Solutions to Homework #5

1. Let U, V and W be vector spaces over the same field F. Construct a natural isomorphism of vector spaces $\varphi : (U \oplus V) \otimes W \to (U \otimes W) \oplus (V \otimes W)$ and prove your φ is indeed an isomorphism.

Solution: Define the map $f : (U \oplus V) \times W \to (U \otimes W) \oplus (V \otimes W)$ by

$$f((u, v), w) = (u \otimes w, v \otimes w).$$

By straightforward verification, f is bilinear, and hence by the universal property of tensor products (Theorem 10.1), there exists a linear map $\varphi : (U \oplus V) \otimes W \to (U \otimes W) \oplus (V \otimes W)$ such that $\varphi((u, v) \otimes w) = (u \otimes w, v \otimes w)$ for all $u \in U, v \in V, w \in W$. We will prove that φ is an isomorphism by constructing the inverse map ψ.

First consider the maps $g_U : U \times W \to (U \oplus V) \otimes W$ and $g_V : V \times W \to (U \oplus V) \otimes W$ defined by

$$g_U(u, w) = (u, 0) \otimes w \text{ and } g_V(v, w) = (0, v) \otimes w.$$

Again g_U and g_V are bilinear, so by Theorem 10.1 there exist linear maps $\psi_U : U \otimes W \to (U \oplus V) \otimes W$ and $\psi_V : V \otimes W \to (U \oplus V) \otimes W$ such that $\psi_U(u \otimes w) = (u, 0) \otimes w$ and $g_V(v \otimes w) = (0, v) \otimes w$ for all $u \in U, v \in V, w \in W$.

Now define $\psi : (U \otimes W) \oplus (V \otimes W) \to (U \oplus V) \times W$ by $\psi(x, y) = (\psi_U(x), \psi_U(y))$ for all $x \in U \otimes W$ and $y \in V \otimes W$ (here we do not assume that x and y are simple tensors, so we do not need to justify that ψ is well defined).

Clearly, ψ is linear, and for all $u \in U, v \in V, w \in W$ we have

$$\psi((u \otimes w, v \otimes w)) = \psi_U(u \otimes w) + \psi_V(v \otimes w) = (u, 0) \otimes w + (0, v) \otimes w = (u, v) \otimes w.$$

It follows that $\psi\varphi(a) = a$ for all a of the form $(u, v) \otimes w$ and $\varphi\psi(b) = b$ for all b of the form $(u \otimes w, v \otimes w)$. Since $(U \oplus V) \otimes W$ and $(U \otimes W) \oplus (V \otimes W)$ are spanned by such elements a and b, respectively, and since φ and ψ are linear (hence so are their compositions), we conclude that $\psi\varphi$ and $\varphi\psi$ are both identity maps, as desired.

2. Let V_1, V_2, W_1 and W_2 be vector spaces over the same field F, and let $\varphi : V_1 \to V_2$ and $\psi : W_1 \to W_2$ be linear maps. Prove that there exists a unique linear map $\varphi \otimes \psi : V_1 \otimes W_1 \to V_2 \otimes W_2$ such that $(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w)$ for all $v \in V_1$ and $w \in W_1$ (here $\varphi \otimes \psi$ is just the notation for the map being defined).

Solution: Define the map $f : V_1 \times W_1 \to V_2 \otimes W_2$ by $f(v, w) = \varphi(v) \otimes \psi(w)$. It is straightforward to check that f is bilinear. Hence by Theorem 10.1 there exists a linear map $\varphi\otimes\psi : V_1\otimes W_1 \to V_2\otimes W_2$ such that $(\varphi\otimes\psi)(v\otimes w) = \varphi(v) \otimes \psi(w)$ for all $v \in V_1$ and $w \in W_1$. Since $V_1 \otimes W_1$ is spanned by simple tensors, a linear map satisfying the above equation is unique.

3. Let V and W be finite-dimensional vector spaces over the same field F, and let $\varphi : V \to V$ and $\psi : W \to W$ be linear maps.

(a) Prove that $\mathrm{Tr}(\varphi \otimes \psi) = \mathrm{Tr}(\varphi)\mathrm{Tr}(\psi)$

(b) Assume that φ and ψ are both diagonalizable. Prove that $\varphi \otimes \psi$ is also diagonalizable and express the eigenvalues of $\varphi\otimes\psi$ in terms of the eigenvalues of φ and ψ.

Solution: (a) Choose any bases $\beta = \{v_1, \ldots, v_n\}$ of V and $\gamma = \{w_1, \ldots, w_m\}$ of W, and let $A = [\varphi]_\beta$ and $B = [\psi]_\gamma$. Thus, if $A = (a_{ij})$ and $B = (b_{kl})$, then $\varphi(v_j) = \sum_{i=1}^n a_{ij} v_i$ for all $1 \le j \le n$ and $\varphi(w_l) = \sum_{k=1}^n b_{kl} w_k$ for all $1 \le l \le m$.

We know that $\beta \otimes \gamma = \{v_j \otimes w_l\}$ is a basis of $V \otimes W$. For all $1 \le j \le n$ and $1 \le l \le m$ we have

$$(\varphi \otimes \psi)(v_j \otimes w_l) = (\sum_{i=1}^{n} a_{ij} v_i) \otimes (\sum_{k=1}^{m} b_{kl} w_k) = \sum_{i=1}^{n}\sum_{k=1}^{m} a_{ij} b_{kl}\ v_i \otimes w_k.$$

Let $A \otimes B = [\varphi \otimes \psi]_{\beta\otimes\gamma}$. We can think of rows and columns of $A \otimes B$ as being indexed by pairs of integers (i, k) with $1 \le i \le n$ and $1 \le k \le m$. The above computation shows that the $((i, k), (j, l))$-entry of the matrix $A \otimes B$ is equal to $a_{ij} b_{kl}$. The trace of $A\otimes B$ (which is equal to the trace of $\varphi\otimes\psi$) is the sum of the diagonal entries (that is, $((i, k), (i, k))$-entries) and thus equals

$$\sum_{i=1}^{n}\sum_{k=1}^{m} a_{ii} b_{kk} = \sum_{i=1}^{n} a_{ii} \sum_{k=1}^{m} b_{kk} = \mathrm{Tr}(A)\mathrm{Tr}(B) = \mathrm{Tr}(\varphi)\mathrm{Tr}(\psi).$$

(b) Let $n = \dim(V)$, $m = \dim(W)$, and let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$ be the eigenvalues of φ and ψ, respectively, listed with multiplicities. Since φ and ψ are diagonalizable, there exist bases $\{v_1, \ldots, v_n\}$ of V and $\{w_1, \ldots, w_m\}$ of W such that $\varphi(v_i) = \lambda_i v_i$ and $\psi(w_k) = \mu_k w_k$ for all $1 \le i \le n$ and $1 \le k \le m$.

Then $(\varphi \otimes \psi)(v_i \otimes w_k) = \lambda_i \mu_k v_i \otimes w_k$. Since the vectors $\{v_i \otimes w_k\}$ form a basis of $V \otimes W$, we conclude that $\varphi \otimes \psi$ is diagonalizable, and its eigenvalues are $\{\lambda_i \mu_k\}$.

4. Let $\rho : S_3 \to GL(\mathbb{C}^3)$ be the representation of S_3 introduced in HW#1.7, and let $W = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\}$. Recall that W is S_3-invariant, and let $\rho_W : S_3 \to GL(W)$ be the corresponding subrepresentation. Find a basis β of W such that the matrix $[\rho_W(g)]_\beta$ has integer entries for all $g \in S_3$ and compute those matrices explicitly (for each $g \in S_3$).

Let $\beta = \{e_1 - e_2, e_2 - e_3\}$. For $g \in G$ let $A_g = [\rho_W(g)]_\beta$. By direct computation we have

$$A_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_{(1,2)} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A_{(1,3)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$
$$A_{(2,3)} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \qquad A_{(1,2,3)} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad A_{(1,3,2)} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

5. Let G be a cyclic group and (ρ, V) an irreducible complex representation of G. Prove that $\dim(V) = 1$.

Solution: Let a be a generator of G. Since $\mathbb{C}$ is algebraically closed, $\rho(a)$ has (at least one) eigenvalue $\lambda \in \mathbb{C}$ and hence there is a nonzero $v \in V$ with $\rho(a)v = \lambda v$. By straightforward induction $\rho(a^k)v = \rho(a)^k v = \lambda^k v$ for all $k \in \mathbb{N}$. Also $v = \rho(a^{-1})\rho(a)v = \rho(a^{-1})(\lambda v) = \lambda \rho(a^{-1})v$. Thus, $\rho(a^{-1})v = \lambda^{-1}v$ and likewise $\rho(a^{-k})v = \lambda^{-k}v$ for all $k \in \mathbb{N}$.

Since also $\rho(a^0)v = Iv = v = \lambda^0 v$, we conclude that $\rho(a^k)v = \lambda^k v$ for all $k \in \mathbb{Z}$. Thus, v is an eigenvector for $\rho(g)$ for every $g \in G$ and hence $\mathbb{C}v$ is G-invariant. Since (ρ, V) is irreducible, we conclude that $\mathbb{C}v = V$ and hence $\dim(V) = 1$.

6. Let G be a group. Prove that the external and internal direct sums are equivalent as representations of G in the following sense. Let (ρ, V) be a representations of G, and let V_1 and V_2 be subrepresentations of V such that $V = V_1 \oplus V_2$. Prove that (ρ, V) is equivalent to the (external) direct sum of the representations (ρ_1, V_1) and (ρ_2, V_2) where $\rho_i(g) \in GL(V_i)$ is simply the restriction of $\rho(g)$ to V_i.

Solution: Below we assume that V is an internal direct sum of V_1 of V_2 (so that V_1 and V_2 are actually subspaces of V, without any identification involved). For the clarity of the argument we denote the external direct sum of V_1 and V_2 (considered as a representation of G) by (ρ^{ext}, V^{ext}). Thus by definition

$$\rho^{ext}(g)((v_1, v_2)) = (\rho_1(g)(v_1), \rho_2(g)(v_2)) \text{ for all } v_1 \in V_1, v_2 \in V_2, g \in G.$$

Define the map $T : V^{ext} \to V$ by $T((v_1, v_2)) = v_1 + v_2$. Since V is a direct sum of V_1 and V_2, the map T is an isomorphism of vector spaces. For all $g \in G$, $v_1 \in V_1$ and $v_2 \in V_2$ we have

$$\begin{aligned} \rho(g)T((v_1, v_2)) = \rho(g)(v_1 + v_2) = \rho(g)(v_1) + \rho(g)(v_2) = \rho_1(g)(v_1) + \rho_2(g)(v_2) \\ = T((\rho_1(g)(v_1), \rho_2(g)(v_2))) = T(\rho^{ext}(g)((v_1, v_2))) = T\rho^{ext}(g)((v_1, v_2)). \end{aligned}$$

Thus, $\rho(g)T = T\rho^{ext}(g)$ as maps and hence T is a homomorphism (and hence an isomorphism) of representations (ρ, V) and (ρ^{ext}, V^{ext}).