Partial solutions to practice problems for Midterm 2

1. As usual for a natural number n let $[n] = \{1, 2, \ldots, n\}$. A derangement of $[n]$ is a **bijective** function $f : [n] \to [n]$ such that $f(i) \neq i$ for all $i \in [n]$ (that is, f does not fix any element of $[n]$).

(i) Let d_n be the total number of derangements of $[n]$. Use the inclusion-exclusion principle to prove that

$$d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} = \sum_{i=2}^{n} (-1)^i \frac{n!}{i!}$$

Note: By definition $0! = 1$. The above two sums are indeed equal since the terms corresponding to $i = 0$ and $i = 1$ in the first sum cancel each other.

(ii) What is the limit $\lim\limits_{n\to\infty} \frac{d_n}{n!}$?

Hint: For each $1 \leq i \leq n$ define A_i to be the set of all bijective functions $f : [n] \to [n]$ such that $f(i) = i$.

Solution: (i) Let B be the set of all bijective functions from $[n]$ to $[n]$. Then each A_i is a subset of B, and $\cup_{i=1}^{n} A_i$ is the set of bijective functions $f : [n] \to [n]$ such that $f(i) = i$ for at least one i. Hence the set of derangements of $[n]$ is precisely the complement of $\cup_{i=1}^{n} A_i$ in B and thus $d_n = |B| - |\cup_{i=1}^{n} A_i|$. By the inclusion-exclusion principle we get

$$d_n = |B| - \sum_{1\leq i\leq n} |A_i| + \sum_{1\leq i<j\leq n} |A_i \cap A_j| - \ldots \qquad (***)$$

Next observe that a function $f : [n] \to [n]$ is bijective $\iff$ f is injective. Indeed, bijective always implies injective; on the other hand, if f is injective, the numbers $f(1), \ldots, f(n)$ are n distinct elements of $[n]$, and since $|[n]| = n$, the sequence $f(1), \ldots, f(n)$ must include all elements of $[n]$, making f is surjective and hence also bijective (since injectivity was assumed from the beginning). This observation is helpful because we can use the method of Problem#7(b) in HW#9 to compute $|B|, |A_i|, |A_i \cap A_j|$ etc.

First we have $|B| = n!$. This is a direct application of the result of Problem#7(b) with $m = n$ (since for functions $f : [n] \to [n]$ bijective is the same as injective). Using the same reasoning, we get $|A_i| = (n-1)!$ for each i. For instance, to choose a bijective function $f \in A_1$ we do not have any choice for $f(1)$ (which must equal 1), $n-1$ choices for $f(2)$, $n-2$ choices for $f(3)$ etc. And since we can order the elements of $[n]$ arbitrarily, starting with any given $i \in [n]$, the same argument shows that $|A_i| = (n-1)!$ for each i. Similarly, $|A_i \cap A_j| = (n-2)!$ for any $1 \leq i < j \leq n$, $|A_i \cap A_j \cap A_k| = (n-3)!$ for any $1 \leq i < j < k \leq n$ etc.

Thus, for each of the summations $\sum_{1\leq i\leq n} |A_i|$, $\sum_{1\leq i<j\leq n} |A_i \cap A_j|$ etc. appearing in (***) above, all the terms are the same, so the sum is equal to the common value of the terms multiplied by the number of terms. The number of terms in $\sum_{1\leq i\leq n} |A_i|$ is $n = \binom{n}{1}$, the number of terms in $\sum_{1\leq i<j\leq n} |A_i \cap A_j|$ is $\binom{n}{2}$ etc. Thus, plugging in the obtained expression into (***), we get

$$d_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \ldots = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

Finally, using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we get $\binom{n}{k}(n-k)! = \frac{n!}{k!}$, which yields the desired formula for d_n.

(ii) Using (i) and the definition of the sum of a series, we get

$$\lim_{n\to\infty}\frac{d_n}{n!}=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{(-1)^i}{i!}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}.$$

Recall from Calculus II that $e^x=\sum\limits_{i=0}^{\infty}\frac{x^i}{i!}$ for all $x\in\mathbb{R}$. Setting $x=-1$, we get $\sum\limits_{i=0}^{\infty}\frac{(-1)^i}{i!}=e^{-1}=\frac{1}{e}$.

The meaning of the result of (ii) is that if we randomly choose a bijective function $f:[n]\to[n]$, the probability that the chosen function is a derangement tends to $\frac{1}{e}$ as $n\to\infty$.

2. Let $A=\mathbb{N}$, and define a relation $\sim$ on A by $x\sim y \iff xy$ is a perfect square.

(i) Prove that $\sim$ is an equivalence relation. To prove transitivity you may want to use the criterion for $\sqrt{n}$ being irrational proved in Lecture 12 (Theorem 12.1).
(ii) Describe explicitly the equivalence classes $[1],[2],[4]$ and $[12]$.

Solution sketch: (i) The only condition that requires work is transitivity. Assume $x\sim y$ and $y\sim z$. By definition there exist $m,n\in\mathbb{N}$ such that $xy=m^2$ and $yz=n^2$. Then $xzy^2=m^2n^2$ and hence $xz=(\frac{mn}{y})^2$. Since xz and $\frac{mn}{y}$ are positive by construction, taking square roots, we get $\sqrt{xz}=\frac{mn}{y}$, so $\sqrt{xz}\in\mathbb{Q}$. By Theorem 12.1, the square root of a natural number N is irrational unless N is a perfect square. Thus, xz must be a perfect square and hence $x\sim z$, as desired.

(ii) The answer is as follows: $[1]=[4]=\{$all perfect squares$\}$,
$[2]=\{$all integers of the form $2n^2$ with $n\in\mathbb{N}\}$,
$[12]=\{$all integers of the form $3n^2$ with $n\in\mathbb{N}\}$.

3. Consider the relation $\sim$ on $\mathbb{R}\times\mathbb{R}$ given by $(x,y)\sim(z,w)$ if and only if $\max\{x,y\}=\max\{z,w\}$.

(i) Prove that $\sim$ is an equivalence relation
(ii) Describe explicitly the equivalence class of $(1,2)$.
(iii) Describe geometrically the partition of $\mathbb{R}\times\mathbb{R}$ into the equivalence classes with respect to $\sim$ (drawing a picture is sufficient).

Answer to (ii) and (iii): (ii) $[(1,2)]=A\cup B$ where A is the set of all points $(x,2)$ with $x\le 2$ and B is the set of all points $(2,y)$ with $y\le 2$.

(iii) Geometrically, each equivalence class is the union of two rays (half-lines) starting from a point of the form (a,a) for some $a\in\mathbb{R}$, where one ray is horizontal moving to the left and the other is vertical moving down.

4. For each of the following functions determine if it is bijective. If it is, find an explicit formula for the inverse.

(a) $f:\mathbb{Z}\to\mathbb{E}$ given by $f(x)=2x$ where $\mathbb{E}$ is the set of all even integers
(b) $f:\mathbb{Z}\to\mathbb{Z}$ given by

$$f(x)=\begin{cases} x-1 & \text{if } x \text{ is even}\\ x+1 & \text{if } x \text{ is odd}\end{cases}$$

(c) $f:\mathbb{Z}\to\mathbb{Z}$ given by

$$f(x)=\begin{cases} x & \text{if } x \text{ is even}\\ x+1 & \text{if } x \text{ is odd}\end{cases}$$

Answer: (c) is not injective since $f(1) = f(2) = 2$. In fact, it is not hard to see that $f(x)$ is always even, so f is not surjective either.

(a) and (b) are bijective (note that (a) is surjective since we have chosen $\mathbb{E}$, not $\mathbb{Z}$, as our codomain). The inverse of f in (a) is $g : \mathbb{E} \to \mathbb{N}$ given by $g(x) = \frac{x}{2}$. The function f in (b) is equal to its own inverse.

5. Let $n \in \mathbb{N}$.

(a) Prove that n is divisible by 20 $\iff$ the last digit of n is 0 and the next-to-last digit of n is even.
(b) State and prove a simple criterion of divisibility by 12.

Solution: (a) Using the same argument as in the proof of the criterion of divisibility by 4, we deduce that n is divisible by 20 $\iff$ the number formed by the last two digits of n is divisible by 20 (in general this type statement about divisibility by N is true $\iff$ N divides 100). The only numbers with at most 2 digits divisible by 20 are clearly 0 (which we should write as 00 here keeping in mind that it comes as the last two digits of another integer), 20, 40, 60 and 80. The numbers on this list are precisely the numbers with at most 2 digits where the last digit is 0 and the next-to-last digit is even.

(b) Since 3 and 4 are relatively prime, Theorem 3.4.3 from the book tells us that n is divisible by 12 $\iff$ n is divisible by both 3 and 4. Thus, using the criteria of divisibility by 3 and 4 proved in class, we get the following criterion of divisibility by 12:

n is divisible by 12 $\iff$ the sum of the digits of n is divisible by 3 and the number formed by the last 2 digits of n is divisible by 4.

6. Let $A = \{1, 2, 3, 4\}$. Find the total number of relations on A which are

(i) reflexive
(ii) symmetric
(iii) antisymmetric
(iv) reflexive and symmetric
(v) symmetric and antisymmetric
(vi) equivalence relations R such that $|R| = 8$.

Answers: (i) 2^{12}; (ii) 2^{10}; (iii) $2^4 \cdot 3^6$; (iv) 2^6; (v) 2^4; (vi) 3.

Brief justifications: In all parts we think of relations as subsets of $A \times A$ and R denotes an arbitrary relation satisfying the required properties.

(i) In order for a relation R to be reflexive, it must contain the 4 diagonal pairs $(1, 1), (2, 2), (3, 3), (4, 4)$ in $A \times A$. There are 12 non-diagonal pairs in $A \times A$; each of them can be included or not included in R, so we have 2 choices for each pair; overall 2^{12} choices by FPC.

(ii) We have 2 choices for each diagonal pair (a, a) (either include it in R or not include it); also for each 2-elements subset $\{a, b\}$ of A we can either include both (a, b) and (b, a) in R or not include either (again 2 choices). Since there are 4 diagonal pairs and 6 two-element subsets, we have the total of 2^{4+6} choices.

(iii) The reasoning is similar to (ii). The only difference is that for each 2-elements subset $\{a, b\}$ of A we have 3 choices: include (a, b) in R (but not (b, a)) or include (b, a) (but not (a, b)) or not include either (a, b) or (b, a) (the only thing we cannot do is include both (a, b) and (b, a)).

(iv) Same reasoning as in (ii) except we do not have any choice for the diagonal pairs (a, a), all of which have to be included.

(v) see the reasoning in the solution to Problem 4 in HW#8.

(vi) To be reflexive R must include the 4 diagonal pairs (this is both necessary and sufficient for reflexivity). Since $|R| = 8$, we need to include 4 more pairs in R, and since R is symmetric, those 4 pairs have to be of the form $(a, b), (b, a), (c, d), (d, c)$ for some $a, b, c, d \in A$. This is enough to make R symmetric. It is not hard to see that the obtained R will be transitive $\iff$ a, b, c and d are all distinct. Such relations correspond bijectively to partitions of $\{1, 2, 3, 4\}$ into two 2-element subsets. Clearly, there are $3 = \frac{1}{2}\binom{6}{2}$ such partitions: $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$ and $\{\{1, 4\}, \{2, 3\}\}$.