Bilinear Forms and Group Representations, Fall 2017. Midterm #1. Due on Thursday, September 28th, in class

Directions: Provide complete arguments (do not skip steps). State clearly and FULLY any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given. If you are unable to solve a problem (or a part of a problem), you may still use its result to solve a later part of the same problem or a later problem in the exam.

Rules: You are NOT allowed to discuss midterm problems with anyone else except me. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use the following resources:

(i) your class notes

(ii) homework solutions (both your solutions and solutions posted on the course webpage)

(iii) Ben Webster's 4657 notes

(iv) the books 'Linear Algebra' by Friedberg, Insel and Spence and 'Linear Algebra Done Wrong' by Treil

The use of other books or online sources is NOT allowed.

1. (10 pts) Let $V = Mat_2(\mathbb{R})$, the vector spaces of 2×2 matrices over $\mathbb{R}$. Given a real number c, define $H_c : V \times V \to V$ by

$$H_c(A, B) = Tr(AB) + c\,Tr(A)Tr(B).$$

Prove that H_c is a symmetric bilinear form, find a basis β such that $[H_c]_\beta$ is diagonal and compute the signature of H_c (both parts of your answer will depend on c).

2. (10 pts) Let V be a finite-dimensional REAL inner product space and $A \in \mathcal{L}(V)$. Prove that $\langle Ax, x\rangle = 0$ for all $x \in V$ if and only if $A^* = -A$.

3. (10 pts) Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $A, B \in \mathcal{L}(V)$ be such that $AB = BA$.

(a) Let λ be an eigenvalue of A and $E_\lambda(A) = \{v \in V : Av = \lambda v\}$ the corresponding eigenspace. Prove that $E_\lambda(A)$ is B-invariant.

(b) Now assume that V is an inner product space and A and B are Hermitian. Use (a) to prove that there exists an orthonormal basis β such that $[A]_\beta$ and $[B]_\beta$ are both diagonal.

(c) (bonus, 2 extra pts) Now prove the assertion of (b) only assuming that A and B are normal.

4. (10 pts)

(a) Let V be a finite-dimensional vector space over $\mathbb{R}$ and let H be a symmetric bilinear form on V. Prove that one can write $H = H^+ - H^-$ such that H^+ and H^- are both symmetric and positive semidefinite and $rk(H) = rk(H^+) + rk(H^-)$.

(b) Let $A \in Mat_n(\mathbb{R})$ be a symmetric positive semidefinitive matrix of rank r. Prove that A can be written as $A = P^T P$ for some $r \times n$ matrix P.

Note: (a) and (b) are not directly related, but a solution to (a) (which is easier) may give you a hint how to proceed with (b).

5. (10 pts) Let H and G be Hermitian forms on $\mathbb{C}^2$ which are not proportional, and let W be the set of linear combinations of H and G with REAL coefficients. Thus, W is a 2-dimensional (real) subspace of the space $\mathbb{H}(\mathbb{C}^2)$ of all Hermitian forms on $\mathbb{C}^2$ (note that $\mathbb{H}(\mathbb{C}^2)$ is a vector space over $\mathbb{R}$, but not over $\mathbb{C}$). Prove that the following three conditions are equivalent:

(i) The forms H and G are simultaneously diagonalizable, that is, there exists a basis β such that $[H]_\beta$ and $[G]_\beta$ are both diagonal

(ii) The subspace W contains a positive definite form

(iii) If $[H]$ and $[G]$ are the matrices of H and G with respect to the standard basis, then there exist $a, b \in \mathbb{R}$ such that $\det(a[H] + b[G]) > 0$.

Note: It is probably easiest to prove the equivalences (i) $\Longleftrightarrow$ (ii) and (ii) $\Longleftrightarrow$ (iii). I do not see a natural way to relate (i) directly to (iii).