Math 7751, Fall 2009. Final exam. Due Tuesday, December 8th

Directions: Each problem is worth 10 points. The best 7 out of 8 scores will be counted with 100% weight, and the lowest score will be counted with 50% weight, so the maximal possible total is 75 points. Provide complete arguments (do not skip steps). State clearly any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given.

Rules: You are not allowed to discuss midterm problems with each other. You may ask me any questions about the problems (e.g. if the formulation is unclear), but I may only provide minor hints. You may use freely your class notes, previous homework assignments and the book by Dummit and Foote. The use of other books is allowed, but not encouraged. If you happen to run across a problem very similar or identical to one on the midterm which is solved in another book, do not consult that solution.

1. (a) Let R and S be rings with 1. Prove that every ideal of $R \times S$ has the form $I \times J$ where I is an ideal of R and J is an ideal of S.

(b) Let G and H be finite groups of relatively prime orders. Prove that every subgroup of $G \times H$ has the form $A \times B$ where A is a subgroup of G and B is a subgroup of H.

2. Let G be a group in which any two conjugate elements commute with each other, that is, x and gxg^{-1} commute for any $x, g \in G$.

(a) Prove that G has a non-trivial abelian normal subgroup (possibly equal to G).

(b) Prove that if G is also finite, then G must be solvable

3. (a) Let R be a commutative ring with 1. The *Krull dimension* of R is the largest integer $n \geq 0$ such that R has an ascending chain of *prime* ideals $P_0 \subset P_1 \subset \ldots \subset P_n \subset R$ where all inclusions are strict. Suppose that R is a PID. What are possible Krull dimensions of R? **Hint:** There are only finitely many possibilities.

(b) Let R be a UFD, P a nonzero prime ideal of R and $S = R/P$. Determine which of the following statements is true:

(i) S is always a PID

(ii) S may not be a PID, but it is always a UFD

(iii) S may not even be a UFD

4. (a) Prove that for any integer $n \geq 2$ the ring $\mathbb{Z}[in] \subset \mathbb{C}$ is not a PID (here i is the complex number i)
(b) Now assume that $n \geq 2$ is odd. Prove that $\mathbb{Z}[in]$ is not a UFD.

5. Prove that the polynomial $f(x) = 32x^6 + 4x + 1$ is irreducible in $\mathbb{Z}[x]$. **Hint:** use an algebraic trick to reduce to Eisenstein.

6. Let F be a field. Prove that the additive group of F and the multiplicative group of F are not isomorphic to each other. **Hint:** Look at the orders of elements in both groups. There will be two very different cases.

7. A group G is called *just-infinite* if G is infinite but all its proper quotients are finite (that is, G/N is finite for any non-trivial normal subgroup N of G). Prove that every finitely generated group has a just-infinite quotient. You may use the following fact without proof:

Fact: If G is a finitely generated group and H is a subgroup of G of finite index, then H is also finitely generated.

Hint: Reduce the problem to showing that certain poset associated to G has a maximal element (you will first need to figure out what that poset is).

8. Let p and q be primes such that p is a generator of the multiplicative group $\mathbb{F}_q^*$. Prove that the cyclotomic polynomial $\Phi_q(x) = \sum_{i=0}^{q-1} x^i$ is irreducible in $\mathbb{F}_p[x]$.
Hint: Let $R = \mathbb{F}_p[x]/(x^q - 1)$. Prove that R is a direct product of fields of p-power order. Then consider the orders of elements in the multiplicative group R^*.