**Homework #1. Solutions to selected problems.**

**2.** In each of the following examples determine if $H$ is a bilinear form on $V$ (make sure to justify your answer):

(a) $V = Mat_n(F)$ for some field $F$ and $n \in \mathbb{N}$ and $H(A,B) = AB$.

(b) $V = Mat_n(F)$ for some field $F$ and $n \in \mathbb{N}$ and $H(A,B) = (AB)_{1,1}$ (the (1,1)-entry of the matrix $AB$).

(c) $V = F^n$ for some field $F$ and $n \in \mathbb{N}$ and $H((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = x_1 + y_1$.

**Solution:** (a) If $n = 1$, this is a bilinear form. If $n > 1$, this is a bilinear MAP, but not a bilinear form since its values are not scalars (do not lie in the field $F$).

(b) this is bilinear which can be checked by straightforward verification.

(c) This is not bilinear. If it were bilinear, for all $x \in V$ we would have had $H(0,x) = H(0+0,x) = H(0,x) + H(0,x)$ and hence $H(0,x) = 0$. But the latter is false, e.g. because $H(0,e_1) = 1 \neq 0$.

**4.** Let $F$ be any field, $n \in \mathbb{N}$ and $V = Mat_n(F)$, the vector space of $n \times n$ matrices over $F$. Let $e_{ij}$ be the matrix whose $(i,j)$-entry is equal to 1 and all other entries are 0. Then $\beta = \{e_{ij} : 1 \leq i,j \leq n\}$ is a basis of $V$ (you do not need to verify this). Define $H : V \times V \to F$ by

$$H(A,B) = Tr(AB^T)$$

(where $B^T$ is the transpose of $B$).

**Solution:** Bilinearity is straightforward. By direct computation $H(e_{ij}, e_{kl}) = Tr(e_{ij}e_{lk}) = Tr(\delta_{ik}e_{jl}) = \delta_{ik}\delta_{jl}$. Thus, $H(e_{ij}, e_{kl}) = 1$ if $(i,j) = (k,l)$ as pairs and $H(e_{ij}, e_{kl}) = 0$ otherwise. Hence $[H]_\beta = I_{n^2}$, the identity $n^2 \times n^2$-matrix. Since this matrix is symmetric, the form $H$ is also symmetric (it is also not hard to check the latter directly).

**5.** Let $F$ be a field with $\mathrm{char}(F) \neq 2$, let $V$ be a finite-dimensional vector space over $F$, and let $H$ be a bilinear form on $V$. Prove that $H$ can be **uniquely** written as $H = H^+ + H^-$ where $H^+$ is a symmetric bilinear form on $V$ and $H^-$ is an antisymmetric bilinear form on $V$.

**Solution:** Define $H^+(x,y) = \frac{1}{2}(H(x,y) + H(y,x))$ and $H^-(x,y) = \frac{1}{2}(H(x,y) - H(y,x))$. Note that we can divide by 2 precisely because $\mathrm{char}(F) \neq 2$. Then it is clear that $H^+$ is symmetric, $H^-$ is antisymmetric and $H = H^+ + H^-$. This proves existence.

For uniqueness assume that we have two representations $H = S_1 + A_1 = S_2 + A_2$ where $S_1$ and $S_2$ are symmetric and $A_1$ and $A_2$ are antisymmetric. Then $S_1 - S_2 = A_2 - A_1$, so the form $G = S_1 - S_2 = A_2 - A_1$ is both symmetric (being the difference of two symmetric forms) and antisymmetric (being the difference of two antisymmetric forms). Then for all $x, y$ we have $G(x,y) = G(y,x) = -G(x,y)$ which implies that $G(x,y) = 0$ (again using that $\mathrm{char}(F) \neq 2$). Thus $G$ is identically zero and hence $S_1 = S_2$ and $A_1 = A_2$.

**6.** Let $F$ be any field and $n \in \mathbb{N}$.

(a) Let $V = F^n$ (the standard $n$-dimensional vector space over $F$). Let $D : V \times V \to F$ be the dot product form. Prove that $D$ is left non-degenerate.

(b) Now $V$ be any $n$-dimensional vector space over $F$, $\beta$ an ordered basis for $V$ and $H$ a bilinear form on $V$. Prove that $H$ is left non-degenerate if and only if $[H]_\beta$ (the matrix of $H$ with respect to $\beta$) is invertible.

**Solution:** (a) Take any $0 \neq x = (x_1, \ldots, x_n) \in V$ and choose any $i$ such that $x_i \neq 0$. Then $D(x, e_i) = x_i \neq 0$, so $D$ is left non-degenerate.

(b) "$\Rightarrow$" We argue by contrapositive (we want to show that if $[H]_\beta$ is not invertible, then $H$ is left degenerate). Suppose that $[H]_\beta$ is not invertible. Then $[H]_\beta^T$ (the transposed matrix) is also not invertible. By elementary linear algebra we know that there exists $0 \neq v \in V$ such that $[H]_\beta^T [v]_\beta = 0$. Transposing both sides of this equation, we get $[v]_\beta^T [H]_\beta = 0$ (note that the 0 on the right is now a row vector). But then $H(v,w) = [v]_\beta^T [H]_\beta [w]_\beta = 0$ for all $w \in V$, so $H$ is left degenerate.

"$\Leftarrow$" Suppose that $[H]_\beta$ is invertible. Take any $0 \neq v \in V$. Then $[v]_\beta \neq 0$ as well. Since the dot product $D$ is left non-degenerate by (a), there exists $y \in V$ such that $D([v]_\beta, [y]_\beta) \neq 0$. Since $[H]_\beta$ is invertible, it is surjective as a map from $F^n$ to $F^n$, so we can find $w \in V$ such that $[H]_\beta [w]_\beta = [y]_\beta$. Thus, $D([v]_\beta, [H]_\beta [w]_\beta) \neq 0$. Since $D([v]_\beta, [H]_\beta [w]_\beta) = [v]_\beta^T [H]_\beta [w]_\beta = H(v,w)$, this implies that $H$ is left non-degenerate.

**7.** Let $F$ be any field, $n \in \mathbb{N}$, $V = F^n$ and $\{e_1, \ldots, e_n\}$ the standard basis of $V$. Define $\rho : S_n \to GL(V)$ by $(\rho(g))(e_i) = e_{g(i)}$. As discussed in Lecture 1, the pair $(\rho, V)$ is a representation of $S_n$.

(a) Let $V_0$ be the subspace of $V$ consisting of all vectors whose sum of coordinates is equal to 0:

$$V_0 = \{(x_1, \ldots, x_n) \in V : x_1 + \ldots + x_n = 0\}.$$

Prove that $V_0$ is an $S_n$-invariant subspace of $V$, and therefore $(\rho, V_0)$ is also a representation of $S_n$.

(b) Now prove that the representation $(\rho, V_0)$ is irreducible, that is, if $W$ is any $S_n$-invariant subspace of $V_0$, then $W = 0$ or $W = V_0$.

**Solution:** (a) Let us see how $\rho(g)$ acts on an arbitrary element of $V$. Take $x = (x_1, \ldots, x_n) \in V$. Then $x = \sum_{i=1}^n x_i e_i$, so $(\rho(g))(x) = \sum_{i=1}^n x_i (\rho(g)) e_i = \sum_{i=1}^n x_i e_{g(i)}$. Hence, the $i^{\text{th}}$ coordinate of $(\rho(g))(x)$ is equal to $x_j$ where $j$ is the unique integer such that $g(j) = i$; in other words $j = g^{-1}(i)$. Thus,

$$(\rho(g))((x_1, \ldots, x_n)) = (x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}).$$

In other words, $\rho(g)$ permutes the coordinates of every $x \in V$, so if the sum of coordinates of $x$ is 0, it will still be 0 after permutation. Thus, $V_0$ is $S_n$-invariant.

(b) The assertion is false in general – we need to assume that $\text{char}(F)$ does not divide $n$. This is equivalent to saying that $n \neq 0$ in $F$.

Let $W$ be any $S_n$-invariant subspace of $V_0$. We shall assume that $W \neq 0$ and deduce that $W = V_0$. Since $W \neq 0$, we can choose a nonzero vector $x = (x_1, \ldots, x_n) \in W$. We claim that at least two coordinates of $x$ are different. Indeed, suppose all coordinates of $x$ are equal to each other. Then the sum of the coordinates is $nx_1$. Since $x \in V_0$, we have $nx_1 = 0$, and since $n \neq 0$ in $F$ (by the extra assumption), we get $x_1 = 0$, which then forces $x = 0$, a contradiction.

Thus, there exist $i \neq j$ such that $x_i \neq x_j$. Let $g = (i, j)$, the transposition that swaps $i$ and $j$. An easy computation then shows that $(\rho(g))(x) - x = (x_i - x_j)(e_j - e_i)$. Since $W$ is $S_n$-invariant, we must have $(x_i - x_j)(e_j - e_i) \in W$, and since $x_i \neq x_j$, dividing by $x_i - x_j$, we get $e_j - e_i \in W$.

Now given any $k \neq l$, with $1 \leq k, l \leq n$, we can find $f \in S_n$ such that $f(j) = k$ and $f(i) = l$. Then $(\rho(f))(e_j - e_i) = e_{f(j)} - e_{f(i)} = e_k - e_l$. Thus, $W$ contains all elements of the form $e_k - e_l$. These elements span $V_0$ (for instance, any $(x_1, \ldots, x_n) \in V_0$ can be written as $\sum_{i=1}^{n-1} x_i(e_i - e_n)$ since $x_n$ must equal $-\sum_{i=1}^{n-1} x_i$). Thus $W$ contains a spanning set for $V_0$ and hence must equal $V_0$.