Solutions to Homework #2

1. List all the partitions of the set $\{1, 2, 3, 4, 5\}$ which do not have any 1-element blocks.

Solution: Since blocks of a partition must be disjoint and non-empty and we are disallowing 1-element blocks, there are only two choices for the block sizes: we can have just one block of size 5 or two blocks, one of size 2 and the other one of size 3. Clearly, there is just one partition with one block, namely $\{\{1, 2, 3, 4, 5\}\}$. To specify a partition of $2 + 3$ type we just need to choose which elements are included in the block of size 2. There are $10 = 5(5 - 1)/2$ ways to do so (counting problems of this type are discussed in Chapter 4). Explicitly, the partitions of type $2 + 3$ are $\{\{1, 2\}, \{3, 4, 5\}\}$, $\{\{1, 3\}, \{2, 4, 5\}\}$, $\{\{1, 4\}, \{2, 3, 5\}\}$, $\{\{1, 5\}, \{2, 3, 4\}\}$, $\{\{2, 3\}, \{1, 4, 5\}\}$, $\{\{2, 4\}, \{1, 3, 5\}\}$, $\{\{2, 5\}, \{1, 3, 4\}\}$, $\{\{3, 4\}, \{1, 2, 5\}\}$, $\{\{3, 5\}, \{1, 2, 4\}\}$ and $\{\{4, 5\}, \{1, 2, 3\}\}$.

2. Problem 7 in 1.3 from the BOOK.

Solution: (a) $\{C \cup D, H \cup S\}$ (here C stands for the set of all clubs etc.); (b) $\{C, D, H, S\}$; (c) $\{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \mathbf{10}, \mathbf{J} \cup \mathbf{Q}, \mathbf{K} \cup \mathbf{A}\}$. Here $\mathbf{2}$ stands for the set of all 2's etc.; (d) There is exactly one partition with this property, namely the one where all the blocks have size 1, that is, each card forms its one block.

3. Problem 1 in 1.3 from the BOOK. Justify your answer.

(a) We claim that $\{R, S \setminus R\}$ is a partition of $R \cup S \iff (R \neq \emptyset$ and $S \not\subseteq R)$, that is, $\{R, S \setminus R\}$ is a partition of $R \cup S \iff R$ is non-empty and S is not contained in R. We will prove this equivalence by establishing implications in both directions.

"$\Rightarrow$" Suppose $\{R, S \setminus R\}$ is a partition of $R \cup S$. We want to show that $R \neq \emptyset$ and $S \not\subseteq R$. Since blocks of the partitions must be non-empty, we must have $R \neq \emptyset$ and $S \setminus R \neq \emptyset$. The set $S \setminus R$ is non-empty $\iff$ there is at least one element of S which does not lie in R, which is the same as saying that S is not contained in R. Thus, we proved that $R \neq \emptyset$ and $S \not\subseteq R$, as desired. Note that there are several conditions in the definition of a partition that we did not have to use for this first part of the proof.

"$\Leftarrow$" Now suppose that $R \neq \emptyset$ and $S \not\subseteq R$. We want to show that $\{R, S \setminus R\}$ is a partition. For this we need to check three properties:

(i) $R \neq \emptyset$ and $S \setminus R \neq \emptyset$

(ii) $R \cap (S \setminus R) = \emptyset$

(iii) $R \cup (S \setminus R) = R \cup S$

For (i), we are given that $R \neq \emptyset$, and condition $S \not\subseteq R$ (also given to us) implies that (in fact, is equivalent to) $S \setminus R \neq \emptyset$, as we observed in the proof of the forward direction.

(ii) By definition an element of $S \setminus R$ cannot lie in R, so the intersection $R \cap (S \setminus R)$ must be empty.

(iii) This is also fairly clear from the definition; alternatively (as usual) one can use truth tables.

(b) **Answer:** The collection $\{B_i \mid i \in I\}$ is a partition of $\cup_{i \in I} B_i$ if and only if the collection is disjoint ($B_i \cap B_j = \emptyset$ for all $i \neq j$) and each B_i is non-empty. The proof is fairly similar to that of part (a).

5. Problem 9(b)(d) in 1.3 from the BOOK. Note that the definition of a *refinement* can be rephrased as follows. Let $\mathcal{C} = \{C_i \mid i \in I\}$ and $\mathcal{D} = \{D_j \mid j \in J\}$ be partitions of the same set. Then $\mathcal{C}$ is a refinement of $\mathcal{D}$ if every block of C is contained in some block of D, that is, for every $i \in I$ there exists $j \in J$ such that $C_i \subseteq D_j$. Justify your answer.

Solution: (b) For simplicity let us use the following notations: E will denote the set of all even natural numbers; O the set of all odd integers; $E_{<99}$ the set of all even natural numbers < 99 and $E_{>99}$ the set of all even natural numbers > 99. Thus, $\mathcal{P} = \{E, O\}$ and $\mathcal{C} = \{E_{<99}, E_{>99}, O\}$. We claim that $\mathcal{P}$ is finer than $\mathcal{C}$ – to prove this we need to show that each block of $\mathcal{P}$ is contained in some block of $\mathcal{C}$. The latter is clear: the first two blocks of $\mathcal{C}$ are both contained in E (which is a block of $\mathcal{P}$), and the third block of $\mathcal{C}$ is itself a block of $\mathcal{P}$ (this is fine, since every set is contained in itself).

(d) Again $\mathcal{P}$ is finer than $\mathcal{C}$. In this case every block of $\mathcal{P}$ has size 1, so $\mathcal{P}$ is actually finer than any partition of $\mathbb{R}$.

6. Problem 2(b)(d)(f) in 1.4 from the BOOK. Your statements should avoid expression like "it is not true that ..." or "it is false that ..."

Solution:

(b) Presidential candidates must be 35 years or older and must be citizens of the United States.

(d) Presidential candidates do not have to be citizens of United States or possess \$27 million. This is the most direct way to phrase the negation, but the obtained statement may be somewhat ambiguous. Using the De Morgan law $\sim (Q \vee R) \equiv (\sim Q) \wedge (\sim R)$, we can restate the negation in a longer (but probably a more transparent) way: Presidential candidates do not have be citizens of United States and presidential candidates do not have to possess \$27 million.

(f) This is a bit challenging to phrase in good English. The best I could come up with is "Presidential candidates do not have to be citizens of United States who are 35 years or older and possess \$27 million."

7. Solve problem 6 in 1.4 from the BOOK using truth tables (here A, B and C are arbitrary statements).

Solution: This is essentially identical to 4(c) in HW#1. The only difference is that here A, B and C are statement and not sets (which is why column labels are A, B, C etc. instead of $x \in A$ etc.).

A	B	C	$B \vee C$	$A \wedge (B \vee C)$	$A \wedge B$	$A \wedge C$	$(A \wedge B) \vee (A \wedge C)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**T**	T	F	**T**
T	F	T	T	**T**	F	T	**T**
T	F	F	F	**F**	F	F	**F**
F	T	T	T	**F**	F	F	**F**
F	T	F	T	**F**	F	F	**F**
F	F	T	T	**F**	F	F	**F**
F	F	F	F	**F**	F	F	**F**

8. Solve problem 13 in 1.4 from the BOOK using just De Morgan and distributive laws (do not compute the truth tables).

Solution: We have $\sim [(P\vee \sim Q) \wedge R] \equiv (\sim [(P\vee \sim Q]) \vee (\sim R) \equiv (\sim P\wedge \sim (\sim Q)) \vee (\sim R) \equiv (\sim P \wedge Q) \vee (\sim R)$. The first two equivalences hold by de Morgan laws, and the third equivalence uses the "double negation law" $\sim (\sim Q) \equiv Q$.

9. Let P, Q and R be statements.

(a) Find a statement S obtained from P, Q and R using negation, conjunction and disjunction (possibly several times) whose truth value is given by the following table:

P	Q	R	S
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

(b) Now let U be any statement whose truth value is completely determined once the truth values of P, Q and R are known. Prove that there exists a statement V obtained from P, Q and R using negation, conjunction and disjunction such that V and U are equivalent statements.

Solution: We will describe the general algorithm for part (b), prove that it does work, and then apply the algorithm to the statement S in part (a).

(b) The algorithm is as follows: We start by looking at the U-column of the truth table. For each line L which has T in the U-column we form a statement V_L of the form $P' \wedge Q' \wedge R'$ where $P' = P$ if L has T in the P-column and $P' =\sim P$ if L has F in the P-column; similarly Q' is Q or $\sim Q$ and R' is R or $\sim R$ depending on the entries in the Q and R columns. For instance, if (P, Q, R) entires are (F, T, F), the statement V_L is $(\sim P) \wedge Q \wedge (\sim R)$. If we create a column corresponding to the statement V_L, it is clear that this column will have T in the line L and F in all other lines.

Now if $\mathcal{L}$ is the set of all lines which have T in the U column and V is the disjunction of all the statements V_L with $L \in \mathcal{L}$ (that is, $V = \vee_{L\in\mathcal{L}} V_L$), then the V-column will have T in all lines from $\mathcal{L}$ and F in all other lines. Thus by construction, the U-column and the V-column coincide and hence $U \equiv V$.

(a) Now we apply the above algorithm to the statement $U = S$ from part (a). The S-column has T in lines $4, 5$ and 8. The corresponding statements V_L are $P \wedge (\sim Q) \wedge (\sim R)$ (for line 4), $(\sim P) \wedge Q \wedge R$ (for line 5) and $(\sim P) \wedge (\sim Q) \wedge (\sim R)$ (for line 8).

Thus the statement

$$V = (P \wedge (\sim Q) \wedge (\sim R)) \vee ((\sim P) \wedge Q \wedge R) \vee ((\sim P) \wedge (\sim Q) \wedge (\sim R))$$

is equivalent to S. Note that the disjunction of the first and third statements above is $(\sim Q)\wedge(\sim R)$ (by distributivity laws and the fact that $P \vee (\sim P)$ is identically true), so the above statement V could be replaced by $((\sim Q)\wedge(\sim R)) \vee ((\sim P) \wedge Q \wedge R)$.