Algebra-I, Fall 2018. Solutions to Midterm #1

1. Let G be a group, H, K subgroups of G and $a, b \in G$.

(a) (6 pts) Suppose that $aH = bK$. Prove that $H = K$.

Solution: (a) Multiplying both sides by b^{-1} on the left, we get $b^{-1}aH = K$. It follows that K is one of the left cosets of H. Since $1 \in K$ and the only left coset of H containing 1 is H itself, we conclude that $K = H$.

Another solution: Taking inverses of both sides of the equality $aH = bK$, we get $K^{-1}b^{-1} = H^{-1}a^{-1}$. Since H and K are subgroups, we have $K^{-1} = K$ and $H^{-1} = H$, and therefore $Ha^{-1} = Kb^{-1}$. Now multiplying this by the original equality on the right, we get $H \cdot H = K \cdot K$. Since H and K are subgroups, we get $H = K$.

(b) (4 pts) Suppose that $aH = Kb$. Prove that H and K are conjugate.

Solution: Let $K' = b^{-1}Kb$ (note that K' is a subgroup). Then $bK' = Kb = aH$. Thus $K' = H$ by (a), so H and K are conjugate.

2. Let G be a group and H a subgroup of G. Recall that H is called **characteristic in** G if $\varphi(H) = H$ for any $\varphi \in \mathrm{Aut}(G)$. The subgroup H is said to be **fully characteristic in** G if $\varphi(H) \subseteq H$ for any $\varphi \in \mathrm{End}(G)$ where $\mathrm{End}(G)$ is the set of all homomorphisms from G to G.

(a) (6 pts) Suppose that K is a subgroup of H and H is a subgroup of G.

(i) Prove that if H is normal in G and K is characteristic in H, then K is normal in G.

(ii) Give an example showing that if H is characteristic in G and K is normal in H, then K may not be normal in G.

Solution: (i) Since H is normal in G, for any $g \in G$, the map ι_g given by $\iota_g(x) = g^{-1}xg$ is an automorphism of H. Since K is characteristic in H, we conclude that $\iota_g(K) = K$, which means that K is normal in G.

(ii) Let $G = A_4$ and $H = V_4 = \{e, (12)(34), (13)(24), (14)(24)\}$ and $K = \{e, (12)(34)\}$. Then H is characteristic in G since it is the only subgroup of G of order 4 (this holds, e.g. since all elements of $G \setminus H$ have order 3). We also know that K is normal in H since H is abelian. On the other hand, K is not normal in G, e.g. since $(123) \cdot (12)(34) \cdot (123)^{-1} = (23)(14) \notin K$.

(b) (4 pts) Give an example of a group G and a subgroup H of G which is characteristic, but not fully characteristic in G.

Solution: We shall give two examples:

Example 1: $G = D_8$ and $H = \langle r \rangle$, the rotation subgroup. Then H is characteristic in G since it is the only subgroup of G isomorphic to $\mathbb{Z}_4$. To see that it is not fully characteristic, take any $s \in G \setminus H$ and $K = \langle r, s^2 \rangle$. Then K is a subgroup of index 2 in G (hence normal) and the quotient G/K is cyclic of order 2. Since the subgroup $\langle s \rangle$ of G is also cyclic of order 2, composing the natural projection from G to G/K with an isomorphism $G/K \to \langle s \rangle$, we obtain a homomorphism $\varphi : G \to G$ with $\operatorname{Ker} \varphi = K$ and $\operatorname{Im} \varphi = \langle s \rangle = \{1, s\}$. Since $r \notin K$, we have $\varphi(r) = s$, so $\varphi(H)$ is not contained in H and thus H is not fully characteristic.

Example 2: Let A and B be any groups such that

(1) A has is abelian,

(2) B has trivial center

(3) there is a non-trivial homomorphism $\varphi : A \to B$

(For instance, we could take $A = \mathbb{Z}_2$, $B = S_3$ and φ the map which sends the non-trivial element of $\mathbb{Z}_2$ to $(1, 2)$).

We claim that if we set $G = A \times B$ and $H = A \times \{1_B\}$, then H is characteristic but not fully characteristic.

It is easy to show that for any groups U and V the center of their direct product is equal to the direct product of the centers: $Z(U \times V) = Z(U) \times Z(V)$. Thus, in our case $Z(G) = Z(A) \times Z(B) = A \times \{1_B\} = H$. It is also easy to see that the center is always characteristic, so H is characteristic in G.

On the other hand, the map $f : G \to G$ given by $f((a, b)) = (1_A, \varphi(a))$ is a homomorphism (where φ comes from (3) above), and $f(H)$ contains an element of the form $(1_A, b)$ with $b \neq 1_B$, so H is not fully characteristic.

3. Let G be a group.

(a) (5 pts) Suppose that H is a cyclic normal subgroup of G. Prove that H commutes with $[G, G]$ elementwise, that is, $hx = xh$ for any $h \in H$ and $x \in [G, G]$.

Solution: (a) Let $\iota : G \to \operatorname{Aut}(H)$ be the conjugation homomorphism, that is, $\iota(g) = (h \mapsto ghg^{-1})_{h \in H}$. Proving that $hx = xh$ for any $h \in H$ and $x \in [G, G]$ is equivalent to showing that for any $x \in [G, G]$ we have $\iota(x) = 1_{\operatorname{Aut}(H)}$, that is, $\iota(x)$ is the identity element of the group $\operatorname{Aut}(H)$.

If H is finite cyclic, then $H \cong \mathbb{Z}_n$ for some n, whence $\operatorname{Aut}(H) \cong \operatorname{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^{\times}$. If H is infinite cyclic, then $H \cong \mathbb{Z}$, whence $\operatorname{Aut}(H) \cong \mathbb{Z}^{\times} \cong \mathbb{Z}_2$. In either case, $\operatorname{Aut}(H)$ is an abelian group, whence $\iota(G)$ is also abelian, being a subgroup of $\operatorname{Aut}(H)$. Since $\iota(G) \cong G/\operatorname{Ker} \iota$, we conclude that $\operatorname{Ker} \iota$ contains

$[G,G]$, which means that $\iota(x) = 1_{\mathrm{Aut}(H)}$ for $x \in [G,G]$.

Another solution (sketch): Take any $x, y \in G$. Then our knowledge of $\mathrm{Aut}(H)$ yields that there exist $n, m \in \mathbb{Z}$ such that $xhx^{-1} = h^n$ and $xhx^{-1} = h^m$ for any $h \in H$. An explicit calculation shows that $[x,y]^{-1}h[x,y] = x^{-1}y^{-1}xyhx^{-1}y^{-1} = h$. Thus, h commutes with every commutator $[x,y]$, whence h commutes with $[G,G]$, the subgroup generated by all commutators.

(b) (5 pts) Let H_1, H_2, H_3 be normal subgroups of G such that $H_iH_j = G$ and $H_i \cap H_j = \{1\}$ for any pair of indices $i \neq j$. Prove that subgroups H_1, H_2, H_3 are isomorphic to each other.

Solution: By the diamond isomorphism theorem, for any $i \neq j$ we have $H_i \cong H_i/H_i \cap H_j \cong H_iH_j/H_j = G/H_j$. In particular, $H_1 \cong G/H_3$ and $H_2 \cong G/H_3$, so H_1 and H_2 are isomorphic. Similarly, H_1 and H_3 are isomorphic.

4. Let G be a finite group and H a normal subgroup of G. Let $\mathcal{K}$ be a G-conjugacy classes (that is, a conjugacy class of G) which is contained in H. Note that every H-conjugacy class is either contained in $\mathcal{K}$ or disjoint from $\mathcal{K}$, so $\mathcal{K}$ is a union of k (distinct) H-conjugacy classes for some k.

(a) (5 pts) Prove that $k = [G : H \cdot C_G(x)]$ where x is an arbitrary element of $\mathcal{K}$.

Solution: Let Ω be the set of H-conjugacy classes contained in $\mathcal{K}$. We claim that G acts on Ω by conjugation – to prove this we just need to show that conjugation by any $g \in G$ sends an H-conjugacy class to an H-conjugacy class. In other words, we need to show the following:

Claim: *Let $a, b \in G$ and $g \in G$. Then a and b are H-conjugate $\iff$ gag^{-1} and gbg^{-1} are H-conjugate.*

Proof of the Claim: We will show the forward implication ($\Rightarrow$); the other direction is analogous. Suppose that a and b are H-conjugate, that is, $b = hah^{-1}$ for some $h \in H$. Then $gbg^{-1} = ghah^{-1}g^{-1} = (ghg^{-1})(gag^{-1})(ghg^{-1})^{-1}$. Since H is normal in G, we have $ghg^{-1} \in H$, whence gag^{-1} and gbg^{-1} are H-conjugate. $\square$

Now we proceed with Problem 4(a). Fix $x \in \mathcal{K}$, and let X be the H-conjugacy class of x. By the orbit-stablizer formula applied to the action of G on Ω constructed above we have $|Orbit(X)| = [G : Stab_G(X)]$. Since this action is clearly transitive, we have $|Orbit(X)| = |\Omega| = k$, so to prove the desired formula for k it is sufficient to show that $Stab_G(X) = H \cdot C_G(x)$.

Since any two H-conjugacy classes either coincide or intersect trivially, we have $Stab_G(X) = \{g \in G : gxg^{-1} \in X\} = \{g \in G : gxg^{-1} = hxh^{-1} \text{ for some } h \in H\}$. Since $gxg^{-1} = hxh^{-1} \iff h^{-1}g \in C_G(x) \iff g \in hC_G(x)$ for some $h \in H$, we conclude that $Stab_G(X) = H \cdot C_G(x)$.

Another solution: As in the first solution, let X be the H-conjugacy class of x. By the orbit-stabilizer formula applied to the conjugation action of H on G we have $|X| = [H : C_H(x)]$, and clearly $C_H(x) = C_G(x) \cap H$. Since H is normal in G, the diamond isomorphism theorem yields

$$[C_G(x) : C_G(x) \cap H] = [HC_G(x) : H],$$

which implies that $[H : C_H(x)] = \frac{|H|}{|C_G(x)\cap H|} = \frac{|HC_G(x)|}{|C_G(x)|}$. On the other hand, by the orbit-stabilizer formula applied to the conjugation action of G on itself we have $|C_G(x)| = |G|/|\mathcal{K}|$, and thus we conclude that

$$|X| = \frac{|HC_G(x)|}{|C_G(x)|} = \frac{|HC_G(x)||\mathcal{K}|}{|G|} = \frac{|\mathcal{K}|}{[G : HC_G(x)]},$$

whence $[G : HC_G(x)] = \frac{|\mathcal{K}|}{|X|}$.

As shown in the first solution, all H-conjugacy classes contained in $\mathcal{K}$ are of the same size (since they are G-conjugate to each other), and there are k of them by definition of k. Thus means that $|X| = \frac{|\mathcal{K}|}{k}$, whence $k = \frac{|\mathcal{K}|}{|X|} = [G : HC_G(x)]$.

Common gap in the exam papers: In many solutions using the second approach it was assumed without proof that all H-conjugacy classes inside $\mathcal{K}$ have the same size.

(b) (2 pts) Let $G = S_n$ and $H = A_n$, and pick some element $\sigma \in \mathcal{K}$. Prove that $k = 1$ if $C_G(\sigma)$ contains an odd permutation and $k = 2$ otherwise.

Solution: If $C_{S_n}(\sigma)$ contains no odd permutations, then $C_{S_n}(\sigma) \subseteq A_n$, whence $C_{S_n}(\sigma)A_n = A_n$. Thus, by part (a), $k = [S_n : C_{S_n}(\sigma)A_n] = [S_n : A_n] = 2$.

If $C_{S_n}(\sigma)$ contains an odd permutations, then $C_{S_n}(\sigma)A_n$ is strictly larger than A_n, and therefore $C_{S_n}(\sigma)A_n = S_n$ since A_n already has index 2 in S_n. Thus, $k = [S_n : S_n] = 1$.

(c) (3 pts) Once again, let $G = S_n$ and $H = A_n$, let $m \leq n$ be an odd number and $\mathcal{K}$ the G-conjugacy class consisting of all m-cycles (note that $\mathcal{K} \subseteq A_n$). Prove that $k = 2$ if $n - m \leq 1$ and $k = 1$ otherwise.

Solution: If σ is an m-cycle, then the conjugacy class of σ in S_n consists of all m-cycles, and the number of distinct m-cylces is easily seen to equal $\frac{n(n-1)\dots(n-m+1)}{m}$. Thus, $|C_{S_n}(\sigma)| = \frac{n!\cdot m}{n(n-1)\dots(n-m+1)} = m \cdot (n-m)!$.

If $m = n$ or $m = n-1$, we conclude that $|C_{S_n}(\sigma)| = m$, whence $C_{S_n}(\sigma) = \langle\sigma\rangle$ since $|\langle\sigma\rangle| = m$ and $\langle\sigma\rangle \subseteq C_{S_n}(\sigma)$. Since $\langle\sigma\rangle \subseteq A_n$, we have $k = 2$ by part (b).

If $m \leq n-2$, there is at least one transposition disjoint from σ. Such transposition is an odd permutation inside $C_{S_n}(\sigma)$, and therefore $k = 1$ by part (b).

5. Let G be a group of order $105 = 3 \cdot 5 \cdot 7$.

(a) (3 pts) Prove that G has a normal Sylow 5-subgroup OR a normal Sylow 7-subgroup.

Solution: Let n_p be the number of Sylow p-subgroups in G for $p = 3, 5, 7$. Standard application of Sylow theorems yields $n_5 = 1$ or 21 and $n_7 = 1$ or 15. If $n_5 = 1$ or $n_7 = 1$, then G has a normal Sylow 5-subgroup or a normal Sylow 7-subgroup.

The only other possibility is that $n_5 = 21$ and $n_7 = 15$. Since distinct Sylow 5-subgroups and distinct Sylow 7-subgroups of G interesect trivially, in this case G will have $(5-1)\cdot 21 = 84$ elements of order 5 and $(7-1)\cdot 15 = 90$ elements of order 7. Since $84 + 90 > 105 = |G|$, we reach a contradiction.

(b) (4 pts) Use (a) to prove that G has a normal subgroup of order 35. Deduce that G has a normal Sylow 5-subgroup AND a normal Sylow 7-subgroup.

Solution: Let P_5 and P_7 be Sylow 5-subgroup and Sylow 7-subgroup of G, respectively. We know that at least one of the subgroups P_5 or P_7 is normal, so P_5P_7 must be a subgroup of G. Furthermore, $|P_5P_7| = |P_5||P_7|/|P_5 \cap P_7| = 35$. Since $[G : P_5P_7] = 3$, by the small index lemma G has a normal subgroup H such that $H \subseteq P_5P_7$ and $[G : H]$ divides $3! = 6$. Since $[G : H] \geq [G : P_5P_7] = 3$ and $[G : H]$ divides 105, the only possibility is that $[G : H] = 3$, so $H = P_5P_7$ is a normal subgroup of order 35. Furthermore, by classification of groups of order pq given in class, we know that $H \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35}$ since $5 \nmid (7-1)$.

Now we shall prove that both P_5 and P_7 are normal. Take any $g \in G$. Since $P_5 \subseteq H$ and H is normal in G, we have $gP_5g^{-1} \subseteq H$. Thus, P_5 and gP_5g^{-1} are both Sylow 5-subgroups of H. On the other hand, being abelian H has just one Sylow 5-subgroup. Therefore, $gP_5g^{-1} = P_5$, so P_5 is normal in G. Similarly, P_7 is normal.

(c) (3 pts) Prove that there are two isomorphism classes of groups of order 105 and describe them (briefly) using semi-direct products. You may use without proof that $\mathrm{Aut}(\mathbb{Z}_n \times \mathbb{Z}_m) \cong \mathrm{Aut}(\mathbb{Z}_n) \times \mathrm{Aut}(\mathbb{Z}_m)$ if $gcd(m, n) = 1$.

Solution: Let H be as in part (b) and let P_3 be a Sylow 3-subgroup of G. Since H is normal in G, $|H||P_3| = |G|$ and $gcd(|H|, |P_3|) = 1$, we have $G = H \rtimes P_3$. Since $H \cong \mathbb{Z}_{35}$ and $P \cong \mathbb{Z}_3$, the group G is isomorphic to the semi-direct product $\mathbb{Z}_{35} \rtimes_{\varphi} \mathbb{Z}_3$ for some homomorphism $\varphi : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_{35})$.

Non-trivial homomorphisms $\varphi : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_{35})$ correspond bijectively to elements of order 3 in $\mathrm{Aut}(\mathbb{Z}_{35})$. Since $\mathrm{Aut}(\mathbb{Z}_{35}) \cong \mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_7) \cong \mathbb{Z}_4 \times \mathbb{Z}_6$, it has exactly two elements of order 3, namely $(0, 2)$ and $(0, 4)$. Thus, there are exactly two non-trivial homomorphisms from $\mathbb{Z}_3$ to $\mathrm{Aut}(\mathbb{Z}_{35})$, call them φ_1 and φ_2.

The group $\mathbb{Z}_3$ has an automorphism θ given by $\theta(x) = -x$, and it is clear that $\varphi_2 = \varphi_1 \circ \theta$. Hence, by Observation 10.1 from class (=Problem 2(a) in HW#5), the semidirect products $\mathbb{Z}_{35} \rtimes_{\varphi_1} \mathbb{Z}_3$ and $\mathbb{Z}_{35} \rtimes_{\varphi_2} \mathbb{Z}_3$ are isomorphic.

Thus, there are (at most) two possible isomorphism classes for G: the direct product $\mathbb{Z}_{35} \times \mathbb{Z}_3 \cong \mathbb{Z}_{105}$ (corresponding to the trivial φ) and $\mathbb{Z}_{35} \rtimes_{\varphi_1} \mathbb{Z}_3 \cong \mathbb{Z}_{105}$. Clearly, these two groups are non-isomorphic because the first one is abelian and the second one is not.