Midterm #2 solutions

1. (8 pts) Let G be a group and H a subgroup of G. Let (ρ, V) be a representation of G, and let $V^H = \{v \in V : \rho(h)v = v \text{ for all } h \in H\}$ be the subspace of H-invariant vectors.

(i) Assume that H is normal in G. Prove that the subspace V^H is G-invariant.

(ii) Give an example showing that if H is not normal, then V^H need not be G-invariant.

Solution: (i) Take any $v \in V^H$. We need to show that $\rho(g) \in V^H$ for all $g \in G$, that is, $\rho(h)(\rho(g)v) = \rho(g)v$ for all $g \in G$ and $h \in H$.

So take any $g \in G, h \in H$. Since H is normal, we have $hg = gh'$ for some $h' \in H$. Since $v \in V^H$, we have $\rho(h')v = v$. Hence (since ρ is a homomorphism), we have

$$\rho(h)(\rho(g)v) = \rho(hg)v = \rho(gh')v = \rho(g)(\rho(h')v) = \rho(g)v.$$

(b) Let $(\rho, \mathbb{C}^3)$ be the defining representation of $G = S_3$, and let $H = \langle (1,2) \rangle$. Then clearly $e_3 \in V^H$, but $\rho((1,3))e_3 = e_1 \notin V^H$ (since $\rho((1,2))e_1 = e_2 \neq e_1$), so we conclude that V^H is not G-invariant.

2. (6 pts) Let (α, V) and (β, W) be representations of a group G. Let (α^*, V^*) be the dual of (α, V), and let $(\alpha^* \otimes \beta, V^* \otimes W)$ be the tensor product of the representations (α^*, V^*) and (β, W). Define the representation $(\gamma, \mathrm{Hom}(V, W))$ of G by

$$(\gamma(g))(f) = \beta(g) \circ f \circ \alpha(g)^{-1}$$

for all $f \in \mathrm{Hom}(V, W)$. Prove that

$$(\gamma, \mathrm{Hom}(V, W)) \cong (\alpha^* \otimes \beta, V^* \otimes W)$$

as representations of G. Make sure to provide all the details.

Solution: For simplicity, we introduce a new notation: given $f \in V^*$ and $w \in W$, define $f \times w$ to be the element of $\mathrm{Hom}(V, W)$ given by $(f \times w)(v) = f(v)w$ for all $v \in V$.

By Lecture 11 there exists an isomorphism of vector spaces $T : V^* \otimes W \to \mathrm{Hom}(V, W)$ such that

$$T(f \otimes w) = f \times w \text{ for all } f \in V^* \text{ and } w \in W.$$

Thus, to prove the assertion of Problem#2 it suffices show that T is a homomorphism of representations, that is, for all $g \in G$,

$$T \circ (\alpha^* \otimes \beta)(g) = \gamma(g) \circ T \qquad (***)$$

as maps from $V^* \otimes W$ to $\mathrm{Hom}(V, W)$.

Since both sides of (***) are clearly linear and since $V^* \otimes W$ is spanned by simple tensors, it suffices to check that

$$(T \circ (\alpha^* \otimes \beta)(g))(f \otimes w) = (\gamma(g) \circ T)(f \otimes w) \text{ for all } f \in V^*, w \in W.$$

The left-hand side of the above expression is equal to

$$T(((\alpha^* \otimes \beta)(g))(f \otimes w)) = T((\alpha^*(g)f) \otimes (\beta(g)w)) = (\alpha^*(g)f) \times (\beta(g)w),$$

and the right-hand side is equal to $\gamma(g)(T(f \otimes w)) = \gamma(g)(f \times w) = \beta(g) \circ (f \times w) \circ \alpha(g)^{-1}$. Thus we are reduced to proving the equality

$$(\alpha^*(g)f) \times (\beta(g)w) = \beta(g) \circ (f \times w) \circ \alpha(g)^{-1}. \qquad (!!!)$$

Both sides are maps from V to W, and we show that they are equal by evaluating them at an arbitrary $v \in V$. The left-hand side of (!!!) is equal to

$$((\alpha^*(g)f) \times (\beta(g)w))(v) = (\alpha^*(g)f)(v) \cdot \beta(g)w = f(\alpha(g)^{-1}v) \cdot \beta(g)w.$$

Here $\cdot$ denotes the scalar multiplication; the first equality holds by the definition of the "cross product" and the second equality holds by the definition of the dual representation.

The right-hand side of (!!!) is equal to

$$(\beta(g) \circ (f \times w) \circ \alpha(g)^{-1})(v) = \beta(g)((f \times w)(\alpha(g)^{-1}v)) = \beta(g)(f(\alpha(g)^{-1}v)w).$$

Since $f(\alpha(g)^{-1}v)$ is a scalar and $\beta(g)$ is linear, the last expression is equal to $f(\alpha(g)^{-1}v) \cdot \beta(g)w$, which is precisely what we got when transforming the left-hand side.

3. (4 pts) Let (ρ, V) and (ρ', V) be complex representations of a finite group G (the vector space V is the same for both representations). Suppose that $\rho'(g)$ is conjugate to $\rho(g)$ in $\mathrm{GL}(V)$ for every $g \in G$. Prove that the representations (ρ, V) and (ρ', V) are equivalent. **Note:** The result is not automatic since the matrix which conjugates $\rho'(g)$ to $\rho(g)$ may depend on g.

Solution: Since conjugate matrices we have the same trace, we conclude that $\chi_{\rho'}(g) = \mathrm{Tr}(\rho'(g)) = \mathrm{Tr}(\rho(g)) = \chi_\rho(g)$ for all $g \in G$. Thus, $\chi_{\rho'} = \chi_\rho$ as functions. Since G is finite and representations are complex, by Theorem 19.1, $\rho' \equiv \rho$ as representations of G.

4. (8 pts) Let G be a finite group, and suppose you are given the character table of G. Give a simple algorithm to determine which elements of G lie in $[G, G]$ based on the character table. Make sure to prove that your algorithm works.

Solution: We will prove the following theorem:
Theorem A: *Let $g \in G$. Then $g \in [G, G] \iff \chi(g) = 1$ for every 1-dimensional character χ.*

Theorem A yields a simple algorithm for characterizing $[G, G]$. First identify the identity element of G (to do this find the unique column of the table where the value of every character is a positive real number). Then identify all the 1-dimensional characters (find all the rows where the value at the identity element is equal to 1). Then find all the columns where every 1-dimensional character is equal to 1. By Theorem A, $[G, G]$ is the union of conjugacy classes corresponding to those columns.

Proof of Theorem A: "$\Rightarrow$" Let $\chi : G \to \mathbb{C}^\times$ be a one-dimensional representation. Then $G/\mathrm{Ker}\chi \cong \chi(G)$ is abelian since $\chi(G)$ is a subgroup of $\mathbb{C}^\times$. Hence by Claim 14.4(c), $\mathrm{Ker}\chi \supseteq [G, G]$. In other words, every one-dimensional representation vanishes at every element of $[G, G]$.

"$\Leftarrow$" We will prove this by contrapositive (assume that $g \notin [G, G]$ and show that there exists a one-dimensional representaion ρ of G with $\rho(g) \neq 1$). We will give two different proofs. The first argument is based on our explicit description of one-dimensional representations of finite groups.

Write G^{ab} as a direct product of cyclic subgroups: $G^{ab} = \langle c_1 \rangle \times \ldots \times \langle c_t \rangle$, and let $n_k = o(c_k)$. Thus every element of G^{ab} can be uniquely written as $c_1^{m_1} \ldots c_k^{m_k}$ with $0 \leq m_k < n_k$ for each k. Now take any $g \in G \setminus [G, G]$, and let $\pi : G \to G/[G, G]$ be the natural projection. Then $\pi(g) \neq 1$, so $\pi(g) = c_1^{u_1} \ldots \times c_t^{u_t}$ where $0 < u_s < n_s$ for at least one index s.

By the discussion in Lecture 14, there is a (well-defined) one-dimensional representation $\rho : G^{ab} \to \mathbb{C}^\times$ given by $\rho(c_1^{m_1} \ldots c_k^{m_k}) = e^{2\pi i \frac{m_s}{n_s}}$ (where s is chosen as above). Then $\rho' = \rho \circ \pi$ is a one-dimensional representation of G, and by construction $\rho'(g) = e^{2\pi i \frac{u_s}{n_s}} \neq 1$, as desired. This completes the first argument.

The second argument will be based on the following general theorem:

Theorem B: *Let Q be any finite group and $q \in Q$ a non-identity element. Then there exists an IRREDUCIBLE complex representation ρ of Q such that $\rho(q) \neq I$ (where I is the identity operator/matrix of suitable size).*

Recall from HW#7.4 that it is not always possible to find ρ as in the theorem which will work for all non-identity elements simultaneously.

Proof of Theorem B: Let $(\rho_{reg}, \mathbb{C}[Q])$ be the regular representation. Then $\rho_{reg}(q) \neq I$ for any $q \neq e$ since in fact $qx \neq x$ for any $x \in Q$. On the other hand, we know that $\mathbb{C}[Q]$ is a direct sum of irreducible subrepresentations (ρ_k, V_k) (where $\rho_k(q)$ is the restriction of $\rho_{reg}(q)$ to V_k). The matrix of $\rho_{reg}(q)$ in a suitable basis is block-diagonal, where the diagonal blocks are matrices of the restrictions $\rho_k(q)$. Since $\rho_{reg}(q) \neq I$, there is at least one k for which $\rho_k(q)$ is not the identity matrix. Since ρ_k is irreducible, the theorem is proved.

Let us apply Theorem B to $Q = G^{ab}$. Since Q is abelian, each of its irreducible complex representations is one-dimensional. Hence Theorem B in this case asserts that for every $e \neq q$ in Q there is a one-dimensional representation $\rho : Q \to \mathbb{C}^{\times}$ such that $\rho(q) \neq 1$.

Finally, given any $g \in G \setminus [G, G]$, let $q = \pi(g)$. If ρ is the representation from the previous paragraph (applied to q), then $\rho' = \rho \circ \pi$ is a a one-dimensional representation of G satisfying $\rho'(g) \neq 1$.

5. (6 pts) Let $G = D_{2n}$, the dihedral group of order $2n$. Let F be an algebraically closed field. Prove that every irreducible representation of G over F has dimension 1 or 2. **Note:** Partial credit for the case $F = \mathbb{C}$ will be given.

Solution: We start with a very general result:

Theorem C: *Let V be a representation of an arbitrary group H over any field, with $V \neq 0$. Then V has an irreducible subrepresentation W.*

Proof: If V is irreducible, set $W = V$; if not, V has a proper nonzero G-invariant subspace V_1. If V_1 is irreducible as a representation of G, set $W = V_1$; otherwise V_1 has a proper nonzero G-invariant subspace V_2 etc. Since $\dim(V) < \infty$, the process will terminate after finitely many steps.

Now let (ρ, V) be an irreducible representation of $G = D_{2n}$ over an algebraically closed field F, and let H be the rotation subgroup of G. By Theorem C, V has an H-invariant subspace W which is irreducible as a representation of H. Since H is abelian and F is algebraically closed, we must have $\dim(W) = 1$. On the other hand, since V is irreducible as a G-representation, by HW#9.1' we have $\dim(V) \leq dim(W)[G : H] = 1 \cdot 2 = 2$.

Remark: Since H is actually cyclic in this case, we could argue that V has a 1-dimensional H-invariant subspace without referring to Theorem C; namely, if h is a generator of H and $w \in V$ is any eigenvector of

$\rho(h)$ (which again exists since F is algebraically closed), then $W = Fw$ is 1-dimensional and H-invariant.

As suggested in the formulation of the problem, there is a completely different proof in the case $F = \mathbb{C}$ which uses our knowledge about the number and dimensions of irreducible complex representations (I$\mathbb{C}$R) of finite groups. In fact, there are two slightly different arguments.

In Lecture 15 we showed that $G = D_{2n}$ has 2 one-dimensional complex representations if n is odd and 4 such representations if n is even. The result of Problem #7.1 shows that G has at LEAST $\frac{n-1}{2}$ irreducible (pairwise non-equivalent) two-dimensional complex representations if n is odd and $\frac{n}{2} - 1$ such representations if n is even. In either case, the sum of squares of the dimensions of these one-dimensional and two-dimensional representations already equals $2n = |G|$, so we cannot have any additional I$\mathbb{C}$R (not equivalent to one of the above).

Alternatively, we can count the number of conjugacy classes in D_{2n}: there are $\frac{n+3}{2}$ conjugacy classes if n is odd and $\frac{n}{2}+3$ conjugacy classes if n is even. Thus, if n is odd, we have 2 one-dimensional and $\frac{n-1}{2}$ higher-dimensional I$\mathbb{C}$R and if n is even, we have 4 one-dimensional and $\frac{n-2}{2}$ higher-dimensional I$\mathbb{C}$R. On the other hand, the sum of squares of the dimensions of higher-dimensional I$\mathbb{C}$R is equal to $2n - 2 = 2(n-1)$ if n is odd and $2n - 4 = 2(n-2)$ if n is even. In either case, this sum is 4 times the number of higher-dimensional I$\mathbb{C}$R, which is only possible if all higher-dimensional I$\mathbb{C}$R are two-dimensional.

6. (8 pts) Let G be the group of order 20 from HW#8.6. Explicitly construct a 4-dimensional irreducible complex representation of G (and prove that your representation has the required property).

First construction: Our goal is to construct a homomorphism $\varphi : G \to S_5$ and then compose it with the standard representation of S_5. Recall that G has a presentation $\langle x, y | x^4 = y^5 = 1, xyx^{-1} = y^2 \rangle$, so to construct φ we want to find $a, b \in S_5$ such that $a^4 = b^5 = 1, aba^{-1} = b^2$.

We do not know at this point if φ we are looking for should be injective, but we can always try to assume that it is. This means that a must have order 4 and b must have order 5, so a must be a 4-cycle and b must be a 5-cycle. Since all 5-cycles are conjugate, WOLOG assume that $b = (1,2,3,4,5)$. We also know that a must have a fixed point, and again WOLOG assume that $a(1) = 1$.

By the conjugation formula $aba^{-1} = (a(1), a(2), a(3), a(4), a(5)) = (1, a(2), a(3), a(4), a(5))$. On the other hand, $b^2 = (1,3,5,2,4)$, so $aba^{-1} = b^2$ holds if and only if $a(2) = 3$, $a(3) = 5$, $a(5) = 4$ and $a(4) = 2$, that is, $a = (2,3,5,4)$.

The obtained elements $a = (2,3,5,4)$ and $b = (1,2,3,4,5)$ satisfy all the required relations, so there exists a homomorphism $\varphi : G \to S_5$ with $\varphi(x) = a$ and $\varphi(y) = b$. If χ_{std} is the character of the standard representation ρ_{std} of S_5, then $\chi_{std}(a) = \chi_{std}(a^2) = \chi_{std}(a^3) = 0$ and $\chi_{std}(b) = -1$. Hence if χ is the character of the representation $\rho_{std} \circ \varphi$ of G, then $\chi(x) = \chi(x^2) = \chi(x^3) = 0$, $\chi(y) = -1$ (and of course, $\chi(e) = 4$). By the character table of G computed in HW#8.6, χ is the character of the unique 4-dimensional I$\mathbb{C}$R of G, which finishes the proof.

Second construction: This time we want to find directly a homomorphism $\rho : G \to \mathrm{GL}(V)$ where V is a 4-dimensional complex vector space. Thus we are looking for matrices A, B satisfying $A^4 = B^5 = I$ and $ABA^{-1} = B^2$.

By an anticipated analogy with the Heisenberg group, let us assume that V has basis $e_{[0]}, e_{[1]}, e_{[2]}, e_{[3]}$ (where the subscripts are integers mod 4), that A cyclically shifts the basis vectors, that is, $Ae_{[k]} = e_{[k+1]}$ (this ensures that $A^4 = 1$) and that B acts diagonally in this basis, that is, $Be_{[k]} = \lambda_k e_{[k]}$ for some $\lambda_k \in \mathbb{C}$. Then $B^5 = I \iff \lambda_{[k]}^5 = 1$ for each k.

With these restrictions, the relation $ABA^{-1} = B^2$ holds if and only if $\lambda_{[k-1]} = \lambda_{[k]}^2$ for each k. If we set, $\omega = e^{\frac{2\pi i}{5}}$ and $\lambda_{[3]} = \omega$, then recursively we get $\lambda_{[2]} = \omega^2$, $\lambda_{[1]} = \omega^4$ and $\lambda_{[0]} = \omega^8 = \omega^3$. Finally, we need to check that the equation holds for $k = 0$ (and fortunately it does): $\lambda_{[0]}^2 = \omega^6 = \omega = \lambda_{[3]} = \lambda_{[-1]}$ (recall that the subscripts are mod 4).

Thus, A defined above and $B = \mathrm{diag}(\omega^3, \omega^4, \omega^2, \omega)$ satisfy the desired relations. It is clear that $\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = \mathrm{Tr}(A^3) = 0$ (since A, A^2 and A^3 will have zeroes on the diagonal) and $\mathrm{Tr}(B) = \sum_{k=1}^4 \omega^k = -1$ (since the sum of all n^{th} roots of unity is equal to 0 for any $n > 1$). Thus, we constructed a representation $\rho : G \to \mathrm{GL}(V)$ which has the desired character.

Closing remark: While both constructions above are rather ad hoc, there are conceptual ways to obtain either representation (with the actual matrices we found above, not just up to equivalence). The representation from the first construction is (naturally equivalent to) the augmentation representation described in HW#10.4. And the representation from the second construction can be naturally obtained as an induced representation (see HW#11.1.)