Practice Problems for Exam 1

0. Problems from the book (note that many problems in the following list were previously assigned as practice problems): 1.4.6, 1.4.13, 1.5.1, 1.5.2, 1.5.3, 1.6.9, 1.6.11, 2.1.9, 2.1.11, 2.1.15, 2.2.2, 2.2.9, 2.2.15, 3.1.6, 3.2.1, 3.2.3, 3.2.9, 3.2.11.

1. Let A be a set, and let $S(x)$ be a statement depending on a free variable x (and no other free variables) where x ranges over A.

(i) Find all sets A for which the implication

$$(\forall x \in A \quad S(x)) \Rightarrow (\exists x \in A \quad S(x))$$

is true, regardless of the statement $S(x)$

(ii) Now find all sets A for which the implication

$$(\exists x \in A \quad S(x)) \Rightarrow (\forall x \in A \quad S(x))$$

is true, regardless of the statement $S(x)$.

2. Prove that $4^n > 3^n + n$ for all integers $n \geq 2$

3. Let $\{f_n\}$ be the Fibonacci sequence defined as follows:

$$f_1 = f_2 = 1 \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3.$$

Use induction to prove the following identities:

(i) $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all $n \in \mathbb{N}$

(ii) $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for all $n \in \mathbb{N}$

4. Prove that $\sqrt{2} + \sqrt{6}$ is irrational.

5. Let $a, b, c \in \mathbb{Z}$, and assume that $a \mid b$ and $b \mid c$. Prove that $a \mid c$ directly using definition of divisibility.

6. Let $a, b, c \in \mathbb{Z}$, and assume that $c \mid ab$. Is it always true that $c \mid a$ or $c \mid b$? If the answer is yes, prove it; if the answer is no, give a specific counterexample.

7. Let $m, n, a, b \in \mathbb{Z}$ be such that

$$am + bn = 3.$$

List all natural numbers which are possible values of $gcd(a, b)$.

For every number you listed, show that this number is indeed a possible value of $gcd(a, b)$ by giving a specific example. For all other natural numbers prove that they cannot equal $gcd(a, b)$.