Solutions to homework #11

1. Problem 1(a)(b) from Section 6.3. Recall that we proved 1(c) in Lecture 22. Give a detailed argument.

Solution: Recall that if A and B are sets, the expression $|A| = |B|$ should not be interpreted as equality of two elements of a certain set – it is just a shortcut for the statement 'There exists a bijection from A to B'. Thus, to prove that $\sim$ is reflexive we need to show that for any set A there exists a bijection $f : A \to A$. An example of such a bijection is the identity function ($f(a) = a$ for all $a \in A$).

To prove that $\sim$ is symmetric we need to show that if there exists a bijection $f : A \to B$, then there exists a bijection $g : B \to A$. An example of such g is the inverse function $f^{-1} : B \to A$ (the inverse f^{-1} exists since f is bijective, and it is clear from the definition that f^{-1} is bijective whenever it exists).

2. Problem 2 from Section 6.4. Note: the problem should say $f^{-1}(\{i\})$ is countable, not $|f^{-1}(\{i\})|$ is countable. **Hint:** Let $A_i = f^{-1}(\{i\})$. How is the collection of sets $\{A_i\}$ related to A?

Solution: First we claim that $\cup_{i=1}^{\infty} A_i = A$. The inclusion $\cup_{i=1}^{\infty} A_i \subseteq A$ is clear since each A_i is a subset of A. Conversely, take any $a \in A$. Since f is a function from A to $\mathbb{N}$, we have $f(a) = i$ for some $i \in \mathbb{N}$, so $a \in f^{-1}(\{i\}) = A_i$ and hence $a \in \cup_{i=1}^{\infty} A_i$. Thus, $A \subseteq \cup_{i=1}^{\infty} A_i$.

Since $A = \cup_{i=1}^{\infty} A_i$ and each A_i is countable by assumption, Theorem 23.5 implies that A is also countable.

3. Let $A_1, \ldots, A_n$ be countable sets. Prove that the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is countable.

Solution: We argue by induction on n.

Base case: $n = 1, 2$. If $n = 1$, there is nothing to prove, and if $n = 2$, the result is true by Corollary 6.3.10 from the book.

Induction step: Take $n \geq 3$, and assume that $A_1 \times \ldots \times A_{n-1}$ is countable. We want to show that $A_1 \times \ldots \times A_n$ is countable.

Let $B = A_1 \times \ldots \times A_{n-1}$. Since B is countable by the induction hypothesis, Corollary 6.3.10 implies that $B \times A_n$ is countable. On the other hand, there is an obvious bijection between $A_1 \times \ldots \times A_n$ and $B \times A_n$ (which sends an n-tuple $(a_1, \ldots, a_n)$ to the pair $((a_1, \ldots, a_{n-1}), a_n)$). Thus, $|A_1 \times \ldots \times A_n| = |B \times A_n|$ and hence $A_1 \times \ldots \times A_n$ is also countable.

4. In Lecture 24 (Tue, April 24) we will prove that $\mathbb{Q}$ is countable using Theorem 23.5 from class (a countable union of countable sets is countable). Give another proof of countability of $\mathbb{Q}$ by constructing a surjective map $g : A \times B \to \mathbb{Q}$ for certain countable sets A and B (and using suitable theorems).

Solution: By definition, the elements of $\mathbb{Q}$ are precisely the numbers of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. This means that if we let $A = \mathbb{Z}$ and $B = \mathbb{Z} \setminus \{0\}$ and define the function $f : A \times B \to \mathbb{Q}$ by $f((a, b)) = \frac{a}{b}$, then f is surjective.

We proved in class that $\mathbb{Z}$ is countable, and $\mathbb{Z} \setminus \{0\}$ is countable being a subset of a countable set. Thus, A and B are both countable and hence $\mathbb{Q}$ is countable by Theorem 23.4(2).

5. Let A be an uncountable set and B a countable subset of A.

(a) Prove that $A \setminus B$ is uncountable.

(b) Prove that A and $A \setminus B$ have the same cardinality.

Hint for (b): Since $A \setminus B$ is infinite by (a), by Theorem 6.3.5 from the book we can choose a countably infinite subset C of $A \setminus B$. Use things proved in class to show that the identity map $f : (A \setminus B) \setminus C \to (A \setminus B) \setminus C$ can be extended to a bijection from $A \setminus B$ and A. Draw a picture!

Solution: (i) We argue by contradiction. Suppose that $A \setminus B$ is countable. Then, $A = (A \setminus B) \cup B$ is a union of two countable sets, hence A is countable, contrary to our hypothesis.

(ii) As suggested in the hint, let C be a countably infinite subset of $A \setminus B$. Note that $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

Since B and C are both countable, their union $B \cup C$ is also countable; moreover $B \cup C$ is infinite since C is infinite. Thus, B and C are both countably infinite, so by definition $|B| = |\mathbb{N}|$ and $|C| = |\mathbb{N}|$, whence $|B| = |C|$ by Problem 1, so there is a bijection $\phi : C \to B \cup C$.

Now define the map $f : A \setminus B \to A$ by

$$f(x) = \begin{cases} x & \text{if } x \in (A \setminus B) \setminus C = A \setminus (B \cup C) \\ \phi(x) & \text{if } x \in C. \end{cases}$$

It is clear that f is a bijection from $A \setminus B$ to A.

6. A real number α is called algebraic if α is a root of a (nonzero) polynomial with **integer** coefficients, that is, if there exist integers $c_0, \ldots, c_n$, not all 0 such that $\sum_{k=0}^{n} c_k \alpha^k = 0$. Note that all rational numbers are algebraic (if $\alpha = \frac{p}{q}$, then α is a root of the polynomial $qx - p$), but many irrational numbers are algebraic as well (e.g. $\sqrt{2}$ is algebraic as $\sqrt{2}$ is a root of $x^2 - 2$).

The goal of this problem is to prove that the set of all algebraic numbers is countable.

(a) For a fixed integer $n \geq 0$, let Z_n be the set of all polynomials of degree at most n with integer coefficients, that is, Z_n is the set of all polynomials of the form $\sum\limits_{k=0}^{n} c_k x^k$ with each $c_i \in \mathbb{Z}$. Prove that each Z_n is countable. **Hint:** Construct a bijection between Z_n and a Cartesian product of finitely many countable sets and use the result of Problem 2.

(b) Now use (a) (and a suitable theorem) to show that the set of polynomials with integer coefficients (of arbitrary degree) is countable.

(c) Finally use (b) and the fact that every polynomial has finitely many roots to show that the set of all algebraic numbers is countable.

Solution: (a) Let $\mathbb{Z}^{n+1} = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{n+1 \text{ times}}$ be the Cartesian product of $n+1$ copies of $\mathbb{Z}$. Define the function $f : \mathbb{Z}^{n+1} \to Z_n$ by $f((c_0, \ldots, c_n)) = \sum\limits_{k=0}^{n} c_k x^k$. Then f is surjective (by definition of Z_n); also, f is injective since two polynomials are equal if and only if they have the same coefficients in every degree. Thus, f is bijective. Since $\mathbb{Z}^{n+1}$ is countable by Problem 3, we conclude that Z_n is also countable.

(b) By definition, the set of all polynomials with integer coefficients is equal to $\cup_{n=1}^{\infty} Z_n$. Since each Z_n is countable by (a), $\cup_{n=1}^{\infty} Z_n$ is also countable by Theorem 23.5.

(c) Let Z be the set of all polynomials with integer coefficients. Since Z is countable, there is a sequence $p_1, p_2, \ldots$ containing all elements of Z (and where each $p_i \in Z$). Thus, if we denote by A_i the set of roots of p_i, then $\cup_{i=1}^{\infty} A_i$ is the set of all algebraic numbers. On the other hand, each A_i is finite, hence countable, and therefore $\cup_{i=1}^{\infty} A_i$ is countable.

The above argument can be made a bit more elegant using a slightly more general form of Theorem 23.5 which says the following:

Theorem 23.5': *If I is any countable set and $\{A_i \mid i \in I\}$ is a collection of sets indexed by I such that each A_i is countable, then $\bigcup\limits_{i \in I} A_i$ is countable.*

Note that Theorem 23.5' is what the statement "A countable union of countable sets is countable" really refers to. Theorem 23.5 corresponds to the case where $I = \mathbb{N}$ or $I = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

Let us apply Theorem 23.5' with $I = Z$. For each polynomial $p \in I$ let A_p be the set of its roots. Then the set of all algebraic numbers is $\bigcup\limits_{p \in I} A_p$.

Since each A_p is finite and I is countable, we can conclude that $\bigcup_{p\in I} A_p$ is countable.

7. Problem 3 from Section 6.3. For parts (a) and (b) construct an explicit bijection between the given sets; one way to solve (c) is to use a suitable theorem from Section 6.4.

Solution: (a) By basic calculus the function $f : \mathbb{R} \to (0,\infty)$ given by $f(x) = e^x$ is a bijection, so $|\mathbb{R}| = |(0,\infty)|$.

(b) It is straightforward to check that the function $f : [0,1] \to [a,b]$ given by $f(x) = a + (b-a)x$ is a bijection, so $|[0,1]| = |[a,b]|$.

(c) We will solve (c) without using the Schroeder-Bernstein theorem, but we will use transitivity of equality for cardinalities. By (a) we know $|\mathbb{R}| = |(0,\infty)|$. In HW#9 we showed that the function $f : [0,\infty) \to [0,1)$ given by $f(x) = \frac{x}{x+1}$ is bijection; the same argument shows that the same f is a bijection from $(0,\infty)$ to $(0,1)$, so $|(0,\infty)| = |(0,1)|$.

Thus, to prove that $|\mathbb{R}| = |[0,1]|$ it suffices to prove that $|(0,1)| = |[0,1]|$. Similarly to the end of Lecture 24, we have the following explicit bijection $f : (0,1) \to [0,1]$:

$f(\frac{1}{2}) = 0$; $f(\frac{1}{3}) = 1$; $f(\frac{1}{n}) = \frac{1}{n-2}$ for all $n \in \mathbb{Z}_{\geq 4}$ and finally $f(x) = x$ for all $x \in (0,1) \setminus \{\frac{1}{n} \mid n \in \mathbb{Z}_{\geq 2}\}$.