Homework #8. Solutions to selected problems

Problems:

1. Let G be a group, N a normal subgroup of G and let $\pi : G \to G/N$ be the natural projection.

(a) Given a representation $\rho : G/N \to GL(V)$ of G/N, define the representation $\widetilde{\rho} : G \to GL(V)$ of G by

$$\widetilde{\rho}(g) = \rho \circ \pi(g) = \rho(gN). \qquad (***)$$

Prove that $\widetilde{\rho}$ is irreducible $\iff$ ρ is irreducible. Also prove that two representations ρ_1 and ρ_2 of G/N are equivalent $\iff$ the corresponding representations $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ of G are equivalent.

(b) Now fix a field F. Let

- $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible representations of G over F;
- $\mathrm{Irr}(G/N)$ the set of equivalence classes of irreducible representations of G/N over F;
- $\mathrm{Irr}(G, N)$ the set of all $[\rho] \in \mathrm{Irr}(G)$ such that $N \subseteq \mathrm{Ker}\rho$.

(here $[\rho]$ is the equivalence class of the representation ρ). Define the map $\Phi : \mathrm{Irr}(G/N) \to \mathrm{Irr}(G)$ by

$$\Phi([\rho]) = [\widetilde{\rho}]$$

(where $\widetilde{\rho}$ is defined by (***)). Explain why Φ is well defined and injective (this follows immediately from (a)) and then prove that $\mathrm{Im}(\Phi) = \mathrm{Irr}(G, N)$.

Solution: (a) Let W be a subspace of V. Then W is a subrepresentation of $(\widetilde{\rho}, V)$ $\iff$ W is $\widetilde{\rho}(g)$-invariant for every $g \in G$. Similarly, W is a subrepresentation of (ρ, V) $\iff$ W is $\rho(x)$-invariant for every $x \in G/N$. Since $\{\widetilde{\rho}(g) : g \in G\}$ and $\{\rho(x) : x \in G/N\}$ are the same sets of operators, we conclude that $(\widetilde{\rho}, V)$ is irreducible $\iff$ (ρ, V) is irreducible.

The representations (ρ_1, V_1) and (ρ_2, V_2) of G/N are equivalent $\iff$ there exists an isomorphism $T : V_1 \to V_2$ such that $\rho_2(gN)T = T\rho_1(gN)$ for all $gN \in G/N$ $\iff$ $\rho_2(gN)T = T\rho_1(gN)$ for all $g \in G$ $\iff$ the representations $(\widetilde{\rho}_1, V_1)$ and $(\widetilde{\rho}_2, V_2)$ of G are equivalent.

(b) If $[\rho_1] = [\rho_2]$, then $\rho_1 \cong \rho_2$ (as representations of G/N), so by (a) $\widetilde{\rho}_1 \cong \widetilde{\rho}_2$ (as representations of G), so $[\widetilde{\rho}_1] = [\widetilde{\rho}_2]$. This shows that Φ

is well defined. Reversing the above chain of implications, we conclude that Φ is injective.

Any element in $\mathrm{Im}(\Phi)$ is equal to $[\widetilde{\rho}]$ for some representation ρ of G/N. If $g \in N$, then $\widetilde{\rho}(g) = \rho(gN) = \rho(N) = I$, so $N \subseteq \mathrm{Ker}\widetilde{\rho}$ and thus $[\widetilde{\rho}] \in \mathrm{Irr}(G, N)$.

Conversely, let $[\alpha] \in \mathrm{Irr}(G, N)$. Define $\rho : G/N \to GL(V)$ (where V is the representation space of α) by $\rho(gN) = \alpha(g)$. Since $N \subseteq \mathrm{Ker}\alpha$ by assumption, ρ is well defined, and it is clear that $\alpha = \widetilde{\rho}$, so $[\alpha] \in \mathrm{Im}(\Phi)$.

5. Compute the character table for the alternating group A_4 (with detailed justification) and explicitly construct its irreducible complex representations. First prove that $[A_4, A_4] = V_4$, the Klein 4-group.

Solution:

I. First we show that $[A_4, A_4] = V_4$. For any distinct a, b, c, d we have

$$\begin{aligned}[(a,b,c),(b,c,d)] &= (a,b,c)^{-1}(b,c,d)^{-1}(a,b,c)(b,c,d) \\ &= (a,c,b)(b,d,c)(a,b,c)(b,c,d) = (a,d)(b,c).\end{aligned}$$

This shows that $[A_4, A_4] \supseteq V_4$. On the other hand, V_4 is normal in A_4 (since it is normal in S_4) and $|A_4/V_4| = 3$, whence $A_4/V_4 \cong \mathbb{Z}_3$. Thus, A_4/V_4 is abelian and hence by Claim 14.4 from class, $V_4 \supseteq [A_4, A_4]$.

II. Next we discuss the conjugacy classes of A_4. We claim that there are four conjugacy classes K_1, K_2, K_3, K_4 with representatives $g_1 = e, g_2 = (1,2,3), g_3 = (1,3,2)$ and $g_4 = (1,2,3,4)$ and sizes $|K_1| = 1$, $|K_2| = |K_3| = 4$ and $|K_4| = 3$. To justify this statement we prove a general statement (Theorem A below) describing how conjugacy classes in A_n are related to conjugacy classes of S_n.

If H is a normal subgroup of a group G, every conjugacy class K of G is either contained in H or does not intersect H. If K is contained in H, it is possible that all elements of K are conjugate to each other in H (and thus K is also a conjugacy class of H), but it is also possible that K splits into several H-conjugacy classes. Theorem A describes exactly when each possibility happens in the case $G = S_n$ and $H = A_n$.

Theorem A: *Let $f \in A_n$. Let $K(f)$ be the conjugacy class of f in S_n, and let $C(f)$ denote the centralizer of f in S_n.*

(1) *If $C(f) \not\subseteq A_n$, then $K(f)$ remains a single conjugacy class in A_n*
(2) *If $C(f) \subseteq A_n$, then $K(f)$ is the union of two A_n-conjugacy classes which have the same size $|K(f)|/2$.*

Proof: Let $K'(f)$ be the conjugacy class of f in A_n. We will show that

(i) $K'(f) = K(f)$ if $C(f) \not\subseteq A_n$ and
(ii) $|K'(f)| = |K(f)|/2$ if $C(f) \subseteq A_n$.

First let us explain why this result would imply Theorem A.

It is clear that (i) implies (1). Suppose now that $C(f) \subseteq A_n$. If g is any element of $K(f)$, it is easy to check that $C(g)$ is conjugate to $C(f)$ (in S_n) and hence is contained in A_n. Thus, applying (ii) to g instead of f, we conclude that $|K'(g)| = |K(g)|/2 = |K(f)|/2$ for all $g \in K(f)$. This clear implies (2).

Let us now prove (i) and (ii). Recall that for any finite group G and $g \in G$ we have the equality $|K_G(g)| = \frac{|G|}{|C_G(g)|}$ where $K_G(g)$ and $C_G(g)$ are the conjugacy class and the centralizer of g in G, respectively.

Thus, if we denote the centralizer of f in A_n by $C'(f)$, we have $|K(f)| = \frac{|S_n|}{|C(f)|}$ and $|K'(f)| = \frac{|A_n|}{|C'(f)|}$, whence

$$\frac{|K(f)|}{|K'(f)|} = \frac{|S_n|}{|A_n|}\frac{|C'(f)|}{|C(f)|} = 2\frac{|C'(f)|}{|C(f)|}. \qquad (***)$$

It is also clear that $C'(f) = C(f) \cap A_n$.

If $C(f) \subseteq A_n$, then $C'(f) = C(f)$ and hence $\frac{|K(f)|}{|K'(f)|} = 2$, as desired. On the other hand, if $C(f) \not\subseteq A_n$, then $C'(f) \neq C(f)$. Since $C'(f)$ is a subgroup of $C(f)$, we have $\frac{|C(f)|}{|C'(f)|} = [C(f) : C'(f)] \geq 2$ and hence (***) implies that $\frac{|K(f)|}{|K'(f)|} \leq 1$, that is, $|K'(f)| \geq |K(f)|$. Since $K'(f)$ is obviously contained in $K(f)$, we conclude that $K'(f) = K(f)$, as desired. □

Let us now use Theorem A with $n = 4$. While computing the centralizer of an element of A_4 is easy, all we need to know for this problem is that every S_n-conjugacy class consisting of even permutations is either a single A_n-conjugacy class or is the union of two A_n-conjugacy classes of equal size.

The conjugacy class of $(1,2)(3,4)$ in S_4 has 3 elements and thus cannot split into two subclasses of equal size, so in our earlier notations $|K_4| = 3$. On the other hand, the conjugacy class of $(1,2,3)$ in S_4 has 8 elements and hence has to split (since 8 is not a divisor of $12 = |A_4|$). This implies that the A_4-conjugacy classes of both $(1,2,3)$ and $(1,3,2)$ have 4 elements, and the only thing left to show is that $(1,2,3)$ and $(1,3,2)$ are not conjugate in A_4.

If elements x and y of a group G are conjugate, then $y = g^{-1}xg$ for some $g \in G$ and hence $x^{-1}y = x^{-1}g^{-1}xg = [x,g] \in [G,G]$. Since

$(1,2,3)^{-1}(1,3,2) = (1,2,3) \notin V_4 = [A_4, A_4]$, we conclude that $(1,2,3)$ and $(1,3,2)$ are not conjugate in A_4.

III. Let us now compute the character table of A_4. From our earlier description $A_4^{ab} \cong \mathbb{Z}_3$, and it is clear that the image of $g_2 = (1,2,3)$ in A_4^{ab} is a generator. Thus, there are three 1-dimensional (complex) representations of A_4 determined by sending g_2 to one of the three third roots of unity (namely, $1, \omega$ or ω^2 where $\omega = e^{\frac{2\pi i}{3}}$). Note that $g_3 = (1,3,2) = g_2^2$ must go to the square of the image of g_1, and $g_4 = (1,2)(3,4)$ lies in $[G,G]$ and hence goes to 1 in any 1-dimensional representation. This allows us to complete the first 3 of the 4 rows of the character table:

	$g_1 = e$	$g_2 = (1,2,3)$	$g_3 = (1,3,2)$	$g_4 = (1,2)(3,4)$
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1
χ_4				

From the formula $\sum_{i=1}^4 \dim(\chi_i)^2 = 4$, we get that $\chi_4(e) = \dim(\chi_4) = 3$. Using the fact that each of the columns of the character table is orthogonal to the first column (with respect to the Hermitian dot product), we conclude that $\chi_4(g_2) = \chi_4(g_3) = 0$, $\chi_4(g_4) = -1$.

IV. Finally, let us explicitly construct an irreducible representation of A_4 whose character is χ_4. Let ρ_4 be the restriction of the standard representation of S_4 to A_4. We claim that ρ_4 has the desired property. Indeed, from our computation of the character table of S_4 we know that χ_{ρ_4} (the character of ρ_4) is equal to χ_4. We also know that there is some irreducible representation ρ with $\chi_\rho = \chi_4$. Since two representations of a finite group having the same character are equivalent (Corollary 19.1), we get that $\rho_4 \cong \rho$ and hence ρ_4 is irreducible. Alternatively, we could simply say that

$$\langle \chi_{\rho_4}, \chi_{\rho_4} \rangle = \frac{1}{12}(|3|^2|K_1| + |0|^2|K_2| + |0|^2|K_3| + |-1|^2|K_4|) = 1$$

and hence ρ_4 is irreducible by Corollary 19.2.

6. Let G be the group of all matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{Z}_5$ and $a \neq 0$.

(a) Prove that G has a presentation $\langle x, y \mid x^4 = y^5 = e, xyx^{-1} = y^2 \rangle$ where $x = \mathrm{diag}(2,1) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $y = E_{12}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (in the notation of HW#7.2).

(b) Prove that G has 5 conjugacy class with sizes $1, 4, 5, 5, 5$. You can use either the original definition or the presentation from part (a).

(c) Now compute the character table of G (with detailed justification).

Solution: (a) This is completely analogous to HW#7.2(b). Note that in the course of the calculation we show that every element of G can be written as $x^i y^j$ with $0 \leq i \leq 3$ and $0 \leq j \leq 4$.

(b) Let $g_1 = e$, $g_2 = x$, $g_3 = x^2$, $g_4 = x^3$ and $g_5 = y$. We claim that $g_1, \ldots, g_5$ are representatives of conjugacy classes of G (which we call $K_1 = K(g_1), \ldots, K_5 = K(g_5)$) and $|K_1| = 1$, $|K_2| = |K_3| = |K_4| = 5$ and $|K_5| = 4$.

First note that conjugate elements of G must have the same eigenvalues and e is only conjugate to itself. This implies that the above 5 elements cannot be conjugate to each other and moreover $|K(g_1)| = 1$, $|K(g_i)| \leq 5$ for $i = 2, 3, 4$ and $|K(g_5)| \leq 4$. On the other hand, it is easy to check that if $A \in \mathrm{GL}_n(F)$ is a diagonal matrix with DISTINCT entries, then A only commutes with diagonal matrices. Hence for $i = 2, 3, 4$ we have $|C(g_i)| \leq 4$, and since $|K(g_i)| \cdot |C(g_i)| = |G| = 20$, we must have $|K(g_i)| = 5$ and $|C(g_i)| = 4$. Finally, by direct computation $y = g_5$ only commutes with its powers, whence $|C(g_5)| = 5$ and hence $|K(g_5)| = 4$.

(c) By a simple direct computation $[G, G] = \langle y \rangle$, so $|G^{ab}| = 20/5 = 4$. Moreover, the image of x in G^{ab} has order 4 (since no power of x lies in $\langle y \rangle$ unless that power is already trivial in G). Thus, G^{ab} is cyclic of order 4 generated by the image of x, and hence G has four one-dimensional representations which send x to $1, i, -1$ or $-i$ (and y to 1 since $y \in [G, G]$). As in Problem 5, we get the first four rows of the character table:

	$g_1 = e$	$g_2 = x$	$g_3 = x^2$	$g_4 = x^3$	$g_5 = y$
χ_1	1	1	1	1	1
χ_2	1	i	-1	$-i$	1
χ_3	1	-1	1	-1	1
χ_4	1	$-i$	-1	i	1
χ_5					

As in Problem 5, using orthogonality relations, we conclude that the fifth row of the table is $(4, 0, 0, 0, -1)$. We will discuss how to explicitly construct a representation of G whose character is χ_5 later in the course.

7. Let G be a finite group and (ρ, V) a cyclic representation of G over an arbitrary field. Prove that $\dim(V) \leq |G|$.

We start by proving a general result:

Theorem B: *Let G be a group, (ρ, V) a representation of G and $v \in V$. Let $S = \{\rho(g)v : g \in G\}$. Then* $\mathrm{Span}(S)$ *is the smallest G-invariant subspace containing v.*

Proof: First we show that $\mathrm{Span}(S)$ is G-invariant, that is, $\rho(x)\mathrm{Span}(S) \subseteq \mathrm{Span}(S)$ for all $x \in G$. By linearity, it suffices to show that $\rho(x)S \subseteq S$ for all $x \in G$.

So take any $x \in G$ and $s \in S$, so that $s = \rho(g)v$ for some $g \in G$. Then $\rho(x)s = \rho(x)\rho(g)v = \rho(xg)v \in S$.

Thus, we proved that $\mathrm{Span}(S)$ is a G-invariant subspace containing v. On the other hand, if W is any G-invariant subspace containing v, then W clearly must contain S and hence also $\mathrm{Span}(S)$. $\square$

Let us now use Theorem B to solve problem 7. Since (ρ, V) is cyclic, there exists $v \in V$ such that the smallest G-invariant of V containing v is V itself. Hence, by Theorem B we have $V = \mathrm{Span}(S)$ where $S = \{\rho(g)v : g \in G\}$. Since $|S| \leq |G|$ (which is clear from definition) and $\dim \mathrm{Span}(S) \leq |S|$, we conclude that $\dim(V) \leq |G|$.