition of standard Lie and group commutators from Hall's commutator calculus [Hall].

Let  $Q$  be a countable set and  $\text{Lie}(Q)$  the free  $\mathbb{F}_p$ -Lie algebra on  $Q$ . The *standard commutators in  $Q$  of degree 1* are simply elements of  $Q$ . Suppose we have defined standard commutators of degrees  $\leq n-1$  and have chosen some (total) order on the set of those commutators so that  $u < v$  whenever  $\deg u < \deg v$ . An element  $u \in \text{Lie}(Q)$  is called a *standard Lie commutator in  $Q$  of degree  $n$*  if  $u = [u_1, u_2]$  where

- (i)  $u_1$  and  $u_2$  are standard Lie commutators with  $\deg u_1 + \deg u_2 = n$ ;
- (ii)  $u_1 > u_2$ ;
- (iii) If  $u_1 = [v_1, v_2]$ , then  $u_2 \geq v_2$ .

Standard group commutators are defined in the analogous way.

#### Proposition 3.1.

- (a) Let  $Q$  be a countable set. The elements  $\{c^{p^k} : c \text{ is a standard Lie commutator in } Q\}$  form a basis of the free restricted Lie algebra on  $Q$ .
- (b) Let  $X$  be a countable set,  $F = F(X)$  the free pro- $p$  group on  $X$  and

$$\mathcal{C} = \{c : c \text{ is a standard group commutator in } X\}.$$

Fix an order on  $\{c^{p^k} : c \in \mathcal{C}, k \geq 0\}$ . Then any element of  $f \in F$  can be written in the form

$$f = \prod_{c \in \mathcal{C}, k \geq 0} c^{\alpha_{c,k} p^k} \quad (***)$$

where each  $\alpha_{c,k} \in \{0, \dots, p-1\}$ .

A factorization as in (\*\*\*) will be called a *power-commutator presentation in  $X$* .

*Proof.* (a) is a well known result which follows, e.g. from [Ba, Theorem 1, p.51] and [Ba, Proposition 14, p.66].

(b) is fairly standard as well, but we will give a proof as we are not aware of a satisfactory reference. First observe that if  $f \in F$  is the limit of a sequence  $\{f_n\}$  and each  $f_n$  has a power-commutator factorization in  $X$ , then by Cantor's diagonal argument  $f$  also has such a factorization. Thus without loss of generality we can (and will) assume that  $X$  is finite.

Let  $\mathcal{PC} = \{c^{p^k} : c \in \mathcal{C}, k \geq 0\}$ , and for each  $t \in \mathbb{N}$  let

$$\mathcal{PC}(t) = \{c^{p^k} \in \mathcal{PC} : k \cdot \deg(c) = t\}, \quad \mathcal{PC}(\geq t) = \cup_{s \geq t} \mathcal{PC}(s) \text{ and } \mathcal{PC}(< t) = \cup_{s < t} \mathcal{PC}(s).$$

Let  $\{D_t F\}_{t \geq 1}$  be the Zassenhaus filtration of  $F$ , that is,  $D_t F = \prod_{ip^j \geq t} (\gamma_i F)^{p^j}$ . Since  $X$  is finite,  $\{D_t F\}$  is a base of neighborhoods of identity for  $F$ .

*Step 1:* We claim that for each  $t \in \mathbb{N}$  any  $f \in D_t F$  can be written as  $f = \prod_{h \in \mathcal{PC}(t)} h^{\alpha_h} f'$  with  $0 \leq \alpha_h \leq p-1$  and  $f' \in D_{t+1} F$ . This is an easy consequence of the Hall-Petrescu formula and the following basic fact: any commutator in  $X$  of degree  $i$  (not necessarily standard) can be written as a product of standard commutators of length  $i$  and commutators of length  $> i$ .

*Step 2:* We claim that for each  $t \in \mathbb{N}$  any  $f \in F$  can be written as  $f = \prod_{h \in \mathcal{PC}(\leq t)} h^{\alpha_h} r_t(f)$  with  $0 \leq \alpha_h \leq p-1$  and  $r_t(f) \in D_{t+1} F$  (where the product is taken in the prescribed order on  $\mathcal{PC}$ ). This claim is established by straightforward induction on  $t$  using Step 1.

Since the sequence  $\{f \cdot r_t(f)^{-1}\}_{t \in \mathbb{N}}$  converges to  $f$ , we are done by the observation made at the beginning of the proof.  $\square$

We are now ready to state our first characterization of weight functions.

**Convention:** If  $G$  is a pro- $p$  group,  $W$  a valuation of  $G$  and  $Y$  a subset of  $G$ , we shall consider  $\{\text{LT}(y) : y \in Y\}$  as a *multiset*.

**Proposition 3.2.** *Let  $F$  be a free pro- $p$  group,  $X$  a generating set of  $F$  and  $W$  a valuation on  $F$ . Let  $S = \{\text{LT}(x) : x \in X\} \subset L_W(F)$  and  $\langle S \rangle$  the restricted Lie subalgebra generated by  $S$ . Then the following are equivalent:*

- (i)  $W$  is a weight function with respect to  $X$
- (ii) For any standard commutator  $c$  in  $X$  we have  $W(c) = \prod_{i=1}^d W(x_i)^{n_i}$  where  $x_1, \dots, x_d$  are the elements of  $X$  which occur in  $c$  and  $n_i$  is the number of occurrences of  $x_i$  in  $c$ , and for any  $f \in F$  given by its power-commutator presentation  $f = \prod_{c \in \mathcal{C}, k \geq 0} c^{\alpha_{c,k} p^k}$  we have  $W(f) = \max\{W(c)^{p^k} : \alpha_{c,k} \neq 0\}$
- (iii)  $\langle S \rangle$  is freely generated by  $S$  (as a restricted Lie algebra)
- (iv)  $L_W(F)$  is a free restricted Lie algebra freely generated by  $S$ .

*Proof.* We shall proceed via the sequence of implications (i)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i) and (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii).

“(i)  $\Rightarrow$  (iii)” Let  $w$  be a weight function on  $\mathbb{F}_p[[F]]$  with respect to  $\{x-1 : x \in X\}$  which is compatible with  $W$ . The definition of weight functions easily implies that  $gr_w(\mathbb{F}_p[[F]])$  is a free associative algebra on  $\overline{Q} = \{\overline{x-1} : x \in X\}$ .

Let  $\pi : \mathcal{U}(L_W(F)) \rightarrow gr_w(\mathbb{F}_p[[F]])$  be the canonical map (see Proposition 2.1). Since  $\pi|_S$  is a bijection between  $S$  and  $\overline{Q}$ , we deduce that the associative subalgebra of  $\mathcal{U}(L_W(F))$  generated by  $S$  is free on  $S$ . Hence by Proposition 2.4 the restricted Lie subalgebra  $\langle S \rangle$  must be free on  $S$ .

“(iii) $\Rightarrow$  (ii)” Let  $c = c(x_1, \dots, x_d)$  be a standard group commutator in  $X$ , and let  $\text{Lie}(c) = \text{Lie}(c)(\text{LT}(x_1), \dots, \text{LT}(x_d))$  be the corresponding standard Lie commutator in  $S$ . Since  $\langle S \rangle$  is free restricted on  $S$ , we have  $\text{Lie}(c)^{p^k} \neq 0$  for any  $k \geq 0$ , and an easy induction on  $\deg(c)$  shows that  $\text{LT}(c^{p^k}) = \text{Lie}(c)^{p^k}$ . In particular, the equality  $\text{LT}(c) = \text{Lie}(c)$  implies the desired formula for  $W(c)$ .

Now take any  $1 \neq f \in F$ . Let  $f = \prod_{c,k} c^{\alpha_{c,k} p^k}$  be its power-commutator presentation in  $X$ , and let  $t = \max\{W(c)^{p^k} : \alpha_{c,k} \neq 0\}$ . Since  $W$  is a valuation, we have  $W(f) \leq t$ , and we need to show that  $W(f) = t$ . Suppose that  $W(f) < t$ . Then

$$W\left(\prod_{W(c)^{p^k}=t} c^{\alpha_{c,k} p^k} \prod_{W(c)^{p^k}<t} c^{\alpha_{c,k} p^k}\right) = W(f) < t$$

and so

$$W\left(\prod_{W(c)^{p^k}=t} c^{\alpha_{c,k} p^k}\right) < t.$$

Hence

$$\sum_{W(c)^{p^k}=t} \alpha_{c,k} \text{Lie}(c)^{p^k} = 0.$$

This contradicts freeness of  $\langle S \rangle$ .

“(ii) $\Rightarrow$  (i)” This is almost obvious. Indeed, let  $W'$  be the unique weight function with respect to  $X$  which coincides with  $W$  on  $X$ . Applying the implication “(i) $\Rightarrow$  (ii)” to  $W'$ , we conclude that the values of  $W$  and  $W'$  on each element of  $F$  must coincide.

“(ii)+(iii) $\Rightarrow$  (iv)” We only need to show that  $L_W(F) = \langle S \rangle$ . Take any  $1 \neq f \in F$ , let  $f = \prod_{c,k} c^{\alpha_{c,k} p^k}$  be its power-commutator representation in  $X$ , and let  $t = W(f)$ . Using (ii) and the equality  $\text{LT}(c^{p^k}) = \text{Lie}(c)^{p^k}$  established in the proof of the implication “(iii) $\Rightarrow$  (ii)”, we get

$$\text{LT}(f) = \sum_{W(c)^{p^k}=t} \alpha_{c,k} \text{Lie}(c)^{p^k} \in \langle S \rangle.$$

Finally, the implication “(iv) $\Rightarrow$  (iii)” is obvious.  $\square$

**Corollary 3.3.** *Let  $F$  be a free pro- $p$  group and  $W$  a valuation on  $F$ . Then  $L_W(F)$  is free restricted Lie algebra if and only if  $W$  is a weight function on  $F$ .*

*Proof.* “ $\Leftarrow$ ” If  $W$  is a weight function then  $L_W(F)$  is free by Proposition 3.2.

“ $\Rightarrow$ ” Assume that  $L_W(F)$  is free. By Proposition 2.6, there exists a free graded generating set  $S_0$  of  $L_W(F)$ . We can write each element  $s \in S_0$  in the form  $\text{LT}(y)$  for some  $y \in F$ , and let  $Y \subset F$  be such that  $S_0 = \{\text{LT}(y) : y \in Y\}$ . Then  $Y$  generates  $F$  since  $L_W(\langle Y \rangle) \supseteq \langle S_0 \rangle = L_W(F)$ , so  $F$  has a free generating set  $X$  contained in  $Y$ . Put  $S = \{\text{LT}(x) : x \in X\}$ . Since  $S \subseteq S_0$ ,  $\langle S \rangle$  is a free restricted Lie algebra freely generated by  $S$ . Thus,  $W$  is a weight function by Proposition 3.2.  $\square$

Since subalgebras of free restricted Lie algebras are free restricted (Proposition 2.3), Corollary 3.3 implies the following important result:

**Corollary 3.4.** *Let  $F$  be a free pro- $p$  group and  $W$  a weight function on  $F$ . If  $H$  is a closed subgroup of  $F$ , then the restriction of  $W$  to  $H$  is also a weight function.*

**3.2.  $W$ -rank and  $W$ -optimal generating sets.** Let  $G$  be a pro- $p$  and  $W$  a valuation on  $G$ . Recall that for each real number  $\alpha \in [0, 1]$  we have defined

$$G_\alpha = \{g \in G : W(g) \leq \alpha\} \quad \text{and} \quad G_{<\alpha} = \{g \in G : W(g) < \alpha\}.$$

We also set  $\Phi(G) = [\overline{G}, \overline{G}] \overline{G}^p$ ,  $\overline{G} = G/\Phi(G)$ ,

$$\overline{G}_\alpha = G_\alpha \Phi(G) / \Phi(G) = G_\alpha / \Phi(G)_\alpha \quad \overline{G}_{<\alpha} = G_{<\alpha} \Phi(G) / \Phi(G) = G_{<\alpha} / \Phi(G)_{<\alpha}$$

$$c_\alpha(G) = \log_p[G_\alpha : G_{<\alpha}] \quad c_\alpha(\overline{G}) = \log_p[\overline{G}_\alpha : \overline{G}_{<\alpha}]$$

**Observation 3.5.** *There is a natural isomorphism*

$$G_\alpha / G_{<\alpha} / (\Phi(G)_\alpha / \Phi(G)_{<\alpha}) \cong \overline{G}_\alpha / \overline{G}_{<\alpha}$$

*In particular,  $c_\alpha(\overline{G}) = c_\alpha(G) - c_\alpha(\Phi(G))$ .*

*Proof.* This is an easy consequence of isomorphism theorems for groups.  $\square$

**Definition.** Let  $G$  be a pro- $p$  group and  $W$  a valuation on  $G$ . The quantity

$$rk_W(G) = \sum_{\alpha \in \text{Im } W} c_\alpha(\overline{G}) \alpha$$

is called the  $W$ -rank of  $G$ .

The following proposition explains why this notion of  $W$ -rank is natural:

**Proposition 3.6.** *Let  $G$  be a pro- $p$  group and  $W$  a valuation on  $G$ .*

- (a) *There exists a minimal generating set  $X$  of  $G$  such that for each  $\alpha \in \text{Im } W$  we have  $|\{x \in X : W(x) = \alpha\}| = c_\alpha(\overline{G})$ . In particular, if  $G$  is of finite  $W$ -rank,*

$$W(X) = rk_W(G).$$

- (b) *If  $Y$  is any generating set of  $G$  then for any  $\alpha \in \text{Im } W$  we have  $|\{y \in Y : W(y) \geq \alpha\}| \geq \sum_{\beta \geq \alpha} c_\beta(\overline{G})$ . In particular, if  $G$  is of finite  $W$ -rank,*

$$W(Y) \geq rk_W(G).$$

*Proof.* (a) For each  $\alpha$  with  $c_\alpha(\overline{G}) \neq 0$  choose a basis  $T_\alpha$  of the  $\mathbb{F}_p$ -vector space  $\overline{G}_\alpha / \overline{G}_{<\alpha}$ . By definition of  $\overline{G}_\alpha$  and  $\overline{G}_{<\alpha}$  we can lift  $T_\alpha$  to a subset  $X_\alpha \subset G$  such that  $W(x) = \alpha$  for each  $x \in X_\alpha$ . Then the set  $X = \sqcup X_\alpha$  clearly has the required property.

(b) Fix  $\alpha \in \text{Im } W$ , and let  $k = \sum_{\beta \geq \alpha} c_\beta(\overline{G})$ . Note that  $k = \log_p[G : \Phi(G)G_{<\alpha}]$ , and thus at least  $k$  elements of the generating set  $Y$  of  $G$  have non-trivial projection onto the space  $G/\Phi(G)G_{<\alpha}$ . On the other hand, any  $y \in Y$  with  $W(y) < \alpha$  will have trivial projection, and thus there must be at least  $k$  elements with  $W$ -weight  $\geq \alpha$ .  $\square$

**Definition.** Let  $G$  be a pro- $p$  group and  $W$  a valuation on  $G$ . We say that a generating set  $X$  of  $G$  is  $W$ -optimal if for all  $\alpha \in \text{Im } W$

$$|\{x \in X : W(x) = \alpha\}| = c_\alpha(\overline{G}).$$

**Corollary 3.7.** *Let  $G$  be a pro- $p$  group and  $W$  is a valuation on  $G$ . Assume that  $G$  is of finite  $W$ -rank. Then a generating set  $X$  of  $G$  is  $W$ -optimal if and only if  $W(X) = rk_W(G)$ .*

We will use the following characterization of  $W$ -optimal generating sets.

**Proposition 3.8.** *Let  $G$  be a pro- $p$  group,  $X \subset G$  a generating set and  $W$  a valuation on  $G$ . Let  $S = \{\text{LT}(x) : x \in X\}$ . Then the image of  $S$  in  $L_W(G)/L_W(\Phi(G))$  forms a basis if and only if  $X$  is  $W$ -optimal.*

*Proof.* Note that  $L_W(G)/L_W(\Phi(G))$  is naturally isomorphic to  $\oplus_\alpha \overline{G}_\alpha/\overline{G}_{<\alpha}$  by Observation 3.5.

“ $\Rightarrow$ ” If the image of  $S$  in  $L_W(G)/L_W(\Phi(G))$  forms a basis, then

$$|\{x \in X : W(x) = \alpha\}| = |\{\text{LT}(x) : x \in X, W(x) = \alpha\}| = \dim \overline{G}_\alpha/\overline{G}_{<\alpha} = c_\alpha(G),$$

so by definition  $X$  is  $W$ -optimal.

“ $\Leftarrow$ ” If  $X$  is  $W$ -optimal then again  $\dim \overline{G}_\alpha/\overline{G}_{<\alpha} = |\{\text{LT}(x) : x \in X, W(x) = \alpha\}|$ . Thus we only have to show that  $\{\text{LT}(x) : x \in X, W(x) = \alpha\} \subseteq G_\alpha/G_{<\alpha}$  is linearly independent modulo the subspace  $\Phi(G)_\alpha/\Phi(G)_{<\alpha}$  for each  $\alpha \in \text{Im } W$ .

If this does not hold, then there exists  $g \in G_{<\alpha}$  and  $\alpha_x \in \{0, \dots, p-1\}$  not all equal to zero such that  $g \prod_{x \in X, W(x)=\alpha} x^{\alpha_x} \in \Phi(G)$ . But this means that for some  $x \in X$  with  $W(x) = \alpha$  the set  $X \setminus \{x\} \cup \{g\}$  still generates  $G$ . Since  $W(g) < W(x)$ , this contradicts the assumption that  $X$  is  $W$ -optimal.  $\square$

Next we will show that if  $W$  is a weight function (on a free pro- $p$  group), the concepts of  $W$ -free and  $W$ -optimal generating sets coincide.

**Proposition 3.9.** *Let  $W$  be a weight function on a free pro- $p$  group  $F$  and  $X$  a free generating set for  $F$ . Then  $X$  is  $W$ -free if and only if  $X$  is  $W$ -optimal.*

*Proof.* “ $\Leftarrow$ ” Assume first that  $X$  is  $W$ -optimal. Let  $S = \{\text{LT}(x) : x \in X\} \subset L_W(F)$  and  $L = \langle S \rangle$ , the restricted Lie subalgebra of generated by  $S$ . By Proposition 2.3,  $L$  is a free restricted Lie algebra. By Proposition 3.8, the image of  $S$  in  $L_W(F)/L_W(\Phi(F))$  is linearly independent and so the image of  $S$  in  $L/([L, L] + L^p)$  is linearly independent. Hence by Lemma 2.5,  $S$  is a free generating set of  $L$ , and so  $X$  is  $W$ -free by Proposition 3.2.

“ $\Rightarrow$ ” Conversely, suppose that  $X$  is  $W$ -free. To prove that  $X$  is  $W$ -optimal, it is enough to show that for any free generating set  $X'$  of  $F$  and any  $\alpha > 0$  we have  $|X'_{\geq \alpha}| \geq |X_{\geq \alpha}|$  where for a set  $Y$  we put

$$Y_{<\alpha} = \{y \in Y : W(y) < \alpha\} \quad \text{and} \quad Y_{\geq \alpha} = \{y \in Y : W(y) \geq \alpha\}.$$

Fix  $\alpha > 0$ . Take any  $y \in (X')_{<\alpha}$ , and choose its power-commutator presentation in  $X$ . By Proposition 3.2 this presentation cannot include factors of the form  $x^n$  with  $0 < n < p$  and  $x \in X_{\geq \alpha}$ ; in other words,  $y$  must be written in the form  $y = \prod_{x \in X_{<\alpha}} x^{n_x} \cdot r$  where  $r \in \Phi(F)$ . It follows that the image of the set  $X'_{<\alpha}$  in  $F/\Phi(F)$  lies in the subspace spanned by the image of  $X_{<\alpha}$ . Since  $X'$  and  $X$  are linearly independent sets mod  $\Phi(F)$ , we conclude that  $|X'_{\geq \alpha}| \geq |X_{\geq \alpha}|$ .  $\square$

Given a pro- $p$  group  $G$ , a valuation  $W$  on  $G$  and some  $W$ -optimal generating of  $G$ , it is natural to ask if one can (explicitly) construct a  $W$ -optimal generating set for a closed subgroup  $H$  of  $G$ . Our last result in this subsection addresses this problem in the case when  $G$  is free,  $W$  is a weight function and  $[G : H] = p$ .

**Lemma 3.10.** *Let  $G$  be a pro- $p$  group,  $W$  a valuation on  $G$  and  $H$  an open subgroup of index  $p$ . The following hold*

- (a) *There exists a  $W$ -optimal generating set  $X$  of  $G$  and  $x \in X$  such that  $X \setminus \{x\} \subset H$*
- (b) *For any  $X$  and  $x$  satisfying (a) the set*

$$(3.1) \quad X' = \cup_{y \in X \setminus \{x\}} \{y, [y, x], [y, x, x], \dots, [y, \underbrace{x, \dots, x}_{p-1 \text{ times}}]\} \cup \{x^p\}$$

*is a generating set of  $H$ . Moreover if  $G$  is free and  $W$  is a weight function, then  $X'$  is  $W$ -optimal.*



*Proof.* (a) Let  $X_0$  be some  $W$ -optimal generating set of  $G$ . Clearly,  $X_0 \setminus H$  is finite, and choose  $x \in X_0 \setminus H$  for which  $W(x)$  is smallest possible. For each  $y \in X_0 \setminus (H \cup \{x\})$  choose  $j(y) \in \mathbb{N}$  such that  $yx^{j(y)} \in H$ , and let  $X = (X_0 \cap H) \cup \{yx^{j(y)} : y \in X_0 \setminus (H \cup \{x\})\} \cup \{x\}$ . Clearly,  $X$  also generates  $G$ , by construction  $X \setminus \{x\} \subset H$  and  $W(X) \leq W(X_0)$ . Since  $X_0$  is  $W$ -optimal,  $X$  is also  $W$ -optimal.

(b) From the Schreier process of writing generators for subgroups of finite index we know that the set  $\cup_{y \in X \setminus \{x\}} \{y^{x^i}, i = 0, \dots, p-1\} \cup \{x^p\}$  generates  $H$ . Clearly  $y^{x^i} \in X'$  for each  $i$ , so  $X'$  also generates  $H$ .

Now, assume that  $G$  is free and  $W$  is a weight function. Let  $L = L_W(G)$ , and let  $S = \{\text{LT}(y) : y \in X\}$ . By Proposition 3.2,  $L$  is a free restricted Lie algebra on  $S$ .

Now let  $s = \text{LT}(x)$ , let  $S' = \cup_{t \in S \setminus \{s\}} \{t, [t, s], [t, s, s], \dots, [t, \underbrace{s, \dots, s}_{p-1 \text{ times}}]\} \cup \{s^p\}$ , and let  $L' = \langle S' \rangle$  be the restricted subalgebra generated by  $S'$ . By [BKS, Lemma 2.1],  $L'$  is a subalgebra of index  $p$  (=codimension 1), and  $S'$  is a free generating set of  $L'$ .

Next note that  $S'$  is precisely the set of leading terms of elements of  $X'$ . Thus, if  $\langle X' \rangle$  is the subgroup generated by  $X'$ , then  $L_W(\langle X' \rangle) \supseteq \langle S' \rangle = L'$ . Since  $[L : L'] = p$ , and on the other hand  $\langle X' \rangle$  is clearly contained in  $H$ , we conclude that  $\langle X' \rangle = H$ . Thus, we have shown that  $L_W(H)$  is free restricted on  $S' = \{\text{LT}(x') : x' \in X'\}$ . By Proposition 3.2  $W$  is a weight function on  $H$  with respect to  $X'$ , so  $X'$  is  $W$ -free and hence  $W$ -optimal.  $\square$

**3.3.  $W$ -index and weighted Schreier formula.** If  $G$  is a pro- $p$  group and  $H$  is a closed subgroup of  $G$ , it is easy to see that the index of  $H$  in  $G$  can be computed by the following formula:

$$[G : H] = \prod_{\alpha \in \text{Im } W} p^{c_\alpha(G) - c_\alpha(H)}.$$

where the integers  $c_\alpha(\cdot)$  are defined as in § 3.2. The notion of  $W$ -index is defined using certain generalization of this formula:

**Definition.** Let  $G$  be a pro- $p$  group,  $H$  a closed subgroup of  $G$  and  $W$  a valuation on  $G$ . The quantity

$$[G : H]_W = \prod_{\alpha \in \text{Im } W} \left( \frac{1 - \alpha^p}{1 - \alpha} \right)^{c_\alpha(G) - c_\alpha(H)}$$

is called the  $W$ -index of  $H$  in  $G$ .

The following properties of  $W$ -index are straightforward:

**Proposition 3.11.** *Let  $G$  be a pro- $p$  group,  $W$  a valuation of  $G$  and  $H$  a closed subgroup of  $G$ . The following hold:*

- (a)  $[G : H]_W = \lim_{\alpha \rightarrow 0} \frac{[G : G_\alpha]_W}{[H : H_\alpha]_W}$ .
- (b)  $H$  is of finite  $W$ -index in  $G$  if and only if

$$\sum_{\alpha \in \text{Im } W} \alpha(c_\alpha(G) - c_\alpha(H)) < \infty.$$

- (c)  $W$ -index is multiplicative, that is, for any closed subgroup  $K$  of  $G$  we have  $[G : K]_W = [G : K]_W \cdot [G : K]_W$ .

Our main goal in this subsection is to prove the following weighted analogue of the Schreier index formula relating  $W$ -index and  $W$ -rank:

**Theorem 3.12** (Weighted Schreier formula). *Let  $G$  be a pro- $p$  group,  $W$  a valuation on  $G$  and  $H$  a closed subgroup for which  $[G : H]_W < \infty$ . Then*

$$rk_W(H) - 1 \leq [G : H]_W \cdot (rk_W(G) - 1).$$

*Moreover, if  $G$  is free and  $W$  is a weight function, then*

$$rk_W(H) - 1 = [G : H]_W \cdot (rk_W(G) - 1).$$

We start with two lemmas dealing with the integers  $c_\alpha(\cdot)$ . The first lemma is straightforward.

**Lemma 3.13.** *Let  $H$  be a closed subgroup of  $G$  and  $\alpha > 0$ .*

- (a) *If  $K$  is a closed subgroup of  $H$ , the natural map  $K_\alpha/K_{<\alpha} \rightarrow H_\alpha/H_{<\alpha}$  is injective. In particular,  $c_\alpha(K) \leq c_\alpha(H)$*
- (b)  *$c_\alpha(H) = c_\alpha(G) - [HG_\alpha : HG_{<\alpha}]$*

The second lemma reduces the computation of the integers  $c_\alpha$  to the case of open subgroups.

**Lemma 3.14.** *Let  $H$  be a closed subgroup of  $G$  and  $\alpha > 0$ , and let  $U$  be an open subgroup with  $H \subseteq U \subseteq HG_{<\alpha}$ .*

- (a) *We have natural isomorphisms*

$$H_\alpha/H_{<\alpha} \cong U_\alpha/U_{<\alpha}, \quad \Phi(H)_\alpha/\Phi(H)_{<\alpha} \cong \Phi(U)_\alpha/\Phi(U)_{<\alpha} \quad \text{and} \quad \overline{H}_\alpha/\overline{H}_{<\alpha} \cong \overline{U}_\alpha/\overline{U}_{<\alpha}.$$

- (b) *The following equalities hold:  $c_\alpha(H) = c_\alpha(U)$  and  $c_\alpha(\overline{H}) = c_\alpha(\overline{U})$ .*

*Proof.* Note that  $UG_\alpha = HG_\alpha$  and  $UG_{<\alpha} = UG_{<\alpha}$ , so  $c_\alpha(H) = c_\alpha(U)$  by Lemma 3.13(b). Thus, the natural map  $H_\alpha/H_{<\alpha} \cong U_\alpha/U_{<\alpha}$ , which is injective by Lemma 3.13(a), must be an isomorphism.

We also have  $\Phi(H) \subseteq \Phi(U) \subseteq \Phi(HG_{<\alpha}) \subseteq \Phi(H)G_{<\alpha}$ , and thus by the same argument the natural map  $\Phi(H)_\alpha/\Phi(H)_{<\alpha} \rightarrow \Phi(U)_\alpha/\Phi(U)_{<\alpha}$  is an isomorphism. The last isomorphism follows from the first two isomorphisms, Observation 3.5 and commutativity of the following diagram (which is straightforward to check):

$$(3.2) \quad \begin{array}{ccc} H_\alpha/H_{<\alpha}/(\Phi(H)_\alpha/\Phi(H)_{<\alpha}) & \longrightarrow & \overline{H}_\alpha/\overline{H}_{<\alpha} \\ \downarrow & & \downarrow \\ U_\alpha/U_{<\alpha}/(\Phi(U)_\alpha/\Phi(U)_{<\alpha}) & \longrightarrow & \overline{U}_\alpha/\overline{U}_{<\alpha} \end{array}$$

This proves (a), and (b) is a direct consequence of (a).  $\square$

Using Lemma 3.14 we can reduce computation of  $W$ -index to the case of open subgroups.

**Lemma 3.15.** *Let  $G$  be a pro- $p$  group,  $W$  a valuation on  $G$  and  $H$  a closed subgroup of  $G$ . Then*

$$[G : H]_W = \lim_{H \leq U \leq_o G} [G : U]_W.$$

*Proof.* First note that the above limit (either finite or infinite) exists by multiplicativity of  $W$ -index. By Lemma 3.14(b) for any  $\alpha > 0$  and any open subgroup  $U$  with  $H \subseteq U \subseteq HF_{<\alpha}$  we have  $c_\beta(H) = c_\beta(U)$  for all  $\beta \geq \alpha$ . This observation immediately implies the desired equality.  $\square$

Our next result establishes the weighted Schreier formula in the case of subgroups of index  $p$ .

**Lemma 3.16.** *Let  $G$  be a pro- $p$  group,  $W$  a valuation on  $G$  and  $U$  an open subgroup of  $G$  of index  $p$ . Let  $X$  be a  $W$ -optimal generating set of  $G$  and  $x \in X$  be such that  $X \setminus \{x\} \subset U$  (such  $X$  and  $x$  exist by Lemma 3.10). We have*

- (a)  $[G : U]_W = \frac{1-\alpha^p}{1-\alpha}$  where  $\alpha = W(x)$
- (b) *If  $rk_W(G) < \infty$ , then  $rk_W(U) - 1 \leq [G : U]_W \cdot (rk_W(G) - 1)$ . Moreover, if  $G$  is free and  $W$  is a weight function, then  $rk_W(U) - 1 = [G : U]_W \cdot (rk_W(G) - 1)$ .*

*Proof.* (a) Since  $p = [G : U] = \prod_{\gamma} p^{c_{\gamma}(G) - c_{\gamma}(U)}$ , there exists unique  $\beta \in (0, 1)$  such that  $c_{\beta}(G) = c_{\beta}(U) + 1$  and  $c_{\gamma}(G) = c_{\gamma}(U)$  for  $\gamma \neq \beta$ . Thus we just have to prove that  $\beta = \alpha$ . Since  $c_{\alpha}(G) - c_{\alpha}(U) = \log_p[UG_{\alpha} : UG_{<\alpha}]$ , it is enough to show that  $x \in UG_{\alpha}$  and  $x \notin UG_{<\alpha}$ .

By definition  $x \in G_{\alpha} \subseteq UG_{\alpha}$ . On the other hand, if  $x \in UG_{<\alpha}$ , we can find  $y \in G_{<\alpha}$  such that  $x \in yU$ . Hence  $\{y\} \cup X \setminus \{x\}$  is also a generating set of  $G$ , which contradicts  $W$ -optimality of  $X$ .

- (b) Note that  $W(x^p) \leq W(x)^p$  and  $W(\underbrace{[y, x, \dots, x]}_{k \text{ times}}) \leq W(y)W(x)^k$  for each  $y \in X \setminus \{x\}$

and  $k \in \mathbb{N}$ , and furthermore these inequalities become equalities if  $G$  is free and  $W$  is a weight function. If  $X'$  is defined as in Lemma 3.10(b), an easy computation shows that  $W(X') - 1 \leq \frac{1-\alpha^p}{1-\alpha}(W(X) - 1)$ , with equality holding when  $G$  is free and  $W$  is a weight function. Thus the desired result follows from part (a) and Lemma 3.10(b).  $\square$

Next we prove the analogue of Lemma 3.15 for  $W$ -rank which requires more work.

**Lemma 3.17.** *Let  $G$  be a pro- $p$  group,  $H$  its closed subgroup and  $W$  a valuation on  $G$ . Assume that  $[G : H]_W < \infty$ . Then*

$$rk_W(H) = \lim_{H \leq U \leq_o G} rk_W(U).$$

*Proof.* Choose a descending chain of subgroups  $G = U_0 \supset U_1 \supset U_2 \supset \dots$  such that  $[U_i : U_{i+1}] = p$  for each  $i$  and  $\cap U_i = H$ . To prove the lemma we will show that

- (a)  $rk_W(H) \leq \liminf_{i \rightarrow \infty} rk_W(U_i)$  and
- (b)  $rk_W(H) \geq \limsup_{i \rightarrow \infty} rk_W(U_i)$ .

For a closed subgroup  $K$  of  $G$  and  $\beta > 0$  we set

$$rk_{W, \geq \beta}(K) = \sum_{\alpha \geq \beta} c_{\alpha}(\overline{K}) \cdot \alpha \quad \text{and} \quad rk_{W, < \beta}(K) = \sum_{\alpha < \beta} c_{\alpha}(\overline{K}) \cdot \alpha = rk_W(K) - rk_{W, \geq \beta}(K).$$

Equivalently, if  $X$  is a  $W$ -optimal generating set of  $K$ , then

$$rk_{W, \geq \beta}(K) = \sum_{x \in X, W(x) \geq \beta} W(x) \quad \text{and} \quad rk_{W, < \beta}(K) = \sum_{x \in X, W(x) < \beta} W(x).$$

By Lemma 3.14(b) for each  $\beta > 0$  the sequence  $rk_{W, \geq \beta}(U_i)$  eventually stabilizes and  $rk_{W, \geq \beta}(H) = \lim_{n \rightarrow \infty} rk_{W, \geq \beta}(U_i)$ . This implies (a).

To establish (b) it suffices to show that for any  $\varepsilon > 0$  there exists  $\beta > 0$  and  $M \in \mathbb{N}$  such that  $rk_{W, < \beta}(U_n) < \varepsilon$  for all  $n > M$ .

Lemmas 3.16 and 3.15 imply that there exist real numbers  $\{\alpha_i\}_{i \geq 0}$  such that

- (i)  $rk_W(U_{i+1}) \leq \frac{1-\alpha_i^p}{1-\alpha_i} \cdot rk_W(U_i)$
- (ii)  $[G : H]_W = \prod_{i \geq 0} \frac{1-\alpha_i^p}{1-\alpha_i}$ .
- (iii)  $rk_{W, \geq \beta}(U_{i+1}) \geq rk_{W, \geq \beta}(U_i) - \alpha_i$  for any  $\beta \in (0, 1)$ .

Recall that  $[G : H]_W < \infty$ . By (i) and (ii) this implies that  $\limsup_{i \rightarrow \infty} rk_W(U_i) < \infty$  and also that the series  $\sum \alpha_i$  converges. Choose  $M$  such that  $\sum_{i > M} \alpha_i < \varepsilon/3$  and  $rk_W(U_M) \geq \limsup_{i \rightarrow \infty} rk_W(U_i) - \varepsilon/3$ . Choose  $\beta > 0$  such that  $rk_{W, \geq \beta}(U_M) \geq rk_W(U_M) - \varepsilon/3$ . Then for all  $n > M$  we have

$$rk_{W, \geq \beta}(U_n) \geq rk_{W, \geq \beta}(U_M) - \sum_{M \leq i < n} \alpha_i \geq \limsup_{i \rightarrow \infty} rk_W(U_i) - \varepsilon,$$

which yields (b).  $\square$

Putting everything together we can now prove Theorem 3.12.

*Proof of Theorem 3.12.* In Lemma 3.16 we have already established  $W$ -Schreier formula for  $[G : H] = p$ , and by multiplicativity of  $W$ -index (Proposition 3.11(c)) the formula extends to arbitrary open subgroups. Finally, the general case follows from Lemma 3.15 and Lemma 3.17.  $\square$

#### 4. WEIGHTED PRESENTATIONS AND WEIGHTED DEFICIENCY

In this section we introduce several variations of the notion of weighted deficiency for pro- $p$  groups and their presentations. We will not give a separate name for each type of weighted deficiency, as it should always be clear from the context and notations what we are talking about. We start in § 4.1 by defining the deficiency of a pro- $p$  presentation with respect to a weight function and the weighted deficiency of a pro- $p$  group. In § 4.2 we consider the more involved notion of the deficiency of a pro- $p$  group with respect to a finite valuation. In § 4.3 we introduce virtual valuations which will play a key role in the proof of Theorem 1.5, and in § 4.4 we define the deficiency associated to a finite virtual valuation. Finally, in § 4.5 we introduce pro- $p$  groups of positive virtual weighted deficiency (PVWD) which naturally arise in the proof of Theorem 1.5.

Before talking about weighted deficiency we need to define what it means for a valuation on a pro- $p$  group to be *finite*. In the case of weight functions, the definition is the obvious one.

**Definition.** A weight function  $W$  on a pro- $p$  group  $F$  is called *finite* if  $rk_W(F) < \infty$ .

Somewhat surprisingly, the extension of this definition to arbitrary valuations is more complex:

**Definition.** A valuation  $W$  on a pro- $p$  group  $G$  will be called *finite* if there exists  $Y \subset G$  such that  $\{LT(y) : y \in Y\}$  generate  $L_W(G)$  and  $W(Y) = \sum_{y \in Y} W(y)$  is finite.

**Remark:** The fact that the two definitions coincide in the case of weight functions follows from Proposition 3.2. In general, if a valuation  $W$  on  $G$  is finite, then  $rk_W(G) < \infty$ , but the converse need not hold.

We will see in § 4.2 why the above definition of a finite valuation is convenient to use.

##### 4.1. Weighted deficiency of pro- $p$ presentations and pro- $p$ groups.

**Definition.** Let  $(X, R)$  be a pro- $p$  presentation by generators and relators. Let  $F = F(X)$  be the free pro- $p$  group on  $X$ .

- (a) Given a finite weight function  $W$  on  $F$  with respect to  $X$ , the quantity

$$def_W(X, R) = W(X) - W(R) - 1$$

is called the  $W$ -deficiency of  $(X, R)$ .

- (b) The weighted deficiency of the presentation  $(X, R)$ , denoted by  $wdef_p(X, R)$  is the supremum of the set  $\{def_W(X, R)\}$  where  $W$  runs over all finite weight functions on  $F$  with respect to  $X$ .

**Remark:** The subscript  $p$  in the notation  $wdef_p(X, R)$  is used to avoid confusion in the case when  $R$  is a subset of  $F_{abs}(X)$  (the free abstract group on  $X$ ) since in this case we can consider  $(X, R)$  as a pro- $p$  presentation for different primes  $p$ .

**Definition.** Let  $G$  be a pro- $p$  group. The *weighted deficiency* of  $G$  denoted by  $wdef(G)$  is the supremum of the set  $\{wdef_p(X, R)\}$  where  $(X, R)$  runs over all (pro- $p$ ) presentations of  $G$ .

**Lemma 4.1.**

- (a) *The weighted deficiency of the trivial group  $E$  (considered as a pro- $p$  group) is equal to  $-1$ .*  
 (b) *Let  $G$  be a finitely generated pro- $p$  group and  $d(G)$  its minimal number of generators. Then  $wdef(G) \leq d(G) - 1$ .*

*Proof.* (a) Clearly,  $wdef(E) \geq -1$ . To prove the reverse inequality, let  $(X, R)$  be any pro- $p$  presentation of  $E$  and  $W$  a finite weight function on  $F = F(X)$  with respect to  $X$ . Then  $R$  generates  $F$  as a (closed) normal subgroup and hence  $R$  also generates  $F$  as a pro- $p$  group. Since  $X$  is a  $W$ -optimal generating set for  $F$ , we have  $W(R) \geq W(X)$ , whence  $W(X) - W(R) - 1 \leq -1$ .

(b) Let  $(X, R)$  be a presentation of  $G$  and  $W$  a weight function on  $F = F(X)$  with respect to  $X$ . Let  $Y \subseteq F$  be a generating set of  $G$  with  $|Y| = d(G)$ . Then  $(X, R \cup Y)$  is a presentation of the trivial group, so by (a)  $W(X) - W(R \cup Y) - 1 \leq -1$ . Since  $W(R \cup Y) = W(R) + W(Y) \leq W(R) + d(G)$ , we obtain that  $W(X) - W(R) - 1 \leq d(G) - 1$ .  $\square$

Next we define weighted deficiency corresponding to a slightly different notion of a (pro- $p$ ) presentation, where we will specify only a free pro- $p$  group and its normal subgroup, but not generators and relations.

**Definition.**

- (i) A *presentation* is a pair  $(F, N)$  where  $F$  is a free pro- $p$  group and  $N$  is a (closed) normal subgroup of  $F$ .  
 (ii) A *weighted presentation* is a triple  $(F, N, W)$  where  $(F, N)$  is a presentation and  $W$  is a finite weight function on  $F$ .  
 (iii) Given a weighted presentation  $(F, N, W)$ , we define  $def_W(F, N)$  to be the supremum of the set  $\{def_W(X, R)\}$  where  $X$  is a  $W$ -free generating set of  $F$  and  $R$  is a set of normal generators of  $N$ .

We have the following “closed” formula for  $def_W(F, N)$ :

**Lemma 4.2.** *Let  $(F, N, W)$  be a weighted presentation. The following hold:*

- (a) *If  $R$  is a set of normal generators for  $N$ , then  $W(R) \geq rk_W(N/[N, F])$ , and there exists  $R$  for which equality holds*  
 (b)  $def_W(F, N) = rk_W(F) - rk_W(N/[N, F]) - 1$ .

*Proof.* (a) is a direct consequence of the following well-known fact: if  $R$  is a subset of  $N$ , then  $R$  generates  $N$  as a normal subgroup of  $F$  if and only if the image of  $R$  in  $N/[N, F]$  generates  $N/[N, F]$  as a subgroup.

(b) follows from (a) and the fact that  $rk_W(F) = W(X)$  for any  $W$ -free generating set  $X$  (since  $W$ -free =  $W$ -optimal).  $\square$

**Proposition 4.3.** *Let  $F$  be a free pro- $p$  group,  $W$  a finite weight function on  $F$  and  $(F, N, W)$  a weighted presentation. Let  $F'$  be a closed subgroup of  $F$  containing  $N$  and assume that  $[F : F']_W < \infty$ . Then*

$$(4.1) \quad \text{def}_W(F', N) \geq \text{def}_W(F, N) \cdot [F : F']_W.$$

*Proof.* *Case 1:*  $F'$  has index  $p$  in  $F$ . In view of Theorem 3.12, we only have to show that for any subset  $R \subseteq N$  with  $\langle R \rangle^F = N$  there exists a subset  $R' \subseteq N$  with  $\langle R' \rangle^{F'} = N$  such that

$$W(R') \leq [F : F']_W \cdot W(R).$$

So assume that  $R$  generates  $N$  as a normal subgroup of  $F$ , and set

$$R' = \{r, [r, x], \dots, [r, \underbrace{x, \dots, x}_{p-1 \text{ times}}]\} : r \in R\}.$$

Then by the Schreier rewriting process  $R'$  generates  $N$  as a normal subgroup of  $F'$ . Since  $W(\underbrace{[r, x, \dots, x]}_{k \text{ times}}) \leq W(r)W(x)^k$ , using Lemma 3.16 we get

$$W(R') \leq W(R) \frac{1 - W(x)^p}{1 - W(x)} = W(R) \cdot [F : F']_W.$$

*Case 2:*  $F'$  has arbitrary finite index in  $F$ . In this case the proposition follows from Case 1 by multiplicativity of  $W$ -index.

*Case 3:*  $F'$  is of infinite index in  $F$ . It is clear that

$$rk_W(N/[N, F']) = \lim_{F' \leq U \leq_o F} rk_W(N/[N, U]).$$

Hence, using Lemma 3.15, Lemma 3.17, Lemma 4.2 and the result in Case 2 we have

$$\begin{aligned} \text{def}_W(F', N) &= rk_W(F') - rk_W(N/[N, F']) - 1 \\ &= \lim_{F' \leq U \leq_o F} (rk_W(U) - rk_W(N/[N, U]) - 1) \\ &= \lim_{F' \leq U \leq_o F} \text{def}_W(U, N) \geq \lim_{F' \leq U \leq_o F} \text{def}_W(F, N) \cdot [F : U]_W \\ &= \text{def}_W(F, N) \cdot [F : F']_W. \end{aligned}$$

□

**Remark:** The part of the argument dealing with the case  $[F : F'] < \infty$  essentially appears in the proof of [EJ, Theorem 3.11].

As an immediate consequence of Proposition 4.3, we deduce that the class of PWD pro- $p$  groups is closed under taking open subgroups, a fact we stated in the introduction. Proposition 4.3 also yields a very simple proof of the infiniteness of PWD groups.

**Corollary 4.4.** *A pro- $p$  group of positive weighted deficiency (PWD) must be infinite.*

*Proof.* Let  $G$  be a pro- $p$  group of PWD. By Lemma 4.1(a)  $G$  is non-trivial, so we can find a proper open subgroup  $H$  of  $G$ . By Proposition 4.3  $H$  also has PWD, and we can apply the same argument to  $H$ . This process can be continued indefinitely, so  $G$  must be infinite. □

**4.2. Presentations of valued pro- $p$  groups.** Let  $G$  be a pro- $p$  group and  $(F, N, \widetilde{W})$  a weighted presentation of  $G$ , and fix an epimorphism  $\pi : F \rightarrow G$  with  $\text{Ker } \pi = N$ . Recall that  $\widetilde{W}$  induces a valuation  $W$  on  $G$  given by

$$W(g) = \min\{\widetilde{W}(f) : \pi(f) = g\}.$$

We shall show (see Proposition 4.7 below) that each valuation  $W$  on a pro- $p$  group  $G$  arises from some weighted presentation  $(F, N, \widetilde{W})$  of  $G$  in such way. If in addition we want  $F$  to be of finite  $\widetilde{W}$ -rank, we need to assume that the valuation  $W$  is finite – this explains the definition of a finite valuation given at the beginning of the section.

We start with two auxiliary lemmas.

**Lemma 4.5.** *Let  $\varphi : K \rightarrow G$  be a homomorphism of pro- $p$  groups,  $\widetilde{W}$  a valuation on  $K$  and  $W$  a valuation on  $G$ . Assume  $\varphi(K_\alpha) \subseteq G_\alpha$  for all  $\alpha$ . Then the induced map  $\overline{\varphi} : L_{\widetilde{W}}(K) \rightarrow L_W(G)$  is a homomorphism of restricted Lie algebras. Moreover if  $\overline{\varphi}$  is surjective, then  $\varphi(K_\alpha) = G_\alpha$ , and so  $\widetilde{W}$  induces  $W$ .*

*Proof.* The first assertion of the lemma is standard. Let us show that  $\varphi(K_\alpha) = G_\alpha$ . Since  $L_{\widetilde{W}}(K)$  maps onto  $L_W(G)$ , we have  $G_\alpha = \varphi(K_\alpha)G_{<\alpha}$ . Let  $\alpha = \alpha_0 > \alpha_1 > \alpha_2 > \dots$  be all possible values of  $W$  which are  $\leq \alpha$ , so that  $G_{\alpha_i} = G_{<\alpha_{i-1}}$  for  $i \geq 1$ . Then we have

$$G_\alpha = \varphi(K_\alpha)G_{<\alpha_0} = \varphi(K_\alpha)G_{\alpha_1} = \varphi(K_\alpha)\varphi(K_{\alpha_1})G_{<\alpha_1} = \varphi(K_\alpha)G_{<\alpha_1} = \dots = \varphi(K_\alpha).$$

□

**Lemma 4.6.** *Let  $\pi : F \rightarrow G$  be a homomorphism of pro- $p$  groups where  $F$  is free,  $\widetilde{W}$  a weight function on  $F$  and  $W$  a valuation on  $G$ . Suppose that  $F$  has a  $\widetilde{W}$ -free generating set  $X$  such that  $\widetilde{W}(x) \geq W(\pi(x))$  for any  $x \in X$ . Then  $\widetilde{W}(f) \geq W(\pi(f))$  for any  $f \in F$  and therefore  $\pi(F_\alpha) \subseteq G_\alpha$  for all  $\alpha$ .*

*Proof.* This is an easy consequence of the implication “(i) $\Rightarrow$ (ii)” in Proposition 3.2. □

**Definition.** Let  $G$  be a pro- $p$  group and  $W$  a valuation on  $G$ . A *presentation* of  $(G, W)$  is a triple  $(F, \pi, \widetilde{W})$  where  $F$  is a free pro- $p$  group,  $\pi : F \rightarrow G$  is an epimorphism and  $\widetilde{W}$  is a weight function on  $F$  which induces  $W$  under  $\pi$ .

**Proposition 4.7.** *Let  $G$  be a pro- $p$  group and  $W$  a valuation on  $G$ . Then there exists a presentation  $(F, \pi, \widetilde{W})$  of  $(G, W)$ . Moreover, the weight function  $\widetilde{W}$  can be chosen finite if and only if  $W$  is finite.*

*Proof.* Let  $Y$  be a countable subset of  $G \setminus \{1\}$  such that  $S = \{\text{LT}(y) : y \in Y\}$  generates  $L_W(G)$ . Then  $Y$  generates  $G$ , and if  $Y$  is infinite, we can assume that  $Y$  converges to 1. Consider the free pro- $p$  group  $F = F(Y)$  and let  $\pi : F \rightarrow G$  be the natural epimorphism that extends the map  $Y \rightarrow G$ . Let  $\widetilde{W}$  be the weight function on  $F(Y)$  with respect to  $Y$  such that  $\widetilde{W}(y) = W(\pi(y))$  for all  $y \in Y$ .

By Lemma 4.6,  $\pi(F_\alpha) \subseteq G_\alpha$  for all  $\alpha$ . Furthermore,  $\pi(\widetilde{\text{LT}}(y)) = \text{LT}(y)$  for all  $y \in Y$  (where  $\widetilde{\text{LT}}$  denotes the leading term in  $L_{\widetilde{W}}(F)$ ), and so by the choice of  $S$  we have  $\pi(L_{\widetilde{W}}(F)) = L_W(G)$ . Hence by Lemma 4.5,  $\widetilde{W}$  induces  $W$  under  $\pi$ .

By our construction, if  $W$  is finite, then the weight function  $\widetilde{W}$  is finite. On the other hand, if  $(F, \pi, \widetilde{W})$  is any presentation  $(G, W)$  with  $\widetilde{W}$  finite, then clearly  $W$  is finite. □

Proposition 4.7 enables us to define the deficiency of a pro- $p$  group with respect to a finite valuation.

**Definition.** Let  $G$  be a pro- $p$  group and  $W$  a finite valuation on  $G$ . The *deficiency of  $G$  with respect to  $W$* , denoted by  $\text{def}_W(G)$ , is defined to be the supremum of the set  $\{\text{def}_{\widetilde{W}}(F, \text{Ker } \pi)\}$ , where  $(F, \pi, \widetilde{W})$  runs over all presentations of  $(G, W)$ , with  $\widetilde{W}$  finite.

**4.3. Virtual valuations.** In our analysis of PWD groups we will need to deal with situations when a pro- $p$  group  $G$  does not necessarily have PWD, but some open subgroup  $U$  of  $G$  does have positive deficiency with respect some valuation  $W$  on  $U$  (note that by Proposition 4.3 we can then assume that  $U$  is normal). As we will see, the group  $G$  will still satisfy key properties of PWD groups under the additional assumption that  $W$  is  $G$ -invariant, as defined below. In fact, we may even allow  $G$  to be a virtually pro- $p$  group (the open subgroup  $U$  must be pro- $p$ ).

**Definition.** Let  $G$  be a profinite group and  $U$  an open normal pro- $p$  subgroup of  $G$ . A valuation  $W$  on  $U$  will be called  *$G$ -invariant* if

$$W(u^g) = W(u) \text{ for any } g \in G \text{ and } u \in U.$$

We will also refer to such  $W$  as a *virtual valuation of  $G$  defined on  $U$* .

**Remark:** If  $G$  is a pro- $p$  group,  $W$  is a valuation on  $G$  and  $U$  is an open normal subgroup of  $G$ , then  $W$  restricted to  $U$  is a  $G$ -invariant valuation on  $U$ , but the reverse transition is not always possible: a  $G$ -invariant valuation on  $U$  may not be extendable to a valuation in  $G$ .

Here is another simple way to produce virtual valuations:

**Observation 4.8.** *Let  $G$  be a finitely generated profinite group,  $U$  an open normal pro- $p$  subgroup of  $G$  and  $W$  a uniform valuation on  $U$ . Then  $W$  is  $G$ -invariant, so  $W$  is a virtual valuation of  $G$ .*

*Proof.* This is a direct consequence of Proposition 2.2. □

In the case of weight functions invariance can often be checked using the following criterion.

**Lemma 4.9.** *Let  $F$  be a profinite group and  $U$  an open normal free pro- $p$  subgroup of  $F$ . Let  $W$  be a weight function on  $U$  and  $X$  a  $W$ -optimal generating set of  $U$ . Then  $W$  is  $F$ -invariant if and only if  $W(x^f) \leq W(x)$  for all  $x \in X$  and  $f \in F$ .*

*Proof.* The forward direction is obvious. For the reverse direction fix  $f \in F$  and let  $X^f = \{x^f : x \in X\}$ . First of all note that we must have equality  $W(x^f) = W(x)$  for all  $x \in X$  for otherwise  $X$  will not be  $W$ -optimal.

Next define  $W' : F \rightarrow [0, 1)$  by  $W'(g) = W(g^{f^{-1}})$ . Then clearly  $W'$  is weight function with respect to the set  $X^f$ . On the other hand, for each  $x \in X$  we have

$$W'(x) = W(x^{f^{-1}}) = W(x) = W'(x^f)$$

Hence,  $X$  is also  $W'$ -optimal and  $W'(x) = W(x)$  for all  $x \in X$ . Therefore,  $W' = W$  as functions, which is equivalent to  $f$ -invariance of  $W$ . □

If  $W$  is a weight function on a free pro- $p$  group  $F$ , one can construct another weight function  $W'$  on  $F$  with respect to the same free generating set  $X$  by dividing the weights of all elements of  $X$  by the same constant  $c \geq 1$ . This simple-minded operation, called the  *$c$ -contraction* proved to be very useful in [EJ] and will again play a key role in this paper.



**Definition.** Let  $F$  be a free pro- $p$  group and  $W$  a weight function on  $F$ . Let  $c \geq 1$  be a real number. Choose any  $W$ -free generating set  $X$  of  $F$ , and let  $W'$  be the unique weight function on  $F$  with respect to  $X$  such that  $W'(x) = W(x)/c$  for all  $x \in X$ . Then we will say that the pair  $(F, W')$  is obtained from  $(F, W)$  by the  $c$ -contraction. It is easy to see that  $W'$  is independent of the choice of  $X$ .

The next result shows that  $c$ -contractions can be applied not only to weight functions, but also to virtual weight functions.

**Lemma 4.10.** *Let  $F$  be a profinite group,  $U$  an open normal free pro- $p$  subgroup of  $F$  and  $W$  an  $F$ -invariant weight function on  $U$ . Let  $c \geq 1$ , and let  $(U, W) \rightarrow (U, W')$  be the  $c$ -contraction. Then the weight function  $W'$  is also  $F$ -invariant.*

*Proof.* It is clear from the definition of  $c$ -contraction that  $W'(g) \leq W(g)/c$  for all  $g \in U$ . Now if  $X$  is a  $W$ -optimal generating set for  $U$ , then for any  $x \in X$  and  $f \in F$  we have

$$W'(x^f) \leq \frac{W(x^f)}{c} = \frac{W(x)}{c} = W'(x).$$

Thus,  $W'$  is  $F$ -invariant by Lemma 4.9.  $\square$

#### 4.4. Presentations of virtually valued virtually pro- $p$ groups.

**Definition.** Let  $G$  be a profinite group and  $W$  a virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$ . A *presentation* of  $(G, W)$  is a triple  $(F, \pi, \widetilde{W})$  where  $F$  is a profinite group,  $\pi : F \rightarrow G$  is an epimorphism such that  $\pi^{-1}(U)$  is a free pro- $p$  group and  $\widetilde{W}$  is an  $F$ -invariant weight function on  $\pi^{-1}(U)$  which induces  $W$  under  $\pi$ .

The following result generalizes Proposition 4.7 to the case of virtual valuations.

**Proposition 4.11.** *Let  $G$  be a virtually pro- $p$  group and  $W$  a virtual valuation of  $G$ . Then there exists a presentation  $(F, \pi, \widetilde{W})$  of  $(G, W)$ . Moreover,  $\widetilde{W}$  can be chosen finite if and only if  $W$  is finite.*

*Proof.* Assume that  $W$  is defined on an open normal pro- $p$  subgroup  $U$  of  $G$ . Let  $Y_1$  be a countable subset of  $U \setminus \{1\}$  such that  $S = \{LT(y) : y \in Y_1\}$  generates  $L_W(U)$  (as in Proposition 4.7 we can assume that  $Y_1$  converges to 1 if  $Y_1$  is infinite). By definition we can make  $W(Y_1)$  finite if and only if  $W$  is finite. Let  $Y_2 \subset G \setminus U$  be a finite set that generates  $G/U$ . Let  $Y = Y_1 \sqcup Y_2$ . The rest of the proof is divided in two steps – constructing a presentation  $(F, \pi)$  for  $G$  (Step 1) and then constructing a weight function  $\widetilde{W}$  on  $\pi^{-1}(U)$  (Step 2).

*Step 1: Constructing  $F$  and  $\pi$ .* Let  $F_{abs} = F_{abs}(Y)$  be the free abstract group on  $Y$  and let  $\pi_{abs} : F_{abs} \rightarrow G$  be the natural homomorphism that extends the map  $Y \rightarrow G$ . For  $i = 1, 2$  let  $F_{i,abs}$  be the subgroup of  $F_{abs}$  generated by  $Y_i$  (so that  $F_{abs} = F_{1,abs} * F_{2,abs}$ ). Let  $V_{abs} = \pi_{abs}^{-1}(U)$  and  $V_{2,abs} = V_{abs} \cap F_{2,abs}$ .

Let  $Z$  be any free generating set of  $V_{2,abs}$ , and let  $T = \{t_1, \dots, t_s\}$  be a Schreier transversal of  $V_{2,abs}$  in  $F_{2,abs}$  with respect to  $Y_2$ . Then  $T$  is also a Schreier transversal of  $V_{abs}$  in  $F_{abs}$ , and it is easy to see that  $V_{abs}$  is freely generated by the set

$$X = (\sqcup_{i=1}^s Y_1^{t_i}) \sqcup Z \quad (***)$$

(simply apply the Schreier rewriting process to the generating set  $Y$  of  $F_{abs}$  and the transversal  $T$  to obtain a free generating set of  $V_{abs}$ ).

Now let  $\widehat{F}$  be the free profinite group on  $Y$  and let  $\widehat{V}$  be the closure of  $V_{abs}$  in  $\widehat{F}$ . Let  $V = \widehat{V}/H$  be the maximal pro- $p$  quotient of  $\widehat{V}$ ; observe that  $V$  is a free pro- $p$  group on  $X$ .

Also note that  $H$  is normal in  $\widehat{F}$  (since  $\widehat{V}$  is normal in  $\widehat{F}$  and  $H$  is characteristic in  $\widehat{V}$ ), and thus we can consider the virtually pro- $p$  group  $F = \widehat{F}/H$  (containing  $V$  as an open subgroup). Let  $F_i$  be the subgroup of  $F$  generated by  $Y_i$  (for  $i = 1, 2$ ) and  $V_2 = V \cap F_2$ . Note that  $T$  is a transversal of  $V$  in  $F$  (and also of  $V_2$  in  $F_2$ ).

Finally, let  $\hat{\pi} : \widehat{F} \rightarrow G$  be the natural epimorphism that extends  $\pi_{abs} : F_{abs} \rightarrow G$ . Since  $\hat{\pi}(\widehat{V}) = U$  is pro- $p$ , we can factor  $\hat{\pi}$  through the map  $\pi : F \rightarrow G$ . Note that  $V = \pi^{-1}(U)$ .

*Step 2: Choosing  $\widetilde{W}$ .* Recall that  $F$  is free pro- $p$  on  $X$ . Let  $\alpha_0 = \max\{W(g) : g \in U\}$ , fix  $\alpha_0 < \alpha < 1$ , and let  $\widetilde{W}$  be the weight function on  $V$  with respect to  $X$  such that  $\widetilde{W}(y) = W(\pi(y))$  for any  $y \in \cup_{i=1}^s Y_1^{t_i}$  and  $\widetilde{W}(z) = \alpha$  for any  $z \in Z$ .

We will now show that  $\widetilde{W}$  is  $F$ -invariant. Let  $f \in F$ , and write  $f = tv$ , where  $t \in T$  and  $v \in V$ . Since  $W$  is  $G$ -invariant and  $\widetilde{W}$  is  $V$ -invariant, for any  $y \in Y_1$  we have

$$\widetilde{W}(y^f) = \widetilde{W}((y^t)^v) = \widetilde{W}(y^t) = W(\pi(y^t)) = W(\pi(y)^{\pi(t)}) = W(\pi(y)) = \widetilde{W}(y).$$

Since  $f$  is arbitrary, we deduce that  $\widetilde{W}(y^f) = \widetilde{W}(y)$  for any  $y \in \cup_{i=1}^s Y_1^{t_i}$  as well.

The restriction of  $\widetilde{W}$  to  $V_2$  is  $F_2$ -invariant by Observation 4.8, so for any  $z \in Z$  we obtain

$$\widetilde{W}(z^f) = \widetilde{W}((z^t)^v) = \widetilde{W}(z^t) = \widetilde{W}(z).$$

Hence  $\widetilde{W}$  is  $F$ -invariant by Lemma 4.9.

By Lemma 4.6 we have  $\pi(V_\gamma) \subseteq U_\gamma$  for all  $\gamma$ , and by the same argument as in Proposition 4.7 we have  $\pi(L_{\widetilde{W}}(F_1)) = L_W(U)$ . Hence by Lemma 4.5  $\widetilde{W}$  induces  $W$ . Thus  $(F, \pi, \widetilde{W})$  is a presentation of  $(G, W)$ , and  $\widetilde{W}$  is finite if and only if  $W$  is finite.  $\square$

We can now define the notion of deficiency with respect to a virtual valuation:

**Definition.** Let  $G$  be a profinite group and  $W$  a finite virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$ . The  $G$ -invariant deficiency of  $U$  with respect to  $W$ , denoted by  $\text{def}_W^G(U)$  is the supremum of the set  $\{\text{def}_{\widetilde{W}}(\pi^{-1}(U), \text{Ker } \pi)\}$ , where  $(F, \pi, \widetilde{W})$  is a presentation of  $(G, W)$ .

The following elementary result describes how weighted deficiency may change when a profinite group  $G$  is replaced by its quotient. It will be applied very frequently in the sequel.

**Lemma 4.12.** *Let  $G$  be a profinite group and  $W$  a finite virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$ . Let  $S$  be a subset of  $U$  and  $N$  the normal subgroup of  $G$  generated by  $S$ . Then*

$$\text{def}_W^{G/N}(U/N) \geq \text{def}_W^G(U) - [G : U]W(S).$$

*Proof.* Let  $T$  be a transversal of  $U$  in  $G$ . Then the set  $S' = \{s^t : s \in S, t \in T\}$  generates  $N$  as a normal subgroup of  $U$  and  $W(S') \leq |T|W(S) = [G : U]W(S)$  since  $W$  is  $G$ -invariant. This yields the assertion of the lemma.  $\square$

We also point out two simple consequences of Lemma 4.12 that will be needed later.

**Lemma 4.13.** *Let  $G$  be a profinite group and  $W$  a finite virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$ . Let  $\Lambda$  and  $\Delta$  be abstract subgroups of  $G$  which have the same closure in  $G$ . Then for any  $\varepsilon > 0$  there exists a normal subgroup  $N$  of  $G$  such that  $\text{def}_W^{G/N}(UN/N) \geq \text{def}_W^G(U) - \varepsilon$  and  $\Lambda N/N = \Delta N/N$ , that is,  $\Lambda$  and  $\Delta$  have the same image in  $G/N$ .*

*Proof.* Let  $\varepsilon' = \frac{\varepsilon}{2[G:U]}$ . Let  $\{a_i\}_{i \geq 1}$  (resp.  $\{b_i\}_{i \geq 1}$ ) be a countable dense subset of  $\Lambda \cap U$  (resp.  $\Delta \cap U$ ). Since  $\Lambda$  and  $\Delta$  have the same closure, for each  $i \geq 1$  we can choose  $b'_i \in \Delta \cap U$  and  $a'_i \in \Lambda \cap U$  such that  $W(a_i(b'_i)^{-1}) < \frac{\varepsilon'}{2^i}$  and  $W(b_i(a'_i)^{-1}) < \frac{\varepsilon'}{2^i}$ . Applying Lemma 4.12 to the set  $S = \{a_i(b'_i)^{-1}, b_i(a'_i)^{-1}\}_{i \geq 1}$ , we get the desired result.  $\square$

**Lemma 4.14.** *Let  $G$  be a profinite group,  $W$  a finite virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$  and  $\Gamma$  a dense abstract subgroup of  $G$ . Then for any  $\varepsilon > 0$  there exists a normal subgroup  $N$  of  $G$  with  $N \subseteq U$  such that*

- (i)  $\text{def}_W^{G/N}(U/N) \geq \text{def}_W^G(U) - \varepsilon$
- (ii)  $\Gamma N/N$  is finitely generated (so  $G/N$  is finitely generated)
- (iii)  $(\Gamma \cap U)N/N$  is  $p$ -torsion

*Proof.* Let  $A$  be a generating set of  $U$  with  $W(A) < \infty$ . By Lemma 4.13 without loss of generality we can assume that the abstract subgroup generated by  $A$  coincides with  $\Gamma$ .

Now let  $\varepsilon' = \frac{\varepsilon}{2[G:U]}$ . We can find a subset  $S_1 \subseteq A$  such that  $A \setminus S_1$  is finite and  $W(S_1) < \varepsilon'$ . Next we enumerate elements of  $\Gamma \cap U : g_1, g_2, \dots$  and choose a subset  $S_2$  of the form  $\{g_i^{p^{n_i}} : i \in \mathbb{N}\}$  with  $W(S_2) < \varepsilon'$ .

Let  $N$  be the normal subgroup of  $G$  generated by  $S = S_1 \cup S_2$ . Condition (i) holds by Lemma 4.12, (ii) holds since  $N$  contains  $S_1$  and (iii) holds since  $N$  contains  $S_2$ .  $\square$

#### 4.5. Pro- $p$ groups of positive virtual weighted deficiency.

**Definition.** Let  $G$  be a virtually pro- $p$  group. We will say that  $G$  has *positive virtual weighted deficiency* if there exist an open normal pro- $p$  subgroup  $U$  of  $G$  and a  $G$ -invariant valuation  $W$  on  $U$  such that  $\text{def}_W^G(U) > 0$ .

Groups of positive virtual weighted deficiency (PVWD) will appear naturally in the analysis of quotients of PWD groups (see Section 5), and their consideration is necessary for the proof of our main results about PWD groups. Moreover, it seems that all the interesting properties of PWD groups extend to PVWD groups. At least this is true for the results formulated in the introduction; in fact, in Section 8 we will restate and prove most of these results for PVWD groups (this requires almost no extra work).

We point out that having PVWD is a stronger condition than being virtually of PWD (that is, having an open subgroup of PWD). For instance, let  $F$  be a non-abelian pro- $p$  group and  $G$  the wreath product of  $F$  and  $\mathbb{Z}/n\mathbb{Z}$  (with  $\mathbb{Z}/n\mathbb{Z}$  being the active subgroup). Then  $G$  has an open subgroup  $\underbrace{F \times \dots \times F}_{n \text{ times}}$  which is clearly of PWD. On the other hand,

$G$  does not have PVWD – this will follow from Proposition 8.11 and Theorem 8.9.

### 5. KEY STEP

The following theorem is the key step in the proof of Theorem 1.5. It can be thought of as a pro- $p$  analogue of Theorem 1.1 with some technicalities added.

**Theorem 5.1.** *Let  $G$  be a profinite group and  $W$  a finite virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$ . Assume that  $\text{def}_W^G(U) > 0$ . Let  $H$  be a closed subgroup of  $G$ . Then the following hold:*

- (a) *If  $[U : (U \cap H)]_W < \infty$ , then there exists a normal subgroup  $N$  of  $G$  such that  $\text{def}_W^{G/N}(UN/N) > 0$  and  $HN/N$  is open in  $G/N$ .*

- (b) If  $[U : (U \cap H)]_W = \infty$  and  $rk_W(U \cap H) < \infty$ , then there exists an open subgroup  $V$  of  $U$  which is normal in  $G$ , a finite  $G$ -invariant valuation  $W'$  on  $V$  and a normal subgroup  $N$  of  $G$  such that  $def_{W'}^{G/N}(VN/N) > 0$  and  $HN/N$  is finite.

The two parts of Theorem 5.1 will be established using rather different arguments. The argument in part (b) is more involved, but the key idea behind it is very old and goes back to the original paper of Golod [Go]. We present it as a separate lemma.

**Lemma 5.2.** *Let  $G$  be a profinite group and  $W$  a finite virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$ . Let  $H$  be a closed subgroup of  $U$  with  $rk_W(H) < 1$ . Then for any  $\varepsilon > 0$  there exists a normal subgroup  $N$  of  $G$  such that  $def_W^{G/N}(U/N) \geq def_W^G(U) - \varepsilon$  and  $HN/N$  is finite.*

*Proof.* Let  $Y$  be a  $W$ -optimal generating set for  $H$ , so that  $W(Y) = rk_W(H) < 1$ .

*Case 1:  $H$  is finitely generated.* For  $m \in \mathbb{N}$  let

$$Y^{(m)} = \{[y_{i_1}, \dots, y_{i_m}] : y_{i_j} \in Y\} \cup \{y^{p^m} : y \in Y\}$$

be the set consisting of all left-normed commutators of length  $m$  in  $Y$  and all  $p^m$ -powers of elements of  $Y$ . Clearly,  $W(Y^{(m)}) \leq W(Y)^m + |Y|\delta^m \rightarrow 0$  as  $m \rightarrow \infty$  (where  $\delta = \max\{W(y) : y \in Y\}$ ).

Choose  $m$  for which  $W(Y^{(m)}) < \frac{\varepsilon}{[G:U]}$ , and let  $N$  be the normal subgroup of  $G$  generated by  $Y^{(m)}$ . Then by Lemma 4.12,

$$def_W^{G/N}(U/N) \geq def_W^G(U) - [G:U]W(Y^{(m)}) \geq def_W^G(U) - \varepsilon.$$

On the other hand,  $HN/N$  is nilpotent and generated by a finite set of torsion elements, hence finite.

*General case:* Fix  $0 < \varepsilon' < \frac{\varepsilon}{[G:U]}$ , and write  $Y$  as disjoint union  $Y_1 \sqcup Y_2$  where  $Y_1$  is finite and  $W(Y_2) < \varepsilon'$ . Let  $K$  be the normal subgroup of  $G$  generated by  $Y_2$ . Then  $def_W^{G/K}(U/K) > def_W^G(U) - \varepsilon$  (again by Lemma 4.12), while  $rk_W(HK/K) \leq rk_W(H) < 1$  and  $HK/K$  is finitely generated. Thus, we are reduced to Case 1.  $\square$

**Remark:** In Section 8 we will need the following generalization of Lemma 5.2, which can be proved by the same argument. Let  $G, W$  and  $U$  be as above, and let  $\{H_i\}$  be a countable collection of closed subgroups of  $U$  with  $rk_W(H_i) < 1$  for each  $i$ . Then for any  $\varepsilon > 0$  there exists a normal subgroup  $N$  of  $G$  such that  $def_W^{G/N}(U/N) \geq def_W^G(U) - \varepsilon$  and  $H_i N/N$  is finite for each  $i$ .

*Proof of Theorem 5.1.* Replacing  $H$  by  $H \cap U$  if needed, we can assume without loss of generality that  $H \subseteq U$ .

*Case (a):  $[U : H]_W < \infty$ .*

Let  $\varepsilon = def_W^G(U)/[G:U]$ . Since  $[U : H]_W < \infty$ , by Lemma 3.15 we can find an open subgroup  $V$  of  $U$  containing  $H$  such that  $\log_2([V : H]_W) < \varepsilon$ . Hence

$$\sum_{\alpha} \alpha(c_{\alpha}(V) - c_{\alpha}(H)) < \varepsilon.$$

By Lemma 3.13,  $|V_{\alpha}H/V_{<\alpha}H| = p^{c_{\alpha}(V) - c_{\alpha}(H)}$  for each  $\alpha \in \text{Im } W$ . Take a subset  $T_{\alpha}$  of  $V_{\alpha}$  such that its image forms a basis in  $V_{\alpha}H/V_{<\alpha}H$ , and put  $T = \cup T_{\alpha}$ . Then  $V = \langle H, T \rangle$  and

$W(T) < \varepsilon$ . Let  $N$  be the normal subgroup of  $G$  generated by  $T$ . By Lemma 4.12,

$$\text{def}_W^{G/N}(U/N) \geq \text{def}_W^G(U) - [G : U]W(T) > 0.$$

On the other hand,  $HN/N = VN/N$  by construction and so  $HN/N$  is open in  $G/N$ .

*Case (b):*  $[U : H]_W = \infty$ .

Let  $(F, \pi, \widetilde{W})$  be a presentation of  $(G, W)$  such that  $\text{def}_{\widetilde{W}}(\pi^{-1}(U), \text{Ker } \pi) > 0$ . Since,  $[U : H]_W = \infty$ , by Lemma 3.11(a) there exists  $\alpha \in \text{Im } W$  such that

$$[H : H_\alpha]_W < \frac{\text{def}_{\widetilde{W}}(\pi^{-1}(U), \text{Ker } \pi)[U : U_\alpha]_W}{rk_W(H) - 1}.$$

Then Theorem 3.12 implies that

$$rk_W(H_\alpha) - 1 \leq [H : H_\alpha]_W(rk_W(H) - 1) < \text{def}_{\widetilde{W}}(\pi^{-1}(U), \text{Ker } \pi)[U : U_\alpha]_W.$$

Thus, by Proposition 4.3,

$$rk_W(H_\alpha) < \text{def}_{\widetilde{W}}(\pi^{-1}(U_\alpha), \text{Ker } \pi) + 1.$$

Choose any  $c$  with

$$rk_W(H_\alpha) < c < \text{def}_{\widetilde{W}}(\pi^{-1}(U_\alpha), \text{Ker } \pi) + 1,$$

and let  $\widetilde{W}'$  be the weight function on  $\pi^{-1}(U_\alpha)$  obtained from  $\widetilde{W}$  by the  $c$ -contraction. Then

$$rk_{\widetilde{W}'}(\pi^{-1}(U_\alpha)) = rk_{\widetilde{W}}(\pi^{-1}(U_\alpha))/c \text{ and } \widetilde{W}'(f) \leq \widetilde{W}(f)/c \text{ for any } f \in \pi^{-1}(U_\alpha), \quad (*)$$

whence

$$\text{def}_{\widetilde{W}'}(\pi^{-1}(U_\alpha), \text{Ker } \pi) + 1 \geq (\text{def}_{\widetilde{W}}(\pi^{-1}(U_\alpha), \text{Ker } \pi) + 1)/c. \quad (**)$$

It is easy to see that  $\pi^{-1}(U_\alpha) = \widetilde{U}_\alpha \text{Ker } \pi$  where  $\widetilde{U}_\alpha = \{f \in \pi^{-1}(U) : \widetilde{W}(f) \leq \alpha\}$ . Since  $\widetilde{W}$  is  $F$ -invariant,  $\widetilde{U}_\alpha$  is normal in  $F$ , whence  $\pi^{-1}(U_\alpha)$  is also normal in  $F$ . Thus we can apply Lemma 4.10 and deduce that  $\widetilde{W}'$  is  $F$ -invariant.

Let  $W'$  be the virtual valuation of  $G$  induced by  $\widetilde{W}'$ . Then  $(**)$  implies that

$$\text{def}_{W'}^G(U_\alpha) \geq \text{def}_{\widetilde{W}'}(\pi^{-1}(U_\alpha), \text{Ker } \pi) \geq (\text{def}_{\widetilde{W}}(\pi^{-1}(U_\alpha), \text{Ker } \pi) + 1)/c - 1 > 0$$

while  $(*)$  yields

$$rk_{W'}(H_\alpha) \leq rk_W(H_\alpha)/c < 1.$$

We can now finish the proof by applying Lemma 5.2 to the virtual valuation  $W'$ .  $\square$

## 6. WEIGHTED DEFICIENCY OF ABSTRACT GROUPS

In this section we define weighted deficiency for abstract groups and explain how results about weighted deficiency of pro- $p$  groups obtained in the previous sections can be applied to abstract groups.

**Definition.** Let  $\Gamma$  be a finitely generated abstract group.

- (i) For each prime  $p$  the *weighted  $p$ -deficiency* of  $\Gamma$  is the quantity  $wdef(\Gamma_{\widehat{p}})$ , the weighted deficiency of the pro- $p$  completion of  $\Gamma$
- (ii) The *weighted deficiency* of  $\Gamma$  is the supremum of its weighted  $p$ -deficiencies over all primes  $p$ .

The reader may find this definition slightly unsatisfactory for two reasons. First, we indeed have to require that  $\Gamma$  is finitely generated for otherwise the pro- $p$  completion  $\Gamma_{\hat{p}}$  need not be countably based (and therefore falls outside of our considerations). Second, it may be desirable to define weighted deficiency of an abstract group  $\Gamma$  using only abstract presentations of  $\Gamma$  (not pro- $p$  presentations). In particular, instead of the weighted  $p$ -deficiency  $wdef(\Gamma_{\hat{p}})$  one may consider the related quantity  $wdef_p(\Gamma)$  defined as follows:

**Definition.** Let  $\Gamma$  be a finitely generated abstract group. For each prime  $p$  define  $wdef_p(\Gamma)$  to be the supremum of the set  $\{wdef_p(X, R)\}$  where  $(X, R)$  runs over all *abstract* presentations of  $\Gamma$  (with  $X$  finite), that is, presentations where  $R$  lies in the abstract group generated by  $X$ .

It is clear that  $wdef_p(\Gamma) \leq wdef(\Gamma_{\hat{p}})$ , but we do not know if the opposite inequality holds. In particular, we do not know if  $wdef_p(\Gamma) > 0$  whenever  $wdef(\Gamma_{\hat{p}}) > 0$ . However, the following weaker statement holds:

**Proposition 6.1.** *Let  $\Gamma$  be a finitely generated abstract group of positive weighted  $p$ -deficiency. Then  $\Gamma$  has a quotient  $\Gamma'$  with  $wdef_p(\Gamma') > 0$ .*

Proposition 6.1 (which will not be used in the sequel) is an easy consequence of the following observation which implies that one can replace a pro- $p$  presentation of a pro- $p$  group  $G$  by an abstract presentation for some quotient of  $G$  without increasing the total weight of relators by more than a given  $\varepsilon > 0$ .

**Observation 6.2.** *Let  $G$  be a pro- $p$  group,  $W$  a pseudo-valuation on  $G$  and  $\Gamma$  a dense abstract subgroup of  $G$ , and  $\varepsilon > 0$  a real number. Then any  $g \in G$  can be written as an infinite product  $g = \prod_{i=0}^{\infty} g_i$  s.t.*

$$(a) \text{ each } g_i \in \Gamma \quad (b) \sum_{i=1}^{\infty} W(g_i) \leq \varepsilon \quad (c) W(g_0) = W(g)$$

**Remark:** Recall that a pseudo-valuation is defined in the same way as a valuation except that we may have  $W(g) = 0$  for  $g \neq 1$ .

*Proof.* Straightforward. □

Suppose now that an abstract group  $\Gamma$  sits densely inside a pro- $p$  group  $G$  of PWD. If we manage to construct a quotient of  $G$  with some prescribed property  $(P)$ , we can consider the corresponding quotient of  $\Gamma$ , which will often have a property similar to  $(P)$ . However, the reverse transition (obtaining quotients of  $G$  from quotients of  $\Gamma$ ) may not be possible unless we know that  $G$  is the pro- $p$  completion of  $\Gamma$ . Our next result essentially resolves this problem.

**Theorem 6.3.** *Let  $G$  be a pro- $p$  group,  $W$  a finite valuation of  $G$  and  $\Gamma$  a dense abstract subgroup of  $G$ . Then for any  $\varepsilon > 0$  there exists a normal subgroup  $N$  of  $G$  such that  $def_W(G/N) \geq def_W(G) - \varepsilon$  and  $G/N$  is naturally isomorphic to the pro- $p$  completion of  $\Gamma N/N$ .*

*Proof.* First, by Lemma 4.14 without loss of generality we can assume that  $\Gamma$  is finitely generated. Let  $\psi : \Gamma_{\hat{p}} \rightarrow G$  be the epimorphism induced by the embedding of  $\Gamma$  into  $G$ . Choose a countable set  $\{r_1, r_2, \dots\}$  of normal generators of  $\text{Ker } \psi$ . Applying Observation 6.2 to the pseudo-valuation  $W \circ \psi$  on  $\Gamma_{\hat{p}}$  we deduce that there are elements  $\{r_{i,j} \in \Gamma\}_{i \geq 1, j \geq 0}$  such that

- (i)  $r_i = r_{i,0}r_{i,1}r_{i,2}\dots$
- (ii)  $\sum_{j=1}^{\infty} W(\psi(r_{i,j})) \leq \frac{\varepsilon}{2^i}$  for  $i \geq 1$ .

Let  $N$  be the normal subgroup of  $G$  generated by  $S = \{\psi(r_{i,j})\}_{i,j \geq 1}$ . Then  $W(S) \leq \varepsilon$  and so  $\text{def}_W(G/N) \geq \text{def}_W(G) - \varepsilon$  by Lemma 4.12.

We claim that the natural epimorphism  $(\Gamma N/N)_{\hat{p}} \rightarrow G/N$  is an isomorphism. Since  $G/N$  is hopfian (being a finitely generated pro- $p$  group), it suffices to construct a (continuous) epimorphism  $G/N \rightarrow (\Gamma N/N)_{\hat{p}}$ . By definition of  $N$  we have

$$G/N \cong \Gamma_{\hat{p}} / \langle \{r_i, r_{i,j}\}_{i,j \geq 1} \rangle^{\Gamma_{\hat{p}}} = \Gamma_{\hat{p}} / \langle \{r_{i,j}\}_{i \geq 1, j \geq 0} \rangle^{\Gamma_{\hat{p}}} \cong (\Gamma/K)_{\hat{p}},$$

where  $K = \langle \{r_{i,j}\}_{i \geq 1, j \geq 0} \rangle^{\Gamma}$ . On the other hand,  $K$  is clearly contained in  $\Gamma \cap N$ , so  $(\Gamma/K)_{\hat{p}}$  surjects onto  $(\Gamma/\Gamma \cap N)_{\hat{p}} \cong (\Gamma N/N)_{\hat{p}}$ .  $\square$

**Corollary 6.4.** *Let  $\Gamma$  be a dense abstract subgroup of a pro- $p$  group of positive weighted deficiency. Then  $\Gamma$  has a quotient with positive weighted deficiency.*

Similarly to pro- $p$  groups, analysis of abstract groups of positive weighted deficiency (PWD) can be easily extended to abstract groups of positive virtual weighted deficiency (PVWD) defined below.

**Definition.** Let  $\Gamma$  be a finitely generated abstract group,  $\Lambda$  a finite index subgroup of  $\Gamma$  and  $p$  a prime. Consider the topology on  $\Gamma$  whose base of neighborhoods of identity consists of (all) normal subgroups of  $\Lambda$  of  $p$ -power index. The completion of  $\Gamma$  in this topology will be called the *virtual pro- $p$  completion of  $\Gamma$  relative to  $\Lambda$*  and denoted by  $\Gamma_{\hat{p}, \Lambda}$ . Note that the canonical image of  $\Lambda$  in  $\Gamma_{\hat{p}, \Lambda}$  is naturally isomorphic to  $\Lambda_{\hat{p}}$ .

**Definition.** Let  $\Gamma$  be a finitely generated abstract group.

- (a) Given a prime  $p$ , we will say that  $\Gamma$  has *positive virtual weighted  $p$ -deficiency* if there is a finite index normal subgroup  $\Lambda$  of  $\Gamma$  such that if  $G = \Gamma_{\hat{p}, \Lambda}$  and  $U = \Lambda_{\hat{p}}$  (considered as a subgroup of  $G$ ), then  $\text{def}_W^G(U) > 0$  for some  $G$ -invariant valuation  $W$  on  $U$ . Any such  $\Lambda$  will be referred to as an *invariant PWD subgroup* of  $\Gamma$ .
- (b) We will say that  $\Gamma$  has *positive virtual weighted deficiency (PVWD)* if  $\Gamma$  has positive virtual weighted  $p$ -deficiency for some  $p$ .

Here is the “virtual” version of Corollary 6.4 (whose proof is analogous):

**Theorem 6.5.** *Let  $\Gamma$  be a dense abstract subgroup of a pro- $p$  group of PVWD. Then  $\Gamma$  has a quotient with PVWD.*

We note that there are many *natural* examples of PVWD abstract groups which do not have PWD. The key to producing such examples is the following observation:

**Observation 6.6.** *Let  $\Gamma$  be a finitely generated abstract group and  $\Lambda$  a finite index normal subgroup of  $\Gamma$ . Suppose that  $\text{def}_W(\Lambda_{\hat{p}}) > 0$  for some uniform valuation  $W$  on  $\Lambda_{\hat{p}}$ . Then  $\Gamma$  has positive virtual weighted  $p$ -deficiency.*

*Proof.* This follows directly from Observation 4.8.  $\square$

**Corollary 6.7.** *The following classes of groups have positive virtual weighted  $p$ -deficiency for every prime  $p$ :*

- (a) *virtually free groups (which are not virtually cyclic)*
- (b) *fundamental groups of (orientable) hyperbolic 3-manifolds*

*Proof.* (a) is clear by Observation 6.6.

(b) This fact is essentially known, but for completeness we give a proof. Fix a prime  $p$ , and let  $\Gamma$  be the fundamental group of a hyperbolic 3-manifold  $M$ . It is well known that  $\Gamma$  has a presentation  $\langle X|R \rangle$  with  $|X| - |R| = 0$  or  $1$ , and the same is true for all finite index subgroups of  $\Gamma$  (see e.g. [Lu1]). Moreover,  $\Gamma$  is linear and not virtually solvable, so [Lu2, Theorem B] implies that for each  $n \in \mathbb{N}$  there is a finite index subgroup  $\Lambda$  of  $\Gamma$  such that  $d(\Lambda_{\hat{p}}) \geq n$ .

Let  $d = d(\Lambda_{\hat{p}})$ . By [Lu1, Lemma 1.1],  $\Lambda_{\hat{p}}$  has a pro- $p$  presentation  $\langle X'|R' \rangle$  where  $|X'| - |R'| = 0$  or  $1$ ,  $|X'| = d$  and all elements of  $R'$  lie in the Frattini subgroup. Now let  $\widetilde{W}$  be the uniform weight function on  $F(X')$  such that  $\widetilde{W}(x) = \frac{1}{2}$  for all  $x \in X'$ , and let  $W$  be the induced valuation on  $G$ . Then  $\text{def}_W(G) \geq \widetilde{W}(X') - \widetilde{W}(R') - 1 \geq d(\frac{1}{2} - \frac{1}{4}) - 1 = \frac{d}{4} - 1$ , so in particular  $\text{def}_W(G) > 0$  if  $d \geq 5$ .

If  $\Lambda$  is normal in  $\Gamma$ , we are done by Observation 6.6. In the general case we proceed as follows. Assume (as we may) that  $d \geq 20$ , so that  $\text{def}_W(G) \geq 4$ , and let  $\Delta$  be a finite index subgroup of  $\Lambda$  which is normal in  $\Gamma$ . Let  $H$  be the closure of the image of  $\Delta$  in  $G$ . Then  $\text{def}_W(H) \geq \text{def}_W(G) \geq 4$  by Proposition 4.1, whence  $d(H) \geq \text{def}_W(H) + 1 \geq 5$  by Lemma 4.1(b). Since  $\Delta_{\hat{p}}$  surjects onto  $H$ , we have  $d(\Delta_{\hat{p}}) \geq 5$ . We can now finish the proof by applying the earlier argument to  $\Delta$  instead of  $\Lambda$ .  $\square$

## 7. LERF QUOTIENTS OF PWD GROUPS

### 7.1. Property LERF.

**Definition.** A finitely generated abstract group  $\Gamma$  is said to be *LERF* if every finitely generated subgroup of  $\Gamma$  is closed in the profinite topology.

Historically, the groups first shown to be LERF were the free groups – this follows from a classical result of Hall. However, property LERF was not formally introduced and studied until very recently when it naturally arose in several problems in geometric group theory and 3-manifold topology. We refer the reader to [LLR] and [LR] for a discussion of some geometric applications of LERF.

Many important classes of groups do not have LERF – for instance, if  $\Gamma$  is any group, which has a proper finitely generated subgroup that is dense in the profinite topology on  $\Gamma$ , then  $\Gamma$  cannot be LERF, and the former property holds, for instance, in any arithmetic group with the congruence subgroup property. On the contrary, proving that a given group does have LERF is usually difficult, and the list known examples of LERF groups is not very long.

In this section we prove Theorem 1.2 whose statement (in a slightly generalized form) is recalled below as Theorem 7.1. It produces a large class of LERF groups constructed as quotients of PWD groups.

**Theorem 7.1.** *Let  $\Gamma$  be a group of positive weighted  $p$ -deficiency. Then  $\Gamma$  has a quotient which is LERF and  $p$ -torsion and still has positive weighted  $p$ -deficiency.*

While Theorem 7.1 probably cannot be used to prove LERF for any naturally defined group, it does have interesting applications. For instance, it yields an answer to a question of Long and Reid about the existence of LERF groups with property (T) (see [LR, Question 4.5]):

**Corollary 7.2.** *There exists a LERF group with property (T).*

Corollary 7.2 is a direct consequence of Theorem 7.1 and the existence of Kazhdan groups of positive weighted deficiency established in [Er].



### 7.2. Properties $p$ -LERF and $p$ -WLERF and a useful corollary of Theorem 7.1.

In this subsection we establish a simple consequence of Theorem 7.1 (see Corollary 7.4 below) which will be needed for the proofs of Corollary 1.6 and Theorem 1.8.

**Definition.** Let  $\Gamma$  be a finitely generated group. We will say that  $\Gamma$  is  $p$ -LERF if every finitely generated subgroup of  $\Gamma$  is closed in the pro- $p$  topology.

In general being  $p$ -LERF is stronger than being LERF; however, a  $p$ -torsion group which is LERF is automatically  $p$ -LERF. Thus, groups in the conclusion of Theorem 7.1 have  $p$ -LERF.

Neither of the properties LERF and  $p$ -LERF is preserved by quotients; however, there is a weaker version of  $p$ -LERF which is preserved by quotients.

**Definition.** Let  $\Gamma$  be a finitely generated group. We will say that  $\Gamma$  is *weakly*  $p$ -LERF (or  $p$ -WLERF) if for any infinite index subgroup  $\Lambda$  of  $\Gamma$  its pro- $p$  closure  $\overline{\Lambda}$  is also of infinite index in  $\Gamma$  (note that  $\Lambda$  is not assumed to be finitely generated).

**Lemma 7.3.** *The following hold:*

- (a) *The property  $p$ -WLERF is inherited by quotients*
- (b) *Every  $p$ -LERF group is  $p$ -WLERF*
- (c) *An infinite  $p$ -WLERF group has infinite pro- $p$  completion.*

*Proof.* (a) Let  $\pi : \Gamma \rightarrow \Gamma'$  be an epimorphism, and assume that  $\Gamma$  is  $p$ -WLERF. Let  $\Lambda$  be an infinite index subgroup of  $\Gamma'$ . Then  $\pi^{-1}(\Lambda)$  is an infinite index subgroup of  $\Gamma$ , so by assumption there exists an infinite strictly descending chain  $H_1 \supset H_2 \supset \dots$  of open (in the pro- $p$  topology) subgroups of  $\Gamma$  with  $\pi^{-1}(\Lambda) \subset H_i$  for all  $i$ . Note that  $\pi$  maps open subgroups to open subgroups (since open subgroups are precisely subnormal subgroups of  $p$ -power index), so  $\pi(H_1) \supset \pi(H_2) \supset \dots$  is an infinite strictly descending chain of open subgroups of  $\Gamma'$  containing  $\Lambda$ . Thus, the pro- $p$  closure of  $\Lambda$  has infinite index in  $\Gamma'$ , and we have shown that  $\Gamma'$  is  $p$ -WLERF.

(b) Let  $\Gamma$  be  $p$ -LERF and  $\Lambda$  a subgroup of  $\Gamma$  whose pro- $p$  closure  $\overline{\Lambda}$  is of finite index in  $\Gamma$ . Then the group  $\overline{\Lambda}/[\overline{\Lambda}, \overline{\Lambda}]\overline{\Lambda}^p$  is finite, so there exists a finitely generated subgroup  $\Lambda'$  of  $\Lambda$  which surjects onto  $\overline{\Lambda}/[\overline{\Lambda}, \overline{\Lambda}]\overline{\Lambda}^p$ . By a standard pro- $p$  argument the latter implies that  $\overline{\Lambda} = \overline{\Lambda}'$ . Since  $\Lambda'$  is finitely generated and  $\Gamma$  is  $p$ -LERF, we must have  $\overline{\Lambda}' = \Lambda'$ . Hence  $\overline{\Lambda} = \overline{\Lambda}' = \Lambda' \subseteq \Lambda$ , and thus  $\Lambda = \overline{\Lambda}$  must be of finite index.

(c) follows by applying the definition of  $p$ -WLERF to the trivial subgroup.  $\square$

**Corollary 7.4.** *Let  $\Gamma$  be a  $p$ -LERF group. Then any infinite quotient of  $\Gamma$  has infinite pro- $p$  completion.*

Combining Theorem 7.1 and Corollary 7.4 we can now prove Theorem 1.3:

*Proof of Theorem 1.3.* Let  $\Gamma$  be an abstract group of positive weighted  $p$ -deficiency. By Theorem 7.1  $\Gamma$  has a  $p$ -LERF quotient  $\Gamma'$  which still has positive weighted  $p$ -deficiency. By [EJ, Theorem 4.6]  $\Gamma'$  has an infinite quotient  $\Omega$  with  $(T)$ , and by Corollary 7.4  $\Omega$  has infinite pro- $p$  completion. Replacing  $\Omega$  by its image in the pro- $p$  completion, we obtain an infinite residually- $p$  group with  $(T)$  which is a quotient of  $\Gamma$ .  $\square$

### 7.3. Proof of Theorem 7.1.

*Proof of Theorem 7.1.* First, by Lemma 4.14 and Corollary 6.4 we can assume that  $\Gamma$  is  $p$ -torsion, so the pro- $p$  topology and profinite topology on  $\Gamma$  (and any of its quotients) coincide. Let  $G = \Gamma_{\hat{p}}$  be the pro- $p$  completion of  $\Gamma$ , so by assumption  $\text{def}_W(G) > 0$

for some finite valuation  $W$ . Replacing  $\Gamma$  by its image in  $G$ , we can assume that  $\Gamma$  is a subgroup of  $G$ .

We start by reformulating the assertion of the theorem in a language more convenient for us. Let  $\mathcal{P}$  be the set of pairs  $(g, \Lambda)$  where  $g \in \Gamma$  and  $\Lambda$  is a finitely generated subgroup of  $\Gamma$ .

If  $H$  is any pro- $p$  group defined as a quotient of  $G$ , let  $\pi_H : G \rightarrow H$  be the natural projection and  $N_H = \text{Ker } \pi_H$ . Given a pair  $(g, \Lambda) \in \mathcal{P}$ , we will say that

- $(g, \Lambda)$  is of *type I* for  $H$  if  $\pi_H(g)$  lies in  $\pi_H(\Lambda)$
- $(g, \Lambda)$  is of *type II* for  $H$  if  $\pi_H(g)$  does not lie in the closure of  $\pi_H(\Lambda)$
- $(g, \Lambda)$  is of *type III* for  $H$  if  $\pi_H(g)$  does not lie in  $\pi_H(\Lambda)$ , but lies in its closure.

To prove Theorem 7.1 it will be enough to establish the following claim:

**Claim 7.5.** *The group  $G$  has a normal subgroup  $N$  s.t.*

- (1)  $\text{def}_W(G/N) > 0$
- (2)  $N$  is normally generated by a subset of  $\Gamma$
- (3) all pairs in  $\mathcal{P}$  are of type I or II for  $G/N$

Indeed, since  $G$  is the pro- $p$  completion of  $\Gamma$ , condition (2) implies that  $G/N$  is the pro- $p$  completion of  $\Gamma N/N$ . Thus,  $\Gamma N/N$  has PWD by (1), and the pro- $p$  topology on  $\Gamma N/N$  is induced from  $G/N$ . Therefore,  $\Gamma N/N$  has LERF by (3).

If all pairs in  $\mathcal{P}$  are of type I or II for  $G$ , there is nothing to prove. Here is the basic technique for eliminating pairs of type III. If a pair  $(g, \Lambda)$  is of type III for some pro- $p$  group  $H$ , then for any  $\varepsilon > 0$  we can write  $g = lnr$  where  $l \in \Lambda$ ,  $n \in N_H$  and  $W(r) < \varepsilon$ . If we now impose the relation  $r = 1$ , that is, consider the group

$$H' = H / \langle \pi_H(r) \rangle^H \cong G / \langle N_H, r \rangle^G,$$

then  $\pi_{H'}(g) = \pi_{H'}(l)$ , and so  $(g, \Lambda)$  is of type I for  $H'$ .

Since there are only countably many pairs in  $\mathcal{P}$ , this technique enables us to replace the group  $G$  with its quotient  $G'$  which still has positive weighted deficiency and such that all type III pairs for  $G$  have type I for  $G'$ . The problem is that this process may create new type III pairs, that is, some of type II pairs for  $G$  may be of type III for  $G'$ . Another problem is that the defining relations in the obtained presentation of  $G'$  do not necessarily come from  $\Gamma$ , and thus  $G'$  may not coincide with the pro- $p$  completion of the abstract group  $\pi_{G'}(\Gamma)$ . To avoid these problems we proceed slightly differently.

First, we need one more notation. Given a pro- $p$  quotient  $H$  of  $G$ , define the pseudo-distance function  $d_H$  on  $G$  by

$$d_H(a, b) = \inf \{ W(g) : \pi_H(g) = \pi_H(a^{-1}b) \}.$$

Given subsets  $A$  and  $B$  of  $G$ , we set  $d_H(A, B) = \inf \{ d_H(a, b) : a \in A, b \in B \}$ . Note that a pair  $(g, \Lambda)$  is of type III for  $H$  if and only if  $\pi_H(g) \notin \pi_H(\Lambda)$  and  $d_H(g, \Lambda) = 0$ .

Fix  $\varepsilon < \text{def}_W(G)$ . We shall construct a sequence of pro- $p$  groups and epimorphisms  $G = G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow \dots$  with the following properties:

- (a)  $G_i = G_{i-1} / \langle R_{i-1} \rangle^{G_{i-1}}$  where  $R_{i-1}$  is a subset of  $\pi_{G_{i-1}}(\Gamma)$
- (b)  $\text{def}_W(G_i) > \text{def}_W(G) - \varepsilon$
- (c) For each pair  $(g, \Lambda) \in \mathcal{P}$  one of the following holds:
  - (i) There exists  $n$  such that  $(g, \Lambda)$  is of type I for  $G_n$  (hence also of type I for  $G_i$  for all  $i \geq n$ )
  - (ii)  $(g, \Lambda)$  is of type II for  $G_n$  for all  $n$ , and  $\inf_n d_{G_n}(g, \Lambda) > 0$ .

We now describe the construction. First we enumerate all pairs in  $\mathcal{P}$ :  $(g_1, \Lambda_1), (g_2, \Lambda_2), \dots$ . Fix  $n \in \mathbb{N}$ , and suppose we have constructed pro- $p$  groups  $G_1, \dots, G_n$  and their presentations such that

- (1) (a) and (b) hold for all  $i \leq n$ ,
- (2) each of the pairs  $(g_1, \Lambda_1), \dots, (g_n, \Lambda_n)$  is of type I or II for  $G_n$ .

We now construct  $G_{n+1}$  so that (1) and (2) still hold with  $n$  replaced by  $n+1$ . If  $(g_{n+1}, \Lambda_{n+1})$  is of type I or II for  $G_n$ , we set  $R_n = \emptyset$  and  $G_{n+1} = G_n$ . If  $(g_{n+1}, \Lambda_{n+1})$  is of type III for  $G_n$ , choose  $\delta > 0$  such that

- (3)  $\text{def}_W(G_n) - \delta > \text{def}_W(G) - \varepsilon$  and
- (4)  $\delta < d_{G_n}(g_i, \Lambda_i)$  for all  $i \leq n$  such that  $(g_i, \Lambda_i)$  has type II for  $G_n$ .

Since  $(g_{n+1}, \Lambda_{n+1})$  is of type III for  $G_n$ , we can write  $g_{n+1} = lnr$  where  $l \in \Lambda_{n+1}$ ,  $n \in N_{G_n}$  and  $W(r) < \delta/2$ . By Observation 6.2 we can write  $r$  as an infinite converging product  $r = \prod_{j=1}^{\infty} r_j$  with  $r_j \in \Gamma$  such that  $\sum_j W(r_j) < \delta$ . Let  $R_n = \cup \{\pi_{G_n}(r_j)\}_{j=1}^{\infty}$  and let

$$G_{n+1} = G_n / \langle R_n \rangle^{G_n}.$$

Condition (a) holds for  $i = n+1$  by construction, and condition (b) holds for  $i = n+1$  by Lemma 4.12. By construction, the pair  $(g_{n+1}, \Lambda_{n+1})$  has type I for  $G_{n+1}$ . Condition (4) ensures that each of the pairs  $(g_i, \Lambda_i)$  with  $i \leq n$  which had type II for  $G_n$  will also have type II for  $G_{n+1}$ , and moreover,  $d_{G_n}(g_i, \Lambda_i) = d_{G_{n+1}}(g_i, \Lambda_i)$  for each such pair. The latter ensures that the obtained sequence of groups  $\{G_n\}$  satisfies condition (c).

Thus, we have constructed a sequence of groups  $\{G_n\}$  satisfying (a)-(c). Let  $N = \bigcup_{n=1}^{\infty} N_{G_n}$ . By construction  $N$  is normally generated by a subset of  $\Gamma$ , condition (b) implies that  $\text{def}_W(G/N) > 0$ , and (c) ensures that all pairs in  $\mathcal{P}$  are of type I or II for  $G/N$ . Thus, we proved Claim 7.5 and hence also proved Theorem 7.1.  $\square$

We finish this section with the “virtual version” of Theorem 7.1:

**Theorem 7.6.** *Let  $\Gamma$  be an abstract group of positive virtual weighted  $p$ -deficiency with invariant PWD subgroup  $\Lambda$ . Then there is an epimorphism  $\pi : \Gamma \rightarrow \Gamma'$  s.t.  $\Gamma'$  has positive virtual weighted  $p$ -deficiency with invariant PWD subgroup  $\pi(\Lambda)$ , and  $\pi(\Lambda)$  is  $p$ -LERF.*

*Proof.* The proof is similar to that of Theorem 7.1, apart from a few minor changes described below.

Let  $G = \Gamma_{p, \Lambda}^{\wedge}$  be the virtual pro- $p$  completion of  $\Gamma$  relative to  $\Lambda$  and  $U = \Lambda_{\hat{p}}$  (thought of as a subgroup of  $G$ ). By definition  $\text{def}_W^G(U) > 0$  for some  $G$ -invariant valuation  $W$  on  $U$ . As usual, we can assume without loss of generality that  $\Lambda$  is  $p$ -torsion.

Essentially repeating the proof of Claim 7.5, we construct a closed normal subgroup  $N$  of  $G$ , with  $N \subset U$ , such that  $\text{def}_W^{G/N}(U/N) > 0$ ,  $U/N \cong (\Lambda N/N)_{\hat{p}}$  and  $\Lambda N/N$  is  $p$ -LERF. Now let  $\Gamma' = \Gamma N/N$ . It is easy to check that  $G/N$  is the virtual pro- $p$  completion of  $\Gamma'$  relative to  $\Lambda N/N$ . Thus, by definition of positive virtual weighted  $p$ -deficiency for abstract groups  $\Gamma'$  has the required property.  $\square$

## 8. PROOFS OF THE MAIN RESULTS

In this section we prove Theorem 1.5, Corollary 1.6 and Theorem 1.7 generalized to groups of positive virtual weighted deficiency.

**8.1. Constructing locally zero-one groups.** We start with a simple corollary of Lemma 4.13.

**Corollary 8.1.** *Let  $G$  be a profinite group and  $W$  a finite virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$ . Let  $\Gamma$  be a dense abstract subgroup of  $G$  and  $\Lambda$  a subgroup of  $\Gamma$  whose closure  $\overline{\Lambda}$  is open in  $G$ . Then for any  $\varepsilon > 0$  there exists a normal subgroup  $N$  of  $G$  such that if  $\pi : G \rightarrow G/N$  is the natural projection, then*

- (i)  $\text{def}_W^{\pi(G)}(\pi(U)) \geq \text{def}_W^G(U) - \varepsilon$
- (ii)  $\pi(\Lambda) = \pi(\overline{\Lambda} \cap \Gamma)$ , so in particular  $\pi(\Lambda)$  is of finite index in  $\pi(\Gamma)$

**Remark:** Note that Theorem 5.1 and Corollary 8.1 immediately imply Theorem 1.1 (in fact, a stronger version of it).

We now prove Theorem 1.5 using iterated applications of Theorem 5.1 and Corollary 8.1.

**Theorem 8.2** (generalization of Theorem 1.5). *Let  $G$  be a virtually pro- $p$  group of positive virtual weighted deficiency and  $\Gamma$  a dense abstract subgroup of  $G$ . Then  $G$  has a quotient  $H$  such that the image  $\Omega$  of  $\Gamma$  in  $H$  is a locally zero-one group.*

**Remark:** Note that  $\Omega$  is automatically residually finite since  $H$  is profinite.

*Proof.* First, by Lemma 4.14, we can assume that  $\Gamma$  is finitely generated and virtually  $p$ -torsion. Let  $\{\Lambda_i\}_{i=1}^{\infty}$  be the set of finitely generated subgroups of  $\Gamma$  (ordered arbitrarily). By Theorem 5.1 we can construct a sequence of pro- $p$  groups  $G = G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow \dots$  such that

- (1)  $G_{i+1}$  is a quotient of  $G_i$  for each  $i$
- (2) Each  $G_i$  has positive virtual weighted deficiency
- (3) If  $\pi_i : G_0 \rightarrow G_i$  is the natural projection, then either  $\pi_i(\Lambda_i)$  is finite or the closure of  $\pi_i(\Lambda_i)$  in  $G_i$  is open.

Corollary 8.1 enables us to replace condition (3) by its stronger version (3') below while preserving (1) and (2):

- (3')  $\pi_i(\Lambda_i)$  is either finite or has finite index in  $\pi_i(\Gamma)$ .

Now let  $N_i = \text{Ker } \pi_i$ , let  $N_{\infty} = \overline{\bigcup_{i=1}^{\infty} N_i}$  and  $G_{\infty} = G_0/N_{\infty}$ . Since each of the groups  $G_i$  is infinite by (2) and  $G_0$  is finitely generated,  $G_{\infty}$  must also be infinite. Let  $\pi_{\infty} : G_0 \rightarrow G_{\infty}$  be the natural projection and  $\Omega = \pi_{\infty}(\Gamma)$ . By construction  $\Omega$  is an infinite torsion group (in particular, it is not virtually cyclic), and condition (3') implies that every finitely generated subgroup of  $\Omega$  is finite or of finite index. Thus  $\Omega$  is a locally zero-one group.  $\square$

We finish this subsection with a brief discussion of the problem of existence of residually finite globally zero-one groups mentioned in the introduction. This is a well known question (see e.g. [SW, Question 1] and [Kou, Problem 12.43]), which appears to be very difficult. As a special case, one may ask if for a given prime  $p$  there exist  $p$ -torsion groups with this property. We note that there are no such groups for  $p = 2$  – this fact is well known to experts and follows, for instance, from Shunkov's theorem [Shu] which asserts that if  $\Gamma$  is a torsion group containing an involution with finite centralizer, then  $\Gamma$  is virtually solvable. For completeness we will give a short self-contained proof.

**Proposition 8.3.** *There is no residually finite globally zero-one group which is 2-torsion.*

*Proof.* We start with a general claim:

**Claim 8.4.** *If  $\Gamma$  is a 2-group and  $f$  is an automorphism of  $\Gamma$  of order 2, then  $\Gamma$  has a (non-trivial) fixed point.*

*Proof.* Take any  $g \in \Gamma \setminus \{1\}$ . If  $f(g) = g$ , we are done. If  $f(g) \neq g$ , then  $f(g)g^{-1}$  has even order  $n$ , and direct computation shows that  $f$  fixes  $(f(g)g^{-1})^{n/2}$   $\square$

Now let  $\Gamma$  be a residually finite globally zero-one group which is 2-torsion. We claim that for any  $g \in \Gamma$  of order 2, the centralizer  $C_\Gamma(g)$  is infinite. If not, then by residual finiteness of  $\Gamma$  we can find a normal subgroup  $\Lambda$  of  $\Gamma$  such that  $\{g\} \cap \Lambda = \{1\}$ . This is impossible by the above claim. Thus, since  $\Gamma$  is globally zero-one,  $C_\Gamma(g)$  has finite index for any  $g \in \Gamma$  of order 2.

Now let  $N$  be the subgroup of  $\Gamma$  generated by all elements of order 2. Then  $N$  is clearly infinite, so it is also of finite index (and finitely generated). It follows that the centralizer of  $N$  in  $\Gamma$  is of finite index. Thus  $\Gamma$  is virtually abelian and hence finite, a contradiction.  $\square$

**8.2. Constructing hereditarily just-infinite quotients.** We start by formulating Grigorchuk's version of Wilson's classification theorem for just-infinite groups (see [Gr2] and [Will] for more details).

**Theorem 8.5.** (a) *Let  $\Gamma$  be a just-infinite abstract group. Then one of the following holds:*

- (i)  $\Gamma$  is virtually simple
- (ii)  $\Gamma$  is hereditarily just-infinite
- (iii)  $\Gamma$  has a finite index subgroup isomorphic to a direct power  $\Lambda^n$  where  $n \geq 2$  and  $\Lambda$  is simple or hereditarily just-infinite
- (iv)  $\Gamma$  is a branch group.

(b) *Let  $G$  be a just-infinite profinite group. Then one of the following holds:*

- (i)  $G$  is hereditarily just-infinite
- (ii)  $G$  has an open subgroup isomorphic to a direct power  $L^n$  where  $n \geq 2$  and  $L$  is hereditarily just-infinite
- (iii)  $G$  is a branch group.

**Remark:** By definition any branch group also has a finite index subgroup which is a (non-trivial) direct power of another group, and thus Theorem 8.5 yields the classification statement from § 1.5.

Theorem 8.5 is a very minor refinement of [Gr2, Theorem 3]. The only difference is that [Gr2, Theorem 3] leaves the possibility that there exist just-infinite groups which are virtually hereditarily just-infinite, but not hereditarily just-infinite. That this cannot happen was shown by C. Reid [Re, Lemma 5].

We proceed with the proofs of (generalized versions of) Corollary 1.6 and Theorem 1.7.

**Observation 8.6.** *Let  $\Gamma$  be a residually finite locally zero-one group. Then  $\Gamma$  is LERF.*

*Proof.* Finite index subgroups are open (hence closed) in the profinite topology by definition, and finite subgroups are closed since  $\Gamma$  is residually finite.  $\square$

**Corollary 8.7** (generalization of Corollary 1.6). *Let  $\Gamma$  be an abstract group of positive virtual weighted  $p$ -deficiency. Then  $\Gamma$  has a hereditarily just-infinite quotient, which is virtually  $p$ -torsion.*

*Proof.* Let  $\Lambda$  be an invariant PWD subgroup of  $\Gamma$  and  $G = \widehat{\Gamma}_{p,\Lambda}$  be the virtual pro- $p$  completion of  $\Gamma$  relative to  $\Lambda$ . Now apply Theorem 8.2 to  $G$  and  $\Gamma$ , and let  $\Omega$  be the locally zero-one quotient of  $\Gamma$  constructed in the proof. Note that  $\Omega$  is virtually  $p$ -torsion by construction and LERF by Observation 8.6. In particular,  $\Omega$  is virtually  $p$ -LERF.

Let  $\Omega'$  be any just-infinite quotient of  $\Omega$ . Then  $\Omega'$  is also locally zero-one, and by Corollary 7.4,  $\Omega'$  has infinite profinite completion.

By Theorem 8.5(a) (and a remark after it)  $\Omega'$  is virtually simple, hereditarily just-infinite, or a finite extension of a direct power of some infinite group. Since  $\Omega'$  is a locally zero-one group, it clearly cannot have the third type, and since  $\Omega'$  has infinite profinite completion, it cannot be virtually simple. Hence  $\Omega'$  is hereditarily just-infinite.  $\square$

In order to prove Theorem 1.7 (the pro- $p$  version of Corollary 1.6), we need one more lemma.

Let  $H$  be a virutally pro- $p$  group. We will say that  $H$  satisfies condition (\*\*) if

- (\*\*) For every positive integer  $k$  there exists an open pro- $p$  subgroup  $V(k)$  of  $H$  and a dense abstract subgroup  $\Lambda(k)$  of  $V(k)$  such all  $k$ -generated subgroups of  $\Lambda(k)$  are finite.

**Lemma 8.8.** *Let  $G$  be a virtually pro- $p$  group of positive virtual weighted deficiency. Then  $G$  has a quotient  $G'$  of positive virtual weighted deficiency such that all quotients of  $G'$  satisfy condition (\*\*).*

*Proof.* Fix a dense abstract subgroup  $\Gamma$  of  $G$ . Let  $W$  be a finite virtual valuation of  $G$  defined on an open normal pro- $p$  subgroup  $U$  for which  $\text{def}_W^G(U) > 0$ .

By the remark following Lemma 5.2 we can find a normal subgroup  $N$  of  $G$  such that  $\text{def}_W^{G/N}(U/N) > 0$  and for every finite set  $Y \subset \Gamma \cap U$ , with  $W(Y) < 1$ , the image of  $\langle Y \rangle$  in  $G/N$  is finite. Let  $G' = G/N$ .

Note that for each  $k$  the group  $U_{<1/k} = \{g \in U : W(g) < 1/k\}$  is open in  $G$ , and for each  $Y \subset U_{<1/k}$ , with  $|Y| = k$ , we have  $W(Y) < 1$ . Hence any quotient  $H$  of  $G'$  satisfies (\*\*) where we let  $V(k)$  be the image of  $U_{<1/k}$  in  $H$  and  $\Lambda(k)$  the image of  $\Gamma \cap U_{<1/k}$  in  $H$ .  $\square$

With the aid of Lemma 8.8, we can now prove Theorem 1.7.

**Theorem 8.9** (generalization of Theorem 1.7). *Let  $G$  be a virtually pro- $p$  group of positive virtual weighted deficiency. Then  $G$  has a hereditarily just-infinite quotient.*

*Proof.* First, by Lemma 8.8 we can assume that all quotients of  $G$  satisfy condition (\*\*). By Theorem 8.2 there exists a quotient  $H$  of  $G$  which contains a dense locally zero-one subgroup. Replacing  $H$  by its quotient (if necessary), we can assume that  $H$  is just-infinite.

Suppose that  $H$  is not hereditarily just-infinite. Then by Theorem 8.5(b) there exists an open subgroup  $M$  of  $H$  which decomposes as  $M = A \times B$  with  $A$  and  $B$  infinite. Let  $\pi_A : M \rightarrow A$  and  $\pi_B : M \rightarrow B$  be the projection maps.

Let  $k = d(A)$  be the minimal number of (topological) generators of  $A$ . Since by construction  $H$  satisfies (\*\*), there exists an open pro- $p$  subgroup  $V$  of  $H$  and a dense abstract subgroup  $\Lambda$  of  $V$  such that

$$(8.1) \quad \text{all } k\text{-generated subgroups of } \Lambda \text{ are finite.}$$

By making  $V$  smaller (if necessary) we can assume that  $V = C \times E$  where  $C \subseteq A$  and  $E \subseteq B$ .

Since  $\Lambda$  is dense in  $V$ , its projection  $\pi_B(\Lambda)$  is dense in  $\pi_B(V) = E$ . Hence

$E$  has no open subgroups which are (topologically) generated by  $k$  elements.

Indeed, if  $E$  had an open  $k$ -generated subgroup  $O$ , then since  $O$  is a pro- $p$  group, we could find a generating  $k$ -tuple for  $O$  inside  $\pi_B(\Lambda)$ , which is impossible by (8.1).

Now consider the group  $Z = A \times E$ . Recall that  $H$  contains a dense locally zero-one subgroup, call it  $\Omega$ . Since  $\Omega \cap Z$  is dense in  $Z$  and  $k = d(A)$ , there exists a  $k$ -generated subgroup  $K \subseteq \Omega \cap Z$  such that  $\pi_A(K)$  is dense in  $A$ . In particular,  $K$  must be infinite. Since  $\Omega$  is locally zero-one, we conclude that  $K$  has finite index in  $\Omega$ , so its closure  $\overline{K}$  must be open in  $H$ . But then  $\overline{\pi_B(K)} \supseteq \pi_B(\overline{K})$  is an open subgroup of  $\pi_B(Z) = E$ . Hence  $\overline{\pi_B(K)}$  is an open subgroup of  $E$  topologically generated by  $k$  elements. This contradicts our earlier conclusion.  $\square$

As an immediate consequence of the proof of Theorem 8.9, we obtain a positive answer to Problem 20 from Chapter I in [dSSS]:

**Corollary 8.10.** *There exist hereditarily just-infinite pro- $p$  groups of infinite lower rank.*

Our last result in this section shows that there is a large class of groups which cannot have hereditarily just-infinite quotients, except for virtually cyclic ones. Combining this result with Theorems 8.7 and 8.9 we deduce that branch groups cannot have positive virtual weighted deficiency.

**Proposition 8.11.**

- (a) *Let  $G$  be a virtually pro- $p$  (resp. abstract) group, and assume that some open (resp. finite index) normal subgroup  $H$  of  $G$  is isomorphic to  $L_1 \times \dots \times L_k$  where  $k \geq 2$ , all  $L_i$ 's are isomorphic to each other, and the conjugation action of  $G$  on  $H$  permutes the  $L_i$ 's transitively. Then all hereditarily just-infinite quotients of  $G$  are virtually procyclic (resp. virtually cyclic).*
- (b) *Now let  $G$  be a virtually pro- $p$  (resp. abstract) branch group. Then open (resp. finite index) subgroups of  $G$  do not have non-virtually procyclic (resp. non-virtually cyclic) hereditarily just-infinite quotients. Therefore, by Theorem 1.7 open (resp. finite index) subgroups of  $G$  cannot have positive weighted deficiency.*

*Proof.* We will deal with the pro- $p$  case; the abstract case is analogous.

(a) Let  $Q$  be a hereditarily just-infinite quotient of  $G$  and  $\pi : G \rightarrow Q$  an epimorphism. Then  $\pi(H)$  is just-infinite, so for each  $i$  the group  $\pi(L_i)$  is either trivial or open in  $Q$ , and since  $L_i$ 's are conjugate,  $\pi(L_i)$  must be open for all  $i$ . On the other hand, the subgroups  $\pi(L_i)$ ,  $1 \leq i \leq k$ , commute with each other. Combining these two properties, we conclude that  $Q$  must be virtually abelian. Since  $Q$  is just-infinite, it must actually be virtually procyclic.

(b) Suppose that  $G$  has an open subgroup  $H$  which has a non-virtually procyclic hereditarily just-infinite quotient  $Q$ . Note that every open subgroup of  $H$  also has such a quotient. Since  $G$  is branch, by making  $H$  smaller we can assume that  $H = L_1 \times \dots \times L_k$  where each  $L_i$  is branch. Then at least one of the subgroups  $L_i$  surjects onto an open subgroup of  $Q$ , which is also hereditarily just-infinite and non-virtually procyclic. This is a contradiction since branch groups satisfy the hypothesis of part (a).  $\square$

The last result naturally brings up the following question:

**Question 8.12.** *What can one say about groups which contain a finite index subgroup of positive weighted deficiency?*

We just proved that such groups cannot be branch. A simple observation is that such groups also cannot be just-infinite. Indeed, Theorem 8.5 implies that if  $G$  is a finite index subgroup of a just-infinite group, then  $G$  has finitely many commensurability classes of normal subgroups, and it is easy to see that a PWD group cannot have such property.

## 9. JUST INFINITE PRO- $p$ GROUPS WITH THE ASSOCIATED GRADED ALGEBRA OF EXPONENTIAL GROWTH

**9.1. Discussion of the result.** In [Sm] Smoktunowicz constructed examples of (graded) Golod-Shafarevich algebras all of whose infinite-dimensional homomorphic image have exponential growth. In this section we produce an analogous result for pro- $p$  groups (Theorem 9.1 below).

**Definition.** Let  $G$  be a finitely generated pro- $p$  group, and let  $\omega$  be the augmentation ideal of  $\mathbb{F}_p[[G]]$ . By  $gr\mathbb{F}_p[[G]]$  we denote the algebra  $\bigoplus_{n=0}^{\infty} \omega^n / \omega^{n+1}$ , the graded algebra of  $\mathbb{F}_p[[G]]$  with respect to powers of  $\omega$ .

**Theorem 9.1.** *Let  $G$  be a pro- $p$  group of positive weighted deficiency. Then  $G$  has a quotient  $G'$  which also has positive weighted deficiency and such that for every infinite quotient  $H$  of  $G'$  the graded algebra  $gr\mathbb{F}_p[[H]]$  has exponential growth. In particular, by Theorem 1.7 there exists a hereditarily just-infinite pro- $p$  group  $H$  for which  $gr\mathbb{F}_p[[H]]$  has exponential growth.*

**Remark:** The last assertion of Theorem 9.1 is the pro- $p$  analogue of Theorem 1.8. Theorem 1.8 itself will be established at the end of the section as an easy consequence of Theorem 9.1 and Corollary 7.4.

In spite of the apparent similarity of Theorem 9.1 to the main result of [Sm], the techniques used in the two papers are completely different. In addition, the construction from [Sm] is only valid for algebras over fields of infinite transcendence degree, and it would be interesting to see if our techniques could be used to extend the result of [Sm] to arbitrary fields.

To prove Theorem 9.1 we relate the growth of graded algebras associated to pro- $p$  groups to the concept of  $W$ -index:

**Lemma 9.2.** *Let  $G$  be a finitely generated pro- $p$  group,  $W$  a finite valuation on  $G$  and  $N$  a normal subgroup of  $G$  of infinite  $W$ -index. Then the algebra  $gr\mathbb{F}_p[[G/N]]$  has exponential growth.*

Lemma 9.2 will be proved at the end of § 9.2. Theorem 9.1 is a consequence of this lemma and the following theorem which is the main result of this section:

**Theorem 9.3.** *Let  $G$  be a pro- $p$  group and  $W$  a finite valuation on  $G$  such that  $def_W(G) > 0$ . Then  $G$  has a quotient  $G'$  such that  $def_W(G') > 0$  and each closed normal subgroup of  $G'$  of infinite index is also of infinite  $W$ -index.*

There is an obvious “naive” approach to Theorem 9.3 – one needs to make sure that all closed normal subgroups of  $G$  of finite  $W$ -index map to finite index subgroups of  $G'$ . According to Case 1 of Theorem 5.1 the latter can be achieved for any single closed normal subgroup of  $G$  and more generally for any countable collection of such subgroups, but this is far from sufficient. Instead, we address the analogous problem on the level of Lie algebras.

One can define  $W$ -index for subalgebras of the Lie algebra  $L_W(G)$  in complete analogy to the group case; then a subgroup  $H$  has finite  $W$ -index in  $G$  if and only if  $L_W(H)$  has finite  $W$ -index in  $L_W(G)$ . The key Lemma 9.5 below implies that every graded ideal<sup>3</sup> of  $L_W(G)$  of finite  $W$ -index contains a finitely generated graded ideal of finite  $W$ -index, and clearly there are countably many of those, call them  $M(1), M(2), \dots$ . Arguing similarly to

---

<sup>3</sup>Recall that by an ideal of a restricted Lie algebra we mean a restricted ideal



Case 1 of Theorem 5.1, we construct a quotient  $G'$  of  $G$  such that  $\text{def}_W(G') > 0$  and each of the ideals  $M(i)$  maps onto a finite index ideal under the induced map  $L_W(G) \rightarrow L_W(G')$ . It follows easily that the group  $G'$  has the required property.

**9.2. Growth in restricted Lie algebras.** In this subsection we prove Lemma 9.2 and two other lemmas needed for the proof of Theorem 9.3.

**Lemma 9.4.** *Let  $\Omega$  be a subsemigroup of  $((0, 1), \cdot)$  s.t.  $\Omega \cap (\alpha, 1)$  is finite for all  $\alpha > 0$ . Let  $L = \bigoplus_{\alpha \in \Omega} L_\alpha$  be a graded restricted  $\mathbb{F}_p$ -Lie algebra with finite-dimensional homogeneous components, and let  $A = \mathbb{F}_p \oplus (\bigoplus_{\alpha \in \Omega} A_\alpha)$  be the universal enveloping algebra of  $L$  with the associated grading by  $\Omega \cup \{1\}$ . Let  $c_\alpha = \dim L_\alpha$  and  $a_\alpha = \dim A_\alpha$  for  $\alpha \in \Omega \cup \{1\}$ . The following hold:*

(i)

$$\sum_{\alpha \in \Omega} a_\alpha \alpha^s = \prod_{\alpha \in \Omega} \left( \frac{1 - \alpha^{ps}}{1 - \alpha^s} \right)^{c_\alpha}.$$

where both sides are considered as Dirichlet series in a formal variable  $s$ .

(ii) For any  $\alpha \in \Omega$  we have  $c_\alpha \leq a_\alpha$

(iii) The numerical series  $\sum_{\alpha \in \Omega} \alpha c_\alpha$  converges if and only if  $\sum_{\alpha \in \Omega} \alpha a_\alpha$  converges.

*Proof.* (i) is obtained from the Poincare-Birkhoff-Witt theorem for graded restricted Lie algebras by dimension counting, and (ii) and (iii) are direct consequences of (i).  $\square$

**Lemma 9.5.** *Let  $L = \bigoplus_{\alpha \in \Omega} L_\alpha$  be a finitely generated  $\Omega$ -graded restricted  $\mathbb{F}_p$ -Lie algebra,*

*where  $\Omega$  is a finitely generated subsemigroup of  $((0, 1), \cdot)$ . Let  $D = \bigoplus_{\alpha \in \Omega} D_\alpha$  be a graded ideal such that*

$$\sum_{\alpha \in \Omega} \alpha (\dim L_\alpha - \dim D_\alpha) < \infty.$$

*Then  $D$  contains a graded ideal  $M = \bigoplus_{\alpha \in \Omega} M_\alpha$  of  $L$  which is finitely generated as an ideal and satisfies the inequality*

$$\sum_{\alpha \in \Omega} \alpha (\dim L_\alpha - \dim M_\alpha) < \infty.$$

*Proof.* Let  $A$  be the universal enveloping algebra of  $L$  with its natural grading by  $\Omega \cup \{1\}$ :

$$A = \mathbb{F}_p \oplus \left( \bigoplus_{\alpha \in \Omega} A_\alpha \right).$$

Let  $I$  be the ideal of  $A$  generated by  $D$ ; it is then a graded ideal:  $I = \bigoplus_{\alpha \in \Omega} I_\alpha$ .

Note that  $A/I$  is the universal enveloping algebra for  $L/D$ . Thus, by Lemma 9.4 the condition

$$\sum_{\alpha \in \Omega} \alpha (\dim L_\alpha - \dim D_\alpha) < \infty$$

is equivalent to the condition

$$\sum_{\alpha \in \Omega} \alpha (\dim A_\alpha - \dim I_\alpha) < \infty.$$

In particular, there exists  $\beta > 0$  such that

$$\sum_{\alpha \leq \beta} \alpha(\dim A_\alpha - \dim I_\alpha) < 1. \quad (***)$$

Since  $L$  is finitely generated,  $A$  is also finitely generated and so

$$B = \mathbb{F}_p \oplus \left( \bigoplus_{\beta \geq \alpha \in \Omega} A_\alpha \right)$$

is finitely generated. Assume that  $B$  is generated by graded elements of degree greater than  $\gamma$ . Let  $M \subseteq D$  be the ideal of  $L$  generated by  $\{D_\alpha : \alpha > \gamma\}$  and  $J = \bigoplus_\alpha J_\alpha \subseteq I$  the ideal of  $A$  generated by  $M$ . Put  $\bar{A} = A/J = \mathbb{F}_p + \bigoplus_{\alpha \in \Omega} \bar{A}_\alpha$  and  $\bar{B} = \mathbb{F}_p + \bigoplus_{\beta \geq \alpha \in \Omega} \bar{A}_\alpha$ .

We shall show that

$$\sum_{\alpha \in \Omega} \alpha \dim \bar{A}_\alpha = \sum_{\alpha \in \Omega} \alpha(\dim A_\alpha - \dim J_\alpha) < \infty.$$

Since  $A/J$  is the universal enveloping algebra for  $L/M$ , this will finish the proof by Lemma 9.4(iii).

Let  $\{\gamma_1, \dots, \gamma_k\} = \{\alpha \in \Omega : \beta \geq \alpha > \gamma\}$ . Since  $\bar{B}$  is generated by  $\{\bar{A}_{\gamma_1}, \dots, \bar{A}_{\gamma_k}\}$ , we obtain that  $\dim \bar{A}_\alpha \leq c_\alpha$  for all  $\alpha \leq \beta$ , where  $c_\alpha$  are the coefficients of the Dirichlet series

$$(9.1) \quad \sum_{\alpha} c_\alpha \alpha^s = \frac{1}{1 - \sum_{i=1}^k \dim \bar{A}_{\gamma_i} \gamma_i^s}.$$

Since  $\gamma_i > \gamma$ , we have  $\dim \bar{A}_{\gamma_i} = \dim A_{\gamma_i} - \dim J_{\gamma_i} = \dim A_{\gamma_i} - \dim I_{\gamma_i}$ . On the other hand  $\gamma_i \leq \beta$ , so (\*\*\*) implies that

$$\sum_{i=1}^k \gamma_i \dim \bar{A}_{\gamma_i} < 1.$$

Hence (9.1) holds as a numerical equality for  $s = 1$ , and we get

$$\sum_{\alpha \in \Omega} \alpha \dim \bar{A}_\alpha \leq \sum_{\beta < \alpha \in \Omega} \alpha \dim \bar{A}_\alpha + \sum_{\beta \geq \alpha \in \Omega} \alpha c_\alpha = \sum_{\beta < \alpha \in \Omega} \alpha \dim \bar{A}_\alpha + \frac{1}{1 - \sum_{i=1}^k \gamma_i \dim \bar{A}_{\gamma_i}} < \infty.$$

□

We finish this subsection with the proof of Lemma 9.2.

*Proof of Lemma 9.2.* First of all note that  $[G : N]_W = [G/N : \{1\}]_W$ , so without loss of generality we can assume that  $N = \{1\}$ .

Consider the Lie algebra  $L_W(G) = \bigoplus_{\alpha \in \text{Im } W} L_\alpha$ , and  $c_\alpha = \dim L_\alpha = \log_p [G_\alpha : G_{<\alpha}]$  for  $\alpha \in \text{Im } W$ . By definition of  $W$ -index we have  $[G : \{1\}]_W = \prod_{\alpha \in \text{Im } W} \left( \frac{1-\alpha^p}{1-\alpha} \right)^{c_\alpha}$ . Hence  $\sum_{\alpha \in \text{Im } W} \alpha c_\alpha = \infty$ .

Let  $q = \max(\text{Im } W) \in (0, 1)$  be the maximal value of  $W$  on  $G$ , and for  $k \in \mathbb{N}$  let  $d_k = \sum_{q^{k+1} < \alpha \leq q^k} c_\alpha$ . Thus  $\sum_{k=0}^{\infty} d_k q^k = \infty$ , whence

$$\limsup_{k \rightarrow \infty} \sqrt[k]{d_k} > 1 \quad (***)$$

Now let  $\omega$  be the augmentation ideal of  $\mathbb{F}_p[[G]]$ , and let  $\{D_k G\}_{k \in \mathbb{N}}$  be the Zassenhaus filtration of  $G$  defined by  $D_k G = \{g \in G : g - 1 \in \omega^k\}$ . It is clear that  $D_k G \subseteq$

$G_{\leq q^k}$ , whence  $\log_p[G : D_k G] \geq \sum_{n < k} d_n$ . On the other hand, we have an embedding  $D_k G / D_{k+1} G \rightarrow \omega^k / \omega^{k+1}$ . Thus if we put  $b_k = \dim \omega^k / \omega^{k+1}$ , then  $\sum_{n < k} b_k \geq \sum_{n < k} d_n$ , so  $\limsup_{k \rightarrow \infty} \sqrt[k]{b_k} > 1$  by (\*\*\*) .

Finally, since the sequence  $\{b_k\}$  is submultiplicative ( $b_{k+l} \leq b_k b_l$ ), the limit  $\lim_{k \rightarrow \infty} \sqrt[k]{b_k}$  must exist. Therefore,  $\lim_{k \rightarrow \infty} \sqrt[k]{b_k} > 1$ , so the algebra  $gr\mathbb{F}_p[[G]] = \bigoplus_{k=0}^{\infty} \omega^k / \omega^{k+1}$  is of exponential growth.  $\square$

### 9.3. Proofs of Theorem 9.3 and Theorem 1.8.

*Proof of Theorem 9.3.* As usual, without loss of generality we can assume that the Lie algebra  $L_W(G)$  is finitely generated.

Let  $\mathcal{I}$  be the set of graded ideals  $M = \bigoplus_{\alpha \in \text{Im } W} M_\alpha$  of infinite index such that  $M$  is finitely generated as an ideal and

$$\sum_{\alpha \in \text{Im } W} \alpha(\dim L_\alpha - \dim M_\alpha) < \infty.$$

It is clear that  $\mathcal{I}$  is countable, so we can enumerate its elements  $\mathcal{I} = \{M(1), M(2), \dots\}$ .

For each  $i$  we can find  $\alpha_i > 0$  such that

$$\sum_{\alpha_i > \alpha \in \text{Im } W} \alpha(\dim L_\alpha - \dim M(i)_\alpha) < \frac{\text{def}_W(G)}{2^i}.$$

Let  $S_i$  be a subset of  $G$  such that  $\{\text{LT}(g) + M(i)_{W(g)} : g \in S_i\}$  is a basis of  $\bigoplus_{\alpha_i > \alpha \in \text{Im } W} L_\alpha / M(i)_\alpha$ .

Note that  $W(S_i) = \sum_{\alpha_i > \alpha \in \text{Im } W} \alpha(\dim L_\alpha - \dim M(i)_\alpha) < \frac{\text{def}_W(G)}{2^i}$ .

Let  $S = \bigcup_{i \geq 1} S_i$  and let  $N$  be the normal subgroup of  $G$  generated by  $S$ . We have that  $W(S) < \text{def}_G(W)$  and so by Lemma 4.12,  $\text{def}_W(G/N) > 0$ .

Now, let us show that all closed normal subgroups of  $G/N$  of finite  $W$ -index are open. By Lemma 9.5 and the choice of  $S$  for every graded ideal  $P = \bigoplus_\alpha P_\alpha$  of  $L_W(G/N) = \bigoplus_\alpha \bar{L}_\alpha$ , either

- (i)  $P$  is of finite index or
- (ii)  $\sum_\alpha \alpha(\dim \bar{L}_\alpha - \dim P_\alpha) = \infty$ .

Let  $K$  be a closed normal subgroup of  $G/N$  of finite  $W$ -index and put  $P = L_W(K)$ . Then  $P$  is an ideal of  $L_W(G/N)$ . Since the  $W$ -index of  $K$  is finite, we have

$$\sum_\alpha \alpha(\dim \bar{L}_\alpha - \dim P_\alpha) = \sum_\alpha \alpha(c_\alpha(G/N) - c_\alpha(K)) < \infty.$$

(where integers  $c_\alpha(\cdot)$  are defined as in Section 3). Thus by the above dichotomy  $P$  is of finite index in  $L_W(G/N)$ , so  $K$  must be open in  $G/N$ .  $\square$

*Proof of Theorem 1.8.* By Theorem 7.1 there exists a PWD group  $\Gamma$  which is  $p$ -LERF. Let  $G = \Gamma_{\hat{p}}$ , and let  $G'$  be a group satisfying the conclusion of Theorem 9.1 applied to  $G$ . By Theorem 6.3 we can replace  $G'$  by another quotient  $G''$  such that  $G''$  has PWD and  $G'' = (\Gamma'')_{\hat{p}}$  where  $\Gamma''$  is the image of  $\Gamma$  in  $G''$ .

Let  $\Delta$  be any infinite quotient of  $\Gamma''$ . Since  $\Gamma$  is  $p$ -LERF and  $\Delta$  is also a quotient of  $\Gamma$ , by Corollary 7.4 the pro- $p$  completion  $\Delta_{\hat{p}}$  is infinite. The natural projection  $\Gamma'' \rightarrow \Delta$  induces the corresponding epimorphism  $G'' = (\Gamma'')_{\hat{p}} \rightarrow \Delta_{\hat{p}}$ , and so by the choice of  $G'$  and by Lemma 9.2 the graded algebra  $gr\mathbb{F}_p[[\Delta_{\hat{p}}]]$  has exponential growth. It is easy to show that  $gr\mathbb{F}_p[\Delta]$  is naturally isomorphic to  $gr\mathbb{F}_p[[\Delta_{\hat{p}}]]$  as graded algebras, and thus  $gr\mathbb{F}_p[\Delta]$

also has exponential growth. Finally, by Theorem 1.6  $\Delta$  can be chosen hereditarily just-infinite.  $\square$

## REFERENCES

- [Ba] Yu. A. Bahturin, *Identical relations in Lie Algebras*, VNU Science Press, b.v., Utrecht, 1987. x+309 pp.
- [Bo] N. Boston, *Galois groups of tamely ramified  $p$ -extensions*. J. Théor. Nombres Bordeaux 19 (2007), no. 1, 59–70
- [BKS] R.M. Bryant, L.G. Kovacs, R. Stohr, *Subalgebras of free restricted Lie algebras*, Bull. Austral. Math. Soc. 72 (2005), 147–156.
- [dSSS] *M. du Sautoy, D. Segal and A. Shalev, New horizons in pro- $p$  groups*, Progress in Mathematics, 184, Birkhäuser Boston, Boston, MA, 2000.
- [Er] M. Ershov, *Golod-Shafarevich groups with property (T) and Kac-Moody groups*, Duke Math. J. 145 (2008), no. 2, 309–339
- [EJ] M. Ershov, *Kazhdan quotients of Golod-Shafarevich groups*, with appendices by A. Jaikin-Zapirain, arXiv:09083734, accepted by Proc. Lond. Math. Soc.
- [Go] E. Golod, *On nil algebras and finitely approximable groups*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964) 273–276
- [GSh] E. Golod and I. Shafarevich, *On the class field tower*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964) 261–272
- [Gr1] R. I. Grigorchuk, *On Burnside’s problem on periodic groups*. Functional Anal. Appl. 14 (1980), no. 1, 41–43.
- [Gr2] R. I. Grigorchuk, *Just-infinite branch groups*, New horizons in pro- $p$  groups, 125–185, Progr. Math., 184, Birkhäuser Boston, Boston, MA, 2000.
- [Hall] M. Hall, Jr. *The theory of groups. Reprinting of the 1968 edition*. Chelsea Publishing Co., New York, 1976. xiii+434 pp.
- [Ko] H. Koch, *Galois theory of  $p$ -extensions*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002. xiv+190 pp.
- [Kou] *The Kourovka notebook (Unsolved problems in group theory), 17th edition*. Russian Academy of Sciences, Novosibirsk, 2010.
- [LLR] M. Lackenby, D. D. Long, A. Reid, *LERF and the Lubotzky-Sarnak conjecture*. Geom. Topol. 12 (2008), no. 4, 2047–2056.
- [LR] D. D. Long, A. Reid, *Subgroup separability and virtual retractions of groups*, Topology 47 (2008), 137–159.
- [Lu1] A. Lubotzky, *Group presentation,  $p$ -adic analytic groups and lattices in  $SL_2(C)$* , Ann. of Math. (2) 118 (1983), no. 1, 115–130.
- [Lu2] A. Lubotzky, *On finite index subgroups of linear groups*, Bull. London Math. Soc. 19 (1987) 325–328.
- [Ol] A. Yu. Ol’shanskii, *An infinite group with subgroups of prime orders*, Izvestia Akad. Nauk SSSR, Ser. Matem., 44 (1980), N 2, 309–321 (in Russian); English translation: Math. USSR-Izv. 16 (1981), no. 2, 279–289.
- [OO] A. Yu. Ol’shanskii, D. Osin, *Large groups and their periodic quotients*. Proc. Amer. Math. Soc. 136 (2008), no. 3, 753–759
- [OS] A. Yu. Ol’shanskii, M. Sapir, *Non-amenable finitely presented torsion-by-cyclic groups*, Publ. Math. IHES 96 (2003), 43–169.
- [Re] C. Reid, *Subgroups of finite index and the just infinite property*, preprint (2009), arXiv:0905.1624.
- [RZ] L. Ribes, P. Zalesskii, *Profinite groups. Second edition*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 40. Springer-Verlag, Berlin, 2010. xvi+464 pp.
- [SP] J.-C. Schlage-Puchta, *A  $p$ -group with positive rank gradient*, preprint (2010).
- [Shu] V. P. Shunkov. *On periodic groups with an almost regular involution*, Algebra i Logika 11 (1972), no. 4, 470–493; English translation: Algebra and Logic 11 (1972), no. 4, 260–272.
- [SW] H. Smith, J. Wiegold, *Locally graded groups with all subgroups normal-by-finite*. J. Austral. Math. Soc. (Series A) 60 (1996), 222–227.
- [Sm] A. Smoktunowicz, *GK-dimension of algebras with many generic relations*. Glasg. Math. J. 51 (2009), no. 2, 253–256.

- [Wil1] J. Wilson, *Groups with every proper quotient finite*. Proc. Cambridge Philos. Soc. 69 (1971), 373–391.
- [Wil2] J. Wilson, *Finite presentations of pro- $p$  groups and discrete groups*. Invent. Math. 105 (1991), no. 1, 177–183.
- [Wil3] J. Wilson, *Profinite groups*, London Mathematical Society Monographs. New Series, 19. The Clarendon Press, Oxford University Press, New York, 1998
- [Wil4] J. Wilson, *Large hereditarily just infinite groups*, J. Algebra 324 (2010), no. 2, 248–255.
- [Wt] E. Witt, *Die Unterringe der freien Lieschen Ringe*, Math. Z. 64 (1956), 195–216.

UNIVERSITY OF VIRGINIA

*E-mail address:* `ershov@virginia.edu`

UNIVERSIDAD AUTÓNOMA DE MADRID AND ICMAT

*E-mail address:* `andrei.jaikin@uam.es`