Bilinear Forms and Group Representations. Fall 2017.
Final exam.

Due on Thursday, December 7th, by 4PM

Directions: Provide complete arguments (do not skip steps). State clearly and FULLY any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given. If you are unable to solve a problem (or a part of a problem), you may still use its result to solve a later part of the same problem or a later problem in the exam.

Rules: You are NOT allowed to discuss midterm problems with anyone else except me. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use the following resources:

(i) your class notes
(ii) homework solutions (both your solutions and solutions posted on the course webpage)
(iii) Ben Webster's 4657 notes
(iv) the books 'Representation Theory of Finite Groups' by Benjamin Steinberg and 'Linear Algebra' by Friedberg, Insel and Spence

The use of other books or online sources is NOT allowed.

1. (10 pts) Let $n \geq 2$ and let $G = \mathrm{SU}(n)$, the group of $n \times n$ unitary matrices with determinant 1. Let D be the subgroup of diagonal matrices inside G.

(a) Prove that every element of G is conjugate (in G) to some element of D.
(b) Prove that if $n = 2$, then every $g \in G$ is conjugate to g^{-1}.
(c) Prove that the assertion of (b) is false for $n > 2$.

2. (12 pts) Let G and H be groups and $G \times H$ their direct product.

(a) Let F be a field, let (ρ_G, V_G) be a representation of G over F and (ρ_H, V_H) a representation of H over F. Prove that there exists a unique representation $(\rho, V_G \otimes V_H)$ of $G \times H$ such that

$$(\rho((g,h)))(u \otimes v) = \rho_G(g)u \otimes \rho_H(h)v$$

for all $g \in G, h \in H, u \in V_G$ and $v \in V_H$. In parts (b)-(d) below we denote the obtained representation ρ by $\rho_G \otimes \rho_H$.

(b) Find the formula for the character of $\rho_G \otimes \rho_H$ in terms of the characters of ρ_G and ρ_H.

In parts (c) and (d) assume that G and H are finite and $F = \mathbb{C}$.

(c) Prove that if ρ_G and ρ_H in (a) are irreducible, then $\rho_G \otimes \rho_H$ is also irreducible.

(d) Now prove that every irreducible representation of $G \times H$ is equivalent to $\rho_G \otimes \rho_H$ for some irreducible representations ρ_G of G and ρ_H of H.

3. (16 pts) Let G be a group given by the presentation

$$\langle x, y \mid x^4 = y^3 = 1, x^{-1}yx = y^{-1} \rangle.$$

You may use without proof that $|G| = 12$.

(a) Let $K = \langle y \rangle$. Prove that $|K| = 3$ and $K = [G, G]$.

(b) Prove that G has 6 conjugacy classes with representatives e, x, x^2, x^3, y and x^2y.

(c) Prove that $g = x^2$ lies in the center of G.

(d) Compute the character table of G. Provide all the details of your argument.

4*. (6 pts) Let G be a finite group such that $|G|$ is a composite number, and let (ρ, V) be an irreducible representation of G over some field F. Prove that $\dim(V) \leq |G| - 2$.

5. (10 pts) Let p be a prime, let $\omega \neq 1$ be a p^{th}-root of unity (in $\mathbb{C}$). Let $G = \mathrm{Heis}(\mathbb{Z}_p)$, the Heisenberg group over $\mathbb{Z}_p$ and let ρ_ω be the p-dimensional representation of G defined in HW#9.2. Construct a proper subgroup H of G and a representation α_ω of H such that the induced representation $\mathrm{Ind}\!\uparrow_H^G \alpha_\omega$ is equivalent to ρ_ω (and prove that your H and α_ω have the desired properties).

6. (4 pts) Let G be a group, let $z \in G$ be a central element, and let (ρ, V) be a representation of G over a field F. Let $\lambda \in F$ be an eigenvalue of $\rho(z)$, and let $V_\lambda(z)$ be the eigenspace of $\rho(z)$ corresponding to λ, that is, $V_\lambda(z) = \{v \in V : \rho(z)v = \lambda v\}$. Prove that $V_\lambda(z)$ is G-invariant.

7*. (12 pts) Let p be a prime, $n \geq 2$ an integer and $G = \mathrm{GL}_n(\mathbb{Z}_p)$. Let $X = \mathbb{Z}_p^n \setminus \{(0, \ldots, 0)\}$ be the set of nonzero vectors in $\mathbb{Z}_p^n$ (so that $|X| = p^n - 1$). The group G has a natural action on X by left multiplication. Let $\mathbb{C}X$ be the corresponding permutation representation of G. Prove that $\mathbb{C}X$ decomposes as a direct sum of p irreducible representations of G:

$$\mathbb{C}X = \bigoplus_{k=1}^{p} V_k$$

where $\dim V_k = \frac{p^n-1}{p-1}$ for $1 \leq k \leq p-2$, $\dim V_{p-1} = \frac{p^n-1}{p-1} - 1$ and $\dim V_p = 1$.

Hint for #4: Use the fact that G has at least one subgroup different from G and $\{e\}$.

Hint for #7: Use Problem 6 to decompose $\mathbb{C}X$ as a direct sum of $p-1$ subrepresentations of dimension $\frac{p^n-1}{p-1}$. You may use without proof that the group $\mathbb{Z}_p^\times$ is cyclic.

Then show that one of these $p-1$ subrepresentations splits into two. Finally, use something else to argue the obtained pieces cannot be decomposed any further.