Bilinear Forms and Group Representations.
Solutions to the first midterm

1. (10 pts) Let $V = Mat_2(\mathbb{R})$, the vector spaces of 2×2 matrices over $\mathbb{R}$. Given a real number c, define $H_c : V \times V \to V$ by

$$H_c(A, B) = \mathrm{Tr}(AB) + c\,\mathrm{Tr}(A)\mathrm{Tr}(B).$$

Prove that H_c is a symmetric bilinear form, find a basis β such that $[H_c]_\beta$ is diagonal and compute the signature of H_c (both parts of your answer will depend on c).

Solution: The fact that H_c is bilinear is straightforward. Symmetry of H_c immediately follows from the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ which was established earlier.

Let us start with the basis $\gamma = \{e_{12}, e_{21}, e_{11}, e_{22}\}$ (ordered in this way). A simple computation shows that $[H_c]_\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & c+1 & c \\ 0 & 0 & c & c+1 \end{pmatrix}$. Since $[H_c]_\gamma$ is block-diagonal, it is enough to diagonalize H_c restricted to subspaces $U = \mathrm{Span}(e_{12}, e_{21})$ and $W = \mathrm{Span}(e_{11}, e_{22})$.

Let $v_1 = e_{12} + e_{21}$. Then $H_c(v_1, v_1) = 2$, so by the proof of Theorem 3.4 to diagonalize H_c on U we just need to find a vector in U orthogonal to v_1. Solving the equation $H_c(v_1, ae_{12} + be_{21}) = 0$ with $a, b \in \mathbb{R}$, we find that $v_2 = e_{12} - e_{21}$ is such a vector, so H_c restricted to U is diagonal in $\{v_1, v_2\}$.

Similarly, if we let $v_3 = e_{11} - e_{22}$, then $H(v_3, v_3) = 2$, and we find that $H(v_3, v_4) = 0$ for $v_4 = e_{11} + e_{22} \in W$. Thus, if we let $\gamma = \{v_1, v_2, v_3, v_4\}$, then $[H_c]_\gamma$ is diagonal. By direct computation of diagonal entries (two of which we already know) we get $[H_c]_\gamma = \mathrm{diag}(2, -2, 2, 4c + 2)$. Thus,

$$sig(H_c) = \begin{cases} (3,1) & \text{if } c > -\frac{1}{2} \\ (2,1) & \text{if } c = -\frac{1}{2} \\ (2,2) & \text{if } c < -\frac{1}{2}. \end{cases}$$

2. (10 pts) Let V be a finite-dimensional REAL inner product space and $A \in \mathcal{L}(V)$. Prove that $\langle Ax, x \rangle = 0$ for all $x \in V$ if and only if $A^* = -A$.

Solution: "$\Leftarrow$" Suppose that $A^* = -A$. Then for all $x \in V$ we have $\langle Ax, x\rangle = \langle x, A^*x\rangle = \langle x, (-A)x\rangle = -\langle Ax, x\rangle$ and hence (since $\mathrm{char}(\mathbb{R}) \neq 2$) we get $\langle Ax, x\rangle = 0$.

"$\Rightarrow$" Suppose now that $\langle Ax, x\rangle = 0$ for all $x \in V$. Then for all $x, y \in V$ we have $0 = \langle A(x+y), x+y\rangle - \langle Ax, x\rangle - \langle Ay, y\rangle = \langle Ax, y\rangle + \langle Ay, x\rangle$. Hence $\langle (-A)x, y\rangle = \langle x, Ay\rangle$ for all $x, y \in V$. This means that the operator $-A$ satisfies the definition of the adjoint of A and hence (by uniqueness of the adjoint) $A^* = -A$.

3. (10 pts) Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $A, B \in \mathcal{L}(V)$ be such that $AB = BA$.

(a) Let λ be an eigenvalue of A and $E_\lambda(A) = \{v \in V : Av = \lambda v\}$ the corresponding eigenspace. Prove that $E_\lambda(A)$ is B-invariant.

(b) Now assume that V is an inner product space and A and B are Hermitian. Use (a) to prove that there exists an orthonormal basis β such that $[A]_\beta$ and $[B]_\beta$ are both diagonal.

(c) (bonus, 2 extra pts) Now prove the assertion of (b) only assuming that A and B are normal.

Solution to (b) and (c): Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A, and let $V_i = E_{\lambda_i}(A)$ for $1 \leq i \leq k$. Since A is normal, we know that $V = \oplus_{i=1}^k V_i$; moreover the subspaces V_i are mutually orthogonal.

By (a) each V_i is B-invariant. So if we choose any basis γ_i of V_i for each i and let $\gamma = \cup_{i=1}^k \gamma_i$, then γ is a basis of V and $[B]_\gamma$ is block-diagonal (with diagonal blocks of size $\dim(V_1), \ldots, \dim(V_k)$).

Let B_i be the restriction of B to V_i. We claim that each B_i is a normal operator. To prove this assume (which we can) that each γ_i is orthonormal. Then γ is also orthonormal. Since B is normal, the matrix $[B]_\gamma$ is normal, that is, $[B]_\gamma^*[B]_\gamma = [B]_\gamma[B]_\gamma^*$. On the other hand, note that the i^{th} diagonal block of $[B]_\gamma$ is precisely $[B_i]_{\gamma_i}$ and hence the i^{th} diagonal block of $[B]_\gamma^*$ is $[B_i]_{\gamma_i}^*$. Using the formula for multiplying block-diagonal matrices (with blocks of the same size), we conclude that $[B_i]_{\gamma_i}^*[B_i]_{\gamma_i} = [B_i]_{\gamma_i}[B_i]_{\gamma_i}^*$ for each i, that is, each matrix $[B_i]_{\gamma_i}$ is normal. Since γ_i is orthonormal, we conclude that the operator B_i is also normal.

Since each B_i is normal, by Theorem 7.1 we can find an orthonormal basis β_i of V_i for each i such that $[B_i]_{\beta_i}$ is diagonal. Then $\beta = \cup_{i=1}^k \beta_i$ is an orthonormal basis of V, by construction $[B]_\beta$ is diagonal, and $[A]_\beta$ is also diagonal (since each element of β is an eigenvector of A, again by construction).

4. (10 pts)

(a) Let V be a finite-dimensional vector space over $\mathbb{R}$ and let H be a symmetric bilinear form on V. Prove that one can write $H = H^+ - H^-$ such that H^+ and H^- are both symmetric and positive semidefinite and $\mathrm{rk}(H) = \mathrm{rk}(H^+) + \mathrm{rk}(H^-)$.

(b) Let $A \in Mat_n(\mathbb{R})$ be a symmetric positive semidefinitive matrix of rank m. Prove that A can be written as $A = P^T P$ for some $m \times n$ matrix P.

Solution: In both parts of the problem we denote by $D(p, q, r)$ the (square) diagonal matrix of size $p+q+r$ whose first p diagonal entires are equal to 1, the next q are equal to -1 and the last r are equal to 0.

(a) By Theorem 3.4 there exists a basis β of V such that $[H]_\beta = D(p, q, r)$ for some p, q, r. Note that $\mathrm{rk}(H) = \mathrm{rk}([H]_\beta) = p+q$. Let H^+ and H^- be the unique bilinear forms such that $[H^+]_\beta = D(p, 0, q+r)$ and $[H^-]_\beta = D(0, q, p+r)$. We claim that H^+ and H^- have the required properties.

The forms H^+ and H^- are symmetric since the matrices $D(p, 0, q+r)$ and $D(0, q, p+r)$ are symmetric. Also $\mathrm{rk}(H^+) = \mathrm{rk}\, D(p, 0, q+r) = p$ and likewise $\mathrm{rk}(H^-) = q$. Finally, $[H^+ + H^-]_\beta = [H^+]_\beta + [H^-]_\beta = D(p, 0, q+r) + D(0, q, p+r) = D(p, q, r) = [H]_\beta$, whence $H = H^+ + H^-$.

(b) Since A is positive semidefinitive, by the matrix version of Theorem 3.4 there exists a matrix $Q \in GL_n(\mathbb{R})$ such that $Q^T A Q = D(p, 0, r)$. Note that $p = \mathrm{rk}(Q^T A Q) = \mathrm{rk}(A) = m$ and $r = n - m$.

Let B be the $m \times n$ matrix obtained from the $m \times m$ identity matrix by adding an $m \times (n-m)$ block of zeroes on the right. An easy direct computation shows that $B^T B = D(p, 0, r) = Q^T A Q$. Hence $A = (Q^T)^{-1} B^T B Q^{-1} = (Q^{-1})^T B^T B Q^{-1} = P^T P$ where $P = BQ^{-1}$. Since Q is invertible, we have $\mathrm{rk}(P) = \mathrm{rk}(B) = m$.

5. (10 pts) Let H and G be Hermitian forms on $\mathbb{C}^2$ which are not proportional, and let W be the set of linear combinations of H and G with REAL coefficients. Thus, W is a 2-dimensional (real) subspace of the space $\mathbb{H}(\mathbb{C}^2)$ of all Hermitian forms on $\mathbb{C}^2$ (note that $\mathbb{H}(\mathbb{C}^2)$ is a vector space over $\mathbb{R}$, but not over $\mathbb{C}$). Prove that the following three conditions are equivalent:

(i) The forms H and G are simultaneously diagonalizable, that is, there exists a basis β such that $[H]_\beta$ and $[G]_\beta$ are both diagonal

(ii) The subspace W contains a positive definite form

(iii) If $[H]$ and $[G]$ are the matrices of H and G with respect to the standard basis, then there exist $a, b \in \mathbb{R}$ such that $\det(a[H] + b[G]) > 0$.

Solution: "(i)$\Rightarrow$ (ii)" Let Z be the real vector space of all 2×2 diagonal matrices with real entires. Clearly $\dim(Z) = 2$. By assumption both $[H]_\beta$ and $[G]_\beta$ are elements of Z; moreover, they are not proportional (and hence linearly independent) since H and G are not proportional. Hence $[H]_\beta$ and $[G]_\beta$ form a basis of Z, so in particular there exist $a, b \in \mathbb{R}$ such that $a[H]_\beta + b[G]_\beta = I_2$, the 2×2 identity matrix. Since $a[H]_\beta + b[G]_\beta = [aH + bG]_\beta$, we conclude that the form $aH + bG$ (which lies in W) is positive-definite.

"(ii)$\Rightarrow$ (i)" Let $aH + bG$ be a postive definite form in W. We know that a and b cannot both be zero, and WOLOG we can assume that $a \neq 0$. By HW#4.4 there is basis β such that $[G]_\beta$ and $[aH + bG]_\beta$ are both diagonal. But then

$$[H]_\beta = \left[\frac{(aH + bG) - bG}{a}\right]_\beta = \frac{1}{a}[aH + bG]_\beta - \frac{b}{a}[G]_\beta$$

is also diagonal, so (i) holds.

Next we prove a general result:

Lemma: *Let B be a Hermitian form on a finite-dimensional vector space V. If $\det([B]_\beta) > 0$ for some basis β of V, then $\det([B]_\gamma) > 0$ for every basis γ of V.*

Proof: We know that $[B]_\gamma = Q^*[B]_\beta Q$ for some $Q \in GL(V)$. Hence

$$\begin{aligned}\det([B]_\gamma) &= \det(Q^*[B]_\beta Q) = \det(Q^*)\det([B]_\beta)\det(Q) \\ &= \overline{\det(Q)}\det([B]_\beta)\det(Q) = |\det(Q)|^2\det([B]_\beta).\end{aligned}$$

Since $\det(Q) \neq 0$, the result follows.

"(ii)$\Rightarrow$ (iii)" Let $aH + bG$ be a postive definite form in W. By the Hermitian version of Theorem 3.4 we know that there is a basis γ such that $[aH + bG]_\beta = I_2$. Then $\det([aH + bG]_\gamma) = 1 > 0$ and hence by Lemma $\det([aH + bG]_{std}) > 0$ as well.

"(iii)$\Rightarrow$ (ii)" Suppose that $\det([aH + bG]_{std}) > 0$. Let γ be a basis such that $[aH + bG]_\gamma = \mathrm{diag}(u, v)$ for some $u, v \in \mathbb{R}$. By Lemma, $\det([aH + bG]_\gamma) > 0$ as well. Thus, $uv > 0$, so u and v are both positive or both negative. If u and v are both positive, then $aH + bG$ is positive definite, and if u and v are both negative, then $(-a)H + (-b)G$ (which also lies in W) is positive definite.