# THE TARSKI NUMBERS OF GROUPS

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ABSTRACT. The Tarski number of a non-amenable group $G$ is the minimal number of pieces in a paradoxical decomposition of $G$. In this paper we investigate how Tarski numbers may change under various group-theoretic operations. Using these estimates and known properties of Golod-Shafarevich groups, we show that the Tarski numbers of 2-generated non-amenable groups can be arbitrarily large. We also use $L^2$-Betti numbers to show that there exist groups with Tarski number 6. These provide the first examples of non-amenable groups without free subgroups whose Tarski number has been computed precisely.

## 1. INTRODUCTION

Recall the definition of a *paradoxical decomposition* of a group.

**Definition 1.1.** A group $G$ admits a *paradoxical decomposition* if there exist positive integers $m$ and $n$, disjoint subsets $P_1, \ldots, P_m, Q_1, \ldots, Q_n$ of $G$ and elements $g_1, \ldots, g_m$, $h_1, \ldots, h_n$ of $G$ such that

$$G = \bigcup_{i=1}^{m} g_i P_i = \bigcup_{j=1}^{n} h_j Q_j. \tag{1.1}$$

It is well known [29] that $G$ admits a paradoxical decomposition if and only if it is non-amenable. The minimal possible value of $m + n$ in a paradoxical decomposition of $G$ is called the *Tarski number* of $G$ and denoted by $\mathcal{T}(G)$.

The definition stated above appears both in [29] and [26]. A slightly different definition of a paradoxical decomposition (see, for example, [1]) requires the sets $P_1, \ldots, P_m$, $Q_1, \ldots, Q_n$ to cover the entire group $G$ and each of the unions $\bigcup_{i=1}^{m} g_i P_i$ and $\bigcup_{j=1}^{n} h_j Q_j$ to be disjoint. This alternative definition leads to the same notion of Tarski number: this follows from the proof of [25, Proposition 1.2] and Remark 2.2 below, but for completeness we will prove the equivalence of the two definitions of Tarski numbers in Appendix A.

It is clear that for any paradoxical decomposition we must have $m \geq 2$ and $n \geq 2$, so the minimal possible value of Tarski number is 4. By a theorem of Jónsson and Dekker (see, for example, [26, Theorem 5.8.38]), $\mathcal{T}(G) = 4$ if and only if $G$ contains a non-Abelian free subgroup.

The problem of describing the set of Tarski numbers of groups has been formulated in [1], and the following results have been proved there:

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2000 *Mathematics Subject Classification.* Primary 43A07, 20F65 Secondary 20E18, 20F05, 20F50.

*Key words and phrases.* Tarski number, paradoxical decomposition, amenability, $L^2$-Betti number, cost, Golod-Shafarevich groups.

The work of the first author was partially supported by the NSF grant DMS-1201452 and the Sloan Research Fellowship grant BR 2011-105. The work of the second author was partially supported by BSF grant T-2012-238 and the Ministry of Science and Technology of Israel. The work of the third author was partially supported by NSF DMS-1318716, DMS-1261557 and BSF 2010295.

**Theorem 1.2.**

(i) *The Tarski number of any torsion group is at least 6.*

(ii) *The Tarski number of any non-cyclic free Burnside group of odd exponent* $\geq 665$ *is between 6 and 14.*

For quite some time it was unknown if the set of Tarski numbers is infinite. That question was asked by Ozawa [22] and answered in the positive by the third author. For every $m \geq 1$ let $\mathrm{Amen}_m$ (resp. $\mathrm{Fin}_m$) be the class of all groups where all $m$-generated subgroups are amenable (resp. finite). For example, $\mathrm{Amen}_1$ is the class of all groups and $\mathrm{Fin}_1$ is the class of all torsion groups. Clearly $\mathrm{Fin}_m \subseteq \mathrm{Amen}_m$ for every $m$. Ozawa noticed [22] that all groups in $\mathrm{Amen}_m$ have Tarski number at least $m+3$, and the third author observed that $\mathrm{Fin}_m$ (for every $m$) contains non-amenable groups. This immediately follows from two results about Golod-Shafarevich groups proved in [4] and [5] (see § 4.1 below). Thus there exist non-amenable groups with arbitrarily large Tarski numbers.

In fact, results of [4, 5] imply the following much stronger statement (see § 4 for details).

**Theorem 1.3.** *There exists a finitely generated non-amenable group $H$ such that for every $m \geq 1$, $H$ has a finite index subgroup $H_m$ lying in* $\mathrm{Fin}_m$ *and hence $H_m$ is a non-amenable group with Tarski number at least $m+3$. Moreover, for every prime $p$ we can assume that $H$ is a residually-$p$ group.*

**Remark 1.4.** Since subgroups of finite index are quasi-isometric to the whole group, Theorem 1.3 implies that for some natural number $t > 4$, the property of having Tarski number $t$ is not invariant under quasi-isometry. We do not know what the number $t$ is, i.e., what is the Tarski number of the group $G$ from Theorem 1.3. The only estimates we have are based on a rough calculation of the isoperimetric constant of the group $G$ which gives an enormous upper bound for $t$ (about $10^{10^8}$). Note that a well-known question of Benson Farb asks whether the property of finitely generated groups of having a non-Abelian free subgroup is invariant under quasi-isometry. In view of the result of Jónsson and Dekker this is equivalent to the question whether the property of having Tarski number 4 is invariant under quasi-isometry.

We now turn to the discussion of our results.

1.1. **Tarski numbers of subgroups and quotients.** If $H$ is a non-amenable group which is either a subgroup or a quotient of a group $G$, it is easy to see that $\mathcal{T}(G) \leq \mathcal{T}(H)$ (for a proof see [26, Theorems 5.8.10, 5.8.13]). Conversely, in many cases it is possible to find an explicit upper bound on $\mathcal{T}(H)$ in terms of $\mathcal{T}(G)$. Our results of this type are collected in the following theorem:

**Theorem 1.** *Let $G$ be a non-amenable group and $H$ a subgroup of $G$.*

(a) *Suppose that $H$ has finite index in $G$. Then*

$$\mathcal{T}(H) - 2 \leq [G : H](\mathcal{T}(G) - 2).$$

(b) *Let $\mathcal{V}$ be a variety of groups where all groups are amenable and relatively free groups are right orderable. Then there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ (depending only on $\mathcal{V}$) with the following property: if $H$ is normal in $G$ and $G/H \in \mathcal{V}$, then $\mathcal{T}(H) \leq f(\mathcal{T}(G))$.*

(c) *Assume that $H$ is normal and amenable. Then $\mathcal{T}(G/H) = \mathcal{T}(G)$.*

(d) *Assume that $G = H \times K$ for some $K$. Then* $\min\{\mathcal{T}(H), \mathcal{T}(K)\} \leq 2(\mathcal{T}(G) - 1)^2$.

**Remarks 1.5.** (i) There is an obvious similarity between Theorem 1(a) and the Schreier index formula. The main difference with the latter is that we do not know whether the above inequality can become an equality (for $H \neq G$), and if not, how large the ratio $\frac{\mathcal{T}(H)-2}{\mathcal{T}(G)-2}$ can be compared to the index $[G : H]$. We do know that dependence on the index cannot be eliminated in view of Theorem 1.3.

(ii) Varieties $\mathcal{V}$ for which the hypotheses of Theorem 1(b) hold include the variety of all Abelian groups and more generally all solvable groups of a given class [19]. In particular, if $G/H$ is cyclic, Theorem 1(b) yields a (non-trivial) lower bound on $\mathcal{T}(G)$ in terms of $\mathcal{T}(H)$ alone (independent of the size of $G/H$). This special case will be used to prove Theorem 2 below.

(iii) We do not know if $\mathcal{T}(H \times K)$ can be strictly smaller than the minimum of $\mathcal{T}(H)$ and $\mathcal{T}(K)$. By Theorem 1(c) the inequality becomes an equality if one of the groups $H$ or $K$ is amenable. For the case $K = H$ see [26, Problem 5.9.23].

1.2. **Unbounded Tarski numbers.** It is clear that non-amenable groups from $\mathrm{Amen}_m$ must have at least $m + 1$ generators. Thus the already mentioned results about groups with arbitrarily large Tarski numbers give rise to the following natural question:

**Question 1.6.** *Is there a relation between the minimal number of generators of a non-amenable group and its Tarski number?*

The next theorem shows that the answer is negative.

**Theorem 2.** *The set of Tarski numbers of* 2*-generated non-amenable groups is infinite. Moreover, the set of Tarski numbers of* 2*-generated infinite groups with property (T) is infinite.*

To prove Theorem 2 we use a construction from [20] to embed any finitely generated group $G$ from $\mathrm{Fin}_m$ into a 2-generated subgroup $H$ of a group which is an extension of a group from $\mathrm{Fin}_m$ by a finite metabelian group. The construction ensures that $H$ has property $(T)$ whenever $G$ does, and the lower bound on the Tarski number of $H$ then follows from Theorem 1(b).

Theorem 2 has an interesting application to cogrowth and spectral radius (for the definition of cogrowth and spectral radius see [1]; note that both quantities are invariants of a marked group, that is, a group with a chosen finite generating set). Recall that the maximal possible cogrowth of an $m$-generated group is $2m - 1$ and the maximal spectral radius is 1. If $m > 1$, an $m$-generated group is amenable if and only if its cogrowth is exactly $2m-1$ and if and only if the spectral radius is 1. The formulas from [1, Section IV] relating the Tarski number with the cogrowth and spectral radius of a group immediately imply the following:

**Corollary 1.7.** *For every $\varepsilon > 0$ there exists a* 2*-generated non-amenable group $G$ such that every* 2*-generated subgroup $\langle x, y\rangle$ of $G$ has cogrowth at least $3 - \varepsilon$ (with respect to the generating set $\{x, y\}$) and spectral radius at least $1 - \varepsilon$.*

Note that since there are 2-generated amenable groups (say, the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$) which are inductive limits of 2-generated non-amenable groups [13], there are 2-generated non-amenable groups with cogrowth arbitrarily close to 3. But all previously known examples of such groups contain non-Abelian free subgroups and so they have 2-generated subgroups with cogrowth 0.

Theorem 1(a) and Theorem 1.3 for $p = 2$ yield another interesting corollary which is somewhat similar to the Bertrand postulate for prime numbers.

**Corollary 1.8** (See Theorem 4.8)**.** *For every sufficiently large natural number $n$, there exists a group with Tarski number between $n$ and $2n$.*

1.3. **Explicit values.** The final problem we address in this paper is precise calculation of Tarski numbers. Prior to this paper there were no examples of non-amenable groups without free subgroups whose Tarski number had been determined. In fact, no integer $> 4$ was known to be the Tarski number of a group. We will show using $L^2$-Betti numbers of groups that there exist groups (in fact, a large class of groups) with Tarski number 6.

**Theorem 3.** *Let $G$ be any 3-generated group with $\beta_1(G) \geq 3/2$ where $\beta_1(G)$ is the first $L^2$-Betti number of $G$. Then $\mathcal{T}(G) \leq 6$. In particular, if $G$ is torsion, then $\mathcal{T}(G) = 6$.*

Note that in [21], Osin showed that for any integer $d \geq 2$ and any $\varepsilon > 0$ there exists a $d$-generated torsion group $G$ with $\beta_1(G) \geq d - 1 - \varepsilon$. Thus, torsion groups satisfying the hypotheses of Theorem 3 (and therefore having Tarski number 6) do exist. Moreover, one can construct such groups with very explicit presentations (see Appendix B). The last assertion of Theorem 3 follows from Theorem 1.2(i).

**Organization.** In § 2 we introduce basic graph-theoretic terminology and give a graph-theoretic interpretation of paradoxical decompositions. In § 3 we prove Theorem 1. In § 4 we prove Theorem 2 and discuss some related results. In § 5 we prove Theorem 3. In Appendix A we prove the equivalence of two definitions of Tarski numbers. In Appendix B we describe an explicit construction of groups with Tarski number 6. Finally, Appendix C contains a brief introduction to Golod-Shafarevich groups.

**Acknowledgments.** The authors would like to thank Rostislav Grigorchuk, Damien Gaboriau, Wolfgang Lück, Russell Lyons, Denis Osin, Dan Salajan and Andreas Thom for useful discussions and Narutaka Ozawa for posting question [22].

## 2. Preliminaries

2.1. **$k$-paradoxical decompositions.** It will be convenient to slightly generalize the notion of paradoxical decomposition defined in the introduction (this generalization is also used, in particular, in [25]).

**Definition 2.1.** Let $G$ be a group and $k \geq 2$ an integer. Suppose that there exist finite subsets $S_1 = \{g_{1,1}, \ldots, g_{1,n_1}\}, \ldots, S_k = \{g_{k,1}, \ldots, g_{k,n_k}\}$ of $G$ and disjoint subsets $\{P_{ij} : 1 \leq i \leq k, 1 \leq j \leq n_i\}$ of $G$ such that for each $1 \leq i \leq k$ we have $G = \bigcup_{j=1}^{n_i} g_{i,j} P_{i,j}$. Then we will say that $G$ admits a *$k$-paradoxical decomposition with translating sets $S_1, \ldots, S_k$*. The set $\cup_{i=1}^{k} S_i$ will be called the *total translating set* of the decomposition.

Note that 2-paradoxical decompositions are paradoxical decompositions in the usual sense. Every $k$-paradoxical decomposition with translating sets $S_1, ..., S_k$ "contains" a 2-paradoxical decomposition with translating sets $S_1, S_2$. Conversely, given a 2-paradoxical decomposition of a group $G$, there is a simple way to construct a $k$-paradoxical decomposition of $G$ for arbitrarily large $k$ (see Lemma 3.3). We will mostly use 2-paradoxical decompositions, but 4-paradoxical decompositions will naturally arise in the proof of Theorem 1(b).

The following result is obvious:

**Remark 2.2.** If $G$ has a $k$-paradoxical decomposition with translating sets $S_1, \ldots, S_k$, then $G$ also has a $k$-paradoxical decomposition with translating sets $g_1 S_1, \ldots, g_k S_k$ for any given $g_1, \ldots, g_k \in G$. In particular, we can always assume that $1 \in S_i$ for each $i$.

Next we introduce some graph-theoretic terminology which is convenient for dealing with paradoxical decompositions. We will mostly work with oriented graphs, which will be allowed to have loops and multiple edges. In some cases edges of our graphs will be colored and/or labeled. The sets of vertices and edges of a graph $\Gamma$ will be denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. If edges of $\Gamma$ are colored using colors $\{1,\ldots,k\}$, we denote by $E_i(\Gamma)$ the set of edges of color $i$.

**Definition 2.3.**

(i) Let $k$ be a positive integer. An oriented graph $\Gamma$ will be called a *$k$-graph* if at each vertex of $\Gamma$ there are (precisely) $k$ outgoing edges and at most one incoming edge.
(ii) A $k$-graph $\Gamma$ will be called *evenly colored* if the edges of $\Gamma$ are colored using $k$ colors and at each vertex of $\Gamma$ the $k$ outgoing edges all have different colors.

Let $G$ be a group and $S$ a subset of $G$. The Cayley graph $Cay(G,S)$ is the **oriented** graph with vertex set $G$ and a directed edge from $g$ to $gs$ for every $g\in G$ and $s\in S$. The edge $(g,gs)$ will be labeled by the element $s$. We will also need the colored version of Cayley graphs.

**Definition 2.4.** Let $S_1,\ldots,S_k$ be subsets of a group $G$. Let $E_i$ be the edge set of $Cay(G,S_i)$, and define $Cay(G,(S_1,\ldots,S_k))$ to be the colored graph with vertex set $G$ and edge set $\sqcup_{i=1}^k E_i$ where edges in $E_i$ are colored with color $i$. Note that if the sets $S_1,\ldots,S_k$ are not disjoint, the graph $Cay(G,(S_1,\ldots,S_k))$ will have multiple edges, but there will be at most one edge of a given color between any two vertices.

One can reformulate the notion of $k$-paradoxical decomposition in terms of $k$-subgraphs of colored Cayley graphs as follows:

**Lemma 2.5.** *Let $S_1,\ldots,S_k$ be finite subsets of a group $G$. The following are equivalent:*

(i) *$G$ admits a $k$-paradoxical decomposition with translating sets $S_1,\ldots,S_k$.*
(ii) *The colored Cayley graph $Cay(G,(S_1,\ldots,S_k))$ contains a spanning evenly colored $k$-subgraph.*

*Proof.* Assume that (ii) holds, and let $\Gamma$ be a spanning evenly colored $k$-subgraph of $Cay(G,(S_1,\ldots,S_k))$. For $1\le i\le k$ choose an ordering $g_{i,1},\ldots,g_{i,n_i}$ of the elements of $S_i$.

For $1\le i\le k$ and $1\le j\le n_i$ let $C_{i,j}$ be the set of head vertices of all edges of $\Gamma$ which have color $i$ and label $g_{i,j}$, that is,

$$C_{i,j}=\{g\in G:(gg_{i,j}^{-1},g)\in E_i(\Gamma)\}.$$

Since every vertex of $\Gamma$ has at most one incoming edge, the sets $C_{i,j}$ are disjoint.

On the other hand note that $C_{i,j}g_{i,j}^{-1}$ is the set of tail vertices of all edges of $\Gamma$ of color $i$ labeled by $g_{i,j}$. Since every vertex of $\Gamma$ has exactly one outgoing edge of any given color, for any $1\le i\le k$ we have $G=\sqcup_{j=1}^{n_i}C_{i,j}g_{i,j}^{-1}$, or equivalently $G=\sqcup_{j=1}^{n_i}g_{i,j}C_{i,j}^{-1}$. Therefore, $G$ has a $k$-paradoxical decomposition with translating sets $S_1,\ldots,S_k$ and pieces $P_{i,j}=C_{i,j}^{-1}$, so (i) holds.

Conversely, suppose that (i) holds. In the notations of the definition of a $k$-paradoxical decomposition we can assume that the unions $\cup_{j=1}^{n_i}g_{i,j}P_{i,j}$ are disjoint (by making the sets $P_{i,j}$ smaller if needed). The rest of the proof is completely analogous to the implication "(ii)$\Rightarrow$ (i)". □

Given an oriented graph $\Gamma$ and a finite subset $A$ of $V(\Gamma)$, we put

$$V_\Gamma^+(A) = \{v \in V(\Gamma) : (a,v) \in E(\Gamma) \text{ for some } a \in A\};$$
$$V_\Gamma^-(A) = \{v \in V(\Gamma) : (v,a) \in E(\Gamma) \text{ for some } a \in A\}.$$

In other words, $V_\Gamma^+(A)$ is the set of head vertices of all edges whose tail vertex lies in $A$, and $V_\Gamma^-(A)$ is the set of tail vertices of all edges whose head vertex lies in $A$.

If in addition $E(\Gamma)$ is colored using colors $\{1,\ldots,k\}$, we put

$$V_\Gamma^{+,i}(A) = \{v \in V(\Gamma) : (a,v) \in E_i(\Gamma) \text{ for some } a \in A\}$$

(recall that $E_i(\Gamma)$ is the set of edges of $\Gamma$ of color $i$).

A convenient tool for constructing $k$-subgraphs is the following version of the P. Hall marriage theorem.

**Theorem 2.6.** *Let $\Gamma$ be a locally finite oriented graph and $k \geq 1$ an integer.*

(i) *Assume that for every finite subset $A$ of $V(\Gamma)$ we have $|V_\Gamma^+(A)| \geq k|A|$. Then $\Gamma$ contains a spanning $k$-subgraph.*

(ii) *Suppose now that edges of $\Gamma$ are colored using colors $\{1,\ldots,k\}$. Assume that for any finite subsets $A_1,\ldots,A_k$ of $V(\Gamma)$ we have*

$$|\cup_{i=1}^k V_\Gamma^{+,i}(A_i)| \geq \sum_{i=1}^k |A_i|. \tag{2.1}$$

*Then $\Gamma$ contains a spanning evenly colored $k$-subgraph.*

*Proof.* A slight modification of the usual form of P. Hall theorem (see, for example, [26, Lemma 5.8.25]) asserts the following:

**Lemma 2.7.** *Let $k \geq 1$ be an integer, $\mathcal{I}$ and $S$ any sets and $\{S_\alpha\}_{\alpha\in\mathcal{I}}$ a collection of finite subsets of $S$. Suppose that for any finite subset $\mathcal{J} \subseteq \mathcal{I}$ we have $|\cup_{\alpha\in\mathcal{J}} S_\alpha| \geq k|\mathcal{J}|$. Then there exist pairwise disjoint $k$-element subsets $\{X_\alpha\}_{\alpha\in\mathcal{I}}$ such that $X_\alpha \subseteq S_\alpha$ for all $\alpha$.*

This immediately implies part (i).

In the setting of (ii), let $S = V(\Gamma)$, $\mathcal{I} = V(\Gamma) \times \{1,\ldots,k\}$, and for $\alpha = (v,i) \in \mathcal{I}$ put $S_\alpha = \{w \in V(\Gamma) : (v,w) \in E_i(\Gamma)\}$, that is, $S_\alpha$ is the set of head vertices of edges of color $i$ in $\Gamma$ whose tail vertex is $v$. Condition (2.1) means precisely that Lemma 2.7 is applicable to this collection for $k = 1$. If $\{x_\alpha\}_{\alpha\in\mathcal{I}}$ is the resulting set of vertices, let $\Lambda$ be the spanning subgraph of $\Gamma$ with edges of the form $(v, x_{(v,i)}) \in E_i(\Gamma)$ for every $(v,i) \in \mathcal{I}$. It is clear that $\Lambda$ is an evenly colored $k$-subgraph. □

Note that part (i) of Theorem 2.6 for arbitrary $k \geq 1$ can be easily deduced from part (ii): starting with an uncolored graph $\Gamma$, we consider the colored graph $\Gamma_k$ with $V(\Gamma_k) = V(\Gamma)$ and $E_i(\Gamma_k) = E(\Gamma)$ for $i \in \{1,\ldots,k\}$. Then the assumption $|V_\Gamma^+(A)| \geq k|A|$ ensures that (ii) is applicable to $\Gamma_k$.

## 3. Tarski numbers and extensions

In this section we will prove Theorem 1. The proofs of parts (a) and (b) of that theorem will be based on Proposition 3.1 below. For the convenience of the reader we will restate all parts of the above theorem in this section.

Throughout the section $G$ will denote a fixed non-amenable group and $H$ a subgroup of $G$. When $H$ is normal, $\rho\colon G \to G/H$ will denote the natural homomorphism. Let $T$ be a right transversal of $H$ in $G$, that is, a subset of $G$ which contains precisely one element from each right coset of $H$. Thus, there exist unique maps $\pi_H\colon G\to H$ and $\pi_T\colon G\to T$ such that $g=\pi_H(g)\pi_T(g)$ for all $g\in G$. We shall also assume that $1\in T$.

**Proposition 3.1.** *Suppose that $G$ has a $k$-paradoxical decomposition with translating sets $S_1,\dots,S_k$ and assume that $1\in S_1$. Let $S=\cup_{i=1}^k S_i$. Let $F$ be a subset of $T$, let $\Phi_i=\pi_T(FS_i)$ and $\Phi=\pi_T(FS)=\cup_{i=1}^k\Phi_i$. Finally, let $S_i'=FS_i\Phi_i^{-1}\cap H$.*

(i) *Suppose that $|\Phi|=|F|$. Then $H$ has a $k$-paradoxical decomposition with translating sets $S_1',\dots,S_k'$. Therefore, $\mathcal{T}(H)\le\sum_{i=1}^k|S_i'|$.*

(ii) *Suppose that $|\Phi|\le\frac{k}{2}|F|$. Then $H$ has a 2-paradoxical decomposition with total translating set $\cup_{i=1}^k S_i'$. Therefore, $\mathcal{T}(H)\le 2\sum_{i=1}^k|S_i'|$.*

*Proof.* (i) By Lemma 2.5, the Cayley graph $Cay(G,(S_1,\dots,S_k))$ contains a spanning evenly colored $k$-subgraph $\Gamma$. Let $\Gamma_0$ be the subgraph of $\Gamma$ with vertex set $HFS=H\Phi$ which contains an edge $(g,g')$ of color $i$ if and only if $g\in HF$ and $g'\in HFS_i=H\Phi_i$. Note that by construction $\Gamma_0$ contains all edges of $\Gamma$ whose tail vertex lies in $HF$.

Let $\Lambda$ be the quotient graph of $\Gamma_0$ in which we glue two vertices $g$ and $g'$ of $\Gamma_0$ if and only if they have the same $H$-component, that is, $\pi_H(g)=\pi_H(g')$. We do not make any edge identifications during this process, so typically $\Lambda$ will have plenty of multiple edges. We can naturally identify the vertex set of $\Lambda$ with $H$.

We claim that the graph $\Lambda$ satisfies the condition of Theorem 2.6(ii). By construction, every vertex of $\Lambda$ has at most $|\Phi|$ incoming edges. Since $|\Phi|=|F|$, it suffices to show that for every $i\in\{1,\dots,k\}$ and every finite subset $A$ of $H$ we have $|E_\Lambda^{+,i}(A)|\ge|F||A|$ where $E_\Lambda^{+,i}(A)$ is the set of edges of color $i$ whose tail vertex lies in $A$. The latter statement is clear since (again by construction) every vertex of $\Lambda$ has $|F|$ outgoing edges of each color.

Thus, by Theorem 2.6(ii) $\Lambda$ contains a spanning evenly colored $k$-subgraph. By Lemma 2.5, to finish the proof of (i) it suffices to check that every edge of $\Lambda$ of color $i$ is labeled by an element of $S_i'$. Indeed, every edge $e=(h,h')\in E_i(\Lambda)$ comes from an edge $(hf,h'f_i)\in E_i(\Gamma)$ for some $f\in F$ and $f_i\in\Phi_i$. Hence

$$\mathrm{label}(e)=h^{-1}h'=f((hf)^{-1}h'f_i)f_i^{-1}\in FS_i\Phi_i^{-1}\cap H=S_i'.$$

The proof of (ii) is almost identical except that we do not keep track of colors. The same counting argument shows that $|V_\Lambda^+(A)|\ge\frac{k|A||F|}{|\Phi|}\ge 2|A|$. Hence by Theorem 2.6(i), $\Lambda$ contains a spanning 2-subgraph, and by the above computation all edges of $\Lambda$ are labeled by elements of $S'=\cup_{i=1}^k S_i'$. Hence $Cay(G,S')$ contains a spanning 2-subgraph, so $Cay(G,(S',S'))$ contains a spanning evenly colored 2-subgraph. □

**Theorem 1(a).** *Let $G$ be a non-amenable group and $H$ a subgroup of finite index in $G$. Then $\mathcal{T}(H)-2\le[G:H](\mathcal{T}(G)-2)$.*

*Proof.* Choose a paradoxical decomposition of $G$ with translating sets $S_1$ and $S_2$ such that $|S_1|+|S_2|=\mathcal{T}(G)$ and $1\in S_1\cap S_2$ (this is possible by Remark 2.2), and apply Proposition 3.1 to that decomposition with $F=T$. Then $\Phi_i=\Phi=T$ for each $i$, so hypotheses of Proposition 3.1(i) hold.

Note that for each $t'\in T$ and $g\in G$ there exists a unique $t\in T$ such that $t'gt^{-1}\in H$. Moreover, if $g=1$, then $t=t'$ and therefore $t'gt^{-1}=1$. Hence for $i=1,2$ we have

$$|S_i'|=|FS_i\Phi_i^{-1}\cap H|=|TS_iT^{-1}\cap H|\le|T|(|S_i|-1)+1$$

(here we use the fact that each $S_i$ contains 1). Hence $\mathcal{T}(H)\le|S_1'|+|S_2'|\le[G:H](|S_1|+|S_2|-2)+2=[G:H](\mathcal{T}(G)-2)+2$. □

For the proof of Theorem 1(b) we will need the following two additional lemmas.

**Lemma 3.2.** *Let a variety $\mathcal{V}$ be as in Theorem 1(b). Then there exists a function $g\colon \mathbb{N} \to \mathbb{N}$ such that for every $Z \in \mathcal{V}$ and every $n$-element subset $U \subseteq Z$ there exists a finite set $F \subseteq Z$ with $|F| \le g(n)$, for which $|FU| \le 2|F|$.*

*Proof.* Fix $n \in \mathbb{N}$. Let $V_n$ be the relatively free group of $\mathcal{V}$ of rank $n$. Let $X = \{x_1, \dots, x_n\}$ be the set of free generators of $V_n$. Since $V_n$ is amenable, by the Følner criterion [29], there exists a finite subset $F' \subseteq V_n$ such that $|F'X| \le 2|F'|$. We can assume that $F'$ has the smallest possible number of elements and define $g(n) = |F'|$. Given a group $Z \in \mathcal{V}$ and an $n$-element subset $U \subseteq Z$, let $\varphi$ be the homomorphism $V_n \to Z$ which sends $x_i$ to the corresponding element of $U$. Let $\prec$ be a total right-invariant order on $V_n$. Let

$$R = \{r \in F' : \text{ there exists } f \in F' \text{ with } f \prec r \text{ and } \varphi(f) = \varphi(r)\}.$$

Clearly $R$ is a proper subset of $F'$, so the minimality of $F'$ implies $|RX| > 2|R|$. Note that $\varphi(F') = \varphi(F' \setminus R)$ and $\varphi$ is injective on $F' \setminus R$. Since $\prec$ is right-invariant, for every $x \in X$ and $r \in R$ there exists $f \in F'$ such that $fx \prec rx$ and $\varphi(fx) = \varphi(rx)$. Hence $\varphi(F')U = \varphi(F'X) = \varphi(F'X \setminus RX)$, so

$$\begin{aligned}|\varphi(F')U| &\le |\varphi(F'X \setminus RX)| \le |F'X| - |RX| \le 2|F'| - 2|R| \\ &= 2|F' \setminus R| = 2|\varphi(F' \setminus R)| = 2|\varphi(F')|.\end{aligned}$$

Thus we can take $F = \varphi(F')$. □

**Lemma 3.3.** *Suppose that a group $G$ has a $k$-paradoxical decomposition with translating sets $S_1, \dots, S_k$ and an $l$-paradoxical decomposition with translating sets $T_1, \dots, T_l$. Then $G$ has a $kl$-paradoxical decomposition with translating sets $\{S_iT_j\}$.*

*Proof.* By Lemma 2.5, the Cayley graphs $Cay(G, (S_1, \dots, S_k))$ and $Cay(G, (T_1, \dots, T_l))$ have spanning evenly colored $k$-subgraph $\Gamma_k$ and $l$-subgraph $\Gamma_l$, respectively. Let $\Gamma = \Gamma_k\Gamma_l$ be the graph with vertex set $G$ and edge set

$$E(\Gamma) = \{(g, g') \in G \times G : \text{there is } x \in G \text{ s.t. } (g, x) \in E(\Gamma_k) \text{ and } (x, g') \in E(\Gamma_l)\}.$$

In other words, edges of $\Gamma$ are (oriented) paths of length 2 where the first edge of the path lies in $\Gamma_k$ and the second lies in $\Gamma_l$. Using the colorings of $\Gamma_k$ and $\Gamma_l$, we can naturally color $E(\Gamma)$ with $kl$ colors. It is clear that $\Gamma$ will become a spanning evenly colored $kl$-graph, representing a $kl$-paradoxical decomposition with translating sets $\{S_iT_j\}$. □

**Theorem 1(b).** *Let $\mathcal{V}$ be a variety of groups where all groups are amenable and relatively free groups are right orderable. Then there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ (depending only on $\mathcal{V}$) with the following property: if a non-amenable group $G$ has a normal subgroup $H$ such that $G/H \in \mathcal{V}$, then $\mathcal{T}(H) \le f(\mathcal{T}(G))$.*

*Proof.* Let $n = \mathcal{T}(G)$ and $Z = G/H$. Recall that $\rho\colon G \to Z$ is the natural homomorphism. By Lemma 3.3, $G$ has a 4-paradoxical decomposition with translating sets $S_1, S_2, S_3, S_4$ such that $1 \in S_1$ and

$$\sum_{i=1}^{4} |S_i| \le n^2.$$

Let $S = \cup_{i=1}^4 S_i$. By Lemma 3.2 applied to the set $U = \rho(S)$, there exists $\overline{F} \subseteq Z$ with $|\overline{F}| \le g(n^2)$ such that $|\overline{F}U| \le 2|\overline{F}|$. Let $F = \rho^{-1}(\overline{F}) \cap T$. Then $|F| = |\overline{F}|$ and $\rho$ bijectively maps $\Phi = \pi_T(FS)$ onto $\overline{F}U$. Therefore, hypotheses of Proposition 3.1(ii) hold, and we deduce that $H$ has a 2-paradoxical decomposition with total translating set $\cup_{i=1}^4 S_i'$ (where

$S_i'$ are defined as in Proposition 3.1). Clearly, $|S_i'| \leq |S_i||F||\Phi| = |S_i||F||\overline{F}U| \leq 2|S_i||F|^2$, and therefore $\mathcal{T}(H) \leq 2\sum_{i=1}^4 2|S_i||F|^2 \leq 4n^2 g(n^2)^2$. □

For the proof of Theorem 1(c) we will need the following lemma.

**Lemma 3.4.** *Assume that $H$ is normal and amenable. Suppose that $U_1, U_2 \subseteq G$ are finite subsets such that for every pair of finite subsets $F_1, F_2 \subseteq G$ we have $|\bigcup_{i=1}^2 F_iU_i| \geq \sum_{i=1}^2 |F_i|$. Let $U_i' = \rho(U_i)$. Then for every pair of finite subsets $F_1', F_2' \subseteq G/H$ we have $|\bigcup_{i=1}^2 F_i'U_i'| \geq \sum_{i=1}^2 |F_i'|$.*

*Proof.* Let $U = U_1 \cup U_2$. Let $\psi\colon G/H \to T$ be the unique map such that $\rho\psi(gH) = gH$ for all $g \in G$. Note that $\psi$ is a bijection and $\psi\rho(g) = \pi_T(g)$ for all $g \in G$.

Fix $\varepsilon > 0$. Let $F_1', F_2' \subseteq G/H$ be finite sets, let $F_i'' = \psi(F_i')$ and $F'' = F_1'' \cup F_2''$. Let $U_H = \pi_H(F''U)$. Since $U_H \subseteq H$ is a finite subset of the amenable group $H$, by Følner's criterion, there exists a finite set $F_H \subseteq H$ such that $|F_HU_H| < (1+\varepsilon)|F_H|$. Define $F_i = F_HF_i'' \subseteq G$. Since $F_H \subseteq H$ and $F_i'' \subseteq T$, we have $|F_i| = |F_H||F_i''| = |F_H||F_i'|$.

Note that $F_iU_i \subseteq (F_HU_H)\psi(F_i'U_i')$. Indeed,

$$F_iU_i = F_HF_i''U_i \subseteq F_H\pi_H(F_i''U_i)\pi_T(F_i''U_i) \subseteq F_HU_H \cdot \psi\rho(F_i''U_i)$$
$$= F_HU_H\psi(\rho(F_i'')\rho(U_i)) = F_HU_H\psi(F_i'U_i').$$

Therefore, $\bigcup_{i=1}^2 F_iU_i \subseteq \bigcup_{i=1}^2 F_HU_H\psi(F_i'U_i') = F_HU_H\psi(\bigcup_{i=1}^2 F_i'U_i')$. Hence, $|\bigcup_{i=1}^2 F_iU_i| \leq |F_HU_H||\bigcup_{i=1}^2 F_i'U_i'|$. Since $|\bigcup_{i=1}^2 F_iU_i| \geq \sum_{i=1}^2 |F_i|$ by the hypotheses of the theorem, we get $|F_HU_H||\bigcup_{i=1}^2 F_i'U_i'| \geq \sum_{i=1}^2 |F_H||F_i'|$. Hence $(1+\varepsilon)|F_H||\bigcup_{i=1}^2 F_i'U_i'| \geq \sum_{i=1}^2 |F_H||F_i'|$ and $(1+\varepsilon)|\bigcup_{i=1}^2 F_i'U_i'| \geq \sum_{i=1}^2 |F_i'|$. Since the inequality holds for every $\varepsilon > 0$, we conclude that $|\bigcup_{i=1}^2 F_i'U_i'| \geq \sum_{i=1}^2 |F_i'|$. □

**Theorem 1(c).** *Let $G$ be a non-amenable group with an amenable normal subgroup $H$. Then $\mathcal{T}(G/H) = \mathcal{T}(G)$.*

*Proof.* Let $n = \mathcal{T}(G)$ and choose a paradoxical decomposition of $G$ with $|S_1| + |S_2| = n$. By Lemma 2.5, the Cayley graph $Cay(G, (S_1, S_2))$ contains a spanning evenly colored 2-subgraph $\Gamma$. In particular, for every pair of finite subsets $F_1, F_2 \subseteq G$ we have $|\bigcup_{i=1}^2 F_iS_i| \geq \sum_{i=1}^2 |F_i|$. By Lemma 3.4, the Cayley graph $\Gamma' = Cay(G/H, (\rho(S_1), \rho(S_2)))$ satisfies the condition of Theorem 2.6(ii) with $k = 2$. Thus, $\Gamma'$ contains a spanning evenly colored 2-subgraph. Therefore $\mathcal{T}(G/H) \leq |\rho(S_1)| + |\rho(S_2)| \leq \mathcal{T}(G)$. □

To prove Theorem 1(d) we will use the following variation of Lemma 3.4.

**Lemma 3.5.** *Let $G = H_1 \times H_2$, and let $U \subseteq G$ be a finite subset such that for each finite subset $F \subseteq G$ we have $|FU| \geq 2|F|$. Let $U_1 = \pi_1(U)$ and $U_2 = \pi_2(U)$ where $\pi_1$ and $\pi_2$ are the projections onto $H_1$ and $H_2$ respectively. Then for some $i \in \{1,2\}$ for any finite subset $F_i \subseteq H_i$ we have $|F_i(U_i)^2| \geq 2|F_i|$.*

*Proof.* If $|F_1U_1| \geq \sqrt{2}|F_1|$ for each finite subset $F_1 \subseteq H_1$, then replacing the finite subset $F_1$ by $F_1U_1$, we get $|F_1(U_1)^2| = |(F_1U_1)U_1| \geq \sqrt{2}|F_1U_1| \geq 2|F_1|$ and we are done. Otherwise, fix $F_1 \subseteq H_1$ such that $|F_1U_1| < \sqrt{2}|F_1|$. Given a finite subset $F_2 \subseteq H_2$, let $F = F_1 \times F_2$. Note that $U \subseteq U_1 \times U_2$ implies that

$$(F_1U_1) \times (F_2U_2) = (F_1 \times F_2)(U_1 \times U_2) \supseteq FU.$$

Therefore $|F_1U_1||F_2U_2| \geq |FU| \geq 2|F| = 2|F_1||F_2|$. Hence $|F_2U_2| \geq 2\frac{|F_1|}{|F_1U_1|}|F_2| \geq \sqrt{2}|F_2|$. As before replacing $F_2$ by $F_2U_2$ yields the required inequality. □

**Theorem 1(d).** *Let $G = H_1 \times H_2$ be a non amenable group. Then* $\min\{\mathcal{T}(H_1), \mathcal{T}(H_2)\} \leq 2(\mathcal{T}(G) - 1)^2$.

*Proof.* Let $n = \mathcal{T}(G)$ and choose a paradoxical decomposition of $G$ with $|S_1| + |S_2| = n$ and $1 \in S_1 \cap S_2$. By Lemma 2.5, the Cayley graph $Cay(G, (S_1, S_2))$ contains a spanning evenly colored 2-subgraph $\Gamma$. In particular, for $U = S_1 \cup S_2$ and every finite $F \subseteq G$ we have $|FU| \geq 2|F|$. Let $U_1 = \pi_1(U)$, $U_2 = \pi_2(U)$. By Lemma 3.5, for $i = 1$ or $i = 2$ the Cayley graph $Cay(H_i, (U_i)^2)$ satisfies the hypotheses of Theorem 2.6(i) with $k = 2$. Thus, $Cay(H_i, (U_i)^2)$ contains a spanning 2-subgraph, so $Cay(H_i, ((U_i)^2, (U_i)^2))$ contains a spanning evenly colored 2-subgraph. Therefore $\mathcal{T}(H_i) \leq 2|U_i|^2 \leq 2|U|^2 \leq 2(n-1)^2$. □

## 4. Unbounded Tarski numbers

In this section we will prove Theorem 2 and discuss Theorem 1.3 and its corollaries.

4.1. **Lower bound on Tarski numbers.** We start with a simple lemma which can be used to bound Tarski numbers from below. Part (a) is an observation of Ozawa from [22] and part (b) is a natural generalization of Theorem 1.2(i).

**Lemma 4.1.** *Assume that $G$ has a paradoxical decomposition with translating sets $S_1$ and $S_2$. Then*

(a) *The subgroup generated by $S_1 \cup S_2$ is non-amenable.*
(b) *The subgroup generated by $S_i$ is infinite for $i = 1, 2$.*

*Proof.* (a) Let $H$ be the subgroup generated by $S_1 \cup S_2$. Intersecting each set in the given paradoxical decomposition of $G$ with $H$ gives a paradoxical decomposition of $H$ with the same sets of translating elements.

(b) By Lemma 2.5, the colored Cayley graph $Cay(G, (S_1, S_2))$ has a spanning evenly colored 2-subgraph $\Gamma$. Choose any edge of $\Gamma$ of color 2, and let $g_0$ be the head vertex of that edge. Let $g_1, g_2, \ldots$ be the sequence of elements of $G$ defined by the condition that $(g_i, g_{i+1})$ is an edge of color 1 in $\Gamma$ for all $i \geq 0$ (such a sequence is unique since each vertex has a unique outgoing edge of color 1 in $\Gamma$).

We claim that all elements $g_i$ are distinct. Indeed, suppose that $g_i = g_j$ for $i < j$, and assume that $i$ and $j$ are the smallest with this property. If $i > 0$, the vertex $g_i$ would have two incoming edges in $\Gamma$, namely $(g_{i-1}, g_i)$ and $(g_{j-1}, g_i)$, a contradiction. If $i = 0$, we get a contradiction with the assumption that $g_0$ has an incoming edge of color 2.

By construction, all elements $g_0^{-1}g_i$ lie in the subgroup generated by $S_1$, so this subgroup must be infinite. By the same argument, the subgroup generated by $S_2$ is infinite. □

Recall that $\mathrm{Amen}_m$ (resp. $\mathrm{Fin}_m$) denotes the class of groups in which all $m$-generated subgroups are amenable (resp. finite). Combining Lemma 4.1 and Remark 2.2, we deduce the following statement:

**Corollary 4.2.** *Let $G$ be a non-amenable group and $m \in \mathbb{N}$. The following hold:*

(i) *If $G$ belongs to $\mathrm{Amen}_m$, then $\mathcal{T}(G) \geq m + 3$.*
(ii) *If $G$ belongs to $\mathrm{Fin}_m$, then $\mathcal{T}(G) \geq 2m + 4$.*

In particular, to prove that groups with unbounded Tarski numbers exist, it suffices to know that $\mathrm{Amen}_m$ contains non-amenable groups for every $m$. As noticed in [22], already $\mathrm{Fin}_m$ contains non-amenable groups for every $m$ – this follows from the next two theorems on Golod-Shafarevich groups:

(i) (see [4]) Every Golod-Shafarevich group has an infinite quotient with property $(T)$. In particular, every Golod-Shafarevich group is non-amenable.
(ii) (see [11], [6, Theorem 3.3]) For every $m$ there exists an $(m+1)$-generated Golod-Shafarevich group in $\mathrm{Fin}_m$.

Since the class $\mathrm{Fin}_m$ is obviously closed under taking quotients, (i) and (ii) actually yield a stronger corollary, which will be needed to prove Theorem 2:

**Corollary 4.3.** *For every $m \in \mathbb{N}$ there exists an infinite property $(T)$ group in $\mathrm{Fin}_m$.*

4.2. **Proof of Theorem 2.** Recall the formulation of the theorem:

**Theorem 2.** *The set of Tarski numbers of 2-generated non-amenable groups is infinite. Moreover, the set of Tarski numbers of 2-generated infinite groups with property $(T)$ is infinite.*

The proof of Theorem 2 is based on Theorem 4.4 below which is proved using results and ideas from the classical paper by Bernhard Neumann and Hanna Neumann [20].

**Theorem 4.4.** *Let $G$ be a finitely generated group. The following hold:*
(a) *The derived subgroup $[G, G]$ of $G$ embeds into a 2-generated subgroup $H$ of a wreath product $G \wr C_n$ for a sufficiently large $n \in \mathbb{N}$ where $C_n$ is the cyclic group of order $n$. Moreover, $H$ contains the derived subgroup $[G^n, G^n] = [G, G]^n$ of the base group of the wreath product.*
(b) *Assume in addition that $G$ is torsion. Then $G$ embeds into a 2-generated subgroup $H$ of a group $L$ which is an extension of a finite direct power $G^n$ of $G$ (for some $n \in \mathbb{N}$) by a finite metabelian group.*
(c) *Assume in addition that $G$ has property $(T)$. Then in both (a) and (b) $H$ has property $(T)$.*

*Proof.* (a) Let $d$ be the number of generators of $G$. Take any $n > 2^{2^{2d}}$. Let $z$ be a generator of $C_n$. For an element $g \in G$ let $\delta(g)\colon C_n \to G$ be the function given by $\delta(g)(1) = g$ and $\delta(h)(c) = 1$ for $c \neq 1$. Let $X = \{x_1, \ldots, x_d\}$ be a generating set of $G$, and define the function $a\colon C_n \to G$ by

$$a(z^k) = \begin{cases} x_i & \text{if } k = 2^{2^i} \text{ for some } 1 \leq i \leq d, \\ 1 & \text{otherwise.} \end{cases}$$

Let $H$ be the subgroup of $G \wr C_n$ generated by $a$ and $z$. Then it is easy to see that $[a^{z^{-2^{2^i}}}, a^{z^{-2^{2^j}}}] = \delta([x_i, x_j])$. For every word $w$ in $x_i^{\pm 1}$, the function $\delta([x_i, x_j]^w)$ can be obtained from $\delta([x_i, x_j])$ by conjugation by a product of elements of the form $a^{z^m}$. Thus, $H$ contains all functions of the form $\delta([x_i, x_j]^w)$, and clearly these functions generate the subgroup $G_1$ of $G^n$ consisting of all functions $f$ with $f(1) \in [G, G], f(c) = 1$ if $c \neq 1$. Since $z \in H$, the subgroup $H$ contains all conjugates $G_1^{z^m}$, hence it contains the derived subgroup of the base group of $G \wr C_n$.

(b) Again let $d$ be the number of generators of $G$. Since $G$ is torsion, by [20, Lemma 4.1] $G$ embeds into the derived subgroup of the $(k+1)$-generated group $W = G \wr C_n$ where $n$ is the least common multiple of the orders of the generators of $G$. By (a), $[W, W]$ embeds into a 2-generated subgroup of the group $L = W \wr C_m$ for some $m$, and the proof is complete.

(c) We will prove the result in the setting of (a); the proof in the setting of (b) is analogous. Since $G$ has property $(T)$, the abelianization $G/[G, G]$ is finite. Therefore $[G, G]^n$ is a finite index subgroup of $G \wr C_n$, so in particular $[G, G]^n$ has finite index in $H$.

Since the direct product of two groups with property $(T)$ has property $(T)$ and property $(T)$ is preserved by finite index subgroups and overgroups (see [2]), we conclude that $H$ has property $(T)$. $\square$

Now let $f_1 \colon \mathbb{N} \to \mathbb{N}$ and $f_2 \colon \mathbb{N} \to \mathbb{N}$ be the functions from Theorem 1(b) corresponding to the varieties of Abelian groups and metabelian groups, respectively, and define $g_1, g_2 \colon \mathbb{N} \to \mathbb{N}$ by $g_i(x) = \min\{t : f_i(t) \geq x\}$. Clearly, $g_i(x) \to \infty$ as $x \to \infty$.

**Corollary 4.5.** *Let $G$ be any finitely generated non-amenable group from* $\mathrm{Fin}_m$. *In the above notations the following hold:*

(a) *$G$ embeds into a 2-generated group $H$ with $\mathcal{T}(H) \geq g_2(2m+4)$.*
(b) *$[G,G]$ embeds into a 2-generated group $H$ with $\mathcal{T}(H) \geq g_1(2m+4)$.*
(c) *If $G$ has property $(T)$, then in both (a) and (b) $H$ also has property $(T)$.*

*Proof.* This follows directly from Theorem 4.4, Lemma 4.1(b), the obvious fact that if $G$ lies in $\mathrm{Fin}_m$, then any finite direct power of $G$ lies in $\mathrm{Fin}_m$, and the fact that free metabelian groups and free Abelian groups are right orderable (Remark 1.5(ii)). $\square$

Theorem 2 now follows immediately from Corollary 4.5 (we can use either (a) or (b) combined with (c)) and Corollary 4.3.

4.3. **The Bertrand-type property of Tarski numbers.** As we already stated, Golod-Shafarevich groups are always non-amenable by [4]. Moreover, if $G$ is a Golod-Shafarevich group with respect to a prime $p$, the image of $G$ in its pro-$p$ completion (which is a residually-$p$ group) is non-amenable. Therefore, Theorem 1.3 is a corollary of the following result:

**Proposition 4.6.** *Let $p$ be a prime and $G$ a Golod-Shafarevich group with respect to $p$. Then there is a quotient $H$ of $G$ which is also Golod-Shafarevich (with respect to $p$), $p$-torsion and such that for every $m \in \mathbb{N}$ there is a finite index subgroup $H_m$ of $H$ which lies in* $\mathrm{Fin}_m$.

Proposition 4.6 follows immediately from the proof (but not quite from the statement) of [5, Lemma 8.8]. For completeness, we will present a self-contained proof of Proposition 4.6 in Appendix C, where we will also define Golod-Shafarevich groups and some related notions.

Note that if $p \geq 67$, one can deduce Theorem 1.3 from Proposition 4.6 without using non-amenability of arbitrary Golod-Shafarevich groups (but using the fact that a Golod-Shafarevich group with respect to $p$ has infinite pro-$p$ completion). Indeed, in that case there exists a Golod-Shafarevich group $G$ with respect to $p$ with property $(T)$ (see [6, Theorem 12.1]), so all infinite quotients of $G$ are automatically non-amenable.

We do not know the answer to the following question, which can be thought of as a "dual" version of Theorem 1.3.

**Problem 4.7.** *Does there exist a sequence of finitely generated non-amenable groups $\{G_n\}_{n\in\mathbb{N}}$ such that $G_{n+1}$ is a quotient of $G_n$ for each $n$ and $\mathcal{T}(G_n) \to \infty$ as $n \to \infty$?*

Note that while by the above argument the group $H$ in Theorem 1.3 (and its subgroups of finite index) can be chosen to have property $(T)$, groups $G_n$ satisfying the hypotheses of Problem 4.7 (if they exist) cannot have property $(T)$. Indeed, the inductive limit $G_\infty$ of a sequence $\{G_n\}$ of such groups cannot have a finite Tarski number. Hence $G_\infty$ is amenable. Suppose that one of the groups $G_n$ has property $(T)$. Then $G_\infty$ also has property $(T)$. Therefore $G_\infty$ is finite, so $G_\infty$ has a finite presentation. The relations of that presentation must follow from the relations of one of the groups $G_n$. Therefore $G_n$ is a homomorphic image of $G_\infty$, whence $G_n$ is finite, a contradiction.

We conclude this section with the proof of Corollary 1.8 restated below:

**Theorem 4.8.** *For every sufficiently large $n$ there exists a group $H$ with $n \le \mathcal{T}(H) \le 2n$.*

*Proof.* Let $H$ be a group satisfying the conclusion of Theorem 1.3 for $p = 2$. Then $H$ has a descending chain of normal subgroups $H = H_1 \supset H_2 \supset \dots$ such that $[H_i : H_{i+1}] = 2$ for all $i$ and $\mathcal{T}(H_i) \to \infty$. Thus, Theorem 4.8 follows from Theorem 1(a). □

## 5. Tarski numbers and $L^2$-Betti numbers of groups

5.1. **Groups with Tarski number** 6. Let $G$ be a group, $S$ a subset of $G$, and let $\Gamma = Cay(G, S)$. Given a finite set $A$ of $G$, define

$$\partial_S^+ A = V_\Gamma^+(A) \setminus A = \{g \in G \setminus A : (a, g) \in E(\Gamma) \text{ for some } a \in A\};$$

Note that if $S$ is symmetric, i.e., $S = S^{-1}$, then $\partial_S^+ A$ is the usual vertex boundary $\partial_S(A)$ of $A$ in $Cay(G, S)$, considered as an unoriented graph.

**Lemma 5.1.** *Suppose that a group $G$ is generated by a set $T = \{a, b, c\}$ of 3 non-identity elements, and suppose that $|\partial_T^+ A| \ge |A|$ for every finite subset $A \subseteq G$. Then $G$ admits a paradoxical decomposition with both translating sets of size 3, and therefore $\mathcal{T}(G) \le 6$.*

*Proof.* Let $S_1 = \{1, a, b\}$, $S_2 = \{1, a, c\}$, $S = \{1, a, b, c\} = S_1 \cup S_2 = T \cup \{1\}$, and let $\Gamma = Cay(G, S)$. Clearly any two-element subset of $S$ can be ordered so that the first element lies in $S_1$ and the second element lies in $S_2$. Therefore, every 2-subgraph of $\Gamma$ can be colored to yield an evenly colored 2-subgraph of $Cay(G, (S_1, S_2))$. Thus, by Lemma 2.5, we just need to construct a spanning 2-subgraph of $\Gamma$.

Note that $\Gamma$ is obtained from $Cay(G, T)$ by adding a loop at each vertex. By assumption for any finite subset $A \subseteq G$ we have $|\partial_T^+ A| \ge |A|$, whence $|V_\Gamma^+(A)| \ge 2|A|$. Thus, by Theorem 2.6(i), $\Gamma$ contains a spanning 2-subgraph, and we are done. □

Let us recall the statement of Theorem 3.

**Theorem 3.** *Let $G$ be any 3-generated group with $\beta_1(G) \ge 3/2$ where $\beta_1(G)$ is the first $L^2$-Betti number of $G$. Then $\mathcal{T}(G) \le 6$. In particular, if $G$ is torsion, then $\mathcal{T}(G) = 6$.*

Theorem 3 immediately follows from Lemma 5.1 and the next proposition:

**Proposition 5.2.** *Let $G$ be a finitely generated group, $S$ a finite generating subset of $G$, and let $k = 2\beta_1(G) - |S| + 1$. Then for any finite $A \subseteq G$ we have $|\partial_S^+ A| \ge k|A|$.*

*Proof of Proposition 5.2.* Fix a finite subset $A$ of $G$. Let $\Gamma = Cay(G, S \cup S^{-1})$, considered as an unoriented graph without multiple edges. The key result we shall use is the theorem of [17, Corollary 4.12] which asserts that the expected degree of a vertex in the free uniform spanning forest on $\Gamma$ is equal to $2\beta_1(G)$ (note that it is independent of $S$).

The precise definition of the free uniform spanning forest will not be important to us. We only need to know that the free uniform spanning forest on $\Gamma$ is a Borel probability measure on spanning subgraphs of $\Gamma$ which is supported on forests and $G$-invariant in the following sense. Let $\Sigma_\Gamma$ be the set of all spanning subgraphs of $\Gamma$ which can be thought of as the space $\{0, 1\}^{E(\Gamma)}$ with product topology. Let $\mu$ be a Borel probability measure on $\Sigma_\Gamma$. We say that $\mu$ is supported on forests if $\mu(\{\Lambda \in \Sigma_\Gamma : \Lambda \text{ is a forest}\}) = 1$. The natural left multiplication action of $G$ on $\Gamma$ induces the corresponding action of $G$ on $\Sigma_\Gamma$. We say that $\mu$ is $G$-invariant if it is invariant under this action. Since the action of $G$ on $V(\Gamma)$ is transitive, if $\mu$ is $G$-invariant, all the vertices of $\Gamma$ have the same expected degree.

An immediate corollary of the above theorem of Lyons is that there exists an ordinary (unoriented) forest $\mathcal{F}$ on $\Gamma$ (depending on $A$) such that

$$\sum_{g\in A} \deg_{\mathcal{F}}(g) \geq (2\beta_1(G)+2)|A|. \tag{5.1}$$

Indeed, consider the function $\varphi\colon \Sigma_\Gamma \to \mathbb{Z}_{\geq 0}$ given by $\varphi(F) = \sum_{g\in A} \deg_F(g)$. Integrating $\varphi$ with respect to the free uniform spanning forest $\mu$, we have

$$\int_{\Sigma_\Gamma} \varphi \, d\mu = \sum_{g\in A} \deg_\mu(g)$$

where $\deg_\mu(g)$ is the expected degree of $g$ in $\mu$. By Lyons's theorem we have $\sum_{g\in A} \deg_\mu(g) = (2\beta_1(G)+2)|A|$. Since $\mu$ is a probability measure supported on forests, we deduce that $\varphi(\mathcal{F}) \geq (2\beta_1(G)+2)|A|$ for some forest $\mathcal{F} \in \Sigma_\Gamma$.

Let $E$ be the set of all directed edges $(g, gs)$ such that $g \in A$, $s \in S \cup S^{-1}$ and the unoriented edge $\{g, gs\}$ lies in $\mathcal{F}$. Let $E_1$ be the subset of $E$ consisting of all edges $(g, gs) \in E$ with $s \in S \setminus S^{-1}$ (the set $E_1$ may be empty, for example, if $S = S^{-1}$). Note that $|E| \geq (2\beta_1(G)+2)|A|$ by (5.1), and it is clear that $|E_1| \geq |E| - |S||A|$, so that $|E_1| \geq (2\beta_1(G)+2-|S|)|A|$.

Since the sets $S \setminus S^{-1}$ and $(S \setminus S^{-1})^{-1}$ are disjoint, $E_1$ does not contain a pair of opposite edges. Also note that endpoints of edges in $E_1$ lie in the set $A \sqcup \partial_S^+ A$. Let $\Lambda$ be the unoriented graph with vertex set $A \sqcup \partial_S^+ A$ and edge set $E_1$ (with forgotten orientation). Then $\Lambda$ is a subgraph of $\mathcal{F}$; in particular $\Lambda$ is a (finite) forest. Hence

$$|A \sqcup \partial_S^+ A| = |V(\Lambda)| > |E(\Lambda)| = |E_1| \geq (2\beta_1(G)+2-|S|)|A|,$$

and therefore $|\partial_S^+ A| > (2\beta_1(G)+1-|S|)|A|$, as desired. □

**Remark 5.3.** Informally speaking, the result of Proposition 5.2 can only be useful if the intersection $S \cap S^{-1}$ is small. In particular, if $S$ is symmetric (that is, $S = S^{-1}$), the proof shows that the set $E_1$ is empty and hence the obtained inequality is vacuous. At the same time, if $S$ is symmetric, one can actually prove a much stronger inequality $|\partial_S^+ A| \geq 2\beta_1(G)|A|$ (see [18, Theorem 4.5]). Note that even if $S$ is not symmetric, $S \cup S^{-1}$ is symmetric. Hence from [18, Theorem 4.5], it follows that for every finite set $A$ either $|\partial_S^+(A)| \geq \beta_1(G)$ or $|\partial_{S^-}^+(A) \geq \beta_1(G)$. Unfortunately this does not help in our situation because we cannot guarantee that one of these inequalities holds for every $A$.

One can construct groups with Tarski number 6 and any given (minimal) number of generators $d \geq 2$. For $d = 2$ this follows from Theorems 3 and 4.4(b). For $d \geq 3$ one can take the direct product of a group $G$ from Theorem 3 and a finite elementary Abelian group $C_2^k$ for a suitable $k$.

5.2. **Further results and open questions.** We begin this subsection with two open problems:

**Problem 5.4.** *Given $m \in \mathbb{N}$, what is the minimal possible Tarski number of a group from* $\mathrm{Fin}_m$ *(resp.* $\mathrm{Amen}_m$*)?*

Theorem 3 shows that $2m+4$ (resp. $m+3$) is a lower bound for group from $\mathrm{Fin}_m$ (resp. $\mathrm{Amen}_m$). By Lemma 4.1(b) this lower bound is exact for $m = 1$, but we do not know a good estimate already for $m = 2$.

**Problem 5.5.** *Let $G$ be a finitely generated group with $\beta_1(G) > 0$.*

(a) *Is it true that $\mathcal{T}(G) \leq 6$?*
(b) *If the answer to (a) is negative, is it at least true that $\mathcal{T}(G) \leq C$ for some absolute constant $C$?*

It is not unreasonable to expect that the answer in part (b) (or even part (a)) is positive at least for torsion-free groups. In this connection we mention a result of Peterson and Thom [23, Corollary 4.4] which asserts that a torsion-free group $G$ which has positive first $L^2$-Betti number and satisfies the Atiyah zero divisor conjecture must contain a non-Abelian free subgroup (and hence has Tarski number 4).

Below we will show that all groups with Tarski numbers $> 6$ obtained using Theorem 1 and our proofs of Theorems 1.3 and 2 have the first $L^2$-Betti number 0. Note that each of these groups $G$ has an amenable normal subgroup $N$ such that $G/N$ is an extension of a group from $\mathrm{Amen}_2$ of unbounded exponent by an amenable group. By [15, Theorem 7.2 (2)] if a group $G$ has an infinite normal subgroup $N$ with $\beta_1(N) = 0$, then $\beta_1(G) = 0$. Since the first $L^2$-Betti number of every infinite amenable group vanishes, and an extension of a finite group by a group from $\mathrm{Amen}_2$ is in $\mathrm{Amen}_2$, it is enough to show that the first $L^2$-Betti number of any group from $\mathrm{Amen}_2$ of unbounded exponent is 0.

In fact we prove the following stronger statement.

**Theorem 5.6.** *Let $G$ be a finitely generated infinite group in* $\mathrm{Amen}_2$*, and assume that $G$ does not have bounded exponent. Then the maximal cost of $G$ is 1, hence $\beta_1(G) = 0$.*

Recall the definition of the cost of a countable group $G$ (see [8]). Let $(X, \mu)$ be a Borel probability measure space and let $G \curvearrowright X$ be an almost surely free (i.e., free outside a subset of measure 0) right Borel action of $G$ on $X$ preserving $\mu$. Let $\Phi = \{\varphi_i, i = 1, 2, \ldots\}$ be at most countable collection of Borel bijections between Borel subsets $A_i$ and $B_i$ of $X$ such that for every $x \in A_i$ the point $\varphi_i(x)$ belongs to the orbit $x \cdot G$. Then we can construct a graph with vertex set $X$ and edges connecting each $x \in A_i$ with $\varphi_i(x)$. If connected components of that graph are (almost surely) the orbits of $G$, then we call $\Phi = \{\varphi_i\}$ a *graphing* of the action $G \curvearrowright X$. The *cost* of the graphing $\Phi$, denoted by $\mathcal{C}(\Phi)$, is the sum of measures $\sum \mu(A_i)$. The *cost of the action* $G \curvearrowright X$, denoted by $\mathcal{C}(G \curvearrowright X)$, is the infimum of costs of all graphings. The minimal (resp. maximal) cost $\mathcal{C}_{\min}(G)$ (resp. $\mathcal{C}_{\max}(G)$) is the infimum (resp. supremum) of the costs of all such actions of $G$. It is one of the outstanding open problems, called the Fixed Price problem, whether $\mathcal{C}_{\max}(G) = \mathcal{C}_{\min}(G)$ for every countable group $G$.

By [9, Corollary 3.23], for every countable group $G$ we have $\beta_1(G) \leq \mathcal{C}_{\min}(G) - 1 \leq \mathcal{C}_{\max}(G) - 1$. Hence $\mathcal{C}_{\max}(G) = 1$ or $\mathcal{C}_{\min}(G) = 1$ implies $\beta_1(G) = 0$ for any group $G$.

The proof of Theorem 5.6 is based on the following lemma. For convenience we shall denote $\mathcal{C}_{\max}(G) - 1$ by $\mathcal{C}'_{\max}(G)$.

**Lemma 5.7.** *Let $A$ and $B$ be subgroups of the same group. Suppose that $A \cap B$ is amenable. Then*

$$\mathcal{C}'_{\max}(\langle A, B\rangle) \leq \mathcal{C}'_{\max}(A) + \mathcal{C}'_{\max}(B) + \frac{1}{|A \cap B|} \tag{5.2}$$

*where $\frac{1}{|A \cap B|} = 0$ if $A \cap B$ is infinite.*

*Proof.* Let $H = \langle A, B\rangle$. Pick any $\epsilon > 0$. Consider an action $H \curvearrowright X$ with cost exceeding $\mathcal{C}_{\max}(H) - \epsilon$. The induced action of $A \cap B$ on $X$ must have cost $1 - \frac{1}{|A \cap B|}$ because $\mathcal{C}_{\max}(G) = \mathcal{C}_{\min}(G) = 1 - \frac{1}{|G|}$ for any amenable group $G$ [8, Corollaries I.10, III.4]. Moreover one can find a graphing $\Phi_0$ for this action of $G$ for which the intersection of the underlying graph with almost every orbit of $G$ is a tree (i.e., the graphing is a *treeing*). Then by [8, Lemma III.5] we can extend the graphing $\Phi_0$ to a graphing $\Phi_A$ of the action $A \curvearrowright X$ such that $\mathcal{C}(\Phi_A) < \mathcal{C}(A \curvearrowright X) + \epsilon$. Similarly we can extend $\Phi_0$ to a graphing $\Phi_B$ of the action $B \curvearrowright X$ such that $\mathcal{C}(\Phi_B) < \mathcal{C}(B \curvearrowright X) + \epsilon$. The union $\Phi_A \cup \Phi_B$ is obviously a graphing for the action $H \curvearrowright X$, and since $\Phi_A$ and $\Phi_B$ both extend $\Phi_0$, we have $\mathcal{C}(\Phi_A \cup \Phi_B) \leq \mathcal{C}(\Phi_A) + \mathcal{C}(\Phi_B) - \mathcal{C}(\Phi_0) \leq \mathcal{C}(\Phi_A) + \mathcal{C}(\Phi_B) - (1 - \frac{1}{|A\cap B|})$. Therefore,

$$\mathcal{C}_{\max}(H) \leq \mathcal{C}(H \curvearrowright X) + \varepsilon \leq \mathcal{C}(\Phi_A \cup \Phi_B) + \epsilon \leq \mathcal{C}(\Phi_A) + \mathcal{C}(\Phi_B) - 1 + \frac{1}{|A \cap B|} + \epsilon$$

$$\leq \mathcal{C}(A \curvearrowright X) + \mathcal{C}(B \curvearrowright X) - 1 + \frac{1}{|A \cap B|} + 3\epsilon \leq \mathcal{C}_{\max}(A) + \mathcal{C}_{\max}(B) - 1 + \frac{1}{|A \cap B|} + 3\epsilon.$$

Since this is true for every $\epsilon > 0$, inequality (5.2) follows. □

**Remark 5.8.** The proof of Lemma 5.7 is an adaptation to our situation of a proof sent to us by Damien Gaboriau [10]. In fact he proved a stronger result: the inequality (5.2) holds even if we remove the assumption that $A \cap B$ is amenable. Moreover the inequality holds if we replace $\mathcal{C}_{\max}$ by $\mathcal{C}_{\min}$. There is also a direct analogue of Lemma 5.7 dealing with $L^2$-Betti numbers: $\beta_1(\langle A, B\rangle) \leq \beta_1(A) + \beta_1(B) + \frac{1}{|A\cap B|}$ for any subgroups $A$ and $B$ of the same group. This inequality is a straightforward consequence of results of [23].

*Proof of Theorem 5.6.* Let $S = \{s_1, \ldots, s_m\}$ be a finite generating set of $G$.

Fix $N \in \mathbb{N}$. By assumption there exists an element $g_N$ of $G$ whose order is at least $N$. Consider the sequence of subgroups $A_1 = \langle s_1, g_N\rangle, \ldots, A_m = \langle s_m, g_N\rangle$. We can apply Lemma 5.7 to each of the subgroups $\langle A_1, A_2\rangle$, $\langle A_1, A_2, A_3\rangle$, ... inductively, because each $A_i$ is amenable by the assumption of the theorem. Since the intersection of $A_i$ with $\langle A_1, \ldots, A_{i-1}\rangle$ contains at least $N$ elements, $i = 2, \ldots, m$, and, as we have mentioned before, the maximal cost of every amenable group does not exceed 1, we conclude that $\mathcal{C}'_{\max}(G) \leq \sum_{i=1}^m \mathcal{C}'_{\max}(A_i) + \frac{m-1}{N} \leq \frac{m-1}{N}$. Therefore letting $N$ tend to $\infty$, we conclude that $\mathcal{C}'_{\max}(G) = 0$, so $\mathcal{C}_{\max}(G) = 1$. □

**Remark 5.9.** We expect that the first $L^2$-Betti number vanishes for every group $G$ in $\mathrm{Amen}_2$. Theorem 5.6 shows that we need to consider only groups from $\mathrm{Amen}_2$ of bounded exponent. A conjecture by Shalom [28, Section 5.IV] says that every finitely generated group of bounded exponent has property (T). If that was the case (which is hard to believe), every such group would have vanishing first $L^2$-Betti number [3, Corollary 6]. Note also that by a result of Zelmanov [30] for every prime $p$ there exists a number $n = n(p)$ such that every group of exponent $p$ in $\mathrm{Fin}_n$ is finite. It is believable that the minimal such $n(p)$ is 2, that one can replace $\mathrm{Fin}_n$ by $\mathrm{Amen}_n$, and that the result holds for non-prime numbers $p$. This would also imply that the first $L^2$-Betti number of any group in $\mathrm{Amen}_2$ vanishes.

## Appendix A. Equivalence of two definitions of Tarski numbers

**Theorem A.1.** *Let $G$ be a group and $k = \mathcal{T}(G)$. Then there exists a paradoxical decomposition of $G$ with pieces $P_1, \ldots, P_n$, $Q_1, \ldots, Q_m$ and translating elements $g_1, \ldots, g_n$, $h_1, \ldots, h_m$, $n + m = k$, as in Definition 1.1, such that the union $\bigcup P_i \cup \bigcup Q_j$ is the whole $G$, the translated sets $g_i P_i$ are disjoint, and the translated sets $h_j Q_j$ are disjoint.*

The following argument is very close to a translation of the proof of [25, Proposition 1.2] into a graph-theoretic language.

*Proof.* Suppose that $G$ has a paradoxical decomposition with translating sets $S_1$ and $S_2$, with $1 \in S_1$ (we can assume that by Remark 2.2). By Lemma 2.5, $\Gamma = Cay(G, (S_1, S_2))$ has a spanning evenly colored 2-subgraph $\Lambda$.

Let $A$ be the set of vertices which have no incoming edge in $\Lambda$. For each $g \in A$ consider the unique oriented path in $\Lambda$ starting from $g$ in which all edges have color 1. All such paths will clearly be disjoint. Let $\Lambda'$ be the graph obtained from $\Lambda$ by first removing all the edges from those paths and then adding a loop of color 1 at all the vertices on those paths. Then $\Lambda'$ is a spanning evenly colored 2-subgraph of $\Gamma$ with exactly one incoming edge at every vertex.

By the same argument as in Lemma 2.5, the graph $\Lambda'$ yields a 2-paradoxical decomposition having the required properties, with the same translating sets, $S_1$ and $S_2$. □

## Appendix B. Explicit construction of groups with Tarski number 6

As explained in the introduction, the problem of finding explicit examples of groups with Tarski number 6 reduces to an explicit construction of $d$-generated torsion groups $G$ with $\beta_1(G) > d - 1 - \varepsilon$ whose existence is proved in [21] (to produce groups with Tarski number 6 we take $d = 3$ and $\varepsilon = 1/2$). Such groups are constructed inductively in [21], but the proof shows that they are given by presentations of the form $\langle x_1, \ldots, x_d \mid r_1^{n_1}, r_2^{n_2}, \ldots \rangle$ where $r_1, r_2, \ldots$ is a sequence of all elements of the free group on $x_1, \ldots, x_d$ listed in some order and $n_1, n_2, \ldots$ is some integer sequence. Moreover, given $\varepsilon > 0$, one can specify explicitly how fast the sequence $\{n_i\}$ must grow to ensure that $\beta_1(G) > d - 1 - \varepsilon$ for the resulting group $G$.

The goal of this section is to show that a group given by such "torsion" presentation has Tarski number 6 under much milder conditions on exponents $\{n_i\}$ (see Theorem B.1 below). Note that we will not be able to control the first $L^2$-Betti number of such a group $G$, but we will estimate the first $L^2$-Betti number of some quotient $Q$ of $G$, which is sufficient for producing groups with Tarski number 6. Note also that since we do not know the exact value of the Tarski number of the free Burnside group of a sufficiently large odd exponent (we only know by Theorem 1.2(ii) that it is between 6 and 14), it is possible that one can have a constant sequence $n_1, n_2, \ldots$, say, $n_i = 665$, $i \in \mathbb{N}$, and still get a group with Tarski number 6. The proof of Theorem B.1 mostly utilizes ideas from [21] and [16] where similar results were proved.

**Theorem B.1.** *Let $X$ be a finite set, $F(X)$ the free group on $X$, $p$ a prime, $r_1, r_2, \ldots$ a finite or infinite sequence of elements of $F(X)$, and $R = \{r_i^{p^{n_i}}\}$ for some integer sequence $n_1, n_2, \ldots$. Let $G = \langle X | R\rangle$. Then $G$ has a quotient $Q$ such that*

$$\beta_1(Q) \geq |X| - 1 - \sum_i \frac{1}{p^{n_i}}.$$

*In particular, if $|X| = 3$, $\sum_i \frac{1}{p^{n_i}} \leq \frac{1}{2}$ and the sequence $\{r_i\}$ ranges over the whole free group $F(X)$ (so that $G$ and $Q$ are torsion), then $\mathcal{T}(G) = \mathcal{T}(Q) = 6$.*

We start by stating (a special case of) a result of Peterson and Thom [23, Theorem 3.2] which is similar to Theorem B.1:

**Theorem B.2** ([23])**.** *Let $G$ be a group given by a finite presentation $\langle X \mid r_1^{m_1}, \ldots, r_k^{m_k}\rangle$ for some $r_1, \ldots, r_k \in F(X)$ and $m_i \in \mathbb{N}$. Assume that for each $1 \leq i \leq k$, the order of $r_i$ in $G$ is equal to $m_i$. Then $\beta_1(G) \geq |X| - 1 - \sum_{i=1}^k \frac{1}{m_i}$.*

In general, the assumption on the orders of $r_i$ cannot be eliminated since, for instance, the trivial group has a presentation $\langle x, y \mid x^m, x^{m+1}, y^m, y^{m+1}\rangle$ for any $m \in \mathbb{N}$. If all $m_i$ are powers of a fixed prime $p$, it is possible that Theorem B.2 holds without any additional restrictions, but we are not able to prove such a statement. What we can prove is the following variation:

**Proposition B.3.** *In the notations of Theorem B.2 assume that each $m_i$ is a power of some fixed prime $p$. Let $G_p$ be the image of $G$ in its pro-$p$ completion $\widehat{G}_p$. Then $\beta_1(G_p) \geq |X| - 1 - \sum_{i=1}^k \frac{1}{m_i}$.*

Before establishing Proposition B.3 we show how Theorem B.1 follows from it.

*Proof of Theorem B.1.* If the sequence $\{r_i\}$ is finite, the assertion holds with $Q = G_p$ by Proposition B.3. If $\{r_i\}$ is infinite, let $R_m = \{r_i^{p^{n_i}}\}_{i=1}^m$ and $G(m) = \langle X \mid R_m\rangle$. Let $\beta = |X| - 1 - \sum\limits_{i=1}^{\infty} \frac{1}{p^{n_i}}$. Then $\beta_1(G(m)_p) \geq \beta$ for each $m$ by Proposition B.3.

Note that $G(m+1)_p$ is a quotient of $G(m)_p$. Let $Q = \varinjlim G(m)_p$, that is, if $G(m)_p = F(X)/N_m$, put $Q = F(X)/\cup_{m\in\mathbb{N}} N_m$. Then $Q$ is clearly a quotient of $G$; on the other hand, the sequence $\{G(m)_p\}$ converges to $Q$ in the space of marked groups, and therefore by a theorem of Pichot [24, Theorem 1.1] we have $\beta_1(Q) \geq \limsup\limits_m \beta_1(G(m)_p) \geq \beta$. □

We proceed with the proof of Proposition B.3. Below $p$ will be a fixed prime. Let $F$ be a free group. Given an element $f \in F$, define $s(f) \in F$ and $e(f) \in \mathbb{N}$ by the condition that $f = s(f)^{p^{e(f)}}$ and $s(f)$ is not a $p^{\text{th}}$-power in $F$. The following definition was introduced by Schlage-Puchta in [27]:

**Definition B.4.** Given a presentation $(X, R)$ by generators and relations with $X$ finite, define its $p$-deficiency $def_p(X, R)$ by $def_p(X, R) = |X| - 1 - \sum_{r\in R} \frac{1}{p^{e(r)}}$.

Proposition B.3 can now be reformulated as follows:

**Proposition B.5.** *Let $(X, R)$ be a finite presentation of a group $G$. Then $\beta_1(G_p) \geq def_p(X, R)$.*

As usual, for a finitely presented group $G$ we define $def(G)$ to be the maximal possible value of the difference $|X| - |R|$ where $(X, R)$ ranges over all finite presentations of $G$.

**Definition B.6** ([16])**.** Given a finitely presented group $G$, define the quantity $vdef_p(G)$ by $vdef_p(G) = \sup_H \frac{def(H)-1}{[G:H]}$ where $H$ ranges over all normal subgroups of $G$ of $p$-power index.

**Definition B.7** ([7])**.** A presentation $(X, R)$ will be called *$p$-regular* if for any $r \in R$ the element $s(r)$ has order (precisely) $p^{e(r)}$ in the group $\langle X|R\rangle_p$.

According to [16, Lemma 3.6], for any finitely presented group $G$ we have $\beta_1(G_p) \geq vdef_p(G)$. On the other hand, by [7, Lemma 5.5], if a group $G$ has a finite $p$-regular presentation $(X, R)$, then $vdef_p(G) \geq def_p(X, R)$. These two results imply Proposition B.3 in the case of $p$-regular presentations. The proof in the general case will be completed via the following lemma.

**Lemma B.8.** *Let $(X, R)$ be a finite presentation. Then there exists a subset $R'$ of $R$ such that the presentation $(X, R')$ is $p$-regular and the natural surjection $\langle X|R'\rangle \to \langle X|R\rangle$ induces an isomorphism of pro-$p$ completions $\widehat{\langle X|R'\rangle}_p \to \widehat{\langle X|R\rangle}_p$ and hence also an isomorphism of $\langle X|R'\rangle_p$ onto $\langle X|R\rangle_p$.*

*Proof of Lemma B.8.* Let $G = \langle X|R\rangle$, and assume that $(X, R)$ is not $p$-regular. Thus there exists $r \in R$ such that the order of $s(r)$ in $G_p$ is strictly smaller than $p^{e(r)}$. We will show that if we set $R' = R \setminus \{r\}$ and $G' = \langle X|R'\rangle$, then the natural map $\widehat{G'}_p \to \widehat{G}_p$ is an isomorphism. Lemma B.8 will follow by multiple applications of this step.

If a discrete group is given by a presentation by generators and relators, its pro-$p$ completion is given by the same presentation in the category of pro-$p$ groups. It follows that

$$\widehat{G}_p \cong \widehat{G'}_p/\langle\!\langle s(r)^{p^{e(r)}}\rangle\!\rangle \tag{B.1}$$

where $\langle\!\langle S\rangle\!\rangle$ is the closed normal subgroup generated by a set $S$. Thus, it is sufficient to show that $s(r)^{p^{e(r)}} = 1$ in $\widehat{G'}_p$. We will show that already $s(r)^{p^{e(r)-1}} = 1$ in $\widehat{G'}_p$

Let $m$ be the order of $s(r)$ in $\widehat{G}_p$. Then by assumption $m < p^{e(r)}$; on the other hand, $m$ must be a power of $p$ (since $\widehat{G}_p$ is pro-$p$), so $m$ divides $p^{e(r)-1}$. Thus, if we let $g$ be the image of $s(r)^{p^{e(r)-1}}$ in $\widehat{G'}_p$, then $g$ lies in the kernel of the homomorphism $\widehat{G'}_p \to \widehat{G}_p$, whence by (B.1), $g$ lies in the normal closed subgroup generated by $g^p$. It is easy to see that this cannot happen in a pro-$p$ group unless $g = 1$. □

## Appendix C. Golod-Shafarevich groups

In this section we introduce Golod-Shafarevich groups and give a self-contained proof of Proposition 4.6.

The definitions of Golod-Shafarevich groups and the related notion of a weight function will be given in a simplified form below since this will be sufficient for the purposes of this paper. For more details the reader is referred to [6].

Let $p$ be a fixed prime number. Given a finitely generated group $G$, let $\{\omega_n G\}_{n\in\mathbb{N}}$ be the *Zassenhaus $p$-filtration* of $G$ defined by $\omega_n G = \prod_{i\cdot p^j\geq n}(\gamma_i G)^{p^j}$. It is easy to see that $\{\omega_n G\}$ is a descending chain of normal subgroups of $p$-power index in $G$ satisfying

$$[\omega_n G, \omega_m G] \subseteq \omega_{n+m} G \quad \text{and} \quad (\omega_n G)^p \subseteq \omega_{np} G. \tag{C.1}$$

Moreover, $\{\omega_n G\}$ is a base for the pro-$p$ topology on $G$, so in particular, $\cap\omega_n G = \{1\}$ if and only if $G$ is a residually-$p$ group.

Now let $F$ be a finitely generated free group. Then $F$ is residually-$p$ for any $p$, so for any $f \in F \setminus \{1\}$ there exists (unique) $n \in \mathbb{N}$ such that $f \in \omega_n F \setminus \omega_{n+1} F$. This $n$ will be called the degree of $f$ and denoted $\deg(f)$. We set $\deg(1) = \infty$

**Definition C.1.** Let $F$ be a finitely generated free group.

(i) A function $W\colon F \to \mathbb{N} \cup \{\infty\}$ will be called a *weight function* if $W(f) = \tau^{\deg(f)}$ where $\tau \in (0, 1)$ is a fixed real number.

(ii) If $W$ is a weight function on $F$ and $\pi\colon F \to G$ an epimorphism, then $W$ induces a function on $G$ (also denoted by $W$) given by

$$W(g) = \inf\{W(f) : \pi(f) = g\}$$

Such $W$ will be called a *valuation* on $G$.

(iii) If $W$ is a valuation on $G$, for any countable subset $S$ of $G$ we put $W(S) = \sum_{s\in S} W(s) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

The following remark is a reformulation of property (C.1) above.

**Remark C.2.** Let $W$ be a valuation on a (finitely generated) group $G$. Then for any $g, h \in G$ we have

(i) $W(gh) \le \max\{W(g), W(h)\}$ and $W(g^{-1}) = W(g)$
(ii) $W([g,h]) \le W(g)W(h)$
(iii) $W(g^p) \le W(g)^p$.

**Definition C.3.**
(i) Let $\langle X|R\rangle$ be a presentation of a group $G$ with $|X| < \infty$ and $W$ a weight function on $F(X)$. Then we will call the triple $(X,R,W)$ a *weighted presentation* of $G$.
(ii) A weighted presentation $(X,R,W)$ will be called *Golod-Shafarevich* if
$$W(X) - W(R) - 1 > 0.$$
(iii) A finitely generated group $G$ is called *Golod-Shafarevich* (with respect to $p$) if it has a Golod-Shafarevich weighted presentation.

As was already proved in 1960's, Golod-Shafarevich groups are always infinite; in fact, they have infinite pro-$p$ completions (see [6, § 2-4][1]). Also by the nature of their definition, any Golod-Shafarevich group has a lot of quotients which are still Golod-Shafarevich, thanks to the following observation:

**Remark C.4.** Let $(X,R,W)$ be a Golod-Shafarevich weighted presentation of a group $G$, and let $\varepsilon = W(X) - W(R) - 1$ (so that $\varepsilon > 0$ by assumption). Then for any $T \subseteq G$ with $W(T) < \varepsilon$, the group $G/\langle\langle T\rangle\rangle$ is also Golod-Shafarevich (and therefore infinite).

The following proposition is a natural generalization of [6, Theorem 3.3]. In fact, it is a special case of a result from [5] (see [5, Lemma 5.2] and a remark after it), but since the setting in [5] is much more general than ours, we present the proof for the convenience of the reader.

**Proposition C.5.** *Let $G$ be a group with weighted presentation $(X,R,W)$. Let $\Sigma$ be a finite or countable collection of finite subsets of $G$ such that $W(S) < 1$ for each $S \in \Sigma$. Then for every $\varepsilon > 0$ there is a subset $R_\varepsilon$ of $G$ with $W(R_\varepsilon) < \varepsilon$ and the following property: if $G' = G/\langle\langle R_\varepsilon\rangle\rangle$, then for each $S \in \Sigma$, the image of $S$ in $G'$ generates a finite group.*

*In particular, by Remark C.4, if the weighted presentation $(X,R,W)$ is Golod-Shafarevich, by choosing small enough $\varepsilon$, we can ensure that $G'$ is Golod-Shafarevich.*

*Proof.* Let $g_1, g_2, \ldots$ be an enumeration of elements of $G$, and choose integers $n_1, n_2, \ldots$ such that $\sum_{i\in\mathbb{N}} W(g_i^{p^{n_i}}) < \varepsilon/2$ – this is possible by Remark C.2(iii).

Let $S_1, S_2, \ldots$ be an enumeration of $\Sigma$. Given $n, k \in \mathbb{N}$, let $S_n^{(k)}$ be the set of all left-normed commutators of length $k$ in elements of $S_n$. Using Remark C.2(ii) we have
$$W(S_n^{(k)}) \;=\; \sum_{h_1,\ldots,h_k\in S_n} W([h_1,\ldots,h_k]) \;\le\; \sum_{h_1,\ldots,h_k\in S_n} W(h_1)\ldots W(h_k) \;=\; W(S_n)^k,$$
so by our assumption $W(S_n^{(k)}) \to 0$ as $k \to \infty$. Therefore, we can find an integer sequence $k_1, k_2, \ldots$ such that $\sum_{n\in\mathbb{N}} W(S_n^{(k_n)}) < \varepsilon/2$.

Now define $G' = G/\langle\langle R_\varepsilon\rangle\rangle$ where $R_\varepsilon = \{g_i^{p^{n_i}}\}_{i\in\mathbb{N}} \cup \bigcup_{n=1}^{\infty} S_n^{(k_n)}$. Then by construction $W(R_\varepsilon) < \varepsilon$. Also by construction, for each $n$ the subgroup generated by the image of $S_n$ in $G'$ is torsion and nilpotent, hence finite. $\square$

We are finally ready to prove Proposition 4.6 restated below.

[1] In the foundational paper [12] the same statement was proved for a different, although very similar, class of groups.

**Proposition C.6.** *Let $G$ be a Golod-Shafarevich group. Then there exists a quotient $H$ of $G$ which is also Golod-Shafarevich and satisfies the following property: for every $n \in \mathbb{N}$ there is a finite index subgroup $H_n$ of $H$ such that all $n$-generated subgroups of $H_n$ are finite.*

*Proof.* Let $(X, R, W)$ be a Golod-Shafarevich weighted presentation of $G$. For every $n \in \mathbb{N}$ let $G_n = \{g \in G : W(g) < \frac{1}{n}\}$. Then $G_n$ is a finite index subgroup of $G$ (more specifically, if $\tau < 1$ is such that $W(f) = \tau^{\deg(f)}$ for every $f \in F(X)$, then $G_n \supseteq \omega_m G$ whenever $\tau^m < \frac{1}{n}$).

Let $\Sigma$ be the collection of all $n$-element subsets of $G_n$, where $n$ ranges over $\mathbb{N}$. By construction $W(S) < 1$ for each $S \in \Sigma$, and applying Proposition C.5 to this collection of subsets, we obtain a group $H$ with desired properties (where $H_n$ is the image of $G_n$ in $H$). $\square$

**Remark C.7.** We finish with a remark about Theorem 2. Our original construction of infinite 2-generated groups with property $(T)$ and unbounded Tarski numbers was explicit apart from the description of examples of infinite property $(T)$ groups in $\mathrm{Fin}_m$. Such groups can also be defined by explicit presentations as explained below.

Given an integer $d \geq 2$ and a prime $p$, let $G_{p,d}$ be the group with presentation $\langle X|R\rangle$ where $X = \{x_1, \ldots, x_d\}$ and $R = \{x_i^p, [x_i, x_j, x_j]\}_{1\leq i\neq j\leq d}$. By [6, Theorem 12.1], $G_{p,d}$ is a Golod-Shafarevich group with property $(T)$ whenever $d \geq 9$ and $p > (d-1)^2$. Applying the proof of Proposition C.5 to the group $G = G_{p,d}$ and suitable $\Sigma$ and $\varepsilon$, one obtains a concrete example of an infinite group with property $(T)$ which lies in $\mathrm{Fin}_m$ for $m < d/2$.

Moreover, observe that the group $G = G_{p,d}$ admits an automorphism $\sigma$ of order $d$ which cyclically permutes the generators. One can show that the set of relators $R_\varepsilon$ in the proof of Proposition C.5 can be chosen $\sigma$-invariant, so that $\sigma$ induces an automorphism $\sigma'$ of the quotient $G' = G/\langle\langle R_\varepsilon\rangle\rangle$. Then the group $G' \rtimes \langle\sigma'\rangle$ is an infinite 2-generated group with property $(T)$ whose Tarski number can be made arbitrarily large by choosing a large enough $d$ (by Theorem 1(b)). This provides an alternative proof of Theorem 2.

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