Solutions to Homework #11.

1. Let G be the group from Homework#8.6, and let $H = \langle y \rangle$ (in the notations of #8.6), that is, H is the subgroup of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Let $\omega \neq 1$ be a 5^{th} root of unity. Let $(\rho_\omega, \mathbb{C})$ be the representation of H on $\mathbb{C}$ where y acts as multiplication by ω, and let (ρ'_ω, Y) be the induced representation of G.

(a) Explicitly compute the matrices $\rho'_\omega(x)$ and $\rho'_\omega(y)$ with respect to a natural basis of Y (here x and y are the generators of G defined in #8.6).

(b) Use the irreducibility criterion from Lecture 25 to show that ρ'_ω is irreducible.

(c) Deduce from (b) that if we start with two non-equivalent representations of H, where H is a subgroup of a group G, then the corresponding induced representations of G may be equivalent.

Solution: (a) The elements e, x, x^2, x^3 are representatives of left cosets of H in G. Thus, according to the definition in class $Y = \bigoplus_{k=0}^{3} x^k \otimes \mathbb{C}$, and $\beta = \{e \otimes 1, x \otimes 1, x^2 \otimes 1, x^3 \otimes 1\}$ is a basis of Y.

Since left multiplication by x cyclically permutes the elements e, x, x^2 and x^3, we have $[\rho'_\omega(x)]_\beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

To compute $[\rho'_\omega(y)]_\beta$, we need to write each of the elements yx^i in the from $x^{j(i)}h_i$ with $h_i \in H$ or, equivalently, $x^{j(i)}y^{k(i)}$. Using the relation $xyx^{-1} = y^2$ (along with $x^4 = y^5 = 1$), we get $yx = xy^3$, $yx^2 = x^2y^4$ and $yx^3 = x^3y^2$. Also $ye = ey$. Hence, by definition of the induced representation we have

$$\begin{array}{lllllll}
\rho'_\omega(y)(e \otimes 1) & = & e \otimes \rho(y)1 & = & e \otimes \omega & = & \omega(e \otimes 1) \\
\rho'_\omega(y)(x \otimes 1) & = & x \otimes \rho(y^3)1 & = & x \otimes \omega^3 & = & \omega^3(x \otimes 1) \\
\rho'_\omega(y)(x^2 \otimes 1) & = & x^2 \otimes \rho(y^4)1 & = & x^2 \otimes \omega^4 & = & \omega^4(x^2 \otimes 1) \\
\rho'_\omega(y)(x^3 \otimes 1) & = & x^3 \otimes \rho(y^2)1 & = & x^3 \otimes \omega^2 & = & \omega^2(x^3 \otimes 1)
\end{array}$$

Hence $[\rho'_\omega(y)]_\beta = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^3 & 0 & 0 \\ 0 & 0 & \omega^4 & 0 \\ 0 & 0 & 0 & \omega^2 \end{pmatrix}$.

(b) The subgroup H is normal in G. This can be seen easily from the matrix realization of G. Alternatively, we can prove that H is normal by computing its normalizer.

In general, if H is a subgroup of a group G, the *normalizer of H in G* is $N_G(H) = \{g \in G : gHg^{-1} = H\}$. The normalizer $N_G(H)$ is always a subgroup containing H, and H is normal if and only if $N_G(H) = H$.

In our case the relations $xyx^{-1} = y^2$ and $y^5 = 1$ imply that $xHx^{-1} = H$, so $x \in N_G(H)$. Since $N_G(H)$ is a subgroup containing x and G is generated by H and x, we conclude that $N_G(H) = H$.

Once we know that H is normal, we can prove that ρ'_ω is irreducible using Proposition 25.4. According to it, we just need to show that the representations $\rho_\omega^{x^i}$ of H are pairwise inequivalent for $i = 0, 1, 2, 3$. Recall that by definition $\rho_\omega^{x^i}(h) = \rho_\omega(x^i h x^{-i})$ for all $h \in H$. In particular (thinking of ρ_ω as a homomorphism from H to $\mathbb{C}$), we get $\rho_\omega^{x^0}(y) = \rho_\omega(y) = \omega$, $\rho_\omega^{x^1}(y) = \rho_\omega(xyx^{-1}) = \rho_\omega(y^2) = \omega^2$, $\rho_\omega^{x^2}(y) = \rho_\omega(x^2yx^{-2}) = \rho_\omega(y^4) = \omega^4$, $\rho_\omega^{x^3}(y) = \rho_\omega(x^3yx^{-3}) = \rho_\omega(y^8) = \omega^8 = \omega^3$.

Since the representations $\rho_\omega^{x^i}$, $i = 0, 1, 2, 3$ are one-dimensional and send y to different elements, we conclude that they are pairwise non-equivalent.

(c) Let $\omega_1 \neq 1$ and $\omega_2 \neq 1$ be two distinct 5^{th} roots of unity. The representations ρ_{ω_1} and ρ_{ω_2} are non-equivalent (since they are one-dimensional and $\rho_{\omega_1}(y) \neq \rho_{\omega_2}(y)$). Part (b) implies that the corresponding induced representations ρ'_{ω_1} and ρ'_{ω_2} are both irreducible (and both 4-dimensional). But by HW#8.6, G has unique 4-dimensional I$\mathbb{C}$R, up to equivalence. Hence $\rho'_{\omega_1} \cong \rho'_{\omega_2}$.

2. Let G be a finite group and H a subgroup of G. Let ρ be a representation of H and let ρ' be the induced representation of G.

(a) Use Lemma 25.1 to show that $\chi_{\rho'}(g) = \frac{1}{|H|} \sum\limits_{x \in T(g)} \chi_\rho(x^{-1}gx)$ where $T(g) = \{x \in G : x^{-1}gx \in H\}$.

(b)* Use (a) to prove the Frobenius Reciprocity Theorem:
Frobenius Reciprocity Theorem: *Let G be a finite group, H a subgroup of G, let ρ be a complex representation of H and π a complex representation of G. Then*

$$\langle \chi_{Ind\uparrow_H^G \rho}, \chi_\pi \rangle = \langle \chi_\rho, \chi_{\mathrm{Res}\downarrow_H^G \pi} \rangle.$$

Solution: (a) Let $k = [G : H]$, and choose elements $g_1, \ldots, g_k \in G$ such that $G = \sqcup_{i=1}^k g_i H$. Recall that in class we proved the following formula:

$$\chi_{\rho'}(g) = \sum_{i:g_i^{-1}gg_i \in H} \chi_\rho(g_i^{-1}gg_i).$$

Let us show that $\frac{1}{|H|}\sum\limits_{x\in T(g)} \chi_\rho(x^{-1}gx)$ is equal to the right-hand side of the above formula.

Since each $x \in G$ can be uniquely written as $g_i h$ with $1 \le i \le k$ and $h \in H$, we have

$$\frac{1}{|H|}\sum_{x\in T(g)} \chi_\rho(x^{-1}gx) = \sum_{i=1}^{k} \sum_{h\in H:(g_ih)^{-1}g(g_ih)\in H} \chi_\rho((g_ih)^{-1}g(g_ih)). \tag{$***$}$$

Let us now fix i and simplify the interior sum above. Note that $(g_ih)^{-1}g(g_ih) = h^{-1}(g_i^{-1}gg_i)h$. This implies that

(i) $(g_ih)^{-1}g(g_ih) \in H \iff g_i^{-1}gg_i \in hHh^{-1} = H$ (the last equality holds since $h \in H$)

(ii) If $g_i^{-1}gg_i \in H$, then $\chi_\rho((g_ih)^{-1}g(g_ih)) = \chi_\rho(g_i^{-1}gg_i)$ since

$$\begin{aligned}\chi_\rho((g_ih)^{-1}gg_ih) &= \mathrm{Tr}(\rho(h^{-1}(g_i^{-1}gg_i)h)) \\ &= \mathrm{Tr}(\rho(h)^{-1}\rho(g_i^{-1}gg_i)\rho(h)) = \mathrm{Tr}(\rho(g_i^{-1}gg_i)) = \chi_\rho(g_i^{-1}gg_i).\end{aligned}$$

Note that we cannot write $\mathrm{Tr}(\rho(g_i^{-1}gg_i)) = \mathrm{Tr}(\rho(g_i)^{-1}\rho(g)\rho(g_i))$ since both g and g_i may lie outside of H, in which case $\rho(g_i)$ and $\rho(g)$ are not even defined.

Properties (i) and (ii) imply that if $g_i^{-1}gg_i \notin H$, the interior sum in (***) is equal to 0 (as each term is equal to 0), and if $g_i^{-1}gg_i \in H$, then each term in the interior sum is $\chi_\rho(g_i^{-1}gg_i)$, and there are precisely $|H|$ terms, so the interior sum is $|H|\chi_\rho(g_i^{-1}gg_i)$.

Hence the right-hand side of (***) equals $\frac{|H|}{|H|}\sum\limits_{i:g_i^{-1}gg_i\in H} \chi_\rho(g_i^{-1}gg_i)$, as desired.

(b) Using (a), we have

$$\begin{aligned}\langle \chi_{Ind\uparrow_H^G \rho}, \chi_\pi\rangle &= \frac{1}{|G|}\sum_{g\in G} \overline{\chi_{Ind\uparrow_H^G\rho}(g)}\chi_\pi(g) \\ &= \frac{1}{|G||H|}\sum_{g\in G}\sum_{x\in G:\ x^{-1}gx\in H} \overline{\chi_\rho(x^{-1}gx)}\chi_\pi(g)\end{aligned}$$

Since π is a representation of G, we have $\chi_\pi(g) = \chi_\pi(x^{-1}gx)$, so the last expression can be rewritten as a double sum

$$\frac{1}{|G||H|} \sum_{(x,g)\in G\times G\colon\, x^{-1}gx\in H} \overline{\chi_\rho(x^{-1}gx)}\chi_\pi(x^{-1}gx).$$

Each term in the above sum is equal to $\overline{\chi_\rho(h)}\chi_\pi(h)$ for some $h \in H$, and the entire sum is equal to $\frac{1}{|G||H|}\sum\limits_{h\in H} n(h)\overline{\chi_\rho(h)}\chi_\pi(h)$ where $n(h)$ is the number of pairs $(x, g) \in G \times G$ such that $x^{-1}gx \in H$.

We claim that $n(h) = |G|$ (regardless of h). Indeed, if h is fixed, then $x^{-1}gx = h \iff g = xhx^{-1}$, so for every $x \in G$ there exists unique $g \in G$ with $x^{-1}gx = h$. Therefore,

$$\begin{aligned}\langle \chi_{Ind\uparrow_H^G \rho}, \chi_\pi\rangle &= \frac{1}{|G||H|}\sum_{h\in H} |G|\overline{\chi_\rho(h)}\chi_\pi(h)\\ &= \frac{1}{|H|}\sum_{h\in H} \overline{\chi_\rho(h)}\chi_{Res\downarrow_H^G \pi}(h) = \langle \chi_\rho, \chi_{\mathrm{Res}\downarrow_H^G \pi}\rangle.\end{aligned}$$

3. Let G be a group, H a subgroup of finite index and G/H the set of left cosets of H in G. Recall that G has a natural action on $X = G/H$ given by

$$g.(xH) = (gx)H.$$

Let F be a field and (π, FX) the permutation representation of G corresponding to this action. Prove that π is equivalent to the induced representation $\mathrm{Ind}\uparrow_H^G \rho_{triv}$ where $(\rho_{triv}, F))$ is the trivial representation of H. **Note:** You may want to start with the special case $H = \{e\}$, the trivial subgroup. In this special case the problem asserts that

Theorem: *The regular representation of a finite group G is induced from the trivial representation of the trivial subgroup.*

Solution: As in Problem 2, let $k = [G : H]$, and choose elements $g_1, \ldots, g_k \in G$ such that $G = \sqcup_{i=1}^k g_i H$. Then $X = \{g_i H : 1 \le i \le k\}$, so $FX = \oplus_{i=1}^k F(g_i H)$.

On the other hand, the representation space of the induced representation $\mathrm{Ind}\uparrow_H^G \rho_{triv}$ is $V' = \bigoplus\limits_{i=1}^k g_i H \otimes F$. Let $T : V' \to FX$ be the unique linear map given by

$$T(g_i H \otimes 1) = g_i H.$$

Since T sends a basis of V' to a basis of FX, it is an isomorphism of vector spaces. It remains to prove that T is a homomorphism of representations. We need to show that

$$(T \circ \rho'(g))(v) = (\pi(g) \circ T)(v) \text{ for all } g \in G, v \in V'. \qquad (***)$$

Here ρ' is the induced representation $\mathrm{Ind}\uparrow_H^G \rho_{triv}$ and recall that π denotes the permutation representation of G on FX.

Since V' is spanned by the set $\{g_i H \otimes 1 : 1 \leq i \leq k\}$, it suffices to check (***) for v of this form. We have

$$(\pi(g) \circ T)(g_i H \otimes 1) = \pi(g)(T(g_i H \otimes 1)) = \pi(g)(g_i H) = g g_i H.$$

To compute $(T \circ \rho'(g))(g_i H \otimes 1)$, we write gg_i in the form $g_{j(i)} h_i$ with $h_i \in H$. Then $\rho'(g)(g_i H \otimes 1) = g_{j(i)} H \otimes \rho_{triv}(h_i) 1 = g_{j(i)} H \otimes 1$, so

$$(T \circ \rho'(g))(g_i H \otimes 1) = T(g_{j(i)} \otimes 1) = g_{j(i)} H.$$

Finally, since $h_i H = H$, we have $g_{j(i)} H = g_{j(i)}(hH) = (g_{j(i)} h) H = (gg_i) H$, which completes the proof.

4. Use the Frobenius Reciprocity Theorem and Problem 3 to give another proof of Proposition 21.3 from class which asserts that if V is an I$\mathbb{C}$R of a finite group G, then the multiplicity of V in the regular representation is equal to $\dim(V)$. You are allowed to use Proposition 21.2.

Solution: By Proposition 21.3, the multiplicity of V in the regular representation $\mathbb{C}[G]$ is equal to $\langle \chi_{\mathbb{C}[G]}, \chi_V \rangle$. By Problem 3, $\mathbb{C}[G] = \mathrm{Ind}\uparrow_{\{e\}}^G V_{triv}$ where V_{triv} is the trivial representation of the trivial subgroup $\{e\}$ of G. Thus, by Frobenius reciprocity we have

$$\langle \chi_{\mathbb{C}[G]}, \chi_V \rangle = \langle \chi_{\mathrm{Ind}\uparrow_{\{e\}}^G V_{triv}}, \chi_V \rangle = \langle \chi_{V_{triv}}, \chi_{\mathrm{Res}\downarrow_{\{e\}}^G V} \rangle.$$

The last expression is the inner product of two characters of the trivial subgroup and thus equals

$$\chi_{V_{triv}}(e) \chi_V(e) = \dim(V_{triv}) \dim(V) = \dim(V).$$

5. The goal of this problem is to justify the conceptual representation of the induced representation sketched at the end of Lecture 25. Recall the setup. Let G be a finite group, H a subgroup of G and (ρ, V) a representation of H over a field F. Let $W = F[G] \otimes V$. Define the representation (π, W) of G by $\pi(g)(a \otimes v) = (ga) \otimes v$.

(a) Prove that π is well-defined and that π is indeed a representation.
(b) Let Y be the subspace of W spanned by all elements of the form $(ah) \otimes v - a \otimes (\rho(h)v)$ with $a \in F[G], h \in H$ and $v \in V$.

Prove that Y is $\pi(G)$-invariant. Deduce that there is a well-defined representation $(\overline{\pi}, W/Y)$ of G given by $\overline{\pi}(g)(w+Y) = \pi(g)(w) + Y$.

(c) Now let $g_1, \ldots, g_k$ be a system of representatives of the left cosets of H in G and let $V' = \bigoplus_{i=1}^{k} g_i \otimes V$. Let $f : V' \to W/Y$ be the map obtained by composing the natural inclusion of V' into W with the natural projection $W \to W/Y$. Prove that f is an isomorphism of representations between the induced representation $(\mathrm{Ind}{\uparrow}_H^G \rho, V')$ and $(\overline{\pi}, W/Y)$

Solution: Part (b) is straightforward, so we will only give solutions to (a) and (c).

(a) We need to show that for every $g \in G$ there exists a well-defined linear map $\pi(g) : W \to W$ which acts on simple tensors in the desired way and that (π, W) is a representation of G. The second part is straightforward, and the first part could be done in the standard way using the universal property of tensor products (Theorem 10.1). However, we can avoid all these verifications and instead recognize π as a tensor product of two representations of G (which we know is well defined by HW#6.4).

Indeed, let $(\rho_{reg}, F[G])$ be the regular representation of G and let (ρ_{triv}, V) be the trivial representation of G on V (that is, $\rho_{triv}(g)v = v$ for all $g \in G$, $v \in V$). Note that so far we are not using the given representation of H on V. The tensor product of $(\rho_{reg}, F[G])$ and (ρ_{triv}, V) is the representation $(\rho_{reg} \otimes \rho_{triv}, W)$ where

$$((\rho_{reg} \otimes \rho_{triv})(g))(a \otimes v) = (\rho_{reg}(g))(a) \otimes (\rho_{triv}(g))(v) = (ga) \otimes v.$$

Thus, if we set $\pi = \rho_{reg} \otimes \rho_{triv}$, then π has the required property.

(c) First we show that f is surjective. As suggested in the hint, let $\overline{a \otimes v}$ denote the natural image of $a \otimes v$ in W/Y, that is, $\overline{a \otimes v} = a \otimes v + Y$. Then $\overline{ah \otimes v} = \overline{a \otimes \rho(h)v}$, with $h \in H$.

Clearly, W is spanned by elements of the form $g \otimes v$ with $g \in G$ and $v \in V$, so it suffices to show that $\mathrm{Im}(f)$ contains $\overline{g \otimes v} = g \otimes v + Y$ for all $g \in G$ and $v \in V$. Any $g \in G$ can be written as $g = g_i h$ with $h \in H$. Then $\overline{g \otimes v} = \overline{g_i h \otimes v} = \overline{g_i \otimes \rho(h)v} = f(g_i \otimes \rho(h)v) \in \mathrm{Im}(f)$.

Once we know that f is surjective, to prove that is bijective, it suffices to show that $\dim(V') \le \dim(W/Y) = \dim(W) - \dim(Y)$ or, equivalently, $\dim(Y) \le \dim(W) - \dim(V')$. We know that $\dim(V') = [G : H]\dim(V)$ and $\dim(W) = \dim(F[G] \otimes V) = |G|\dim(V)$. Thus, we need to show that

$$\dim(Y) \leq (|G| - [G:H])\dim(V) = [G:H](|H|-1)\dim(V).$$

Let β be any basis of V. We claim that Y is spanned by the set $B = \{g_i h \otimes v - g_i \otimes \rho(h)v : 1 \leq i \leq k = [G:H], h \in H\backslash\{e\}, v \in \beta\}$. This will finish the proof since B clearly has (at most) $[G:H](|H|-1)\dim(V)$ elements.

Recall that by definition Y is spanned by elements of the form

$$(ah) \otimes v - a \otimes (\rho(h)v) \text{ with } a \in F[G], h \in H.$$

Moreover, by linearity it suffices to take $a \in G$ and $v \in \beta$ (by linearity). Thus, it remains to show that $(gh) \otimes v - g \otimes (\rho(h)v) \in \mathrm{Span}(B)$ for all $g \in G$, $h \in H$ and $v \in \beta$.

Take any $g \in G, h \in H$, and write $g = g_i h'$ with $h' \in H$. Since $\rho(h'h)) = \rho(h')\rho(h)$, we have

$$\begin{aligned}
(gh) \otimes v - g \otimes (\rho(h)v) &= (g_i h' h) \otimes v - (g_i h') \otimes (\rho(h)v) \\
&= (g_i h' h) \otimes v - g_i \otimes \rho(h'h)v + g_i \otimes \rho(h')\rho(h)v - (g_i h') \otimes (\rho(h)v) \\
&= ((g_i h' h) \otimes v - g_i \otimes \rho(h'h)v) - ((g_i h') \otimes (\rho(h)v) - g_i \otimes \rho(h')\rho(h)v).
\end{aligned}$$

The last expression is a difference of two elements each of which either lies in B or equals 0 (depending on whether $h'h \neq e$ and $h' \neq e$, respectively). In any case, the difference of these elements lies in the span of B, as desired.