**Midterm #2. Due Thursday, November 5th**

**Directions:** Each problem is worth 10 points, and all 4 problems will be counted. Provide complete arguments (do not skip steps). State clearly any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given.

**Rules:** You are not allowed to discuss midterm problems with each other. You may ask me any questions about the problems (e.g. if the formulation is unclear), but I may only provide minor hints. You may use freely your class notes, previous homework assignments and the book by Dummit and Foote. The use of other books is allowed, but not encouraged. If you happen to run across a problem very similar or identical to one on the midterm which is solved in another book, do not consult that solution.

**1.** Let $G$ be a finite group
(a) (5 pts) Prove that $G$ is nilpotent if and only if $G$ contains a normal subgroup of order $m$ for any $m$ dividing $|G|$.
(b) (5 pts) Prove that $G$ is cyclic if and only if $G$ contains a unique subgroup of order $m$ for any $m$ dividing $|G|$.
**Note:** Of course, the forward direction in (b) is well known, so you can assume it without proof. **Hint:** You may find some results in [DF,6.1] useful.

**2.** In all parts of this problem $G$ is a finite group.
(a) (3 pts) Prove that $G$ has a simple quotient (that is, $G$ has a quotient which is a simple group).
(b) (2 pts) Suppose that $G$ is perfect, that is, $[G, G] = G$. Prove that $G$ has a non-abelian simple quotient.
(c) (5 pts) Once again, let $G$ be an arbitrary finite group. Prove that $G$ has a unqiue normal subgroup $K$ such that $K$ is perfect and $G/K$ is solvable. **Hint:** Such $K$ can be easily described in terms of the derived series of $G$.

**3.** (a) (2 pts) We are given groups $G$ and $H$ and homomorphisms $\phi, \psi : G \to \mathrm{Aut}(H)$. Suppose there exists $\alpha \in \mathrm{Aut}(G)$ such that $\psi = \phi\alpha$. Prove that $G \ltimes_\phi H \cong G \ltimes_\psi H$.
(b) (8 pts) Prove that there are precisely 5 isomorphism classes of groups of order 20. Include all the details.

**4.** The purpose of this problem is prove a special case of Schreier's formula for the rank of subgroups of free groups. Let $X = \{x_0, x_1, \ldots, x_d\}$ be a finite set with $d+1$ elements, and let $F = F(X)$ be the free group on $X$. Let $n \in \mathbb{N}$, and let $\phi : F \to \mathbb{Z}_n$ be the unique homomorphism such that $\phi(x_0) = 1$ and $\phi(x_i) = 0$ for $i > 0$ (such homomorphism exists by the universal property of free groups). Let $K = \mathrm{Ker}\phi$.
(a) (2 pts) Prove that $K$ is a subgroup of $F$ of index $n$.
(b) (5 pts) Consider the following subset $Y$ of $F = F(X)$:

$$Y = \{x_0^n\} \cup \{x_0^k x_i x_0^{-k} : 1 \leq i \leq d, 0 \leq k \leq n-1\}.$$

Note that $|Y| = nd+1$. Prove that $Y$ is a generating set for $K$. **Hint:** First prove that any element $w \in F$ can be written as $w = vr$ where $v \in \langle Y \rangle$ and $r = x_0^k$ for some $0 \leq k < n$.
(c) (3 pts) Prove that $Y$ is a *free generating set for* $K$, which informally means that there are no relations between the elements of $Y$. Formally, you need to prove the following: denote elements of $Y$ by $y_1, \ldots, y_m$ (where $m = nd+1$), let $Y' = \{y'_1, \ldots, y'_m\}$ be "another copy" of the set $Y$, and let $\phi : F(Y') \to K$ be the unique homomorphism such that $\phi(y'_i) = y_i$ for each $i$. Show that $\phi$ is an isomorphism (note that $\phi$ is surjective since $Y$ generates $K$ by (b), so you only need to show that $\phi$ is injective). The proof is not difficult, but it may not be easy to write it down formally.
Thus, (c) implies that $K \cong F(Y')$, so by definition $K$ is free of rank $nd+1$ which is consistent with Schreier's formula.