Solutions to Homework #1

2. Problem 2 in 1.2(b)(d)(f)(h)(j)(l) from the BOOK.

Answer: Below C stands for the set of all hearts, D for the set of all diamonds, H for the set of all hearts and S for the set of all spades. Individual cards will be denoted by a pair of symbols; for instance, AH will denote the ace of hearts.

(b) $\emptyset$; cardinality 0

(d) $\{AS, AD, KS, KD\}$; cardinality 4

(f) $C \cup D \cup H$; cardinality 39 since the sets C, D and H are pairwise disjoint and each of them has 13 elements

(h) $\{KC, KH\}$; cardinality 2

(j) The set of all ordered pairs (x, y) where $x \in \{AC, AD, AH, AS\}$ and $y \in \{KC, KD, KH, KS\}$; cardinality $4 \times 4 = 16$

(l) The set of all ordered pairs (x, y) where $x \in S$ and $y \in C \cup D \cup H$; cardinality $13 \cdot 39 = 507$.

3. Let A and B be subsets of the universal set U. Prove that $A \setminus B = A \cap B^c$.

Proof: Of course, this could be proved formally using truth tables, but it is even easier to prove directly that for all $x \in U$ we have $x \in A \setminus B \iff x \in A \cap B^c$.

Indeed, let $x \in U$. Then by definition $x \in A \setminus B \iff (x \in A$ and $x \notin B)$. On the other hand, $x \in A \cap B^c \iff (x \in A$ and $x \in B^c)$ (by definition of the intersection), and $x \in B^c \iff x \notin B$ (by definition of the complement). Thus, $x \in A \cap B^c \iff (x \in A$ and $x \notin B) \iff x \in A \setminus B$, as desired.

4. Let A, B and C be arbitrary sets. Prove each of the following identities in two ways: by drawing the Venn diagram and by using the true-false table:

(a) $(A \cap B)^c = A^c \cup B^c$

(b) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$. The common value of both sides of this equality is called the *symmetric difference of* A *and* B and is usually denoted by $A \triangle B$.

(c) (practice) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(d) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Solution: We prove all the identities using the truth tables. In all cases we can claim equality of sets $U = V$ if the columns of the truth table corresponding to the statements $x \in U$ and $x \in V$ coincide.

(a)

$x \in A$	$x \in B$	$x \in A \cap B$	$x \in (A \cap B)^c$	$x \in A^c$	$x \in B^c$	$x \in A^c \cup B^c$
T	T	T	**F**	F	F	**F**
T	F	F	**T**	F	T	**T**
F	T	F	**T**	T	F	**T**
F	F	F	**T**	T	T	**T**

(b)

$x \in A$	$x \in B$	$x \in A \setminus B$	$x \in B \setminus A$	$x \in (A \setminus B) \cup (B \setminus A)$	$x \in A \cup B$	$x \in A \cap B$	$x \in (A \cup B) \setminus (A \cap B)$
T	T	F	F	**F**	T	T	**F**
T	F	T	F	**T**	T	F	**T**
F	T	F	T	**T**	T	F	**T**
F	F	F	F	**F**	F	F	**F**

(d)

$x \in A$	$x \in B$	$x \in C$	$x \in A \cap B$	$x \in (A \cap B) \cup C$	$x \in A \cup C$	$x \in B \cup C$	$x \in (A \cup C) \cap (B \cup C)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**T**	T	T	**T**
T	F	T	F	**T**	T	T	**T**
T	F	F	F	**F**	T	F	**F**
F	T	T	F	**T**	T	T	**T**
F	T	F	F	**F**	F	T	**F**
F	F	T	F	**T**	T	T	**T**
F	F	F	F	**F**	F	F	**F**

5. Let A, B and C be subsets of the universal set U. Prove the identity

$$A \setminus (B \setminus C) = (A \cap B^c) \cup (A \cap C)$$

without drawing a Venn diagram or computing the true-false table but instead using the identities from problems 3 and 4.

Solution: Applying the result of Problem 3, we get

$$A \setminus (B \setminus C) = A \setminus (B \cap C^c) \qquad (1).$$

Applying Problem 3 again (with $B \cap C^c$ playing the role of B), we get

$$A \setminus (B \cap C^c) = A \cap (B \cap C^c)^c. \qquad (2)$$

By Problem 4(a) we have $(B \cap C^c)^c = B^c \cup (C^c)^c = B^c \cup C$ (the last equality uses the obvious fact that taking the complement twice produces the original set). Thus,

$$A \cap (B \cap C^c)^c = A \cap (B^c \cup C) = (B^c \cup C) \cap A \qquad (3)$$

Finally, by Problem 4(c) we have

$$(B^c \cup C) \cap A = (B^c \cap A) \cup (C \cap A) = (A \cap B^c) \cup (A \cap C) \qquad (4).$$

Combining equalities (1)-(4), we deduce the desired identity $A \setminus (B \setminus C) = (A \cap B^c) \cup (A \cap C)$.

6. In Lecture 2 we will prove that $|A \cup B| = |A| + |B| - |A \cap B|$ for any finite sets A and B (this is also Theorem 1.2.1(a) from the BOOK). Use this result and a suitable part of Problem 4 to prove that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

for any finite sets A, B and C.

Solution: Since $A \cup B \cup C = (A \cup B) \cup C$, applying Theorem 1.2.1 to the sets $A \cup B$ and C, we get $|A \cup B \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$. Again by Theorem 1.2.1, $|A \cup B| = |A| + |B| - |A \cap B|$, and so

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |(A \cup B) \cap C|. \qquad (*)$$

Now by Problem 4(b) we have $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, and applying Theorem 1.2.1 one more time, we get

$$|(A \cup B) \cap C| = |(A \cap C) \cup (B \cap C)| = |A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|.$$

Since $(A \cap C) \cap (B \cap C) = A \cap B \cap C \cap C = A \cap B \cap C$, we have

$$|(A \cup B) \cap C| = |A \cap C| + |B \cap C| - |A \cap B \cap C|.$$

Substituting the expression on the right-hand side for $|(A \cup B) \cap C|$ in (*) yields the desired result.

7. Problem 4(c)(d)(e) in 1.2 from the BOOK. Make sure to justify your answer.

Answer: in all parts the collection is nested and not disjoint. The unions and intersections are as follows: (c) union is $(1, \infty)$; intersection is $\emptyset$; (d) union is $\mathbb{R}$; intersection is $\emptyset$; (e) union is again $\mathbb{R}$; intersection is $(-1, 1)$. We will now give a detailed proof for (d).

For simplicity we put $D_x = (x, \infty)$, so that $\mathcal{D} = \{D_x \mid x \in \mathbb{R}\}$.

1. We first prove that $\mathcal{D}$ is not disjoint. By definition, we just need to find distinct $x, y \in \mathbb{R}$ such that $D_x \cap D_y \neq \emptyset$. The latter is indeed true – for instance, if we let $x = 0$ and $y = 1$, then $2 \in D_x \cap D_y$, so $D_x \cap D_y \neq \emptyset$ (note that actually $D_x \cap D_y \neq \emptyset$ for ALL $x, y \in \mathbb{R}$, but we do not need this stronger fact to prove disjointness).

2. Next we prove that $\mathcal{D}$ is nested. By definition we need to show that for all $x, y \in \mathbb{R}$ we have $D_x \subseteq D_y$ or $D_y \subseteq D_x$. We claim that

(i) if $x \leq y$, then $D_y \subseteq D_x$
(ii) if $y \leq x$, then $D_x \subseteq D_y$

Since for all $x, y \in \mathbb{R}$ we have $x \leq y$ or $y \leq x$, (i) and (ii) above would imply that the collection $\mathcal{D}$ is nested. Further, it is clear that (i) and (ii) are actually equivalent statements, so it suffices to prove (i).

So assume that $x \leq y$. We need to prove that $D_y \subseteq D_x$, that is, we need to prove the implication $z \in D_y \Rightarrow z \in D_x$ for all $z \in \mathbb{R}$. So let $z \in \mathbb{R}$ and assume that $z \in D_y = (y, \infty)$. By definition of (y, ∞), this means that $y < z$. Since $x \leq y$, by transitivity of inequalities we get $x < z$ and hence $z \in (x, \infty) = D_x$. This finishes the proof of (i).

3. Next we prove that $\cup_{x \in \mathbb{R}} D_x = \mathbb{R}$. We will prove the equality of two sets by showing that they are contained in each other. The containment $\cup_{x \in \mathbb{R}} D_x \subseteq \mathbb{R}$ is clear since each D_x is contained in $\mathbb{R}$ by construction. To prove the reverse containment, take any $y \in \mathbb{R}$. We need to show that $y \in \cup_{x \in \mathbb{R}} D_x$ or, equivalently, there exists $x \in \mathbb{R}$ such that $y \in D_x$. The latter is indeed true: if we set $x = y - 1$, then $x \in \mathbb{R}$ and $x < y$, so $y \in (x, \infty) = D_x$.

4. Finally, we prove that $\cap_{x \in \mathbb{R}} D_x = \emptyset$. Proving that two sets are equal is the same as proving that their complements are equal. Thus, we are reduced to proving that $(\cap_{x \in \mathbb{R}} D_x)^c = \mathbb{R}$. By generalized deMorgan laws, $(\cap_{x \in \mathbb{R}} D_x)^c = \cup_{x \in \mathbb{R}} D_x^c$, so we are reduced to showing that $\cup_{x \in \mathbb{R}} D_x^c = \mathbb{R}$. This equality can be proved in the same way as part 3 above after observing that $D_x^c = \{y \in \mathbb{R} \mid y \leq x\} = (-\infty, x]$.

Finally, we make a comment on (e). There was some confusion as to which subsets lie in the collection $\mathcal{E}$ and what it is indexed by. The collection $\mathcal{E}$ here is indexed by $\mathbb{N}$. For each $n \in \mathbb{N}$ the corresponding member of the collection is $E_n = \{x \in \mathbb{R} \mid x^2 < n\}$, that is, E_n is the set of all reals whose square is less than n. As easy computation shows that actually $E_n = (-\sqrt{n}, \sqrt{n})$, so one could define $\mathcal{E}$ simply as $\mathcal{E} = \{(-\sqrt{n}, \sqrt{n}) \mid n \in \mathbb{N}\}$.