KAZHDAN GROUPS WHOSE FC-RADICAL IS NOT VIRTUALLY ABELIAN

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1. Introduction

If G is a group, the *FC-radical* of G, denoted by $FC(G)$, is the set of all elements of G which centralize a finite index subgroup of G. Equivalently, $FC(G)$ is the set of all elements of G with finite conjugacy class.

In [PV] Popa and Vaes asked the following question:

Question 1.1. *Does there exist a residually finite (discrete) group with Kazhdan's property (T) whose FC-radical is not virtually abelian?*

This question was motivated by [PV, Theorem 6.4(b)], which asserts that any group satisfying these conditions admits a free ergodic profinite action whose associated II_1 factor has all positive real numbers in its fundamental group. No Kazhdan group with the latter property was previously known.

In this short note we give a positive answer to Question 1.1 using Golod-Shafarevich groups.

We shall prove the following theorem:

Theorem 1.2. *Every Golod-Shafarevich group has a residually finite quotient whose FC-radical is not virtually abelian.*

In [Er] it was shown that there exist Golod-Shafarevich groups with property (T). Since property (T) is preserved by quotients, applying Theorem 1.2 to any Golod-Shafarevich group with (T), we obtain a group which settles Question 1.1.

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2. Construction

Informally speaking, a finitely generated group G is Golod-Shafarevich if it has a presentation with a "small" set of relators, where relators are counted with suitable weights. The formal definition is given below.

Definition. Let G be a finitely generated group. Given a prime p let $\mathbb{F}_p[G]$ be the $\mathbb{F}_p$-group algebra of G and I the augmentation ideal of $\mathbb{F}_p[G]$. Let $\{\omega_n G\}_{n\geq 1}$ be the Zassenhaus p-filtration of G defined by $\omega_n G = \{g \in G : g-1 \in I^n\}$. For each $g \in G \setminus \bigcap_{n\in\mathbb{N}} \omega_n G$ we put $\deg_p(g)$ to be the largest n such that $g \in \omega_n G$.

It is well known that the subgroups $\{\omega_n G\}$ are of finite index in G. Moreover, they form a base for the pro-p topology on G, and thus $\bigcap_{n\in\mathbb{N}} \omega_n G$ is the kernel of the natural map from G to its pro-p completion $G_{\widehat{p}}$. In particular, if G is residually-p, then $\bigcap_{n\in\mathbb{N}} \omega_n G = \{1\}$, so $\deg_p(g)$ is defined for any $g \in G \setminus \{1\}$.

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Definition. Fix a prime number p.

(a) A group presentation $\langle X|R\rangle$, with X finite, is said to satisfy the *Golod-Shafarevich* (GS) condition (with respect to p), if there is a real number $t \in (0,1)$ such that

$$1 - H_X(t) + H_R(t) < 0 \text{ where } H_X(t) = |X|t \text{ and } H_R(t) = \sum_{r\in R} t^{\deg_p(r)}.$$

(b) A group G is called *Golod-Shafarevich* if it has a presentation satisfying the Golod-Shafarevich condition.

Any Golod-Shafarevich group G is infinite. In fact, its pro-p completion $G_{\widehat{p}}$ must be infinite and moreover satisfies a number of largeness properties (see, e.g., [Ze], [Er] and references therein for precise statements). We shall only use a very weak statement about Golod-Shafarevich groups:

Proposition 2.1. *If a group G is Golod-Shafarevich with respect to p, then its pro-p completion $G_{\widehat{p}}$ is not virtually abelian.*

Proof of Theorem 1.2. Let G be a Golod-Shafarevich group, so that G has a presentation $\langle X|R\rangle$ with $1 - H_X(t) + H_R(t) < 0$ for some $t \in (0,1)$, and let $\varepsilon = -(1 - H_X(t) + H_R(t))$.

Let $k_0 \in \mathbb{N}$ be such that $t^{k_0} < \frac{\varepsilon}{8}$. By Proposition 2.1 we can choose $x_1, y_1 \in \omega_{k_0} G$ which do not commute in $G_{\widehat{p}}$. Then there exists $k_1 > k_0$ such that x_1 and y_1 do not commute in $G/\omega_{k_1} G$. By making k_1 larger we can also assume that $t^{k_1} < \frac{\varepsilon}{16}$.

Let S_1 be a finite generating set for $\omega_{k_1} G$, let $R_1 = \{[x_1, s], [y_1, s] : s \in S_1\}$ and $G_1 = G/\langle R_1\rangle^G$. Note that if $\bar{x}_1$ and $\bar{y}_1$ are the images of x_1 and y_1 in G_1, then

(i) $G/\omega_{k_1} G \cong G_1/\omega_{k_1} G_1$, so $\bar{x}_1$ and $\bar{y}_1$ do not commute modulo $\omega_{k_1} G_1$;
(ii) $\bar{x}_1$ and $\bar{y}_1$ lie in the FC-radical of G_1 (and the same is true for any quotient of G_1).

The group G_1 need not be Golod-Shafarevich, but it surjects onto the group $\widehat{G}_1 = G_1/\langle x_1, y_1\rangle^{G_1}$ which is Golod-Shafarevich by construction.

Thus, the pro-p completion of G_1 is not virtually abelian, so we can find elements $x_2, y_2 \in \omega_{k_1} G$ and $k_2 > k_1$ such that x_2 and y_2 do not commute in $G_2/\omega_{k_2} G_2$ and $t^{k_2} < \frac{\varepsilon}{32}$.

Let S_2 be a finite generating set for $\omega_{k_2} G$, let $R_2 = \{[x_2, s], [y_2, s] : s \in S_2\}$ and $G_2 = G/\langle R_1 \cup R_2\rangle^G$. By construction we have $G_1/\omega_{k_2} G_1 \cong G_2/\omega_{k_2} G_2$, the images of x_2 and y_2 in G_2 lie in the FC-radical of G_2, and G_2 surjects onto a Golod-Shafarevich group.

Continuing this process indefinitely we obtain a sequence of groups $G = G_0 \to G_1 \to G_2 \to \ldots$, elements $\{x_i, y_i\}_{i\in\mathbb{N}}$ of G and integers $k_0 < k_1 < k_2 < \ldots$ s.t.

(i) G_{i+1} is a quotient of G_i for all i
(ii) G_i surjects onto the group $G/\langle \cup_{j=1}^{i} \{x_j, y_j\}\rangle^G$
(iii) x_i and y_i lie in $\omega_{k_{i-1}} G$, and $t^{k_{i-1}} < \frac{\varepsilon}{2^{i+2}}$
(iv) The images of x_i and y_i in $G_{i-1}/\omega_{k_i} G_{i-1}$ do not commute
(v) $G_{i-1}/\omega_{k_i} G_{i-1} \cong G_j/\omega_{k_i} G_j$ for all $j \geq i$
(vi) The images of x_i and y_i in G_i lie in the FC-radical of G_i

Now let G_∞ be the inductive limit of $\{G_i\}$; in other words, if $G_i = G/N_i$, we let $N_\infty = \cup_{i\in\mathbb{N}} N_i$ and $G_\infty = G/N_\infty$. Let Q be the image of G_∞ in its pro-p completion, that is, $Q = G_\infty / \cap_{n\in\mathbb{N}} \omega_n G_\infty$.

Condition (ii) implies that G surjects onto the group $G/\langle \cup_{j=1}^{\infty} \{x_j, y_j\}\rangle^G$ which is Golod-Shafarevich by (iii). Thus, by Proposition 2.1 the group Q is infinite. Since Q is a subset of $(G_\infty)_{\widehat{p}}$, it is also residually finite.

Let $\pi : G \to Q$ be the natural projection. By condition (vi) the FC-radical of Q contains the subgroup H generated by the elements $\{\pi(x_i), \pi(y_i)\}_{i\in\mathbb{N}}$. It remains to show that H is not virtually abelian. Suppose not, so H contains a finite index abelian subgroup A. Then there exists integers $i < j$ such that $\pi(x_i x_j^{-1}) \in A$ and $\pi(y_i y_j^{-1}) \in A$. Conditions (iv) and (v) imply that $\pi(x_i)$ and $\pi(y_i)$ do not commute modulo $\omega_{k_i}Q$. Thus, if φ_i is the projection map $Q \to Q/\omega_{k_i}Q$, then $\varphi_i\pi([x_i, y_i]) \neq 1$. On the other hand, by construction $x_j, y_j \in \omega_{k_i}G$, so $\varphi_i\pi(x_j) = \varphi_i\pi(y_j) = 1$. Therefore,

$$\varphi_i\pi([x_i, y_i]) = \varphi_i\pi([x_i x_j^{-1}, y_i y_j^{-1}]) \in \varphi_i([A, A]) = \{1\}.$$

The obtained contradiction shows that the FC-radical of Q is not virtually abelian, which finishes the proof. □

References

[Er] M. Ershov, *Golod-Shafarevich groups with property* (T) *and Kac-Moody groups,* Duke Math. J. 145 (2008), no. 2, 309–339

[PV] S. Popa and S. Vaes, *On the fundamental group of* II_1 *factors and equivalence relations arising from group actions,* arXiv:0810.0706, to appear in the Proceedings of the Conference in Honor of Alain Connes' 60th birthday, "*Non-Commutative Geometry* " April 2-6, 2007, IHP Paris.

[Ze] E. Zelmanov, *Infinite algebras and pro-p groups.* Infinite groups: geometric, combinatorial and dynamical aspects, 403–413, Progr. Math., 248, Birkhäuser, Basel, 2005.

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