Algebra-I, Fall 2018. Midterm #1. Due Thursday, October 4th, in class

Directions: Each problem is worth 10 points. The best 4 out of 5 problems will be counted (but you are encouraged to do all 5). Provide complete arguments (do not skip steps). State clearly any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given.

Rules: You are not allowed to discuss midterm problems with each other. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use your class notes, previous homework assignments and the book by Dummit and Foote. The use of other books is allowed, but not encouraged. If you happen to run across a problem very similar or identical to one on the midterm which is solved in another book, do not consult that solution. The use of any online resources (except the class webpage) is absolutely prohibited.

1. Let G be a group, H, K subgroups of G and $a, b \in G$.
(a) (6 pts) Suppose that $aH = bK$. Prove that $H = K$.
(b) (4 pts) Suppose that $aH = Kb$. Prove that H and K are conjugate.

2. Let G be a group and H a subgroup of G. Recall that H is called **characteristic in** G if $\phi(H) = H$ for any $\phi \in \mathrm{Aut}(G)$. The subgroup H is said to be **fully characteristic in** G if $\phi(H) \subseteq H$ for any $\phi \in \mathrm{End}(G)$ where $\mathrm{End}(G)$ is the set of all homomorphisms from G to G.
(a) (6 pts) Suppose that K is a subgroup of H and H is a subgroup of G.

(i) Prove that if H is normal in G and K is characteristic in H, then K is normal in G.

(ii) Give an example showing that if H is characteristic in G and K is normal in H, then K may not be normal in G.

(b) (4 pts) Give an example of a group G and a subgroup H of G which is characteristic, but not fully characteristic in G. **Hint:** You can find such a subgroup inside a familiar group.

3. Let G be a group.
(a) (5 pts) Suppose that H is a cyclic normal subgroup of G. Prove that H commutes with $[G,G]$ elementwise, that is, $hx = xh$ for any $h \in H$ and $x \in [G,G]$.
(b) (5 pts) Let H_1, H_2, H_3 be normal subgroups of G such that $H_iH_j = G$ and $H_i \cap H_j = \{1\}$ for any pair of indices $i \neq j$. Prove that subgroups H_1, H_2, H_3 are isomorphic to each other.

Hint: For (a) – Use the conjugation homomorphism $G \to \mathrm{Aut}(H)$, the description of automorphisms of cyclic groups and some basic results about $[G,G]$ discussed in class. For (b) – use a suitable isomorphism theorem.

4. Let G be a finite group and H a normal subgroup of G. Let $\mathcal{K}$ be a G-conjugacy classes (that is, a conjugacy class of G) which is contained in H. Note that every H-conjugacy class is either contained in $\mathcal{K}$ or disjoint from $\mathcal{K}$, so $\mathcal{K}$ is a union of k (distinct) H-conjugacy classes for some k.
(a) (5 pts) Prove that $k = [G : H \cdot C_G(x)]$ where x is an arbitrary element of $\mathcal{K}$
(b) (2 pts) Let $G = S_n$ and $H = A_n$, and pick some element $\sigma \in \mathcal{K}$. Prove that $k = 1$ if $C_G(\sigma)$ contains an odd permutation and $k = 2$ otherwise.
(c) (3 pts) Once again, let $G = S_n$ and $H = A_n$, let $m \leq n$ be an odd number and $\mathcal{K}$ the G-conjugacy class consisting of all m-cycles (note that $\mathcal{K} \subseteq A_n$). Prove that $k = 2$ if $n - m \leq 1$ and $k = 1$ otherwise.

Hint for (a): You may deduce (a) directly from the orbit-stablizer formula applied to a suitable group action. An alternative hint is given in Dummit and Foote (see Problem 19, page 131).

5. Let G be a group of order $105 = 3 \cdot 5 \cdot 7$.
(a) (3 pts) Prove that G has a normal Sylow 5-subgroup OR a normal Sylow 7-subgroup.
(b) (4 pts) Use (a) to prove that G has a normal subgroup of order 35. Deduce that G has a normal Sylow 5-subgroup AND a normal Sylow 7-subgroup.
(c) (3 pts) Prove that there are two isomorphism classes of groups of order 105 and describe them (briefly) using semi-direct products. You may use without proof that $\mathrm{Aut}(\mathbb{Z}_n \times \mathbb{Z}_m) \cong \mathrm{Aut}(\mathbb{Z}_n) \times \mathrm{Aut}(\mathbb{Z}_m)$ if $gcd(m,n) = 1$.