Homework #9. Solutions to selected problems

1'. Let G be a group, H a subgroup of G of finite index and V an irreducible representation of G over an arbitrary field. Let W be a nonzero subspace of V which is H-invariant. Prove that

$$\dim(V) \leq \dim(W)[G:H].$$

Solution: We start with a general statement:

Theorem A: *Let G be a group, H a subgroup of G and T a left transversal of H in G, that is, T is a subset of G which contains exactly one element from each left coset of H (so that $G = \sqcup_{t\in T} tH$). Let (ρ, V) be a representation of G over an arbitrary field, and let W be a subspace of V which is H-invariant. Define*

$$Z = \sum_{t\in T} \rho(t)W.$$

Then Z is the smallest G-invariant subspace of V containing W.

Proof: Let $Y = \sum_{g\in G} \rho(g)W$. By the same argument as in HW#8.7, the smallest G-invariant subspace of V containing W is equal to Y, so we only need to show that $Y = Z$. The inclusion $Z \subseteq Y$ is obvious. Since Z is a subspace of V (being the sum of a collection of subspaces), to prove that $Y \subseteq Z$, it suffices to show that $\rho(g)W \subseteq Z$ for every $g \in G$. By definition of T we can write $g = th$ for some $t \in T$, $h \in H$. Since W is H-invariant, we have $\rho(h)W \subseteq W$. Hence

$$\rho(g)W = \rho(th)W = \rho(t)\rho(h)W = \rho(t)(\rho(h)W) \subseteq \rho(t)W \subseteq Z. \quad \square$$

Let us now assume that H has finite index in G, that V is irreducible and $W \neq 0$. Then the set T in Theorem A has size $[G : H]$ and the subspace Z must equal V (since Z is G-invariant and clearly nonzero since $W \neq 0$). Thus, $V = \sum_{t\in T} \rho(t)W$ and hence

$$\dim(V) \leq \sum_{t\in T} \dim(\rho(t)W) = \sum_{t\in T} \dim(W) = |T|\dim(W) = \dim(W)[G : H].$$

2. Let p be a prime and $G = \mathrm{Heis}(\mathbb{Z}_p)$, the Heisenberg group over $\mathbb{Z}_p$ defined in HW#7.2

(a) Determine the number of conjugacy classes of G and their sizes. As in HW#8.6, you can work directly with matrices or with their expressions in terms of the generators x, y, z introduced in HW#7.2.

(b) Let $\omega \neq 1$ be a p^{th} root of unity, that is, $\omega = e^{\frac{2\pi k i}{p}}$ with $1 \leq k \leq p-1$. Let V be a p-dimensional complex vector space with basis $e_{[0]}, e_{[1]}, \ldots, e_{[p-1]}$ where we think of indices as elements of $\mathbb{Z}_p$. Prove that there exists a representation (ρ_ω, V) of G such that
   - $\rho_\omega(z)e_{[k]} = \omega e_{[k]}$ for each k (that is, $\rho_\omega(z)$ is just the scalar multiplication by ω),
   - $\rho_\omega(y)e_{[k]} = e_{[k+1]}$ for each k (that is, $\rho_\omega(y)$ cyclically permutes the basis vectors) and finally
   - $\rho_\omega(x)e_{[k]} = \omega^k e_{[k]}$ for each k.

(c) Prove that every representation in (b) is irreducible (do not do this directly from definition) and every irreducible complex representation of G is either one-dimensional or equivalent to (ρ_ω, V) for some ω.

Solution: As usual, given $g \in G$ we denote by $K(g)$ its conjugacy class and by $C(g)$ its centralizer.

First, by direct computation we check that $Z(G) = \{E_{13}(c) : c \in \mathbb{Z}_p\} = \langle z \rangle$. Since elements of the center are precisely conjugacy classes of size 1, we deduce that G has p conjugacy classes of size 1.

Now take any $g \in G \setminus Z(G)$. We claim that $|K(g)| = p$. Indeed, since $|C(g)| \cdot |K(g)| = |G| = p^3$, the only possible values of $|K(g)|$ are $1, p, p^2$ and p^3. Since $g \notin Z(G)$, we cannot have $|K(g)| = 1$. Also, the centralizer $C(g)$ contains $Z(G)$ (which has p elements) and g (which does not lie in $Z(G)$), so $|C(g)| > p$. Hence $|K(g)| < p^2$, and the only possibility left is $|K(g)| = p$.

Thus, each non-central conjugacy class of G has size p. Since $|G \setminus Z(G)| = p^3 - p$, there are $\frac{p^3-p}{p} = p^2 - 1$ conjugacy classes of size p, and overall G has $p^2 + p - 1$ conjugacy classes.

In order to simplify our computation in (c), let us also find explicit representatives for non-central conjugacy classes. We claim that the elements $\{x^a y^b\}$ where $0 \leq a, b \leq p-1$ and $(a, b) \neq (0, 0)$ form such a system of representatives. Since there are $p^2 - 1$ of them, we just need to check that these elements are not conjugate to each other. The latter holds, for instance, since $x^a y^b = \begin{pmatrix} 1 & a & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$; on the other hand, by direct computation the $(1, 2)$ and $(2, 3)$-entries of an element of G do not change under conjugation.

(b) By the discussion in HW#7, to prove the existence of a representation ρ_ω with the given properties, it suffices to show that the given operators $\rho_\omega(x), \rho_\omega(y), \rho_\omega(z) \in \mathrm{GL}(V)$ satisfy the defining relations of the Heisenberg group. This is easy to check by a direct computation.

(c) Let $\chi_\omega = \chi_{\rho_\omega}$. We will prove that ρ_ω is irreducible by showing that $\langle \chi_\omega, \chi_\omega \rangle = 1$. By definition,

$$\langle \chi_\omega, \chi_\omega \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\omega(g)} \chi_\omega(g) = \frac{1}{|G|} \sum_{g \in G} |\chi_\omega(g)|^2.$$

We claim that $|\chi_\omega(g)| = p$ for all $g \in Z(G)$ and $\chi_\omega(g) = 0$ for all $g \notin Z(G)$. Since $|Z(G)| = p$, this would imply that $\langle \chi_\omega, \chi_\omega \rangle = 1$.

Given $g \in G$, let A_g be the matrix of $\rho_\omega(g)$ with respect to the standard basis. If $g \in Z(G)$, then $g = z^k$ for some k and hence A_g is the scalar matrix with ω^k on the diagonal. Hence $|\chi_\omega(g)| = |\mathrm{Tr}(A_g)| = |p\omega^k| = p$.

Let us now show that $\chi_\omega(g) = 0$ if $g \notin Z(G)$. Since characters are constant on conjugacy classes, by our computation in (a) it suffices to consider $g = x^a y^b$ where $0 \le a, b \le p-1$ and $(a, b) \neq (0, 0)$. If $1 \le b \le p-1$, the operator $\rho_\omega(g)$ acts on the standard basis by a non-trivial cyclic shift (by b positions) followed by multiplying each basis vector by some scalar. Hence all the diagonal entires of A_g are equal to zero and so trivially $\chi_\omega(g) = 0$.

Now suppose that $b = 0$, that is, $g = x^a$ with $1 \le a \le p-1$. Then A_g is a diagonal matrix with diagonal entries $1, \omega^a, \ldots, (\omega^a)^{p-1}$. Since $\omega^a \neq 1$, the sum of these entries is equal to $\frac{(\omega^a)^p - 1}{\omega^a - 1} = \frac{(\omega^p)^a - 1}{\omega^a - 1} = 0$.

Thus, we proved that each ρ_ω is irreducible. It remains to show that every I$\mathbb{C}$R of G is either one-dimensional or equivalent to some ρ_ω. By (a) we know that the total number of equivalence classes of I$\mathbb{C}$Rs of G is p^2+p-1. By HW#7.2 we have $|G^{ab}| = p^2$, so G has p^2 one-dimensional representations. If $\omega \neq \omega'$ are p^{th} roots of unity different from 1, the representations ρ_ω and $\rho_{\omega'}$ are non-equivalent as they have different characters (for instance, the value of the characters at z are already different). Thus, one-dimensional representations together with $\{\rho_\omega\}$ give us $p^2 + p - 1$ pairwise non-equivalent I$\mathbb{C}$Rs and hence any other I$\mathbb{C}$R of G must be equivalent to one of these.

4. Let $G = S_n$ for some $n \ge 2$ and χ a complex character of G. Prove that χ is real-valued, that is, $\chi(g) \in \mathbb{R}$ for all $g \in G$.

Solution: We start with a general theorem:

Theorem B: Let G be a finite group in which g is conjugate to g^{-1} for every $g \in G$. Then $\chi(g) \in \mathbb{R}$ for every complex character χ of G and every $g \in G$.

Proof: Let (ρ, V) be a complex representation of G and β a basis of V. By HW#9.3 (=Claim 18.1) we have $[\rho^*(g)]_{\beta^*} = ([\rho(g)]_\beta^{-1})^T$. Hence, for all $g \in G$

$$\begin{aligned}
\chi_{\rho^*}(g) &= \mathrm{Tr}(([\rho(g)]_\beta^{-1})^T) = \mathrm{Tr}(([\rho(g)]_\beta^{-1})) \\
&= \mathrm{Tr}(([\rho(g)^{-1}]_\beta)) \quad \text{since } ([A]_\beta)^{-1} = [A^{-1}]_\beta \text{ for every } A \in \mathrm{GL}(V) \\
&= \mathrm{Tr}(([\rho(g^{-1})]_\beta)) \quad \text{since } \rho \text{ is a homomorphism} \\
&= \chi_\rho(g^{-1})
\end{aligned}$$

Since g^{-1} is conjugate to g, we have $\chi_\rho(g^{-1}) = \chi_\rho(g)$. On the other hand, since (ρ, V) is unitarizable, $\chi_{\rho^*}(g) = \overline{\chi_\rho(g)}$ by Claim 18.1. Putting everything together, we get $\overline{\chi_\rho(g)} = \chi_\rho(g)$ and hence $\chi_\rho(g) \in \mathbb{R}$. $\square$

To deduce Problem 4 from Theorem B we just need to show that every element of S_n is conjugate to its inverse. To see this, write $g \in S_n$ as a product of disjoint cycles. Then g^{-1} is obtained from g by reversing the order of entries in each cycle. In particular, g^{-1} has the same cycle type as g and therefore g^{-1} is conjugate to g.

5. Give an example of two representations V and W of the same group which are not equivalent, but have the same character. Recall that by Corollary 19.2 from class this cannot happen if G is finite and representations are complex.

Solution: Consider the following 2-dimensional complex representations of $G = \mathbb{Z}$ (which will be given in the matrix form): the "trivial 2-dimensional representation" ρ_{triv} given by $\rho_{triv}(k) = I_2$ for all $k \in \mathbb{Z}$ and the representation ρ given by $\rho(k) = A^k$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (such a representation exists by the discussion in Lecture 12). Since $A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, we have $\chi_{\rho_{triv}}(k) = \chi_\rho(k) = 2$ for all $k \in \mathbb{Z}$. On the other hand, ρ and ρ_{triv} are not equivalent since the identity matrix is only conjugate to itself.