Solutions to Homework #10

Problems:

1. Recall that in HW#7.5 we proved that if (ρ, V) is any cyclic representation of a finite group G over an arbitrary field, then $\dim(V) \leq |G|$. Now prove that if (ρ, V) is irreducible, then $\dim(V) \leq |G| - 1$.

Solution: In this problem we have to assume that $|G| > 1$ (otherwise the assertion is false). Let (ρ, V) be an irreducible representation of G. By HW#7.5 we know that $\dim(V) \leq |G|$. So we shall assume that $\dim(V) = |G|$ and try to reach a contradiction.

Fix any nonzero $v \neq 0$. The solution to HW#7.5 shows that if $S = \{\rho(g)v : g \in G\}$, then $V = \mathrm{Span}(S)$. Clearly $|S| \leq |G|$, so the equality $\dim(\mathrm{Span}(S)) = |G|$ forces S to be linearly independent.

Now let $w = \sum\limits_{g \in G} \rho(g)v$; in other words, w is the sum of all the elements of S. Then $w \neq 0$ since S is linearly independent. On the other hand, w is clearly G-invariant, so $W = Fw$ is a nonzero (since $w \neq 0$) G-invariant subspace of V. Since V is irreducible, we must have $W = V$, so $|G| = \dim(V) = \dim(W) = 1$, a contradiction.

2. The goal of this problem is to explicitly decompose the regular representation $(\rho_{reg}, \mathbb{C}[S_3])$ as a direct sum of irreducible representations of S_3.

Solution: We will give an argument which does not follow the outline suggested in the problem. Let $G = S_3$ and $H = (1,2,3)$. By Lecture 21, $\mathbb{C}[H]$ considered as the regular representation of H decomposes as $\mathbb{C}[H] = W_1 \oplus W_2 \oplus W_3$ where

$$\begin{aligned} W_1 &= \mathbb{C}(e + (1,2,3) + (1,3,2)) \\ W_2 &= \mathbb{C}(e + \omega(1,2,3) + \omega^2(1,3,2)) \\ W_3 &= \mathbb{C}(e + \omega^2(1,2,3) + \omega(1,3,2)), \end{aligned}$$

with $\omega = e^{\frac{2\pi i}{3}}$, and each W_i is an H-subrepresentation. Moreover, $\mathbb{C}[H]$ is naturally a subspace of $\mathbb{C}[G]$, and if we consider $\mathbb{C}[G]$ as an H-representation (via the restriction of the regular representation of G), then W_1, W_2 and W_3 are H-subrepresentations of $\mathbb{C}[G]$.

Since $G = H \sqcup (1,2)H$, the solution to HW#9.1 shows that if we set $U_i = W_i + (1,2)W_i$ for $i = 1,2,3$, then each U_i is a G-subrepresentation of $\mathbb{C}[G]$. It is straightforward to check that $\mathbb{C}[G] = U_1 \oplus U_2 \oplus U_3$;

moreover, U_1 can be decomposed further as a G-representation: $U_1 = A \oplus B$ where

$$\begin{aligned} A &= \mathbb{C}(e + (1,2,3) + (1,3,2) + (1,2) + (2,3) + (1,3)) \\ B &= \mathbb{C}(e + (1,2,3) + (1,3,2) - (1,2) - (2,3) - (1,3)). \end{aligned}$$

The vector $a = e + (1,2,3) + (1,3,2) + (1,2) + (2,3) + (1,3)$ is G-invariant, so A is equivalent to the trivial representation. The vector $b = e + (1,2,3) + (1,3,2) - (1,2) - (2,3) - (1,3)$ is H-invariant, and left multiplication by an element of $G \setminus H$ sends b to $-b$, so B is equivalent to the sign representation.

Finally, we claim that U_2 and U_3 are both irreducible and equivalent to the standard representation. This can be checked by computing their characters, but we can also deduce the result from our general knowledge of how the regular representation decomposes into irreducibles.

Indeed, we know that $\mathbb{C}[S_3]$ is a direct sum of 4 irreducible subrepresentations: a copy of the trivial representation, a copy of the sign representation and two copies of the standard representation. On the other hand, we know that $\mathbb{C}[S_3] = A \oplus B \oplus U_1 \oplus U_2$ (where each piece is a G-subrepresentation). Finally, note that if V is a complex representation of a finite group G and W is an irreducible complex representation of G, then $m_W(V)$, the multiplicity of W in V, does not depend on the choice of decomposition of V into irreducibles (I did not emphasize this point in class, but it follows immediately from Proposition 21.2 in class which asserts that $m_W(V) = \langle \chi_V, \chi_W \rangle$).

This independence implies that each of the pieces A, B, U_1 and U_2 must be irreducible; also since we already showed that A and B are equivalent to the trivial and sign representations, respectively, the other two pieces U_2 and U_3 must be equivalent to the standard representation.

3. (Steinberg, Exercise 7.9). Suppose that G is a finite group of order n with s conjugacy classes. Suppose that one chooses a pair $(g,h) \in G \times G$ uniformly at random. Prove that the probability g and h commute is $\frac{s}{n}$. **Hint:** Apply Burnside's counting lemma to a suitable action of G. (If this is not enough, read the hint in Steinberg).

Solution: For each $g \in G$ the number of elements which commute with g is $|C(g)|$, the order of the centralizer of g. Thus the total number of (ordered) commuting pairs of elements of G is $\sum\limits_{g \in G} |C(g)|$. The total number of pairs is $|G|^2 = n^2$, so the probability that two elements commute is $\frac{\sum_{g\in G} |C(g)|}{n^2}$.

Now consider the action of G on itself by conjugation. The number of orbits is precisely s, and for each $g \in G$ its fixed set $\mathrm{Fix}(g)$ is $C(g)$. Hence by the Burnside counting lemma $s = \frac{\sum_{g\in G} |C(g)|}{n}$, and therefore the probability that two elements of G commute is $\frac{s}{n}$.

4. (Steinberg, Exercise 7.5, reformulated). Before doing this problem read Chapter 7 of Steinberg. Let p be a prime, and let G be the set of all functions from $\mathbb{Z}_p$ to $\mathbb{Z}_p$ which have the form $x \mapsto ax + b$ for some $a \in \mathbb{Z}_p^\times$ and $b \in \mathbb{Z}_p$.

(a) Prove that G is a group (with respect to composition) isomorphic to the group of matrices $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z}_p^\times, b \in \mathbb{Z}_p \right\}$. Note that for $p = 5$ this is the group from HW#8.6.

(b) The group G has a natural action on $\mathbb{Z}_p$ (given by $g.x = g(x)$ for all $g \in G$ and $x \in \mathbb{Z}_p$). Prove that this action is 2-transitive (see Steinberg 7.1 for the definition).

(c) By Lemma 22.1(2) the action from (b) yields a homomorphism $\varphi : G \to S_p$. Composing φ with the standard representation of S_p, we obtain a $(p-1)$-dimensional representation of G. Deduce from (b) (and a suitable result from Steinberg 7.2) that this representation is irreducible.

Solution: (b) Take any $a \neq b$ in $\mathbb{Z}_p$, and let $g \in G$ be given by $g(x) = (b - a)x + a$. Then g sends 0 to a and 1 to b. Hence the action of G on $\{(a, b) \in \mathbb{Z}_p \times \mathbb{Z}_p : a \neq b\}$ is transitive, as every element is in the orbit of $(0, 1)$.

(c) For notational convenience, we will consider φ as a map from G to $\mathrm{Sym}(X)$. "Define" the defining representation of $\mathrm{Sym}(X)$ to be the representation $(\rho_{def}, \mathbb{C}X)$ given by $(\rho_{def}(g))(x) = g(x)$ for all $g \in G$ and $x \in X$ (clearly, in the case $X = \{1, \ldots, p\}$ this representation is equivalent to the defining representation of S_p in the usual sense).

Now consider the composition $\rho_{def} \circ \varphi : G \to \mathrm{GL}(\mathbb{C}X)$. This is a representation of G which is EQUAL to the permutation representation (corresponding to the given group action) since for all $g \in G$ and $x \in X$ we have $((\rho_{def} \circ \varphi)(g))(x) = (\rho_{def}(\varphi(g)))(x) = (\varphi(g))(x) = g.x$. Here the second equality holds by the definition of ρ_{def} and the last equality holds by the definition of φ in Lemma 22.1(2).

The subspace $(\mathbb{C}X)_0 = \{\sum_{x\in X} \lambda_x x : \sum \lambda_x = 0\}$ is G-invariant, and the restriction of the permutation representation to $(\mathbb{C}X)_0$, called the *augmentation representation* in Steinberg, is exactly the composition $\rho_{std} \circ \varphi$ where ρ_{std} is the standard representation. Since the action of G on X is 2-transitive by (b), the augmentation representation is irreducible by Theorem 7.2.11 in Steinberg.

5. Let C be the cube in $\mathbb{R}^3$ whose vertices have coordinates $(\pm1, \pm1, \pm1)$. Let G be the group of rotations of C, that is rotations in $\mathbb{R}^3$ which preserve the cube (you may assume that G is a group without proof). Let X be the set of 4 main diagonals of C (diagonals connecting the opposite vertices). Note that G naturally acts on X and therefore we have a homomorphism $\pi : G \to Sym(X) \cong S_4$.

(a) Prove that π is an isomorphism.
(b) Note that G is naturally a subgroup of $\mathrm{GL}_3(\mathbb{R})$ and hence also a subgroup of $\mathrm{GL}_3(\mathbb{C})$, and let $\iota : G \to \mathrm{GL}_3(\mathbb{C})$ be the inclusion map. By (a) we get a representation $\iota \circ \pi^{-1} : S_4 \to \mathrm{GL}_3(\mathbb{C})$. Prove that this representation is equivalent to the tensor product of the standard and sign representations.

Solution: (a) If v and w are adjacent vertices, there is a $90°$ rotation (around one of the coordinate axes) which moves v to w. For any two vertices v and w there is a sequence which starts with v and ends with w and in which any two consecutive vertices are adjacent. This implies that G acts transitively on the set of vertices. If v is a fixed vertex, there are at least 3 elements of G which fix v, namely the rotations by $120°$ in either direction around the main diagonal joining v with $-v$ and the trivial element. Thus, the stabilizer of v has at least 3 elements and hence G has at least $8 \cdot 3 = 24$ elements.

As suggested in the hint, once we know that $|G| \geq 24$, to prove that π is an isomorphism it suffices to show that π is injective. So suppose that $R \in \operatorname{Ker} \pi$. Note that we CANNOT say right away that R fixes every vertex (the latter of course would imply that $R = \mathrm{id}$). All we can say is that R fixes each main diagonal as a SET, that is, it either fixes both of the endpoints or swaps them. Suppose that R fixes at least one vertex v. Then it must permute the three vertices v_1, v_2, v_3 adjacent to v. Since neither two of these three vertices are opposite to each other (and on the other hand, R must send each vertex to itself or its opposite), the only possibility is that R fixes v_1, v_2 and v_3, which implies that $R = \mathrm{id}$.

Thus, we showed that any $R \in \operatorname{Ker} \pi$ is either the identity map or does not fix any vertex, and the latter means that R sends each vertex to its opposite. But then R is a reflection with respect to the origin (in other words, multiplication by -1) and hence not a rotation (e.g. because it has determinant -1), so $R \notin G$. Thus we proved that $\operatorname{Ker} \pi = \{\mathrm{id}\}$ and hence π is injective.

(b) Let $\varphi = \iota \circ \pi^{-1}$. We will prove that φ is equivalent to the "standard tensor sign" representation by computing its character. While it is not difficult to compute the matrices of the individual elements of G in this representation, we will show below that this computation can be completely avoided.

Indeed, let R be a rotation in $\mathbb{R}^3$ with respect to some line passing through the origin (so that R is a linear map). Then in a suitable basis R has the matrix $\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where φ is the angle of the rotation and we can assume that $0 \leq \varphi \leq 180°$. Hence $\operatorname{Tr}(R) = 2\cos\varphi + 1$.

Note that if R has finite order k (as an element of $\mathrm{GL}_3(\mathbb{R})$), then k is the smallest positive integer such that $k\varphi$ is an integer multiple $360°$. In the case $k \leq 4$ the angle φ (with the above restriction $0 \leq \varphi \leq 180°$) is uniquely determined by k: $\varphi = 0°$ if $k = 1$ and $\varphi = \frac{360°}{k}$ if $k = 2, 3$ or 4. Hence $\operatorname{Tr}(R) = 3$ if $k = 1$, $\operatorname{Tr}(R) = -1$ if $k = 2$, $\operatorname{Tr}(R) = 0$ if $k = 3$ and $\operatorname{Tr}(R) = 1$ if $k = 4$.

Since every element of S_4 has order at most 4, the values of the character χ_φ are completely determined by the orders of elements: $\chi_\varphi(e) = 3$, $\chi_\varphi((1,2)) = \chi_\varphi((1,2)(3,4)) = -1$, $\chi_\varphi((1,2,3)) = 0$ and $\chi_\varphi((1,2,3,4)) = 0$. From the character table of S_4 this is precisely the character of the "standard tensor sign" representation.