Solutions to selected practice problems

1. Let A be a set, and let $S(x)$ be a statement depending on a free variable x (and no other free variables) where x ranges over A.

(i) Find all sets A for which the implication

$$(\forall x \in A \quad S(x)) \Rightarrow (\exists x \in A \quad S(x))$$

is true, regardless of the statement $S(x)$

(ii) Now find all sets A for which the implication

$$(\exists x \in A \quad S(x)) \Rightarrow (\forall x \in A \quad S(x))$$

is true, regardless of the statement $S(x)$.

Solution: (i) This implication is true for all $S(x) \iff A$ is non-empty. Indeed, the statement $\forall x \in A \quad S(x)$ asserts that $S(x)$ is true for every $x \in A$, while $\exists x \in A \quad S(x)$ means that $S(x)$ is true for at least one $x \in A$. Obviously, the first statement implies the second as long as A has at least one element, that is, as long as A is non-empty. On the other hand, if A is empty, then $\forall x \in A \quad S(x)$ is true, no matter what $S(x)$ is while $\exists x \in A \quad S(x)$ is false, no matter what $S(x)$ is, so the implication $(\forall x \in A \quad S(x)) \Rightarrow (\exists x \in A \quad S(x))$ is false in this case.

(ii) This implication is true for all $S(x) \iff |A| \leq 1$, that is, A is empty or has one element. Indeed, as we already said, if $A = \emptyset$, then $\exists x \in A \quad S(x)$ is always false and $\forall x \in A \quad S(x)$ is always true, so the implication in (ii) is true. If $|A| = 1$, the statements $\exists x \in A \quad S(x)$ and $\forall x \in A \quad S(x)$ are equivalent, so again the implication in (ii) is true. Finally, if $|A| \geq 2$, we can always find a statement $S(x)$ for which the implication in (ii) is false. For instance, fix some element $a \in A$, and let $S(x)$ be the statement $x = a$. Then $\exists x \in A \quad S(x)$ is true (since $S(a)$ is true), but $\forall x \in A \quad S(x)$ is false since by assumption A has at least one element b besides a, and $S(b)$ is false.

3. Let $\{f_n\}$ be the Fibonacci sequence defined as follows:

$$f_1 = f_2 = 1 \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3.$$

Use induction to prove the following identities:

(i) $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all $n \in \mathbb{N}$

(ii) $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for all $n \in \mathbb{N}$

Solution: (i) Base case $n = 1$:

We have $f_3 = f_1 + f_2 = 2$, so $\sum_{i=1}^{1} f_i = f_1 = 1 = 2 - 1 = f_3 - 1$.

Induction step: Assume that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for some n. Adding f_{n+1} to both sides and observing that $f_{n+2} + f_{n+1} = f_{n+3}$ (by the recursive definition), we get $\sum_{i=1}^{n+1} f_i = f_{n+2} - 1 + f_{n+1} = f_{n+3} - 1 = f_{(n+1)+2} - 1$, as desired.

(ii) Base case $n = 1$: $\sum_{i=1}^{1} f_i^2 = 1^2 = 1 \cdot 1 = f_1 f_2$

Induction step: Assume that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for some n. Adding f_{n+1}^2 to both sides, we get $\sum_{i=1}^{n+1} f_i^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2} = f_{n+1} f_{(n+1)+1}$, as desired.

6. Let $a, b, c \in \mathbb{Z}$, and assume that $c \mid ab$. Is it always true that $c \mid a$ or $c \mid b$? If the answer is yes, prove it; if the answer is no, give a specific counterexample.

Solution: By Euclid's lemma we know that the statement is true if c is prime; however, it is false in general – for instance, take $a = b = 2$ and $c = 4$. Then $c \mid ab$, but $c \nmid a$ and $c \nmid b$.

7. Let $m, n, a, b \in \mathbb{Z}$ be such that

$$am + bn = 3.$$

List all natural numbers which are possible values of $gcd(a, b)$.

For every number you listed, show that this number is indeed a possible value of $gcd(a, b)$ by giving a specific example. For all other natural numbers prove that they cannot equal $gcd(a, b)$.

Answer: 1 and 3.

Solution: 1. As in HW#5.6 let $L_{a,b}$ denote the set $\{ax + by \mid x, y \in \mathbb{Z}\}$, and let let $d = gcd(a, b)$. By the result of HW#5.6 we know that $L_{a,b} = d\mathbb{Z}$, that is, elements of $L_{a,b}$ are precisely the multiples of d. On the other hand, the assumption in our problem is that 3 is an element of $L_{a,b}$. Combining these two facts, we deduce that 3 is a multiple of d or, equivalently, $d \mid 3$. Since we also know that $d > 0$, we deduce that $d = 1$ or $d = 3$.

2. It remains to show that 1 and 3 are possible values of $gcd(a, b)$ by providing specific examples. Indeed, if we set $a = b = 1$, $m = 1$, $n = 2$, then $gcd(a, b) = 1$ and $am + bn = 3$. If we set $a = b = 3$, $m = 1$, $n = 0$, then $gcd(a, b) = 3$ and $am + bn = 3$.