PROPERTY (T) FOR KAC-MOODY GROUPS OVER RINGS

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Abstract. Let R be a finitely generated commutative ring with 1, let A be an indecomposable 2-spherical generalized Cartan matrix of size at least 2 and $v = v(A)$ the largest absolute value of a non-diagonal entry of A. We prove that there exists an integer $n = n(A)$ such that the Kac-Moody group $\mathbb{G}_A(R)$ has property (T) whenever R has no proper ideals of index less than n and all positive integers less than or equal to v are invertible in R.

1. Introduction

Kac-Moody groups can be thought of as infinite-dimensional analogues of Chevalley groups or algebraic groups. Over the last two decades they have attracted a lot of attention from mathematicians working in many different areas. Kac-Moody groups were shown to have a deep and interesting structure theory; at the same time they provided an excellent source of test examples for various conjectures and helped settle open problems in geometric group theory, Lie theory and related areas (see, e.g., [CR2, Re] and references therein). Most of the results obtained so far deal with Kac-Moody groups over fields, with the case of finite fields proving to be particularly interesting. At the same time, the subject of Kac-Moody groups over rings remains a largely uncharted territory, so much so that even the "right" definition over non-fields is yet to be agreed on.

Kac-Moody groups over rings in the sense of this paper are defined very explicitly by generators and relations. This is the definition used, for instance, in recent papers of Allcock [Al1, Al2]; see § 2 for a brief discussion of other possible definitions and connections between them. Given a generalized Cartan matrix A, let $\Phi = \Phi(A)$ be the associated system of real roots. For an arbitrary commutative ring R (with 1), the corresponding simply-connected Kac-Moody group $\mathbb{G}_A(R)$ is the group generated by the root subgroups $\{X_\alpha\}_{\alpha\in\Phi}$, each of which is isomorphic to the additive group of F, modulo certain Steinberg-type relations.[1] In particular, if A is a matrix of finite (spherical) type and F is a field, $\mathbb{G}_A(F)$ is the corresponding Chevalley group with its standard Steinberg presentation. It is easy to see that the obtained correspondence $R \mapsto \mathbb{G}_A(R)$ is functorial and for any epimorphisms of rings $R \to S$ the corresponding map $\mathbb{G}_A(R) \to \mathbb{G}_A(S)$ is also an epimorphism.

The main goal of this paper is to establish a sufficient condition for Kac-Moody groups over rings to have Kazhdan's property (T). The first results on property (T) for (non-spherical and non-affine) Kac-Moody groups are due to Dymara and Januszkiewicz [DJ] who proved that the group $\mathbb{G}_A(F)$ has property (T) for any indecomposable 2-spherical generalized Cartan matrix A and any finite field F satisfying $|F| > 1764^{d(A)}$, where $d(A)$ denotes the size of A. In [Op1], Oppenheim obtained a quantitative improvement of this

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[1] In this paper we will only discuss simply-connected Kac-Moody groups, so for brevity we will restrict all the definitions to the simply-connected case.

result, replacing exponential bound in $d(A)$ by a polynomial (in fact, quadratic) one; see also [Op2] where a generalization of property (T) dealing with affine isometric actions on Banach spaces is established under similar restrictions on F. It does not seem possible to extend those proofs to groups over non-fields since both arguments make essential use of the action of Kac-Moody groups on the associated buildings, which can only be constructed in the case of fields.

In [EJ], an algebraic counterpart of the method from [DJ] was used to establish property (T) for "unipotent subgroups" of Kac-Moody groups over finite rings. The "positive unipotent subgroup" $\mathbb{G}_A^+(R)$ of the Kac-Moody group $\mathbb{G}_A(R)$ is defined to be the subgroup of $\mathbb{G}_A(R)$ generated by positive root subgroups. In [EJ], it was shown that the group $\mathbb{G}_A^+(R)$ has property (T) for any indecomposable 2-spherical $d \times d$ generalized Cartan matrix A with simply-laced Dynkin diagram and any finite commutative ring R with 1 which has no proper ideals of index less than $(d-1)^2$. The abelianization of the group $\mathbb{G}_A^+(R)$ is isomorphic to a direct sum of several copies of $(R,+)$, whence $\mathbb{G}_A^+(R)$ cannot possibly have property (T) if R is infinite. However, it turns out that the techniques from [EJ] can be used to prove property (T) for the full Kac-Moody group $\mathbb{G}_A(R)$ over an arbitrary finitely generated commutative ring R under a similar restriction on indices of ideals in R:

Theorem 1.1. *Let $A = (a_{ij})$ be an indecomposable 2-spherical generalized Cartan matrix of size $d \geq 2$, let $v = v(A) = \max\{|a_{ij}| : i \neq j\} = \max\{-a_{ij} : i \neq j\}$ be the largest absolute value of a non-diagonal entry of A (the assumptions on A imply that $1 \leq v \leq 3$). Define the integer $n = n(A)$ as follows:*

(i) $n = (2d-2)^2$ *if $v = 1$ (equivalently, the Dynkin diagram of A is simply-laced)*
(ii) $n = 3(2d-2)^4$ *if $v = 2$ (equivalently, the Dynkin diagram of A has a double edge, but no triple edges)*
(iii) $n = 188(2d-2)^{16}$ *if $v = 3$ (equivalently, the Dynkin diagram of A has a triple edge)*

Let R be any finitely generated commutative ring with 1 which does not have proper ideals of index less than n and such that every positive integer $\leq v$ is invertible in R. Then the Kac-Moody group $\mathbb{G}_A(R)$ has Kazhdan's property (T).

As a simple corollary of this theorem, we deduce a uniform bound for Kazhdan constants (with respect to generating sets of bounded size) for Kac-Moody groups of a fixed indecomposable 2-spherical type over finite fields of sufficiently large characteristic:

Corollary 1.2. *Let A be an indecomposable 2-spherical generalized Cartan matrix of size $d \geq 2$, and let $n = n(A)$ be defined as in Theorem 1.1. There exist an integer $k = k(A)$ and a real number $\varepsilon = \varepsilon(A) > 0$ with the following property: for any finite field F with $\mathrm{char}(F) > n$ there exists a finite generating set $S(F)$ of $\mathbb{G}_A(F)$ with $|S| \leq k$ such that $\kappa(\mathbb{G}_A(F), S(F)) \geq \varepsilon$ where $\kappa(\mathbb{G}_A(F), S(F))$ is the Kazhdan constant of $\mathbb{G}_A(F)$ with respect to $S(F)$.*

We do not know if the above result remains true if a lower bound on characteristic of F is replaced by a lower bound on its size, even if we assume that A is simply-laced. To prove the corollary observe that the ring $R_n = \mathbb{Z}[\frac{1}{n!}, t]$ (the ring of polynomials in one variable over $\mathbb{Z}[\frac{1}{n!}]$) has no ideals of index less than n and surjects onto any finite field of characteristic larger than n. Therefore, for any finite field F with $\mathrm{char}(F) > n$ there exists a natural epimorphism $\pi : \mathbb{G}_A(R_n) \to \mathbb{G}_A(F)$. If S is any finite generating set for $\mathbb{G}_A(R_n)$ and $S(F)$ is its image in $\mathbb{G}_A(F)$, we get $\kappa(\mathbb{G}_A(F), S(F)) \geq \kappa(\mathbb{G}_A(R_n), S)$, and $\kappa(\mathbb{G}_A(R_n), S) > 0$ since $\mathbb{G}_A(R_n)$ has property (T) by Theorem 1.1.

Another interesting result on property (T) for Kac-Moody groups was obtained by Hartnick and Köhl [HK] who proved that for any local field F and any indecomposable 2-spherical $d \times d$ generalized Cartan matrix A, with $d \geq 2$, the group $\mathbb{G}_A(F)$ has property (T) when considered as a topological group with Kac-Peterson topology. We will give an alternative proof of this theorem in § 5.

Organization: In § 2 we recall basic properties of Kac-Moody root systems and define Kac-Moody groups over commutative rings. In § 3 we collect background information on property (T). In § 4 we establish some auxiliary results on orthogonality constants in Chevalley groups of rank 2 (see § 3 for the definition of orthogonality constants). In § 5 we prove Theorem 1.1 as well as its variation dealing with "pseudo-parabolic" subgroups and give a new proof of the theorem of Hartnick and Köhl mentioned above.

Convention: All rings considered in this paper are assumed to be unital.

2. Preliminaries on Kac-Moody groups

In this section we recall basic properties of Kac-Moody root systems, define Kac-Moody groups over rings and state some basic facts about them. Note that Kac-Moody groups can be defined without an explicit reference to Kac-Moody Lie algebras even though Lie algebras are needed to establish some key properties of Kac-Moody groups. For more details we refer the reader to the book of Kac [Kac] and recent paper of Allcock [Al1].

2.1. **Kac-Moody root systems.** Let $A = (a_{ij})$ be a *generalized Cartan matrix* (abbreviated below as GCM), that is, a square matrix satisfying the following conditions:

(a) $a_{ij} \in \mathbb{Z}$ for all i, j (b) $a_{ii} = 2$ for all i

(c) $a_{ij} \leq 0$ if $i \neq j$ (d) $a_{ij} = 0 \iff a_{ji} = 0$.

For the rest of the section we fix a GCM A with entries a_{ij} and let d denote its size.

Denote by $\Delta = \Delta(A)$ the root system of the complex Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated to A and by $\Phi = \Phi(A)$ the set of real roots in Δ. We shall not use Δ or $\mathfrak{g}$ in this paper, so we will define Φ directly in terms of A.

Let $Q = \oplus_{i=1}^d \mathbb{Z}\alpha_i$ and $Q^\vee = \oplus_{i=1}^d \mathbb{Z}\alpha_i^\vee$ be free abelian groups of rank d with bases $\Pi = \{\alpha_1, \dots \alpha_d\}$ and $\Pi^\vee = \{\alpha_1^\vee, \dots \alpha_d^\vee\}$. Define the bilinear pairing $\langle \cdot, \cdot \rangle : Q^\vee \times Q \to \mathbb{Z}$ by $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. For each $1 \leq i \leq d$ define the map $s_i \in \mathrm{Aut}\,(Q)$ by $s_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i$; in particular $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$.

Let $W = \langle s_1, \dots, s_d \rangle$ be the subgroup of $\mathrm{Aut}\,(Q)$ generated by $\{s_i\}_{i=1}^d$, and define $\Phi = W(\Pi)$ to be the set of W orbits of Π. It is not hard to check that the group W is a Coxeter group; it is called the *Weyl group* of A. The elements of Π are called *simple roots*, and elements of Φ are called *real roots*; we will refer to real roots just as *roots* in this paper since we will never deal with imaginary roots. As in the case of finite root systems, every root is a linear combination of simple roots with all coefficients non-negative or all coefficients non-positive; roots are called positive and negative, accordingly. The sets of positive and negative roots will be denoted by Φ^+ and Φ^-, respectively.

The Weyl group W has unique action on $Q^\vee$ such that $\langle w\alpha_i^\vee, w\alpha_j \rangle = \langle \alpha_i^\vee, \alpha_j \rangle$. For each root $\alpha \in \Phi$ define $\alpha^\vee \in Q^\vee$ as follows: choose $1 \leq i \leq d$ and $w \in W$ such that $\alpha = w\alpha_i$ and define $\alpha^\vee = w\alpha_i^\vee$; this definition does not depend on the representation of α as $w\alpha_i$. Define $s_\alpha \in \mathrm{Aut}\,(Q)$ by $s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha$. It is easy to see that $s_{w\alpha} = w s_\alpha w^{-1}$ for all $\alpha \in \Phi, w \in W$; in particular this implies that each $s_\alpha \in W$.

Definition 2.1. Given a subset I of $\{1, \dots, d\}$, we let A_I be the $|I| \times |I|$ matrix $(a_{ij})_{i,j \in I}$. Matrices of the form A_I will be called *submatrices of* A.

Definition 2.2. Let A be a GCM.

(i) A is called *indecomposable* if there is no partition $\{1,\dots,d\} = I \sqcup J$ with $I, J \neq \emptyset$ such that $a_{ij} = 0$ for all $i \in I, j \in J$.
(ii) A is called *spherical* (or finite) if it is positive definite.
(iii) A is called *affine* if A is positive semi-definite and all of its proper indecomposable submatrices are spherical.
(iv) Given an integer $2 \leq k \leq d$, the matrix A is called *k-spherical* if for every $I \subseteq \{1,\dots,k\}$ with $|I| = k$, the submatrix A_I is spherical. This is equivalent to saying that for any such I the subgroup $\langle s_i : i \in I\rangle$ of W is finite.

It is easy to see that $A = (a_{ij})$ is *2-spherical* if and only if $a_{ij}a_{ji} \leq 3$ for all $i \neq j$.

Definition 2.3. A pair of roots α, β in Φ is called *prenilpotent* if there exist $w, w' \in W$ such that $w\alpha, w\beta \in \Phi^+$ and $w'\alpha, w'\beta \in \Phi^-$.

In the following proposition we collect some well-known properties of prenilpotent pairs which will be used in this paper.

Proposition 2.4. *Let $\alpha, \beta \in \Phi$ with $\beta \neq -\alpha$. The following hold:*

(a) *$\{\alpha, \beta\}$ is prenilpotent if and only if the set $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Phi$ is finite.*
(b) *The numbers $\langle \alpha^\vee, \beta\rangle$ and $\langle \beta^\vee, \alpha\rangle$ are both positive, both negative or both zero.*
(c) *If $\langle \alpha^\vee, \beta\rangle \geq 0$, then $\{\alpha, \beta\}$ is prenilpotent, and moreover $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Phi \subseteq \{\alpha + \beta\}$, that is, $\alpha + \beta$ is the only possible root in $\mathbb{N}\alpha + \mathbb{N}\beta$.*
(d) *Assume that $\langle \alpha^\vee, \beta\rangle < 0$. Then the following are equivalent:*
  (i) *$\{\alpha, \beta\}$ is prenilpotent;*
  (ii) *$\langle s_\alpha, s_\beta\rangle$ is finite;*
  (iii) *$\langle \alpha^\vee, \beta\rangle \cdot \langle \beta^\vee, \alpha\rangle \leq 3$.*

Proof. (a) holds by [KP2, Proposition 4.7]. Note that in [KP2] the result is proved under the additional hypothesis that $\alpha, \beta \in \Phi^+$; however, (a) easily follows from this special case. Indeed, both conditions in (a) do not change if we replace $\{\alpha, \beta\}$ by $\{w\alpha, w\beta\}$ for some $w \in W$. Since $\beta \neq -\alpha$, it is easy to see that there exists $w \in W$ such that $w\alpha$ and $w\beta$ are both positive or both negative. In the former case we are reduced to the situation in [KP2], and in the latter case we use the obvious fact that $\{\gamma, \delta\}$ is prenilpotent if and only if $\{-\gamma, -\delta\}$ is prenilpotent.

(b) holds, for example, by [KP2, p.139] (see the argument after Lemma 1.2).

(c) follows from (a) and [KP2, Lemma 2.1(c)(ii)] (using the same remark as in the proof of (a)). Finally, (d) follows from the proof of [KP2, Proposition 4.7]. □

Finally, recall the notion of the *Dynkin diagram* of A, which we will denote by $Dyn(A)$. We define $Dyn(A)$ to be a graph with vertex set $\Pi = \{\alpha_1, \dots, \alpha_d\}$, where α_i and α_j (for $i \neq j$) are connected by $a_{ij}a_{ji}$ edges. Given a subset I of $\{1,\dots,d\}$, we denote by $Dyn_I(A)$ the full subgraph of $Dyn(A)$ on the vertex set $\{\alpha_i : i \in I\}$. We will refer to $Dyn_I(A)$ as a *Dynkin subdiagram.*

2.2. **Definition of Kac-Moody groups over rings.** In this subsection we define Kac-Moody groups over rings by certain presentations by generators and relators. Our definition is easily seen to be equivalent to the one in Allcock's paper [Al1].

Let A be a GCM and R a commutative ring with 1. Define the Kac-Moody group $\mathbb{G}_A(R)$ to be the group with generators $\{x_\alpha(r) : \alpha \in \Phi, r \in R\}$ subject to relations (R1)-(R7) below. In those relations $r, u \in R$ and $\alpha, \beta \in \Phi$ are arbitrary unless a restriction is explicitly imposed. The signs in the relations (R3) are not canonical and depend on the

choice of a Chevalley basis for the Kac-Moody Lie algebra associated to A (see [Ti, 3.2] for details).

(R1) $x_\alpha(r+u) = x_\alpha(r)x_\alpha(u)$

(R2) If $\{\alpha, \beta\}$ is a prenilpotent pair, then

$$[x_\alpha(r), x_\beta(u)] = \prod_{i,j\geq 1} x_{i\alpha+j\beta}(C_{ij\alpha\beta} r^i u^j)$$

where the product on the right hand side is over all pairs $(i,j) \in \mathbb{N}\times\mathbb{N}$ such that $i\alpha+j\beta \in \Phi$, in some fixed order, and $C_{ij\alpha\beta}$ are integers independent of R (but depending on the order). Note that the product is finite by Proposition 2.4(a).

For $1 \leq i \leq d$ and $r \in R^\times$ set

$$\tilde{s}_i(r) = x_{\alpha_i}(r)x_{-\alpha_i}(-r^{-1})x_{\alpha_i}(r), \quad \tilde{s}_i = \tilde{s}_i(1) \quad \text{and} \quad h_i(r) = \tilde{s}_i(r)\tilde{s}_i^{-1}.$$

The remaining relations are

(R3) $\tilde{s}_i x_\alpha(r)\tilde{s}_i^{-1} = x_{s_i\alpha}(\pm r)$

(R4) $h_i(r)x_\alpha(u)h_i(r)^{-1} = x_\alpha(ur^{\langle\alpha_i^\vee,\alpha\rangle})$ for $r \in R^\times$

(R5) $\tilde{s}_i h_j(r)\tilde{s}_i^{-1} = h_j(r)h_i(r^{-a_{ji}})$ for $r \in R^\times$

(R6) $h_i(ru) = h_i(r)h_i(u)$ for $r, u \in R^\times$

(R7) $[h_i(r), h_j(u)] = 1$ for $r, u \in R^\times$

Remark 2.5. (a) Even though it is not obvious from the above relations, one can show that the Kac-Moody group $\mathbb{G}_A(R)$ can be defined directly in terms of the root system Φ, without explicit reference to A. In view of this, we will occasionally write $\mathbb{G}_\Phi(R)$ instead of $\mathbb{G}_A(R)$.

(b) The subgroup $\widetilde{W} = \langle \tilde{s}_i : 1 \leq i \leq d\rangle$ of $\mathbb{G}_A(F)$ is usually not isomorphic to W; however, there is always an epimorphism $\widetilde{W} \to W$ which sends $\tilde{s}_i$ to s_i for each i, whose kernel is a finite group of exponent ≤ 2.

Groups $\mathbb{G}_A(R)$ given by the above presentation first appeared in Tits' paper [Ti]; however, they are not called Kac-Moody groups in [Ti]. Instead Tits defines a *Kac-Moody functor* (corresponding to a fixed GCM A), which we denote by $\mathbb{G}_A^{Tits}$, to be a functor from commutative rings to groups satisfying certain axioms and shows that, when restricted to fields, such a functor is unique (up to natural equivalence), and for a field F, the group $\mathbb{G}_A^{Tits}(F)$ is isomorphic to the group $\mathbb{G}_A(F)$ given by the above presentation. If R is a ring which is not a field, it is not known whether the group $\mathbb{G}_A^{Tits}(R)$ is uniquely determined up to isomorphism by the functor axioms or whether the group $\mathbb{G}_A(R)$ satisfies these axioms. What is known (and already established in [Ti]) is that there is a natural homomorphism from $\mathbb{G}_A(R)$ to $\mathbb{G}_A^{Tits}(R)$ mapping root subgroups onto root subgroups. Another possible definition of Kac-Moody groups over rings, which uses highest weight modules, is briefly discussed in [Al2] and generalizes the analogous definition in the case of fields [CG]. Again, it can be shown that there are natural homomorphisms from the groups $\mathbb{G}_A(R)$ to those "highest-weight module Kac-Moody groups".

Since property (T) is preserved by homomorphic images, in view of the above remarks, the statement of Theorem 1.1 remains true if the groups $\mathbb{G}_A(R)$ are replaced by either highest-weight module Kac-Moody groups or subgroups of Tits' Kac-Moody groups generated by root subgroups (with the same restrictions on A and R).

Before proceeding, we define several Steinberg-type groups which project onto $\mathbb{G}_A(R)$. Let $St_A^{(2)}(R)$ be the group generated by the same set of symbols $\{x_\alpha(r) : \alpha \in \Phi, r \in R\}$ but only subject to relations (R1) and (R2) and by $St_A^{(3)}(R)$ the group with the same

generating set and relations (R1), (R2) and (R3), so that we have natural epimorphisms

$$St_A^{(2)}(R) \to St_A^{(3)}(R) \to \mathbb{G}_A(R).$$

The group $St_A^{(2)}(R)$ is called the Steinberg group (corresponding to the pair (A, R)) in [Ti]. The group $St_A^{(3)}(R)$ does not seem to have a specific name in the literature; we point it out since it is the largest quotient of $St_A^{(2)}(R)$ for which we will be able to prove property (T) in the setting of our main theorem (thus, relations (R4)-(R7) will not be important for us). Finally, [MR] and [Al1] use a different notion of Steinberg group – in their terminology Steinberg group is certain quotient of $St_A^{(3)}(R)$ which projects onto $\mathbb{G}_A(R)$. We refer the reader to [Al1] for the precise definition and detailed discussion about the relationship between different Steinberg-type groups.

2.3. **Some examples and facts about Kac-Moody groups.** For every root $\alpha \in \Phi$ and every subset S of R we will set $X_\alpha(S) = \{x_\alpha(s) : s \in S\}$ (considered as a subgroup of $\mathbb{G}_A(R)$). If S is a subgroup of $(R, +)$, then $X_\alpha(S)$ is a subgroup isomorphic to S. We will write $X_\alpha = X_\alpha(R)$ whenever R is clear from the context.

The groups $\{X_\alpha\}$ are called the root subgroups of $\mathbb{G}_A(R)$. Define $\mathbb{G}_A^+(R) = \langle X_\alpha : \alpha \in \Phi^+\rangle$ to be the subgroup of $\mathbb{G}_A(R)$ generated by all positive root subgroups.

We proceed with two basic examples of Kac-Moody groups.

Example 1. *Let A be a GCM of spherical type (that is, A is a Cartan matrix) and $\Phi = \Phi(A)$.*

Given a commutative ring R, let $\mathbb{E}_\Phi(R)$ denote the elementary subgroup of the Chevalley group of type Φ over R. Then there exists a natural epimorphism $\mathbb{G}_A(R) = \mathbb{G}_\Phi(R) \to \mathbb{E}_\Phi(R)$, which is an isomorphism whenever R is a field.

Example 2. *Let $d \geq 2$ be an integer and define the $d \times d$ matrix $A = (a_{ij})$ by $a_{ij} = -1$ if $i - j \equiv \pm 1 \mod d$, $a_{ij} = 2$ if $i = j$ and $a_{ij} = 0$ otherwise. Then $\Phi = \Phi(A)$ is an affine root system of type $\widetilde{A}_d$.*

Given a commutative ring R, there exists an epimorphism $\pi : \mathbb{G}_A(R) \to EL_d(R[t, t^{-1}])$ given by

$$\pi(x_{\alpha_i}(r)) = E_{i,i+1}(r) \text{ for } 1 \leq i \leq d-1, \quad \pi(x_{\alpha_d}(r)) = E_{d,1}(rt),$$
$$\pi(x_{-\alpha_i}(r)) = E_{i+1,i}(r) \text{ for } 1 \leq i \leq d-1, \quad \pi(x_{-\alpha_d}(r)) = E_{d,1}(rt^{-1}).$$

As in Example 1, π is an isomorphism if R is a field. The group $\pi(\mathbb{G}_A^+(R))$ coincides with the subgroup of $EL_d(R[t])$ consisting of matrices which have upper-unitriangular image under the projection $EL_d(R[t]) \to EL_d(R)$ which sends t to 0.

We now return to the general case. The following theorem is part of the assertion of [Al1, Theorem 1.1]:

Theorem 2.6. *Let A be a 2-spherical GCM and $v = \max\{-a_{ij} : i \neq j\}$ (thus, $v \leq 3$). Let R be a commutative ring which does not have proper ideals of index $\leq v$. Then $\mathbb{G}_A^+(R)$ is generated by $\{X_{\alpha_i}\}_{i=1}^d$, the root subgroups corresponding to simple roots.*

A key part of the proof of Theorem 2.6 is establishing the result when A has rank 2 (so that Φ is a finite root system of rank 2), in which case the result is deduced directly from commutation relations between root subgroups. Below we list those relations since most of them will be explicitly used. In all four cases below we denote the simple roots of Φ by

α and β with α being the long root. We shall only list non-trivial commutator relations, that is, relations between root subgroups which do not commute.

Case 1: $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\Phi = A_1 \times A_1$, $\Phi^+ = \{\alpha, \beta\}$. In this case X_α and X_β commute.

Case 2: $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\Phi = A_2$, $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$. In this case

$$[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(rs)$$

(for suitable choice of Chevalley basis).

Case 3: $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, $\Phi = B_2$, $\Phi^+ = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta\}$. In this case we have the following relations:

$$\begin{aligned} [x_\alpha(r), x_\beta(s)] &= x_{\alpha+\beta}(rs)x_{\alpha+2\beta}(rs^2) \\ [x_{\alpha+\beta}(r), x_\beta(s)] &= x_{\alpha+2\beta}(2rs). \end{aligned}$$

Case 4: $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, $\Phi = G_2$, $\Phi^+ = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}$. In this case we have the following relations:

$$\begin{aligned} [x_\alpha(r), x_\beta(s)] &= x_{\alpha+\beta}(rs)x_{\alpha+2\beta}(rs^2)x_{\alpha+3\beta}(rs^3)x_{2\alpha+3\beta}(r^2s^3), \\ [x_{\alpha+\beta}(r), x_\beta(s)] &= x_{\alpha+2\beta}(2rs)x_{\alpha+3\beta}(3rs^2)x_{2\alpha+3\beta}(3r^2s), \\ [x_{\alpha+2\beta}(r), x_\beta(s)] &= x_{\alpha+3\beta}(3rs), \\ [x_{\alpha+2\beta}(r), x_{\alpha+\beta}(s)] &= x_{2\alpha+3\beta}(3rs), \\ [x_{\alpha+3\beta}(r), x_\alpha(s)] &= x_{2\alpha+3\beta}(-rs). \end{aligned}$$

3. Property (T)

We start by recalling the definition of property (T) as well as the definition of relative property (T) in the sense of [Co]. In the definitions below we allow G to be an arbitrary topological group; however, we will deal primarily with discrete groups, with Theorem 5.4 being the only exception. For a general introduction to property (T) we refer the reader to [BHV].

Definition 3.1. Let G be a group and S a subset of G.

(a) Let V be a unitary representation of G and $\varepsilon > 0$. A vector $v \in V$ is called (S, ε)-invariant if $\|sv - v\| < \varepsilon\|v\|$ for all $s \in S$
(b) The Kazhdan constant $\kappa(G, S)$ is the largest $\varepsilon \geq 0$ such that if V is any unitary representation of G which contains an (S, ε)-invariant vector, then V contains a nonzero G-invariant vector.
(c) S is called a *Kazhdan subset* of G if $\kappa(G, S) > 0$.
(d) G has *property* (T) if it has a compact Kazhdan subset.

Definition 3.2. Let G be a group and B a subset of G. The pair (G, B) is said to have *relative property* (T) if there exist a compact subset S of G and a function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that if V is any unitary representation of G and $v \in V$ satisfies $\|sv - v\| \leq f(\varepsilon)\|v\|$ for every $s \in S$, then $\|bv - v\| \leq \varepsilon\|v\|$ for every $b \in B$.

In the case when B is a normal subgroup, the above definition is equivalent to the following one (which is the original definition of relative property (T)):

Definition 3.3. Let B be a normal subgroup of G. The pair (G, B) has *relative property* (T) if there exists a compact subset S of G such that whenever a unitary representation V of G contains an (S, ε)-invariant vector, it must also contain a nonzero B-invariant vector.

The only result about relative property (T) which we will explicitly use in this paper is the following straightforward lemma.

Lemma 3.4. *Let G be a group and K a subgroup of G with property (T). Then for any subset H of K, the pair (G, H) has relative property (T).*

Proof. Since K has (T), it is clear from the second definition of relative property (T) that the pair (K, K) has relative (T). And it is clear from the first definition that if a pair (A, B) has relative (T), then for any overgroup $A' \supseteq A$ and subset $B' \subseteq B$, the pair (A', B') has relative (T). In particular, (G, H) has relative (T). □

In order to prove our main theorem we will use the "almost orthogonality" criterion for property (T) based on the notion of orthogonality constant between subgroups of the same group. This method was originally introduced by Dymara and Januskieiwcz in [DJ] and developed further in [EJ, EJK, Ka2, Op2].

Definition 3.5. Let H and K be subgroups of the same group, and let $G = \langle H, K \rangle$ be the group generated by them.

(i) Given a unitary representation V of G, let V^H and V^K denote the subspaces of H-invariant (resp. K-invariant) vectors in V. The orthogonality constant $orth(H, K; V)$ is defined by

$$orth(H, K; V) = \sup\{|\langle v, w \rangle| : v \in V^H, w \in V^K, \|v\| = \|w\| = 1\}.$$

(ii) The orthogonality constant $orth(H, K)$ is the supremum of the quantities $orth(H, K; V)$ where V ranges over all unitary representations of G without invariant vectors.

We will use the following form of the almost orthogonality criterion.

Theorem 3.6. ([EJ, Theorem 1.2]) Let G be a discrete group and $H_1, \ldots, H_d$ subgroups of G. Suppose that $orth(H_i, H_j) < \frac{1}{d-1}$ for any $i \neq j$ and the pair (G, H_i) has relative property (T) for each i. Then G has property (T).

The following lemma collects two important cases where the orthogonality constant is small.

Lemma 3.7. *The following hold:*

(a) *(see* [EJ, Lemma 3.4]*) If H and K are subgroups of the same group, and one of them normalizes the other (e.g., if they commute), then* $\mathrm{orth}(H, K) = 0$.

(b) *(special case of* [EJ, Proposition 4.7]*) Let R be a countable associative ring and $m(R)$ the smallest index of a proper ideal of R. Let $G = Heis(R)$ be the Heisenberg group over R, that is, the group of upper-unitriangular matrices over R, let $H = E_{12}(R)$ and $K = E_{23}(R)$. Then* $\mathrm{orth}(H, K) \leq \frac{1}{\sqrt{m(R)}}$.

Here are two basic examples where property (T) follows immediately from Theorem 3.6 and Lemma 3.7:

Example 3. *Let A be a 2-spherical $d \times d$ GCM with simply-laced Dynkin diagram and R any finite commutative ring which has no proper ideals of index at most $(d-1)^2$. Then the positive unipotent subgroup $\mathbb{G}_A^+(R)$ has property (T).*

Here we let $\{H_1, \dots, H_d\}$ be the simple root subgroups of $G = \mathbb{G}_A^+(R)$. The groups H_i are finite since R is finite, so the assumption that (G, H_i) has relative property (T) holds trivially. Since $Dyn(A)$ is simply-laced, any two of the subgroups H_i and H_j either commute (if $a_{ij} = 0$) or generate the Heisenberg group $Heis(R)$ (if $a_{ij} = -1$).

Example 4. *Now let $d \geq 3$ be any integer and R any finitely generated associative ring with no proper ideals of index at most $(d-1)^2$. Let $G = EL_d(R)$, the subgroup of $GL_d(R)$ generated by elementary matrices. Then G has property (T).*

As proved in [EJ], the group $EL_d(R)$ actually has property (T) without any restrictions on indices of ideals in R, but this result requires a much more general version of the almost orthogonality method. To deduce the result stated in Example 4 directly from Theorem 3.6 and Lemma 3.7 we set $H_i = E_{i,i+1}(R)$ for $1 \leq i \leq d-1$ and $H_d = E_{d,1}(R)$. Then H_i and H_j either commute (if $i - j \not\equiv \pm 1 \mod d$) or generate the Heisenberg group, whence $\mathrm{orth}(H_i, H_j) < \frac{1}{d-1}$ by Lemma 3.7. The fact that the pairs (G, H_i) have relative property (T) follows from Kassabov's theorem [Ka1, Theorem 1.2].

Note that the argument in Example 4 remains valid if we replace $EL_d(R)$ by the Steinberg group $St_d(R)$. Thus it also proves that the Kac-Moody group $\mathbb{G}_{A_{d-1}}(R)$ has property (T) since $\mathbb{G}_{A_{d-1}}(R)$ is a quotient of $St_d(R)$ as observed in § 2. As we will see in § 5, our general argument for property (T) for Kac-Moody groups over rings will in some sense generalize Example 4 even though the collection of subgroups $\{H_i\}$ to which Theorem 3.6 will be applied in the case of groups of type A_{d-1} is different from the one in Example 4.

4. Orthogonality constants in Chevalley groups of rank 2

Notation: Let G be a group generated by subgroups X and Y and let H be another subgroup of G. Let $m(H, X, Y) \in \mathbb{N} \cup \{\infty\}$ be the minimal dimension of an irreducible representation V of G such that H acts non-trivially on V and the subspaces V^X and V^Y are both nonzero.

The following result is a variation of [EJK, Theorem 10.8] and is proved by essentially the same argument.

Theorem 4.1. *Let G be a countable group generated by subgroups X and Y. Let H be a subgroup of $Z(G)$, denote by X' and Y' the images of X and Y in G/H, respectively, and let $m = m(H, X, Y)$. Then* $\mathrm{orth}(X, Y) \leq \sqrt{\mathrm{orth}(X', Y') + \frac{1}{m}}$.

Proof. Let $\varepsilon = \mathrm{orth}(X', Y')$. By [EJK, Claim 10.7] is suffices to prove that $\mathrm{orth}(V^X, V^Y) \leq \sqrt{\varepsilon + \frac{1}{m}}$ for every non-trivial *irreducible* representation V of G.

Let us fix such a representation V. We can assume that V^X and V^Y are both nonzero (otherwise the result is trivial). If H acts trivially on V, then V is a representation of G/H, so there is nothing to prove. Thus, we can assume that H acts non-trivially, and therefore $\dim(V) \geq m$ by assumption.

Let $HS(V)$ denote the set of all Hilbert-Schmidt operators on V, that is, linear operators $A : V \to V$ such that $\sum_i \|A(e_i)\|^2$ is finite where $\{e_i\}$ is an orthonormal basis of V. Then $HS(V)$ is a Hilbert space with the inner product given by

$$\langle A, B \rangle = \sum_i \langle A(e_i), B(e_i) \rangle.$$

It is easy to see that the set $HS(V)$ and the inner product do not depend on the choice of $\{e_i\}$.

The representation of G on V yields the corresponding representation of G on $HS(V)$, where an element $g \in G$ acts on $A \in HS(V)$ by $(gA)(v) = gA(g^{-1}v)$. Since $H \subseteq Z(G)$ and V is an irreducible representation of G, by Schur's lemma H acts by scalars on V and hence trivially on $HS(V)$. Thus, $HS(V)$ becomes a representation of G/H.

For a nonzero vector $v \in V$ let $P_v \in HS(V)$ denote the (orthogonal) projection onto $\mathbb{C}v$. For a subspace W of $HS(V)$ let $\pi_W : HS(V) \to HS(V)$ denote the projection onto W. The following results are proved in [EJK]:

(a) [EJK, Lemma 10.1]) $\langle P_u, P_w\rangle = |\langle u, w\rangle|^2$ for any unit vectors $u, w \in V$;
(b) [EJK, Lemma 10.3]) $\|\pi_{HS(V)^G}(P_v)|^2 = \frac{1}{\dim(V)}$ for any unit vector $v \in V$.

The assertion of Theorem 4.1 follows easily from these two results. Indeed, take any unit vectors $u \in V^X$ and $w \in V^Y$. Then by (a)

$$|\langle u, w\rangle|^2 = \langle P_u, P_w\rangle = \langle \pi_{HS(V)^G}(P_u), \pi_{HS(V)^G}(P_w)\rangle + \langle \pi_{(HS(V)^G)^\perp}(P_u), \pi_{(HS(V)^G)^\perp}(P_w)\rangle.$$

By (b) we have

$$|\langle \pi_{HS(V)^G}(P_u), \pi_{HS(V)^G}(P_w)\rangle| \le \|\pi_{HS(V)^G}(P_u)\| \cdot \|\pi_{HS(V)^G}(P_w)\| = \frac{1}{\dim(V)} \le \frac{1}{m}.$$

On the other hand, $(HS(V)^G)^\perp$ is a representation of G/H without invariant vectors, so $|\langle \pi_{(HS(V)^G)^\perp}(P_u), \pi_{(HS(V)^G)^\perp}(P_w)\rangle| \le \varepsilon \|\pi_{(HS(V)^G)^\perp}(P_u)\| \cdot \|\pi_{(HS(V)^G)^\perp}(P_w)\| \le \varepsilon$, which finishes the proof. □

Lemma 4.2. *Let R be a countable commutative ring and $m(R)$ the smallest index of a proper ideal of R. Let Φ be an irreducible finite root system of rank 2, and let $\{\alpha, \beta\}$ be a base of Φ, with α a long root.*

Define the group G and its subgroups X, Y and H by one of the following:

(a) $\Phi = A_2$, $G = \mathbb{G}^+_\Phi(R)$, $X = X_\alpha, Y = X_\beta$ *and* $H = X_{\alpha+\beta}$
(b) $\Phi = B_2$, $G = \mathbb{G}^+_\Phi(R)$, $X = X_\alpha, Y = X_\beta$ *and* $H = X_{\alpha+2\beta}$.
(c) $\Phi = G_2$, $G = \mathbb{G}^+_\Phi(R)$, $X = X_\alpha, Y = X_\beta$ *and* $H = X_{\alpha+3\beta}X_{2\alpha+3\beta}$

In case (b) assume that 2 is invertible in R, and in case (c) assume that 3 is invertible in R. Then (in each case) $m(H, X, Y) \ge m(R)$

Remark 4.3. Case (a) of Lemma 4.2 has already been established in [EJ]; however, we have chosen to reproduce the proof as the arguments in other cases are similar, with additional technicalities involved.

Proof. In each case we start with an arbitrary irreducible representation V of G with $V^X \neq \{0\}$ and $V^Y \neq \{0\}$ on which H acts non-trivially. Our goal is to show that $\dim(V) \ge m(R)$. Since $H \subseteq Z(G)$ by assumption, there exists a non-trivial character $\lambda : H \to S^1$ such that each $h \in H$ acts on V as the scalar $\lambda(h)$.

(a) For brevity we set $\lambda(r) = \lambda(x_{\alpha+\beta}(r))$ for $r \in R$. By assumption there exists nonzero $v \in V^X$. The commutator relation $[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(rs)$ (with $r, s \in R$ arbitrary) implies that

$$x_\alpha(r)x_\beta(s)v = x_\beta(s)x_\alpha(r)[x_\alpha(r), x_\beta(s)]v = \lambda(rs)x_\beta(s)x_\alpha(r)v = \lambda(rs)x_\beta(s)v.$$

Thus, for every $s \in R$, the vector $x_\beta(s)v$ is an eigenvector for X with character $\lambda_s : X \to S^1$ given by $\lambda_s(x_\alpha(r)) = \lambda(rs)$. Let $I = \{s \in R : \lambda(rs) = 0 \text{ for all } r \in R\}$. Then it is clear that I is an ideal of R; moreover, $I \neq R$ since λ is non-trivial, and therefore, $|R/I| \ge m(R)$. On the other hand, $\lambda_s = \lambda_t$ if and only if $s \equiv t \mod I$, and therefore the number of distinct characters of the form λ_s is at least $m(R)$. Since eigenvectors corresponding to distinct characters must be linearly independent, we conclude that $\dim(V) \ge m(R)$ as desired.

(b) This time we set $\lambda(r) = \lambda(x_{\alpha+2\beta}(r))$ for $r \in R$. Again choose any nonzero $v \in V^X$. Since 2 is invertible in R, the set $\{s \in R : \lambda(2rs) = 0 \text{ for all } r \in R\}$ is a proper ideal of R, and arguing as in (a), this time using the relations $[x_{\alpha+\beta}(r), x_\beta(s)] = x_{\alpha+2\beta}(2rs)$, we conclude that $\dim(V) \geq m(R)$.

(c) First assume that $X_{2\alpha+3\beta}$ acts non-trivially on V. Then the result follows directly from (a) since there is an isomorphism between $\langle X_\alpha, X_{\alpha+3\beta}, X_{2\alpha+3\beta}\rangle$ and $St_{A_2^+}(R)$ which sends $X_{2\alpha+3\beta}$ to $X_{\alpha+\beta}$.

Assume now that $X_{2\alpha+3\beta}$ acts trivially on V. Then by assumption $X_{\alpha+3\beta}$ must act non-trivially, and moreover V is a representation of $G' = G/X_{2\alpha+3\beta}$. The following relation holds in G':

$$[x_{\alpha+2\beta}(r), x_\beta(s)] = x_{\alpha+3\beta}(3ru).$$

Since 3 is invertible in R and $X_{\alpha+3\beta}$ is a central subgroup of G' which acts non-trivially, arguing as in (a), we conclude that $\dim(V) \geq m(R)$. □

Combining Theorem 4.1 and Corollary 4.4, we can now estimate orthogonality constants between simple root subgroups in Chevalley groups of rank 2.

Given a positive real number m, define the sequence $s_0(m), s_1(m), \ldots$ by $s_0(m) = 0$ and $s_i(m) = \sqrt{s_{i-1}(m) + \frac{1}{m}}$ for all $i \geq 1$.

Corollary 4.4. *Let R be a countable commutative ring and $m = m(R)$ the smallest index of a proper ideal of R. Let Φ be a finite root system of rank 2, and let $\{\alpha, \beta\}$ be a base of Φ, with α a long root. Let $G = \mathbb{G}^+_\Phi(R)$, $X = X_\alpha(R)$ and $Y = X_\beta(R)$. The following hold:*

(a) *If $\Phi = A_1 \times A_1$, then* $\operatorname{orth}(X, Y) = 0$
(b) *If $\Phi = A_2$, then* $\operatorname{orth}(X, Y) \leq s_1(m) = \frac{1}{\sqrt{m}}$
(c) *If $\Phi = B_2$ and 2 is invertible in R, then* $\operatorname{orth}(X, Y) \leq s_2(m) < \sqrt[4]{\frac{3}{m}}$
(d) *If $\Phi = G_2$ and 2 and 3 are invertible in R, then* $\operatorname{orth}(X, Y) \leq s_4(m) < \sqrt[16]{\frac{188}{m}}$

Proof. In case (a) X and Y commute, so we are done by Lemma 3.7(a). Each of the subsequent cases follows from the previous one using Theorem 4.1, Lemma 4.2 and the following isomorphisms which send simple root groups to simple root groups: $\mathbb{G}^+_{A_2}(R)/X_{\alpha+\beta} \cong \mathbb{G}^+_{A_1\times A_1}(R)$, $\mathbb{G}^+_{B_2}(R)/X_{\alpha+2\beta} \cong \mathbb{G}^+_{A_2}(R)$, $\mathbb{G}^+_{G_2}(R)/\langle X_{\alpha+3\beta}, X_{2\alpha+3\beta}\rangle \cong \mathbb{G}^+_{B_2}(R)$ (for the reduction of G_2 to B_2 we need to apply Theorem 4.1 twice). □

5. Proof of the main theorem and some variations

In this section we will establish Theorem 1.1 and briefly discuss some of its variations. Theorem 1.1 The result will be obtained as an easy consequence of Corollary 4.4 and the following theorem:

Theorem 5.1. *Let A be a 2-spherical $d \times d$ GCM whose indecomposable components have size at least two (equivalently, the Dynkin diagram of A has no isolated vertices). Let R be a commutative ring and $m(R)$ the minimal index of a proper ideal of R, and assume that $m(R) > \max\{-a_{ij} : i \neq j\}$. Let $G = \mathbb{G}_A(R)$ and $\Phi = \Phi(A)$ the associated real root system. Then there exists a subset Σ of Φ with $|\Sigma| < 2d$ such that*

(a) *$\gamma + \delta \neq 0$ for any $\gamma, \delta \in \Sigma$;*
(b) *for any $\gamma, \delta \in \Sigma$ either X_γ and X_δ commute or there exist $\alpha_i, \alpha_j \in \Pi$ and $w \in W$ such that $w\gamma, w\delta \in \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$.*
(c) *the set $\cup_{\gamma\in\Sigma} X_\gamma$ generates $G = \mathbb{G}_A(R)$.*

We will first prove Theorem 1.1 assuming Theorem 5.1 and then prove Theorem 5.1.

Proof of Theorem 1.1. We will prove that G has property (T) by applying Theorem 3.6 to the collection of subgroups $\{H_\gamma\}_{\gamma\in\Sigma}$, where Σ satisfies the conclusion of Theorem 5.1.

First we show that the pair (G, X_α) has relative property (T) for every $\alpha \in \Sigma$. By relations (R3) in the definition of $\mathbb{G}_A(R)$, replacing X_α by a conjugate, we can assume that $\alpha = \alpha_i$ is a simple root. Since A is indecomposable, there exists $j \neq i$ such that $a_{ij} \neq 0$. Let $K = \langle X_{\pm\alpha_i}, X_{\pm\alpha_j}\rangle \subseteq G$. Let $\Phi_{i,j} = \Phi(A_{\{i,j\}})$ (where $A_{\{i,j\}} = \begin{pmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{pmatrix}$). Since A is 2-spherical and $a_{ij} \neq 0$, $\Phi_{i,j}$ is a root system of type A_2, B_2 or G_2, and it is clear from the defining relations that K is a quotient of the Steinberg group $St_{\Phi_{i,j}}(R)$. The group $St_{\Phi_{i,j}}(R)$ has property (T) by [EJK], whence (G, X_α) has relative property (T) by Lemma 3.4.

It remains to check the required upper bounds on orthogonality constants. Take any $\gamma, \delta \in \Sigma$. By Theorem 5.1 either X_γ and X_δ commute, in which case $\mathrm{orth}(X_\gamma, X_\delta) = 0$, or there exist $\alpha_i, \alpha_j \in \Pi$ and $w \in W$ such that $w\gamma, w\delta \in \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j$. In the latter case, after conjugation in G, we can assume that $w = 1$. Since $\gamma + \delta \neq 0$, an easy case-by-case verification using commutator relations in Chevalley groups shows that there exists a finite root system Ψ of rank 2 and an epimorphism $\mathbb{G}^+_\Psi(R) \to \langle X_{\alpha_i}, X_{\alpha_j}\rangle$ which sends simple root subgroups of $\mathbb{G}^+_\Psi(R)$ to X_{α_i} and X_{α_j}; moreover, $\Psi \neq B_2$ if $a_{ij}a_{ji} \leq 1$ and $\Psi \neq G_2$ if $a_{ij}a_{ji} \leq 2$. Applying Corollary 4.4 (and recalling the assumption on R in the Theorem 1.1), we deduce that $\mathrm{orth}(X_{\alpha_i}, X_{\alpha_j}) < \frac{1}{2d-2} \leq \frac{1}{|\Sigma|-1}$. □

Proof of Theorem 5.1. Recall that $\Pi = \{\alpha_1, \dots, \alpha_d\}$ denotes the set of simple roots and $Dyn(A)$ is the Dynkin diagram of A. Let Π_1 be a maximal subset of Π with the property that no two roots in Π_1 are connected to each other in $Dyn(A)$. Let $\Pi_2 = \Pi \setminus \Pi_1$, $k = |\Pi_1|$ and $l = |\Pi_2| = d - k$. Without loss of generality we can assume that $\Pi_1 = \{\alpha_1, \dots, \alpha_k\}$. For brevity set $\beta_i = \alpha_{k+i}$ for $1 \leq i \leq l$, so that $\Pi_2 = \{\beta_1, \dots, \beta_l\}$.

Let $w_0 = s_{\alpha_1} \dots s_{\alpha_k}$, set $\gamma_i = w_0(-\beta_i)$ for $1 \leq i \leq l$, and let $\Sigma = \Pi \sqcup \{\gamma_i\}_{i=1}^l$. Clearly $|\Sigma| < 2d$ and Σ has property (a). We will now prove that Σ also satisfies (b) and (c).

(b) Let $\gamma, \delta \in \Sigma$. If $\gamma, \delta \in \Pi$ or if $\gamma, \delta \in \{\gamma_i\}$, there is nothing to prove. Thus, after possibly swapping γ and δ, we can assume that $\delta = \alpha_i$ for some $1 \leq i \leq d$ and $\gamma = \gamma_j = w_0(-\beta_j)$ for some $1 \leq j \leq l$

Case 1: $\alpha_i \neq \beta_j$ (that is, $i \neq j+k$). In this case any element of $\mathbb{N}\gamma + \mathbb{N}\delta$ clearly has both positive and negative coefficients (when expressed as a linear combination of simple roots), hence cannot be a root. Therefore, the pair $\{\gamma, \delta\}$ is prenilpotent by Proposition 2.4(a), and X_γ and X_δ commute by relations (R2).

Case 2: $\alpha_i = \beta_j$. We have $\gamma_j = -\beta_j - \sum_{t=1}^k n_t\alpha_t$ where each $n_t \geq 0$; moreover $n_t > 0$ if and only if α_t is connected to β_j in $Dyn(A)$. Thus, $\langle \gamma, \delta^\vee\rangle = \langle \gamma_j, \beta_j^\vee\rangle \geq -2 + m$ where m is the number of roots in Π_1 connected to β_j (by assumption $m \geq 1$). If $m = 1$, then $\gamma, \delta \in \mathbb{Z}\alpha_t \oplus \mathbb{Z}\beta_j$ where α_t is the unique root in Π_1 which is connected to β_j. Assume now that $m \geq 2$. Then $\langle \gamma, \delta^\vee\rangle \geq 0$, whence by Proposition 2.4(c), $\{\gamma, \delta\}$ is prenilpotent and the intersection $(\mathbb{N}\gamma + \mathbb{N}\delta) \cap \Phi$ is either empty or equals $\{\gamma + \delta\}$. In the former case we are done as in case 1, and the latter case is actually impossible. Indeed $\gamma + \delta = -\sum_{t=1}^k n_t\alpha_t$ with at least two coefficients n_t positive, whence $\gamma + \delta$ is not a root since the roots $\{\alpha_t\}_{t=1}^k$ are pairwise disjoint (as elements of $Dyn(A)$).

(c) Let H be the subgroup generated by $\cup_{\gamma\in\Sigma}X_\gamma$. We need prove that $H = G$, for which it is sufficient to check that H contains $X_{-\omega}$ for every simple root ω. We will argue by induction on d.

Base case $d = 2$. In this case $\Phi = A_2, B_2$ or G_2 and, following our earlier convention, we denote the elements of Π by α and β, with α being a long root.

Note that if $\{\gamma, \delta\}$ is any base of Φ, then $\{\gamma, \delta\}$ (considered as an unordered pair) is conjugate to Π; hence by Theorem 2.6 and our assumption on R, for any root $\varepsilon \in \mathbb{N}\gamma+\mathbb{N}\delta$ the root subgroup X_ε lies in $\langle X_\gamma, X_\delta\rangle$.

Subcase 1: $\Phi = A_2$, $\alpha_1 = \alpha$, $\beta_1 = \beta$. In this case $\gamma_1 = s_\alpha(-\beta) = -(\alpha + \beta)$. Since $\{\gamma_1, \beta\}$ is a base of Φ and $-\alpha = \gamma_1 + \beta$, we have $X_{-\alpha} \in H$.

Once we know that H contains X_α and $X_{-\alpha}$, we conclude that $\widetilde{s}_\alpha \in H$, whence $X_{-\beta} = X_{s_\alpha(\gamma_1)} = \widetilde{s}_\alpha^{-1} X_\gamma \widetilde{s}_\alpha \in H$, and we are done.

Subcase 2: $\Phi = B_2$, $\alpha_1 = \alpha$, $\beta_1 = \beta$. In this case $\gamma_1 = -(\alpha + \beta)$.

Since $\{\gamma_1, \alpha\}$ is a base and $-(\alpha + 2\beta) = 2\gamma_1 + \alpha$, the subgroup H contains $X_{-(\alpha+2\beta)}$. Since $\{-(\alpha + 2\beta), \beta\}$ is a base and $-\alpha = -(\alpha + 2\beta) + 2\beta$, we conclude that H contains $\widetilde{s}_\alpha$, and we can finish the proof as in subcase 1.

Subcase 3: $\Phi = B_2$, $\alpha_1 = \beta$, $\beta_1 = \alpha$. In this case $\gamma_1 = s_\beta(-\alpha) = -(\alpha + 2\beta)$.

Since $\{-(\alpha + 2\beta), \beta\}$ is a base and $-(\alpha + \beta) = -(\alpha + 2\beta) + \beta$, we have $X_{-(\alpha+\beta)} \in H$, hence we are done by subcase 2.

Subcase 4: $\Phi = G_2$, $\alpha_1 = \alpha$, $\beta_1 = \beta$. In this case $\gamma_1 = -(\alpha + \beta)$, and the argument is analogous to subcase 2 with $-(\alpha + 2\beta)$ replaced by $-(\alpha + 3\beta)$.

Subcase 5: $\Phi = G_2$, $\alpha_1 = \beta$, $\beta_1 = \alpha$. In this case $\gamma_1 = -(\alpha + 3\beta)$, and the argument is analogous to subcase 3, again with $-(\alpha + 2\beta)$ replaced by $-(\alpha + 3\beta)$.

We proceed with the induction step. Let $d > 2$, and assume (b) has been established for all matrices of size less than d. If $Dyn(A)$ is disconnected, the induction step is trivial, so we can assume that $Dyn(A)$ is connected.

Case 1: Each simple root in Π_1 is connected to a root of Π_2 different from β_1. In this case the Dynkin subdiagram with vertex set $\Pi \setminus \{\beta_1\}$ has no isolated vertices and Π_1 is a maximal subset of pairwise disconnected vertices in $\Pi \setminus \{\beta_1\}$. Hence, by induction hypothesis H contains $X_{-\gamma}$ for every $\gamma \in \Pi$ except possibly $\gamma = \beta_1$. In particular, $X_{-\alpha_i} \in H$ for all $1 \le i \le k$. Since $-\beta_1 = s_{\alpha_1} \dots s_{\alpha_k}(\gamma_1)$ and $X_{\gamma_1} \in H$, arguing as in subcase 1 of the base step, we conclude that $X_{-\beta_1} \in H$, and we are done.

Case 2: There exists a simple root in Π_1 which is only connected to β_1. Without loss of generality assume that α_1 is the root with this property. Since we assume that $Dyn(A)$ is connected, β_1 must be connected to a simple root other than α_1. Hence the Dynkin subdiagram with vertex set $\Pi \setminus \{\alpha_1\}$ has no isolated vertices, and it is clear that $\Pi_1 \setminus \{\alpha_1\}$ is a maximal subset of pairwise disconnected vertices in $\Pi \setminus \{\alpha_1\}$.

Since α_1 is only connected to β_1, we have $s_{\alpha_1} \dots s_{\alpha_k}(-\beta_j) = s_{\alpha_2} \dots s_{\alpha_k}(-\beta_j)$ for all $j \ne 1$. Hence, by induction hypothesis H contains $X_{-\gamma}$ for every $\gamma \in \Pi$ except possibly $\gamma = \beta_1$ and $\gamma = \alpha_1$.

In particular, H contains $X_{\pm\alpha_i}$ for all $2 \le i \le k$. Since $X_{\gamma_1} \subseteq H$ and $s_{\alpha_2} \dots s_{\alpha_k}(\gamma_1) = s_{\alpha_1}(-\beta_1)$, it follows that that H contains $X_{s_{\alpha_1}(-\beta_1)}$. Applying the result in the base case to the Dynkin subdiagram with vertex set $\{\alpha_1, \beta_1\}$, we conclude that H contains $X_{-\alpha_1}$ and $X_{-\beta_1}$. The proof is complete. □

5.1. **Some variations.** As we already saw in Example 3, if R is a finite commutative ring and A is a 2-spherical GCM with simply-laced Dynkin diagram, then the positive subgroup $\mathbb{G}_A^+(R)$ of the Kac-Moody group $\mathbb{G}_A(R)$ has property (T) whenever R has no proper ideals

of small index. The proof of Theorem 1.1 shows that the result remains true without the assumption that A is simply-laced. As already mentioned in the introduction, if R is infinite, the group $\mathbb{G}_A^+(R)$ has infinite abelianization and thus cannot have property (T); however, as we will prove below, one can still construct many subgroups with property (T) which lie between $\mathbb{G}_A^+(R)$ and $\mathbb{G}_A(R)$, at least when the Dynkin diagram of A is simply-laced.

Let A be a GCM of size d and I a subset of $\{1, 2, \ldots, d\}$. Given a commutative ring R, define $P_{A,I}(R)$ to the subgroup of $\mathbb{G}_A(R)$ generated by all simple root subgroups $X_{\alpha_i}(R), 1 \leq i \leq d$ as well as the negative root subgroups $X_{-\alpha_i}(R), i \in I$ (thus, $\mathbb{G}_A(R) = P_{A,I}(R)$ for $I = \{1, 2, \ldots, d\}$ and $\mathbb{G}_A^+(R) = P_{A,\emptyset}(R)$). We will call the groups $P_{A,I}(R)$ the *pseudo-parabolic* subgroups of $\mathbb{G}_A(R)$ (the parabolic subgroups, which we will not consider, are defined in the same way except that they must also contain the standard torus).

Theorem 5.2. *Let $A = (a_{ij})$ be an indecomposable 2-spherical GCM of size $d \geq 2$ with simply-laced Dynkin diagram, and let I be a subset of $\{1, 2, \ldots, d\}$ such that for every $1 \leq i \leq d$ there exists $j \in I$ such that $j \neq i$ and $a_{ij} \neq 0$. If R is any finitely generated commutative ring which does not have proper ideals of index less than $(2d-2)^2$, then the pseudo-parabolic subgroup $P_{A,I}(R)$ has property (T).*

Remark 5.3.

1. We believe that Theorem 5.2 remains valid without the assumption that $Dyn(A)$ is simply-laced (under stronger restrictions on R, as in Theorem 1.1); however, a proof of such theorem would require results on relative property (T) which do not seem to be known at the moment.

2. If R surjects onto $\mathbb{Z}$, it is easy to show that the condition on I in the statement of Theorem 5.2 is necessary for $P_{A,I}(R)$ to have property (T). We do not know if there are any infinite rings for which $P_{A,I}(R)$ has property (T) without the above condition on I.

Proof. The proof of Theorem 5.2 is essentially the same as that of Theorem 1.1, requiring just small modifications. Let I_1 be a maximal subset of I with the property that any two simple roots α_i, α_j with $i, j \in I_1$, are not connected. Let $I_2 = I \setminus I_1$, $I_3 = \{1, \ldots, d\} \setminus I$ and $\Pi_k = \{\alpha_i : i \in I_k\}$ for $k = 1, 2, 3$.

Let $w = \prod_{i \in I_1} s_{\alpha_i}$, let $\Lambda = w(-\Pi_2) = \{-w(\alpha_j) : j \in I_2\}$ and $\Sigma = \Pi \cup \Lambda$. We claim that Σ satisfies conditions (a) and (b) of Theorem 5.1 and the set $\{X_\gamma : \gamma \in \Sigma\}$ generates $P_{A,I}(R)$. Condition (a) is obvious. Next we check (b) – it is clear if $\gamma, \delta \in \Pi$ or $\gamma, \delta \in \Lambda$ and holds by the proof of Theorem 5.2 if $\gamma \in \Pi_1 \cup \Pi_2$ and $\delta \in \Lambda$ (or vice versa). If $\gamma \in \Pi_3$ and $\delta \in \Lambda$ (or vica versa), then no simple root appears in the expansion of both γ and δ; since γ and δ have opposite signs, X_γ and X_δ commute by the argument from Case 1 of the proof of Theorem 5.1(b). Finally, applying the proof of Theorem 5.1 to the Dynkin subdiagram $Dyn_I(A)$, we conclude that the subgroup $\langle X_\gamma : \gamma \in \Pi_1 \cup \Pi_2 \cup \Lambda \rangle$ is equal to $\langle X_\gamma : \gamma \in \pm(\Pi_1 \cup \Pi_2) \rangle$, whence $\{X_\gamma : \gamma \in \Sigma\}$ must generate $P_{A,I}(R)$.

To finish the proof it remains to show that the pair $(P_{A,I}(R), X_\gamma)$ has relative property (T) for every $\gamma \in \Sigma$ (once this is done, we simply repeat the argument in the proof of Theorem 1.1), and this is where our hypothesis on I comes into play. First of all, by assumption the subdiagram $Dyn_I(A)$ has no isolated vertices, hence we can apply Theorem 1.1 to the submatrix A_I to conclude that the group $\langle X_\gamma : \gamma \in \Pi_1 \cup \Pi_2 \cup \Lambda \rangle$ has property (T). In particular, this implies that $(P_{A,I}(R), X_\gamma)$ has relative (T) for $\gamma \in \Pi_1 \cup \Pi_2 \cup \Lambda$.

If $\gamma \in \Pi_3$, we choose $\delta \in \Pi_1 \cup \Pi_2$ such that γ and δ are connected (such δ exists by assumption on I), and consider the subgroup $H = \langle X_\gamma, X_\delta, X_{-\delta} \rangle$. Then $H \subseteq P_{A,I}(R)$,

and there exists an epimorphism $EL_2(R) \ltimes R^2 \to H$ which sends $EL_2(R)$ to $\langle X_\delta, X_{-\delta}\rangle$ and R^2 to $\langle X_\gamma, X_{\gamma+\delta}\rangle = X_\gamma X_{\gamma+\delta}$. Since the pair $(EL_2(R) \ltimes R^2, R^2)$ has relative property (T) by [Sh, Theorem 3.4], it follows that $(H, X_\gamma X_{\gamma+\delta})$ (and hence also $(P_{A,I}(R), X_\gamma)$) has relative property (T). □

If R is a ring which is not finitely generated, Kac-Moody groups over R cannot possibly have property (T) as discrete groups; however, they may still have property (T) when endowed with suitable topology. In particular, the following theorem was proved by Hartnick and Köhl in [HK]:[2]

Theorem 5.4. *Let A be an indecomposable 2-spherical $d \times d$ GCM, with $d \geq 2$, and F a local field. Then the Kac-Moody group $\mathbb{G}_A(F)$ endowed with Kac-Peterson topology has property (T).*

We finish the paper by providing a new proof of this theorem. We refer the reader to [HKM, HK] for the definition of the Kac-Peterson topology.

Proof. Let Q be a countable dense subfield of F, and let Σ be a subset of $\Phi(A)$ satisfying the hypotheses of Theorem 5.1. Since Q has no proper ideals of finite index, Corollary 4.4 shows that $\mathrm{orth}(X_\alpha(Q), X_\beta(Q)) = 0$ for all $\alpha, \beta \in \Sigma$. We cannot use [EJ, Theorem 1.2] since $(\mathbb{G}_A(Q), X_\alpha(Q))$ may not have relative property (T); however, we can still apply [EJ, Corollary 5.3] to deduce that the set $X_Q = \cup_{\alpha\in\Sigma} X_\alpha(Q)$ is a Kazhdan subset of $\mathbb{G}_A(Q)$, that is, $\kappa(\mathbb{G}_A(Q), X_Q) > 0$. Let $\varepsilon = \kappa(\mathbb{G}_A(Q), X_Q)$.

For each $1 \leq i \leq d = size(A)$ choose $j = j(i) \neq i$ such that $a_{ij} \neq 0$, and let $G(i) = \langle X_{\pm\alpha_i}(F), X_{\pm\alpha_j}(F)\rangle$. By definition of Kac-Peterson topology, $G(i)$ is isomorphic (as a topological group) to a simple Lie group of rank 2 over F, hence has property (T). Since every root subgroup of $\mathbb{G}_A(F)$ is conjugate to X_{α_i}, it follows that every root group X_α is contained in some subgroup $G(\alpha)$ with property (T). Thus we can find compact generating sets K_α of $G(\alpha)$ and $\delta > 0$ such that for any continuous unitary representation (π, V) of $\mathbb{G}_A(F)$, any unit vector v which is $(\cup_{\alpha\in\Sigma} K_\alpha, \delta)$-invariant must also be $(\cup_{\alpha\in\Sigma} G(\alpha), \varepsilon)$-invariant; in particular, v is (X_Q, ε)-invariant. By the choice of ε, if V has an (X_Q, ε)-invariant vector, then V must also have a $\mathbb{G}_A(Q)$-invariant vector, and since $\mathbb{G}_A(Q)$ is clearly dense in $\mathbb{G}_A(F)$, any $\mathbb{G}_A(Q)$-invariant vector is $\mathbb{G}_A(F)$-invariant. Hence by definition, the group $\mathbb{G}_A(F)$ with Kac-Peterson topology has property (T). □

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