Solutions to the practice problems for the final

1. Let $f : A \to B$ be a function. Prove that the following two conditions are equivalent:

(a) f is injective
(b) $f^{-1}(f(C)) = C$ for every subset C of A.

Hint: Use contrapositive for the proof in one of the two directions.

Solution: "(a)$\Rightarrow$(b)" Suppose that f is injective. First we prove that $C \subseteq f^{-1}(f(C))$ (this inclusion does not use injectivity). Take any $c \in C$. Then $f(c) \in f(C)$. Hence c is an element of the set $\{x \in A \mid f(x) \in f(C)\}$, and by definition this set is $f^{-1}(f(C))$. Thus, we proved the implication $(c \in C) \Rightarrow (c \in f^{-1}(f(C)))$, so $C \subseteq f^{-1}(f(C))$.

Now we prove that $f^{-1}(f(C)) \subseteq C$ (this time using injectivity). Take any $x \in f^{-1}(f(C))$. This means that $f(x) \in f(C)$, that is, $f(x) = f(c)$ for some $c \in C$. But f is injective, so $f(x) = f(c)$ implies $x = c$, so $x \in C$.

"(b)$\Rightarrow$ (a)" We will prove the contrapositive: if f is not injective, then there exists a subset C of A such that $f^{-1}(f(C)) \neq C$. So suppose f is not injective. This means that there exist $a_1, a_2 \in A$ such that $a_1 \neq a_2$ but $f(a_1) = f(a_2)$. Let $C = \{a_1\}$. Then $f^{-1}(f(C))$ definitely contains both a_1 and a_2 and hence $f^{-1}(f(C)) \neq C$.

2. Give a short proof of the furthermore part of Theorem 6.2.8(d) from the book using Problem 1 above, the first assertion of Theorem 6.2.8(d) and Theorem 6.2.9(d)

Note: I realized that there is no nice way to solve this problem using Theorem 6.2.8(d) and Theorem 6.2.9(d), so we will just give some proof of the furthermore part of Theorem 6.2.8(d). Recall that it asserts the following: Let $f : A \to B$ be a function. Then f is injective $\iff$ $f(C \setminus D) = f(C) \setminus f(D)$ for any subsets C, D of A.

For the backwards direction see solutions to HW#10.6. Let us prove the forward direction. We assume that f is injective (note that we will not be using this assumption until the end of the proof), and let C and D be subsets of A. We need to show that $f(C \setminus D) = f(C) \setminus f(D)$. The inclusion $f(C \setminus D) \supseteq f(C) \setminus f(D)$ holds for any f (by the first part of Theorem 6.2.8(d)).

Thus we just need to prove the opposite inclusion $f(C \setminus D) \subseteq f(C) \setminus f(D)$. Equivalently, we need to prove the implication

$$x \in f(C \setminus D) \Rightarrow x \in f(C) \setminus f(D) \qquad (***)$$

So take any $x \in f(C \setminus D)$. By definition of the image this means that $x = f(a)$ for some $a \in C \setminus D$. Since $a \in C \setminus D$, we have $a \in C$ and $a \notin D$. Since $x = f(a)$ and $a \in C$, we have $x \in f(C)$.

Next we prove that $x \notin f(D)$ by contradiction. Assume, by way of contradiction, that $x \in f(D)$. This means that $x = f(d)$ for some $d \in D$. Since $x = f(a)$ and $x = f(d)$, we have $f(a) = f(d)$. At this point, we can use injectivity of f to conclude that $a = d$. But this is impossible since by assumption $a \notin D$ and $d \in D$. The obtained contradiction proves that $x \notin f(D)$.

Thus, we proved that $x \in f(C)$ and $x \notin f(D)$, and therefore $x \in f(C) \setminus f(D)$. This finishes the proof of (***).

3. Let A and B be non-empty sets. Prove that the following are equivalent:

(i) there is an injection $f : A \to B$
(ii) there is a surjection $g : B \to A$

Hint: If you are not sure how to start, take a look at the proof of Theorem 23.4. This should help you at least with the proof of the implication "(ii)$\Rightarrow$(i)".

Note: An extra assumption $A \neq \emptyset$ is needed for (i) and (ii) to be equivalent (this assumption was missing from the original formulation).

"(i)$\Rightarrow$" Assume there is an injection $f : A \to B$. Fix some element $a_0 \in A$ (this is possible since $A \neq \emptyset$). Now define a function $g : B \to A$ (which we will show to be surjective) as follows: Take any $b \in B$. We define $g(b)$ by cases:

Case 1: $b \in f(A)$. Then there exists $a \in A$ such that $f(a) = b$ (by definition of $f(A)$). Moreover, such a is unique (for a given b) since f is injective. In this case we define $g(b) = a$.

Case 2: $b \notin f(A)$. In this case we define $g(b) = a_0$.

By construction $g(f(a)) = a$ for all $a \in A$. In particular, for every $a \in A$ there exists $b \in B$ such that $g(b) = a$ (namely $b = f(a)$), so g is surjective.

Note that surjectivity of g follows entirely from Case 1. Case 2 is only needed to ensure that g is defined on the entire B.

"(i)$\Leftarrow$" Now assume there is a surjection $g : B \to A$. Define $f : A \to B$ (which we will show to be injective) as follows:

Take any $a \in A$. Since $g : B \to A$ is surjective, we can choose some element $b_a \in B$ such that $g(b_a) = a$ (there may be more than one choice for b_a). Define $f(a) = b_a$.

Now we prove that f is injective. Suppose that $f(a) = f(c)$ for some $a, c \in A$. By definition of f this means $b_a = b_c$. Applying g to both sides of the last equality we get $g(b_a) = g(b_c)$. But by definition of b_* we have $g(b_a) = a$ and $g(b_c) = c$, and so $a = c$.

5. Let $A = [-1, 1]$ and $B = (-1, 1)$. Define the functions $f : A \to B$ and $g : B \to A$ by $f(a) = \frac{a}{2}$ and $g(b) = b$ for all $a \in A$ and $b \in B$. It is straightforward to check that f and g are both injective, so one can apply the proof of the Schroeder-Bernstein theorem to those f and g to construct a bijection $\Phi : A \to B$. Find an explicit formula for such Φ (this is a good way to test if you understand the proof of the Schroeder-Bernstein theorem).

Solution: We use the proof of the Schroeder-Bernstein theorem given in class and notations from that proof. First we compute the function $h : A \to A$. By definition $h = g \circ f$, so we get $h(a) = \frac{a}{2}$ for all $a \in A$ (note that in our case h and f have the same domains and the same output values and only differ in their codomains).

Next we compute the sets $A_1, B_1, A_2, B_2, \ldots$. We have $A_1 = A = [-1, 1]$ and $B_1 = g(B) = (-1, 1)$. For each $n \geq 2$ we have $A_n = h(A_{n-1})$ and $B_n = h(B_{n-1})$. Since $h(a) = \frac{a}{2}$ for all $a \in A$, we have $A_2 = [-\frac{1}{2}, \frac{1}{2}]$, $B_2 = (-\frac{1}{2}, \frac{1}{2})$, $A_3 = [-\frac{1}{4}, \frac{1}{4}]$, $B_3 = (-\frac{1}{4}, \frac{1}{4})$, $A_4 = [-\frac{1}{8}, \frac{1}{8}]$, $B_4 = (-\frac{1}{8}, \frac{1}{8})$ etc. A simple induction shows that $A_n = [-\frac{1}{2^{n-1}}, \frac{1}{2^{n-1}}]$, $B_n = (-\frac{1}{2^{n-1}}, \frac{1}{2^{n-1}})$ for all $n \in \mathbb{N}$.

Now according to the proof of the the Schroeder-Bernstein theorem, the function H defined below is a bijection from A to $g(B)$:

$$H(x) = \begin{cases} h(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

Note that in the proof in class we use the auxiliary notations P_n (recall that $P_1 = A_1 \setminus B_1$, $P_2 = B_1 \setminus A_2$, $P_3 = A_2 \setminus B_2$, $P_4 = B_2 \setminus A_3$ etc.) These notations were helpful for proving injectivity of H, but are not really needed to define H (which is why we are not using them here).

From the above description of the sets $\{A_n\}$ and $\{B_n\}$ it is clearly that for each n the set $A_n \setminus B_n$ has exactly two elements: $-\frac{1}{2^{n-1}}$ and $\frac{1}{2^{n-1}}$.

Thus, the explicit formula for H is as follows:

$$H(x) = \begin{cases} \frac{x}{2} & \text{if } x = \pm\frac{1}{2^{n-1}} \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

Remark 1: Note that the above H is a bijection from A to $g(B)$. In general, to get a bijection from A to B, we would need to compose H with a bijection from $g(B)$ to B (which we know exists since g is injective). However, this is not necessary here since we have $g(B) = B$.

Remark 2: Note that H restricted to each of the sets $C_+ = \{\frac{1}{2^{n-1}} \mid n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$ and $C_- = \{-\frac{1}{2^{n-1}} \mid n \in \mathbb{N}\}$ shifts all elements by one to the right: $H(1) = \frac{1}{2}, H(\frac{1}{2}) = \frac{1}{4}, H(\frac{1}{4}) = \frac{1}{8}$ etc. Thus, H happens to be very similar to the bijection between $[0, 1]$ and $[0, 1)$ from the end of Lecture 24 even though the general idea we used to construct it was completely different.

6. Let A be a set. Prove that $|A| \neq |\mathcal{P}(A)|$, that is, there is no bijection $\phi : A \to \mathcal{P}(A)$. **Hint:** Argue by contradiction: suppose that such ϕ exists. Consider the set $B = \{a \in A \mid a \notin \phi(a)\} \in \mathcal{P}(A)$, and show that B cannot lie in the image of ϕ. The contradiction you should get is very similar to the one in Russell's paradox.

Solution: Suppose that the set B defined above lies in $\phi(A)$, so that $B = \phi(x)$ for some $x \in A$. Then one of the two things must be true: $x \in B$ or $x \notin B$. We will obtain a contradiction in each case.

Case 1: $x \notin B$. This means that $x \notin \phi(x)$, so x is an element of the set $\{a \in A \mid a \notin \phi(a)\}$. But this set is B (by definition), so $x \in B$, a contradiction.

Case 2: $x \in B$. This means that $x \in \phi(x)$, so x is NOT an element of the set $\{a \in A \mid a \notin \phi(a)\}$. Thus, $x \notin B$, again a contradiction.