Math 3000. Spring 2018. Information about the second midterm.

The second midterm will be given on Thursday, April 12, in class. It will contain 4-6 regular problems and possibly a bonus problem. The exam will be based on the material from Lectures 11-19 and the first part of Lecture 20 (before inverse images) and the following sections of the book: 3.3, 3.4, 4.1, 4.2, 4.3, 5.1, 5.2, 6.1 and the first part of 6.2 (before inverse images). The relevant homework assignments are 6-9. The main topics that will be covered are prime factorization, basic counting techniques (including the inclusion-exclusion principle), relations (with likely emphasis on equivalence relations) and functions.
Here are some basic suggestions for preparing for the test:

(i) Go over your homework assignments; whenever you lost points on a problem, make sure to understand why. If you made a minor mistake, realizing what the mistake was is probably sufficient; if your solution was not close to being correct, redo the problem.
(ii) Work on the practice problems that appeared in homeworks 6-9. Many of those problems are solved in the solution manual
(iii) Go over the class notes and prepare questions about things you do not understand.
(iv) Go over the sections of the book that you did not fully understand on the first reading.

Additional practice problems

1. As usual for a natural number n let $[n] = \{1, 2, \ldots, n\}$. A derangement of $[n]$ is a **bijective** function $f : [n] \to [n]$ such that $f(i) \neq i$ for all $i \in [n]$ (that is, f does not fix any element of $[n]$).

(i) Let d_n be the total number of derangements of $[n]$. Use the inclusion-exclusion principle to prove that
$$d_n = \sum_{i=0}^{n} (-1)^n \frac{n!}{i!} = \sum_{i=2}^{n} (-1)^n \frac{n!}{i!}$$
Note: By definition $0! = 1$. The above two sums are indeed equal since the terms corresponding to $i = 0$ and $i = 1$ in the first sum cancel each other.
(ii) What is the limit $\lim\limits_{n\to\infty} \frac{d_n}{n!}$?

Hint: For each $1 \le i \le n$ define A_i to be the set of all bijective functions $f : [n] \to [n]$ such that $f(i) = i$.

2. Let $A = \mathbb{N}$, and define a relation $\sim$ on A by $x \sim y \iff xy$ is a perfect square.

(i) Prove that $\sim$ is an equivalence relation. To prove transitivity you may want to use the criterion for $\sqrt{n}$ being irrational proved in Lecture 12 (Theorem 12.1).
(ii) Describe explicitly the equivalence classes $[1], [2], [4]$ and $[12]$.

3. Consider the relation $\sim$ on $\mathbb{R} \times \mathbb{R}$ given by $(x, y) \sim (z, w)$ if and only if $\max\{x, y\} = \max\{z, w\}$.

(i) Prove that $\sim$ is an equivalence relation
(ii) Describe explicitly the equivalence class of $(1, 2)$.
(iii) Describe geometrically the partition of $\mathbb{R} \times \mathbb{R}$ into the equivalence classes with respect to $\sim$ (drawing a picture is sufficient).

4. For each of the following functions determine if it is bijective. If it is, find an explicit formula for the inverse.

(a) $f : \mathbb{Z} \to \mathbb{E}$ given by $f(x) = 2x$ where $\mathbb{E}$ is the set of all even integers

(b) $f : \mathbb{Z} \to \mathbb{Z}$ given by

$$f(x) = \begin{cases} x - 1 & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$$

(c) $f : \mathbb{Z} \to \mathbb{Z}$ given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$$

5. Let $n \in \mathbb{N}$.

(a) Prove that n is divisible by 20 $\iff$ the last digit of n is 0 and the next-to-last digit of n is even.

(b) State and prove a simple criterion of divisibility by 12.

6. Let $A = \{1, 2, 3, 4\}$. Find the total number of relations on A which are

(i) reflexive
(ii) symmetric
(iii) antisymmetric
(iv) reflexive and symmetric
(v) symmetric and antisymmetric
(vi) equivalence relations R such that $|R| = 8$.