PROPERTY (T) FOR GROUPS GRADED BY ROOT SYSTEMS

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Abstract. We introduce and study the class of groups graded by root systems. We prove that if Φ is an irreducible classical root system of rank ≥ 2 and G is a group graded by Φ, then under certain natural conditions on the grading, the union of the root subgroups is a Kazhdan subset of G. As the main application of this theorem we prove that for any reduced irreducible classical root system Φ of rank ≥ 2 and a finitely generated commutative ring R with 1, the Steinberg group $\mathrm{St}_{\Phi}(R)$ and the elementary Chevalley group $\mathbb{E}_{\Phi}(R)$ have property (T). We also show that there exists a group with property (T) which maps onto all finite simple groups of Lie type and rank ≥ 2, thereby providing a "unified" proof of expansion in these groups.

Contents

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1. Introduction

1.1. **The main result.** In this paper by a *ring* we will always mean an associative ring with 1. In a recent work of the first two authors [EJ] it was shown that for any integer $n \geq 3$ and a finitely generated ring R, the elementary linear group $\mathrm{EL}_n(R)$ and the Steinberg group $\mathrm{St}_n(R)$ have Kazhdan's property (T) (in fact property (T)

for $\mathrm{EL}_n(R)$ is a consequence of property (T) for $\mathrm{St}_n(R)$ since $\mathrm{EL}_n(R)$ is a quotient of $\mathrm{St}_n(R)$). In this paper we extend this result to elementary Chevalley groups and Steinberg groups corresponding to other classical root systems of rank ≥ 2 (see Theorem 1.1 below).

We will use the term *root system* in a very broad sense (see § 4). By a *classical root system* we mean the root system of some semisimple algebraic group (such root systems are often called *crystallographic*).

If Φ is a reduced irreducible classical root system and R a commutative ring, denote by $\mathbb{G}_\Phi(R)$ the corresponding simply-connected Chevalley group over R and by $\mathbb{E}_\Phi(R)$ the *elementary subgroup* of $\mathbb{G}_\Phi(R)$, that is, the subgroup generated by the root subgroups with respect to the standard torus. For instance, if $\Phi = A_{n-1}$, then $\mathbb{G}_\Phi(R) = \mathrm{SL}_n(R)$ and $\mathbb{E}_\Phi(R) = \mathrm{EL}_n(R)$. For brevity we will refer to $\mathbb{E}_\Phi(R)$ as an *elementary Chevalley group.* There is a natural epimorphism from the Steinberg group $\mathrm{St}_\Phi(R)$ onto $\mathbb{E}_\Phi(R)$.

Theorem 1.1. *Let Φ be a reduced irreducible classical root system of rank ≥ 2. Let R be a finitely generated ring, which is commutative if Φ is not of type A_n. Then the Steinberg group $\mathrm{St}_\Phi(R)$ and the elementary Chevalley group $\mathbb{E}_\Phi(R)$ have Kazhdan's property (T).*

Remark. There are many cases when $\mathbb{E}_\Phi(R) = \mathbb{G}_\Phi(R)$. For instance, this holds if $R = \mathbb{Z}[x_1, \ldots, x_k]$ or $R = F[x_1, \ldots, x_k]$, where F is a field, and Φ is of type A_n (see [Su]) or C_n (see [GMV]), with $n \geq 2$.

Steinberg groups and elementary Chevalley groups over rings are typical examples of *groups graded by root systems* which are introduced and studied in this paper. Our central result asserts that if G is any group graded by a (finite) root system Φ of rank ≥ 2, the grading satisfies certain non-degeneracy condition, and $\{X_\alpha\}_{\alpha\in\Phi}$ are the root subgroups, then $\cup X_\alpha$ is a Kazhdan subset of G (see Theorem 1.2 below). Theorem 1.1 follows primarily from this result and relative property (T) for the pair $(\mathrm{St}_2(R) \ltimes R^2, R^2)$ established by Shalom when R is commutative [Sh1] and by the third author for general R [Ka1]; however, additional considerations are needed in the case when Φ is not simply laced. Before discussing the proofs of these results, we briefly comment on the previous work on property (T) for Chevalley and Steinberg groups and the proof of the main theorem in [EJ].

1.2. **Property (T) for $\mathrm{EL}_n(R)$: summary of prior work.** By the 1967 foundational paper of Kazhdan [Kazh] and the subsequent work of Vaserstein [Va], the Chevalley groups $\mathbb{G}_\Phi(\mathbb{Z}) = \mathbb{E}_\Phi(\mathbb{Z})$ and $\mathbb{G}_\Phi(F[t]) = \mathbb{E}_\Phi(F[t])$, where F is a finite field, have property (T) for any reduced irreducible classical root system of rank ≥ 2. The question of whether the groups $\mathbb{E}_\Phi(R)$, with $\mathrm{rk}(\Phi) \geq 2$ (in particular, the groups $\mathrm{EL}_n(R)$, $n \geq 3$) have property (T) for "larger" rings R remained completely open for a long time.[1]

In 2005, Kassabov and Nikolov [KN1] showed that the group $\mathrm{EL}_n(R)$, $n \geq 3$, has property (τ) (a certain weak form of property (T)) for any finitely generated commutative ring R, which gave an indication that these groups might also have

[1]The situation in rank 1 is completely different. It is easy to see that $\mathrm{EL}_2(R)$ does not have (T) whenever R (possibly noncommutative) surjects onto $\mathbb{Z}$ or $F[t]$, with F a finite field. A more delicate argument shows that $\mathrm{EL}_2(R)$ does not have (T) for any infinite commutative ring R (with 1).

property (T). This indication was partially confirmed by Shalom in 2006 who proved in [Sh2] that the groups $\mathrm{EL}_n(R)$ have property (T) whenever R is commutative and $n \geq \mathrm{Kdim}(R)+2$ (where $\mathrm{Kdim}(R)$ is the Krull dimension of R). In 2007, Vaserstein [Va2] eliminated this restriction on the Krull dimension by showing that $\mathrm{EL}_n(R)$, $n \geq 3$, has property (T) for any finitely generated commutative ring R. Finally, in [EJ] the result was extended to arbitrary finitely generated (associative) rings, and the method of proof was very different from the one used by Shalom and Vaserstein. To explain the idea behind this method, we recall some standard terminology.

Let G be a discrete group and S a subset of G. Following the terminology in [BHV], we will say that S is a *Kazhdan subset* of G if every unitary representation of G containing almost S-invariant vectors must contain a G-invariant vector. By definition, G has property (T) if it has a finite Kazhdan subset; however, one can prove that G has (T) by finding an infinite Kazhdan subset K such that the pair (G, K) has relative property (T) (see § 2 for details).

If $G = \mathrm{EL}_n(R)$, where $n \geq 3$ and R is a finitely generated ring, the aforementioned results of Shalom and Kassabov yield relative property (T) for the pair (G, X) where $X = \cup_{i\neq j} E_{ij}(R)$ is the union of root subgroups. Thus, establishing property (T) for G is reduced to showing that X is a Kazhdan subset. An easy way to prove the latter is to show that G is boundedly generated by X — this is the so-called bounded generation method of Shalom [Sh1]. However, G is known to be boundedly generated by X only in a few cases, namely, when R is a finite extension of $\mathbb{Z}$ or $F[t]$, with F a finite field. In [EJ], it was proved that X is a Kazhdan subset of $G = \mathrm{EL}_n(R)$ for any ring R using a different method, described in the next subsection.

1.3. **Almost orthogonality, codistance and a spectral criterion from** [EJ]**.** Suppose we are given a group G and a finite collection of subgroups $H_1, \ldots, H_k$ which generate G, and we want to know whether the union of these subgroups $X = \cup H_i$ is a Kazhdan subset of G. By definition, this will happen if and only if given a unitary representation V of G without (nonzero) invariant vectors, a unit vector $v \in V$ cannot be arbitrarily close to each of the subspaces V^{H_i} (where as usual V^K denotes the subspace of K-invariant vectors). In the simplest case $k = 2$ the latter property is equivalent to asserting that the angle between subspaces V^{H_1} and V^{H_2} must be bounded away from 0. For an arbitrary k, the closeness between the subspaces $V_1, \ldots, V_k$ of a Hilbert space V can be measured using the notion of a codistance introduced in [EJ]. We postpone the formal definition until § 2; here we just say that the codistance between $\{V_i\}$, denoted by $\mathrm{codist}(V_1, \ldots, V_k)$ is a real number in the interval $[1/k, 1]$, and in the above setting $\cup H_i$ is a Kazhdan subset of G if and only if $\sup\{\mathrm{codist}(V^{H_1}, \ldots, V^{H_k})\} < 1$ where V runs over all representations of G with $V^G = \{1\}$.

Let H and K be subgroups of the same group. We define $\sphericalangle(H, K)$, the angle between H and K, to be the infimum of the set $\{\sphericalangle(V^H, V^K)\}$ where V runs over all representations of $\langle H, K\rangle$ without invariant vectors; we shall also say that H and K are ε-orthogonal if $\cos \sphericalangle(H, K) \leq \varepsilon$. The idea of using such angles to prove property (T) already appears in 1991 paper of Burger [Bu] and is probably implicit in earlier works on unitary representations. However, this idea has not been fully exploited until the paper of Dymara and Januszkiewicz [DJ] which shows that for a group G generated by k subgroups $H_1, \ldots, H_k$, property (T) can be established by controlling "local information", the angles between H_i and H_j. More precisely, in [DJ] it is proved that if H_i and H_j are ε-orthogonal for $i \neq j$ for sufficiently small ε, then $\cup H_i$ is a Kazhdan subset of G (so if in addition each H_i is finite, then G has property (T)). This "almost orthogonality method" was generalized in [EJ] using the notion of codistance. As a result, a new spectral criterion for property (T) was obtained, which is applicable to groups with a given graph of groups decomposition, as defined below.

Let G be a group and Γ a finite graph. A decomposition of G over Γ is a choice of a vertex subgroup $G_\nu \subseteq G$ for each vertex ν of Γ and an edge subgroup $G_e \subseteq G$ for each edge e of Γ such that

(i) G is generated by the vertex subgroups $\{G_\nu\}$
(ii) Each vertex group G_ν is generated by edge subgroups $\{G_e\}$, with e incident to ν
(iii) If an edge e connects ν and ν', then G_e is contained in $G_\nu \cap G_{\nu'}$.

The spectral criterion [EJ, Theorem 5.1] asserts that if a group G has a decomposition $(\{G_\nu\}, \{G_e\})$ over a graph Γ, and for each vertex ν of Γ the codistance between subgroups $\{G_e\}$, with e incident to ν, is sufficiently small with respect to the spectral gap of Γ, then the codistance between the vertex subgroups $\{G_\nu\}$ is less than 1 (and thus the union of vertex subgroups is a Kazhdan subset of G).

1.4. **Groups graded by root systems and associated graphs of groups.** Property (T) for the groups $\mathbb{G}_\Phi(R)$ and $\mathrm{St}_\Phi(R)$ will be proved using certain generalization of the spectral criterion from [EJ]. First we shall describe the relevant graph decompositions of those groups, for simplicity concentrating on the case $G = \mathrm{EL}_n(R)$.

For each $n \geq 2$ consider the following graph denoted by $\Gamma(A_n)$. The vertices of $\Gamma(A_n)$ are labeled by the elements of the symmetric group $\mathrm{Sym}(n+1)$, and two vertices σ and σ' are connected if any only if they are not opposite to each other in the Cayley graph of $\mathrm{Sym}(n+1)$ with respect to the generating set $\{(12), (23), \ldots, (n, n+1)\}$.

Now let R be ring and $n \geq 3$. The group $G = \mathrm{EL}_n(R)$ has a natural decomposition over $\Gamma(A_{n-1})$ defined as follows. For each $\sigma \in \mathrm{Sym}(n)$ the vertex group G_σ is defined to be the subgroup of G generated by $\{X_{ij} : \sigma(i) < \sigma(j)\}$ where $X_{ij} = \{e_{ij}(r) : r \in R\}$. Thus, vertex subgroups are precisely the maximal unipotent subgroups of G normalized by the diagonal subgroup; in particular, the vertex subgroup corresponding to the identity permutation is the upper-unitriangular subgroup of $\mathrm{EL}_n(R)$. If e is the edge connecting vertices σ and σ', we define the edge subgroup G_e to be the intersection $G_\sigma \cap G_{\sigma'}$ (note that this intersection is non-trivial precisely when σ and σ' are connected in $\Gamma(A_{n-1})$).

As already discussed in the last paragraph of § 1.2, property (T) for $G = \mathrm{EL}_n(R)$ is reduced to showing that $\cup X_{ij}$ is a Kazhdan subset of G. By the standard bounded generation argument [Sh1] it suffices to show that the larger subset $\cup_{\sigma \in \mathrm{Sym}(n)} G_\sigma$ (the union of vertex subgroups in the above decomposition) is Kazhdan, and one might attempt to prove the latter by applying the spectral criterion from [EJ] to the decomposition of G over $\Gamma(A_{n-1})$ described above. This almost works. More precisely, the attempted application of this criterion yields a "boundary case", where equality holds in the place of the desired inequality. In order to resolve this problem, a slightly generalized version of the spectral criterion must be used. The precise conditions entering this generalized spectral criterion are rather technical (see § 3), but these conditions are consequences of a very transparent property satisfied by $\mathrm{EL}_n(R)$, namely the fact that $\mathrm{EL}_n(R)$ is strongly graded by a root system of type A_{n-1} (which has rank ≥ 2) as defined below.

Let G be a group and Φ a classical root system. Suppose that G has a family of subgroups $\{X_\alpha\}_{\alpha\in\Phi}$ such that

$$[X_\alpha, X_\beta] \subseteq \prod_{\gamma\in\Phi\cap(\mathbb{Z}_{>0}\alpha\oplus\mathbb{Z}_{>0}\beta)} X_\gamma \tag{1.1}$$

for any $\alpha, \beta \in \Phi$ such that $\alpha \neq -\lambda\beta$ with $\lambda \in \mathbb{R}_{>0}$. Then we will say that G is graded by Φ and $\{X_\alpha\}$ is a Φ-grading of G.

Clearly, for any root system Φ the Steinberg group $\mathrm{St}_\Phi(R)$ and the elementary Chevalley group $\mathbb{E}_\Phi(R)$ are graded by Φ (with $\{X_\alpha\}$ being the root subgroups). On the other hand, any group G graded by Φ has a natural graph decomposition. We already discussed how to do this for $\Phi = A_n$ (in which case the underlying graph is $\Gamma(A_n)$ defined above). For an arbitrary Φ, the vertices of the underlying graph $\Gamma(\Phi)$ are labeled by the elements of W_Φ, the Coxeter group of type Φ, and given $w \in W_\Phi$, the vertex subgroup G_w is defined to be $\langle X_\alpha : w\alpha \in \Phi^+\rangle$ where Φ^+ is the set of positive roots in Φ (with respect to some fixed choice of simple roots). The edges of $\Gamma(\Phi)$ and the edge subgroups of G are defined as in the case $\Phi = A_n$.

Once again, the above decomposition of G over $\Gamma(\Phi)$ corresponds to the boundary case of the spectral criterion from [EJ], and the generalized spectral criterion turns out to be applicable under the addition assumption that the grading of G by Φ is *strong*. Informally a grading is strong if the inclusion in (1.1) is not too far from being an equality (see § 4 for precise definition). For instance, if Φ is a simply-laced system, a sufficient condition for a Φ-grading $\{X_\alpha\}$ to be strong is that $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ whenever $\alpha + \beta$ is a root.

We can now formulate the central result of this paper.

Theorem 1.2. *Let Φ be an irreducible classical root system of rank ≥ 2, and let G be a group which admits a strong Φ-grading $\{X_\alpha\}$. Then $\cup X_\alpha$ is a Kazhdan subset of G.*

Theorem 1.2 in the case $\Phi = A_n$ was already established in [EJ]; however, this was achieved by only considering the graph $\Gamma(A_2)$, called the *magic graph on six vertices* in [EJ]. This was possible thanks to an observation that every group strongly graded by A_{n-1}, $n \geq 3$. must also be strongly graded by A_2; for simplicity we illustrate the latter for $G = \mathrm{EL}_n(R)$. If $n = 3k$, the A_2 grading follows from the well-known isomorphism $\mathrm{EL}_{3k}(R) \cong \mathrm{EL}_3(\mathrm{Mat}_k(R))$, and for arbitrary n one should think of $n \times n$ matrices as "3×3 block-matrices with blocks of uneven size." This observation is a special case of the important concept of a *reduction of root systems* discussed in the next subsection (see § 6 for full details).

The proof of Theorem 1.2 for arbitrary Φ follows the same general approach as in the case $\Phi = A_2$ done in [EJ], although some arguments which are straightforward for $\Phi = A_2$ require delicate considerations in the general case. Perhaps more importantly, the proof presented in this paper provides a conceptual explanation of certain parts of the argument from [EJ] and shows that there was really nothing "magic" about the graph $\Gamma(A_2)$.

1.5. **Further examples.** So far we have discussed important, but very specific examples of groups graded by root systems – Chevalley and Steinberg groups. We shall now describe two general methods of constructing new groups graded by root systems. Thanks to Theorem 1.2 and suitable results on relative property (T), in this way we will also obtain new examples of Kazhdan groups.

The first method is simply an adaptation of the construction of twisted Chevalley groups to a slightly different setting. Suppose we are given a group G together with a grading $\{X_\alpha\}$ by a root system Φ and a finite group Q of automorphisms of G which permutes the root subgroups between themselves. Under some additional conditions we can use this data to construct a new group graded by a (different) root system. Without going into details, we shall mention that the new group, denoted by $\widehat{G^Q}$, surjects onto certain subgroup of G^Q, the group of Q-fixed points of G, and the new root system often coincides with the set of orbits under the induced action of Q on the original root system Φ. As a special case of this construction, we can take $G = \mathrm{St}_\Phi(R)$ for some ring R and let Q be the cyclic subgroup generated by an automorphism of G of the form $d\sigma$ where σ is a ring automorphism of $\mathrm{St}_\Phi(R)$ and d is a diagram automorphism of $\mathrm{St}_\Phi(R)$ having the same order as σ.

In this way we will obtain "Steinberg covers" of the usual twisted Chevalley groups over commutative rings of type ${}^2A_n$, ${}^2D_n$, ${}^3D_4$ and ${}^2E_6$. The Steinberg covers for the groups of type ${}^2A_n$ (which are unitary groups over rings with involution) can also be defined over non-commutative rings; moreover, the construction allows additional variations leading to a class of groups known as *hyperbolic unitary Steinberg groups* (see [HO], [Bak2]). Using this method one can also construct interesting families of groups which do not seem to have direct counterparts in the classical theory of algebraic groups; for instance, we will define Steinberg groups of type ${}^2F_4$ – these correspond to certain groups constructed by Tits [Ti] which, in turn, generalize twisted Chevalley groups of type ${}^2F_4$. We will show that among these Steinberg-type groups the ones graded by a root system of rank ≥ 2 have property (T) under some natural finiteness conditions on the data used to construct the twisted group.

The second method is based on the notion of a reduction of root systems defined below. This method does not directly produce new groups graded by root systems, but rather shows how given a group G graded by a root system Φ, one can construct a new grading of G by another root system of smaller rank.

If Φ and Ψ are two root systems, a reduction of Φ to Ψ is just a surjective map $\eta : \Phi \to \Psi \cup \{0\}$ which extends to a linear map between the real vector spaces spanned by Φ and Ψ, respectively. Now if G is a group with a Φ-grading X_α, for each $\beta \in \Psi$ we set $Y_\beta = \langle X_\alpha : \eta(\alpha) = \beta \rangle$. If the groups $\{Y_\beta\}_{\beta\in\Psi}$ happen to generate G (which is automatic, for instance, if η does not map any roots to 0), it is easy to see that $\{Y_\beta\}$ is a Ψ-grading of G. Furthermore, if the original Φ-grading was strong, then under some natural assumptions on the reduction η the new Ψ-grading will also be strong (reductions with this property will be called 2-good). We will show that every classical root system of rank ≥ 2 has a 2-good reduction to a classical root system of rank 2 (that is, A_2, B_2, BC_2 or G_2). [2] In this way we reduce the proof of Theorem 1.1 to Theorem 1.2 for classical root systems of rank 2. While Theorem 1.2 is not any easier to prove in this special case, using such

[2]The reduction of A_n to A_2 was already implicitly used in the proof of property (T) for $\mathrm{St}_{n+1}(R)$ in [EJ].

reductions we obtain much better Kazhdan constants for the groups $\mathrm{St}_\Phi(R)$ and $\mathbb{E}_\Phi(R)$ than what we are able to deduce from direct application of Theorem 1.2.

So far our discussion was limited to groups graded by classical root systems, but our definition of Φ-grading makes sense for any finite subset Φ. In this paper by a *root system* we mean any finite subset of $\mathbb{R}^n$ symmetric about the origin, and Theorem 1.2 remains true for groups graded by any root system satisfying a minor technical condition (these will be called *regular root systems*). There are plenty of regular root systems, which are not classical, but there is no easy recipe for constructing interesting groups graded by them. What we know is that reductions can be used to obtain some exotic gradings on familiar groups – for instance, the groups $\mathrm{St}_n(R)$ and $\mathrm{EL}_n(R)$ naturally graded by A_{n-1} can also be strongly graded by certain two-dimensional root system I_n (see the end of § 6). We believe that the analysis of this and other similar gradings can be used to construct new interesting groups, but we leave such an investigation for a later project.

1.6. **Application to expanders.** The subject of expansion in finite simple groups has seen a burst of activity over the past decade. One of the fundamental results in this area, established in a combination of several papers [Ka2, KLN, BGT], asserts that all (non-abelian) finite simple groups form a family of expanders (formally this means that the Cayley graphs of those groups with respect to certain generating sets of uniformly bounded size form a family of expanders). Note that proofs for different families of finite simple groups use different techniques and also vary a lot in level of complexity – for instance, the proof in the case of alternating groups [Ka2] is very involved, while expansion for the groups $\mathrm{SL}_n(\mathbb{F}_p)$, $n \geq 3$ is merely a consequence of the fact that $\mathrm{SL}_n(\mathbb{Z})$ has property (T).

In this paper we address the question of which families $\mathcal{F}$ of finite simple groups can be realized as quotients of a single group G with property (T). We will refer to a group which surjects onto every group in a family $\mathcal{F}$ as a *mother group* for $\mathcal{F}$. The main result of § 9 (see Theorem 9.3) asserts that the collection of all finite simple groups of Lie type and rank ≥ 2 admits a mother group with property (T) (thereby providing a unified proof of expansion for these groups). It is known that this result cannot be extended to all finite simple groups (even those of Lie type) since the family $\{\mathrm{SL}_2(\mathbb{F}_p)\}$ does not have a mother group with (T); however, it is still possible that all finite simple groups have a mother group with property (τ) (which would be sufficient for expansion). To prove Theorem 9.3 we first divide all finite simple groups of Lie type and rank ≥ 2 into finitely many subfamilies. Then for each subfamily $\mathcal{F}$ we construct a strong Ψ-grading for each group $G \in \mathcal{F}$ by a suitable classical root system Ψ (depending only on $\mathcal{F}$). Finally we show that all groups in $\mathcal{F}$ are quotients of a (possibly twisted) Steinberg group associated to Ψ, which has property (T) by one of the criteria established in § 7 or 8.

A concluding remark. To the best of our knowledge, the notion of a group graded by a root system (as defined in this paper) did not previously appear in the literature, but closely related classes of groups were considered by several authors. These include groups with Steinberg relations studied by Faulkner [Fa2] and groups graded by root systems in the sense of Shi [Shi]. The latter class can be defined as groups which are graded by classical root systems in our sense and endowed with a suitable action of the corresponding Weyl group. These groups are investigated further by Zhang [Zh2] where they are called root graded groups. A more general class of groups which includes groups graded by Kac-Moody root systems has been studied in [LN].

2. Preliminaries

In this section we shall define the notions of property (T), relative property (T), Kazhdan constants and Kazhdan ratios, orthogonality constants, angles and codistances between subspaces of Hilbert spaces, and recall some basic facts about them. We shall also state two new results on Kazhdan constants for nilpotent groups and group extensions, which will be established at the end of the paper (in § 10).

All groups in this paper will be assumed discrete, and we shall consider their unitary representations on Hilbert spaces. By a *subspace* of a Hilbert space we shall mean a closed subspace unless indicated otherwise.

2.1. **Property** (T).

Definition. Let G be a group and S a subset of G.

(a) Let V be a unitary representation of G. A nonzero vector $v \in V$ is called (S,ε)-invariant if
$$\|sv - v\| \le \varepsilon\|v\| \text{ for any } s \in S.$$
(b) Let V be a unitary representation of G without nonzero invariant vectors. The *Kazhdan constant* $\kappa(G,S,V)$ is the infimum of the set
$$\{\varepsilon > 0 : V \text{ contains an } (S,\varepsilon)\text{-invariant vector}\}.$$
(c) The *Kazhdan constant* $\kappa(G,S)$ of G with respect to S is the infimum of the set $\{\kappa(G,S,V)\}$ where V runs over all unitary representations of G without nonzero invariant vectors.

(d) S is called a *Kazhdan subset of* G if $\kappa(G,S) > 0$.

(e) A group G has *property* (T) if G has a finite Kazhdan subset, that is, if $\kappa(G,S) > 0$ for some finite subset S of G.

Remark. The Kazhdan constant $\kappa(G,S)$ may only be nonzero if S is a generating set for G (see, e.g., [BHV, Proposition 1.3.2]). Thus, a group with property (T) is automatically finitely generated. Furthermore, if G has property (T), then $\kappa(G,S) > 0$ for any finite generating set S of G, but the Kazhdan constant $\kappa(G,S)$ depends on S.

We note that if S is a "large" subset of G, positivity of the Kazhdan constant $\kappa(G,S)$ does not tell much about the group G. In particular, the following holds (see, e.g., [BHV, Proposition 1.1.5] or [Sh1, Lemma 2.5] for a slightly weaker version):

Lemma 2.1. *For any group G we have $\kappa(G,G) \ge \sqrt{2}$.*

2.2. **Relative property** (T) **and Kazhdan ratios.**

Relative property (T) has been originally defined for pairs (G,H) where H is a normal subgroup of G:

Definition. Let G be a group and H a normal subgroup of G. The pair (G,H) has *relative property* (T) if there exist a finite set S and $\varepsilon > 0$ such that if V is any unitary representation of G with an (S,ε)-invariant vector, then V has a (nonzero) H-invariant vector. The largest ε with this property (for a fixed set S) is called the *relative Kazhdan constant* of (G,H) with respect to S and denoted by $\kappa(G,H;S)$.

Remark. In the computation of relative Kazhdan constants it is enough to consider cyclic representations of G. Indeed, if V is any unitary representation of G which does not have H-invariant vectors, but has an (S, ε)-invariant vector v, then $W := \overline{\operatorname{span}(Gv)}$, the cyclic subrepresentation generated by v, has the same property. Of course, the same remark applies to the computation of usual Kazhdan constants.

The generalization of the notion of relative property (T) to pairs (G, B), where B is an arbitrary subset of a group G, has been given by de Cornulier [Co] and can be defined as follows (see also a remark in [EJ, § 2]):

Definition. Let G be a group and B a subset of G. The pair (G, B) has *relative property* (T) if for any $\varepsilon > 0$ there are a finite subset S of G and $\mu > 0$ such that if V is any unitary representation of G and $v \in V$ is (S, μ)-invariant, then v is (B, ε)-invariant.

In this more general setting it is not clear how to "quantify" the relative property (T) using a single real number. However, this is possible under the additional assumption that the dependence of μ on ε in the above definition may be expressed by a linear function. In this case we can define the notion of a Kazhdan ratio:

Definition. Let G be a group and B and S subsets of G. The *Kazhdan ratio* $\kappa_r(G, B; S)$ is the largest $\delta \in \mathbb{R}$ with the following property: if V is any unitary representation of G and $v \in V$ is $(S, \delta\varepsilon)$-invariant, then v is (B, ε)-invariant.

Somewhat surprisingly, there is a simple relationship between Kazhdan ratios and relative Kazhdan constants:

Observation 2.2. *Let G be a group, and let B and S be subsets of G. The following hold:*

(i) *If $\kappa_r(G, B; S) > 0$ and S is finite, then (G, B) has relative (T).*

(ii) *If B is a normal subgroup of G, then*

$$\sqrt{2}\kappa_r(G, B; S) \leq \kappa(G, B; S) \leq 2\kappa_r(G, B; S).$$

In particular, (G, B) has relative (T) if and only if $\kappa_r(G, B; S) > 0$ for some finite set S.

(iii) $\kappa(G, S) \geq \kappa(G, B)\kappa_r(G, B; S)$

Proof. (i) and (iii) follow immediately from the definition. The first inequality in (ii) holds by Lemma 2.1 applied to B (here we just need B to be a subgroup, not necessarily normal). Finally, the second inequality in (ii) is a standard fact proved, for instance, in [Sh1, Corollary 2.3]), but for completeness we reproduce the argument here.

Consider any unitary representation V of G. We have $V = V^B \oplus (V^B)^\perp$, where both V^B and $(V^B)^\perp$ are G-invariant since B is normal in G. Now take any nonzero $v \in V$, and assume that v is (S, δ)-invariant for some δ. Write $v = v_b + v_b^\perp$ where $v_b \in V^B$ and $v_b^\perp \in (V^B)^\perp$.

For any $s \in S$ we have $\|sv - v\| \leq \delta\|v\|$. Since both V^B and $(V^B)^\perp$ are G-invariant, $\|sv - v\|^2 = \|sv_b - v_b\|^2 + \|sv_b^\perp - v_b^\perp\|^2$, and thus

$$\|sv_b^\perp - v_b^\perp\| \leq \delta\|v\|.$$

On the other hand, $(V^B)^\perp$ has no B-invariant vectors, so there exists $s \in S$ such that $\|sv_b^\perp - v_b^\perp\| \geq \mu\|v_b^\perp\|$ where $\mu = \kappa(G, B; S)$ (we can assume that $\mu > 0$

since otherwise there is nothing to prove). Combining the two inequalities, we get $\|v_b^{\perp}\| \leq \frac{\delta}{\mu}\|v\|$.

Now take any $b \in B$. Since b fixes v_b, we have

$$\|bv - v\| = \|bv_b^{\perp} - v_b^{\perp}\| \leq 2\|v_b^{\perp}\| \leq \frac{2\delta}{\mu}\|v\|,$$

so by definition of the Kazhdan ratio, $\kappa_r(G, B; S) \geq \frac{\mu}{2}$, as desired. □

2.3. **Using relative property (T).** A typical way to prove that a group G has property (T) is to find a subset K of such that

(a) K is a Kazhdan subset of G
(b) the pair (G, K) has relative property (T).

Clearly, (a) and (b) imply that G has property (T). Note that (a) is easy to establish when K is a large subset of G, while (b) is easy to establish when K is small, so to obtain (a) and (b) simultaneously one typically needs to pick K of intermediate size.

In all our examples, pairs with relative property (T) will be produced with the aid of the following fundamental result: if R is any finitely generated ring, then the pair $(\mathrm{EL}_2(R) \ltimes R^2, R^2)$ has relative property (T). This has been proved by Burger [Bu] for $R = \mathbb{Z}$, by Shalom [Sh1] for commutative R and by Kassabov [Ka1] in general. In fact, we shall use what formally is a stronger result, although its proof is identical to the one given in [Ka1]:

Theorem 2.3. *Let $R \star R$ denote the free product of two copies of the additive group of R, and consider the semi-direct product $(R \star R) \ltimes R^2$, where the first copy of R acts by upper-unitriangular matrices, that is, $r \in R$ acts as left multiplication by the matrix $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, and the second copy of R acts by lower-unitriangular matrices. Then the pair $((R \star R) \ltimes R^2, R^2)$ has relative property (T).*

In § 7 we shall state a higher-dimensional generalization of this theorem along with an explicit bound for the relative Kazhdan constant (see Theorem 7.10). For completeness, we will present proofs of both Theorem 2.3 and 7.10 in Appendix.

Given a group G, once we have found some subsets B of G for which (G, B) has relative property (T), it is easy to produce many more subsets with the same property. First it is clear that if (G, B_i) has relative (T) for some finite collection of subsets $B_1, \ldots, B_k$, then $(G, \cup B_i)$ also has relative (T). Indeed, suppose that given $\varepsilon > 0$ there exist finite subsets $S_1, \ldots, S_k$ of G and $\mu_1, \ldots, \mu_k > 0$ such that for each $1 \leq i \leq k$, if a vector v in some unitary representation of G is (S_i, μ_i)-invariant, then v is also (B_i, ε)-invariant. If we set $S = \cup S_i$ and $\mu = \min\{\mu_i\}$, then any v which is (S, μ)-invariant must also be (B, ε)-invariant.

Here is a more interesting result of this kind, on which the bounded generation method is based.

Lemma 2.4 (Bounded generation principle)**.** *Let G be a group, S a subset of G and $B_1, \ldots, B_k$ a finite collection of subsets of G. Let $B_1 \ldots B_k$ be the set of all elements of G representable as $b_1 \ldots b_k$ with $b_i \in B_i$.*

(a) *Suppose that (G, B_i) has relative (T) for each i. Then $(G, B_1 \ldots B_k)$ also has relative (T).*

(b) *Suppose in addition that* $\kappa_r(G, B_i; S) > 0$ *for each* i. *Then*

$$\kappa_r(G, B_1 \dots B_k; S) \geq \frac{1}{\sum\limits_{i=1}^{k} \frac{1}{\kappa_r(G, B_i; S)}} \geq \frac{\min\{\kappa_r(G, B_i; S)\}_{i=1}^{k}}{k} > 0.$$

Proof. Let V be a unitary representation of G. For any $b_1 \in B_1, \dots, b_k \in B_k$ and $v \in V$ we have

$$\left\| \left(\prod_{i=1}^{k} b_i \right) v - v \right\| = \left\| \sum_{i=1}^{k} \left(\prod_{j=1}^{i} b_j v - \prod_{j=1}^{i-1} b_j v \right) \right\| = \left\| \sum_{i=1}^{k} \prod_{j=1}^{i-1} b_j (b_i v - v) \right\| \leq \sum_{i=1}^{k} \|b_i v - v\|.$$

Thus, if v is (B_i, ε_i)-invariant for each i, then v is $(B_1 \dots B_k, \sum_{i=1}^{k} \varepsilon_i)$-invariant. Both (a) and (b) follow easily from this observation. □

For convenience, we introduce the following terminology:

Definition. Let G be a group and $B_1, \dots, B_k$ a finite collection of subsets of G. Let H be another subset of G. We will say that H *lies in a bounded product of* $B_1, \dots, B_k$ if there exists $N \in \mathbb{N}$ such that any element $h \in H$ can be written as $h = h_1 \dots h_N$ with $h_i \in \cup_{j=1}^{k} B_j$ for all i.

By Lemma 2.4 if a group G has subsets $B_1, \dots, B_k$ such that (G, B_i) has relative (T) for each i, then (G, H) has relative (T) for any subset H which lies in a bounded product of $B_1, \dots, B_k$. This observation will be frequently used in the proof of property (T) for groups graded by non-simply laced root systems.

2.4. **Orthogonality constants, angles and codistances.** The notion of the orthogonality constant between two subspaces of a Hilbert space was introduced and successfully applied in [DJ] and also played a key role in [EJ]:

Definition. Let V_1 and V_2 be two (closed) subspaces of a Hilbert space V.

(i) The *orthogonality constant* $\mathrm{orth}(V_1, V_2)$ between V_1 and V_2 is defined by

$$\mathrm{orth}(V_1, V_2) = \sup\{|\langle v_1, v_2 \rangle| : \ ||v_i|| = 1, v_i \in V_i \text{ for } i = 1, 2\}$$

(ii) The quantity $\sphericalangle(V_1, V_2) = \arccos(\mathrm{orth}(V_1, V_2))$ will be called the *angle* between V_1 and V_2. Thus, $\sphericalangle(V_1, V_2)$ is the infimum of angles between a nonzero vector from V_1 and a nonzero vector from V_2.

These quantities are only of interest when the subspaces V_1 and V_2 intersect trivially. In general, it is more adequate to consider the corresponding quantities after factoring out the intersection. We call them the *reduced orthogonality constant and reduced angle.*

Definition. Let V_1 and V_2 be two subspaces in a Hilbert space V, and assume that neither of the subspaces V_1 and V_2 contains the other.

(i) The *reduced orthogonality constant* $\mathrm{orth}_{red}(V_1, V_2)$ between V_1 and V_2 is defined by

$$\mathrm{orth}_{red}(V_1, V_2) = \sup\{\langle v_1, v_2 \rangle \mid ||v_i|| = 1, v_i \in V_i, v_i \perp V_1 \cap V_2 \text{ for } i = 1, 2\}.$$

(ii) The quantity $\sphericalangle_{red}(V_1, V_2) = \arccos(\mathrm{orth}_{red}(V_1, V_2))$ will be called the *reduced angle between* V_1 *and* V_2. Thus, $\sphericalangle_{red}(V_1, V_2)$ is the infimum of the angles between nonzero vectors v_1 and v_2, where $v_i \in V_i$ and $v_i \perp V_1 \cap V_2$.

Remark. If V_1 and V_2 are two distinct planes in a three-dimensional Euclidean space, then the reduced angle $\sphericalangle_{red}(V_1, V_2)$ coincides with the usual (geometric) angle between V_1 and V_2.

Reduced angles play a key role in Kassabov's paper [Ka3] (where they are called just 'angles'), but in the present paper the case of subspaces with trivial intersections will suffice. In fact, in the subsequent discussion we shall operate with orthogonality constants rather than angles.

The following simple result will be very important for our purposes.

Lemma 2.5. *Let V_1 and V_2 be two subspaces of a Hilbert space V. Then the reduced angle between the orthogonal complements $V_1^\perp$ and $V_2^\perp$ is equal to the reduced angle between V_1 and V_2. Equivalently,*

$$\mathrm{orth}_{red}(V_1^\perp, V_2^\perp) = \mathrm{orth}_{red}(V_1, V_2).$$

Proof. This result appears as [Ka3, Lemma 3.9] as well as [EJ, Lemma 2.4] (in a special case), but it has apparently been known to functional analysts for a long time (see [BGM] and references therein). □

The notion of codistance introduced in [EJ] generalizes orthogonality constants to the case of more than two subspaces.

Definition. Let V be a Hilbert space, and let $\{U_i\}_{i=1}^n$ be subspaces of V. Consider the Hilbert space V^n and its subspaces $U_1 \times U_2 \times \ldots \times U_n$ and $\mathrm{diag}(V) = \{(v, v, \ldots, v) : v \in V\}$. The quantity

$$\mathrm{codist}(\{U_i\}) = (\mathrm{orth}(U_1 \times \ldots \times U_n, \mathrm{diag}(V)))^2$$

will be called the *codistance* between the subspaces $\{U_i\}_{i=1}^n$. It is easy to see that

$$\mathrm{codist}(\{U_i\}) = \sup\left\{\frac{\|u_1 + \cdots + u_n\|^2}{n(\|u_1\|^2 + \cdots + \|u_n\|^2)} : u_i \in U_i\right\}.$$

For any collection of n subspaces $\{U_i\}_{i=1}^n$ we have $\frac{1}{n} \le \mathrm{codist}(\{U_i\}) \le 1$, and $\mathrm{codist}(\{U_i\}) = \frac{1}{n}$ if and only if $\{U_i\}$ are pairwise orthogonal. In the case of two subspaces we have an obvious relation between $\mathrm{codist}(U, W)$ and $\mathrm{orth}(U, W)$:

$$\mathrm{codist}(U, W) = \frac{1 + \mathrm{orth}(U, W)}{2}.$$

Similarly one can define the *reduced codistance*, but we shall not use this notion. The closely related notion of (reduced) angle between several subspaces is investigated in [Ka3].

We now define the orthogonality constants and codistances for subgroups of a given group.

Definition. (a) Let H and K be subgroups of the same group and let $G = \langle H, K\rangle$, the group generated by H and K. We define $\mathrm{orth}(H, K)$ to be the supremum of the quantities $\mathrm{orth}(V^H, V^K)$ where V runs over all unitary representations of G without nonzero invariant vectors.

(b) Let $\{H_i\}_{i=1}^n$ be subgroups of the same group, and let $G = \langle H_1, \ldots, H_n\rangle$. The *codistance* between $\{H_i\}$, denoted by $\mathrm{codist}(\{H_i\})$, is defined to be the supremum of the quantities $\mathrm{codist}(V^{H_1}, \ldots, V^{H_n})$, where V runs over all unitary representations of G without nonzero invariant vectors.

The basic connection between codistance and property (T), already discussed in the introduction, is given by the following lemma (see [EJ, Lemma 3.1]):

Lemma 2.6 ([EJ]). *Let G be a group and $H_1, H_2, \ldots, H_n$ subgroups of G such that $G = \langle H_1, \ldots, H_n \rangle$. Let $\rho = \mathrm{codist}(\{H_i\})$, and suppose that $\rho < 1$. The following hold:*

(a) $\kappa(G, \bigcup H_i) \geq \sqrt{2(1-\rho)}$.
(b) *Let S_i be a generating set of H_i, and let $\delta = \min\{\kappa(H_i, S_i)\}_{i=1}^n$. Then*
$$\kappa(G, \bigcup S_i) \geq \delta\sqrt{1-\rho}.$$
(c) *Assume in addition that each pair (G, H_i) has relative property (T). Then G has property (T).*

2.5. **Kazhdan constants for nilpotent groups and group extensions.** We finish this section by formulating two theorems and one simple lemma which provide estimates for Kazhdan constants. These results are new (although they have been known before in some special cases [BHV, Ha, NPS]). The two theorems will be established in § 10 of this paper, while the lemma will be proved right away.

The first theorem concerns relative Kazhdan constants in central extensions of groups:

Theorem 2.7. *Let G be a group, N a normal subgroup of G and Z a subgroup of $Z(G) \cap N$. Put $H = Z \cap [N, G]$. Assume that A, B and C are subsets of G satisfying the following conditions:*

(1) *A and N generate G,*
(2) $\kappa(G/Z, N/Z; B) \geq \varepsilon$,
(3) $\kappa(G/H, Z/H; C) \geq \delta$.

Then
$$\kappa(G, N; A \cup B \cup C) \geq \frac{1}{\sqrt{3}} \min\left\{\frac{12\varepsilon}{5\sqrt{72\varepsilon^2|A| + 25|B|}}, \delta\right\}.$$

In a typical application of this theorem the following additional conditions will be satisfied:

(a) The group G/N is finitely generated
(b) The group Z/H is finite

In this case (3) holds automatically, and (1) holds for some finite set A. Therefore, Theorem 2.7 implies that under the additional assumptions (a) and (b), relative property (T) for the pair $(G/Z, N/Z)$ implies relative property (T) for the pair (G, N).

The second theorem that we shall use gives a bound for the codistance between subgroups of a nilpotent group. It is not difficult to see that if G is an abelian group generated by subgroups $X_1, \ldots, X_k$, then $\mathrm{codist}(X_1, \ldots, X_k) \leq 1 - \frac{1}{k}$ (and this bound is optimal). We shall prove a similar (likely far from optimal) bound in the case of countable nilpotent groups:

Theorem 2.8. *Let G be a countable nilpotent group of class c generated by subgroups $X_1, \ldots, X_k$. Then*
$$\mathrm{codist}(X_1, \ldots, X_k) \leq 1 - \frac{1}{4^{c-1}k}.$$

We finish with a technical lemma which yields certain supermultiplicativity property involving Kazhdan ratios. It can probably be applied in a variety of situations, but in this paper it will only be used to obtain a better Kazhdan constant for the Steinberg groups of type G_2:

Lemma 2.9. *Let G be a group, H a subgroup of G and N a normal subgroup of H. Suppose that there exists a subset Σ of G and real numbers $a, b > 0$ such that*

(1) $\kappa_r(G, N; \Sigma) \geq \frac{1}{a}$
(2) $\kappa(H/N, \overline{\Sigma \cap H}) \geq \frac{1}{b}$ *where $\overline{\Sigma \cap H}$ is the image of $\Sigma \cap H$ in H/N.*

Then $\kappa_r(G, H; \Sigma) \geq \dfrac{1}{\sqrt{2a^2 + 4b^2}}$.

Proof. Fix $\varepsilon > 0$, and let V be a unitary representation of G and $v \in V$ such that

$$\|sv - v\| \leq \varepsilon\|v\| \text{ for any } s \in \Sigma.$$

Write $v = v_1 + v_2$, where $v_1 \in V^N$ and $v_2 \in (V^N)^\perp$. Since $\kappa_r(G, N; \Sigma) \geq \frac{1}{a}$, we obtain that for every $n \in N$

$$\|nv_2 - v_2\| = \|nv - v\| \leq a\varepsilon\|v\|.$$

On the other hand, by Lemma 2.1 there exists $n \in N$ such that $\|nv_2 - v_2\| \geq \sqrt{2}\|v_2\|$. Hence $\|v_2\| \leq \frac{a\varepsilon}{\sqrt{2}}\|v\|$.

Since N is normal in H, the subspaces V^N and $(V^N)^\perp$ are H-invariant. Hence for any $s \in \Sigma \cap H$ we have $\|sv_1 - v_1\| \leq \|sv - v\| \leq \varepsilon\|v\|$.

By Observation 2.2(ii) we have

$$\kappa_r(H/N, H/N; \overline{\Sigma \cap H}) \geq \frac{\kappa(H/N, H/N; \overline{\Sigma \cap H})}{2} = \frac{\kappa(H/N, \overline{\Sigma \cap H})}{2} \geq \frac{1}{2b}.$$

Thus, considering V^N as a representation of H/N, we obtain that $\|hv_1 - v_1\| \leq 2b\varepsilon\|v_1\| \leq 2b\varepsilon\|v\|$ for any $h \in H$. Hence for any $h \in H$,

$$\|hv - v\|^2 = \|hv_1 - v_1\|^2 + \|hv_2 - v_2\|^2 \leq 4b^2\varepsilon^2\|v\|^2 + 4\|v_2\|^2 \leq \varepsilon^2(2a^2 + 4b^2)\|v\|^2. \quad \square$$

3. Generalized spectral criterion

3.1. **Graphs and Laplacians.** Let Γ be a finite graph without loops. We will denote the set of vertices of Γ by $\mathcal{V}(\Gamma)$ and the set of edges by $\mathcal{E}(\Gamma)$. For any edge $e = (x, y) \in \mathcal{E}(\Gamma)$, we let $\bar{e} = (y, x)$ be the inverse of e. We assume that if $e \in \mathcal{E}(\Gamma)$, then also $\bar{e} \in \mathcal{E}(\Gamma)$. If $e = (x, y)$, we let $e^- = x$ be the initial vertex of e and by $e^+ = y$ the terminal vertex of e.

Now assume that the graph Γ is connected, and fix a Hilbert space V. Let $\Omega^0(\Gamma, V)$ be the Hilbert space of functions $f : \mathcal{V}(\Gamma) \to V$ with the scalar product

$$\langle f, g \rangle = \sum_{y \in \mathcal{V}(\Gamma)} \langle f(y), g(y) \rangle$$

and let $\Omega^1(\Gamma, V)$ be the Hilbert space of functions $f : \mathcal{E}(\Gamma) \to V$ with the scalar product

$$\langle f, g \rangle = \frac{1}{2} \sum_{e \in \mathcal{E}(\Gamma)} \langle f(e), g(e) \rangle.$$

Define the linear operator

$$d : \Omega^0(\Gamma, V) \to \Omega^1(\Gamma, V) \text{ by } (df)(e) = f(e^+) - f(e^-).$$

We will refer to d as the *difference operator of* Γ.
The adjoint operator $d^* : \Omega^1(\Gamma, V) \to \Omega^0(\Gamma, V)$ is given by formula

$$(d^* f)(y) = \sum_{y=e^+} \frac{1}{2} \left(f(e) - f(\bar{e}) \right).$$

The symmetric operator $\Delta = d^* d : \Omega^0(\Gamma, V) \to \Omega^0(\Gamma, V)$ is called the *Laplacian of* Γ and is given by the formula

$$(\Delta f)(y) = \sum_{y=e^+} (f(y) - f(e^-)) = \sum_{y=e^+} df(e).$$

The smallest positive eigenvalue of Δ is commonly denoted by $\lambda_1(\Delta)$ and called the *spectral gap* of the graph Γ (clearly, it is independent of the choice of V).

3.2. **Spectral criteria.**

Definition. Let G be a group and Γ a finite graph without loops. A *graph of groups decomposition (or just a decompoisition) of* G *over* Γ is a choice of a vertex subgroup $G_\nu \subseteq G$ for every $\nu \in \mathcal{V}(\Gamma)$ and an edge subgroup $G_e \subseteq G$ for every $e \in \mathcal{E}(\Gamma)$ such that

(a) The vertex subgroups $\{G_\nu : \nu \in \mathcal{V}(\Gamma)\}$ generate G;
(b) $G_e = G_{\bar{e}}$ and $G_e \subseteq G_{e^+} \cap G_{e^-}$ for every $e \in \mathcal{E}(\Gamma)$.

We will say that the decomposition of G over Γ is *regular* if for each $\nu \in \mathcal{V}(\Gamma)$ the vertex group G_ν is generated by the edge subgroups $\{G_e : e^+ = \nu\}$

The following criterion is proved in [EJ]:

Theorem 3.1. *Let Γ be a connected k-regular graph and let G be a group with a given regular decomposition over Γ. For each $\nu \in \mathcal{V}(\Gamma)$ let p_ν be the codistance between the subgroups $\{G_e : e^+ = \nu\}$ of G_ν, and let $p = \max_\nu p_\nu$. Let Δ be the Laplacian of Γ, and assume that*

$$p < \frac{\lambda_1(\Delta)}{2k}.$$

Then $\cup_{\nu \in \mathcal{V}(\Gamma)} G_\nu$ is a Kazhdan subset of G, and moreover

$$\kappa(G, \cup G_\nu) \geq \sqrt{\frac{2(\lambda_1(\Delta) - 2pk)}{\lambda_1(\Delta)(1-p)}}.$$

In [EJ] a slight modification of this criterion was applied to groups graded by root systems of type A_2 with their canonical graph of groups decompositions (as described in the introduction). In this case one has $p = \frac{\lambda_1(\Delta)}{2k}$, and thus Theorem 3.1 is not directly applicable; however this problem was bypassed in [EJ] using certain trick. We shall now describe a generalization of Theorem 3.1 which essentially formalizes that trick and allows us to handle the “boundary” case $p = \frac{\lambda_1(\Delta)}{2k}$.

First, we shall use extra data – in addition to a decomposition of the group G over the graph Γ, we choose a normal subgroup CG_ν of G_ν for each vertex ν of Γ, called the *core subgroup* of G_ν. We shall assume that if a representation of the vertex group G_ν does not have any CG_ν-invariant vectors, then the codistance between the fixed subspaces of the edge groups is bounded above by $\frac{\lambda_1(\Delta)}{2k} - \varepsilon$ for some $\varepsilon > 0$ (independent of the representation). In order for this extra assumption to be useful, we need to know that there are sufficiently many representations of

G_ν without CG_ν-invariant vectors (for instance, if $CG_\nu = \{1\}$, there are no such non-trivial representations), and thus we shall also require that the core subgroups CG_ν are not too small.

Let us now fix a group G, a regular decomposition of G over a graph Γ, and a normal subgroup CG_ν of G_ν for each $\nu \in \mathcal{V}(\Gamma)$.

Let V be a unitary representation of G. Let $\Omega^0(\Gamma, V)^{\{G_\nu\}}$ denote the subspace of $\Omega^0(\Gamma, V)$ consisting of all function $g : V(\Gamma) \to V$ such that $g(\nu) \in V^{G_\nu}$ for any $\nu \in \mathcal{V}(\Gamma)$. Similarly we define the subspace $\Omega^1(\Gamma, V)^{\{G_e\}}$ of $\Omega^1(\Gamma, V)$.

For each vertex ν we consider the decomposition of V into a direct sum of three subspaces:

$$V = V^{G_\nu} \oplus ((V^{G_\nu})^\perp \cap V^{CG_\nu}) \oplus (V^{CG_\nu})^\perp.$$

Note that $(V^{G_\nu})^\perp \cap V^{CG_\nu}$ (resp. $V^{CG_\nu})^\perp$) is a representation of G_ν without G_ν-invariant (resp. CG^ν-invariant) vectors, so we can apply codistance bounds from Theorem 3.3 (i) and (ii) below to those representations of G_ν.

Combining these decompositions over all vertices, we obtain the corresponding decomposition of $\Omega^0(\Gamma, V)$ into a direct sum of three subspaces, and denote by ρ_1, ρ_2, ρ_3 the projection maps onto those subspaces.

Explicitly, the projections ρ_1, ρ_2, ρ_3 are defined as follows: For a function $g \in \Omega^0(\Gamma, V)$ and $\nu \in \mathcal{V}(\Gamma)$ we set

$$\rho_1(g)(\nu) = \pi_{V^{G_\nu}}(g(\nu)) \qquad \rho_2(g)(\nu) = \pi_{(V^{G_\nu})^\perp \cap V^{CG_\nu}}(g(\nu))$$
$$\rho_3(g)(\nu) = \pi_{(V^{CG_\nu})^\perp}(g(\nu)),$$

that is, the values of $\rho_i(g)$ for $i = 1, 2$ and 3 at the vertex ν are the projections of the vector $g(\nu) \in V$ onto the subspaces V^{G_ν}, $(V^{G_\nu})^\perp \cap V^{CG_\nu}$ and $(V^{CG_\nu})^\perp$, respectively.

By construction ρ_1 is just the projection onto $\Omega^0(\Gamma, V)^{\{G_\nu\}}$, and we have

$$\|\rho_1(g)\|^2 + \|\rho_2(g)\|^2 + \|\rho_3(g)\|^2 = \|g\|^2 \text{ for every } g \in \Omega^0(\Gamma, V).$$

Similarly, we define the projections ρ_1, ρ_2, ρ_3 on the space $\Omega^1(\Gamma, V)$: for a function $g \in \Omega^1(\Gamma, V)$ and an edge $e \in \mathcal{E}(\Gamma)$ we set

$$\rho_1(g)(e) = \pi_{V^{G_{e^+}}}(g(e)) \qquad \rho_2(g)(e) = \pi_{(V^{G_{e^+}})^\perp \cap V^{CG_{e^+}}}(g(e))$$
$$\rho_3(g)(e) = \pi_{(V^{CG_{e^+}})^\perp}(g(e)),$$

Again we have

$$\|\rho_1(g)\|^2 + \|\rho_2(g)\|^2 + \|\rho_3(g)\|^2 = \|g\|^2 \text{ for every } g \in \Omega^1(\Gamma, V).$$

Claim 3.2. *The projections ρ_i, $i = 1, 2, 3$, preserve the subspace $\Omega^1(\Gamma, V)^{\{G_e\}}$.*

Proof. The projection ρ_1 preserve the space $\Omega^1(\Gamma, V)^{\{G_e\}}$ because G_{e^+} contains the group G_e. The other two projections preserve this space because CG_{e^+} is a normal subgroup of G_{e^+}. □

Now we are ready to state the desired generalization of Theorem 3.1.

Theorem 3.3. *Let Γ be a connected k-regular graph. Let G be a group with a chosen regular decomposition over Γ, and choose a normal subgroup CG_ν of G_ν for each $\nu \in \mathcal{V}(\Gamma)$. Let $\bar{p} = \frac{\lambda_1(\Delta)}{2k}$, where Δ is the Laplacian of Γ. Suppose that*

(i) *For each vertex ν of Γ the codistance between the subgroups $\{G_e : e^+ = \nu\}$ of G_v is bounded above by $\bar{p}$.*

(ii) *There exists $\varepsilon > 0$ such that for any $\nu \in \mathcal{V}(\Gamma)$ and any unitary representation V of the vertex group G_ν without CG_ν invariant vectors, the codistance between the fixed subspaces of G_e, with $e^+ = \nu$, is bounded above by $\bar{p}(1-\varepsilon)$;*

(iii) *There exist constants A, B such that for any unitary representation V of G and for any function $g \in \Omega^0(\Gamma, V)^{\{G_\nu\}}$ one has*
$$\|dg\|^2 \leq A\|\rho_1(dg)\|^2 + B\|\rho_3(dg)\|^2.$$

Then $\cup G_\nu$ is a Kazhdan subset of G and
$$\kappa(G, \cup G_\nu) \geq \sqrt{\frac{4\varepsilon k}{\varepsilon\lambda_1(\Delta)A + (2k - \lambda_1(\Delta))\, B}} > 0.$$

Remark. (a) Theorem 3.3 implies Theorem 3.1 as (ii) clearly holds with $\varepsilon = 1 - p/\bar{p} > 0$, and if we put $CG_\nu = G_\nu$ for each ν, then the projection ρ_2 is trivial, and thus (iii) holds with $A = B = 1$.

(b) If ε is sufficiently large, one can show that the conclusion of Theorem 3.3 holds even if $\bar{p}$ is slightly larger than $\frac{\lambda_1(\Delta)}{2k}$, but we are unaware of any interesting applications of this fact.

(c) The informal assumption that the core subgroups are not "too small" discussed above is "hidden" in the condition (iii).

3.3. **Proof of Theorem 3.3.** The main step in the proof is to show that the image of $\Omega^0(\Gamma, V)^{\{G_\nu\}}$ under the the Laplacian Δ is sufficiently far from $\left(\Omega^0\left(\Gamma, V\right)^G\right)^\perp$:

Theorem 3.4. *Let A and B be as in Theorem 3.3. Then for any $g \in \Omega^0(\Gamma, V)^{\{G_\nu\}}$ we have*
$$\|\rho_1(\Delta g)\|^2 \geq \frac{\varepsilon}{B(1-\bar{p}) + \varepsilon A\bar{p}}\|\Delta g\|^2.$$

Proof. Let g be an element of $\Omega^0(\Gamma, V)^{\{G_\nu\}}$. This implies that $dg \in \Omega^1(\Gamma, V)^{\{G_e\}}$ and therefore $\rho_i(dg) \in \Omega^1(\Gamma, V)^{\{G_e\}}$ for $i = 1, 2, 3$ by Claim 3.2. We have
$$\rho_i(\Delta g)(\nu) = \sum_{e^+=\nu} \rho_i(dg)(e).$$

For $i = 1$ we just use the triangle inequality:
$$\|\rho_1(\Delta g)(\nu)\|^2 = \left\|\sum_{e^+=\nu} \rho_1(dg)(e)\right\|^2 \leq k \sum_{e^+=\nu} \|\rho_1(dg)(e)\|^2.$$

Summing over all vertices we get
$$\|\rho_1(\Delta g)\|^2 \leq k \sum_{\nu\in\mathcal{V}(\Gamma)} \sum_{e^+=\nu} \|\rho_1(dg)(e)\|^2 = 2k\,\|\rho_1(dg)\|^2.$$

If $i = 2$ and $i = 3$ the vectors $\rho_i(dg)(e)$ are in V^{G_e} and they are orthogonal to the spaces V^{G_ν} and V^{CG_ν}, respectively. Since $(V^{G_\nu})^\perp$ (resp. $(V^{CG_\nu})^\perp$) is a representation of G_ν without invariant (resp. CG_ν-invariant) vectors, we can use the bounds for codistances from (i) and (ii):
$$\|\rho_2(\Delta g)(\nu)\|^2 = \left\|\sum_{e^+=\nu} \rho_2(dg)(e)\right\|^2 \leq k\bar{p} \sum_{e^+=\nu} \|\rho_2(dg)(e)\|^2,$$

and

$$\|\rho_3(\Delta g)(\nu)\|^2 = \left\| \sum_{e^+=\nu} \rho_3(dg)(e) \right\|^2 \le k\bar{p}(1-\varepsilon) \sum_{e^+=\nu} \|\rho_3(dg)(e)\|^2 .$$

Again, summing over all vertices ν yields

$$\|\rho_2(\Delta g)\|^2 \le 2k\bar{p}\, \|\rho_2(dg)\|^2$$

$$\|\rho_3(\Delta g)\|^2 \le 2k\bar{p}\, (1-\varepsilon)\, \|\rho_3(dg)\|^2.$$

Thus we have

$$\begin{aligned}\bar{p}\left(1-\frac{\varepsilon A}{B}\right)\|\rho_1(\Delta g)\|^2 + \|\rho_2(\Delta g)\|^2 + \|\rho_3(\Delta g)\|^2 \le \\ \le 2k\bar{p}\left(\left(1-\frac{\varepsilon A}{B}\right)\|\rho_1(dg)\|^2 + \|\rho_2(dg)\|^2 + (1-\varepsilon)\,\|\rho_3(dg)\|^2\right) = \\ = 2k\bar{p}\left(\|dg\|^2 - \frac{\varepsilon A}{B}\|\rho_1(dg)\|^2 - \varepsilon\|\rho_3(dg)\|^2\right).\end{aligned}$$

Combining this inequality with the norm inequality from (iii) and the fact that by definition of Δ and $\lambda_1(\Delta)$ we have

$$\|dg\|^2 = \langle \Delta g, g\rangle \le \frac{1}{\lambda_1(\Delta)}\|\Delta g\|^2,$$

we get

$$\begin{aligned}\left(\bar{p}\left(1-\frac{\varepsilon A}{B}\right)-1\right)\|\rho_1(\Delta g)\|^2 + \|(\Delta g)\|^2 \le \\ \le 2k\bar{p}\left(1-\frac{\varepsilon}{B}\right)\|dg\|^2 \le \frac{2k\bar{p}}{\lambda_1(\Delta)}\left(1-\frac{\varepsilon}{B}\right)\|\Delta g\|^2 = \left(1-\frac{\varepsilon}{B}\right)\|\Delta g\|^2.\end{aligned}$$

and so

$$\|\rho_1(\Delta g)\|^2 \ge \frac{\varepsilon}{B(1-\bar{p})+\varepsilon A\bar{p}}\|\Delta g\|^2. \quad \square$$

Proof of Theorem 3.3. Let V be a unitary representation of G without invariant vectors. Let U denote the subspace of $\Omega^0(\Gamma, V)$ consisting of all constant functions, let $W = \Omega^0(\Gamma, V)^{\{G_\nu\}}$ and $V' = \overline{U+W}$. Define $\widetilde{\Delta} : V' \to V'$ by $\widetilde{\Delta} = \mathrm{proj}_{V'} \circ \Delta_{|V'}$. Note that if $\iota : V' \to V$ is the inclusion map, then $\iota^* = \mathrm{proj}_{V'}$, considered as a map from V to V'. Hence

$$\widetilde{\Delta} = \mathrm{proj}_{V'}\, \Delta_{|V'} = \iota^*\Delta\iota = \iota^* d^* d\iota = (d\iota)^* d\iota.$$

Thus, $\widetilde{\Delta}$ is self-adjoint and $\operatorname{Ker}\widetilde{\Delta} = \operatorname{Ker}(d\iota) = U$. Hence the image of $\widetilde{\Delta}$ is dense in $U^{\perp V'}$; in fact, one can show that the image of $\widetilde{\Delta}$ is closed (see [EJ, p.325]), but we will not need this fact.

Let $P = \dfrac{\varepsilon}{B(1-\bar{p})+\varepsilon A\bar{p}}$. By Theorem 3.4 we have $\|\mathrm{proj}_W \Delta g\|^2 \ge P\|\Delta g\|^2$ for all $g \in W$. In fact, the same inequality holds for all $g \in V'$ since $U = \operatorname{Ker}\Delta$. Notice that when $g \in V'$ we also have $\|\mathrm{proj}_W \widetilde{\Delta} g\|^2 = \|\mathrm{proj}_W \mathrm{proj}_{V'} \Delta g\|^2 = \|\mathrm{proj}_W \Delta g\|^2$ (since $W \subseteq V'$), and therefore

$$\|\mathrm{proj}_W \widetilde{\Delta} g\|^2 = \|\mathrm{proj}_W \Delta g\|^2 \ge P\|\Delta g\|^2 \ge P\|\widetilde{\Delta} g\|^2.$$

Since the image of $\widetilde{\Delta}$ is dense in $U^{\perp V'}$, the obtained inequality can be restated in terms of codistance:

$$\mathrm{codist}(U^{\perp V'}, W^{\perp V'}) \leq 1 - P.$$

Note that $U \cap W = \{0\}$ (since V has no G-invariant vectors and G is generated by the vertex subgroups $\{G_\nu\}$), so Lemma 2.5 implies that $\mathrm{codist}(U, W) \leq 1 - P$. But by definition of the codistance and the definition of subspaces U and W this implies that the codistance between the vertex subgroups $\{G_\nu\}$ of G is bounded above by $1 - P$. Finally, by Lemma 2.5 we have

$$\kappa(G, \cup G_\nu) \geq \sqrt{2(1 - \mathrm{codist}(\{G_\nu\}))} \geq \sqrt{2P} = \sqrt{\frac{4\varepsilon k}{\varepsilon\lambda_1(\Delta)A + (2k - \lambda_1(\Delta))\,B}} > 0. \qquad \square$$

4. Root systems

The definition of a root system used in this paper is much less restrictive than that of a classical root system. However, most constructions associated with root systems we shall consider are naturally motivated by the classical case.

Definition. Let E be a real vector space. A finite non-empty subset Φ of E is called a *root system in E* if

(a) Φ spans E;
(b) Φ does not contain 0;
(c) Φ is closed under inversion, that is, if $\alpha \in \Phi$ then $-\alpha \in \Phi$.

The dimension of E is called the *rank of* Φ.

Remark. Sometimes we shall refer to the pair (E, Φ) as a root system.

Definition. Let Φ be a root system in E.

(i) Φ is called *reduced* if any line in E contains at most two elements of Φ;
(ii) Φ is called *irreducible* if it cannot be represented as a disjoint union of two non-empty subsets, whose $\mathbb{R}$-spans have trivial intersection.
(iii) A subset Ψ of Φ is called a *root subsystem of* Φ if $\Psi = \Phi \cap \mathbb{R}\Psi$, where $\mathbb{R}\Psi$ is the $\mathbb{R}$-span of Ψ.

The importance of the following definition will be explained later in this section.

Definition. A root system will be called *regular* if any root is contained in an irreducible subsystem of rank 2.

4.1. **Classical root systems.** In this subsection we define classical root systems and state some well-known facts about them. The reader is referred to [Bou, Ch.VI] and [Hu1, Ch.III] for more details.

Definition. A root system Φ in a space E will be called *classical* if E can be given the structure of a Euclidean space with inner product $(\cdot, \cdot)$ such that for any $\alpha, \beta \in \Phi$

(a) $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$;
(b) $\alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta \in \Phi$.

Any inner product on E satisfying (a) and (b) will be called *admissible* .

Fact 4.1. (a) *Every irreducible classical root system is isomorphic to one of the following:* A_n, $B_n (n \geq 2)$, $C_n (n \geq 3)$, $BC_n (n \geq 1)$, $D_n (n \geq 4)$, E_6, E_7, E_8, F_4, G_2. *The only non-reduced systems in this list are those of type* BC_n.

(b) *Every irreducible classical root system of rank* ≥ 2 *is regular.*

Proof. (a) is well known (see, e.g., [Bou, VI.4.2, VI.4.14]), and (b) can be proved by straightforward case-by-case verification. □

If Φ is a classical irreducible root system in a space E, then an admissible inner product $(\cdot, \cdot)$ on E is uniquely defined up to rescaling. In particular, we can compare *lengths* of different roots in Φ without specifying the Euclidean structure. Furthermore, the following hold:

(i) If $\Phi = A_n$, D_n, E_6, E_7 or E_8, all roots in Φ have the same length;
(ii) If $\Phi = B_n$, C_n, F_4 or G_2, there are two different root lengths in Φ;
(iii) If $\Phi = BC_n$, there are three different root lengths in Φ.

As usual, in case (ii), roots of smaller length will be called *short* and the remaining ones called *long* . In case (iii) roots of smallest length will be called *short*, roots of intermediate length called *long* and roots of largest length called *double* . The latter terminology is due to the fact that double roots in BC_n are precisely roots of the form 2α where α is also a root.

Definition. A subset Π of a classical root system Φ is called a *base* (or a *system of simple roots*) if every root in Φ is an integral linear combination of elements of Π with all coefficients positive or all coefficients negative. Thus every base Π of Φ determines a decomposition of Φ into two disjoint subsets $\Phi^+(\Pi)$ and $\Phi^-(\Pi) = -\Phi^+(\Pi)$, called the sets of *positive (resp. negative) roots with respect to* Π.

Example 4.2. Figure 4.1 illustrates each irreducible classical root system of rank 2 with a chosen base $\Pi = \{\alpha, \beta\}$.

If $\Phi = A_2$, then $\Phi^+(\Pi) = \{\alpha, \beta, \alpha + \beta\}$
If $\Phi = B_2$, then $\Phi^+(\Pi) = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$
If $\Phi = BC_2$, then $\Phi^+(\Pi) = \{\alpha, \beta, \alpha + \beta, 2\beta, \alpha + 2\beta, 2\alpha + 2\beta\}$
If $\Phi = G_2$, then $\Phi^+(\Pi) = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}$

Every classical root system Φ in a space E has a base; in fact, the number of (unordered) bases is equal to the order of the Weyl group of Φ. If Π is a base of Φ, it must be a basis of E. Observe that if $f : E \to \mathbb{R}$ is any functional which takes positive values on Π, then

$$\Phi^+(\Pi) = \{\alpha \in \Phi : f(\alpha) > 0\}.$$

Conversely, if $f : E \to \mathbb{R}$ is any functional which does not vanish on any of the roots in Φ, one can show that the set $\Phi_f = \{\alpha \in \Phi : f(\alpha) > 0\}$ coincides with $\Phi^+(\Pi)$ for some base Π. In fact, Π can be characterized as the elements $\alpha \in \Phi_f$ which are not representable as $\beta + \gamma$, with $\beta, \gamma \in \Phi_f$.

4.2. **General root systems.** In this subsection we extend the notions of a base and a set of positive roots from classical to arbitrary root systems. By the discussion at the end of the last subsection, if Φ is a classical root system, the sets of positive roots with respect to different bases of Φ are precisely the *Borel subsets* of Φ as defined below. The suitable generalization of the notion of a base, called

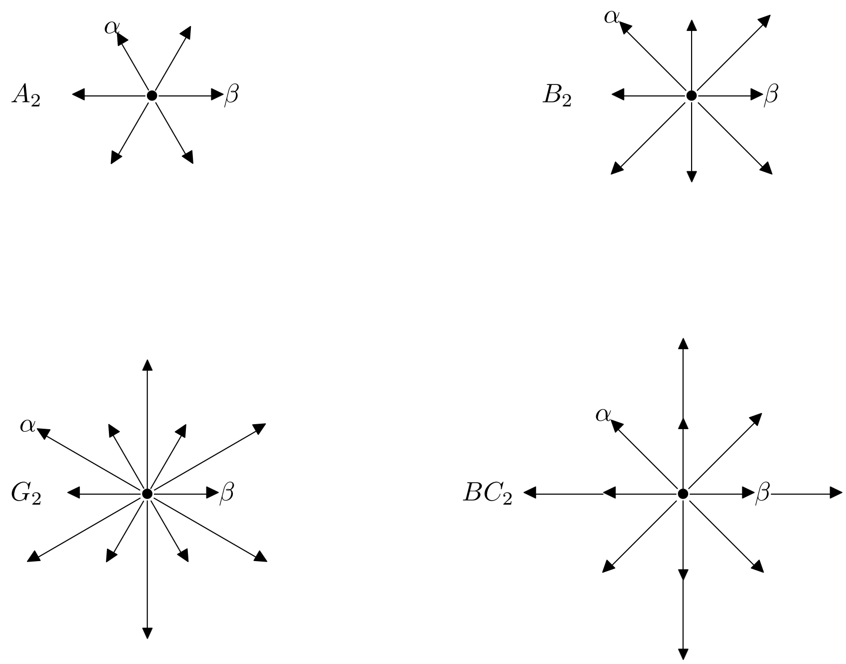


FIGURE 1. Classical irreducible root systems of rank 2.

the *boundary of a Borel set*, is less straightforward and will be given later. The terminology 'Borel subset' will be explained in §4.3.

Remark. If Φ is a reduced classical root system, the notions of boundary and base for Borel subsets of Φ coincide. However, if Φ is not reduced, the boundary of a Borel subset will be larger than its base.

Definition. Let Φ be a root system in a space E. Let $\mathfrak{F} = \mathfrak{F}(\Phi)$ denote the set of all linear functionals $f : E \to \mathbb{R}$ such that

(1) $f(\alpha) \neq 0$ for all $\alpha \in \Phi$;
(2) $f(\alpha) \neq f(\beta)$ for any distinct $\alpha, \beta \in \Phi$.

For $f \in \mathfrak{F}$, the set $\Phi_f = \{\alpha \in \Phi \mid f(\alpha) > 0\}$ is called the *Borel set of* f. The sets of this form will be called *Borel subsets of* Φ. We will say that two elements $f, f' \in \mathfrak{F}$ are equivalent and write $f \sim f'$ if $\Phi_f = \Phi_{f'}$.

Remark. Note that condition (2) implies condition (1) (if $f(\alpha) = 0$, then $f(-\alpha) = f(\alpha)$), but we will not use this fact.

Remark. Observe that for any $f \in \mathfrak{F}$ we can order the elements in Φ_f as follows:

$$\Phi_f = \{\alpha_{f,1}, \alpha_{f,2}, \ldots \alpha_{f,k}\}$$

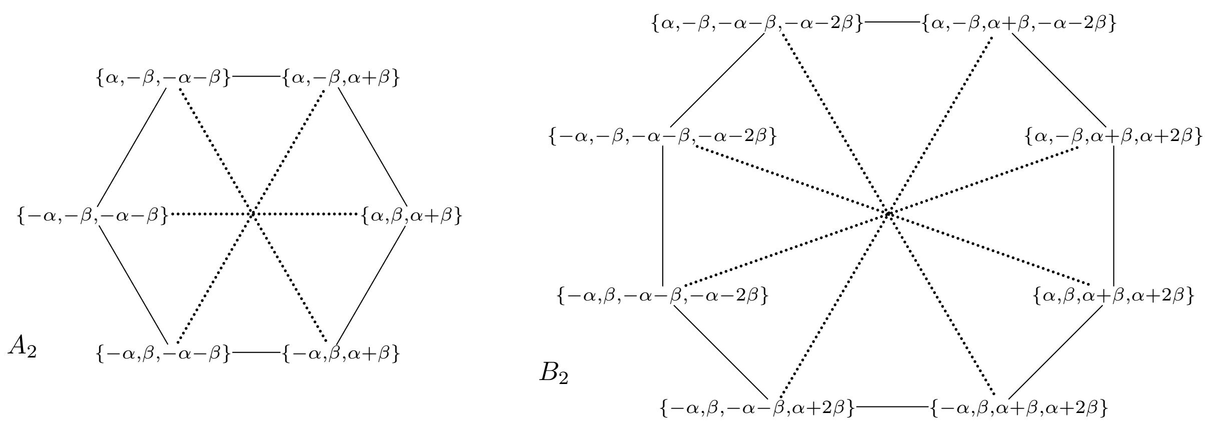


FIGURE 2. Borel Sets in root systems of type A_2 and B_2.

where $k = |\Phi_f| = |\Phi|/2$ and

$$f(\alpha_{f,1}) > f(\alpha_{f,2}) > \cdots > f(\alpha_{f,k}) > 0.$$

If f and g are equivalent functionals, their Borel sets Φ_f and Φ_g coincide, but the orderings on $\Phi_f = \Phi_g$ induced by f and g may be different.

For instance, if $\Phi = A_2$ and $\{\alpha, \beta\}$ is a base of Φ, the functionals f and f' defined by $f(\alpha) = f'(\alpha) = 2$, $f(\beta) = 1$ and $f'(\beta) = 3$ define the same Borel set consisting of the roots α, β and $\alpha + \beta$, however the ordering induced by f and f' are different

$$f(\beta) < f(\alpha) < f(\alpha + \beta) \qquad f'(\alpha) < f'(\beta) < f'(\alpha + \beta).$$

Definition. Let Φ be a root system. Two Borel sets Φ_f and Φ_g will be called

- *opposite* if $\Phi_f \cap \Phi_g = \emptyset$ or, equivalently, $\Phi_g = \Phi_{-f}$;
- *co-maximal* if an inclusion $\Phi_h \supset \Phi_f \cap \Phi_g$ implies that $\Phi_h = \Phi_f$ or $\Phi_h = \Phi_g$;

- *co-minimal* if Φ_f and Φ_{-g} are co-maximal.

Example 4.3. Figure 4.2 shows the Borel sets in root systems of type A_2 and B_2. Pairs of opposite Borel sets are connected with a dotted line and co-maximal ones are connected with a solid line.

Lemma 4.4. *Let Φ be a root system in a space E, and let Φ_f and Φ_g be distinct Borel sets. The following are equivalent:*

(i) *$\Phi_f \cap \Phi_{-g}$ spans one-dimensional subspace*
(ii) *Φ_f and Φ_g are co-maximal*
(iii) *If $h \in \mathfrak{F}$ is such that $\Phi_h \supset \Phi_f \cap \Phi_g$, then $\Phi_h \cap \Phi_g = \Phi_f \cap \Phi_g$ or $\Phi_h \cap \Phi_f = \Phi_f \cap \Phi_g$.*

Proof. (i)$\Rightarrow$ (ii) Since $\Phi_f \cap \Phi_{-g}$ spans one-dimensional subspace, there exists a vector $v \in E$ such that

$$\Phi_f = (\Phi_f \cap \Phi_g) \cup (\Phi_f \cap \Phi_{-g}) \subset (\Phi_f \cap \Phi_g) \cup \mathbb{R}_{>0} v$$

and

$$\Phi_g = (\Phi_f \cap \Phi_g) \cup (\Phi_{-f} \cap \Phi_g) \subset (\Phi_f \cap \Phi_g) \cup \mathbb{R}_{<0} v.$$

Thus, if Φ_h contains $\Phi_f \cap \Phi_g$, then $\Phi_h = \Phi_f$ in the case $h(v) > 0$ or $\Phi_h = \Phi_g$ in the case $h(v) < 0$. Hence Φ_f and Φ_g are co-maximal.

(ii)$\Rightarrow$ (iii) is obvious.

(iii)$\Rightarrow$ (i) Let U be the subspace spanned by $\Phi_f \cap \Phi_{-g}$, and suppose that $\dim U > 1$. Since any sufficiently small perturbation of f does not change its equivalence class, we may assume that the the restrictions of f and g to U are linearly independent.

For any $x \in [0,1]$ consider $h_x = xf + (1-x)g$. Note that $h_0 = g$ is negative on $\Phi_f \cap \Phi_{-g}$ and $h_1 = f$ is positive on $\Phi_f \cap \Phi_{-g}$. Since the restrictions of f and g to U are linearly independent, by continuity there exists $x \in (0,1)$ such that $h = h_x$ is positive on some but not all roots from $\Phi_f \cap \Phi_{-g}$. Thus there exist $\alpha, \beta \in \Phi$ such that $f(\alpha) > 0$, $g(\alpha) < 0$, $h(\alpha) > 0$ and $f(\beta) > 0$, $g(\beta) < 0$, $h(\beta) < 0$, so $\Phi_f \cap \Phi_g \neq \Phi_f \cap \Phi_h$ and $\Phi_f \cap \Phi_g \neq \Phi_h \cap \Phi_g$. On the the hand, since $h = xf + (1-x)g$, it is clear that $\Phi_f \cap \Phi_g$ is contained in Φ_h. This contradicts (iii).

$\square$

Part (a) of the next definition generalizes the notion of a base of a root system.

Definition. (a) *The boundary of a Borel set* Φ_f is the set

$$\partial \Phi_f = \bigcup_g (\Phi_f \setminus \Phi_g) = \bigcup_g (\Phi_f \cap \Phi_{-g}), \text{ where } \Phi_g \text{ and } \Phi_f \text{ are co-maximal.}$$

Equivalently,

$$\partial \Phi_f = \Phi_f \cap (\bigcup_g \Phi_g), \text{ where } \Phi_g \text{ and } \Phi_f \text{ are co-minimal.}$$

(b) *The core of a Borel set* Φ_f is the set

$$C_f = \Phi_f \setminus \partial \Phi_f = \bigcap_g (\Phi_f \cap \Phi_g), \text{ where } \Phi_g \text{ and } \Phi_f \text{ are co-maximal.}$$

If Φ is a classical reduced system and Φ_f is a Borel subset of Φ, it is easy to see that the boundary of Φ_f is precisely the base Π for which $\Phi^+(\Pi) = \Phi_f$. However for non-reduced systems this is not the case and the boundary also contains all roots which are positive multiples of the roots in the base.

Example 4.5. In each of the following examples we consider a classical rank 2 root system Φ, its base $\Pi = \{\alpha, \beta\}$ and the Borel set $\Phi^+(\Pi)$.

1. If $\Phi = A_2$, the core of the Borel set $\{\alpha, \beta, \alpha+\beta\}$ is $\{\alpha+\beta\}$ and the boundary is $\{\alpha, \beta\}$.
2. If $\Phi = B_2$, the core of the Borel set $\{\alpha, \beta, \alpha+\beta, \alpha+2\beta\}$ is $\{\alpha+\beta, \alpha+2\beta\}$ and the boundary is again $\{\alpha, \beta\}$.
3. If $\Phi = BC_2$, the core of the Borel set $\{\alpha, \beta, 2\beta, \alpha+\beta, 2\alpha+2\beta, \alpha+2\beta\}$ is $\{\alpha+\beta, 2\alpha+2\beta, \alpha+2\beta\}$, while the boundary is $\{\alpha, \beta, 2\beta\}$.

Lemma 4.6. *Let Φ be a root system, $f \in \mathfrak{F} = \mathfrak{F}(\Phi)$, and let $\alpha, \beta \in \Phi_f$ be linearly independent. Then any root in Φ of the form $a\alpha + b\beta$, with $a, b > 0$, lies in C_f.*

Proof. Assume the contrary, in which case $a\alpha + b\beta \in \partial\Phi_f$. Thus, there exists $g \in \mathfrak{F}(\Phi)$ such that Φ_f and Φ_g are co-minimal and $a\alpha + b\beta \in \Phi_f \cap \Phi_g$. But then $g(\beta) > 0$ or $g(\alpha) > 0$, so $\Phi_f \cap \Phi_g$ contains linearly independent roots $a\alpha + b\beta$ and α or β. This contradicts Lemma 4.4. □

Lemma 4.7. *Let Φ be a root system, Ψ a subsystem, $f \in \mathfrak{F} = \mathfrak{F}(\Phi)$ and f_0 the restriction of f to $\mathbb{R}\Psi$. Then the core C_{f_0} of Ψ_{f_0} is a subset of the core C_f of Φ_f.*

Proof. Let $v \in C_{f_0}$. We have to show that for any $g \in \mathfrak{F}$ such that Φ_f and Φ_g are co-maximal, $g(v) > 0$. Assume the contrary, that is, $g(v) < 0$. Then by Corollary 4.4, $\Phi_f \cap \Phi_{-g} \subset \mathbb{R}v$. Let g_0 be the restriction of g on Ψ. Then $\emptyset \neq \Psi_{f_0} \cap \Psi_{-g_0} \subseteq \Phi_f \cap \Phi_{-g} \subset \mathbb{R}v$. Again by Corollary 4.4, Ψ_{f_0} and Ψ_{-g_0} are co-minimal, so Ψ_{f_0} and Ψ_{g_0} are co-maximal. Since $v \in C_{f_0}$, we have $g(v) = g_0(v) > 0$, a contradiction. □

Lemma 4.8. *Every root in an irreducible rank 2 system Φ is contained in the core of some Borel set.*

Proof. Let E be the vector space spanned by Φ, and take any $\alpha \in \Phi$. Since Φ is irreducible, there exist $\beta, \gamma \in \Phi$ such that α, β and γ are pairwise linearly independent. Replacing β by $-\beta$ and γ by $-\gamma$ if necessary, we can assume that $\alpha = b\beta + c\gamma$ with $b, c > 0$. If we now take any $f \in \mathfrak{F}(\Phi)$ such that $f(\beta) > 0$ and $f(\gamma) > 0$, then $\alpha \in C_f$ by Lemma 4.6. □

Corollary 4.9. *Every root in a regular root system is contained in the core of some Borel set.*

Proof. This follows from Lemmas 4.7 and 4.8 and the fact that if Ψ is a subsystem of Φ, then any element of $\mathfrak{F}(\Psi)$ is the restriction of some element of $\mathfrak{F}(\Phi)$ to $\mathbb{R}\Psi$. □

4.3. **Weyl graphs.** To each root system Φ we shall associate two graphs $\Gamma_l(\Phi)$ and $\Gamma_s(\Phi)$, called the *large Weyl graph* and the *small Weyl graph*, respectively. Both Weyl graphs $\Gamma_l = \Gamma_l(\Phi)$ and $\Gamma_s = \Gamma_s(\Phi)$ will have the same vertex set:

$$\mathcal{V}(\Gamma_l) = \mathcal{V}(\Gamma_s) = \mathfrak{F}(\Phi)/\sim .$$

Thus vertices of either graph are naturally labeled by Borel subsets of Φ: to each vertex $f \in \mathcal{V}(\Gamma_l) = \mathcal{V}(\Gamma_s)$ we associate the Borel set Φ_f.

- Two vertices f and g are connected in the large Weyl graph Γ_l if and only if their Borel sets are not opposite;
- Two vertices f and g are connected in the small Weyl graph Γ_s if and only if there exists functionals f' and g' such that $\Phi_f \subset C_{f'} \cup \Phi_g$ and $\Phi_g \subset C_{g'} \cup \Phi_f$.

To each (oriented) edge e in $\mathcal{E}(\Gamma_l)$ or $\mathcal{E}(\Gamma_s)$ we associate the set $\Phi_e = \Phi_{e^+} \cap \Phi_{e^-}$. Note that Φ_e is always non-empty by construction.

Remark. If Φ is an irreducible classical root system, both Weyl graphs of Φ are Cayley graphs of $W = W(\Phi)$, the Weyl group of Φ, but with respect to different generating sets. The large Weyl graph $\Gamma_l(\Phi)$ is the Cayley graph with respect to the set $W \setminus \{\alpha_{long}(\Phi)\}$ where $\alpha_{long}(\Phi)$ is the longest element of W relative to the (standard) Coxeter generating set S_Φ.

The generating set corresponding to the small Weyl graph $\Gamma_s(\Phi)$ is harder to describe. At this point we will just mention that it always contains the Coxeter generating set S_Φ, but it is equal to S_Φ only for systems of type A_2.

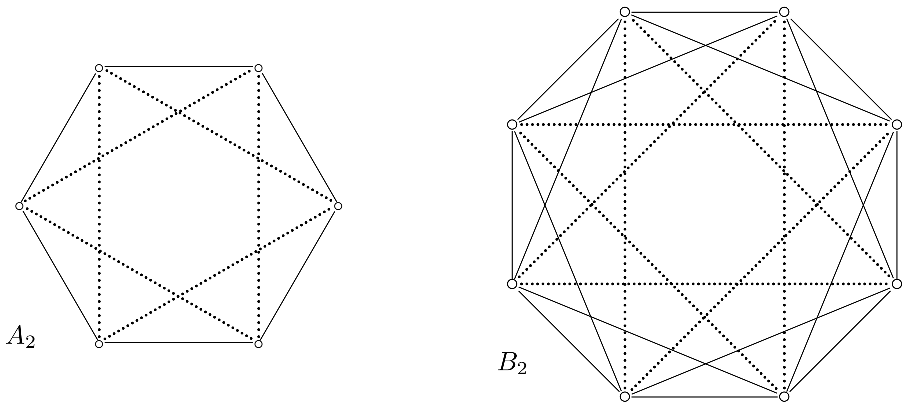


FIGURE 3. Weyl graphs corresponding to root systems of type A_2 and B_2.

Example 4.10. Figure 4.3 shows the Weyl graphs in the root systems of type A_2 and B_2. The edges of the small Weyl graph are denoted by solid lines and ones in the large Weyl graph are either by solid or by dotted lines.

The structure of the large Weyl graph is very transparent.

Lemma 4.11. *Let Φ be a root system.*

(a) *The large Weyl graph $\Gamma_l = \Gamma_l(\Phi)$ is a regular graph with N vertices and degree $N-2$, where N is the number of distinct Borel sets of Φ.*
(b) *The eigenvalues of the adjacency matrix of Γ_l are $N-2$ with multiplicity 1, 0 with multiplicity $N/2$ and -2 with multiplicity $N/2-1$. Therefore the spectral gap of the Laplacian of Γ_l is equal to the degree of Γ_l.*

Proof. (a) is clear. (b) A constant function is an eigenvector with eigenvalue $N-2$, any "antisymmetric" function (one with $F(x) = -F(\bar{x})$ where x and $\bar{x}$ are opposite vertices) has eigenvalue 0, and the space of antisymmetric functions has dimension $N/2$. Finally, any "symmetric" function with sum 0 is an eigenfunction with eigenvalue -2, and the space of such functions has dimension $N/2-1$. □

The role played by the small Weyl graph in this paper will be discussed at the end of this section. The key property we shall use is the following lemma:

Lemma 4.12. *Let Φ be a regular root system. Then the graph $\Gamma_s(\Phi)$ is connected.*

Proof. Let $f, g \in \mathfrak{F}$ be two functionals such that Φ_f and Φ_g are distinct. We prove that there exists a path in Γ_s from f to g by downward induction of $|\Phi_f \cap \Phi_g|$. If Φ_f and Φ_g are co-maximal, then f and g are connected (by an edge) in the small Weyl graph Γ_s by Lemma 4.4 and Corollary 4.9. If Φ_f and Φ_g are not co-maximal, then by Lemma 4.4 there exists h such that $\Phi_f \cap \Phi_g$ is properly contained in $\Phi_h \cap \Phi_f$ and $\Phi_h \cap \Phi_g$. By induction, there are paths that connects h with both f and g. Hence f and g are connected by a path in Γ_s. □

Corollary 4.13. *Both large and small Weyl graphs of any irreducible classical root system of rank ≥ 2 are connected.*

We have computed the diameter of the small Weyl graph for some root systems, and in all these examples the diameter is at most 3. We believe that this is true in general.

Conjecture 4.14. *If Φ is an irreducible classical root system of rank ≥ 2, then the diameter of Γ_s is at most* 3.

4.4. **Groups graded by root systems.**

Definition. Let Φ be a root system and G a group. A Φ-*grading of G* (or just *grading of G*) is a collection of subgroups $\{X_\alpha\}_{\alpha\in\Phi}$ of G, called *root subgroups* such that

(i) G is generated by $\cup X_\alpha$;
(ii) For any $\alpha, \beta \in \Phi$, with $\alpha \notin \mathbb{R}_{<0}\beta$, we have

$$[X_\alpha, X_\beta] \subseteq \langle X_\gamma \mid \gamma = a\alpha + b\beta \in \Phi,\ a, b \geq 1\rangle$$

If $\{X_\alpha\}_{\alpha\in\Phi}$ is a collection of subgroups satisfying (ii) but not necessarily (i), we will simply say that $\{X_\alpha\}_{\alpha\in\Phi}$ is a Φ-*grading* (without specifying the group).

Each grading of a group G by a root system Φ determines canonical graph of groups decompositions of G over the large and small Weyl graphs of Φ. The vertex and edge subgroups in these decompositions are defined as follows.

Definition. Let Φ be a root system, G a group and $\{X_\alpha\}_{\alpha\in\Phi}$ a Φ-grading of G. For each $f \in \mathcal{V}(\Gamma_l) = \mathcal{V}(\Gamma_s)$ we set

$$G_f = \langle X_\alpha \mid \alpha \in \Phi_f\rangle,$$

and for each $e \in \mathcal{E}(\Gamma_l) \supset \mathcal{E}(\Gamma_s)$ we set

$$G_e = \langle X_\alpha \mid \alpha \in \Phi_e\rangle.$$

We will call G_f the *Borel subgroup of G corresponding to f*.

Remark. We warn the reader that our use of the term 'Borel subgroup' is potentially misleading. Assume that Φ is classical, irreducible and reduced. Let F be a field and $G = \mathbb{E}_\Phi(F) = \mathbb{G}_\Phi(F)$ the corresponding simply-connected Chevalley group over F. Let $\{X_\alpha\}_{\alpha\in\Phi}$ be the root subgroups (relative to the standard torus H), so that $\{X_\alpha\}$ is a Φ-grading of G. Then Borel subgroups of G in our sense are smaller than Borel subgroups in the sense of Lie theory. In fact, Borel subgroups in our sense are precisely the unipotent radicals of those Borel subgroups in the sense of Lie theory which contain H. Equivalently, our Borel subgroups are maximal unipotent subgroups of G normalized by H.

Example 4.15. If G is a group graded by a root system of type A_2, Figure 4.4 shows the canonical decomposition of G over the large Weyl graph of A_2, called the "magic graph" in [EJ].

Definition. Let $\{X_\alpha\}$ be a Φ-grading of a group G. For each $f \in \mathfrak{F}(\Phi)$, the *core subgroup G_{C_f}* of G_f is the subgroup generated by the root subgroups in the core, that is,

$$G_{C_f} = \langle X_\alpha \mid \alpha \in C_f\rangle.$$

Lemma 4.16. *In the above notations, for each $f \in \mathfrak{F}$ the core subgroup G_{C_f} is a normal subgroup of G_f.*

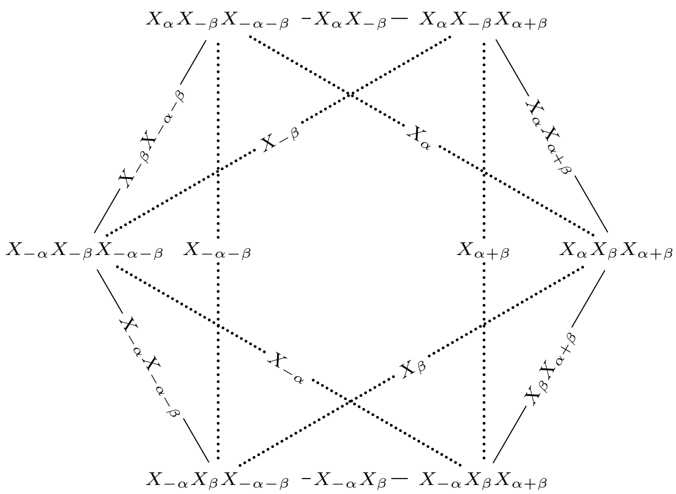


FIGURE 4. Weyl graph of groups for a groups graded by a root system of type A_2.

Proof. This is an immediate consequence of Lemma 4.6. □

Definition. Let Φ be a root system and $\{X_\alpha\}_{\alpha\in\Phi}$ a Φ-grading.

(i) Take any Borel subset Φ_f (with $f \in \mathfrak{F}(\Phi)$) and any root $\gamma \in C_f$. We will say that the grading $\{X_\alpha\}$ is *strong at the pair* (γ, Φ_f) if

$$X_\gamma \subseteq \langle X_\beta \mid \beta \in \Phi_f \text{ and } \beta \notin \mathbb{R}\gamma\rangle.$$

(ii) We will say that the grading $\{X_\alpha\}$ is *strong* if $\{X_\alpha\}$ is strong at every pair (γ, Φ_f) (with $\gamma \in C_f$).

(iii) Given an integer k, we will say that the grading $\{X_\alpha\}_{\alpha\in\Phi}$ is *k-strong* if for any irreducible subsystem Ψ of rank k of Φ the grading $\{X_\alpha\}_{\alpha\in\Psi}$ is strong.

Remark. In § 7 and § 8 we will need to verify that the natural gradings of certain Steinberg groups and twisted Steinberg groups are strong. With the exception of § 8.8, all examples we will consider satisfy the following property:

For any two functionals $f, f' \in \mathfrak{F}(\Phi)$ there exists an automorphism $w \in \mathrm{Aut}(G)$ which permutes the root subgroups $\{X_\alpha\}_{\alpha\in\Phi}$ between themselves, and the induced action of w on Φ sends the Borel subset Φ_f to the Borel subset $\Phi_{f'}$.

In the presence of this property, in order to prove that the grading $\{X_\alpha\}$ is strong it suffices to check that it is strong at (γ, Φ_f) where $f \in \mathfrak{F}(\Phi)$ is a fixed functional and γ runs over C_f. In each of our examples, we shall use a functional f such that Φ_f is the set of positive roots (with respect to a fixed system of simple roots). To simplify the terminology, we shall say that the grading is *strong at* γ if it is strong at (γ, f) for the f that we fixed.

Example 4.17. Let Φ be a root system of type A_2 and let $\{X_\gamma\}$. A sufficient condition for the grading to be strong is that $[X_\alpha, X_\beta] = X_{\alpha+\beta}$ for any pairs of roots α and β such that $\alpha+\beta$ is also a root. This condition is also necessary under the additional assumption that every element in a Borel subgroup can be expressed uniquely as a product of elements in the 3 root subgroups (put in some fixed order).

Example 4.18. Let Φ be a root system of type B_2 and let $\{X_\gamma\}$ be a Φ-grading. Assume that there exists an abelian group R such that each of the root subgroups $\{X_\gamma\}$ is isomorphic to R; thus we can denote the elements of X_γ by $\{x_\gamma(r) : r \in R\}$ so that $x_\gamma(r+s) = x_\gamma(r)x_\gamma(s)$.

Now let $\{\alpha, \beta\}$ be a base of Φ, with α a long root. Let f be any functional such that $\partial\Phi_f = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta\}$ (in which case $C_f = \{\alpha+\beta, \alpha+2\beta\}$). By definition of grading there exist functions $p, q : R \times R \to R$ such that

$$[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(p(r,s))x_{\alpha+2\beta}(q(r,s)) \text{ for all } r, s \in S$$

Then the grading is strong at the pair $(\alpha+\beta, f)$ (resp. $(\alpha+2\beta, f)$) whenever the image of p (resp. q) generates R as a group.

If Φ is a non-reduced root system, it is sometimes useful to slightly modify a given Φ-grading using a simple operation called fattening:

Definition. Let Φ be a non-reduced root system and $\{X_\alpha\}_{\alpha\in\Phi}$ a Φ-grading of some group G. For each $\alpha \in \Phi$ we set $\widetilde{X}_\alpha = \langle X_{a\alpha} : a \geq 1\rangle$. We will say that $\{\widetilde{X}_\alpha\}_{\alpha\in\Phi}$ is the fattening of the grading $\{X_\alpha\}_{\alpha\in\Phi}$.

It is easy to see that the fattening $\{\widetilde{X}_\alpha\}$ is also Φ-grading. Moreover, $\{\widetilde{X}_\alpha\}$ is strong whenever $\{X_\alpha\}$ is strong.

4.5. **A few words about the small Weyl graph.** We end this section explaining how the small Weyl graph and the notion of the core of a Borel set will be used in this paper. Unlike the large Weyl graph, which plays a central role in the proof of Theorem 5.1, the small Weyl graph is just a convenient technical tool.

As discussed in § 3, given a group G graded by a regular root system Φ, Theorem 5.1 for G will be proved by applying the generalized spectral criterion (Theorem 3.3) to the canonical decomposition of G over the large Weyl graph Γ_l. The small Weyl graph Γ_s will be used to verify hypothesis (iii) in Theorem 3.3.

In fact, for many root systems we could use a different definition of the core of a Borel subset (leading to a different small Weyl graph and different core subgroups) which would still work for applications in § 5. We could not make the cores any larger than we did (otherwise hypothesis (ii) in Theorem 3.3 would not hold), but we could often make then smaller – the only properties we need is that the small Weyl graph is connected (Lemma 4.12) and the core subgroups are normal in the ambient vertex groups.

For instance, if Φ is a simply-laced classical root system, we could let C_f consist of just one root, namely, the root of maximal height in the Borel set Φ_f (this definition coincides with ours for $\Phi = A_2$). In this case the small Weyl graph would become the Cayley graph of the Weyl group $W(\Phi)$ with respect to the standard Coxeter generating set. If Φ is a non-simply-laced classical root system, this seemingly more natural definition of the core does not work, although we could still make the core smaller except when $\Phi = B_2$ or BC_2.

5. Property (T) for groups graded by root systems

In this section we prove Theorem 1.2, in fact a slightly generalized version of it dealing with groups graded by regular (not necessarily classical) root systems.

Theorem 5.1. *Let Φ be a regular root system, and let G be a group which admits a strong Φ-grading $\{X_\alpha\}$. Then $\cup X_\alpha$ is a Kazhdan subset of G, and moreover the Kazhdan constant $\kappa(G, \cup X_\alpha)$ is bounded below by a constant κ_Φ which depends only on the root system Φ.*

Theorem 5.1 will be established by applying the generalized spectral criterion from Theorem 3.3 to the canonical decomposition of G over the large Weyl graph of Φ. Thus, we need to show that the hypotheses (i)-(iii) in Theorem 3.3 are satisfied in this setting.

5.1. **Estimates of codistances in nilpotent groups.** We start by proving an upper bound on codistances between certain families of subgroups in nilpotent groups (Lemma 5.2 below). While this result is quite technical, once it is established, verification of conditions (i) and (ii) in the proof of Theorem 5.1 will be rather straightforward.

Lemma 5.2. *Let N be a nilpotent group, and let $\{X_i\}_{i=1}^n$ be a finite family of subgroups of N such that for each $1 \le i \le n$, the product set $N_i = \prod_{j=i}^n X_j$ is a normal subgroup of N, $N_1 = N$ and $[N_i, N] \subseteq N_{i+1}$ for each i.*

Suppose now that we are given another family $\{K_j\}_{j=1}^m$ of subgroups of N and an integer l such that for each i, the inclusion $X_i \subseteq K_j$ holds for at least l distinct indices j. The following hold:

(a) $\mathrm{codist}(K_1, \dots, K_m) \le \frac{m-l}{m}$

(b) *For each $1 \le i \le n$ let H_i be the subgroup generated by $\{\cup K_j : X_i \not\subseteq K_j\}$, and let C be a normal subgroup of N which is contained in the intersection $\bigcap_{i=1}^n H_i$. Then for any representation V of N without C-invariant vectors we have*
$$\mathrm{codist}(V^{K_1}, \dots, V^{K_m}) \le \frac{m-l}{m} \cdot (1-\delta),$$
where $\delta = \frac{8}{(m-2)4^c}$ and c is the nilpotency class of N.

Proof. Let V be a unitary representation of N without invariant vectors. For each $1 \le i \le n$ let $V_i = V^{N_i}$ and $V_i^\perp$ the orthogonal complement of V_i in V. Since N_i is normal in N, both V_i and $V_i^\perp$ are N-invariant. Finally, let $V_{(i)} = V_{i-1} \cap V_i^\perp$. Since $V^N = \{0\}$ by assumption, we clearly have the decomposition
$$V = \bigoplus_i V_{(i)}.$$

Let $\pi_i : V \to V_{(i)}$ be the orthogonal projection. Thus for any $v \in V$ we have $v = \sum_{i=1}^n \pi_i(v)$. Let $\Omega_i = \{1 \le j \le m : X_i \subseteq K_j\}$, and note that by assumption $|\Omega_i| \ge l$ for each i.

Claim 5.3. *Let $v \in V^{K_j}$ for some j. Then $\pi_i(v) \in V^{K_j}$ for all i, and moreover $\pi_i(v) = 0$ if $j \in \Omega_i$.*

Proof. Since each of the groups N_i is normalized by K_j and v is K_j-invariant, its projection $\pi_i(v)$ is also K_j-invariant, which proves the first assertion. On the other hand, by construction $V_{(i)}$ has no X_i-invariant vectors. Hence if $j \in \Omega_i$, then $V_{(i)}$ has no K_j-invariant vectors, and thus $\pi_i(v) = 0$. □

We are now ready to prove both assertions of Lemma 5.2.

(a) By definition of codistance, we need to show that given any vectors $v_j \in V^{K_j}$ for $1 \leq j \leq m$, we have

$$\left\| \sum_{j=1}^{m} v_j \right\|^2 \leq (m-l) \sum \|v_j\|^2.$$

Using the decomposition of V as the direct sum of $V_{(i)}$ we obtain

$$\left\| \sum_{j=1}^{m} v_j \right\|^2 = \left\| \sum_i \sum_{j=1}^{m} \pi_i(v_j) \right\|^2 = \sum_i \left\| \sum_{j=1}^{m} \pi_i(v_j) \right\|^2$$

Using Claim 5.3 and the fact that $|\Omega_i| \geq l$, we get

$$\begin{aligned} \sum_i \left\| \sum_{j=1}^{m} \pi_i(v_j) \right\|^2 &= \sum_i \left\| \sum_{j \notin \Omega_i} \pi_i(v_j) \right\|^2 \leq \sum_i (m-l) \sum_{j \notin \Omega_i} \|\pi_i(v_j)\|^2 \\ &= (m-l) \sum_{j=1}^{m} \sum_i \|\pi_i(v_j)\|^2 = (m-l) \sum_{j=1}^{m} \|v_j\|^2. \end{aligned} \tag{5.1}$$

(b) Suppose now that V has no C-invariant vectors, and fix i with $1 \leq i \leq n$. By assumption, $C \subseteq H_i$, so V has no H_i-invariant vectors as well. Recall that $H_i = \langle K_j : j \notin \Omega_i \rangle$, and of course, H_i is nilpotent of class $\leq c$. Therefore, by Theorem 2.8 we have $\mathrm{codist}(\{K_j : j \notin \Omega_i\}) \leq 1 - \delta$ where δ is as in the statement of Lemma 5.2(b). Equivalently, given vectors $v_j \in K_j$ for $j \notin \Omega_i$ we have

$$\left\| \sum_{j \notin \Omega_i} \pi_i(v_j) \right\|^2 \leq (1-\delta)(m-l) \sum_{j \notin \Omega_i} \|\pi_i(v_j)\|^2.$$

The result of (b) now follows by combining this bound with the same calculation as in (5.1). □

5.2. **Estimates of codistances in Borel subgroups.** For the rest of this section we fix a regular root system Φ and a group G with a strong Φ-grading $\{X_\alpha\}$. Let $\Gamma_l = \Gamma_l(\Phi)$ be the large Weyl graph of Φ. For each vertex of Γ_l we fix a functional $f \in \mathfrak{F}$ representing that vertex. The vertex itself will also be denoted by f, and the associated vertex subgroup will be denoted by G_f.

In this subsection the vertex f of Γ_l will be fixed, and let E_f denote the set of all edges $e \in \mathcal{E}(\Gamma_l)$ with $e^+ = f$. We shall use Lemma 5.2 to obtain the following bound on codistances between edge subgroups of G_f:

Proposition 5.4. *The following hold:*

(a) $\mathrm{codist}(\{G_e : e \in E_f\}) \leq \frac{1}{2}$.

(b) *Let CG_f be the core subgroup of G_f and V a unitary representation of G_f without CG_f-invariant vectors. Then*

$$\operatorname{codist}(\{V^{G_e} : e \in E_f\}) \leq \frac{1-\varepsilon_\Phi}{2},$$

where $\varepsilon_\Phi = \frac{8}{(bor(\Phi)-2)4^{|\Phi|/2}}$ and $bor(\Phi)$ is the number of Borel subsets in Φ.

Proof. Let $\{\alpha_{f,i}\}_{i=1}^{|\Phi|/2}$ be a (fixed) ordering of the roots in the Borel set Φ_f induced by f, that is, we assume that

$$f(\alpha_{f,1}) < f(\alpha_{f,2}) < \ldots < f(\alpha_{f,|\Phi/2|}).$$

We shall apply Lemma 5.2 by letting $N = G_f$, $C = CG_f$, $n = |\Phi|$, $m = |E_f|$, $X_i = X_{\alpha_{f,i}}$ for $1 \leq i \leq n$ and $K_1, \ldots, K_m$ be the edge subgroups $\{G_e : e \in E_f\}$ listed in an arbitrary order.

The required conditions on the subgroups N_i introduced in the statement of Lemma 5.2 hold by our ordering of roots.

For each $1 \leq i \leq n$ set

$$E_{f,i} = \{e \in E_f : \alpha_{f,i} \notin \Phi_e\}$$

Claim 5.5. *For any i we have $|E_{f,i}| = |E_f|/2$.*

Proof. The neighbors of the vertex f in Γ_l can be grouped in pairs consisting of two opposite Borel sets. For any pair of opposite Borel subsets the root $\alpha_{f,i}$ lies in exactly one of them, which yields the claim. □

Note that $X_i \subseteq G_e$ if and only if $e \notin E_{f,i}$, so by Claim 5.5, we can take $l = m/2 = |E_f|/2$ in the statement of Lemma 5.2. Thus, Proposition 5.4(a) follows from Lemma 5.2(a).

To deduce Proposition 5.4(b) from Lemma 5.2(b) we only need to check that for each $1 \leq i \leq n$, the core subgroup CG_f is contained in the group $G_{f,i}$ defined by

$$G_{f,i} = \langle \cup G_e : e \in E_{f,i} \rangle.$$

This is established in Claim 5.7 below.

Claim 5.6. *The set*

$$\bigcup \{\Phi_e : \ e \in E_{f,i}\}$$

contains all roots from Φ_f, which are not multiples of $\alpha_{f,i}$.

Proof. Let $\beta \in \Phi_f$, and assume that β is not a multiple of $\alpha_{f,i}$. Then there exists another functional $f' \in \mathfrak{F}$ such that $\beta \in \Phi_{f'}$ but $\alpha_{f,i} \notin \Phi_{f'}$. Hence Φ_f and $\Phi_{f'}$ are connected by an edge $e \in E_f$ (because $\Phi_f \neq \Phi_{f'}$ and $\Phi_f \cap \Phi_{f'} \neq \emptyset$). Then $\beta \in \Phi_e$ but $\alpha_{f,t} \notin \Phi_e$. Thus by definition $e \in E_{f,i}$ and β lies in the set defined above. □

Claim 5.7. *The core subgroup CG_f is contained in $G_{f,i}$ for each i.*

Proof. By Claim 5.6 the group $G_{f,i}$ contains the root subgroup X_β for each $\beta \in \Phi_f$ which is not a multiple of $\alpha_{f,i}$.

If the root $\alpha_{f,i}$ lies on the boundary of Φ_f, then the set $\Phi_f \setminus \mathbb{R}\alpha_{f,i}$ contains the core $C\Phi_f$, and thus $G_{f,i}$ contains the core subgroup CG_f. If the root $\alpha_{f,t}$ lies in the core $C\Phi_f$, the inclusion $CG_f \subseteq G_{f,i}$ follows from the assumption that the grading is strong.

This finishes the proof of Claim 5.7 and thus also the proof of Proposition 5.4. □

5.3. **Norm estimates.** In this subsection we establish the "norm inequality" (Corollary 5.13) which is needed to verify hypothesis (iii) in Theorem 3.3. This inequality will be proved by considering both the small and the large Weyl graphs. We note that this is the only part of the paper where the small Weyl graph is used. In this subsection we assume that V is a *representation of the whole group* G.

Recall that the large Weyl graph Γ_l and the small Weyl graph Γ_s have the same sets of vertices. Also recall that $\Omega^0(\Gamma_l, V) = \Omega^0(\Gamma_s, V)$ is the set of all functions from $\mathcal{V}(\Gamma_l) = \mathcal{V}(\Gamma_s)$ to V and $\Omega^0(\Gamma_l, V)^{\{G_\nu\}}$ the set of all functions $g \in \Omega^0(\Gamma_l, V)$ such that $g(f) \in V^{G_f}$ for each vertex f. Denote by d_l and d_s the difference operators of Γ_l and Γ_s, respectively.

Lemma 5.8. *Let $g \in \Omega^0(\Gamma_l, V)^{\{G_\nu\}} = \Omega^0(\Gamma_s, V)^{\{G_\nu\}}$. If Φ is a regular root system, then*

(a) $\|d_s g\|^2 \leq \|d_l g\|^2$
(b) $\|d_l g\|^2 \leq C_\Phi \|d_s g\|^2$,

where the constant C_Φ depends only on the root system.

Proof. (a) is clear since Γ_s is a subgraph of Γ_l and (b) holds since Γ_s is connected. Indeed, for each edge e in $\mathcal{E}(\Gamma_l)$ we can find a path in Γ_s connecting the endpoints and write $g(e^+) - g(e^-) = \sum_i (g(e_i^+) - g(e_i^-))$. Use the triangle inequality we get

$$\|g(e^+) - g(e^-)\|^2 \leq k \sum_i \|g(e_i^+) - g(e_i^-)\|^2,$$

where k is the length of the path. □

Example 5.9. The constant C_Φ can be easily computed for "small" root systems, e.g., one can take $C_{A_2} = 5$, $C_{B_2} = C_{BC_2} = 3$ and $C_{G_2} = 2$. It is unclear how the constant C_Φ depends on the rank of the root system.

For the rest of this subsection, for a subgroup H of G we denote by

$$\pi_H : V \to V^H \qquad \text{and} \qquad \pi_{H^\perp} : V \to (V^H)^\perp$$

the projections onto V^H and its orthogonal complement $(V^H)^\perp$, respectively.

For an edge e of Γ_l we let $GR_e = \langle X_\alpha : \alpha \in \Phi_{e^+} \setminus \Phi_e \rangle$

Claim 5.10. *For any edge e of Γ_l and any $v \in V^{G_e}$ we have $\pi_{G_{e^+}}(v) = \pi_{GR_e}(v)$.*

Proof. Let $k = |\Phi_e|$ and let $\{\beta_j\}_{1 \leq j \leq k}$ be the roots in Φ_e ordered so that

$$f(\beta_1) > f(\beta_2) > \ldots > f(\beta_k).$$

For $1 \leq i \leq k$ let H_i be the subgroup generated by GR_e and $\{X_{\beta_j}\}_{1 \leq j \leq i}$. By construction $H_0 = GR_e$, $H_k = G_{e^+}$ and each H_i is normalized by $X_{\beta_{i+1}}$.

By assumption, the vector v is $X_{\beta_{i+1}}$-invariant for each $0 \leq i \leq k-1$. Hence $\pi_{H_i}(v)$ is also $X_{\beta_{i+1}}$-invariant, so $\pi_{H_i}(v) = \pi_{H_{i+1}}(v)$. Combining these equalities for all i we get $\pi_{G_{e^+}}(v) = \pi_{H_k}(v) = \pi_{H_0}(v) = \pi_{GR_e}(v)$. □

Now recall that $\Omega^1(\Gamma_l, V)$ (resp. $\Omega^1(\Gamma_s, V)$) is the space of all functions from $\mathcal{E}(\Gamma_l)$ (resp. $\mathcal{E}(\Gamma_s)$) to V. Notice that these two spaces are different unlike the spaces $\Omega^0(\Gamma_l, V) = \Omega^0(\Gamma_s, V)$.

As in § 3 we have projections ρ_1, ρ_2, ρ_3 defined on each of those spaces with $\|\rho_1(g)\|^2 + \|\rho_2(g)\|^2 + \|\rho_3(g)\|^2 = \|g\|^2$ for all g. Note that in our new notations, for any $g \in \Omega^1(\Gamma_l, V)$ (resp. $g \in \Omega^1(\Gamma_s, V)$) and $e \in \mathcal{E}(\Gamma_l)$ (resp. $e \in \mathcal{E}(\Gamma_s)$) we have

$$\rho_1(g)(e) = \pi_{G_{e^+}}(g(e)) \qquad \rho_3(g)(e) = \pi_{CG^{\perp}_{e^+}}(g(e)).$$

The following is the main result of this subsection:

Theorem 5.11. *For any $g \in \Omega^0(\Gamma_l, V)^{\{G_\nu\}}$ we have*

$$\|d_s g\|^2 \leq \|\rho_1(d_s g)\|^2 + D_\Phi \|\rho_3(d_l g)\|^2,$$

where the constant D_Φ depends only on the root system.

Remark. Notice that the second term on the right hand side involves the differential of the large Borel graph, while the other two terms involve the differential of the small Borel graph.

Proof. Let e be an edge in the small Borel graph Γ_s. Since $g(e^+) = \pi_{G_{e^+}}(g(e^+)) = \pi_{GR_e}(g(e^+))$ and $g(e^-) \in V^{G_e}$, by Claim 5.10 we have

$$\|g(e^+) - g(e^-)\|^2 = \|\pi_{GR_e}\big(g(e^+) - g(e^-)\big)\|^2 + \|\pi_{GR_e{}^\perp}\big(g(e^+) - g(e^-)\big)\|^2 =$$
$$\|\pi_{G_{e^+}}\big(g(e^+) - g(e^-)\big)\|^2 + \|\pi_{GR_e{}^\perp}(g(e^-))\|^2.$$

By the definition of the small Borel graph, we can find another vertex f' such that

$$\Phi_{e^+} \setminus \Phi_{e^-} = \Phi_{e^+} \setminus \Phi_e \subset C_{f'}.$$

This implies that $GR_e \subseteq CG_{f'} \subset G_{f'}$, and so $\pi_{GR_e{}^\perp}(g(f')) = 0$. Therefore

$$\|g(e^+) - g(e^-)\|^2 = \|\pi_{G_{e^+}}\big(g(e^+) - g(e^-)\big)\|^2 + \|\pi_{GR_e{}^\perp}\big(g(f') - g(e^-)\big)\|^2.$$

Since clearly $f' \neq e^-$, there is an edge e' in Γ_l connecting f' and e^-, with $(e')^+ = f'$ and $(e')^- = e^-$. The inclusion $GR_e \subseteq CG_{f'}$ implies that

$$\|\pi_{GR_e{}^\perp}\big(g(f') - g(e^-)\big)\|^2 \leq \|\pi_{CG_{f'}{}^\perp}\big(g(f') - g(e^-)\big)\|^2 =$$
$$\|\rho_3(d_l g)(e')\|^2 \leq \|\rho_3(d_l g)\|^2.$$

Summing over all edges e of Γ_s we get

$$\|d_s g\|^2 = \|\rho_1(d_s g)\|^2 + \sum_{e \in \mathcal{E}(\Gamma_s)} \|\pi_{GR_e{}^\perp}\big(g(f') - g(e^-)\big)\|^2 \leq$$
$$\|\rho_1(d_s g)\|^2 + \frac{|\mathcal{E}(\Gamma_s)|}{2}\|\rho_3(d_l g)\|^2. \quad \square$$

Example 5.12. Carefully doing the above estimates in the case of some "small" root systems gives that once can take $D_\Phi = 1$ if Φ is of type A_2, B_2, BC_2 or G_2.

Combining Lemma 5.8(b), Theorem 5.11 and the obvious inequality $\|\rho_1(d_s g)\| \leq \|\rho_1(d_l g)\|$, we obtain the desired inequality, which verifies hypothesis (iii) of Theorem 3.3 in our setting. Its statement only involves the large Weyl graph:

Corollary 5.13. *Let g be a function in $\Omega^0(\Gamma_l, V)^{\{G_\nu\}}$. Then*

$$\|d_l g\|^2 \leq A_\Phi \|\rho_1(d_l g)\|^2 + B_\Phi \|\rho_3(d_l g)\|^2,$$

where A_Φ and B_Φ are constants which depend on the root system Φ.

Example 5.14. In the case $\Phi = A_2$, the above estimates show that one can take $A_{A_2} = 5$ and $B_{A_2} = 5$. These bounds are not optimal — in [EJ] it is shown that one can use $A_{A_2} = 3$ and $B_{A_2} = 5$.

5.4. **Proof of Theorem 5.1.**

Proof of Theorem 5.1. As explained at the beginning of this section, we shall apply Theorem 3.3 to the canonical decomposition of G over the large Weyl graph $\Gamma = \Gamma_l \Phi$. Let us check that inequalities (i)-(iii) are satisfied.

By Lemma 4.11 the spectral gap of the Laplacian $\lambda_1(\Delta)$ is equal to the degree of Γ. Hence in the notations of Theorem 3.3 we have $\bar{p} = 1/2$. Thus, (i) and (ii) hold by Proposition 5.4. Finally, (iii) holds by Corollary 5.13.

Since all parameters in these inequalities depend only on Φ, Theorem 3.3 yields that $\kappa(G, \cup G_f) \geq \mathcal{K}_\Phi > 0$, with $\mathcal{K}_\Phi$ depending only on Φ. Finally, to obtain the same conclusion with G_f's replaced by root subgroups X_α's, we only need to observe that each G_f lies in a bounded product of root subgroups, where the number of factors does not exceed $|\Phi|/2$. □

6. Reductions of root systems

6.1. **Reductions.**

Definition. Let Φ be a root system in a space $V = \mathbb{R}\Phi$. A *reduction of* Φ is a surjective linear map $\eta : V \to V'$ where V' is another nonzero real vector space. The set $\Phi' = \eta(\Phi) \setminus \{0\}$ is called the *induced root system*. We will also say that η is a *reduction of* Φ *to* Φ' and symbolically write $\eta : \Phi \to \Phi'$.

Lemma 6.1. *Let Φ be a root system, η a reduction of Φ, and Φ' the induced root system. Let $\{X_\alpha\}_{\alpha\in\Phi}$ be a Φ-grading. For any $\alpha' \in \Phi'$ put*

$$Y_{\alpha'} = \langle X_\alpha \mid \eta(\alpha) = \alpha' \rangle.$$

Then $\{Y_{\alpha'}\}_{\alpha'\in\Phi'}$ is a Φ'-grading, which will be called the coarsened *grading.*

Proof. This is a direct consequence of the following fact: if $A = \langle S_1 \rangle$ and $B = \langle S_2 \rangle$ are two subgroups of the same group, then $[A, B]$ is contained in the subgroup generated by all possible commutators in $S_1 \cup S_2$ of length at least 2 with at least one entry from S_1 and S_2. □

A reduction $\eta : \Phi \to \Phi'$ enables us to replace a grading of a given group G by the "large" root system Φ by the coarsened grading by the "small" root system Φ' which may be easier to analyze. Note that the root subgroups of the coarsened grading need not generate G since we "lose" root subgroups of the initial grading which lie in $\ker \eta$. Likewise, since different roots of Φ may map to the same root of Φ', the coarsened grading need not be strong even if the initial grading is strong.

Since we are mostly interested in strong gradings, we would like to have a natural sufficient condition on η and the initial Φ-grading which ensures that the coarsened Φ'-grading is strong. If the only assumption on the initial Φ-grading $\{X_\alpha\}$ is that it is strong, we would limit ourselves to reductions with trivial kernel (which are not interesting). However, if we assume that $\{X_\alpha\}$ is k-strong for some $k < \mathrm{rk}(\Phi)$, we can let η be any k-good reduction, as defined below, which is much less restrictive.

Definition. Let $k \geq 2$ be an integer. A reduction η of Φ to Φ' is called *k-good* if

(a) for any $\gamma \in \ker \eta \cap \Phi$, there exists an irreducible regular subsystem Ψ of Φ of rank k such that $\gamma \in \Psi$ and $\ker \eta \cap \Psi \subseteq \mathbb{R}\gamma$;
(b) for any $f \in \mathfrak{F}(\Phi')$, $\gamma' \in C_f$ and $\gamma \in \Phi$ with $\eta(\gamma) = \gamma'$, there exists an irreducible subsystem Ψ of Φ of rank k and $g \in \mathfrak{F}(\Psi)$ such that $\gamma \in C_g$, $\eta(\Psi_g) \subseteq \Phi'_f$ and $\Psi \cap \eta^{-1}(\mathbb{R}\gamma') \subseteq \mathbb{R}\gamma$.

Lemma 6.2. *Let Φ be a root system, let η be a k-good reduction of Φ, and $\Phi' = \eta(\Phi) \setminus \{0\}$ the induced root system. Let $\{X_\alpha\}_{\alpha\in\Phi}$ be a k-strong grading of a group G. Then the coarsened grading $\{Y_{\alpha'}\}_{\alpha'\in\Phi'}$ is a strong grading of G.*

Proof. First let us show that G is generated by $\{Y_{\alpha'}\}$. Since the subgroups $\{X_\alpha\}_{\alpha\in\Phi}$ generate G, it is enough to show that every X_γ lies in the subgroup generated by $\{Y_{\alpha'}\}$. This is clear if $\eta(\gamma) \neq 0$. Assume now that $\eta(\gamma) = 0$. Since the reduction η is k-good, Φ has an irreducible regular subsystem Ψ of rank k such that $\gamma \in \Psi$ and $\ker\eta \cap \Psi \subseteq \mathbb{R}\gamma$. Since Ψ is regular, $\gamma \in C_f$ for some $f \in \mathfrak{F}(\Psi)$ by Corollary 4.9. Since the grading $\{X_\alpha\}_{\alpha\in\Phi}$ is k-strong, X_γ lies in the group generated by $\{X_\alpha\}_{\alpha\in\Psi_f\setminus\mathbb{R}\gamma} \subseteq \{X_\alpha\}_{\alpha\in\Phi\setminus\ker\eta}$, and so X_γ lies in the subgroup generated by $\{Y_{\alpha'}\}$.

The fact that $\{Y_{\alpha'}\}$ is a strong grading of G follows directly from part (b) of the definition of a k-good reduction and the assumption that the grading $\{X_\alpha\}_{\alpha\in\Phi}$ is k-strong. □

Corollary 6.3. *Let Φ be a system such that any root lies in an irreducible subsystem of rank k. If a Φ-grading of a group G is k-strong, then it is also strong.*

Proof. The identity map $\mathbb{R}\Phi \to \mathbb{R}\Phi$ is clearly a reduction. It is k-good if and only if any root of Φ lies in an irreducible subsystem of rank k. □

6.2. **Examples of good reductions.** In this subsection we present several examples of good reductions that will be used in this paper. In particular, we will establish the following result:

Proposition 6.4. *Every irreducible classical root system of rank > 2 admits a 2-good reduction to an irreducible classical root system of rank 2.*

The following elementary fact can be proved by routine case-by-case verification.

Claim 6.5. *Let Φ be a classical irreducible root system of rank ≥ 2. For $\alpha \in \Phi$ let $N_\Phi(\alpha)$ be the set of all $\beta \in \Phi$ such that* $\mathrm{span}\{\alpha, \beta\} \cap \Phi$ *is an irreducible rank 2 system. Then for any $\alpha \in \Phi$, the set $\{\alpha\} \cup N_\Phi(\alpha)$ spans $\mathbb{R}\Phi$.*

Now let $\eta : \Phi \to \Phi'$ be a reduction of root systems, where Φ is classical irreducible of rank ≥ 2. Claim 6.5 implies that η always satisfies condition (a) in the definition of a 2-good reduction. Indeed, $\ker\eta \neq \mathbb{R}\Phi$ (since $\Phi' \neq \emptyset$), so given $\gamma \in \ker\eta \cap \Phi$, by Claim 6.5 there exists $\beta \in N_\Phi(\gamma) \setminus \ker\eta$. Then clearly the subsystem $\Psi = (\mathbb{R}\gamma + \mathbb{R}\beta) \cap \Phi$ has the required property.

In order to speed up verification of condition (b) in the examples below we shall use symmetries of the "large" root system Φ which project to symmetries of the "small" root system Φ' under η. Formally, we shall use the following observation:

Suppose that a group Q acts linearly on $\mathbb{R}\Phi$ preserving Φ and the subspace $\ker\eta$. Thus we have the induced action of Q on Φ' given by

$$q\,\eta(\alpha) = \eta(q\,\alpha) \text{ for all } \alpha \in \Phi, q \in Q.$$

Then to prove that η satisfies condition (b) in the definition of a k-good reduction it suffices to check that condition for one representatives from each Q-orbit in $\{(f, \gamma) : f \in \mathfrak{F}(\Phi')/\sim, \eta(\gamma) \in C'_f\}$.

In the following examples $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$.

Reduction 6.6. The system A_n (or rather its canonical realization) is defined to be the set of vectors of $\mathbb{R}^{n+1}$ of length $\sqrt{2}$ with integer coordinates that sum to 0 (note that A_n spans a proper subspace of $\mathbb{R}^{n+1}$). Thus

$$A_n = \{e_i - e_j : 1 \leq i, j \leq n+1, i \neq j\}.$$

Choose non-empty pairwise disjoint subsets I_1, I_2, I_3 such that $I_1 \sqcup I_2 \sqcup I_3 = \{1, \ldots, n-1\}$. Then the map $\eta : \mathbb{R}^{n+1} \to \mathbb{R}^3$ defined by

$$\eta(x_1, \ldots, x_{n+1}) = \left(\sum_{i \in I_1} x_i, \sum_{i \in I_2} x_i, \sum_{i \in I_3} x_i \right),$$

is a reduction of A_n to A_2. Let us show that it is 2-good directly from definition (recall that we only need to check condition (b)).

Without loss of generality we can assume that Φ'_f is the Borel subset with the base $\{(1,-1,0), (0,1,-1)\}$. The only root in C_f is $\gamma' = (1, 0, -1)$, and any root γ satisfying $\eta(\gamma) = \gamma'$ has the form $\gamma = e_{i_1} - e_{i_3}$ for some $i_1 \in I_1$ and $i_3 \in I_3$. Now choose any $i_2 \in I_2$, let $\Psi = \{\pm(e_{i_1} - e_{i_2}), \pm(e_{i_2} - e_{i_3}), \pm(e_{i_1} - e_{i_3})\}$, and let $g \in \mathcal{F}(\Psi)$ be such that $\Psi_g = \{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, e_{i_1} - e_{i_3}\}$. Then condition (b) is clearly satisfied.

Reduction 6.7. The system B_n consists of all integer vectors in $\mathbb{R}^n$ of length 1 or $\sqrt{2}$. Thus

$$B_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm e_i : 1 \leq i \leq n\}.$$

A natural reduction of B_n to B_2 is given by the map $\eta \colon \mathbb{R}^n \to \mathbb{R}^2$ defined by

$$\eta(x_1, \ldots, x_n) = (x_1, x_2).$$

Let us show that this reduction is 2-good. Let Q be the dihedral group of order 8, acting naturally on the first two coordinates of $\mathbb{R}^n$. This action preserves $\ker \eta$, and the induced Q-action on $\{(f, \gamma') : f \in \mathfrak{F}(B_2)/\sim, \gamma' \in C_f\}$ has two orbits. The following table shows how to verify condition (b) from the definition of a good reduction for one representative in each orbit (using the notations from that definition). We do not specify the functionals f and g themselves; instead we list the bases of the corresponding Borel sets Φ'_f and Ψ_g.

γ'	γ	base of Φ'_f	base of Ψ_g
(1,0)	$e_1 + xe_i \; (i \geq 3)$	(1,-1), (0,1)	$e_1 - e_2, e_2 + xe_i$
(1,1)	$e_1 + e_2$	(1,-1),(0,1)	$e_1 - e_2, e_2$

Reduction 6.8. The system D_n consists of all integer vectors in $\mathbb{R}^n$ of length $\sqrt{2}$. Thus

$$D_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}.$$

A natural reduction of $D_n (n \geq 3)$ to B_2 is given by the map $\eta \colon \mathbb{R}^n \to \mathbb{R}^2$ defined by

$$\eta(x_1, \ldots, x_n) = (x_1, x_2).$$

This reduction is 2-good – the proof is similar to the case of B_n.

Reduction 6.9. The system C_n consists of all integer vectors in $\mathbb{R}^n$ of length $\sqrt{2}$ together with all vectors of the form $2e$, where e is an integer vector of length 1. Thus

$$C_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm 2e_i : 1 \leq i \leq n\}.$$

A natural reduction of $C_n (n \geq 3)$ to BC_2 is given by the map $\eta\colon \mathbb{R}^n \to \mathbb{R}^2$ defined by

$$\eta(x_1, \dots, x_n) = (x_1, x_2).$$

Let us show that this reduction is 2-good. We use the same action of the dihedral group of order 8 as in the example $B_n \to B_2$, but this time there are three orbits in $\{(f, \gamma') : f \in \mathfrak{F}(BC_2)/\sim,\ \gamma' \in C_f\}$. The following table covers all the cases.

γ'	γ	base of Φ'_f	base of Ψ_g
(1,0)	$e_1 + xe_i\ (i \geq 3)$	(1,-1), (0,1)	$e_1 - e_2, e_2 + xe_i$
(1,1)	$e_1 + e_2$	(1,-1),(0,1)	$e_1 - e_2, 2e_2$
(2,0)	$2e_1$	(1,-1), (0,1)	$e_1 - e_2, 2e_2$

Reduction 6.10. The root system C_n also admits a natural reduction to B_2.

Fix $1 \leq i < n$. The map $\eta_i\colon \mathbb{R}^n \to \mathbb{R}^2$ given by

$$\eta_i(x_1, \dots, x_n) = (x_1 + \ldots + x_i, x_{i+1} + \ldots + x_n)$$

is a reduction of C_n to C_2. Composing η_i with some isomorphism $C_2 \to B_2$, we obtain an explicit reduction of C_n to B_2.

For instance, in the case $i = n - 1$ we obtain the following reduction η from C_n to B_2:

$$\eta(x_1, \dots, x_n) = \left(\frac{x_1 + \ldots + x_{n-1} - x_n}{2}, \frac{x_1 + \ldots + x_{n-1} + x_n}{2} \right).$$

Let us show that it is 2-good. This time we take $Q = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, acting on C_n via the maps $\varphi_{\varepsilon_1,\varepsilon_2}$, with $\varepsilon_1, \varepsilon_2 = \pm 1$, defined by

$$\varphi_{\varepsilon_1,\varepsilon_2}(x_i) = \varepsilon_1 x_i \text{ for } 1 \leq i \leq n-1, \quad \varphi_{\varepsilon_1,\varepsilon_2}(x_n) = \varepsilon_2 x_n.$$

There are four Q-orbits in $\{(f, \gamma') : f \in \mathfrak{F}(B_2)/\sim,\ \gamma' \in C_f\}$. All the cases are described in the following table:

γ'	γ	base of Φ'_f	base of Ψ_g
(1,0)	$e_i - e_n\ (i \leq n-1)$	(1,-1), (0,1)	$-2e_n, e_i + e_n$
(1,1)	$2e_i\ (i \leq n-1)$	(1,-1),(0, 1)	$-2e_n, e_i + e_n$
(1,1)	$e_i + e_j\ (1 \leq i < j \leq n-1)$	(1,-1),(0, 1)	$e_j - e_n, e_i + e_n$
(0,1)	$e_i + e_n\ (i \leq n-1)$	(1,1),(-1,0)	$2e_i, -e_i + e_n$
(-1,1)	$2e_n$	(1,1),(-1,0)	$2e_i, -e_i + e_n$

Reduction 6.11. The system BC_n is the union of B_n and C_n (in their standard realizations). Thus

$$BC_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm e_i, \pm 2e_i : 1 \leq i \leq n\}.$$

Once again, the map $\eta : \mathbb{R}^n \to \mathbb{R}^2$ given by

$$\eta(x_1, \dots, x_n) = (x_1, x_2)$$

is a reduction of BC_n to BC_2. To show that this reduction is 2-good we use the same action of the dihedral group of order 8 as in the reductions $B_n \to B_2$ and $C_n \to BC_2$. There are three Q-orbits in $\{(f, \gamma') : f \in \mathfrak{F}(BC_2)/\sim,\ \gamma' \in C_f\}$, whose representatives are listed in the following table.

γ'	γ	base of Φ'_f	base of Ψ_g
(1,0)	$e_1 + xe_i\,(i \geq 3)$	(1,-1), (0,1)	$e_1 - e_2, e_2 + xe_i$
(1,1)	$e_1 + e_2$	(1,-1),(0,1)	$e_1 - e_2, e_2$
(2,0)	$2e_1$	(1,-1), (0,1)	$e_1 - e_2, e_2$

Reduction 6.12. The system G_2 consists of 12 vectors of lengths $\sqrt{2}$ and $\sqrt{6}$ of $\mathbb{R}^3$ with integer coordinates that sum to 0. Thus

$$G_2 = \{e_i - e_j : 1 \leq i, j \leq 3, i \neq j\} \cup \{\pm(2e_i - e_j - e_k) : 1 \leq i, j, k \leq 3, i \neq j \neq k \neq i\}$$

The system F_4 consists of vectors v of length 1 or $\sqrt{2}$ in $\mathbb{R}^4$ such that the coordinates of $2v$ are all integers and are either all even or all odd. Thus

$$F_4 = \{\pm e_i : 1 \leq i \leq 4\} \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\} \cup \{\pm e_i \pm e_j : 1 \leq i \leq j \leq 4\}.$$

A reduction of F_4 to G_2 is given by the map $\eta : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$\eta(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_2 - x_3, x_3 - x_1).$$

Let us show that this reduction is 2-good. This time we use an action of $S_3 \times \mathbb{Z}/2\mathbb{Z}$ where S_3 permutes the first three coordinates, and the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ sends (x_1, x_2, x_3, x_4) to $(-x_1, -x_2, -x_3, x_4)$. This reduces all the calculations to the following cases.

γ'	γ	base of Φ'_f	base of Ψ_g
(0,1,-1)	$(1, 1, 0, 0)$	(1,-1,0), (-1,2,-1)	$(1, 0, 1, 0), (0, 1, -1, 0)$
(0,1,-1)	$(0, 0, -1, x)$	(1,-1,0), (-1,2,-1)	$(0, -1, 0, x), (0, 1, -1, 0)$
(0,1,-1)	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, x)$	(1,-1,0), (-1,2,-1)	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, x), (0, 1, -1, 0)$
(1,0,-1)	$(0, -1, -1, 0)$	(1,-1,0), (-1,2,-1)	$(0, -1, 0, 0), (0, 1, -1, 0)$
(1,0,-1)	$(1, 0, 0, x)$	(1,-1,0), (-1,2,-1)	$(1, 0, 1, 0), (0, 0, -1, x)$
(1,0,-1)	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, x)$	(1, -1,0), (-1,2,-1)	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, x), (0, 0, -1, 0)$
(1,1,-2)	$(1, 0, -1, 0)$	(1, -1,0), (-1,2,-1)	$(1, -1, 0, 0), (0, 1, -1, 0)$
(2,-1,-1)	$(1, -1, 0, 0)$	(1, -1,0), (-1,2,-1)	$(0, -1, 0, 1), (1, 0, 0, -1)$

Reduction 6.13. The root system E_8 consists of the vectors of length $\sqrt{2}$ in $\mathbb{Z}^8$ and $(\mathbb{Z} + \frac{1}{2})^8$ such that the sum of all coordinates is an even number. The system E_7 is the intersection of E_8 with the hyperplane of vectors orthogonal to $(0, 0, 0, 0, 0, 0, 1, -1)$ in E_8 and the system E_6 is the intersection of E_7 with the hyperplane of vectors orthogonal to $(0, 0, 0, 0, 1, -1, 0, 0)$. The map $\eta \colon \mathbb{R}^8 \to \mathbb{R}^3$

$$\eta(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_1 - x_2, x_2 - x_3, -x_3 - x_1).$$

is a reduction of E_8 to G_2, and the restriction of η to $\mathbb{R}^7$ (resp. $\mathbb{R}^6$) is a reduction of E_7 (resp. E_6) to G_2.

Each of these reductions is 2-good, and the proof is similar to the case $F_4 \to G_2$.

Reduction 6.14. Let $n \geq 3$. Then the map $\eta : \mathbb{R}^n \to \mathbb{R}^3$ defined by

$$\eta(x_1, \ldots, x_n) = (x_1, x_2, x_3)$$

is a 3-good reduction of BC_n to BC_3. The proof is analogous to the previous examples.

Reduction 6.15. Let $n \geq 3$ be a natural number and let $a_k = (\cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n}) \in \mathbb{R}^2$. Define the root system $I_n \subset \mathbb{R}^2$ by

$$I_n = \{a_k - a_l : \ 1 \leq l \neq k \leq n\}.$$

It is easy to see that $I_3 = A_2$, $I_4 = C_2$ and $I_6 = G_2$. For any n, if we normalize all roots in I_n (that is, replace, each $v \in I_n$ by $\frac{v}{\|v\|}$), we obtain the 2-dimensional root system whose elements connect the origin with vertices of a regular $2n$-gon. This root system arises in the classification of finite Coxeter groups and is sometimes denoted by $I_2(n)$.

The map $\eta\colon \mathbb{R}^{n+1} \to \mathbb{R}^2$ defined by

$$\eta(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} x_i a_i$$

is a reduction of A_n to I_{n+1}. We shall prove that it is 2-good.

First, we need to describe the boundary of Borel subsets of I_n. If $f \in \mathfrak{F}(I_n)$ then, since f is different from 0 on I_n, we can find a permutation $i_1, \dots, i_n$ of $\{1, \dots, n\}$ such that

$$f(a_{i_1}) > f(a_{i_2}) > \cdots > f(a_{i_n}), \tag{6.1}$$

so that the Borel set I_f is equal to $\{a_{i_s} - a_{i_t} : s < t\}$. We claim that the root $a_{i_s} - a_{i_t}$ lies in ∂I_f, the boundary of I_f, if and only if $t = s + 1$.

The forward direction is clear since if $t \geq s + 2$, then $a_{i_s} - a_{i_t} = (a_{i_s} - a_{i_{s+1}}) + (a_{i_{s+1}} - a_{i_t})$, whence $a_{i_s} - a_{i_t} \in C_f$ by Lemma 4.6.

To prove the converse, first observe that $\{i_2, i_3\} = \{i_1 - 1, i_1 + 1\}$ and for $1 \leq k \leq n/2$ we have that $i_{2k} = i_1 + k(i_2 - i_1)$ and $i_{2k-1} = i_1 + (k-1)(i_3 - i_1)$. It is then easy to see (algebraically or geometrically) that

$$a_{i_{2k-1}} - a_{i_{2k}} \in \mathbb{R}(a_1 - a_2) \text{ and } a_{i_{2k}} - a_{i_{2k+1}} \in \mathbb{R}(a_2 - a_3). \tag{6.2}$$

Combined with what we already showed, this implies that $\partial I_f \subseteq (\mathbb{R}(a_1 - a_2) \cup \mathbb{R}(a_2 - a_3)) \cap I_f$. On the other hand, it is clear that the boundary of any Borel set in any root system of rank 2 is a union of two half-lines. Therefore, $\partial I_f = (\mathbb{R}(a_1 - a_2) \cup \mathbb{R}(a_2 - a_3)) \cap I_f$, and from (6.2) we deduce that $a_{i_s} - a_{i_{s+1}} \in \partial I_f$ for all $1 \leq s \leq n - 1$.

Now let $\gamma' \in C_f$ and take any $\gamma \in A_{n+1}$ with $\eta(\gamma) = \gamma'$. If $\gamma = e_{i_t} - e_{i_s}$, then $\gamma' = a_{i_t} - a_{i_s}$, and since $\gamma' \in C_f$, by the above argument $t - s \geq 2$. Hence we can take Ψ_g to be $\{e_{i_t} - e_{i_{t+1}}, e_{i_{t+1}} - e_{i_s}, \gamma\}$. This proves that the reduction is 2-good.

7. Steinberg groups over commutative rings

In this section we prove property (T) for Steinberg groups of rank ≥ 2 over finitely generated commutative rings and estimate asymptotic behavior of Kazhdan constants.

7.1. **Graded covers.** Let Γ be a finite graph and G a group with a chosen decomposition $(\{G_\nu\}_{\nu \in \mathcal{V}(\Gamma)}, \{G_e\}_{e \in \mathcal{E}(\Gamma)})$ over Γ. If H is another group with a decomposition $(\{H_\nu, \{H_e\}\})$ over Γ, we will say that the decomposition $(\{H_\nu\}, \{H_e\})$ is isomorphic to $(\{G_\nu\}, \{G_e\})$ if there are isomorphisms $\iota_\nu : G_\nu \cong H_\nu$ for each $\nu \in \mathcal{V}(\Gamma)$ and $\iota_e : G_e \cong H_e$ for each $e \in \mathcal{E}(\Gamma)$ such that $\iota_{e^+|G_e} = \iota_e$ and $\iota_{\bar{e}} = \iota_e$.

Among all groups which admit a decomposition over the graph Γ isomorphic to $(\{G_\nu\}, \{G_e\})$ there is the "largest" one, which surjects onto any other group with this property. This group will be called the *cover of G corresponding to* $(\{G_\nu\}_{\nu \in \mathcal{V}(\Gamma)}, \{G_e\}_{e \in \mathcal{E}(\Gamma)})$ and can be defined as the free product of the vertex subgroups $\{G_\nu\}_{\nu \in \mathcal{V}(\Gamma)}$ amalgamated along the edge subgroups $\{G_e\}_{e \in \mathcal{E}(\Gamma)}$. We will be particularly interested in the special case of this construction dealing with decompositions associated to gradings by root systems.

Definition. Let G be a group, Φ a root system and $\{X_\alpha\}_{\alpha\in\Phi}$ a Φ-grading of G. Let $\Gamma_l\Phi$ be the large Weyl graph of Φ, and consider the canonical decomposition of G over $\Gamma_l\Phi$. The cover of G corresponding to this decomposition will be called the *graded cover of G with respect to the grading* $\{X_\alpha\}$.

Graded covers may be also defined using generators and relations. Assume that $G_f = \langle \cup_{\alpha\in\Phi_f} X_\alpha | \, R_f\rangle$ for each $f \in \mathcal{F}(\Phi)$. Then the graded cover of G with respect to $\{X_\alpha\}_{\alpha\in\Phi}$ is isomorphic to

$$\langle \cup_{\alpha\in\Phi} X_\alpha | \; \cup_{f\in\mathcal{F}(\Phi)} R_f\rangle.$$

Observe that if $\pi : G \to G'$ is an epimorphism, and $\{X_\alpha\}$ is a Φ-grading of G, then $\{\pi(X_\alpha)\}$ is a Φ-grading of G'. If in addition π is injective on all the Borel subgroups of G, then the graded covers of G and G' coincide.

Here is a simple observation about automorphisms of graded covers recorded here for later use.

Definition. If G is a group and $\{X_\alpha\}_{\alpha\in\Phi}$ is a grading of G, an automorphism $\pi \in \mathrm{Aut}(G)$ will be called *graded* (with respect to $\{X_\alpha\}$) if π permutes the root subgroups $\{X_\alpha\}$ between themselves, and the induced action of π on Φ sends Borel sets to Borel sets. The group of all graded automorphisms of G will be denoted by $\mathrm{Aut}_{gr}(G)$.

Lemma 7.1. *Let G be a group, $\{X_\alpha\}$ a grading of G and $\widetilde{G}$ the graded cover of G with respect to $\{X_\alpha\}$. Then each graded automorphism of G naturally lifts to a graded automorphism of $\widetilde{G}$, and the obtained map $\mathrm{Aut}_{gr}(G) \to \mathrm{Aut}_{gr}(\widetilde{G})$ is a monomorphism.*

7.2. **Steinberg groups over commutative rings.** In this subsection we sketch the definition of Steinberg groups and show that the natural grading of these groups is strong. Our description of Steinberg groups follows Steinberg's lecture notes on Chevalley groups [St] and Carter's book [Ca].

We begin by recalling a few basic facts about simple complex Lie algebras (see [Hu1] for more details). Let Φ be a reduced irreducible classical root system of rank l and $\mathcal{L}$ a simple complex Lie algebra corresponding to Φ. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{L}$. Then $\mathcal{H}$ is abelian, $\dim \mathcal{H} = l$, and we have the following decomposition of $\mathcal{L}$:

$$\mathcal{L} = \mathcal{H} \oplus (\oplus_{\alpha\in\Phi}\mathcal{L}_\alpha),$$

where $\mathcal{L}_\alpha = \{l \in \mathcal{L} : \; [h,l]_L = \alpha(h)l \text{ for all } \; h \in \mathcal{H}\}$ (as usual we consider Φ as a subset of $\mathcal{H}^*$). Moreover, each $\mathcal{L}_\alpha$ is one dimensional.

For any $\alpha,\beta \in \Phi$ we put $\langle\beta,\alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$, where $(\cdot,\cdot)$ is an admissible scalar product on Φ (since $(\cdot,\cdot)$ is unique up to scalar multiples, the pairing $\langle\cdot,\cdot\rangle$ is well defined). For any $\alpha \in \Phi$ let $h_\alpha \in \mathcal{H}$ be the unique element such that

$$\beta(h_\alpha) = \langle\beta,\alpha\rangle. \tag{7.1}$$

Note that if $\{\alpha_1,\dots,\alpha_l\}$ is a base of Φ, then $h_{\alpha_1},\dots,h_{\alpha_l}$ is a basis of $\mathcal{H}$.

Proposition 7.2. *There exist nonzero elements $x_\alpha \in \mathcal{L}_\alpha$ for $\alpha \in \Phi$ such that*

$$[x_\alpha, x_{-\alpha}]_L = h_\alpha, \tag{7.2}$$

$$[x_\alpha, x_\beta]_L = \begin{cases} \pm(r+1)x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \\ 0 & \text{if } \alpha+\beta \notin \Phi \end{cases}, \tag{7.3}$$

where $r = \max\{s \in \mathbb{Z} : \ \beta - s\alpha \in \Phi\}$.

Any basis $\{h_{\alpha_i}, x_\alpha : 1 \le i \le l, \alpha \in \Phi\}$ of $\mathcal{L}$ with this property (for some choice of Cartan subalgebra) is called a *Chevalley basis*. It is unique up to sign changes and automorphisms of $\mathcal{L}$. We will also need the following characterization of simple Lie algebras in terms of Chevalley bases:

Lemma 7.3. *Let the root system Φ and pairing $\langle \cdot, \cdot \rangle$ be as above, and let $\{\alpha_1, \dots, \alpha_l\}$ be a base of Φ. Let L be a finite-dimensional complex Lie algebra with basis $\{x_\alpha\}_{\alpha\in\Phi} \sqcup \{h_i\}_{i=1}^l$ such that*

- (i) $[x_{\alpha_i}, x_{-\alpha_i}] = h_i$ *for* $1 \le i \le l$;
- (ii) $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$ *for* $1 \le i \le l$ *and* $\alpha \in \Phi$;
- (iii) $[h_i, h_j] = 0$ *for* $1 \le i, j \le l$;
- (iv) *elements* $\{x_\alpha\}$ *satisfy* (7.3).

Then L is a simple Lie algebra with root system Φ, that is, $L \cong \mathcal{L}$.

From now on we fix a Chevalley basis $\mathcal{B}$ of $\mathcal{L}$. Denote by $\mathcal{L}_\mathbb{Z}$ the subset of $\mathcal{L}$ consisting of all linear combinations of the elements of $\mathcal{B}$ with integer coefficients. By definition of a Chevalley basis, $\mathcal{L}_\mathbb{Z}$ is a Lie subring of $\mathcal{L}$. For any commutative ring R put $\mathcal{L}_R = R \otimes_\mathbb{Z} \mathcal{L}_\mathbb{Z}$. The Lie bracket of $\mathcal{L}_\mathbb{Z}$ naturally extends to a Lie bracket of $\mathcal{L}_R$.

Proposition 7.4. *Let $S = \mathbb{Q}[t, s]$ and $T = \mathbb{Z}[t, s]$ be the polynomial rings in two variables over $\mathbb{Q}$ and $\mathbb{Z}$, respectively. Then for every $\alpha, \beta \in \Phi$, with $\alpha \ne \beta$, we have*

- (i) *the derivation $ad(tx_\alpha) = t\,\mathrm{ad}(x_\alpha)$ of $\mathcal{L}_S$ is nilpotent;*
- (ii) *$x_\alpha(t) = \exp(t\,\mathrm{ad}(x_\alpha)) = \sum_{i=0}^\infty \frac{t^i}{i!}\mathrm{ad}(x_\alpha)^i$ is a well defined automorphism of $\mathcal{L}_S$ which preserves $\mathcal{L}_T$,*
- (iii) *there exist $c_{ij}(\alpha, \beta) \in \mathbb{Z}$ such that the following equality holds:*

$$[x_\alpha(t), x_\beta(s)] = \prod_{i,j} x_{i\alpha+j\beta}(c_{ij}(\alpha,\beta)t^i s^j), \tag{7.4}$$

where the product on the right is taken over all roots $i\alpha + j\beta \in \Phi$, with $i, j \in \mathbb{N}$, arranged in some fixed order. Moreover, the constants $c_{ij}(\alpha, \beta)$ depend only on the set $\{(i, j) \in \mathbb{Z}\times\mathbb{Z} : i\alpha + j\beta \in \Phi\}$ and the chosen order.
- (iv) *If $\alpha+\beta \notin \Phi$, then the product in* (7.4) *is empty (and thus $[x_\alpha(t), x_\beta(s)] = 1$). If $\alpha + \beta \in \Phi$, then $c_{11}(\alpha, \beta) = \pm(r + 1)$ where r is given by* (7.3). *In particular, $c_{11}(\alpha, \beta)$ does not depend on the chosen order.*

Remark. Up to sign, the constants $c_{ij}(\alpha, \beta)$ are independent of the choice of the Chevalley basis (once the order in (7.4) has been fixed).

Let $A_\alpha(t)$ be the matrix representing $x_\alpha(t)$ with respect to $\mathcal{B}$ (note that $\mathcal{L}_S$ is a free S-module). By Proposition 7.4(ii), the entries of $A_\alpha(t)$ are in T.

Now let R be a commutative ring R and let $r \in R$. Let $x_\alpha(r)$ be the automorphism of $\mathcal{L}_R$ represented by $A_\alpha(r)$ (the matrix obtained from $A_\alpha(t)$ by replacing t by r) with respect to $\mathcal{B}$. We denote by $X_\alpha = X_\alpha(R)$ the set $\{x_\alpha(r) : \ r \in R\}$. This is a subgroup of $\mathrm{Aut}(\mathcal{L}_R)$ isomorphic to $(R,+)$. By Proposition 7.4(iii), $\{X_\alpha\}_{\alpha\in\Phi}$ is a Φ-grading.

Definition. Let Φ be a reduced irreducible classical root system and R a commutative ring.

(a) The subgroup of $\mathrm{Aut}(\mathcal{L}_R)$ generated by $\cup_\alpha X_\alpha$ is called the *adjoint elementary Chevalley group over R corresponding to* Φ and will be denoted by $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$.
(b) The *Steinberg group* $\mathrm{St}_\Phi(R)$ is the graded cover of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ with respect to the grading $\{X_\alpha\}_{\alpha\in\Phi}$.

Remark. Elementary Chevalley groups of simply-connected type (and other non-adjoint types) can be constructed in a similar way, except that the adjoint representation of the Lie algebra $\mathcal{L}_R$ should be replaced by a different representation. The graded cover for each such group is isomorphic to $\mathrm{St}_\Phi(R)$.

The Steinberg group $\mathrm{St}_\Phi(R)$ can also be defined as the group generated by the elements $\{x_\alpha(r) : \alpha\in\Phi, r\in R\}$ subject to the following relations for every $\alpha\neq-\beta\in\Phi$ and $t,u\in R$:

$$\begin{aligned} x_\alpha(t)x_\alpha(u) &= x_\alpha(t+u)\\ [x_\alpha(t),x_\beta(u)] &= \prod_{i,j\in\mathbb{N},i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(c_{ij}(\alpha,\beta)t^iu^j), \end{aligned}$$

where the constants $c_{ij}(\alpha,\beta)$ come from (7.4).

Note that while the second definition of Steinberg groups has an advantage of being explicit, the first one shows that the isomorphism class of $\mathrm{St}_\Phi(R)$ does not depend on the choice of Chevalley basis.

Remark. Note that according to our definition the Steinberg group $\mathrm{St}_{A_1}(R)$ is the free product of two copies of $(R,+)$. This definition does not coincide with the usual definition in the literature, but it is convenient for the purposes of this paper.

Remark. If we do not assume that R is commutative, then $\mathcal{L}_R$ does not have a natural structure of a Lie algebra over R, and so the above construction of $\mathrm{St}_\Phi(R)$ is not valid. However in the case $\Phi=A_n$, we can still define the Steinberg group as the graded cover of $\mathrm{EL}_{n+1}(R)$. When $\Phi\neq A_n$ we are not aware of any natural way to define the Steinberg group $\mathrm{St}_\Phi(R)$ when R is noncommutative.

In the special case $\Phi=A_2$, the Steinberg group $\mathrm{St}_{A_2}(R)$ can even be defined for any *alternative* ring R (see [Fa, Appendix]).

The following proposition will be used frequently in the rest of the paper. It shows that some "natural" subgroups of Steinberg groups are quotients of Steinberg groups.

Definition. Let Φ be a root system. A subset Ψ of Φ is called a **weak subsystem** if $\Phi\cap(\sum_{\gamma\in\Psi}\mathbb{Z}\gamma)=\Psi$.

Proposition 7.5. *Let Φ be a reduced irreducible classical root system and Ψ an irreducible weak subsystem. Then Ψ is classical and the subgroup H of* $\mathrm{St}_\Phi(R)$ *generated by $\{X_\gamma :\ \gamma\in\Psi\}$ is a quotient of* $\mathrm{St}_\Psi(R)$.

Proof. Note that Ψ is a root system. Moreover, Ψ is classical, since if $(\cdot,\cdot)$ is an admissible scalar product on Φ, then $(\cdot,\cdot)$ restricted to Ψ is an admissible scalar product on Ψ.

Let $\mathcal{L}$ be a simple complex Lie algebra corresponding to Φ, choose a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$, and define $\{h_\alpha\}_{\alpha\in\Phi}$ by (7.1). Let $\{h_{\alpha_i}, x_\alpha : 1 \le i \le l, \alpha \in \Phi\}$ a Chevalley basis of $\mathcal{L}$ (relative to $\mathcal{H}$). We claim that the Lie subalgebra $\mathcal{L}^\Psi$ generated by $\{x_\alpha : \alpha \in \Psi\}$ is a simple complex Lie algebra corresponding to the root system Ψ, and moreover, if $\{\beta_1,\ldots,\beta_m\}$ is a base of Ψ, then $\{x_\alpha : \alpha \in \Psi\} \sqcup \{h_{\beta_i}\}$ is a Chevalley basis of $\mathcal{L}^\Psi$.

By Lemma 7.3 (to be applied to Ψ) and definition of Chevalley basis, to prove both statements it suffices to check that

(i) the pairing $\langle\cdot,\cdot\rangle_\Psi$ on Ψ is obtained from the pairing $\langle\cdot,\cdot\rangle_\Phi$ on Φ by restriction;
(ii) If $\alpha,\beta \in \Psi$, then the value of r in relation (7.3) does not change if Φ is replaced by Ψ.

Assertion (i) is clear since the scalar product on Ψ is obtained from the scalar product on Φ by restriction. Assertion (ii) holds since Ψ is a weak subsystem and the value of r in (7.3) depends only on the structure of the $\mathbb{Z}$-lattice generated by α and β.

Thus, in view of Proposition 7.4(iii), the values of the coefficients $c_{ij}(\alpha,\beta)$, with $\alpha,\beta \in \Psi$, do not depend on whether we consider α,β as roots of Φ or Ψ. It follows that the defining relations of $\mathrm{St}_\Psi(R)$ hold in H, so H is a quotient of $\mathrm{St}_\Psi(R)$. □

Remark. In most cases a weak subsystem of a classical root system is also a subsystem, but not always. For instance, the long roots of G_2 form a weak subsystem of type A_2, but they do not form a subsystem. Note that the short roots of G_2 do not even form a weak subsystem.

Next we explicitly describe some relations in the Steinberg groups corresponding to root systems of rank 2.

Proposition 7.6. *There exists a Chevalley basis such that*

(A) *if* $\Phi = A_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, *then*
$$[x_\alpha(t), x_\beta(t)] = x_{\alpha+\beta}(tu), \quad [x_{-\alpha}(t), x_{\alpha+\beta}(u)] = x_\beta(tu).$$
(B) *if* $\Phi = B_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta)\}$, *then*
$$[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(tu)x_{\alpha+2\beta}(tu^2), \quad [x_{-\alpha}(t), x_{\alpha+\beta}(u)] = x_\beta(tu)x_{\alpha+2\beta}(-tu^2),$$
$$[x_{\alpha+\beta}(t), x_\beta(u)] = x_{\alpha+2\beta}(2tu).$$
(G) *if* $\Phi = G_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta), \pm(\alpha+3\beta), \pm(2\alpha+3\beta)\}$, *then*
$$[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(tu)x_{\alpha+2\beta}(tu^2)x_{\alpha+3\beta}(tu^3)x_{2\alpha+3\beta}(t^2u^3),$$
$$[x_\alpha(t), x_{\alpha+3\beta}(u)] = x_{2\alpha+3\beta}(tu),$$
$$[x_{\alpha+\beta}(t), x_\beta(u)] = x_{\alpha+2\beta}(2tu)x_{\alpha+3\beta}(3tu^2)x_{2\alpha+3\beta}(3t^2u).$$

Proof. In each of those cases $\{\alpha,\beta\}$ is a base of Φ. The above relations which involve only positive roots with respect to this base hold by [Hu2, Prop 33.3,33.4,33.5], for a suitable choice of Chevalley basis. Moreover, the Chevalley basis for type B_2 constructed in [Hu2, Prop 33.4] satisfies the additional conditions

$$w_\alpha x_\alpha(t) w_\alpha^{-1} = x_{-\alpha}(-t), \qquad w_\alpha x_\beta(t) w_\alpha^{-1} = x_{\alpha+\beta}(t),$$
$$w_\alpha x_{\alpha+\beta}(t) w_\alpha^{-1} = x_\beta(-t), \qquad w_\alpha x_{\alpha+2\beta}(t) w_\alpha^{-1} = x_{\alpha+2\beta}(t),$$

where $w_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ is the Weyl group element corresponding to α. Conjugating the relation $[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(tu)x_{\alpha+2\beta}(tu^2)$ by w_α and using the above conditions, we conclude that $[x_{-\alpha}(t), x_{\alpha+\beta}(u)] = x_\beta(tu)x_{\alpha+2\beta}(-tu^2)$. The desired relation for type A_2 involving $-\alpha$ can be obtained similarly.

We warn the reader that notations in [Hu2] are different from ours, with the roles of α and β switched for types B_2 and G_2. □

Proposition 7.7. *Let Φ be a reduced irreducible classical root system of rank $l \geq 2$ and R a commutative ring, let $G = \mathrm{St}_\Phi(R)$ and $\{X_\alpha : \alpha \in \Phi\}$ the root subgroups of G. Then $\{X_\alpha : \alpha \in \Phi\}$ is a k-strong grading of G for any $2 \leq k \leq l$, and in particular, it is strong.*

Proof. Let Ψ be an irreducible subsystem of Φ of rank ≥ 2. Then by Proposition 7.5, Ψ is classical, and the subgroup H generated by $\{X_\alpha : \alpha \in \Psi\}$ is a quotient of the Steinberg group $\mathrm{St}_\Psi(R)$. Thus, the grading $\{X_\alpha : \alpha \in \Psi\}$ of H is strong if the natural Ψ-grading of $\mathrm{St}_\Psi(R)$ is strong. Since Ψ is regular, using Corollary 6.3 with $k = 2$, we deduce that it is enough to prove Proposition 7.7 when $l = k = 2$. In this case the result easily follows from Proposition 7.6. We illustrate this for $\Phi = G_2$.

Consider a functional f and let $\{\alpha, \beta\}$ be a base on which f takes positive values, with α a long root. Then the core C_f is equal to $\{\alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}$. Since each of the maps $(t,u) \mapsto tu$, $(t,u) \mapsto tu^2$, $(t,u) \mapsto tu^3$ from $R \times R$ to R is clearly surjective, the first two relations in Proposition 7.6(G) imply that

$$\begin{aligned}
X_{\alpha+\beta} &\subseteq [X_\alpha, X_\beta] X_{\alpha+2\beta} X_{\alpha+3\beta} X_{2\alpha+3\beta},\\
X_{\alpha+2\beta} &\subseteq [X_\alpha, X_\beta] X_{\alpha+\beta} X_{\alpha+3\beta} X_{2\alpha+3\beta},\\
X_{\alpha+3\beta} &\subseteq [X_\alpha, X_\beta] X_{\alpha+\beta} X_{\alpha+2\beta} X_{2\alpha+3\beta},\\
X_{2\alpha+3\beta} &= [X_\alpha, X_{\alpha+3\beta}].
\end{aligned}$$

Hence the grading $\{X_\alpha\}_{\alpha \in G_2}$ is strong. □

7.3. **Standard sets of generators of Steinberg groups.** Let R be a commutative ring generated by $T = \{t_0 = 1, t_1, \ldots, t_d\}$. We denote by T^* the set

$$\{t_{i_1} \cdots t_{i_k} : 0 \leq i_1 < \ldots < i_k \leq d\}.$$

In the following proposition we describe a set of generators of $\mathrm{St}_\Phi(R)$ that we will call *standard*.

Proposition 7.8. *Let Φ be a reduced irreducible classical root system of rank at least 2 and R a commutative ring generated by $T = \{t_0 = 1, t_1, \ldots, t_d\}$. Let $\Sigma = \Sigma_\Phi(T)$ be the following set:*

(1) *if $\Phi = A_n$, $B_n (n \geq 3)$, D_n, E_6, E_7, E_8, F_4,*

$$\Sigma = \{x_\alpha(t) : \ \alpha \in \Phi, \ t \in T\},$$

(2) *if* $\Phi = B_2$, C_n,

$$\Sigma = \left\{ \begin{array}{ll} x_\alpha(t),\ t \in T & \alpha \in \Phi \textit{ is a short root} \\ x_\alpha(t),\ t \in T^* & \alpha \in \Phi \textit{ is a long root} \end{array} \right\},$$

(3) *if* $\Phi = G_2$,

$$\Sigma = \left\{ \begin{array}{ll} x_\alpha(t),\ t \in T & \alpha \in \Phi \textit{ is a long root} \\ x_\alpha(t),\ t \in T^* & \alpha \in \Phi \textit{ is a short root} \end{array} \right\}.$$

Then Σ *generates* $\mathrm{St}_\Phi(R)$.

Proof. First we consider the case $\Phi = A_n$, $B_n (n \geq 3)$, D_n, E_6, E_7, E_8, F_4. We prove by induction on k that for any $\gamma \in \Phi$ and any monomial m in variables from T of degree k, the element $x_\gamma(m)$ lies in $\langle \Sigma \rangle$, the subgroup generated by Σ. This statement clearly implies the proposition.

The base of induction is clear. Assume that the statement is true for monomials of degree $\leq k$. Let m be a monomial of degree $k+1$.

If $\Phi = A_n$, D_n, E_6, E_7, E_8, F_4, then we can find a subsystem Ψ of Φ isomorphic to A_2 which contains γ. We write $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1, \gamma_2 \in \Psi$, and $m = m_1 m_2$, where m_1, m_2 are monomials of degree $\leq k$. Then $x_\gamma(m) = [x_{\gamma_1}(m_1), x_{\gamma_2}(m_2)]^{\pm 1}$ and we can apply the inductive hypothesis.

If $\Phi = B_n, n \geq 3$, then any long root lies in an irreducible subsystem isomorphic to A_2, whence the statement holds when γ is a long root. Assume γ is a short root. Then there are a long root α and a short root β such that $\gamma = \alpha + \beta$. Note that α and β generate a subsystem of type B_2. Without loss of generality we may assume that the relations of Proposition 7.6 hold. Then we obtain that

$$x_\gamma(m) = x_{\alpha+\beta}(m) = [x_\alpha(m), x_\beta(1)] x_{\alpha+2\beta}(-m).$$

Since the roots α and $\alpha + 2\beta$ are long, by induction $x_\gamma(m)$ lies $\langle \Sigma \rangle$.

In the case $\Phi = B_2$ the proposition is an easy consequence of the following lemma (we do not need the second part of this lemma now; it will be used later).

Lemma 7.9. *Let* $\{\alpha, \beta\}$ *be a base of* B_2 *with* α *a long root. Consider the semidirect product* $St_{A_1}(R) \ltimes N$, *where* $N = \langle X_\beta, X_{\alpha+\beta}, X_{\alpha+2\beta} \rangle \subset St_{B_2}(R)$ *and the action of* $St_{A_1}(R)$ *on* N *comes from the conjugation action of* $\langle X_\alpha, X_{-\alpha} \rangle \subset St_{B_2}(R)$ *on* N. *Let*

$S_1 = \{x_\alpha(t), x_{-\alpha}(t) :\ t \in T^* \cup T^2\}$; $\quad S_2 = \{x_\beta(t), x_{\alpha+\beta}(t) :\ t \in T\}$ *and*

$S_3 = \{x_{\alpha+2\beta}(t) :\ t \in T^*\}$.

Let G *be the subgroup of* $St_{A_1}(R) \ltimes N$ *generated by the set* $S = S_1 \cup S_2 \cup S_3$. *Then the following hold:*

(1) G *contains* N;
(2) $X_{\alpha+2\beta}/([N, G] \cap X_{\alpha+2\beta})$ *is of exponent 2 and generated by* S_3.

Proof. Without loss of generality we may assume that the relations from Proposition 7.6 hold in $\mathrm{St}_{B_2}(R)$.

We prove by induction on k that for any $\gamma \in \{\alpha+\beta, \alpha+2\beta, \beta\}$ and any monomial m in T of degree k, the element $x_\gamma(m)$ lies in $\langle S \rangle$. This clearly implies the first statement.

The base of induction is clear. Assume that the statement holds for all monomials of degree $\leq k$. Let m be a monomial of degree $k+1$.

Case 1: $\gamma = \alpha + 2\beta$. If $m \in T^*$, then $x_{\alpha+2\beta}(m) \in S_3$. If $m \notin T^*$, we can write $m = m_1 m_2^2$ with $m_1 \in T^*$ and $m_2 \neq 1$, and we obtain that

$$x_{\alpha+2\beta}(m_1 m_2^2) = [x_\alpha(m_1), x_\beta(m_2)] x_{\alpha+\beta}(-m_1 m_2).$$

Thus, by induction, $x_{\alpha+2\beta}(m) \in G$.

Case 2: $\gamma = \alpha + \beta$. If $m \in T^*$, then

$$x_{\alpha+\beta}(m) = [x_\alpha(m), x_\beta(1)] x_{\alpha+2\beta}(-m)$$

and we are done. If $m \notin T^*$, we write $m = t^2 m_1$, where $t \in T \setminus \{1\}$. Then we have

$$x_{\alpha+\beta}(m) = [x_\alpha(t^2), x_\beta(m_1)][x_\alpha(1), x_\beta(-tm_1)] x_{\alpha+\beta}(tm_1) \in G. \tag{7.5}$$

Case 3: $\gamma = \beta$. This case is analogous to Case 2, but this time we use the relation $x_\beta(vu) = [x_{-\alpha}(v), x_{\alpha+\beta}(u)] x_{\alpha+2\beta}(vu^2)$ for $u, v \in R$.

The proof of the second part is an easy exercise based on Proposition 7.6 and equality (7.5). □

We now go back to the proof of Proposition 7.8. In the case $\Phi = B_2$ the result follows from Lemma 7.9(1) since for any $t \in T$ and long root $\gamma \in B_2$, the element $x_\gamma(t^2)$ can be expressed as a product of elements from Σ. For instance, in the case $\gamma = \alpha + 2\beta$ we have

$$x_{\alpha+2\beta}(t^2) = [x_\alpha(1), x_\beta(t)] x_{\alpha+\beta}(-t). \tag{7.6}$$

If $\Phi = C_n$, then any root lies in a subsystem of type B_2, and so the result follows from the previous case.

Finally, consider the case $\Phi = G_2$. The long roots of G_2 form a weak subsystem of type A_2, so Proposition 7.8 for type A_2 (which we already established) and Proposition 7.5 imply that long root subgroups of $\mathrm{St}_{G_2}(R)$ lie in $\langle \Sigma \rangle$. It remains to show that $x_\gamma(m) \in \langle \Sigma \rangle$ for any short root γ and any monic monomial m in T, which we will do by induction on the degree of m. By symmetry, it suffices to prove that $x_{\alpha+2\beta}(m) \in \langle \Sigma \rangle$.

If m is square-free, then $m \in T^*$, so $x_\gamma(m) \in \Sigma$ by definition; otherwise $m = tu^2$ for some monomials t and u, with $u \neq 1$. By Proposition 7.6(C) we have

$$x_{\alpha+2\beta}(tu^2) = x_{2\alpha+3\beta}(-tu^3) x_{\alpha+3\beta}(-t^2u^3) [x_{\alpha+\beta}(u), x_{-\alpha}(t)] x_\beta(tu).$$

All factors on the right-hand side lie in $\langle \Sigma \rangle$, namely $x_{\alpha+\beta}(u), x_\beta(tu) \in \langle \Sigma \rangle$ by the induction hypothesis and $x_{-\alpha}(t), x_{2\alpha+3\beta}(-tu^3), x_{\alpha+3\beta}(-t^2u^3) \in \langle \Sigma \rangle$ since the roots $-\alpha$, $\alpha + 3\beta$ and $2\alpha + 3\beta$ are long. Thus, $x_{\alpha+2\beta}(m) \in \langle \Sigma \rangle$, as desired. □

7.4. **Property (T) for Steinberg groups.** In this subsection we establish property (T) for Steinberg groups (of rank ≥ 2) over finitely generated rings and obtain asymptotic lower bounds for the Kazhdan constants. It will be convenient to use the following notation.

If κ is some quantity depending on $\mathbb{N}$-valued parameters $n_1, \ldots, n_r$ (and possibly some other parameters) and $f : \mathbb{N}^r \to \mathbb{R}_{>0}$ is a function, we will write

$$\kappa \succcurlyeq f(n_1, \ldots, n_r)$$

if there exists an absolute constant $C > 0$ such that $\kappa \geq Cf(n_1, \ldots, n_r)$ for all $n_1, \ldots, n_r \in \mathbb{N}$.

Let Φ be a reduced irreducible classical root system of rank ≥ 2 and R a finitely generated ring (which is commutative if Φ is not of type A). By Proposition 7.7 the standard grading $\{X_\alpha\}$ of $\mathrm{St}_\Phi(R)$ is strong, so to prove property (T) it suffices to check relative property (T) for each of the pairs $(\mathrm{St}_\Phi(R), X_\alpha)$. However, in order to obtain a good bound for the Kazhdan constant of $\mathrm{St}_\Phi(R)$ with respect to a finite generating set of the form $\Sigma_\Phi(T)$ (as defined in Proposition 7.8), we need to proceed slightly differently.

We shall use a good reduction of Φ to a root system of rank 2 (of type A_2, B_2, BC_2 or G_2) described in § 6.2. Proposition 7.7 implies that the coarsened grading $\{Y_\beta\}$ of $\mathrm{St}_\Phi(R)$ is also strong, so Theorem 5.1 can be applied to this grading, this time yielding a much better bound for the Kazhdan constant.

To complete the proof of property (T) for $\mathrm{St}_\Phi(R)$ we still need to establish relative property (T), this time for the pairs $(\mathrm{St}_\Phi(R), Y_\beta)$. Qualitatively, this is not any harder than proving relative (T) for $(\mathrm{St}_\Phi(R), X_\alpha)$; however, we also need to explicitly estimate the corresponding Kazhdan ratios (which will affect the eventual bound for the Kazhdan constant of $\mathrm{St}_\Phi(R)$ with respect to a finite generating set).

Terminology: For brevity, in the sequel instead of saying *relative property* (T) *for the pair* (G, H) we will often say *relative property* (T) *for* H if G is clear from the context.

Our main tool for proving relative property (T) is the following result of Kassabov [Ka1] which generalizes Theorem 2.3 and will be proved in the appendix in a slightly extended form (see Theorem 11.8).

Theorem 7.10 (Kassabov)**.** *Let $n \geq 2$ and R a ring generated by $T = \{1 = t_0, t_1, \ldots, t_d\}$. Let $\{\alpha_1, \cdots, \alpha_n\}$ be a system of simple roots of A_n. Consider the semidirect product $G = \mathrm{St}_{A_{n-1}}(R) \ltimes N$, where $N \cong R^n$ is the subgroup of $\mathrm{St}_{A_n}(R)$ generated by $X_{\alpha_1}, X_{\alpha_1+\alpha_2}, \ldots, X_{\alpha_1+\cdots+\alpha_n}$ and the action of $\mathrm{St}_{A_{n-1}}(R)$ on N comes from the action of the subgroup $\langle X_\gamma : \gamma \in \mathbb{R}\alpha_2 + \ldots + \mathbb{R}\alpha_n\rangle$ of $\mathrm{St}_{A_n}(R)$ on N (we will refer to this action as the standard action of $\mathrm{St}_{A_{n-1}}(R)$ on R^n). Let $S = \Sigma_{A_{n-1}}(T) \cup (\Sigma_{A_n}(T) \cap N)$ and let $G' = \langle S, N\rangle$. Then*

$$\kappa_r(G', N; S) \succcurlyeq \frac{1}{\sqrt{d+n}}.$$

Remark. Note that if $n \geq 3$ we have $G' = G$.

Combining Theorems 7.10 and 2.7, we obtain the following result which immediately implies relative property (T) for the root subgroups of $\mathrm{St}_{B_2}(R)$.

Corollary 7.11. *Let $\{\alpha, \beta\}$ be a base of B_2, with α a long root, and define the semidirect product $\mathrm{St}_{A_1}(R) \ltimes N$ and the set S as in Lemma 7.9. Then*

$$\kappa(\mathrm{St}_{A_1}(R) \ltimes N, N; S) \succcurlyeq \frac{1}{2^{d/2}}, \quad \text{and therefore} \quad \kappa_r(\mathrm{St}_{B_2}(R), \cup_{\gamma\in B_2} X_\gamma; \Sigma) \succcurlyeq \frac{1}{2^{d/2}},$$

where Σ is the standard generating set of $\mathrm{St}_{B_2}(R)$.

Proof. Let $A = \{x_\alpha(t), x_{-\alpha}(t) : t \in T^* \cup T^2\}$, $B = \{x_\alpha(t), x_{-\alpha}(t), x_\beta(t), x_{\alpha+\beta}(t) : t \in T\}$ and let $C = \{x_{\alpha+2\beta}(t) : t \in T^*,\}$. so that $S = A \cup B \cup C$. Let G be the subgroup of $\mathrm{St}_{A_1}(R) \ltimes N$ generated by S.

By Lemma 7.9(1), N is a subgroup of G. Let $Z = X_{\alpha+2\beta}$. The relations of $\mathrm{St}_{B_2}(R)$ described in Propositions 7.6 imply that $(\mathrm{St}_{A_1}(R) \ltimes N)/Z \cong \mathrm{St}_{A_1}(R) \ltimes (N/Z)$ is isomorphic to $\mathrm{St}_{A_1}(R) \ltimes R^2$ (with the standard action of $\mathrm{St}_{A_1}(R)$ on R^2), and the image of G/Z under this isomorphism contains the subgroup denoted by G' in Theorem 7.10. Hence $\kappa(G/Z, N/Z; B) \succcurlyeq \frac{1}{\sqrt{d}}$.

Now let $H = Z \cap [N, G]$. By Lemma 7.9(2), Z/H is an elementary abelian 2-group generated by C, whence $\kappa(G/H, Z/H; C) \succcurlyeq \frac{1}{\sqrt{|C|}} \succcurlyeq \frac{1}{2^{d/2}}$. Since $Z \subseteq Z(G) \cap N$ and AN generates G/N, by Theorem 2.7 we have $\kappa(G, N; S) \succcurlyeq \frac{1}{2^{d/2}}$. Since G is a subgroup of $\mathrm{St}_{A_1}(R) \ltimes N$, we deduce that $\kappa(\mathrm{St}_{A_1}(R) \ltimes N, N; S) \succcurlyeq \frac{1}{2^{d/2}}$.

To prove the second assertion, note that for any root $\gamma \in B_2$ there is a homomorphism $\varphi : \mathrm{St}_{A_1}(R) \ltimes N \to \mathrm{St}_{B_2}(R)$ such that $\varphi(N) \supset X_\gamma$ and $\varphi(S) \subseteq \Sigma \cup \Sigma'$, where $\Sigma' = \{X_\delta(t^2) : t \in T, \delta \text{ is a long root}\}$. Thus, $\kappa_r(\mathrm{St}_{B_2}(R), X_\gamma; \Sigma \cup \Sigma') \succcurlyeq \frac{1}{2^{d/2}}$. Finally, by (7.6), the same asymptotic inequality holds with $\Sigma \cup \Sigma'$ replaced by Σ. □

Before turning to the case-by-case verification of relative property (T) (which is the main part of the proof of Theorem 7.12 below), we briefly summarize how this will be done for different root systems. Let Φ be a reduced irreducible classical root system with $rk(\Phi) \geq 2$.

(1) If Φ is simply-laced, relative property (T) for the root subgroups of $\mathrm{St}_\Phi(R)$ will follow almost immediately from Theorem 7.10 (with the aid of Proposition 7.5).
(2) If Φ is non-simply-laced and $\Phi \neq B_2$, relative property (T) for some of the root subgroups (either short ones or long ones) will again follow from Theorem 7.10. To prove relative property (T) for the remaining root subgroups, we will show that each of them is contained in a bounded product of a finite set and root subgroups for which relative (T) has already been established.
(3) Finally, the most difficult case $\Phi = B_2$ has almost been established in Corollary 7.11.

Explicit bounds for the Kazhdan constants and Kazhdan ratios will follow from Observation 2.2 and Lemma 2.4; these results will be often used without further mention.

We are now ready to prove the main result of this section.

Theorem 7.12. *Let Φ be a reduced irreducible classical root system of rank at least 2 and R a ring (which is commutative if Φ is not of type A) generated by a finite set $T = \{1 = t_0, t_1, \ldots, t_d\}$. Let Σ be the corresponding standard generating set of* $\mathrm{St}_\Phi(R)$ *(as defined in Proposition 7.8). Then* $\mathrm{St}_\Phi(R)$ *has property* (T) *and*

$$\kappa(\mathrm{St}_\Phi(R), \Sigma) \succcurlyeq \mathcal{K}(\Phi, d)$$

where

$$\mathcal{K}(\Phi, d) = \begin{cases} \frac{1}{\sqrt{n+d}} & \text{if } \Phi = A_n, B_n (n \geq 3), D_n \\ \frac{1}{\sqrt{d}} & \text{if } \Phi = E_6, E_7, E_8, F_4 \\ \frac{1}{2^{d/2}} & \text{if } \Phi = B_2, G_2 \\ \frac{1}{\sqrt{n+2^d}} & \text{if } \Phi = C_n \end{cases}$$

Remark. As we mentioned earlier, the Steinberg group $\mathrm{St}_{A_2}(R)$ can be defined for any alternative ring R with 1. Recently Zhang [Zh] proved that all such groups, with R finitely generated, have property (T) as well.

Proof. Let $G = \mathrm{St}_\Phi(R)$. We will show using case-by-case analysis that there exists a root system Φ' of type A_2, B_2, BC_2 or G_2 and a strong Φ'-grading $\{Y_\beta\}$ of G such that

$$\kappa_r(G, \cup Y_\beta; \Sigma) \succcurlyeq \mathcal{K}(\Phi, d).$$

This will imply the assertion of the theorem since $\kappa(G,\Sigma) \geq \kappa(G, \cup Y_\beta)\kappa_r(G, \cup Y_\beta; \Sigma)$ and $\kappa(G, \cup Y_\beta) \geq C$ for some absolute constant $C > 0$ by Theorem 5.1.

Case $\Phi = A_n$. This case is covered by [EJ]; however, we include the argument for completeness.

We use the reduction of A_n to A_2 given by the map

$$(x_1, \dots, x_{n+1}) \mapsto (x_1, x_2, \sum_{i\geq 3} x_i).$$

The coarsened grading is strong by Reduction 6.6, Lemma 6.2 and Proposition 7.7. The new root subgroups are

$$Y_{(-1,1,0)} = X_{e_2-e_1}, \quad Y_{(1,0,-1)} = \prod_{i=3}^{n+1} X_{e_1-e_i}, \quad Y_{(-1,0,1)} = \prod_{i=3}^{n+1} X_{e_i-e_1},$$

$$Y_{(1,-1,0)} = X_{e_1-e_2}, \quad Y_{(0,1,-1)} = \prod_{i=3}^{n+1} X_{e_2-e_i}, \quad \text{and} \quad Y_{(0,-1,1)} = \prod_{i=3}^{n+1} X_{e_i-e_2}.$$

We shall prove that $\kappa_r(G, Y_{(1,0,-1)}; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}$; the other cases are similar. The roots $\{e_i - e_j : 2 \leq i \neq j \leq n+1\}$ form a subsystem of type A_{n-1}. Thus, if H is the subgroup generated by $\{X_{e_i-e_j} : 2 \leq i \neq j \leq n+1\}$ and

$$E = \prod_{i=2}^{n+1} X_{e_1-e_i} \supset Y_{(1,0,-1)},$$

then by Proposition 7.5 there is a natural epimorphism $\mathrm{St}_{A_{n-1}}(R) \ltimes R^n \to H \ltimes E$. Hence using Theorem 7.10 we have

$$\kappa_r(\mathrm{St}_{A_n}(R), Y_{(1,0,-1)}; \Sigma) \geq \kappa_r(H \ltimes E, E; \Sigma) \geq \kappa_r(\mathrm{St}_{A_{n-1}}(R) \ltimes R^n, R^n; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}.$$

Case $\Phi = B_n$, $n \geq 3$. We use the reduction of B_n to B_2 given by the map

$$(x_1, \dots, x_{n+1}) \mapsto (x_1, x_2).$$

The coarsened grading is strong by Reduction 6.7, Lemma 6.2 and Proposition 7.7. The new root subgroups are

$$Y_{(\pm 1,0)} = X_{\pm e_1} \prod_{i=3}^{n} X_{\pm e_1 - e_i} \prod_{i=3}^{n} X_{\pm e_1 + e_i}, \quad Y_{(\pm 1, \pm 1)} = X_{\pm e_1 \pm e_2},$$

$$Y_{(0,\pm 1)} = X_{\pm e_2} \prod_{i=3}^{n} X_{\pm e_2 - e_i} \prod X_{\pm e_2 + e_i}.$$

We shall prove that $\kappa_r(G, Y_{(1,0)}; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}$; the other cases are similar. Since $\{e_i - e_j : 1 \leq i \neq j \leq n\}$ form a subsystem of type A_{n-1}, arguing as in the previous case, we obtain that $\kappa_r(\mathrm{St}_{B_n}(R), \prod_{i=3}^n X_{e_1-e_i}; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}$. The same argument applies to $\kappa_r(\mathrm{St}_{B_n}(R), \prod_{i=3}^n X_{e_1+e_i}; \Sigma)$. It remains to show that $\kappa_r(\mathrm{St}_{B_n}(R), X_{e_1}; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$.

By the same argument as above, $\kappa_r(G, X_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$ for any long root γ. From Proposition 7.6 it follows that

$$X_{e_1} \subseteq [x_{e_2}(1), X_{e_1-e_2}]X_{e_1+e_2} \subseteq \Sigma X_{e_1-e_2}\Sigma X_{e_1-e_2}X_{e_1+e_2}.$$

Therefore $\kappa_r(G, X_{e_1}; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$ by Lemma 2.4.

Case $\Phi = D_n$, $n \geq 3$. We use the reduction of D_n to B_2 given by the map

$$(x_1, \ldots, x_n) \mapsto (x_1, x_2).$$

The coarsened grading is strong by Reduction 6.8, Lemma 6.2 and Proposition 7.7. The new root subgroups are

$$Y_{(\pm 1,0)} = \prod_{i=3}^{n} X_{\pm e_1 - e_i} \prod_{i=3}^{n} X_{\pm e_1 + e_i}, \quad Y_{(\pm 1, \pm 1)} = X_{\pm e_1 \pm e_2},$$

$$Y_{(0,\pm 1)} = \prod_{i=3}^{n} X_{\pm e_2 - e_i} \prod_{i=3}^{n} X_{\pm e_2 + e_i}.$$

The proof of relative property (T) is the same as in the case of B_n.

Case $\Phi = F_4$, E_6, E_7 *or* E_8. In this case we do not have to do any reduction to a root system of bounded rank, since the rank is already bounded. The grading is strong by Proposition 7.7. However, if one wants to obtain explicit estimates for the Kazhdan constants, one can use Reductions 6.12 and 6.13 (again the coarsened gradings are strong from Lemma 6.2 and Proposition 7.7).

In order to prove that $\kappa_r(\mathrm{St}_\Phi(R), X_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$ for any root $\gamma \in \Phi$, we simply observe that γ lies in a subsystem of type A_2, and the same argument as in the case A_2 can be applied.

Case $\Phi = B_2$. The grading is strong by Proposition 7.7, and the inequality $\kappa_r(\mathrm{St}_{B_2}(R), \cup_\gamma X_\gamma; \Sigma) \succcurlyeq \frac{1}{2^{d/2}}$, holds by Corollary 7.11.

Case $\Phi = C_n$, $n \geq 3$. We use the reduction of C_n to BC_2 given by the map

$$(x_1, \ldots, x_n) \mapsto (x_1, x_2).$$

The coarsened grading is strong by Reduction 6.9, Lemma 6.2 and Proposition 7.7. The new root subgroups are

$$Y_{(\pm 1,0)} = \langle \prod_{i=3}^{n} X_{\pm e_1 - e_i}, \quad \prod_{i=3}^{n} X_{\pm e_1 + e_i} \rangle, \quad Y_{(\pm 1, \pm 1)} = X_{\pm e_1 \pm e_2}, \quad Y_{(\pm 2, 0)} = X_{\pm 2e_1},$$

$$Y_{(0,\pm 1)} = \langle \prod_{i=3}^{n} X_{\pm e_2 - e_i}, \prod_{i=3}^{n} X_{\pm e_2 + e_i} \rangle \quad \text{and} \quad Y_{(0,\pm 2)} = X_{\pm 2e_2}.$$

Note that $Y_{(1,0)} \neq AB$ for $A = \prod_{i=3}^{n} X_{e_1 - e_i}$ and $B = \prod_{i=3}^{n} X_{e_1 + e_i}$; however, it is easy to see that $Y_{(1,0)} = ABABAB$. Similar factorization exists for other short root subgroups.

Arguing as in the case $\Phi = B_n, n \geq 3$, we conclude that $\kappa_r(\mathrm{St}_\Phi(R), Y_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+d}}$ when γ is a short or a long root. If γ is a double root, then γ lies in a weak subsystem of type B_2. Thus, from Corollary 7.11 and Proposition 7.5 we obtain that $\kappa_r(\mathrm{St}_\Phi(R), Y_\gamma; \Sigma) \succcurlyeq \frac{1}{2^{d/2}}$. Hence $\kappa_r(\mathrm{St}_\Phi(R), \cup Y_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{n+2^d}}$.

Case $\Phi = G_2$. The grading is strong by Proposition 7.7. If γ is a long root, then $\kappa_r(\mathrm{St}_{G_2}(R), X_\gamma; \Sigma) \succcurlyeq \frac{1}{\sqrt{d}}$, because the long roots form a weak subsystem of type A_2.

Now, we will show that if γ is a short root, then $\kappa_r(\mathrm{St}_{G_2}(R), X_\gamma; \Sigma) \succcurlyeq \frac{1}{2^{d/2}}$. Without loss of generality we may assume that the relations from Proposition 7.6 for $\mathrm{St}_{G_2}(R)$ hold and $\gamma = \alpha + 2\beta$.

Calculating $[x_\alpha(r), x_\beta(2)][x_\alpha(2r), x_\beta(1)]^{-1}$ we see that

$$X_\gamma(2R) = \{x_\gamma(2r) :\ r \in R\} \subseteq X_\alpha X_\alpha^{x_\beta(2)} X_\alpha X_\alpha^{x_\beta(1)} X_{\alpha+3\beta} X_{2\alpha+3\beta},$$

so $X_\gamma(2R)$ lies inside a bounded product of long root subgroups and 2 fixed elements of Σ.

Similarly, calculating $[x_\alpha(r), x_\beta(t)][x_\alpha(tr), x_\beta(1)]^{-1}$ for any $1 \neq t \in T$ we obtain that $X_\gamma((t^2 - t)R) = \{x_\gamma((t^2 - t)r) :\ r \in R\}$ lies inside a bounded product of long root subgroups and 2 fixed elements of Σ.

Let $I = 2R + \sum_{t \in T}(t^2 - t)R$. Since $|T| = d + 1$, using our previous observations and Lemma 2.4(b), we conclude that

$$\kappa_r(\mathrm{St}_{G_2}(R), X_\gamma(I); \Sigma) \succcurlyeq \frac{1}{d\sqrt{d}}.$$

The group $X_\gamma(R)/X_\gamma(I)$ is an elementary abelian 2-group generated by $S = \{x_\gamma(t) :\ t \in T^*\}$. Since $|S| = 2^{d+1}$, we have

$$\kappa(X_\gamma(R)/X_\gamma(I), S) \succcurlyeq \frac{1}{2^{d/2}}.$$

By Lemma 2.9, these two inequalities imply that $\kappa_r(\mathrm{St}_{G_2}(R), X_\gamma(R); \Sigma) \succcurlyeq \frac{1}{2^{d/2}}$. $\square$

8. Twisted Steinberg groups

8.1. **Constructing twisted groups.** In this subsection we introduce a general method for constructing new groups graded by root systems from old ones using the machinery of twists. The method generalizes the construction of twisted Chevalley groups [St].

Let Φ be a root system, G a group and $\{X_\alpha\}_{\alpha\in\Phi}$ a Φ-grading of G. Let $Q \subset \mathrm{Aut}(G)$ be a group of automorphisms of G such that

(i) Each element of Q is a graded automorphism (as defined in § 7.1), so that there is an induced action of Q on Φ.
(ii) Q acts linearly on Φ, that is, the action of Q on Φ extends to an $\mathbb{R}$-linear action of Q on the real vector space spanned by Φ.

Remark. In all our applications Q will be a finite group (in fact, usually a cyclic group).

Let $V = \mathbb{R}\Phi$ be the $\mathbb{R}$-vector space spanned by Φ. Suppose we are given another $\mathbb{R}$-vector space W and a reduction $\eta : V \to W$ such that

$$\eta(q\alpha) = \eta(\alpha) \text{ for any } \alpha \in \Phi \text{ and } q \in Q. \tag{8.1}$$

Let $\Psi = \eta(\Phi) \setminus \{0\}$ be the induced root system and let $\{Y_\beta\}_{\beta\in\Psi}$ be the coarsened grading, that is, $Y_\beta = \langle X_\alpha : \eta(\alpha) = \beta\rangle$. Finally, let $Z_\beta = Y_\beta^Q$ be the set of Q-fixed points in Y_β, and assume that the following additional condition holds:

(iii) $\{Z_\alpha\}_{\alpha\in\Psi}$ is a Ψ-grading.

Then we define the **twisted group** $\widehat{G^Q}$ to be the graded cover of $\langle Z_\alpha : \alpha \in \Psi\rangle$ with respect to the Ψ-grading $\{Z_\alpha\}_{\alpha\in\Psi}$.

Here is a slightly technical but easy-to-use criterion which ensures that condition (iii) holds. This criterion will be applicable in all of our examples.

Proposition 8.1. *Let Ψ' be the set of all roots in Ψ which are not representable as $a\gamma$ for some other $\gamma \in \Psi$ and $a > 1$ (in particular, $\Psi' = \Psi$ if Ψ is reduced). Assume that*

(a) *For any $\gamma, \delta \in \Psi$ such that $\delta = a\gamma$ with $a \geq 1$, we have $Y_\delta \subseteq Y_\gamma$.*
(b) *For any Borel subset B of Ψ, any element of $\langle Y_\gamma\rangle_{\gamma\in B}$ can be* **uniquely** *written as $\prod_{i=1}^k y_{\gamma_i}$ where $\gamma_1, \ldots, \gamma_k$ are the roots in $B \cap \Psi'$ taken in some fixed order and $y_{\gamma_i} \in Y_{\gamma_i}$ for all i.*

Then $\{Z_\alpha\}_{\alpha\in\Psi}$ is a Ψ-grading.

Proof. Let $\alpha, \beta \in \Psi$ such that $\beta \notin \mathbb{R}_{<0}\alpha$, in which case there exists a Borel subset B containing both α and β. Let $\Omega = \{\gamma \in \Psi : \gamma = a\alpha + b\beta \text{ with } \alpha, \beta \geq 1\}$, and let $\gamma_1, \ldots, \gamma_k$ be the roots in $B \cap \Psi'$ (as in condition (b)). Let I be the set of all $i \in \{1, \ldots, k\}$ such that $\mathbb{R}_{\geq 1}\gamma_i \cap \Omega \neq \emptyset$. For each $i \in I$ let $\delta_i = a_i\gamma_i$ such that $\delta_i \in \Omega$ and $a_i \in \mathbb{R}_{\geq 1}$ is smallest possible.

Now take any $z \in Z_\alpha$ and $w \in Z_\beta$. Since $\{Y_\gamma\}_{\gamma\in\Psi}$ is a Ψ-grading, using condition (a) it is easy to show that $[z, w] = \prod_{i\in I} y_{\delta_i}$ where $y_{\delta_i} \in Y_{\delta_i}$ for each i. Since z and w are fixed by Q, for any $q \in Q$ we have $[z, w] = \prod_{i\in I} q(y_{\delta_i})$.

Thus, we have obtained two factorizations for $[z, w]$, and since $Y_{\delta_i} \subseteq Y_{\gamma_i}$, they both satisfy the requirement in (b). Therefore, (b) implies that $q(y_{\delta_i}) = y_{\delta_i}$ for each i. Hence $y_{\delta_i} \in Y_{\delta_i}^Q = Z_{\delta_i}$ for each $i \in I$, and so $[z, w] \in \langle Z_\gamma : \gamma \in \Omega\rangle$. □

If Q is finite, one always has a natural choice for the pair (W, η) satisfying (8.1). Indeed, by condition (ii) the action of Q on Φ extends to a linear action of Q on V. Then we can take $W = V^Q$, the subspace of Q-invariant vectors and $\eta : V \to W$ the natural projection, that is,

$$\eta(v) = \frac{1}{|Q|}\sum_{q\in Q} qv. \tag{8.2}$$

In fact, in all our examples the pair (W, η) will be of this form up to isomorphism, but it will be more convenient to define W and η first and then check (8.1) rather than realize W as the subspace of Q-invariant vectors in V.

Remark. If Q and Q' are conjugate in the group $\mathrm{Aut}_{gr}(G)$ of graded automorphisms of G, then the corresponding twisted groups $\widehat{G^Q}$ and $\widehat{G^{Q'}}$ are easily seen to be isomorphic.

We will discuss in detail six families of twisted groups. The first five families will all be of the following form, while the construction of the sixth family will involve minor modifications. Let Φ be classical, reduced and irreducible of rank ≥ 2. We take $G = \mathrm{St}_\Phi(R)$ for some ring R, which is commutative if Φ is not of type A. The acting group Q will be a finite (usually cyclic) subgroup of $\mathrm{Aut}(G)$ whose elements are compositions of diagram, ring and diagonal automorphisms (as defined in § 8.2).

The latter restriction on Q implies that it naturally acts on the corresponding adjoint elementary Chevalley group $\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)$. Let $\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)^Q$ be the subgroup of Q-fixed points of $\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)$, and let $\widetilde{\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)^Q}$ be the group generated by the intersections of $\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)^Q$ with the root subgroups of the Ψ-grading of $\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)$.

Thus, if $\widehat{\mathrm{St}_{\Phi}(R)^Q}$ is the twisted group obtained via the above procedure, there is a natural epimorphism from $\widehat{\mathrm{St}_{\Phi}(R)^Q}$ onto $\widetilde{\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)^Q}$. We will refer to $\widehat{\mathrm{St}_{\Phi}(R)^Q}$ as a *twisted Steinberg group*, and to $\widetilde{\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)^Q}$ as a *twisted Chevalley group*, and we will also say that $\widehat{\mathrm{St}_{\Phi}(R)^Q}$ is a *Steinberg cover* of $\mathbb{E}_{\Phi}^{\mathrm{ad}}(R)^Q$.

We note that the term 'twisted Chevalley group' usually has a more restricted meaning – instead of all possible finite groups of automorphisms Q as above, one only considers those which are used in (canonical) realizations of finite simple groups of twisted Lie type. In this section we will mostly deal with the Steinberg covers for these types of twisted Chevalley groups. The obtained Steinberg groups are summarized below and will be studied in Examples 1-5.

1. Groups $\mathrm{St}_{C_n}^{\omega}(R,*)$ where R is a ring, $*$ is an involution on R and ω is a central unit of R satisfying $\omega^* = \omega^{-1}$. These groups are Steinberg covers for *hyperbolic unitary groups* (see [HO]). The special case $\omega = 1$ corresponds to twisted Chevalley groups of type ${}^2A_{2n-1}$ (unitary groups in even dimension).
2. Groups $\mathrm{St}_{BC_n}(R,*)$ where R is a ring and $*$ is an involution on R. These groups are Steinberg covers for twisted Chevalley groups of type ${}^2A_{2n}$ (unitary groups in odd dimension).
3. Groups $\mathrm{St}_{B_n}(R,\sigma)$ where R is a commutative ring and σ is an involution on R. These groups are Steinberg covers for twisted Chevalley groups of type ${}^2D_n$.
4. Groups $\mathrm{St}_{G_2}(R,\sigma)$ where R is a commutative ring and σ an automorphism of R of order 3. These groups are Steinberg covers for twisted Chevalley groups of type ${}^3D_4$.
5. Groups $\mathrm{St}_{F_4}(R,\sigma)$ where R is a commutative ring and σ an involution of R. These groups are Steinberg covers for twisted Chevalley groups of type ${}^2E_6$.

Remark. Our notations for the twisted Steinberg groups are chosen in such a way that the subscript indicates the root system by which this twisted Steinberg group is naturally graded.

In Example 3 we shall define a more general family of twisted groups (which will include the groups $\mathrm{St}_{B_n}(R,\sigma)$ as a special case) using an observation that the (classical) Steinberg group $\mathrm{St}_{D_n}(R)$ arises as the twisted Steinberg group $\mathrm{St}_{C_n}^{-1}(R,*)$ (from Example 1) in the special case when $*$ is trivial (and R is commutative).

In the last example (Example 6) we construct certain groups $\mathrm{St}_{{}^2F_4}(R,*)$, where R is a commutative ring of characteristic 2 and $* : R \to R$ is an injective homomorphism such that $(r^*)^* = r^2$. These groups are graded by a root system in $\mathbb{R}^2$ with 24 roots and can be defined as graded covers of certain "algebraic-like" groups constructed by Tits [Ti]. We note that the standard definition of twisted Chevalley groups of type ${}^2F_4$ is only valid when R is a perfect field, in which case they coincide with Tits' groups. Unlike Examples 1-5, in the construction of the groups $\mathrm{St}_{{}^2F_4}(R,*)$, the initial group G will not be the entire Steinberg group $\mathrm{St}_{F_4}(R)$, but certain subgroup of it. The general twisting procedure will also be slightly modified in this example, as we will have to apply the *fattening* operation (defined in § 4.4) to the coarsened grading $\{Y_\beta\}$.

8.2. **Graded automorphisms of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ and $\mathrm{St}_\Phi(R)$.** In this section we describe some natural families of graded automorphisms of (non-twisted) adjoint elementary Chevalley groups and Steinberg groups. Each automorphism will be defined via its action on the root subgroups, and we will need to justify that it can be extended to the entire group.

In the case of elementary Chevalley groups and Steinberg groups over commutative rings the following observation will provide the justification:

Observation 8.2. *Let Φ be a reduced irreducible classical root system, R a commutative ring and $\mathcal{L}_R$ the R-Lie algebra of type Φ, as defined in § 7. Let $f \in \mathrm{Aut}(\mathcal{L}_R)$ be an automorphism which permutes the root subspaces of $\mathcal{L}_R$. Then $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$, considered as a subgroup of $\mathrm{Aut}(\mathcal{L}_R)$, is normalized by f, and moreover, the conjugation by f permutes the root subgroups $\{X_\alpha(R)\}$ of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$. Thus f naturally induces a graded automorphism of $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ and hence also induces a graded automorphism of $\mathrm{St}_\Phi(R)$ by Lemma 7.1.*

If $G = \mathrm{St}_{A_m}(R)$, with R noncommutative, the existence of the automorphism of G with a given action on the root subgroups is easy to establish using the standard presentation of $\mathrm{St}_{A_m}(R)$ recalled below.

As usual, we realize A_m as the subset $\{e_i - e_j : \ 1 \le i \ne j \le m+1\}$ of $\mathbb{R}^{m+1}$. The group $\mathrm{St}_{A_m}(R)$ has generators $\{x_{e_i-e_j}(r) : 1 \le i \ne j \le m+1, r \in R\}$ and relations

$$x_{e_i-e_j}(r+s) = x_{e_i-e_j}(r)x_{e_i-e_j}(s) \text{ and}$$
$$[x_{e_i-e_j}(r), x_{e_k-e_l}(s)] = \begin{cases} x_{e_i-e_l}(rs) & \text{if } j=k \text{ and } i \ne l \\ x_{e_k-e_j}(-sr) & \text{if } j \ne k \text{ and } i = l \\ 0 & \text{if } j \ne k \text{ and } i \ne l \end{cases}$$

In each of the following examples we fix a ring R and a reduced irreducible classical root system Φ, and G will denote one of the groups $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ or $\mathrm{St}_\Phi(R)$, unless additional restrictions are imposed.

Type I: *ring automorphisms.* Let σ be an automorphism of the ring R. Then we can define the automorphism φ_σ of G by

$$\varphi_\sigma(x_\alpha(r)) = x_\alpha(\sigma(r)), \ \alpha \in \Phi, \ r \in R.$$

If R is commutative, φ_σ is well defined since it is induced (as in Observation 8.2) by the automorphism of $\mathcal{L}_R$ which sends $r \otimes l$ (where $l \in \mathcal{L}_\mathbb{Z}$ and $r \in R$) to $\sigma(r) \otimes l$.

If R is arbitrary and $G = \mathrm{St}_{A_n}(R)$, the automorphism φ_σ is well defined since it clearly respects the defining relations of G.

Type II: *diagonal automorphisms.* Let $Z(R)^\times$ be the group of invertible elements of $Z(R)$, let $\mathbb{Z}\Phi$ denote the $\mathbb{Z}$-span of Φ, and let $\mu : \mathbb{Z}\Phi \to Z(R)^\times$ be a homomorphism. Then we can define the automorphism χ_μ of G given by

$$\chi_\mu(x_\alpha(r)) = x_\alpha(\mu(\alpha)r), \ \alpha \in \Phi, \ r \in R.$$

If R is commutative, χ_μ is well defined since it is induced by the automorphism of $\mathcal{L}_R$ which fixes h_α and sends x_α to $\mu(\alpha)x_\alpha$ for any $\alpha \in \Phi$.

As for type I, if R is arbitrary and $G = \mathrm{St}_{A_n}(R)$, the automorphism χ_μ is well defined since it respects the defining relations.

Type III: *root system automorphisms (commutative case).* In this example we assume that R is commutative. Let $V = \mathbb{R}\Phi$ be the $\mathbb{R}$-span of Φ, and let π be an automorphism of V which stabilizes Φ (equivalently, we can start with an automorphism of Φ and uniquely extend it to an automorphism of V). Then there are constants $\gamma_\alpha = \pm 1 (\alpha \in \Phi)$ such that the map

$$\lambda_\pi(x_\alpha(r)) = x_{\pi(\alpha)}(\gamma_\alpha r),\ \alpha \in \Phi,\ r \in R,$$

can be extended to an automorphism of G.

The existence of an automorphism of $\mathcal{L}_R$ which induces λ_π is a consequence of the Isomorphism Theorem for simple Lie algebras ([Ca, Theorem 3.5.2], see also [Ca, Proposition 12.2.3]).

Note that there is no canonical choice for the constants γ_α (except when $\Phi = A_n$), so in our notation λ_π is only unique up to a diagonal automorphism (which acts as multiplication by ± 1 on each root subgroup).

Type III': *mixed automorphisms of* $\mathrm{St}_{A_m}(R)$. In this example we assume that $G = \mathrm{St}_{A_m}(R)$ and R is arbitrary. Let V be the $\mathbb{R}$-span of A_m. It is well known that every automorphism of A_m has the form

$$a(\pi, \delta) : e_i - e_j \mapsto (-1)^\delta (e_{\pi(i)} - e_{\pi(j)})$$

for some permutation $\pi \in \Sigma_{m+1}$ and $\delta = 0, 1$. In particular, $\mathrm{Aut}(A_m)$ has order $2(m+1)!$ (if $m \geq 2$), and it is easy to see that the automorphisms with $\delta = 0$ are precisely the elements of the Weyl group of A_m.

If R is commutative, we have already associated an automorphism of G of type III to each element of $\mathrm{Aut}(A_m)$. The type III automorphism of G corresponding to $a(\pi, 0) \in \mathrm{Aut}(A_m)$ can be defined even if R is not commutative. It will be denoted by λ_π^+ and is given by

$$\lambda_\pi^+(x_{e_i - e_j}(r)) = x_{e_{\pi(i)} - e_{\pi(j)}}(r),\ \alpha \in \Phi,\ r \in R.$$

Similarly, if R is commutative, we will denote by λ_π^- the type III automorphism of G corresponding to $a(\pi, 1) \in \mathrm{Aut}(A_m)$. It is given by

$$\lambda_\pi^-(x_{e_i - e_j}(r)) = x_{e_{\pi(j)} - e_{\pi(i)}}(-r),\ \alpha \in \Phi,\ r \in R.$$

The formula for λ_π^- will not define an automorphism of G if R is noncommutative. However, if we are given an anti-automorphism $*$ of R, for each $\pi \in \Sigma_{m+1}$ we can define an automorphism $\lambda_{\pi,*}^-$ of G by setting

$$\lambda_{\pi,*}^-(x_{e_i - e_j}(r)) = x_{e_{\pi(j)} - e_{\pi(i)}}(-r^*),\ \alpha \in \Phi,\ r \in R.$$

These automorphisms will be called *mixed*.

Note that if R is commutative, then $\lambda_{\pi,*}^-$ is just the composition of λ_π^- and the ring automorphism φ_*.

The collection of twisted groups that can be constructed using these four types of automorphisms and their compositions is clearly too large for case-by-case analysis and is beyond the scope of this paper. We shall concentrate on automorphisms which yield natural analogues of twisted Chevalley groups listed at the end of § 8.1.

Among all root system automorphisms of particular importance are *diagram automorphisms* – the ones induced by an automorphism of the Dynkin diagram of Φ. For instance, in the case $\Phi = A_m$, there is unique (non-trivial) diagram automorphism (for a given choice of simple roots) – in the above notations it is the automorphism λ_π^- where $\pi \in \Sigma_{m+1}$ is given by $\pi(i) = m+2-i$. Each of the twisted

Chevalley groups of type ${}^k\Phi$, where $\Phi = A_n, D_n$ or E_6 and $k = 2$, or $\Phi = D_4$ and $k = 3$, is obtained from $\mathbb{E}_\Phi^{\mathrm{ad}}(R)$ using the twisting by the composition of a diagram automorphism and a ring automorphism of the same order k.

8.3. **Unitary Steinberg groups over non-commutative rings with involution.** In this subsection we shall define (twisted) Steinberg groups corresponding to (quasi-split) unitary groups and, in the case of even dimension, their generalizations, called hyperbolic unitary Steinberg groups. We shall establish property (T) for most of those groups. To simplify the exposition, we will not provide explicit estimates for the Kazhdan constants, although in most cases reasonably good estimates can be obtained by adapting the arguments from § 7.

Throughout this subsection we fix a ring R, and let $* : R \to R$ be an involution, that is, an anti-automorphism of order ≥ 2.

As we already stated, in the classical setting unitary groups are obtained from Chevalley groups of type A_m via twisting by the order 2 automorphism

$$Dyn_* = \lambda^-_{\pi,*} \text{ where } \pi \text{ is the permutation } i \mapsto m + 2 - i.$$

In even dimension (that is, if m is odd), there is an interesting generalization of this construction, where instead of Dyn_* one uses the composition of Dyn_* with a suitable diagonal automorphism of order 2.

To each $\omega \in Z(R)^\times$ we can associate a homomorphism $T_\omega : \mathbb{Z}A_m \to Z(R)^\times$ given by

$$T_\omega(e_i - e_j) = \begin{cases} 1 & \text{if } i, j \leq (m+1)/2 \text{ or } i, j > (m+1)/2 \\ \omega & \text{if } i \leq (m+1)/2 < j \\ \omega^{-1} & \text{if } j \leq (m+1)/2 < i \end{cases}$$

Note that the homomorphism $T_1 : \mathbb{Z}A_m \to Z(R)^\times$ is the trivial homomorphism.

For each such ω we define the automorphism q_ω of $\mathrm{St}_{A_m}(R)$ given by

$$q_\omega = Dyn_* \chi_{T_\omega}. \tag{8.3}$$

(recall that χ_{T_ω} is a diagonal automorphism, defined in § 8.2).

Now let

$$U(R) = \{r \in R^\times : rr^* = 1\} \quad \text{and} \quad U(Z(R)) = U(R) \cap Z(R).$$

It is easy to see that if m is odd and $\omega \in U(Z(R))$, then q_ω has order 2.

The groups obtained from Chevalley groups of type A_m via twisting by q_ω (with m odd and $\omega \in U(Z(R))$) are called *hyperbolic unitary groups*. These groups have been originally defined by Bak [Bak1] and are discussed in detail in the book by Hahn and O'Meara [HO] (see also [Bak2]).

Remark. It is easy to show that if χ is any diagonal automorphism of the Chevalley group $\mathrm{SL}_{m+1}(R)$ such that the composition $Dyn_*\chi$ has order 2, then χ is graded conjugate (in fact, conjugate by a diagonal automorphism) to q_ω for some $\omega \in U(Z(R))$ if m is odd, and graded conjugate to Dyn_* if m is even; see also Observation 8.4 below. This yields a simple characterization of hyperbolic unitary groups among all twisted Chevalley groups.

Before turning to Example 1, we introduce some additional terminology from [HO] (we note that our notations are different from [HO]).

Definition. Let $\omega \in U(Z(R))$. Put

$$\mathrm{Sym}_{-\omega}(R) = \{r \in R : \ r^*\omega = -r\} \quad \text{and} \quad \mathrm{Sym}_{-\omega}^{\min}(R) = \{r - r^*\omega : \ r \in R\}.$$

A *form parameter* of the triple $(R, *, \omega)$ is a subgroup I of $(R, +)$ such that

(i) $\mathrm{Sym}_{-\omega}^{\min}(R) \subseteq I \subseteq \mathrm{Sym}_{-\omega}(R)$
(ii) For any $u \in I$ and $s \in R$ we have $s^*us \in I$.

The following simplified terminology will be used in the case $\omega = \pm 1$.

- The set $\mathrm{Sym}_1(R)$ will be denoted by $\mathrm{Sym}(R)$, and its elements will be called symmetric.
- The set $\mathrm{Sym}_{-1}(R)$ will be denoted by $\mathrm{Asym}(R)$, and its elements will be called antisymmetric

For a subset A of $\mathrm{Sym}_{-\omega}(R)$ we let $\langle A\rangle_{-\omega}$ be the form parameter generated by A, that is,

$$\langle A\rangle_{-\omega} = \{x \in \mathrm{Sym}_{-\omega}(R) : \ x = \sum_{i=1}^{k} s_i a_i s_i^* + (r - r^*\omega) \text{ with } a_i \in A, s_i, r \in R\}.$$

Remark. If $A = \{a_1, \ldots, a_m\}$ is finite, then any element $x \in \langle A\rangle_{-\omega}$ has an expansion in the form

$$x = \sum_{i=1}^{m} s_i a_i s_i^* + (r - r^*\omega),$$

that is, with only one term sas^* for each $a \in A$. This is because $uau^* + vav^* = (u+v)a(u+v)^* + (r - r^*\omega)$ for $r = -uav^*$.

Example 1: *Hyperbolic unitary Steinberg groups.* Let $\Phi = A_{2n-1}$ and $G = \mathrm{St}_\Phi(R)$. Fix $\omega \in U(Z(R))$, let $q = q_\omega \in \mathrm{Aut}(G)$ and $Q = \langle q\rangle$.

The twisted group $\widehat{G^Q}$ constructed in this example will be denoted by $\mathrm{St}_{C_n}^{\omega}(R, *)$. This group is graded by the root system C_n and corresponds to the group of transformations preserving the sesquilinear form

$$f(u, v) = \sum_{i=1}^{n} u_i v_{\bar{i}}^* + \omega u_{\bar{i}} v_i^* \text{ on } R^{2n} \text{ where } \bar{i} = 2n + 1 - i.$$

Remark. This form is ω-hermitian, that is, $f(v, u) = \omega(f(u, v)^*)$. For more information on groups fixing this form see [HO, Chapter 5.3].

We shall use the standard realization for both A_{2n-1} and C_n, and to avoid confusion we shall denote the roots of A_{2n-1} by $e_i - e_j$, with $1 \le i \ne j \le 2n$, and the roots of C_n by $\pm\varepsilon_i \pm \varepsilon_j$ and $\pm 2\varepsilon_i$, with $1 \le i \ne j \le n$.

The action of q on the root subgroups of G is given by

$$q(x_{e_i - e_j}(r)) = \begin{cases} x_{e_{\bar{j}} - e_{\bar{i}}}(-r^*) & \text{if } i, j \le n \text{ or } i, j > n \\ x_{e_{\bar{j}} - e_{\bar{i}}}(-\omega r^*) & \text{if } i \le n < j \\ x_{e_{\bar{j}} - e_{\bar{i}}}(-\omega^* r^*) & \text{if } j \le n < i \end{cases}$$

Define $\eta : \bigoplus_{i=1}^{2n} \mathbb{R}e_i \to \bigoplus_{i=1}^{n} \mathbb{R}\varepsilon_i$ by $\eta(e_i) = \varepsilon_i$ if $i \le n$ and $\eta(e_i) = -\varepsilon_{\bar{i}}$ if $i > n$. It is straightforward to check that η is q-invariant. Then

$$\eta(e_i - e_j) = \begin{cases} \varepsilon_i - \varepsilon_j & \text{if } i, j \le n \\ \varepsilon_i + \varepsilon_{\bar{j}} & \text{if } i \le n,\ j > n \\ -\varepsilon_{\bar{i}} - \varepsilon_j & \text{if } i > n,\ j \le n \\ \varepsilon_{\bar{j}} - \varepsilon_{\bar{i}} & \text{if } i, j > n \end{cases}$$

so the root system $\Psi = \eta(\Phi) \setminus \{0\}$ is indeed of type C_n (with standard realization).

Let $\{Y_\gamma\}_{\gamma\in\Psi}$ denote the coarsened Ψ-grading of G. If $\gamma \in \Psi$ is a short root, the corresponding root subgroup Y_γ consists of elements $\{y_\gamma(r, s) : r, s \in R\}$ where

$$y_{\varepsilon_i-\varepsilon_j}(r, s) = x_{e_i-e_j}(r)x_{e_{\bar{j}}-e_{\bar{i}}}(s)$$

$$y_{\pm(\varepsilon_i+\varepsilon_j)}(r, s) = x_{\pm(e_i-e_{\bar{j}})}(r)x_{\pm(e_j-e_{\bar{i}})}(s)$$

If $\gamma \in \Psi$ is a long root, the corresponding root subgroup Y_γ consists of elements $\{y_\gamma(r) : r \in R\}$ where

$$y_{2\varepsilon_i}(r) = x_{e_i-e_{\bar{i}}}(r).$$

Computing q-invariants and letting $Z_\gamma = Y_\gamma^q$, we get

$$Z_{\varepsilon_i-\varepsilon_j} = \{z_{\varepsilon_i-\varepsilon_j}(r) = x_{e_i-e_j}(r)x_{e_{\bar{j}}-e_{\bar{i}}}(-r^*) :\ r \in R\} \text{ for } i < j$$

$$Z_{\varepsilon_i-\varepsilon_j} = \{z_{\varepsilon_i-\varepsilon_j}(r) = x_{e_i-e_j}(-r^*)x_{e_{\bar{j}}-e_{\bar{i}}}(r) :\ r \in R\} \text{ for } i > j$$

$$Z_{\varepsilon_i+\varepsilon_j} = \{z_{\varepsilon_i+\varepsilon_j}(r) = x_{e_i-e_{\bar{j}}}(r)x_{e_j-e_{\bar{i}}}(-\omega r^*) :\ r \in R\} \text{ for } i < j$$

$$Z_{-\varepsilon_i-\varepsilon_j} = \{z_{-\varepsilon_i-\varepsilon_j}(r) = x_{-e_i+e_{\bar{j}}}(-r^*)x_{-e_j+e_{\bar{i}}}(\omega^* r) :\ r \in R\} \text{ for } i < j$$

$$Z_{2\varepsilon_i} = \{z_{2\varepsilon_i}(r) = x_{e_i-e_{\bar{i}}}(r) :\ r \in \mathrm{Sym}_{-\omega}(R)\}$$

$$Z_{-2\varepsilon_i} = \{z_{-2\varepsilon_i}(r) = x_{-e_i+e_{\bar{i}}}(-r^*) :\ r \in \mathrm{Sym}_{-\omega}(R)\}$$

Note that

$Z_\gamma \cong (R, +)$ if γ is a short root and
$Z_\gamma \cong (\mathrm{Sym}_{-\omega}(R), +)$ if γ is a long root

Remark. When γ is short, there is no "canonical" isomorphism between Z_γ and $(R, +)$, so a choice needs to be made in the definition of $z_\gamma(r)$.

It is easy to see that the hypothesis of Proposition 8.1 holds in this example. Hence $\{Z_\gamma\}_{\gamma\in\Psi}$ is a Ψ-grading, and we can form the graded cover $\widehat{G^Q}$.

Thus, by definition $\widehat{G^Q} = \langle Z \,|\, E\rangle$ where $Z = \sqcup_{\gamma\in\Psi} Z_\gamma$ and E is the set of commutation relations (inside G) expressing the elements of $[Z_\gamma, Z_\delta]$ in terms of $\{Z_{a\gamma+b\delta} : a, b \ge 1\}$ (where $\delta \notin \mathbb{R}_{<0}\gamma$). These relations are obtained by straightforward calculation.

Below we list the non-trivial commutation relations between the positive root subgroups (omitting the relations where the commutator is equal to 1).

$$(E1)\qquad [z_{\varepsilon_i-\varepsilon_j}(r), z_{\varepsilon_j-\varepsilon_k}(s)] = z_{\varepsilon_i-\varepsilon_k}(rs) \text{ for } i<j<k$$
$$(E2)\qquad [z_{\varepsilon_i-\varepsilon_j}(r), z_{\varepsilon_i+\varepsilon_j}(s)] = z_{2\varepsilon_i}(sr^*-\omega rs^*) \text{ for } i<j$$
$$(E3)\qquad [z_{2\varepsilon_j}(r), z_{\varepsilon_i-\varepsilon_j}(s)] = z_{\varepsilon_i+\varepsilon_j}(-sr)z_{2\varepsilon_i}(srs^*) \text{ for } i<j$$
$$(E4)\qquad [z_{\varepsilon_i-\varepsilon_j}(r), z_{\varepsilon_j+\varepsilon_k}(s)] = \begin{cases} z_{\varepsilon_i+\varepsilon_k}(rs) & \text{for } i<j<k \\ z_{\varepsilon_i+\varepsilon_k}(sr^*) & \text{for } k<i<j \\ z_{\varepsilon_i+\varepsilon_k}(-\omega rs^*) & \text{for } i<k<j \end{cases}$$

The remaining relations (involving negative root subgroups) are analogous. We list just those relations which will be explicitly used later in the paper.

$$(E5)\qquad [z_{-2\varepsilon_i}(r), z_{\varepsilon_i+\varepsilon_j}(s)] = \begin{cases} z_{\varepsilon_j-\varepsilon_i}(-r^*s)z_{2\varepsilon_j}(s^*rs) & \text{for } i<j \\ z_{\varepsilon_j-\varepsilon_i}(sr^*)z_{2\varepsilon_j}(srs^*) & \text{for } i>j \end{cases}$$
$$(E6)\qquad [z_{-2\varepsilon_i}(r), z_{\varepsilon_i-\varepsilon_j}(s)] = \begin{cases} z_{-\varepsilon_i-\varepsilon_j}(rs)z_{-2\varepsilon_j}(s^*rs) & \text{for } i<j \\ z_{-\varepsilon_i-\varepsilon_j}(-sr)z_{-2\varepsilon_j}(srs^*) & \text{for } i>j \end{cases}$$
$$(E7)\qquad [z_{\varepsilon_i-\varepsilon_j}(r), z_{\varepsilon_i+\varepsilon_j}(s)] = z_{2\varepsilon_i}(\omega s^*r-r^*s) \text{ for } i>j$$
$$(E8)\qquad [z_{\varepsilon_i-\varepsilon_j}(r), z_{-\varepsilon_i-\varepsilon_j}(s)] = \begin{cases} z_{-2\varepsilon_j}(\omega s^*r-r^*s) & \text{for } i<j \\ z_{-2\varepsilon_j}(sr^*-\omega rs^*) & \text{for } i>j \end{cases}$$
$$(E9)\qquad [z_{2\varepsilon_j}(r), z_{\varepsilon_i-\varepsilon_j}(s)] = z_{\varepsilon_i+\varepsilon_j}(rs)z_{2\varepsilon_i}(s^*rs) \text{ for } i>j$$

The group $\widehat{G^Q}$ we just constructed will be denoted by $\mathrm{St}^{\omega}_{C_n}(R,*)$.

Variations of $\mathrm{St}^{\omega}_{C_n}(R,*)$ involving form parameters. The defining relations show that $\mathrm{St}^{\omega}_{C_n}(R,*)$ admits a natural family of subgroups also graded by C_n, obtained by decreasing long root subgroups.

Let J be a form parameter of $(R,*,\omega)$. Given $\gamma\in C_n$, let

$$Z_{J,\gamma} = \begin{cases} Z_\gamma & \text{if } \gamma \text{ is a short root} \\ \{z_\gamma(r) : r\in J\} & \text{if } \gamma \text{ is a long root.} \end{cases}$$

The defining relations of $\mathrm{St}^{\omega}_{C_n}(R,*)$ imply that $\{Z_{J,\gamma}\}_{\gamma\in C_n}$ is a grading. Define $\overline{\mathrm{St}}^{\omega}_{C_n}(R,*,J)$ to be the subgroup of $\mathrm{St}^{\omega}_{C_n}(R,*)$ generated by $Z_J := \cup Z_{J,\gamma}$, and let $\mathrm{St}^{\omega}_{C_n}(R,*,J)$ be the graded cover of $\overline{\mathrm{St}}^{\omega}_{C_n}(R,*,J)$. It is not hard to show that $\overline{\mathrm{St}}^{\omega}_{C_n}(R,*,J)$ has the presentation $\langle Z_J|E_J\rangle$ where $E_J\subseteq E$ is set of those commutation relations of $\overline{\mathrm{St}}^{\omega}_{C_n}(R,*)$ which only involve generators from Z_J.

Here are two important observations. The first one is that non-twisted Steinberg groups of type C_n and D_n are special cases of the groups $\{\mathrm{St}^{\omega}_{C_n}(R,*,J)\}$. The second observation describes some natural isomorphisms between these groups.

Observation 8.3. *Assume that the ring R is commutative, so that the identity map $id: R\to R$ is an involution. The following hold:*

(1) *The group $\mathrm{St}^{-1}_{C_n}(R,id)$ coincides with $\mathrm{St}_{C_n}(R)$, the usual (non-twisted) Steinberg group of type C_n.*
(2) *$J=\{0\}$ is a possible form parameter of $(R,id,1)$, and the group $\mathrm{St}^{1}_{C_n}(R,id,\{0\})$ coincides with $\mathrm{St}_{D_n}(R)$, the usual Steinberg group of type D_n. This happens because the long root subgroups in the C_n-grading on $\mathrm{St}^{1}_{C_n}(R,id,\{0\})$ are trivial, and we can "remove" those roots to obtain a D_n-grading.*

Observation 8.4. *Let $\omega \in U(Z(R))$, and let $\omega' = \omega\mu^{-1}\mu^*$ for some invertible element $\mu \in Z(R)$. Then the automorphisms q_ω and $q_{\omega'}$ are graded-conjugate and so $\mathrm{St}^{\omega}_{C_n}(R,*)$ and $\mathrm{St}^{\omega'}_{C_n}(R,*)$ are isomorphic. In particular, $\mathrm{St}^{\omega}_{C_n}(R,*) \cong \mathrm{St}^{-\omega}_{C_n}(R,*)$ whenever $Z(R)$ contains an invertible antisymmetric element.*

Remark. An explicit isomorphism is constructed as follows. If $Z_\gamma = \{z_\gamma(r)\}$ are the root subgroups of $\mathrm{St}^{\omega}_{C_n}(R)$ and $Z'_\gamma = \{z'_\gamma(r)\}$ are the root subgroups of $\mathrm{St}^{\omega'}_{C_n}(R)$, then the map φ defined on root subgroups as

$$\varphi(z_\gamma(r)) = \begin{cases} z'_\gamma(r) & \text{if } \gamma = \varepsilon_i - \varepsilon_j \\ z'_\gamma(\mu^* r) & \text{if } \gamma = \varepsilon_i + \varepsilon_j \\ z'_\gamma(\mu^{-1} r) & \text{if } \gamma = -\varepsilon_i - \varepsilon_j \end{cases}$$

is an isomorphism.

We now turn to the proof of property (T) for hyperbolic unitary Steinberg groups.

Lemma 8.5. *Let R be a ring with involution $*$, let $\omega \in U(Z(R))$, and let J be a form parameter of $(R,*,\omega)$.*

(a) *If $n \geq 3$, the C_n-grading on $\mathrm{St}^{\omega}_{C_n}(R,*,J)$ is strong.*
(b) *Assume that the left ideal of R generated by J equals R. Then the C_n-grading on $\mathrm{St}^{\omega}_{C_n}(R,*,J)$ is 2-strong (in particular, the grading is strong if $n = 2$).*

Proof. (a) By definition, we need to check that the grading is strong at (γ, B) for every Borel subset B and $\gamma \in C(B)$, the core of B, and by symmetry it suffices to consider the case when B is the standard Borel. If $\gamma \in C(B)$ is a long root, the grading is strong at (γ, B) by relations (E3) with $s = 1$. If $n \geq 3$ and $\gamma \in C(B)$ is a short root, the grading is strong at (γ, B) by relations (E1) or (E4).

(b) If $n = 2$, the grading is strong at short root subgroups by relations (E3). The same argument shows that the grading is 2-strong for any $n \geq 2$. □

Proposition 8.6. *Let R be a finitely generated ring with involution $*$, $\omega \in U(Z(R))$ and J a form parameter of $(R,*,\omega)$. Assume that J is finitely generated as a form parameter. The following hold:*

(a) *The group $H = \mathrm{St}^{\omega}_{C_n}(R,*,J)$ has property (T) for any $n \geq 3$.*
(b) *Assume in addition that $\omega = -1$, $1_R \in J$ (so, in particular, the left ideal of R generated by J equals R), and R is a finitely generated right module over its subring generated by a finite set of elements from J. Then the group $\mathrm{St}^{-1}_{C_2}(R,*,J)$ has property (T).*

Proof. Lemma 8.5 ensures that the C_n-grading is strong, so we only need to check relative property (T) for root subgroups.

(a) Relations (E1) ensure that any short root subgroup Z_γ can be put inside a group which is a quotient of $\mathrm{St}_{A_2}(R) = \mathrm{St}_3(R)$ and hence the pair (H, Z_γ) has relative property (T). To prove relative (T) for long root subgroups we realize each of them as a subset of a bounded product of short root subgroups and some finite set. Without loss of generality, we will establish the required factorization for the long root $\gamma = 2\varepsilon_1$.

Let T be a finite set which generates J as a form parameter of $(R,*,\omega)$. By the remark following the definition of a form parameter, any $r \in J$ can be written as

$r = \sum_{t\in T} s_t t s_t^* + (u - \omega u^*)$ for some $s_t, u \in R$. Relations (E2) and (E3) yield the following identity:

$$z_{2\varepsilon_1}(r) = \prod_{t\in T}[z_{2\varepsilon_2}(t), z_{\varepsilon_1-\varepsilon_2}(s_t)]z_{\varepsilon_1+\varepsilon_2}(\sum_{t\in T} s_t t)[z_{\varepsilon_1-\varepsilon_2}(1), z_{\varepsilon_1+\varepsilon_2}(u)]$$

It follows that

$$Z_{2\varepsilon_1} \subseteq \prod_{t\in T} Z_{\varepsilon_1-\varepsilon_2}^{z_{2\varepsilon_2}(t)} Z_{\varepsilon_1-\varepsilon_2} Z_{\varepsilon_1+\varepsilon_2} Z_{\varepsilon_1-\varepsilon_2} Z_{\varepsilon_1+\varepsilon_2} Z_{\varepsilon_1-\varepsilon_2} Z_{\varepsilon_1+\varepsilon_2}.$$

The set $\{z_{2\varepsilon_2}(t) : t \in T\}$ of conjugating elements is finite, so we obtained the desired factorization.

(b) Relative property (T) in this case will be established in Proposition 8.16 in § 8.6. □

Example 2: *Unitary Steinberg groups in odd dimension.* Let $\Phi = A_{2n}$ and $G = \mathrm{St}_\Phi(R)$. Let $q = Dyn_* \in \mathrm{Aut}(G)$ and $Q = \langle q\rangle$.

The twisted group $\widehat{G^Q}$ constructed in this example will be denoted by $\mathrm{St}_{BC_n}(R,*)$ and graded by the root system BC_n. It corresponds to the group of transformations preserving the Hermitian form

$$f(u,v) = u_{n+1}v_{n+1}^* + \sum_{i=1}^n (u_i v_{\bar i}^* + u_{\bar i} v_i^*) \text{ on } R^{2n+1} \text{ where } \bar i = 2n+2-i.$$

The action of $q = Dyn_*$ on the root subgroups of G is given by

$$q : x_{e_i-e_j}(r) \mapsto x_{e_{\bar j}-e_{\bar i}}(-r^*).$$

Define $\eta : \bigoplus_{i=1}^{2n+1} \mathbb{R}e_i \to \bigoplus_{i=1}^n \mathbb{R}\varepsilon_i$ by $\eta(e_i) = \varepsilon_i$ if $i \le n$ and $\eta(e_i) = -\varepsilon_{\bar i}$ if $i \ge n+2$ and $\eta(e_{n+1}) = 0$. Similarly to Example 1, we check that η is q-invariant and the root system $\Psi = \eta(\Phi) \setminus \{0\}$ is indeed of type BC_n.

If $\gamma \in \Psi$ is a long root, the corresponding root subgroup Y_γ consists of elements $\{y_\gamma(r,s) : r,s \in R\}$ where

$$y_{\varepsilon_i-\varepsilon_j}(r,s) = x_{e_i-e_j}(r)x_{e_{\bar j}-e_{\bar i}}(s)$$
$$y_{\pm(\varepsilon_i+\varepsilon_j)}(r,s) = x_{\pm(e_i-e_{\bar j})}(r)x_{\pm(e_j-e_{\bar i})}(s).$$

If $\gamma \in \Psi$ is a short root, the root subgroup Y_γ consists of elements $\{y_\gamma((r,s,t)) : r,s,t \in R\}$ where

$$y_{\pm\varepsilon_i}(r,s,t) = x_{\pm(e_i-e_{n+1})}(r)x_{\pm(e_{n+1}-e_{\bar i})}(s)x_{\pm(e_i-e_{\bar i})}(t).$$

Note that the groups $Y_{\pm\varepsilon_i}$ are not abelian, and multiplication in them is determined by

$$\begin{aligned} y_{\varepsilon_i}(r_1,s_1,t_1)y_{\varepsilon_i}((r_2,s_2,t_2)) &= y_{\varepsilon_i}(r_1+r_2, s_1+s_2, t_1+t_2-r_2s_1) \\ y_{-\varepsilon_i}(r_1,s_1,t_1)y_{-\varepsilon_i}(r_2,s_2,t_2) &= y_{-\varepsilon_i}(r_1+r_2, s_1+s_2, t_1+t_2+s_1r_2) \end{aligned}$$

Finally, the double root subgroup $Y_{\pm 2\varepsilon_i}$ is the subgroup of $Y_{\pm\varepsilon_i}$ consisting of all elements of the form $y_{\pm\varepsilon_i}((0,0,t))$ where $t \in R$.

Now, calculating $Z_\alpha = Y_\alpha^{\langle q\rangle}$ we obtain that

$$
\begin{array}{lcl}
Z_{\varepsilon_i-\varepsilon_j} & = & \{z_{\varepsilon_i-\varepsilon_j}(r) = x_{e_i-e_j}(r)x_{e_{\bar{j}}-e_{\bar{i}}}(-r^*) :\ r\in R\} \ \text{ for } i<j,\\
Z_{\varepsilon_i-\varepsilon_j} & = & \{z_{\varepsilon_i-\varepsilon_j}(r) = x_{e_i-e_j}(-r^*)x_{e_{\bar{j}}-e_{\bar{i}}}(r) :\ r\in R\} \ \text{ for } i>j,\\
Z_{\varepsilon_i+\varepsilon_j} & = & \{z_{\varepsilon_i+\varepsilon_j}(r) = x_{e_i-e_{\bar{j}}}(r)x_{e_j-e_{\bar{i}}}(-r^*) :\ r\in R\} \ \text{ for } i<j,\\
Z_{-(\varepsilon_i+\varepsilon_j)} & = & \{z_{-(\varepsilon_i+\varepsilon_j)}(r) = x_{-e_i+e_{\bar{j}}}(-r^*)x_{-e_j+e_{\bar{i}}}(r) :\ r\in R\} \ \text{ for } i<j,\\
Z_{\varepsilon_i} & = & \{z_{\varepsilon_i}(r,t) = x_{e_i-e_{n+1}}(r)x_{e_{n+1}-e_{\bar{i}}}(-r^*)x_{e_i-e_{\bar{i}}}(t) :\ r,t\in R,\ rr^*=t+t^*\},\\
Z_{-\varepsilon_i} & = & \{z_{-\varepsilon_i}(r,t) = x_{-e_i+e_{n+1}}(-r^*)x_{-e_{n+1}+e_{\bar{i}}}(r)x_{-e_i+e_{\bar{i}}}(-t^*) :\ r,t\in R,\ rr^*=t+t^*\},\\
Z_{\pm 2\varepsilon_i} & = & \{z_{\pm\varepsilon_i}(0,t)\in Z_{\pm\varepsilon_i}\}.
\end{array}
$$

Clearly,

$Z_\gamma \cong (R,+)$ if γ is a long root and
$Z_\gamma \cong (\mathrm{Asym}(R),+)$ if γ is a double root.

Define

$$P(R,*) = \{(r,t)\ : r,t\in R \text{ and } t+t^* = rr^*\}.$$

and introduce the group structure on $P(R,*)$ by setting

$$(r_1,t_1)(r_2,t_2) = (r_1+r_2, t_1+t_2+r_2r_1^*).$$

Then for any short root $\gamma\in\Psi$, the root subgroup Z_γ is canonically isomorphic to $P(R,*)$ via the map $(r,t)\mapsto z_\gamma(r,t)$. Thus, the subgroup Z_γ is usually not abelian, and there is a natural injection $Z_\gamma/Z_{2\gamma}\to(R,+)$ which need not be an isomorphism.

Applying Proposition 8.1 we obtain that $\{Z_\gamma\}_{\gamma\in\Psi}$ is a grading. The corresponding graded cover $\widehat{G^Q}$ will be denoted by $\mathrm{St}_{BC_n}(R,*)$.

Below we list the non-trivial commutation relations between the positive root subgroups (again the the remaining relations are similar).

$$
\begin{array}{ll}
(E1) & [z_{\varepsilon_i-\varepsilon_j}(r), z_{\varepsilon_j-\varepsilon_k}(s)] = z_{\varepsilon_i-\varepsilon_k}(rs) \ \text{ for } i<j<k\\
(E2) & [z_{\varepsilon_i-\varepsilon_j}(r), z_{\varepsilon_i+\varepsilon_j}(s)] = z_{\varepsilon_i}(0, sr^*-rs^*) \ \text{ for } i<j\\
(E3) & [z_{\varepsilon_j}(r,t), z_{\varepsilon_i-\varepsilon_j}(s)] = z_{\varepsilon_i}(-sr, sts^*)z_{\varepsilon_i+\varepsilon_j}(-st) \ \text{ for } i<j\\
(E4) & [z_{\varepsilon_i}((r,t)), z_{\varepsilon_j}(s,q)] = z_{\varepsilon_i+\varepsilon_j}(-rs^*) \ \text{ for } i<j\\
(E5) & [z_{\varepsilon_i-\varepsilon_j}(r), z_{\varepsilon_j+\varepsilon_k}(s)] = \begin{cases} z_{\varepsilon_i+\varepsilon_k}(rs) & \text{for } i<j<k\\ z_{\varepsilon_i+\varepsilon_k}(sr^*) & \text{for } k<i<j\\ z_{\varepsilon_i+\varepsilon_k}(-rs^*) & \text{for } i<k<j\end{cases}
\end{array}
$$

As in Example 1 we can construct a family of generalizations of $\mathrm{St}_{BC_n}(R,*)$, this time by decreasing the short root subgroups. Let I be a left ideal of R. Define

$$P(R,I,*) = \{(r,t)\in P(R,*) : r\in I\} = \{(r,t)\ : r\in I, t\in R \text{ and } t+t^*=rr^*\}.$$

For each $\gamma\in BC_n$ we put

$$Z_{I,\gamma} = \begin{cases} Z_\gamma & \text{if } \gamma \text{ is a long or a double root}\\ \{z_\gamma(r,t) :\ (r,t)\in P(R,I,*)\} & \text{if } \gamma \text{ is a short root.}\end{cases}$$

We define $\mathrm{St}_{BC_n}(R,*,I)$ to be the graded cover of the subgroup of $\mathrm{St}_{BC_n}(R,*)$ generated by $\cup_{\gamma\in BC_n} Z_{I,\gamma}$.

Observation 8.7. *The group $\mathrm{St}_{BC_n}(R,*,\{0\})$ is isomorphic to $\mathrm{St}^1_{C_n}(R,*)$.*

Proposition 8.8. *Let R be a finitely generated ring with involution $*$. Assume that $\{r\in I:\ \exists t\in R, rr^*=t+t^*\}$ is finitely generated as a left ideal and $\mathrm{Asym}(R)$ is finitely generated as a form parameter of $(R,*,1)$. The following hold:*

(a) *The group $\mathrm{St}_{BC_n}(R,*,I)$ has property (T) for any $n\geq 3$.*
(b) *Assume in addition that there exists an invertible antisymmetric element $\mu\in Z(R)$, and R is a finitely generated module over a ring generated by a finite set of symmetric elements. Then the group $\mathrm{St}_{BC_2}(R,*,I)$ has property (T).*

Proof. We shall prove (a) and (b) simultaneously. The fact that the grading is strong in both cases is verified as in Lemma 8.5, so we only need to check relative property (T). Observe that the set Ψ of long and double roots in BC_n is a weak subsystem of type C_n, so the corresponding root subgroups generate a quotient of $\mathrm{St}^1_{C_n}(R,*)$ (this is proved similarly to Proposition 7.5). Hence relative property (T) for long and double root subgroups follows directly from Proposition 8.6(a) if $n\geq 3$. Note that the existence of an invertible antisymmetric element $\mu\in Z(R)$ implies that $\mathrm{St}^1_{C_2}(R,*)\cong \mathrm{St}^{-1}_{C_2}(R,*)$ by Observation 8.4. Thus, relative property (T) in the case $n=2$ follows from Proposition 8.6(b).

Finally, we claim that every short root subgroup lies in a bounded product of fixed conjugates of long and double root subgroups. This follows easily from relations (E3) and the fact that $\{r\in I:\ \exists t\in R, rr^*=t+t^*\}$ is finitely generated as a left ideal. □

The result of Proposition 8.8 is not entirely satisfactory since its hypotheses may be hard to verify in specific examples. Things becomes much easier under the additional assumption that R contains a central element a such that $a+a^*=1$:

Lemma 8.9. *Assume that there exists $a\in Z(R)$ such that $a+a^*=1$. The following hold:*

(1) $\mathrm{Sym}(R)=\mathrm{Sym}^{\min}(R)$ *and* $\mathrm{Asym}(R)=\mathrm{Asym}^{\min}(R)$.
(2) $P(R,*,I)=\{(r,rar^*+t):r\in I,t\in\mathrm{Asym}(R)\}$. *In particular, the set $\{r\in I:\ \exists t\in R, rr^*=t+t^*\}$ is equal to I.*

Proof. (1) For any $x\in\mathrm{Sym}(R)$ we have $x=xa+xa^*=xa+x^*a^*=xa+(xa)^*\in\mathrm{Sym}^{\min}(R)$ where the last equality holds since a is central. The equality $\mathrm{Asym}(R)=\mathrm{Asym}^{\min}(R)$ is proved similarly.

(2) By direct computation, any element of the form (r,rar^*+t) with $r\in I,t\in\mathrm{Asym}(R)$, lies in $P(R,*,I)$. Conversely, given $(r,u)\in P(R,*,I)$, we can write $u=rar^*+t$ for some t, and then we must have $t+t^*=0$. □

Thanks to this lemma, we obtain the following variation of Proposition 8.8:

Proposition 8.10. *Let R be a finitely generated ring with involution $*$. Assume that there exists $a\in Z(R)$ such that $a+a^*=1$, and let I be a finitely generated left ideal of R. The following hold:*

(a) *The group $\mathrm{St}_{BC_n}(R,*,I)$ has property (T) for any $n\geq 3$.*

(b) *Assume in addition that there exists an invertible antisymmetric element $\mu \in Z(R)$, and R is a finitely generated module over a ring generated by a finite set of symmetric elements. Then* $\mathrm{St}_{BC_2}(R, *, I)$ *has property* (T).

8.4. **Twisted groups of types ^2D_n and $^{2,2}A_{2n-1}$.** Recall that the next family on our agenda were the Steinberg covers of the twisted Chevalley groups of type ^2D_n ($n \geq 4$). These groups can be constructed using our general twisting procedure by taking $G = \mathrm{St}_{D_n}(R)$, where R is a commutative ring endowed with involution σ and $Q \subseteq \mathrm{Aut}(G)$ the subgroup of order 2 generated by Dyn_σ, the composition of the ring automorphism φ_σ and the Dynkin involution of D_n. However, we shall present a more general construction, making use of Observation 8.3(2).

Recall that the Steinberg group $\mathrm{St}_{D_n}(R)$, for R commutative, was realized as the group $\mathrm{St}^1_{C_n}(R, *, \{0\})$ where $*$ is the trivial involution. It turns out that if we start with any ring R (not necessarily commutative) endowed with an involution $*$ and an automorphism σ of order ≤ 2 which commutes with $*$, then the analogous twisting on $\mathrm{St}^1_{C_n}(R, *)$ can be constructed.

Example 3: *Steinberg groups* $\mathrm{St}^1_{BC_n}(R, *, \sigma)$. Let R be a ring endowed with an involution $*$ and an automorphism σ of order ≤ 2 which commutes with $*$. The fixed subring of σ will be denoted by R^σ. In this example we will construct the group $\mathrm{St}^1_{BC_n}(R, *, \sigma)$ graded by the root system BC_n.

Let $\Phi = C_{n+1}$ and $G = \mathrm{St}^1_{C_{n+1}}(R, *)$, the group constructed in Example 1 with $\omega = 1$. Denote the roots of Φ by $\pm\varepsilon_i \pm \varepsilon_j$ and $\pm 2\varepsilon_i$, and let $\{Z_\gamma\}_{\gamma\in\Phi}$ be the grading of G constructed in Example 1.

Let ρ be the automorphism of $\oplus_{i=1}^{n+1}\mathbb{R}\varepsilon_i$ given by

$$\rho(\varepsilon_i) = \varepsilon_i \text{ for } 1 \leq i \leq n \text{ and } \rho(\varepsilon_{n+1}) = -\varepsilon_{n+1}.$$

Clearly ρ stabilizes Φ. We claim that there exists an automorphism $q = q_\sigma \in \mathrm{Aut}(G)$ of order 2 such that

$$q(z_\gamma(r)) = z_{\rho(\gamma)}(\pm\sigma(r)) \text{ for all } \gamma \in \Phi, r \in R \tag{8.4}$$

(for some choice of signs). Unlike Examples 1 and 2, we cannot prove the existence of such q by referring to general results from § 8.2. One (rather tedious) way to prove this is first to define q as an automorphism of the free product $\star_{\gamma\in\Phi} Z_\gamma$ (using (8.4)), and then show that for a suitable choice of signs in (8.4), q respects the defining relations of G established in Example 1 and hence induces an automorphism of G. However, we will also give a conceptual argument for the existence of q at the end of this example.

Now let $\eta : \oplus_{i=1}^{n+1}\mathbb{R}\varepsilon_i \to \oplus_{i=1}^{n}\mathbb{R}\alpha_i$ be the reduction given by $\eta(\varepsilon_i) = \alpha_i$ for $i \leq n$ and $\eta(\varepsilon_{n+1}) = 0$. It is clear that η is q-invariant and the induced root system $\Psi = \eta(\Phi) \setminus \{0\} = \{\pm\alpha_i \pm \alpha_j\} \cup \{\pm\alpha_i\} \cup \{\pm 2\alpha_i\}$ is of type BC_n.

Let $\{W_\alpha\}_{\alpha\in\Phi}$ be the q-invariants of the Φ-grading of G. By Proposition 8.1 $\{W_\alpha\}$ is a grading, and thus we can form the graded cover $\widehat{G^{\langle q\rangle}}$ which will be denoted by $\mathrm{St}^1_{BC_n}(R, *, \sigma)$.

An easy calculation shows that

$$W_{\pm\alpha_i\pm\alpha_j} = \{w_{\pm\alpha_i\pm\alpha_j}(r) = z_{\pm\varepsilon_i\pm\varepsilon_j}(r) : \ r \in R^\sigma\}$$

$$W_{\pm\alpha_i} = \{w_{\pm\alpha_i}(r,t) = z_{\pm(\varepsilon_i-\varepsilon_{n+1})}(r) z_{\pm(\varepsilon_i+\varepsilon_{n+1})}(\sigma(r)) z_{\pm 2\varepsilon_i}(t) : \ t \in \mathrm{Asym}(R), \ t - r\sigma(r^*) \in R^\sigma\}$$

$$W_{\pm 2\alpha_i} = \{w_{\pm 2\alpha_i}(t) = z_{\pm 2\varepsilon_i}(t) : \ t \in \mathrm{Asym}(R^\sigma)\} = \{w_{\pm\alpha_i}(0,t) : \ t \in \mathrm{Asym}(R^\sigma)\}$$

Thus, $W_\gamma \cong (R^\sigma, +)$ if γ is a long root, and $W_\gamma \cong \mathrm{Asym}(R^\sigma, +)$ if γ is a double root. Let

$$Q(R,*,\sigma) = \{(r,t) : t \in \mathrm{Asym}(R) \text{ and } t - r\sigma(r^*) \in R^\sigma\},$$

and define the group structure on $Q(R,*,\sigma)$ by setting

$$(r_1,t_1)\cdot(r_2,t_2) = (r_1+r_2, t_1+t_2+r_2\sigma(r_1)^* - \sigma(r_1)r_2^*).$$

It is straightforward to check that if γ is a short root, W_γ is isomorphic to $Q(R,*,\sigma)$ via the map $(r,t) \mapsto w_\gamma(r,t)$.

The commutation relations between the positive root subgroups of the grading $\{W_\alpha\}$ are as follows.

$(E1)$ $[w_{\alpha_i-\alpha_j}(r), w_{\alpha_j-\alpha_k}(s)] = w_{\alpha_i-\alpha_k}(rs)$ for $i<j<k$

$(E2)$ $[w_{\alpha_i-\alpha_j}(r), w_{\alpha_i+\alpha_j}(s)] = w_{\alpha_i}(0, sr^* - rs^*)$ for $i<j$

$(E3)$ $[w_{\alpha_j}(r,t), w_{\alpha_i-\alpha_j}(s)] = w_{\alpha_i}(-sr, sts^*) w_{\alpha_i+\alpha_j}(s(r\sigma(r^*)-t))$ for $i<j$

$(E4)$ $[w_{\alpha_i}(r,t), w_{\alpha_j}(s,q)] = w_{\alpha_i+\alpha_j}(-r\sigma(s^*) - \sigma(r)s^*)$ for $i<j$

$(E5)$ $[w_{\alpha_i-\alpha_j}(r), w_{\alpha_j+\alpha_k}(s)] = \begin{cases} w_{\alpha_i+\alpha_k}(rs) & \text{for } i<j<k \\ w_{\alpha_i+\alpha_k}(sr^*) & \text{for } k<i<j \\ w_{\alpha_i+\alpha_k}(-rs^*) & \text{for } i<k<j \end{cases}$

Variations of the groups $\mathrm{St}_{BC_n}(R,*,\sigma)$. Let $I \subseteq R$ be a left R^σ-submodule and $J \subseteq \mathrm{Asym}(R)$ a form parameter of $(R,*,1)$. Then we can define the group $\mathrm{St}^1_{BC_n}(R,*,\sigma,I,J)$ by decreasing the short and double root subgroups. Define

$$Q(R,*,\sigma,I,J) = \{(r,t) \in Q(R,*,\sigma) : r \in I, t \in J\}.$$

For a root $\alpha \in BC_n$ we put

$$W_{I,J,\alpha} = \begin{cases} W_\alpha & \text{if } \alpha \text{ is a long root} \\ \{w_\alpha(r,t) : (r,t) \in Q(R,*,\sigma,I,J)\} & \text{if } \alpha \text{ is a short root} \\ \{w_\alpha(t) \in W_\alpha : \ t \in J \cap R^\sigma\} & \text{if } \alpha \text{ is a double root} \end{cases}$$

It is straightforward to check that $\{W_{I,J,\alpha}\}_{\alpha\in BC_n}$ is a grading. We define

$$\mathrm{St}^1_{BC_n}(R,*,\sigma,I,J)$$

to be the graded cover of the subgroup of $\mathrm{St}^1_{BC_n}(R,*,\sigma)$ generated by $\cup_{\alpha\in BC_n} W_{I,J,\alpha}$.

Now assume that R is commutative, so that the identity map **id** is an involution. Then $J = \{0\}$ is a valid form parameter of $(R, \mathrm{id}, 1)$, and the double root subgroups of $\mathrm{St}^1_{BC_n}(R,\mathrm{id},\sigma,I,\{0\})$ are trivial. Hence we obtain a group graded by a system of type B_n. This group will be denoted by $\mathrm{St}_{B_n}(R,\sigma,I)$.

We let $\mathrm{St}_{B_n}(R,\sigma) = \mathrm{St}_{B_n}(R,\sigma,R) = \mathrm{St}^1_{BC_n}(R,\mathrm{id},\sigma,R,\{0\})$ This is the Steinberg cover for the twisted Chevalley group of type ${}^2D_{n+1}$ over R, which we discussed at the beginning of this example.

We now state a sufficient condition for the groups $\mathrm{St}^1_{BC_n}(R,*,\sigma,I,J)$ to have property (T).

Proposition 8.11. *Assume that*

(i) *R^σ is finitely generated as a ring*
(ii) *$\{r \in I : \exists t \in J, t - r\sigma(r^*) \in R^\sigma\}$ is finitely generated as an R^σ-module*
(iii) *$J \cap R^\sigma$ is finitely generated as a form parameter of $(R^\sigma,*,1)$.*

Then the group $\mathrm{St}^1_{BC_n}(R,*,\sigma,I,J)$ has property (T) for any $n \geq 3$.

Proof. The proof is analogous to that of Proposition 8.8(a). □

If R is commutative, involution $*$ is trivial and $J = \{0\}$, the set defined in (ii) above coincides with I (since $r\sigma(r)$ always lies in R^σ), and condition (iii) is of course vacuous. Thus, as a special case of Proposition 8.11, we have the following:

Proposition 8.12. *Assume that*

(i) *R^σ is finitely generated as a ring*
(ii) *I is finitely generated as an R^σ-module*

Then the group $\mathrm{St}_{B_n}(R,\sigma,I)$ has property (T) for any $n \geq 3$.

As in Example 2, the hypotheses necessary to prove property (T) can be simplified in the presence of a nice element. This time we wish to assume that R contains a (not necessarily central) element a such that $a + \sigma(a^*) = 1$.

Lemma 8.13. *Let $R,*,\sigma,I$ and J be as above, and suppose that there exists $a \in R$ is such that $a + \sigma(a)^* = 1$. Then*

$$Q(R,*,\sigma,I,J) = \{(r, ra\sigma(r^*) - (ra\sigma(r^*))^* + t) : r \in I, t \in J\}.$$

In particular, the set $\{r \in I : \exists t \in J, t - r\sigma(r^*) \in R^\sigma\}$ (appearing in condition (ii) of Proposition 8.11) is equal to I.

Proof. The proof of this lemma is analogous to that of Lemma 8.9. □

Another definition of the groups $\mathrm{St}^1_{BC_n}(R,*,\sigma)$. There is a less intuitive, but in some sense more convenient, way to construct the groups $\mathrm{St}^1_{BC_n}(R,*,\sigma)$. The construction we described uses the twist by q_σ on the group $\mathrm{St}^1_{C_{n+1}}(R,*)$ which, in turn, was itself constructed using the twist by Dyn_* on $\mathrm{St}_{A_{2n+1}}(R)$. It is easy to see that $\mathrm{St}^1_{BC_n}(R,*,\sigma)$ can also be obtained directly from $\mathrm{St}_{A_{2n+1}}(R)$ as follows.

Let π' be the permutation $(n+1,n+2)$ and τ the automorphism of $\mathrm{St}_{A_{2n+1}}(R)$ defined by

$$\tau(x_{e_i-e_j}(r)) = x_{e_{\pi'(i)}-e_{\pi'(j)}}(\sigma(r)).$$

Note that τ commutes with Dyn_*, and let Q be the group generated by τ and Dyn_* (so that $Q \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$). Then $\mathrm{St}^1_{BC_n}(R,*,\sigma)$ can be obtained from $\mathrm{St}_{A_{2n+1}}(R)$ using the twist by Q. One advantage of this approach is that the existence of the automorphism q_σ defined above follows automatically, without case-by-case verification.

Summary of Examples 1-3. For the reader's convenience below we list all the twisted Steinberg groups constructed in Examples 1-3, including the key special cases and relations between them. In all examples, $n \geq 2$ is an integer, R is a ring and $*$ is an involution on R.

1. The groups $\mathrm{St}^{\omega}_{C_n}(R,*,J)$ where ω is an element of $U(Z(R))$ and J is a form parameter of $(R,*,\omega)$.
Special cases:

(i) $\mathrm{St}^{\omega}_{C_n}(R,*) = \mathrm{St}^{\omega}_{C_n}(R,*,R)$;
(ii) $\mathrm{St}_{C_n}(R) = \mathrm{St}^{-1}_{C_n}(R,id)$ where R is commutative;
(iii) $\mathrm{St}_{D_n}(R) = \mathrm{St}^{1}_{C_n}(R,id,\{0\})$ where R is commutative.

2. The groups $\mathrm{St}_{BC_n}(R,*,I)$ where I is a left ideal of R.
Special cases:

(i) $\mathrm{St}^{1}_{C_n}(R,*) = \mathrm{St}_{BC_n}(R,*,\{0\})$.

3. The groups $\mathrm{St}^{1}_{BC_n}(R,*,\sigma,I,J)$ where σ is an automorphism of order ≤ 2 commuting with $*$, $I\subseteq R$ is a left R^{σ}-submodule and $J\subseteq \mathrm{Asym}(R)$ is a form parameter of $(R,*,1)$.
Special cases:

(i) $\mathrm{St}_{B_n}(R,\sigma,I) = \mathrm{St}^{1}_{BC_n}(R,id,\sigma,I,\{0\})$ where R is commutative;
(ii) $\mathrm{St}_{B_n}(R,\sigma) = \mathrm{St}_{B_n}(R,\sigma,R)$;
(iii) $\mathrm{St}_{B_n}(R) = \mathrm{St}_{B_n}(R,id)$.

8.5. **Further twisted examples.** In this subsection we prove property (T) for twisted Steinberg groups of type ${}^3D_4$ and ${}^2E_6$. In both examples R is a commutative ring and $\sigma : R\to R$ is a finite order automorphism.

Example 4: *Steinberg groups of type* ${}^3D_4$. The group in this example will be denoted by $\mathrm{St}_{G_2}(R,\sigma)$ and is graded by the root system G_2. It is the Steinberg cover for the twisted Chevalley group of type ${}^3D_4$ over R.

We use the standard realization of D_n in $\mathbb{R}^n$: $D_n = \{\pm e_i \pm e_j : 1\leq i\neq j\leq n\}$. For a suitable choice of Chevalley basis, the commutation relations in $\mathrm{St}_{D_n}(R)$ are as follows:

$$[x_{e_i-e_j}(r), x_{e_j-e_k}(s)] = x_{e_i-e_k}(rs)$$
$$[x_{e_i-e_j}(r), x_{e_j+e_k}(s)] = \begin{cases} x_{e_i+e_k}(rs) & \text{if } i,j<k \text{ or } i,j>k \\ x_{e_i+e_k}(-rs) & \text{if } j>k>i \text{ or } i>k>j \end{cases}$$
$$[x_{e_i-e_j}(r), x_{-e_i-e_k}(s)] = \begin{cases} x_{-e_j-e_k}(-rs) & \text{if } i,j<k \text{ or } i,j>k \\ x_{-e_j-e_k}(rs) & \text{if } j>k>i \text{ or } i>k>j \end{cases}$$
$$[x_{e_k+e_i}(r), x_{-e_j-e_k}(s)] = \begin{cases} x_{e_i-e_j}(rs) & \text{if } i,j<k \text{ or } i,j>k \\ x_{e_i-e_j}(-rs) & \text{if } j>k>i \text{ or } i>k>j \end{cases}$$

We realize G_2 as the set of vectors $\pm(\varepsilon_i-\varepsilon_j)$ and $\pm(2\varepsilon_i-\varepsilon_j-\varepsilon_k)$ where $i,j,k\in\{1,2,3\}$ are distinct. We let

$$\alpha = 2\varepsilon_2-\varepsilon_1-\varepsilon_3 \quad \text{and} \quad \beta=\varepsilon_1-\varepsilon_2$$

and take $\{\alpha,\beta\}$ as our system of simple roots.

Let $\Phi = D_4$ (with standard realization) and $G=\mathrm{St}_{\Phi}(R)$. Let $\sigma : R\to R$ be an automorphism satisfying $\sigma^3=\mathrm{id}$ and π the isometry of $\mathbb{R}^4$ represented by the following matrix with respect to the basis $\{e_1,e_2,e_3,e_4\}$:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

Then it is clear that π stabilizes D_4. Let $q = \varphi_\sigma \lambda_\pi \in \mathrm{Aut}(G)$ (as defined in § 8.2). With a suitable choice of signs in the definition of λ_π, we can assume that q maps $x_{e_2-e_3}(r)$ to $x_{e_2-e_3}(\sigma(r))$, $x_{e_1-e_2}(r)$ to $x_{e_3-e_4}(\sigma(r))$, $x_{e_3-e_4}(r)$ to $x_{e_3+e_4}(\sigma(r))$ and $x_{e_3+e_4}(r)$ to $x_{e_1-e_2}(\sigma(r))$. Then it easy to see that q is an automorphism of order 3.

Define $\eta : \bigoplus_{i=1}^4 \mathbb{R}e_i \to \bigoplus_{i=1}^3 \mathbb{R}\varepsilon_i$ by $\eta(e_1) = \varepsilon_1 - \varepsilon_3$, $\eta(e_2) = \varepsilon_2 - \varepsilon_3$, $\eta(e_3) = \varepsilon_1 - \varepsilon_2$ and $\eta(e_4) = 0$. It is easy to see that the root system $\Psi = \eta(\Phi)$ is of type G_2. Furthermore,

$$\begin{array}{ll} \eta^{-1}(\beta) = \{e_1 - e_2, e_3 - e_4, e_3 + e_4\} & \eta^{-1}(\alpha) = \{e_2 - e_3\}, \\ \eta^{-1}(\alpha+\beta) = \{e_1 - e_3, e_2 - e_4, e_2 + e_4\} & \eta^{-1}(\alpha+3\beta) = \{e_1 + e_3\}, \\ \eta^{-1}(\alpha+2\beta) = \{e_1 - e_4, e_1 + e_4, e_2 + e_3\} & \eta^{-1}(2\alpha+3\beta) = \{e_1 + e_2\} \end{array} .$$

If $\gamma \in \Psi$ is a short root, the corresponding root subgroup Y_γ consists of elements

$$\{y_\gamma(r,s,t) = x_{\gamma_1}(r) x_{\gamma_2}(s) x_{\gamma_3}(t) :\ \eta^{-1}(\gamma) = \{\gamma_1, \gamma_2, \gamma_3\},\ r,s,t \in R\}.$$

If $\gamma \in \Psi$ is a long root, then

$$Y_\gamma = \{y_\gamma(r) = x_{\gamma_1}(r) :\ \eta(\gamma_1) = \gamma,\ r \in R\}.$$

If $\gamma \in \Psi$ is a short root, the corresponding root subgroup $Z_\gamma = Y_\gamma^q$ is isomorphic to $(R, +)$, and if $\gamma \in \Psi$ is a long root, then $Z_\gamma \cong (R^\sigma, +)$. Positive root subgroups can be explicitly described as follows (we define $z_\gamma(r)$ in such a way that the relations in the non-twisted case coincide with the relations (G) from Proposition 7.6):

$$Z_\beta = \{z_\beta(r) = x_{e_1-e_2}(r) x_{e_3-e_4}(\sigma(r)) x_{e_3+e_4}(\sigma^2(r)) :\ r \in R\},$$
$$Z_{\alpha+\beta} = \{z_{\alpha+\beta}(r) = x_{e_1-e_3}(-r) x_{e_2-e_4}(\sigma(r)) x_{e_2+e_4}(\sigma^2(r)) :\ r \in R\},$$
$$Z_{\alpha+2\beta} = \{z_{\alpha+2\beta}(r) = x_{e_1-e_4}(-r) x_{e_2+e_3}(-\sigma(r)) x_{e_1+e_4}(-\sigma^2(r)) :\ r \in R\},$$
$$Z_\alpha = \{z_\alpha(r) = x_{e_2-e_3}(r) : r \in R^\sigma\}, \quad Z_{\alpha+3\beta} = \{z_{\alpha+3\beta}(r) = x_{e_1+e_3}(r) :\ r \in R^\sigma\},$$
$$Z_{2\alpha+3\beta} = \{z_{2\alpha+3\beta}(r) = x_{e_1+e_2}(r) :\ r \in R^\sigma\}.$$

Below we list the commutation relations between positive root subgroups which will be used in the sequel:

$(E1)$ $\quad [z_\alpha(t), z_\beta(u)] = z_{\alpha+\beta}(tu) \cdot z_{\alpha+2\beta}(tu\sigma(u)) \cdot z_{\alpha+3\beta}(tu\sigma(u)\sigma^2(u)) \cdot z_{2\alpha+3\beta}(t^2 u\sigma(u)\sigma^2(u))$

$(E2)$ $\quad [z_\alpha(t), z_{\alpha+3\beta}(u)] = z_{2\alpha+3\beta}(tu)$

$(E3)$ $\quad [z_{\alpha+\beta}(t), z_\beta(u)] = z_{\alpha+2\beta}(t\sigma(u) + u\sigma(t)) \cdot z_{\alpha+3\beta}(u\sigma(u)\sigma^2(t) + u\sigma(t)\sigma^2(u) + t\sigma(u)\sigma^2(u))$
$\qquad z_{2\alpha+3\beta}(t\sigma(t)\sigma^2(u) + t\sigma(u)\sigma^2(t) + u\sigma(t)\sigma^2(t))$

Proposition 8.14. *The group $G = \mathrm{St}_{G_2}(R, \sigma)$ has property (T) provided*
- (i) *R^σ is finitely generated as a ring*
- (ii) *R is a finitely generated module over R^σ*

Proof. As usual, we need to check two things
- (a) The Ψ-grading of G is strong at each root subgroup
- (b) The pair (G, Z_γ) has relative (T) for each $\gamma \in \Psi$

Relations (E2) imply condition (a) for the root subgroup $Z_{2\alpha+3\beta}$. Condition (a) for the root subgroups $Z_{\alpha+3\beta}$ and $Z_{\alpha+\beta}$ follows from relations (E1) as we can take $u = 1$ and let t be an arbitrary element of R^σ in the case of $Z_{\alpha+3\beta}$ and take $t = 1$ and let u be an arbitrary element of R in the case of $Z_{\alpha+\beta}$. In the non-twisted case there is no problem with $Z_{\alpha+2\beta}$ either as we can take $u = 1$ and arbitrary $t \in R$ in (E1) (in general we cannot do this as t must come from R^σ).

To check the required property for the subgroup $Z_{\alpha+2\beta}$ in the general (twisted) case we need to show that elements of the form $u\sigma(u)t$ with $u \in R, t \in R^\sigma$ span R. Indeed, denote the span of those elements by M. Then M contains all elements of R^σ, in particular all elements of the form $u + \sigma(u) + \sigma^2(u)$. It also contains all elements of the form $(u+1)(\sigma(u+1)) - u\sigma(u) - 1 = u + \sigma(u)$. Since $u = u + \sigma(u) + \sigma^2(u) - (\sigma(u) + \sigma(\sigma(u)))$, we are done with (a).

We now prove (b). The subgroup of G generated by long root subgroups is isomorphic to a quotient of $\mathrm{St}_3(R^\sigma)$ (and R^σ is finitely generated), so condition (b) for long root subgroups holds by Theorem 7.12. It remains to check (b) for short root subgroups. We shall show that any short root subgroup lies in a bounded product of long root subgroups and finite sets. By symmetry, it is enough to establish this property for $Z_{\alpha+2\beta}$. For any set S we put $Z_{\alpha+2\beta}(S) = \{z_{\alpha+2\beta}(s) : s \in S\}$.

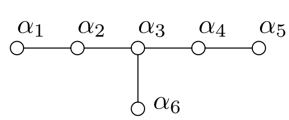
Put $A = \{u\sigma(u) : u \in R\}$. In the proof of (a) we showed that A generates R as an R^σ-module. Thus by our assumption there is a finite subset $U \subseteq A$ which generates R as an R^σ-module. Let S be a finite generating set of R^σ.

Now fix $s \in S$ and $u \in U$, and let $t \in R^\sigma$ be arbitrary. Similarly to the case of non-twisted G_2, if we calculate the quantity $[z_\alpha(t), z_\beta(su)][z_\alpha(ts), z_\beta(u)]^{-1}$ using relations (E1), we obtain that $\{z_{\alpha+2\beta}(t(s^2 - s)u\sigma(u)) : t \in R^\sigma\}$ lies in a bounded product of long root subgroups and fixed elements of short root subgroups. Similarly, this property holds for the set $\{z_{\alpha+2\beta}(2tu\sigma(u)) : t \in R^\sigma\}$ and hence also for the set $Z_{\alpha+2\beta}(IU)$ where $I = 2R^\sigma + \sum_{s\in S}(s^2 - s)R^\sigma$ and $IU = \{\sum_{u\in U} r_u u : r_u \in I\}$.

As we have already seen (in the case of non-twisted G_2), I is a finite index ideal of R^σ whence IU has finite index in $R^\sigma U = R$. Hence $Z_{\alpha+2\beta}$ can be written as a product of $Z_{\alpha+2\beta}(IU)$ and some finite set. This finishes the proof of (b). $\square$

Example 5: *Steinberg groups of type ${}^2E_6$/twisted Steinberg group of type F_4.* The group in this example will be denoted by $\mathrm{St}_{F_4}(R, \sigma)$ and is graded by the root system F_4. We will only sketch the details of the construction.

Let $\Phi = E_6$ and $G = \mathrm{St}_\Phi(R)$. Let $\{\alpha_1, \dots, \alpha_6\}$ be a system of simple roots of Φ ordered as shown below:



Let $\sigma : R \to R$ be an automorphism of order ≤ 2, let π be the automorphism of Φ given by $\pi(\alpha_i) = \alpha_{6-i}$ for $i = 1, 2, 4, 5$ and $\pi(\alpha_i) = \alpha_i$ for $i = 3, 6$, and let $q = \lambda_\pi \varphi_\sigma \in \mathrm{Aut}(G)$. With a suitable choice of signs in the definition of λ_π, we can assume that q is an automorphism of G of order 2 and is given by

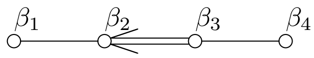
$$q(x_{\pm\alpha_i}(r)) = \begin{cases} x_{\pm\alpha_{6-i}}(-\sigma(r)) & \text{for } i = 1, 2, 4, 5 \\ x_{\pm\alpha_i}(-\sigma(r)) & \text{for } i = 3, 6. \end{cases}$$

Let V be the $\mathbb{R}$-span of Φ, and consider the induced action of q on V (so that $q(\alpha_i) = \pi(\alpha_i)$ for each i). Let $W = V^q$ be the subspace of q-invariants and define

$\eta : V \to W$ by (8.2), that is,

$$\eta(v) = \frac{v + qv}{2}.$$

It is easy to see that $\Psi = \eta(\Phi)$ is a root system of type F_4 with base $\beta_1, \beta_2, \beta_3, \beta_4$ where $\beta_1 = \frac{\alpha_1+\alpha_5}{2}, \beta_1 = \frac{\alpha_2+\alpha_4}{2}, \beta_3 = \alpha_3$ and $\beta_4 = \alpha_6$:



As in Example 4, if $\gamma \in \Psi$ is a short root, the root subgroup Z_γ is isomorphic to $(R, +)$, and if $\gamma \in \Psi$ is a long root, then $Z_\gamma \cong (R^\sigma, +)$.

Proposition 8.15. *Assume that both R and R^σ are finitely generated rings. Then the group $\mathrm{St}_{F_4}(R, \sigma)$ has property (T).*

Proof. The proof is identical to the case of classical (non-twisted) F_4. □

8.6. **Proof of relative property (T) for type C_2.** In this subsection we prove relative property (T) for the pairs (G, Z_γ) where G is a twisted Steinberg group of the form $\mathrm{St}^{-1}_{C_2}(R, *, J)$ for a suitable triple $(R, *, J)$ and Z_γ is one of its root subgroups. The main ingredient in the proof is Theorem 2.7.

Proposition 8.16. *Let R be a ring with involution $*$ and J a form parameter of $(R, *, -1)$ containing 1_R. Assume that*

(1) *There is a finite subset $T = \{t_1, \ldots, t_d\}$ of J and $a_1, \ldots, a_l \in R$ such that $R = \sum_{i=1}^{l} a_i R_0$, where R_0 is the ring generated by T.*
(2) *J is generated as a form parameter by a finite set $U = \{u_1, \ldots, u_D\}$.*

Let

$$W_1 = \{w \in R : w \text{ is a monomial in } T \text{ of degree } \leq d\} \cup \{1\}$$

and

$$W_2 = \{w + w^* \in R : w \text{ is a monomial in } T \text{ of degree } \leq 2d\}.$$

Let

$$W_{short} = \bigcup_{i=1}^{l} \{a_i w : w \in W_1\}$$

and

$$W_{long} = W_2 \cup \bigcup_{i=1}^{l} \{a_i w a_i^* : w \in W_2\} \cup U \cup T \cup \{t^2 : t \in T\} \cup \{1\}.$$

For a short root $\gamma \in C_2$ set $S_\gamma = \{z_\gamma(r) : r \in W_{short} \cup W^*_{short}\}$, and for a long root $\gamma \in C_2$ set $S_\gamma = \{z_\gamma(r) : r \in W_{long}\}$, and let $S = \cup_{\gamma \in C_2} S_\gamma$. Then for every $\gamma \in C_2$ we have

$$\kappa_r(\mathrm{St}^{-1}_{C_2}(R, *, J); Z_\gamma, S) > 0.$$

Proof. In the notations of Example 1 with $\omega = -1$, let $\alpha = 2\varepsilon_2$ and $\beta = \varepsilon_1 - \varepsilon_2$. Then $\{\alpha, \beta\}$ is a base for C_2, and as established in Example 1, we have the following relations:

$$[z_\beta(r), z_{\alpha+\beta}(s)] = z_{\alpha+2\beta}(rs^* + sr^*) \tag{8.5}$$

$$[z_\alpha(r), z_\beta(s)] = z_{\alpha+\beta}(-sr) z_{\alpha+2\beta}(srs^*) \tag{8.6}$$

$$[z_{-\alpha}(r), z_{\alpha+\beta}(s)] = z_\beta(sr) z_{\alpha+2\beta}(srs^*). \tag{8.7}$$

Let $N = \langle Z_{\alpha+\beta}, Z_\beta, Z_{\alpha+2\beta}\rangle$, $S^+ = \cup\{S_\gamma\}_{\gamma\in\pm\alpha,\alpha+\beta,\beta,\alpha+2\beta}$, $G = \langle S^+\rangle$, $Z = Z_{\alpha+2\beta}$ and $H = Z\cap[N,G]$. We claim that Proposition 8.16 follows from Lemma 8.17 below and Theorem 2.7.

Lemma 8.17. *The following hold:*

(a) *N is contained in G;*
(b) *Z/H is a group of exponent 2 generated by (the image of) $S_{\alpha+2\beta}$.*

Indeed, let E be the subgroup of the Steinberg group $\mathrm{St}_2(R_0)$ generated by $\{x_{12}(r), x_{21}(r) : r \in W_{long}\}$, and consider the semi-direct product $E \ltimes (\oplus_{i=1}^l R_0^2)$ where E acts on each copy of R_0^2 by right multiplication. Since W_{long} contains $T\cup\{1\}$ and T generates R_0 as a ring, the pair $(E\ltimes(\oplus_{i=1}^l R_0^2), \oplus_{i=1}^l R_0^2)$ has relative (T) by Theorem 11.1 and a remark after it.

Relations (8.6) and (8.7) and Lemma 8.17(a) imply that $(G/Z, N/Z)$ as a pair is a quotient of $(E\ltimes(\oplus_{i=1}^l R_0^2), \oplus_{i=1}^l R_0^2)$, that is, there exists an epimorphism $\pi : E\ltimes(\oplus_{i=1}^l R_0^2) \to G/Z$ such that $\pi(\oplus_{i=1}^l R_0^2) = N/Z$. Therefore, the pair $(G/Z, N/Z)$ has relative (T) as well.

This result and Lemma 8.17(b) imply that the hypotheses of Theorem 2.7 hold if we put $A = B = S^+$, $C = S_{\alpha+2\beta}$, $\varepsilon = \kappa(G/Z, N/Z)$ and $\delta = \frac{1}{\sqrt{|C|}}$. Applying this theorem we get that $\kappa(G, N; S^+) > 0$, so in particular $\kappa_r(\mathrm{St}_{C_2}^{-1}(R,*,J), Z_\gamma; S) > 0$ for $\gamma \in \{\beta, \alpha+\beta, \alpha+2\beta\}$.

To prove Theorem 8.16 for the remaining root subgroups it is sufficient to know that for any root $\gamma \in C_2$ there is a graded automorphism φ of $\mathrm{St}_{C_2}^{-1}(R,*,J)$ which sends Z_γ to Z_β or $Z_{\alpha+2\beta}$ and leaves the set $S\cup S^{-1}$ invariant (of course, replacing S by $S\cup S^{-1}$ does not affect Kazhdan ratio). The existence of such automorphism φ easily follows from the definition of the group $\mathrm{St}_{C_2}^{-1}(R,*,J)$. In fact, we can choose φ such that

(i) for any short root δ we have $\varphi(z_\delta(r)) = z_{\varphi(\delta)}(r)$ for all $r \in R$ or $\varphi(z_\delta(r)) = z_{\varphi(\delta)}(-r^*)$ for all $r \in R$ and
(ii) for any long root δ we have $\varphi(z_\delta(r)) = z_{\varphi(\delta)}(r)$ for all $r \in J$ or $\varphi(z_\delta(r)) = z_{\varphi(\delta)}(-r)$ for all $r \in J$.

This completes the proof of Theorem 8.16 modulo Lemma 8.17.

Before proving Lemma 8.17, we establish another auxiliary result, from which Lemma 8.17 will follow quite easily.

Lemma 8.18. *For any $r \in R$ the following hold:*

(i) *$z_{\alpha+\beta}(r) \in G$ and $z_\beta(r) \in G$*
(ii) *$z_{\alpha+\beta}(r) \in [N,G]\langle S_{\alpha+2\beta}\rangle$*

Proof. Note that it suffices to prove both statements when r is of the form $r = a_i w$, where w is a monomial in T^+. Let us prove that both (i) and (ii) hold for such r by induction on $m = length(w)$.

If $m \le d$, then $z_{\alpha+\beta}(r), z_\beta(r) \in S^+ \subset G$ by definition of S^+. Also by (8.5) we have $z_{\alpha+\beta}(r) = z_{\alpha+2\beta}(rr^*)[z_\alpha(1), z_\beta(r)]^{-1}$, so both (i) and (ii) hold.

Now fix $m > d$, and assume that for any monomial $w' \in T$ of length less than m both (i) and (ii) hold for $r = a_i w'$.

Claim 8.19. *Let q be some tail of w with $2 \le length(q) \le d+1$ so that $w = pq$ for some p. Then (i) and (ii) hold for $r = a_i pq = a_i w$ if and only if (i) and (ii) hold for $r = a_i pq^*$.*

Remark. Note that if $q = t_{i_1} \dots t_{i_s}$, then $q^* = t_{i_s} \dots t_{i_1}$ is the monomial obtained from q by reversing the order of letters.

Proof. Consider the element $v = p(q+q^*)$. Then

$$[z_\alpha(q+q^*), z_\beta(a_i p)] = z_{\alpha+\beta}(-a_i v) z_{\alpha+2\beta}(a_i p(q+q^*)p^* a_i^*). \tag{8.8}$$

Notice that $p(q+q^*)p^* = u + u^*$ for $u = pqp^*$. Furthermore, $length(u) \leq 2m-2$, so we can write $u = w_1 w_2^*$ where w_1 and w_2 are monomials of length $< m$. Then

$$\begin{aligned} z_{\alpha+2\beta}(a_i(u+u^*)a_i^*) &= z_{\alpha+2\beta}(a_i w_1 (a_i w_2)^* + a_i w_2 (a_i w_1)^*) \\ &= [z_\beta(a_i w_1), z_{\alpha+\beta}(a_i w_2)] \in G \cap [N,G] \quad \text{by induction.} \end{aligned}$$

Since $z_\alpha(q+q^*) \in S^+$ and $z_\beta(a_i p) \in G$ by induction, from (8.8) we get

$$z_{\alpha+\beta}(a_i v) = z_{\alpha+2\beta}(a_i(u+u^*)a_i^*)[z_\alpha(q+q^*), z_\beta(a_i p)]^{-1} \in G \cap [N,G].$$

Since $z_{\alpha+\beta}(a_i v) = z_{\alpha+\beta}(a_i pq) z_{\alpha+\beta}(a_i pq^*)$, we conclude that $z_{\alpha+\beta}(a_i pq) \in G \iff z_{\alpha+\beta}(a_i pq^*) \in G$ and $z_{\alpha+\beta}(a_i pq) \in [N,G]\langle S_{\alpha+2\beta}\rangle \iff z_{\alpha+\beta}(a_i pq^*) \in [N,G]\langle S_{\alpha+2\beta}\rangle$. A similar argument shows that $z_\beta(a_i pq) \in G \iff z_\beta(a_i pq^*) \in G$. □

By Claim 8.19, in order to prove that $z_{\alpha+\beta}(a_i w), z_\beta(a_i w) \in G$ we are allowed to replace w by another word obtained by reversing some tail of w of length $\leq d+1$, and this operation can be applied several times. The corresponding permutations clearly generate the full symmetric group on $d+1$ letters, and since T has d elements, we can assume that w has a repeated letter at the end: $w = pt^2$ where $t \in T$. But then we have

$$[z_\alpha(t^2), z_\beta(a_i p)] = z_{\alpha+\beta}(-a_i w) z_{\alpha+2\beta}(a_i p t^2 p^* a_i^*)$$

and

$$[z_\alpha(1), z_\beta(a_i pt)] = z_{\alpha+\beta}(-a_i pt) z_{\alpha+2\beta}(a_i p t^2 p^* a_i^*),$$

whence

$$z_{\alpha+\beta}(a_i w) = z_{\alpha+\beta}(a_i pt)[z_\alpha(1), z_\beta(a_i pt)][z_\alpha(t^2), z_\beta(a_i p)]^{-1}.$$

Since $z_\beta(a_i pt), z_\beta(a_i p) \in G$ and $z_{\alpha+\beta}(a_i pt) \in G \cap [N,G]\langle S_{\alpha+2\beta}\rangle$ by induction, and $z_\beta(a_i pt), z_\beta(a_i p) \in N$ and $z_\alpha(1), z_\alpha(t^2) \in G$ by definition, we conclude that $z_{\alpha+\beta}(a_i w) \in G \cap [N,G]\langle S_{\alpha+2\beta}\rangle$. A similar argument shows that $z_\beta(a_i w) \in G$. □

Proof of Lemma 8.17. By Lemma 8.18, G contains the root subgroups Z_β and $Z_{\alpha+\beta}$. Hence, by relations (8.5), G contains all elements of the form $z_{\alpha+2\beta}(r+r^*)$. Since by definition G also contains all elements of the form $z_{\alpha+2\beta}(u)$ with $u \in U$ and U generates J as a form parameter, we conclude that G contains $Z = Z_{\alpha+2\beta}$. This completes the proof of Lemma 8.17(a).

From relations (8.6) we have $Z \subseteq [N,G]Z_{\alpha+\beta}$, which together with Lemma 8.18(ii) shows that $Z \subseteq [N,G]\langle S_{\alpha+2\beta}\rangle$. This proves the second assertion of Lemma 8.17(b). Finally, by (8.5), for every $r \in J$ we have $z_{\alpha+2\beta}(2r) = [z_\beta(r), z_{\alpha+\beta}(1)] \in [N,G]$, which proves the first assertion of Lemma 8.17(b). □

□

8.7. **Twisted groups of type $^2F_4$.** Let R be a commutative ring of characteristic 2 and $*: R \to R$ an injective homomorphism such that $(r^*)^* = r^2$ for any $r \in R$. We will use a standard realization of the root system F_4 inside $\mathbb{R}^4$:

$$F_4 = \{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) : 1 \le i \ne j \le 4\}.$$

Let $\overline{F_4}$ by the root system, obtained from F_4 by normalizing all the roots:

$$\overline{F_4} = \{\bar{v} = \frac{v}{|v|} : v \in F_4\} = \{\pm e_i, \frac{1}{\sqrt{2}}(\pm e_i \pm e_j), \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) : 1 \le i \ne j \le 4\}.$$

Let $G = \mathrm{St}_{F_4}(R)$ and let $\{X_\gamma\}_{\gamma \in F_4}$ denote the standard grading of G. For each $\bar\gamma \in \overline{F_4}$ define the subgroup $\widetilde{X}_{\bar\gamma}$ by

$$\widetilde{X}_{\bar\gamma} = \{\widetilde{x}_{\bar\gamma}(r) \ : r \in R\} \text{ where } \widetilde{x}_{\bar\gamma}(r) = \begin{cases} x_\gamma(r) & \text{if } \gamma \text{ is a short root} \\ x_\gamma(r^*) & \text{if } \gamma \text{ is a long root} \end{cases}$$

Since R has a characteristic 2, it is easy to show that $\{\widetilde{X}_{\bar\gamma}\}_{\bar\gamma \in \overline{F_4}}$ is an $\overline{F_4}$-grading, and the elements of its root subgroups satisfy the following commutation relations: for any $\alpha, \beta \in \overline{F_4}$ we have

$$[\widetilde{x}_\alpha(r), \widetilde{x}_\beta(s)] = \begin{cases} 1 & \text{if the angle between } \alpha \text{ and } \beta \text{ is } \frac{\pi}{4}, \frac{\pi}{3}, \text{ or } \frac{\pi}{2} \\ \widetilde{x}_{\alpha+\beta}(rs) & \text{if the angle between } \alpha \text{ and } \beta \text{ is } \frac{2\pi}{3} \\ \widetilde{x}_{\sqrt{2}\alpha+\beta}(r^*s)\widetilde{x}_{\alpha+\sqrt{2}\beta}(rs^*) & \text{if the angle between } \alpha \text{ and } \beta \text{ is } \frac{3\pi}{4} \end{cases}.$$

For instance, consider the case when the angle between α and β is $3\pi/4$ and α is a long root. Then $\{\sqrt{2}\alpha, \beta\} \subset F_4$ is a base for a subsystem of type B_2, and therefore by Proposition 7.6(B) we have

$$[x_{\sqrt{2}\alpha}(r), x_\beta(s)] = x_{\sqrt{2}\alpha+\beta}(rs)x_{\sqrt{2}\alpha+2\beta}(rs^2) \text{ for all } r, s \in R.$$

(Note that the choice of Chevalley basis does not affect the relations since R has characteristic 2). Hence for any $r, s \in R$ we have

$$\begin{aligned}[\widetilde{x}_\alpha(r), \widetilde{x}_\beta(s)] &= [x_{\sqrt{2}\alpha}(r^*), x_\beta(s)] = x_{\sqrt{2}\alpha+\beta}(r^*s)x_{\sqrt{2}\alpha+2\beta}(r^*s^2) \\ &= x_{\sqrt{2}\alpha+\beta}(r^*s)x_{\sqrt{2}(\alpha+\sqrt{2}\beta)}((rs^*)^*) = \widetilde{x}_{\sqrt{2}\alpha+\beta}(r^*s)\widetilde{x}_{\alpha+\sqrt{2}\beta}(rs^*).\end{aligned}$$

We denote by $\widetilde{G}$ the graded cover of the group generated by $\{\widetilde{X}_\alpha\}_{\alpha \in \overline{F_4}}$. Since the commutation relation between elements of two root subgroups is determined entirely by the angle between the corresponding roots, we can construct a graded automorphism of $\widetilde{G}$ from any isometry of the root system $\overline{F_4}$ as follows. Let ρ be an isometry of $\mathbb{R}^4$ which preserves $\overline{F_4}$. Then we can define an automorphism of $\widetilde{G}$, denoted by the same symbol ρ:

$$\rho(\widetilde{x}_\alpha(r)) = \widetilde{x}_{\rho(\alpha)}(r).$$

Let q be the isometry represented by the following matrix with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

and τ be the isometry represented by the following matrix with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

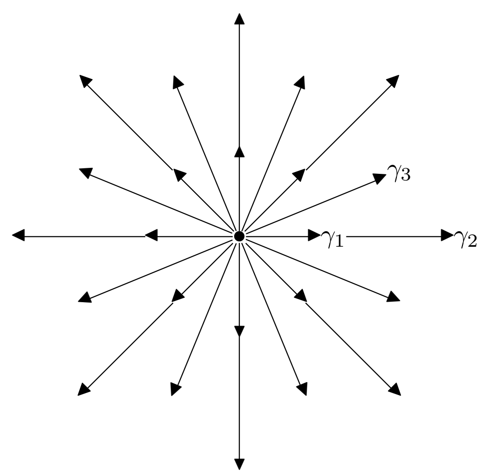
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then q has order 2 and τ has order 8. Moreover they commute.

Example 6: *Twisted groups of type* ${}^2F_4$. The group in this example is denoted by $\mathrm{St}_{{}^2F_4}(R)$ and is obtained from $\widetilde{G}$ using the twist by the automorphism q. Define

$$\eta : \bigoplus_{i=1}^{4} \mathbb{R}e_i \to \bigoplus_{i=1}^{2} \mathbb{R}\varepsilon_i$$

by $\eta(e_1) = (1+\sqrt{2})\varepsilon_1$, $\eta(e_2) = \varepsilon_1$, $\eta(e_3) = (1+\sqrt{2})\varepsilon_2$ and $\eta(e_4) = \varepsilon_2$. It is easy to see that the root system $\Psi = \eta(\overline{F_4})$ is as in the diagram below.



We see that there are three types of roots. We shall call them short, long and double by analogy with BC_2 even though this time the double roots are $(\sqrt{2}+1)$ times longer than the short ones.

The short roots are

$$\{\pm\varepsilon_i, \frac{1}{\sqrt{2}}(\pm\varepsilon_i \pm \varepsilon_j) : \ 1 \le i \ne j \le 2\},$$

the double roots are

$$\{\pm(\sqrt{2}+1)\varepsilon_i, \frac{\sqrt{2}+1}{\sqrt{2}}(\pm\varepsilon_i \pm \varepsilon_j) : \ 1 \le i \ne j \le 2\}$$

and the long roots are

$$\{\pm\frac{1}{\sqrt{2}}((1+\sqrt{2})\varepsilon_i \pm \varepsilon_j) : \ 1 \le i \ne j \le 2\}.$$

Since τ commutes with q, it permutes the roots of Ψ. It is easy to see that this action has 3 orbits corresponding to short, double and long roots. In fact, τ acts on $\mathbb{R}\Psi$ simply as the counterclockwise rotation by $\frac{\pi}{4}$ (with respect to the basis $\varepsilon_1, \varepsilon_2$).

In this example we denote by $\{Y_\alpha\}_{\alpha\in\Psi}$ not the coarsened grading on $\widetilde{G}$, but its fattening (see § 4.5), where the short root subgroups are the ones being fattened.

If $\alpha\in\Psi$ is a long or double root, the corresponding root subgroup Y_α consists of elements

$$\{y_\alpha(r,s)=\widetilde{x}_{\alpha_1}(r)\widetilde{x}_{\alpha_2}(s):\ \eta^{-1}(\alpha)=\{\alpha_1,\alpha_2\},\ r,s,t\in R\}.$$

The chosen order on the set $\{\alpha_1,\alpha_2\}$ is not important since the root subgroups $\widetilde{X}_{\alpha_1}$ and $\widetilde{X}_{\alpha_2}$ commute.

If $\alpha\in\Psi$ is a short root, then Y_α consists of elements

$$y_\alpha(r,s,t,u)=\widetilde{x}_{\alpha_1}(r)\widetilde{x}_{\alpha_2}(s)\widetilde{x}_{\alpha_3}(t)\widetilde{x}_{\alpha_4}(u),$$

where $\eta^{-1}(\alpha)=\{\alpha_1,\alpha_2\}$, $\eta^{-1}((\sqrt{2}+1)\alpha)=\{\alpha_3,\alpha_4\}$ and $r,s,t,u\in R$.

Here parameterization does depend on how elements of $\eta^{-1}(\alpha)$ are ordered, so we shall specify the order as follows. By the above discussion there exists unique $0\le i\le 7$ such that $\alpha=\tau^i\varepsilon_1$. Then we shall put $\alpha_1=\tau^i(\frac{e_1-e_2}{\sqrt{2}})$ and $\alpha_2=\tau^i(e_2)$.

Next we describe the subgroups $Z_\alpha=Y_\alpha^{\langle q\rangle}$. Since τ acts on Ψ and commutes with q, it also permutes the subgroups $\{Z_\alpha\}$, so it suffices to describe Z_α for one root in each $\langle\tau\rangle$-orbit, that is, one root of each length. We shall use the roots $\gamma_1=\varepsilon_1$, $\gamma_2=(1+\sqrt{2})\varepsilon_1$ and $\gamma_3=\frac{1}{\sqrt{2}}((1+\sqrt{2})\varepsilon_1+\varepsilon_2)$. We have

$$\begin{aligned}Z_{\gamma_1}&=\{z_{\gamma_1}(r,s)=\widetilde{x}_{\frac{e_1-e_2}{\sqrt{2}}}(r)\widetilde{x}_{e_2}(r)\widetilde{x}_{e_1}(r^*r+s)\widetilde{x}_{\frac{e_1+e_2}{\sqrt{2}}}(s):\ r,s\in R\},\\ Z_{\gamma_2}&=\{z_{\gamma_2}(r)=z_{\gamma_1}(0,r)=\widetilde{x}_{e_1}(r)\widetilde{x}_{\frac{e_1+e_2}{\sqrt{2}}}(r):\ r\in R\},\\ Z_{\gamma_3}&=\{z_{\gamma_3}(r)=\widetilde{x}_{\frac{e_1+e_4}{\sqrt{2}}}(r)\widetilde{x}_{\frac{1}{2}(e_1+e_2+e_3-e_4)}(r):\ r\in R\}.\end{aligned}$$

If $\gamma\in\Psi$ is a long or double root, the corresponding root subgroup Z_γ is isomorphic to $(R,+)$, and if $\gamma\in\Psi$ is a short root, then Z_γ is nilpotent of class 2.

To simplify the notation we denote $\tau^i(\gamma_k)$ by $\gamma_{k,i}$ and $z_{\gamma_{k,i}}(r)$ by $z_{k,i}(r)$. (Recall that τ acts on the root system Ψ as a counterclockwise rotation by $\frac{\pi}{4}$.)

We list the relevant commutation relations between the elements of the root subgroups. All the commutation relations may be found in the Tits paper [Ti] (note that our notation is slightly different).

$(E1)$ $[z_{1,i}(r,s),z_{1,i+1}(t,u)]=z_{3,i}(rt)$

$(E2)$ $[z_{1,i}(r,s),z_{3,i+1}(t)]=z_{1,i+1}(0,rt)=z_{2,i+1}(rt)$

$(E3)$ $[z_{1,0}(r,s),z_{3,3}(t)]\equiv z_{1,3}(tr,0)z_{1,2}(t^*s,0)z_{1,1}(tr^*r+ts,0)\mod \prod_{i=1}^{3}Z_{\gamma_{3,i-1}}Z_{\gamma_{2,i}}$

$(E4)$ $[z_{3,i+4}(r),z_{2,i+2}(s)]=z_{2,i+3}(rs)$ and $[z_{3,i}(r),z_{2,i+3}(s)]=z_{2,i+2}(rs)$

$(E5)$ $[z_{3,i}(r),z_{2,i+2}(s)]=[z_{3,i+4}(r),z_{2,i+3}(s)]=1$

$(E6)$ $[z_{2,i}(r),z_{2,i+3}(s)]\equiv z_{3,i+1}(rs)\mod Z_{\gamma_{2,i+1}}Z_{\gamma_{2,i+2}}$

Proposition 8.20. *Let R be a finitely generated ring. Then the group $\mathrm{St}_{{}^2F_4}(R)$ has property (T).*

Proof. As usual, we need to check two things:

(a) The Ψ-grading of $\mathrm{St}_{{}^2F_4}(R)$ is strong at (γ,B) for each Borel subset B and each root γ in the core of B;

(b) The pair $(\mathrm{St}_{{}^2F_4}(R),Z_\gamma)$ has relative (T) for each $\gamma\in\Psi$.

As usual, it suffices to check (a) for the Borel set with boundary $\{\gamma_{1,0}, \gamma_{2,0}, \gamma_{3,3}\}$. Relations (E1) imply condition (a) for the long root subgroups (that is, the root subgroups $Z_{\gamma_{3,i}}$ with $0 \le i \le 2$). From relations (E2) and (E3) we obtain condition (a) for the short root subgroups. This also implies condition (a) for the double root subgroups since they are contained in the short root subgroups.

Now let us prove (b). Relations (E4) and (E5) imply that the pair

$$(\langle Z_{\gamma_{3,i}}, Z_{\gamma_{3,i+4}}, Z_{\gamma_{2,i+2}}, Z_{\gamma_{2,i+3}} \rangle, Z_{\gamma_{2,i+2}} Z_{\gamma_{2,i+3}})$$

is a quotient of $(\mathrm{St}_2(R) \ltimes R^2, R^2)$. This yields relative property (T) for the double root subgroups. It follows from relations (E6) that any long root subgroup lies in a bounded product of double root subgroups, so relative (T) also holds for the long root subgroups.

It remains to prove relative property (T) for the short root subgroups. By symmetry, it suffices to treat the subgroup Z_{γ_1}. Fix $s \in R$, and consider relation (E3) with $r = 0$ and t arbitrary. It implies that the set

$$P_0 = \{z_{1,2}(t^* s, 0) z_{1,1}(ts, 0) : t \in R\}$$

lies in a bounded product of the double and long root subgroups. The same holds for each of the sets $P_1 = \{z_{1,2}(t^*, 0) z_{1,1}(t, 0) : t \in R\}$ (setting $s = 1$ in P_0), $P_2 = \{z_{1,2}(t^* s^*, 0) z_{1,1}(ts, 0) : t \in R\}$ (replacing t by ts in P_1) and $P_3 = \{z_{1,2}(t^*(s + s^*), 0) z_{1,1}(t(s + s^*), 0) : t \in R\}$ (replacing s by $s + s^*$ in P_0). Considering the product $P_0 P_2 P_3$ and using relations (E1) with $i = 1$, we conclude that the set $P = \{z_{1,1}(t(s + s^*), 0) : t \in R\}$ lies in a bounded product of the double and long root subgroups (for any fixed s).

Let I be the ideal of R generated by $\{s + s^* : \ s \in R\}$. Since R is finitely generated, it is Noetherian, and so I is generated by a finite subset of $\{s + s^* : s \in R\}$. Hence, by what we just proved the set $\{z_{1,1}(r, 0) : \ r \in I\}$ also lies in a bounded product of the double and long root subgroups. On the other hand, every element of the quotient ring R/I is idempotent (since $r^2 + r = ((r^*)^* + r^*) + (r^* + r)$), whence I has finite index in R. Thus, the set $\{z_{1,1}(r, 0) : \ r \in R\}$ lies in a bounded product of the double and long root subgroups and fixed elements of G. Since the short root subgroup Z_{γ_1} is a product of $\{z_{1,1}(r, 0) : \ r \in R\}$ and Z_{γ_2}, we have proved relative property (T) for Z_{γ_1}. □

8.8. **More groups graded by root systems.** The families of groups graded by root systems which were described in this section can be generalized in several different ways, and for many of those generalizations one can prove property (T) by similar methods. However, in the absence of a specific application, it is not clear which results of this kind are useful and which are not, so we will not try to achieve the utmost generality in this subsection. Instead we will describe three new families of groups with property (T) which will be needed in § 9 where we will prove that the class of all finite simple groups of Lie type of rank at least 2 admits a mother group with property (T).

All these families are slight variations of Steinberg groups described earlier in this section and will be defined by the same procedure as, for instance, the groups $\mathrm{St}^{\omega}_{C_n}(R, *, J)$ were obtained from $\mathrm{St}^{\omega}_{C_n}(R, *)$, that is, by decreasing (some of) the root subgroups, so that the decreased subgroups still form a grading (by the same root system) and then taking the graded cover of the subgroup generated by these decreased root subgroups. The main difference is that in the previously considered examples, how much a root subgroup X_γ was decreased depended only on the root length of γ, while in the examples in this subsection the procedure will not be "symmetric".

It will be convenient to use the following notation and terminology. Let G be one of the types of Steinberg groups considered in this section associated to a ring R endowed with some set of operations, that is, $G = \mathrm{St}_{A_n}(R)$, $G = \mathrm{St}^{\omega}_{C_n}(R,*)$, etc., and let $\{X_\gamma\}_{\gamma\in\Phi}$ be its standard grading. By definition, each root subgroup X_γ is isomorphic to certain group R_γ^{full} (which in many cases is defined as a subgroup of $(R,+)$), and in the course of our construction we have chosen a specific isomorphism between X_γ and R_γ^{full}. Below we recall the explicit description of R_γ^{full} in the cases which will be used in this section:

(i) if $G = \mathrm{St}_{A_n}(R)$, then $R_\gamma^{full} = (R,+)$ for all γ;
(ii) if $G = \mathrm{St}^{\omega}_{C_n}(R,*)$, then $R_\gamma^{full} = (R,+)$ if γ is short and $R_\gamma^{full} = (\mathrm{Sym}_\omega(R),+)$ if γ is long
(iii) if $G = \mathrm{St}_{BC_n}(R,*)$, then $R_\gamma^{full} = (R,+)$ if γ is long, $R_\gamma^{full} = (\mathrm{Asym}(R),+)$ if γ is double, and $R_\gamma^{full} = P(R,*)$ if γ is short (where $P(R,*)$ is defined in Example 2),
(iv) if $G = \mathrm{St}^{1}_{BC_n}(R,*,\sigma)$, then $R_\gamma^{full} = (R^\sigma,+)$ if γ is long, $R_\gamma^{full} = (\mathrm{Asym}(R^\sigma),+)$ if γ is double, and $R_\gamma^{full} = Q(R,*,\sigma)$ if γ is short (where $Q(R,*,\sigma)$ is defined in Example 3)

Definition. In the above setting, choose a subgroup R_γ of R_γ^{full} for each $\gamma\in\Phi$, and let $X_\gamma(R_\gamma)$ be the image of R_γ under the chosen isomorphism $R_\gamma^{full}\to X_\gamma$. We will say that $\{R_\gamma\}$ is a *root content for G* if $\{X_\gamma(R_\gamma)\}_{\gamma\in\Phi}$ is a Φ-grading.

Example 8.21. Let $n\geq 2$ be an integer, R a ring, and let $G = \mathrm{St}_{A_n}(R)$. Choose a subgroup R_γ of $(R,+)$ for each $\gamma\in A_n$. Then $\{R_\gamma\}_{\gamma\in A_n}$ is a root content for G if and only if $R_\alpha R_\beta\subseteq R_{\alpha+\beta}$ whenever $\alpha+\beta$ is a root.

If $\{R_\gamma\}$ is a root content for G, we can consider the subgroup $\langle\cup X_\gamma(R_\gamma)\rangle$ generated by decreased root subgroups $X_\gamma(R_\gamma)$ and take its graded cover. This graded cover will be denoted by $G(\{R_\gamma\})$.

Example 8.22. Let $G = \mathrm{St}^{\omega}_{C_n}(R,*)$, let J be a form parameter of $(R,*,\omega)$, and define $\{R_\gamma\}_{\gamma\in C_n}$ by $R_\gamma = R$ if γ is short and $R_\gamma = J$ if γ is long. Then $\{R_\gamma\}_{\gamma\in C_n}$ is a root content for G, and the associated group $G(\{R_\gamma\})$ is the group $\mathrm{St}^{\omega}_{C_n}(R,*,J)$ as defined in Example 2.

We are now ready to describe the three families we will be interested in. In all statements below, the following convention will be used: given additive subgroups A and B of a ring R, by AB we denote the additive subgroup of R generated by the set $\{ab : a\in A, b\in B\}$.

Proposition 8.23. *Let R be a ring, let $n\geq 4$ be an integer and $G = \mathrm{St}_{A_{n-1}}(R)$. Let M be a left ideal of R and N a right ideal of R. Define $\{R_\alpha\}_{\alpha\in A_{n-1}}$ by*

$$R_{e_i-e_j} = \begin{cases} R & \text{if } 1\leq i\neq j\leq n-1 \\ M & \text{if } j = n \\ N & \text{if } i = n. \end{cases}$$

Then $\{R_\alpha\}_{\alpha\in A_{n-1}}$ is a root content for G, and the associated group $G(\{R_\alpha\})$ has property (T) if the following conditions hold:

(i) *R is finitely generated as a ring*
(ii) *M is finitely generated as a left ideal*
(iii) *N is finitely generated as a right ideal*
(iv) *$MN = R$.*

The group $G(\{R_\alpha\})$ will be denoted by $\mathrm{St}_{A_{n-1}}(R; M, N)$.

Proof. It is straightforward to check that $\{R_\alpha\}_{\alpha\in A_{n-1}}$ is a root content for G. For the rest of the proof we will denote the group $G(\{R_\alpha\})$ by H, and we let $\{Y_\alpha = X_\alpha(R_\alpha)\}_{\alpha\in A_{n-1}}$ be the canonical A_{n-1}-grading of H.

In the proof of property (T) for H, the relative property (T) part is virtually identical to the case of $\mathrm{St}_{A_{n-1}}(R)$ and uses conditions (i)-(iii) above, but for completeness we provide the argument. The subgroup generated by $\{Y_{e_i-e_j} : i, j < n\}$ is isomorphic to a quotient of $\mathrm{St}_{A_{n-2}}(R)$ and therefore relative property (T) holds for $(H, Y_{e_i-e_j})$ with $i, j < n$.

There is an epimorphism from $\mathrm{St}_{A_{n-2}}(R) \ltimes M^{n-1}$ onto the subgroup generated by $\{Y_{e_i-e_j} : i < n\}$ which maps M^{n-1} onto $\langle\{Y_{e_i-e_n} : i < n\}\rangle$. Since the pair $(\mathrm{St}_{A_{n-2}}(R) \ltimes M^{n-1}, M^{n-1})$ has relative property (T) by Theorem 11.8, it follows that $(H, Y_{e_i-e_n})$ has relative (T) for all $i < n$. Similarly, $(H, Y_{e_n-e_i})$ has relative property (T).

Finally, we need to verify that the grading $\{Y_\alpha\}$ is strong. Here things behave differently from the case $\mathrm{St}_{A_{n-1}}(R)$ and condition (iv) must be used, as we now demonstrate. Since Borel subgroups are no longer isomorphic to each other, we cannot restrict ourselves to checking that the grading is strong at a pair (γ, B) where B is the standard Borel, as we did in all the previous examples. Let us introduce the following notations: given a subset S of A_{n-1}, let

$$H_S = \langle Y_\alpha : \alpha \in S\rangle.$$

Thus, we need to check that for any Borel subset B and any root $\gamma \in C(B)$, the core of B, we have

$$Y_\gamma \subseteq H_{B\setminus\mathbb{R}\gamma}. \tag{8.9}$$

So, let us take any root $\gamma = e_i - e_j$ and any Borel B such that $\gamma \in C(B)$. Then there exists $k \neq i, j$ such that $e_i - e_k$ and $e_k - e_j$ both lie in B. Then $H_{B\setminus\mathbb{R}\gamma}$ contains $[x_{e_i-e_k}(r), x_{e_k-e_j}(s)] = x_{e_i-e_j}(rs)$ for every $r \in R_{e_i-e_k}$ and $s \in R_{e_k-e_j}$, and so we just have to check that the additive subgroup generated by all products of this form coincides with $R_{e_i-e_j}$. If $k \neq n$, this is clearly true (in fact, in this case it suffices to take just products, not their sums), and if $k = n$, this holds precisely because of condition (iv). □

Proposition 8.24. *Let R be a ring, $*$ an involution of R, ω an element of $Z(R)$ satisfying $\omega^* = \omega^{-1}$, let $n \geq 4$ and $G = \mathrm{St}^{\omega}_{C_n}(R, *)$. Let J be a form parameter of $(R, *, \omega)$, let M be a left ideal of R, and let $M^* = \{m^* : m \in M\}$. Let J_M be the additive subgroup of J generated by the set*

$$\{m^*xm : x \in J, m \in M\} \cup \{m^*n - \omega n^*m : m, n \in M\}.$$

Define $\{R_\alpha\}_{\alpha\in C_n}$ *by*

$$\begin{aligned} R_{\pm e_i \pm e_j} &= R \text{ for } 1 \le i \ne j \le n-1 \\ R_{\pm e_i \pm e_n} &= M \text{ for } 1 \le i \le n-1 \\ R_{\pm 2e_i} &= J \text{ for } 1 \le i \le n-1 \\ R_{\pm 2e_n} &= J_M \end{aligned}$$

Then $\{R_\alpha\}_{\alpha\in C_n}$ *is a root content for* G*, and the associated group* $G(\{R_\alpha\})$ *has property* (T) *if the following conditions hold:*

(i) R *is finitely generated as a ring*
(ii) J *is finitely generated as a form parameter of* $(R,*,\omega)$
(iii) M *is finitely generated as a left ideal of* R
(iv) $MM^* = R$
(v) *There exists* $d \in \mathbb{N}$ *such that every element of* M^*M *can be written as a sum* $\sum_{i=1}^d m_i^* n_i$ *with* $m_i, n_i \in M$*.*

The group $G(\{R_\alpha\})$ *will be denoted by* $\mathrm{St}^{\omega}_{C_n}(R,*,J;M)$*.*

Remark. Condition (v) automatically holds if M in principal which will be the case in all applications of Proposition 8.24 in § 9.

Proof. We will use the same general notations as in Proposition 8.23, that is $H = G(\{R_\alpha\})$ and $\{Y_\alpha\}$ the canonical C_n-grading of H. As in Example 1, we will denote the elements of root subgroups of G by $z_\alpha(r)$ (not $x_\alpha(r)$); on the other hand we denote the roots of C_n by $\pm e_i \pm e_j$ and $\pm 2e_i$ (not $\pm\varepsilon_i \pm \varepsilon_j$ and $\pm 2\varepsilon_i$ used in Example 1).

First, we establish relative property (T); as in the case of type A_n, the argument is rather similar to Proposition 8.6.

If α is a short root, the root subgroup Y_α lies inside a quotient of the Steinberg group $\mathrm{St}_{A_{n-1}}(R,M,M^*)$ which has property (T) by Proposition 8.23. If $\alpha = \pm 2e_i$ with $i < n$, then Y_α lies inside a quotient of the Steinberg group $G = \mathrm{St}^{\omega}_{C_{n-1}}(R,*,J)$ and thus has property (T) by Proposition 8.6. Thus, the pair (H, Y_α) has relative (T) in all these cases.

It remains to consider $\alpha = \pm 2e_n$. We will treat the case $\alpha = 2e_n$; the case $\alpha = -2e_n$ is analogous.

Let T be a finite set which generates J as a form parameter of $(R,*,\omega)$. It is easy to see that any element $r \in J_M$ can be written as

$$r = \sum_{t\in T} m_t^* t m_t + (u - \omega u^*) \qquad (***)$$

for some $m_t \in M$ and $u \in M^*M$. By condition (v), we can write $u = \sum_{i=1}^d n_i^* p_i$ with $n_i, p_i \in M$ and d independent of r.

Relations (E2) and (E3) yield the following identity:

$$z_{2e_n}(r) = \prod_{t\in T} [z_{2e_1}(t), z_{e_n-e_1}(m_t)] z_{e_n+e_1}(\sum_{t\in T} m_t t) \prod_{i=1}^{d} [z_{e_n-e_1}(n_i), z_{e_1+e_n}(p_i)]$$

It follows that

$$Y_{2e_n} \subseteq \prod_{t\in T} (Y_{e_n-e_1}^{z_{2e_1}(t)} Y_{e_n-e_1} Y_{e_1+e_n})(Y_{e_n-e_1} Y_{e_1+e_n})^{2d}.$$

The set $\{z_{2e_1}(t) : t \in T\}$ of conjugating elements is finite, so Y_{2e_n} lies in a bounded product of subgroups for which relative (T) has already been established.

Finally, we check that the grading is strong, that is, (8.9) holds for each root γ and Borel subset B containing γ in its core. If γ is a short root, this is checked precisely as in Proposition 8.23, so we only need to consider long roots. We shall treat the case when γ is positive, that is, $\gamma = 2e_i$ for some i; the case of negative γ is analogous.

First consider the case $\gamma = 2e_n$. Since $\gamma \in C(B)$, γ is representable as a sum of two short roots in B, so there exists $1 \le k \le n-1$ s.t. $e_n - e_k$ and $e_n + e_k$ both lie in B. In addition, either $2e_k$ or $-2e_k$ lies in B. Both cases are analogous, so we shall assume that $2e_k \in B$.

By relations (E7) in Example 1 with $i = n$ and $j = k$, the group $H_{B\setminus\mathbb{R}\gamma}$ contains all elements of the form $z_\gamma(m^*n - \omega n^*m)$ with $m, n \in M$, and by relations (E9) with $i = n$ and $j = k$, $H_{B\setminus\mathbb{R}\gamma}$ contains all elements of the form $z_\gamma(m^*xm)$ with $m \in M$ and $x \in J$. Thus, in view of (***), (8.9) holds for (γ, B).

Now consider the case $\gamma = 2e_k$ where $k < n$. Again B contains $e_k - e_l$ and $e_k + e_l$ for some $l \ne k$ and without loss of generality we can assume that $2e_l \in B$. If $l \ne n$, (8.9) obviously holds, so we will only consider the case $l = n$. First, by relations (E2), $H_{B\setminus\mathbb{R}\gamma}$ contains all elements of the form $z_\gamma(mn^* - \omega nm^*)$ with $m, n \in M$ which, by condition (iv), account for all elements of the form $z_\gamma(r - \omega r^*)$ with $r \in R$. By relations (E3) with $i = k$ and $j = n$, $H_{B\setminus\mathbb{R}\gamma}$ contains all elements of the form $z_\gamma(mxm^*)$ with $x \in J_M$, $m \in M$, so in particular, all elements of the form $z_\gamma(mn^*xnm^*)$ with $x \in J$, $m, n \in M$.

Now fix $x \in J$, and choose $m_i, n_i \in M$ such that $\sum_i m_i n_i^* = 1$. Then

$$x = (\sum_i m_i n_i^*) x (\sum_i m_i n_i^*)^* = \sum_i m_i n_i^* x n_i m_i^* + y - \omega y^*$$

where $y = \sum_{i<j} m_i n_i^* x n_j m_j^*$. Thus, $z_\gamma(x) \in H_{B\setminus\mathbb{R}\gamma}$, so (8.9) holds for (γ, B). $\square$

Before describing our last family, we introduce the following notations which generalize analogous notations used in Example 3. Let R be a ring, $*$ an involution of R and σ an automorphism of R of order ≤ 2 which commutes with σ. For any subset S of R we put

$$S^\sigma = \{s \in S : \sigma(s) = s\} = S \cap R^\sigma.$$

Given additive subgroups S, I and J of $(R, +)$, define

$$Q(S, *, \sigma, I, J) = \{(r, t) : r \in I, t \in J \text{ and } t - r\sigma(r)^* \in S^\sigma\}$$

Proposition 8.25. *Let R be a ring, $*$ an involution on R and σ an automorphism of order ≤ 2 which commutes with $*$. Let $n \ge 4$ and $G = \mathrm{St}^1_{BC_n}(R, *, \sigma)$. Let $I \subseteq R$ be a left R^σ-submodule, J a form parameter of $(R, *, 1)$, M be a left ideal of R and*

$M^* = \{m^* : m \in M\}$. Define $\{R_\alpha\}_{\alpha \in BC_n}$ by

$$\begin{aligned} R_{\pm e_i \pm e_j} &= R^\sigma \textit{ for } 1 \le i \ne j \le n-1 \\ R_{\pm e_i \pm e_n} &= M^\sigma \textit{ for } 1 \le i \le n-1 \\ R_{\pm e_i} &= Q(R, *, \sigma, I, J) \textit{ for } 1 \le i \le n-1 \\ R_{\pm 2e_i} &= J^\sigma \textit{ for } 1 \le i \le n-1 \\ R_{\pm e_n} &= Q(M^*M, *, \sigma, (M^*)^\sigma I, J) \\ R_{\pm 2e_n} &= (J \cap M^*M)^\sigma \end{aligned}$$

Then $\{R_\alpha\}_{\alpha \in BC_n}$ is a root content for G, and the associated group $G(\{R_\alpha\})$ has property (T) if the following conditions hold:

(i) *R^σ is finitely generated as a ring*
(ii) *There exists $a \in R$ such that $a + \sigma(a)^* = 1$*
(iii) *I is finitely generated as an R^σ-module*
(iv) *J^σ is finitely generated as a form parameter of $(R^\sigma, *, 1)$*
(v) *M^σ is finitely generated as a left ideal of R^σ*
(vi) *$M^\sigma (M^*)^\sigma = R^\sigma$*
(vii) *There exists $d \in \mathbb{N}$ such that every element of $(M^*)^\sigma M^\sigma$ can be written as a sum $\sum_{i=1}^d m_i^* n_i$ with $m_i, n_i \in M^\sigma$.*
(viii) *$(J \cap M^*M)^\sigma$ is equal to the additive subgroup generated by the set*

$$\{m^* x m : x \in J^\sigma, m \in M^\sigma\} \cup \{m^* n - n^* m : m, n \in M^\sigma\}.$$

The group $G(\{R_\alpha\})$ will be denoted by $\mathrm{St}^1_{BC_n}(R, *, \sigma, I, J; M)$.

Condition (viii) above may be rather difficult to check in general, but it always holds if M is principal and generated by a σ-invariant idempotent:

Observation 8.26. *In the setting of Proposition 8.25, assume that there exists $z \in M^\sigma$ such that $M = Rz$ and $z^2 = z$. Then condition (viii) holds.*

Proof. Denote the additive subgroup generated by the set $\{m^* x m : x \in J^\sigma, m \in M^\sigma\} \cup \{m^* n - n^* m : m, n \in M^\sigma\}$ by $J_{\sigma,M}$. It is clear that $J_{\sigma,M} \subseteq (J \cap M^*M)^\sigma$.

To prove the reverse inclusion, take any $x \in M^*M$. Then, by assumption on M, we have $x = z^* y z$ for some $y \in R$. Since $z^2 = z$, we have $x = (z^2)^* y z^2 = z^* x z$. Thus, if also assume that $x \in J^\sigma$, then $x = z^* x z \in J_{\sigma,M}$ by definition. □

Proof of Proposition 8.25. Similarly to Propositions 8.23 and 8.24, let $H = G(\{R_\alpha\})$ and $\{Y_\alpha\}$ the canonical BC_n-grading of H.

Thanks to condition (viii), the subgroup of H generated by long and double root subgroups is isomorphic to a quotient of $\mathrm{St}^1_{C_n}(R^\sigma, *, J^\sigma)$, which has property (T) by Proposition 8.24 thanks to conditions (i),(iv),(v),(vi) and (vii). Thus, relative (T) holds for (H, Y_α) whenever α is a long or double root. If α is a short root, relative (T) for (H, Y_α) is verified exactly as in Proposition 8.8(a), using condition (iii).

Now we check that the grading $\{Y_\alpha\}$ is strong at each pair (γ, B) with $\gamma \in C(B)$. If $\gamma = \pm e_i \pm e_j$ is a long root, the proof is analogous to Proposition 8.23. If $\gamma = \pm 2e_i$ is a double root, one can argue as in Proposition 8.24.

Finally, consider the case when γ is a short root. As in the proof of Proposition 8.24, without loss of generality we can assume that $\gamma = e_i$ and that B contains $e_i - e_k$ and e_k for some $k \ne i$.

For a short root α denote by I_α the projection of R_α onto the first component. By condition (ii) and Lemma 8.13, $I_{\pm e_j} = I$ for $j < n$ and $I_{\pm e_n} = (M^*)^\sigma I$ (more precisely, for $\alpha = \pm e_n$ we need a suitable generalization of Lemma 8.13 whose proof is analogous). Since we already know that the grading is strong at (δ, B) for each double root δ, by relations (E3) and analogous relations dealing with negative root subgroups in Example 3, it suffices to check that

$$R_{e_i - e_k} I_{e_k} = I_{e_i} \text{ if } i < k \quad \text{and} \quad R^*_{e_i - e_k} I_{e_k} = I_{e_i} \text{ if } i > k. \tag{8.10}$$

If i and k are both different from n, then $R_{e_i - e_k} = R^\sigma$ and $I_{e_i} = I_{e_k} = I$, so (8.10) holds. If $i = n$, then $R^*_{e_i - e_k} = (M^*)^\sigma$, $I_{e_k} = I$ and $I_{e_i} = (M^*)^\sigma I$, so again (8.10) is clear. Finally, if $k = n$, by condition (vi) we have $R_{e_i - e_k} I_{e_k} = M^\sigma (M^*)^\sigma I = I = I_{e_i}$. This completes the proof. □

9. Application: Mother group with property (T)

Let Γ be a finite graph and let $\varepsilon > 0$ be a real number. We say that Γ is an *ε-expander* if for every subset A consisting of at most half of vertices of Γ we have $|\partial A| \geq \varepsilon |A|$. Here ∂A is the edge boundary of A, that is, the set of edges of Γ which join a vertex in A with a vertex outside of A. Expander graphs play an important role in computer science and combinatorics, and many efforts have been devoted to their constructions (see, e.g., [HLW]). Particular attention has been paid to the case of Cayley graphs. Recall that given a group G and a symmetric generating set S, one defines $\mathrm{Cay}(G, S)$ to be the graph with vertex set G, in which two vertices x and y are connected by an edge if and only if $y = xs$ for some $s \in S$. Note that if $|S| = k$ then the Cayley graph is k-regular. We will say that an infinite family $\mathcal{F}$ of groups is a *family of expanders* if there exists $k \in \mathbb{N}$ and $\varepsilon > 0$ such that every group $G \in \mathcal{F}$ has a symmetric generating set S with $|S| = k$ such that $\mathrm{Cay}(G, S)$ is an ε-expander. As a consequence of several works (see [Ka2, KLN, BGT]) the following remarkable result was recently established.

Theorem 9.1. *Any family of (non-abelian) finite simple groups is a family of expanders.*

Definition. Let $\mathcal{F}$ be a family of groups and G a group. We say that G is a *mother group* for $\mathcal{F}$ if every group in $\mathcal{F}$ is a quotient of G.

As discussed in the introduction, one of the conceptually simplest ways to prove that a family $\mathcal{F}$ of finite groups is a family of expanders is to find a mother group for $\mathcal{F}$ with property (T) or (τ). Recall that a group G has *property* (τ) if there exists $\mu > 0$ and a finite subset S of G such that $\kappa(G, S, V) \geq \mu$ for every non-trivial irreducible unitary representation V of G which factors through a finite quotient of G.

In view of Theorem 9.1, it is natural to ask which families of non-abelian finite simple groups admit a mother group with (T) or (τ).

Conjecture 9.2.

(a) *The family of all non-abelian finite simple groups has a mother group with property (τ).*
(b) *A family $\mathcal{F}$ of non-abelian finite simple groups has a mother group with property (T) if and only if $\mathcal{F}$ contains only finitely many finite simple groups of Lie type of rank* 1.

The main result of this section partially confirms part (b) of Conjecture 9.2:

Theorem 9.3. *The family of all finite simple groups of Lie type and rank ≥ 2 has a mother group with property (T).*

Before starting the proof of Theorem 9.3, let us mention other known facts related to Conjecture 9.2. It is a folklore result that a group with property (T) cannot map onto $\mathrm{PSL}_2(\mathbb{F}_q)$ for infinitely many q. Since we are unaware of the proof of this fact in the literature, we include it at the end of this section. However some infinite families of finite simple groups of Lie type and rank 1 have a mother group with property (τ). For instance, $\mathrm{SL}_2(\mathbb{Z}[1/2])$ has property (τ) (see, e.g. [LZ, p. 60]) and clearly maps onto $\mathrm{PSL}_2(\mathbb{F}_p)$ when p is odd.

With regard to Conjecture 9.2(a), we will also prove that alternating groups satisfy the conjecture:

Theorem 9.4. *The family of all alternating groups* $\mathrm{Alt}(n)$ *has a mother group with property (τ).*

Theorem 9.4 will be established in § 9.6.

9.1. **Some general reductions.** Recall that finite simple groups of Lie type can be realized as (possibly twisted) Chevalley groups over finite fields. To simplify the exposition below we shall only discuss groups of rank at least two.

Given a reduced irreducible classical root system Φ of rank ≥ 2, the (untwisted) finite simple group of type Φ over the field of order q will be denoted by $\Phi(q)$. Twisted finite simple groups will be denoted by symbols of the form ${}^l\Phi(q)$ – by definition ${}^l\Phi(q)$ is the subgroup of elements of $\Phi(q^l r)$ fixed by certain automorphism of order l, where $r = 1$ in all cases except ${}^l\Phi = {}^2F_4$, in which case $r = 2$. If ${}^l\Phi \neq {}^2F_4$, the parameter q can be any prime power, and if ${}^l\Phi = {}^2F_4$, we can take $q = 2^k$ for any $k \geq 0$. The groups ${}^l\Phi(q)$ are simple with the exception of ${}^2B_2(2) \cong {}^2C_2(2)$, ${}^2G_2(2)$ and ${}^2F_4(1)$, and in those three cases ${}^l\Phi(q)$ contains a simple subgroup of index 2 (which is equal to the commutator subgroup of ${}^l\Phi(q)$). All these groups are naturally graded by root systems and some of them are classical groups.

It will be convenient to use the following terminology: given two groups of Lie type ${}^lX_m(q)$ and ${}^{l'}X'_{m'}(q')$ (where X and X' stand for A, B, C, D, E, F or G, and a non-twisted group $\Phi(q)$ is temporarily denoted by ${}^1\Phi(q)$), we will say that

(i) ${}^lX_m(q)$ and ${}^{l'}X'_{m'}(q')$ *have the same Lie type* if $X = X'$, $l = l'$ and $m = m'$;
(ii) ${}^lX_m(q)$ and ${}^{l'}X'_{m'}(q')$ *lie in the same Lie family* if $X = X'$, $l = l'$ and in addition m and m' have the same parity if $X = A$ and $l = 2$.

In Table 9.1 below we recollect all this information, with groups sorted by their Lie family (we only list groups of Lie rank ≥ 2). The first column contains the notation for the group, in the second we put its interpretation as a classical group (if such exists) and in the third we describe the graded cover of this group as defined in § 7 and § 8. All these correspondences can be found in [Ca] (see Theorems 11.1.2, 11.3.2, 14.4.1, 14.5.1 and 14.5.2). The notation for classical groups is taken from [KL].

Observe that if $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are families of finite groups and G_i is a mother group for $\mathcal{F}_i$, then $\prod_{i=1}^k G_i$ is a mother group for $\cup_{i=1}^k \mathcal{F}_i$. Thus, to prove Theorem 9.3,

Lie type	**Classical group**	**Graded cover**
$A_n(q)$	$\mathrm{PSL}_{n+1}(\mathbb{F}_q)$	$\mathrm{St}_{A_n}(\mathbb{F}_q)$
$B_n(q)$ (q is odd)	$\mathrm{P}\Omega_{2n+1}(\mathbb{F}_q)$	$\mathrm{St}_{B_n}(\mathbb{F}_q) = \mathrm{St}_{BC_n}(\mathbb{F}_q, \mathrm{id})$
$C_n(q)$ ($q \neq 2$ if $n = 2$)	$\mathrm{PSp}_{2n}(\mathbb{F}_q)$	$\mathrm{St}_{C_n}(\mathbb{F}_q) = \mathrm{St}^{-1}_{C_n}(\mathbb{F}_q, \mathrm{id})$
$D_n(q)$	$\mathrm{P}\Omega^+_{2n}(\mathbb{F}_q)$	$\mathrm{St}_{D_n}(\mathbb{F}_q) = \mathrm{St}^1_{C_n}(\mathbb{F}_q, \mathrm{id}, \{0\})$
$\Phi(q)$ $\Phi = E_n$ or F_4		$\mathrm{St}_\Phi(\mathbb{F}_q)$
$G_2(q)$ $q \neq 2$		$\mathrm{St}_{G_2}(\mathbb{F}_q)$
${}^2A_{2n-1}(q)$	$\mathrm{PSU}_{2n}(\mathbb{F}_q)$	$\mathrm{St}^1_{C_n}(\mathbb{F}_{q^2}, {}^-)$, $\bar{x} = x^q$
${}^2A_{2n}(q)$	$\mathrm{PSU}_{2n+1}(\mathbb{F}_q)$	$\mathrm{St}_{BC_n}(\mathbb{F}_{q^2}, {}^-)$, $\bar{x} = x^q$
${}^2D_n(q)$	$P\Omega^-_{2n}(\mathbb{F}_q)$	$\mathrm{St}^1_{BC_{n-1}}(\mathbb{F}_{q^2}, \mathrm{id}, {}^-, \mathbb{F}_{q^2}, \{0\})$ $= \mathrm{St}_{B_{n-1}}(\mathbb{F}_{q^2}, {}^-)$, $\bar{x} = x^q$
${}^3D_4(q)$		$\mathrm{St}_{G_2}(\mathbb{F}_{q^3}, \theta)$, $\theta(x) = x^q$
${}^2E_6(q)$		$\mathrm{St}_{F_4}(\mathbb{F}_{q^2}, {}^-)$, $\bar{x} = x^q$
${}^2F_4(2^k)$ $(k \geq 1)$		$\mathrm{St}_{{}^2F_4}(\mathbb{F}_{2^{2k+1}}, *)$, $x^* = x^{2^{k+1}}$

TABLE 1. Simple groups of Lie type of rank ≥ 2

it suffices to find a mother group with property (T) for all finite simple groups of rank ≥ 2 within a given Lie family. For the same reason we can exclude any finite set of groups from consideration, so we do not have to worry about the commutator subgroups of $C_2(2)$, $G_2(2)$ and ${}^2F_4(1)$ not included in the above table. Further, it will be convenient to split all groups in a given Lie family into two subfamilies – those of rank $\geq c$ and those of rank $< c$ (but ≥ 2), where c is chosen separately for each Lie family. We will refer to the corresponding two cases as unbounded rank case and bounded rank case, and the construction of a mother group with (T) in these two cases will be rather different. In the bounded rank case we will treat all groups of a given Lie type separately (there are finitely many of such Lie types), and the argument will work for any value of c, while in the unbounded rank case certain minimum value of c is required (we will often choose not the smallest possible c to avoid unnecessary technicalities). Note that we need to consider unbounded rank case only for groups in the families A, B, C, D, ${}^2A$ or ${}^2D$.

The existence of covering epimorphisms from Steinberg groups onto finite simple groups claimed in the above table is a direct consequence of the construction of finite simple groups of Lie type as (possibly twisted) Chevalley groups, as defined in [Ca] and the definition of (possibly twisted) Steinberg groups given in this paper. The case of the groups of type ${}^2F_4$ is somewhat exceptional – in the definition of finite simple groups of type ${}^2F_4$ given in [Ca] the twisting involution on $F_4(2^k)$ is constructed as a composition of a graph automorphism and a field automorphism, while in our definition the twisting involution q on Steinberg groups of type F_4 is defined directly. The fact that the involution on $F_4(2^k)$ corresponding to q coincides with the one defined in [Ca] is easy to check by a direct computation; alternatively, the reader may consult Tits' paper [Ti], where both definitions are discussed.

We now start discussing the proof of Theorem 9.3.

9.2. **Bounded rank case.** Fix a reduced irreducible classical root system Φ. In order to establish Theorem 9.3 in the bounded rank case we need to prove the following:

(1) (untwisted case) If $rk(\Phi) \geq 2$, the family of all finite simple groups of the form $\Phi(q)$ (where q is an arbitrary prime power) admits a mother group with (T).
(2) (twisted case) If $l = 2$ or 3 is such that the Lie type ${}^l\Phi$ is defined and has rank ≥ 2, the family of all finite simple groups of the form ${}^l\Phi(q)$ admits a mother group with (T).

We start with the untwisted case where the argument is very simple. According to our table, it is enough to find a mother group with (T) for the family $\mathrm{St}_\Phi(\mathbb{F}_q)$. Since a finite field is generated by one element (as a ring), it is a quotient of $\mathbb{Z}[t]$ and thus $\mathrm{St}_\Phi(\mathbb{F}_q)$ is a quotient of $\mathrm{St}_\Phi(\mathbb{Z}[t])$, which has property (T) since $rk(\Phi) \geq 2$.

Now we turn to twisted groups. Although $\mathbb{Z}[t]$ can still be used as the "covering ring" in many cases, to simplify the arguments, we will use slightly larger rings. In cases 1-4 below we let $R = \mathbb{Z}[t_1, t_2]$, the ring of polynomials in two (commuting) variables, and let $* : R \to R$ be the involution (also an automorphism of order 2 since R is commutative) which permutes t_1 and t_2.

Lemma 9.5. *Let $S = \mathrm{Sym}(R, *)$, the set of elements of R fixed by $*$. The following hold:*

(1) $S = \mathbb{Z}[t_1 + t_2, t_1 t_2]$
(2) *R is a finitely generated left module over S.*
(3) *S is generated by 1 as a form parameter of $(R, -1)$.*
(4) $\mathrm{Asym}(R, *) = \mathrm{Asym}^{\min}(R, *)$

Proof. (1) is, of course, a standard result about symmetric functions, and (2) holds since R is finitely generated and integral over S.

We now prove (3). Let J be the form parameter of $(R,-1)$ generated by 1. It suffices to show that $(t_1+t_2)^n(t_1t_2)^m \in J$ for any $n,m>0$. If $n=0$ this is clear since $(t_1t_2)^m = t_1^m(t_1^m)^*$, and if $n>0$, this follows from equality $(t_1+t_2)^n(t_1t_2)^m = r+r^*$ where $r = t_1(t_1+t_2)^{n-1}(t_1t_2)^m$.

Finally, to prove (4), note that any $r \in \mathrm{Asym}(R,*)$ cannot contain monomials of the form $t_1^i t_2^i$ with nonzero coefficients, and for any $i \neq j$ the coefficients of $t_1^i t_2^j$ and $t_1^j t_2^i$ must be opposite, so $r = s - s^*$ for some $s \in R$. □

Now we begin case-by-case proof.

Case 1: ${}^l\Phi = {}^2A_{2n-1}(n \geq 2)$. As before, let ${}^- : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ be the automorphism of $\mathbb{F}_{q^2}$ of order 2. As we see from the table, ${}^2A_{2n-1}(q)$ is a quotient of $\mathrm{St}^1_{C_n}(\mathbb{F}_{q^2}, {}^-)$, and by Observation 8.4, $\mathrm{St}^1_{C_n}(\mathbb{F}_{q^2}, {}^-)$ is isomorphic to $\mathrm{St}^{-1}_{C_n}(\mathbb{F}_{q^2}, {}^-)$. Let β be a generator of $\mathbb{F}_q$, and choose $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\alpha + \bar{\alpha} = \beta$. Such α exists since there are q elements of $\mathbb{F}_{q^2}$ whose trace in $\mathbb{F}_q$ is equal to β and not all these elements lie in $\mathbb{F}_q$. Note that $\bar{\alpha} = \alpha^q$, so the subfield generated by α properly contains $\mathbb{F}_q$ and thus must equal $\mathbb{F}_{q^2}$.

Now define a homomorphism $\pi : R \to \mathbb{F}_{q^2}$ by setting $\pi(t_1) = \alpha$ and $\pi(t_2) = \bar{\alpha}$. By construction, π is involution preserving, that is, $\pi(r^*) = \overline{\pi(r)}$ for all $r \in R$, π is surjective by the choice of α and $\pi(\mathrm{Sym}(R,*)) = \mathrm{Sym}(\mathbb{F}_{q^2}, {}^-) = \mathbb{F}_q$ by the choice of β. Therefore, from the definition of Steinberg groups of type C_n, it is clear that π induces an epimorphism from $\mathrm{St}^{-1}_{C_n}(R,*)$ to $\mathrm{St}^{-1}_{C_n}(\mathbb{F}_{q^2}, {}^-)$. The group $\mathrm{St}^{-1}_{C_n}(R,*)$ has property (T) by Proposition 8.6 and Lemma 9.5(2)(3), with part (2) only needed for $n=2$. Note that the only reason we had to use isomorphism between $\mathrm{St}^1_{C_n}(\mathbb{F}_{q^2}, {}^-)$ and $\mathrm{St}^{-1}_{C_n}(\mathbb{F}_{q^2}, {}^-)$ is to cover the case $n=2$ since we do not know any sufficient condition for property (T) for Steinberg groups of type $\mathrm{St}^1_{C_2}$.

Case 2: ${}^l\Phi = {}^2A_{2n}(n \geq 2)$. For $n \geq 3$ we can again use the ring $R = \mathbb{Z}[t_1,t_2]$ with the same involution $*$. Let α be a generator of $\mathbb{F}_{q^2}$, and define an epimorphism $\pi : R \to \mathbb{F}_{q^2}$ by setting $\pi(t_1) = \alpha$ and $\pi(t_2) = \bar{\alpha}$.

Note that $\mathrm{Asym}(\mathbb{F}_{q^2}, {}^-)$ is precisely the set of elements of $\mathbb{F}_{q^2}$ whose $\mathbb{F}_q$-trace is equal to zero, and therefore $\mathrm{Asym}(\mathbb{F}_{q^2}, {}^-) = \mathrm{Asym}^{\min}(\mathbb{F}_{q^2}, {}^-)$ by Hilbert's Theorem 90. Therefore,

$$\pi(\mathrm{Asym}(R,*)) = \mathrm{Asym}(\mathbb{F}_{q^2}, {}^-).$$

Indeed, the left-hand side is clearly contained in the right-hand side, but on the other hand $\pi(\mathrm{Asym}(R,*)) \supseteq \pi(\mathrm{Asym}^{\min}(R,*)) = \mathrm{Asym}^{\min}(\mathbb{F}_{q^2}, {}^-) = \mathrm{Asym}(\mathbb{F}_{q^2}, {}^-)$. This implies that π induces an epimorphism from $\mathrm{St}^1_{BC_n}(R,*)$ to $\mathrm{St}^1_{BC_n}(\mathbb{F}_{q^2}, {}^-)$, and the group $\mathrm{St}^1_{BC_n}(R,*)$ has property (T) by Proposition 8.8 (note that the hypothesis about finite generation as a left ideal holds automatically since R is Noetherian).

If $n=2$, Proposition 8.8 is not applicable in this setting since R does not have invertible antisymmetric elements. To fix this problem, we consider the larger ring $R' = R[s, 1/s]$, and extend the involution $*$ to R' by setting $s^* = -s$. Extend the map $\pi : R \to \mathbb{F}_{q^2}$ to an epimorphism $\pi' : R' \to \mathbb{F}_{q^2}$ by sending s to any nonzero element with zero $\mathbb{F}_q$-trace. By the same argument as above, π' induces an epimorphism form $\mathrm{St}^1_{BC_2}(R',*)$ to $\mathrm{St}^1_{BC_2}(\mathbb{F}_{q^2}, {}^-)$, and it remains to show that the pair $(R',*)$ satisfies the hypotheses of Proposition 8.8 for $n=2$.

By construction, s is an invertible antisymmetric element of R'. A straightforward computation shows that $\mathrm{Asym}(R', *)$ is generated by s as a form parameter of $(R', *, -1)$. Finally, R' is clearly a finitely generated module over $\mathbb{Z}[t_1+t_2, t_1t_2, s^2, 1/s^2]$, which is a finitely generated subring of $\mathrm{Sym}(R', *)$ (in fact, it is easy to show that this subring coincides with $\mathrm{Sym}(R', *)$).

Case 3: ${}^l\Phi = {}^2D_n (n \geq 4)$. The proof in case is completely analogous to Case 1. This time we use $\mathrm{St}_{B_{n-1}}(R, *, R)$ as a mother group. It has property (T) by Proposition 8.12.

Case 4: ${}^l\Phi = {}^2E_6$. The proof in this case is also analogous to Case 1.

Case 5: ${}^l\Phi = {}^3D_4$. Here we need a slight modification of the argument in Case 1. Let $R = \mathbb{Z}[t_1, t_2, t_3]$, and let σ be the automorphism of order 3 which cyclically permutes t_1, t_2 and t_3. As in Lemma 9.5, $R^\sigma = \mathbb{Z}[t_1+t_2+t_3, t_1t_2+t_1t_3+t_2t_3, t_1t_2t_3]$ and R is a finitely generated module over R^σ, so the group $\mathrm{St}_{G_2}(R, \sigma)$ has property (T) by Proposition 8.14.

Now let β be a generator of $\mathbb{F}_q$, and choose any $\alpha \in \mathbb{F}_{q^3} \backslash \mathbb{F}_q$ with $\alpha + \theta(\alpha) + \theta^2(\alpha) = \beta$. As in Case 1, α generates $\mathbb{F}_{q^3}$, and the map $\pi : R \to \mathbb{F}_{q^3}$ given by $\pi(t_i) = \theta^{i-1}(\alpha)$ for $i = 1, 2, 3$ induces an epimorphism from $\mathrm{St}_{G_2}(R, \sigma)$ to $\mathrm{St}_{G_2}(\mathbb{F}_{q^3}, \theta)$.

Case 6: ${}^l\Phi = {}^2F_4$. Let $R = \mathbb{F}_2[t_1, t_2]$ and let $p : R \to R$ be the homomorphism which sends t_1 to t_2 and t_2 to t_1^2, and define $* : \mathbb{F}_{2^{2k+1}} \to \mathbb{F}_{2^{2k+1}}$ by $x^* = x^{2^{k+1}}$. Let α be a generator of $\mathbb{F}_{2^{2k+1}}$, and define $\pi : R \to \mathbb{F}_{2^{2k+1}}$ by $\pi(t_1) = \alpha$ and $\pi(t_2) = \alpha^{2^{k+1}}$. Then $\pi(p(r)) = r^*$ for any $r \in R$, so $G = \mathrm{St}_{{}^2F_4}(R, p)$ surjects onto $\mathrm{St}_{{}^2F_4}(\mathbb{F}_{2^{2k+1}}, *)$ which, in turn, surjects onto ${}^2F_4(2^k)$ according to our table. Since G has property (T) by Proposition 8.20, the proof is complete.

9.3. **Unbounded rank case: overview.** Now we produce a mother group with property (T) for finite simple groups of Lie type of sufficiently high rank, considering separately the families A_n, C_n, D_n, ${}^2A_n$ (n odd), B_n, ${}^2A_n$ (n even) and ${}^2D_n$ (in this order).

Our general procedure is as follows. Let $\mathbf{\Phi} = \{\Phi_n\}$ be a Lie family that we are considering. The groups $\Phi_n(q)$ have a description as classical groups. We will define a Ψ-grading for each $\Phi_n(q)$ with n sufficiently large, where Ψ is a root system depending only on $\mathbf{\Phi}$ (and not on n). This grading can be obtained by coarsening the canonical Φ_n-grading of $\Phi_n(q)$ using a suitable reduction $\eta : \Phi_n \to \Psi$, but this fact will not be essential for the proof. Then we will construct a Ψ-graded group $G_{\mathbf{\Phi}}$ which maps onto all $\Phi_n(q)$ with n sufficiently large and use of one of the criteria from § 8 to prove that $G_{\mathbf{\Phi}}$ has property (T). In order to simplify arguments (in particular, to show that $G_{\mathbf{\Phi}}$ has property (T)), we will sometimes make Ψ larger than it could be.

We start with a brief outline of the proof in the case $\mathbf{\Phi} = \{A_{n-1}\}$, which should also give the reader some idea on how other cases will be handled. Let us first treat a special case when for a fixed integer $d \geq 3$, we consider only n which are multiples of d. If $n = dk$, then as we already observed in the introduction, considering $n \times n$ matrices as $d \times d$ block matrices with each block being a $k \times k$ matrix, we obtain a natural isomorphism $\mathrm{SL}_n(\mathbb{F}_q) = \mathrm{EL}_n(\mathbb{F}_q) \cong \mathrm{EL}_d(\mathrm{Mat}_k(\mathbb{F}_q))$. It is well known (see Lemma 9.6 below) that $\mathrm{Mat}_k(\mathbb{F}_q)$ can be generated as a ring by two matrices, so it is a quotient of $\mathbb{Z}\langle x, y\rangle$, the free associative ring in two variables. Therefore,

$\mathrm{EL}_d(\mathrm{Mat}_k(\mathbb{F}_q))$ is a quotient of $\mathrm{EL}_d(\mathbb{Z}\langle x,y\rangle)$, and the latter group has property (T).

In general, as long as $n \geq d$, we can consider an $n \times n$ matrix as a $d \times d$ block matrix, but we can no longer guarantee that blocks will have the same size. We will use block decompositions with $d = 4$ in which the first three diagonal blocks have the same size k and the last diagonal block has size $l \leq k$ (so that $n = 3k + l$). The isomorphism $\mathrm{EL}_{dk}(\mathbb{F}_q) \cong \mathrm{EL}_d(\mathrm{Mat}_k(\mathbb{F}_q))$ has natural analogue in this setting, which yields the corresponding A_{d-1}-grading on $\mathrm{EL}_n(\mathbb{F}_q)$. In fact, this grading is the precisely the coarsened A_{d-1}-grading of $\mathrm{EL}_n(\mathbb{F}_q)$ corresponding to a suitable reduction $A_{n-1} \to A_{d-1}$.

The obtained A_{d-1}-grading shows that the group $\mathrm{SL}_n(\mathbb{F}_q) = \mathrm{EL}_n(\mathbb{F}_q)$ is a quotient of the Steinberg group $\mathrm{St}_{A_3}(R, M, N)$ described in Proposition 8.23 where $R = \mathrm{Mat}_k(\mathbb{F}_q)$, $M = \mathrm{Mat}_{k \times l}(\mathbb{F}_q)$ and $N = \mathrm{Mat}_{l \times k}(\mathbb{F}_q)$. If $\widehat{R}$ is a finitely generated ring, $\pi : \widehat{R} \to R$ an epimorphism and we choose a left ideal $\widehat{M}$ of $\widehat{R}$ and a right ideal $\widehat{N}$ of $\widehat{R}$ such that $\pi(\widehat{N}) = N$ and $\pi(\widehat{M}) = M$, then it is clear from the definitions that $\mathrm{St}_{A_3}(R, M, N)$ becomes a quotient of $\mathrm{St}_{A_3}(\widehat{R}, \widehat{M}, \widehat{N})$. However, to prove that the latter group has property (T), we need to know that $\widehat{M}$ and $\widehat{N}$ are finitely generated and that $\widehat{M}\widehat{N} = \widehat{R}$. We do not know how to achieve these conditions if we simply take $\widehat{R} = \mathbb{Z}\langle x, y\rangle$ (most likely it is impossible), so instead we will make the ring $\widehat{R}$ a little larger by adding suitable generators and relations.

Define R_{main} to be the associative ring on four generators x, y, z, w subject to one relation

$$z^2 + wz^2w = 1, \tag{9.1}$$

that is,

$$R_{main} = \mathbb{Z}\langle x, y, z, w\rangle/(z^2 + wz^2w - 1).$$

In the case when Φ is of type A, we will take $\widehat{R} = R_{main}$, and the epimorphism $\pi : \widehat{R} \to \mathrm{Mat}_k(\mathbb{F}_q)$ will be constructed so that we can take $\widehat{M} = \widehat{R}z$ and $\widehat{N} = z\widehat{R}$. Thus, $\widehat{M}\widehat{N} = \widehat{R}z^2\widehat{R}$, the two-sided ideal generated by z^2, which is equal to $\widehat{R}$ by the relation (9.1) we imposed. The precise form of this relation is chosen so that we can use R_{main} as a model for the "covering ring" $\widehat{R}$ in other cases (Φ is not of type A) when an involution with suitable properties will need to be defined on $\widehat{R}$.

Some notations. Before proceeding, we introduce some general notations and state two results on generation of matrix rings that will be repeatedly used below without further mention.

If S is a ring (possibly non-commutative), we denote by $S\langle t\rangle$ the (ring-theoretic) free product of S with $\mathbb{Z}[t]$ and by $S[t]$ the largest quotient of $S\langle t\rangle$ in which t is central, that is, the ring of polynomials over S in one variable t. For instance, $\mathbb{Z}\langle x\rangle = \mathbb{Z}[x]$ and $(\mathbb{Z}\langle x\rangle)\langle y\rangle = \mathbb{Z}\langle x, y\rangle$, the free associative ring in two variables.

If R is a ring, r an element of R and i, j are positive integers, by $(r)_{i,j}$ we will denote the matrix whose (i, j)-entry is equal to r and all other entries are equal to 0 – the size of the matrix is not specified in the notation, but will always be clear from the context. Using this notation, we put

$$E_{i,j} = (1)_{i,j} \text{ and } \mathrm{Id}_k = \sum_{i=1}^{k} E_{i,i}.$$

The matrix Id_k is, of course, the identity element of $\mathrm{Mat}_k(R)$, but may also be considered as an element of $\mathrm{Mat}_{i\times j}(R)$ for any $i, j \geq k$.

If $(k_1, \ldots, k_d)$ is a sequence of positive integers, by a block matrix of type $(k_1, \ldots, k_d)$, we will mean a $d \times d$ block-diagonal matrix whose (i, j)-block is a $k_i \times k_j$ matrix. Thus, any $n \times n$ matrix can be considered as a block matrix of type $(k_1, \ldots, k_d)$ whenever $\sum k_i = n$. For $1 \leq i \leq n$ denote by $\mathrm{block}(i)$ the block into which index i falls under this decomposition, that is, $\mathrm{block}(i) = j$ if $\sum_{t<j} k_t < i \leq \sum_{t\leq j} k_t$.

We will frequently use the notation $(r)_{i,j}$ introduced above in this setting of block matrices – for instance, if we consider 10×10 matrices as block matrices of type $(5, 3, 2)$, then for any 5×3 matrix A, the block matrix $\begin{pmatrix} 0_{5\times5} & A & 0_{5\times2} \\ 0_{3\times5} & 0_{3\times3} & 0_{3\times2} \\ 0_{2\times5} & 0_{2\times3} & 0_{2\times2} \end{pmatrix}$ will be denoted by $(A)_{1,2}$.

We will need the following result on generation in matrix rings, which will be proved at the end of § 9.4.

Lemma 9.6. *Let F be a finite field and k an integer. If $k \geq 2$, then $\mathrm{Mat}_k(F)$ can be generated by two symmetric matrices. If $|F| = q^2$, σ is the automorphism of F of order 2 and $k \geq 3$, then $\mathrm{Mat}_k(F)$ can be generated by two hermitian (with respect to σ) matrices.*

9.4. **Unbounded rank case: proof.** We now begin the formal case-by-case proof. In each of the seven cases considered below we will define integers k and l satisfying $l \leq k \leq 2l$ and let

$$Z = \mathrm{Id}_l \text{ and } W = \sum_{j=1}^{k-l} (E_{j,j+l} + E_{j+l,j}).$$

By direct computation we have

$$\mathrm{Id}_k = Z^2 + WZ^2W \tag{9.2}$$

This equation reveals where the relation (9.1) in the definition of the ring R_{main} comes from.

Case 1: $\mathbf{\Phi} = A$. There exists a group G_A with property (T) which maps onto $\mathrm{SL}_n(\mathbb{F}_q)$ for $n \geq 18$.

Since $n \geq 18$, it is easy to see that we can write $n = 3k + l$ where $l \leq k \leq 2l$. Considering $n \times n$ matrices as 4×4 block matrices of type (k, k, k, l), we obtain a natural A_3-grading $\{X_\gamma\}$ of $\mathrm{SL}_n(\mathbb{F}_q)$ described below. Recall that Id_k denotes the unit $k \times k$ matrix. The corresponding positive root subgroups are

$$X_{e_1-e_2} = \begin{pmatrix} \mathrm{Id}_k & \mathrm{Mat}_k(\mathbb{F}_q) & 0 & 0 \\ 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & \mathrm{Id}_l \end{pmatrix}, \ X_{e_1-e_4} = \begin{pmatrix} \mathrm{Id}_k & 0 & 0 & \mathrm{Mat}_{k\times l}(\mathbb{F}_q) \\ 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & \mathrm{Id}_l \end{pmatrix},$$

$$X_{e_1-e_3} = \begin{pmatrix} \mathrm{Id}_k & 0 & \mathrm{Mat}_k(\mathbb{F}_q) & 0 \\ 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & \mathrm{Id}_l \end{pmatrix}, \ X_{e_2-e_4} = \begin{pmatrix} \mathrm{Id}_k & 0 & 0 & 0 \\ 0 & \mathrm{Id}_k & 0 & \mathrm{Mat}_{k\times l}(\mathbb{F}_q) \\ 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & \mathrm{Id}_l \end{pmatrix},$$

$$X_{e_2-e_3} = \begin{pmatrix} \mathrm{Id}_k & 0 & 0 & 0 \\ 0 & \mathrm{Id}_k & \mathrm{Mat}_k(\mathbb{F}_q) & 0 \\ 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & \mathrm{Id}_l \end{pmatrix}, \ X_{e_3-e_4} = \begin{pmatrix} \mathrm{Id}_k & 0 & 0 & 0 \\ 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & \mathrm{Id}_k & \mathrm{Mat}_{k\times l}(\mathbb{F}_q) \\ 0 & 0 & 0 & \mathrm{Id}_l \end{pmatrix}.$$

If $\gamma \in A_3$ is a negative root, we put $X_\gamma = (X_{-\gamma})^{\mathrm{tr}}$, where tr denotes transposition.

It is easy to check that this grading is the coarsened grading corresponding to the reduction $\eta : A_{n-1} \to A_3$ given by $\eta(e_i) = e_{\mathrm{block}(i)}$.

If we set $k_1 = k_2 = k_3 = k$ and $k_4 = l$, then using our shortcut notations, we can rewrite the definition of the above root subgroups as follows:

$$X_{e_i-e_j} = \{\mathrm{Id}_n + (A)_{i,j} \ : A \in \mathrm{Mat}_{k_i\times k_j}(\mathbb{F}_q)\}.$$

Let $R = \mathrm{Mat}_k(\mathbb{F}_q)$. By Lemma 9.6, R is generated by two matrices, say X and Y, and observe that $\mathrm{Mat}_{k\times l}(\mathbb{F}_q) = RZ$ and $\mathrm{Mat}_{k\times l}(\mathbb{F}_q) = ZR$ where as we recall $Z = \mathrm{Id}_l \in R$.

Now let $\widehat{R} = R_{main}$, and define G_A to be the subgroup of $\mathrm{EL}_4(\widehat{R})$ generated by the subgroups $\{\widehat{X}_\gamma\}_{\gamma\in A_3}$ described below:

$$\widehat{X}_{e_1-e_2} = \begin{pmatrix} 1 & \widehat{R} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widehat{X}_{e_1-e_3} = \begin{pmatrix} 1 & 0 & \widehat{R} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widehat{X}_{e_1-e_4} = \begin{pmatrix} 1 & 0 & 0 & \widehat{R}z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{X}_{e_2-e_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \widehat{R} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widehat{X}_{e_2-e_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \widehat{R}z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widehat{X}_{e_3-e_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \widehat{R}z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{X}_{e_2-e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \widehat{R} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widehat{X}_{e_3-e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \widehat{R} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widehat{X}_{e_4-e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \widehat{R}z & 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{X}_{e_3-e_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \widehat{R} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \widehat{X}_{e_4-e_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \widehat{R}z & 0 & 1 \end{pmatrix}, \ \widehat{X}_{e_4-e_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \widehat{R}z & 1 \end{pmatrix}.$$

Recall that $W \in \mathrm{Mat}_k(\mathbb{Z})$ satisfies $Z^2 + WZ^2W = 1_R$. Let $\pi : \widehat{R} \to R$ be the (unique) epimorphism such that $\pi(x) = X$, $\pi(y) = Y$, $\pi(z) = Z$ and $\pi(w) = W$. Note that π induces an epimorphism from G_A onto $\mathrm{SL}_n(\mathbb{F}_q)$. On the other hand, it is clear from the above definition that G_A is a quotient of the Steinberg group $\mathrm{St}_{A_3}(\widehat{R}; \widehat{R}z, z\widehat{R})$ defined in Proposition 8.23. The product $\widehat{R}z \cdot z\widehat{R} = \widehat{R}z^2\widehat{R}$ is equal to $\widehat{R}$ thanks to the relation $z^2 + wz^2w = 1$. Hence by Proposition 8.23, the group $\mathrm{St}_{A_3}(\widehat{R}; \widehat{R}z, z\widehat{R})$ has property (T), and so does its quotient G_A.

Case 2: $\mathbf{\Phi} = C$. There exists a group G_C with property (T) which maps onto $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ for $n \geq 18$.

Write $n = 3k + l$ where $l \le k \le 2l$. Let

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_l & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_l & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{F}_q).$$

We use the following realization of $Sp_{2n}(\mathbb{F}_q)$:

$$\mathrm{Sp}_{2n}(\mathbb{F}_q) = \{A \in \mathrm{Mat}_{2n}(\mathbb{F}_q) : \ A^{\mathrm{tr}} J A = J.\}$$

Considering $2n \times 2n$ matrices as 8×8 block matrices of type (k, k, k, l, l, k, k, k), we obtain a natural C_4-grading $\{X_\gamma\}_{\gamma \in C_4}$ of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$.

Below $1 \le i < j \le 4$, $\bar{i} = 9 - i$, $\bar{j} = 9 - j$, and we set $k_1 = k_2 = k_3 = k$ and $k_4 = l$. The positive root subgroups X_γ are defined as follows:

$$\begin{aligned} X_{e_i - e_j} &= \{\mathrm{Id}_{2n} + (A)_{i,j} - (A^{\mathrm{tr}})_{\bar{j},\bar{i}} : \ A \in \mathrm{Mat}_{k_i \times k_j}(\mathbb{F}_q)\} \\ X_{e_i + e_j} &= \{\mathrm{Id}_{2n} + (A)_{i,\bar{j}} + (A^{\mathrm{tr}})_{j,\bar{i}} : \ A \in \mathrm{Mat}_{k_i \times k_j}(\mathbb{F}_q)\} \\ X_{2e_i} &= \{\mathrm{Id}_{2n} + (A)_{i,\bar{i}} : \ A = A^{\mathrm{tr}} \in \mathrm{Mat}_{k_i \times k_i}(\mathbb{F}_q)\} \end{aligned}$$

The negative root subgroups can be obtained by the formulas $X_{-\gamma} = (X_\gamma)^{\mathrm{tr}}$.

This grading is the coarsened grading corresponding to the reduction $\eta : C_n \to C_4$ given by $\eta(e_i) = e_{\mathrm{block(i)}}$ for $1 \le i \le n$.

As in case 1, let $R = \mathrm{Mat}_k(\mathbb{F}_q)$, $Z = \mathrm{Id}_l \in R$, recall that $\mathrm{Mat}_{k\times l}(\mathbb{F}_q) = RZ$ and $\mathrm{Mat}_{l\times k}(\mathbb{F}_q) = ZR$, and note that $\mathrm{Mat}_l(\mathbb{F}_q) = ZRZ$. Also observe that the set

$$I = \{A = A^{\mathrm{tr}} \in \mathrm{Mat}_k(\mathbb{F}_q)\}$$

of symmetric matrices in R is a form parameter of $(R, \mathrm{tr}, -1)$ and that

$$\{A = A^{\mathrm{tr}} \in \mathrm{Mat}_l(\mathbb{F}_q)\} = ZIZ.$$

It is also easy to see that I is generated by $U = E_{11}$ (as a form parameter).

Now let $\widehat{R} = R_{main}\langle u\rangle = \mathbb{Z}\langle x, y, z, w, u\rangle/(z^2 + wz^2w - 1)$. Let $*$ be the involution of $\widehat{R}$ that fixes x, y, z, w and u (such involution certainly exists on the free associative ring $\mathbb{Z}\langle x, y, z, w, u\rangle$, and since that involution preserves the element $z^2 + wz^2w - 1$, it induces an involution on $\widehat{R}$ with required properties).

Let $\widehat{I}$ be the form parameter of $(\widehat{R}, *, -1)$ generated by u. Define the subsets $\{\widehat{R}_\gamma\}_{\gamma \in C_4}$ of $\widehat{R}$ by

$$\widehat{R}_{e_i - e_j} = \begin{cases} \widehat{R}, & \text{if } 1 \le i \ne j \le 3 \\ \widehat{R}z, & \text{if } j = 4 \\ z\widehat{R}, & \text{if } i = 4 \end{cases}$$

$$\widehat{R}_{e_i + e_j} = \begin{cases} \widehat{R}, & \text{if } 1 \le i < j \le 3 \\ \widehat{R}z, & \text{if } 1 \le i \le 3, j = 4 \end{cases}$$

$$\widehat{R}_{-(e_i + e_j)} = \begin{cases} \widehat{R}, & \text{if } 1 \le i < j \le 3 \\ z\widehat{R}, & \text{if } 1 \le i \le 3, j = 4 \end{cases}$$

$$\widehat{R}_{2e_i} = \begin{cases} \widehat{I}, & \text{if } 1 \le i \le 3 \\ z\widehat{I}z, & \text{if } i = 4, \end{cases}$$

and define G_C to be the subgroup of $\mathrm{EL}_8(\widehat{R})$ generated by the following subgroups $\{\widehat{X}_\gamma\}_{\gamma\in C_4}$:

$$\begin{aligned}\widehat{X}_{e_i-e_j} =&\{\mathrm{Id}_8 + (r)_{i,j} - (r^*)_{\bar{j},\bar{i}}:\ r \in \widehat{R}_{e_i-e_j}\}\\ \widehat{X}_{e_i+e_j} =&\{\mathrm{Id}_8 + (r)_{i,\bar{j}} + (r^*)_{j,\bar{i}}:\ r \in \widehat{R}_{e_i+e_j}\}\\ \widehat{X}_{2e_i} =&\{\mathrm{Id}_8 + (r)_{i,\bar{i}}:\ r \in \widehat{R}_{2e_i}\}\end{aligned}$$

Choose two symmetric matrices X and Y which generate R, and let $\pi : \widehat{R} \to R$ be the epimorphism given by $\pi(x) = X, \pi(y) = Y, \pi(z) = Z$, $\pi(u) = U$ and $\pi(w) = W$. By construction, π is involution-preserving: $\pi(r^*) = (\pi(r))^{\mathrm{tr}}$ for any $r \in \widehat{R}$, and from the above description it is clear that π induces an epimorphism from G_C to $Sp_{2n}(\mathbb{F}_q)$.

On the other hand, by construction, G_C is a quotient of the group $\mathrm{St}^{-1}_{C_4}(\widehat{R}, *, \widehat{I}; \widehat{R}z)$ described in Proposition 8.24. Since $\widehat{R}z(\widehat{R}z)^* = \widehat{R}z^2\widehat{R} = \widehat{R}$ and $\widehat{I}$ is finitely generated by construction, we conclude that G_C has property (T).

Case 3: $\mathbf{\Phi} = D$. There exists a group G_D with property (T) which maps onto $\Omega^+_{2n}(\mathbb{F}_q)$ for $n \geq 18$.

Again write $n = 3k + l$, where $l \leq k \leq 2l$. We shall consider $n \times n$ matrices as 8×8 block matrices of type $\overrightarrow{k} = (k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8) = (k, k, k, l, l, k, k, k)$ and use the same notational convention as in case 2.

Let

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k\\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0\\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 & 0\\ 0 & 0 & 0 & 0 & \mathrm{Id}_l & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \sum_{i=1}^{4} (\mathrm{Id}_{k_i})_{i,\bar{i}} \in \mathrm{Mat}_{2n}(\mathbb{F}_q).$$

We realize $O^+_{2n}(\mathbb{F}_q)$ as the group of matrices $M \in \mathrm{GL}_{2n}(\mathbb{F}_q)$ that preserve the quadratic form $q(u, u) = u^{\mathrm{tr}}Ju$, where $u \in \mathrm{Mat}_{2n\times 1}(\mathbb{F}_q) \cong \mathbb{F}_q^{2n}$:

$$O^+_{2n}(\mathbb{F}_q) = \{M \in \mathrm{GL}_{2n}(\mathbb{F}_q)\ : q(Mu, Mu) = q(u, u) \text{ for all } u \in \mathbb{F}_q^{2n}\}$$

Note that $O^+_{2n}(\mathbb{F}_q)$ is a subgroup of the group

$$\{M \in GL_{2n}(\mathbb{F}_q)\ : M^{\mathrm{tr}}(J^{\mathrm{tr}} + J)M = J^{\mathrm{tr}} + J\},$$

and the two groups coincide if q is odd.

For a positive integer m, define $\mathrm{Asym}_0(m, q)$ to be the set of antisymmetric matrices in $\mathrm{Mat}_m(\mathbb{F}_q)$ with diagonal entries equal to zero. The group $\Omega^+_{2n}(\mathbb{F}_q)$ has the following C_4-grading $\{X_\gamma\}_{\gamma\in C_4}$:

$$\begin{aligned}X_{e_i-e_j} =&\{\mathrm{Id}_{2n} + (A)_{i,j} - (A^{\mathrm{tr}})_{\bar{j},\bar{i}}:\ A \in \mathrm{Mat}_{k_i\times k_j}(\mathbb{F}_q)\}\\ X_{e_i+e_j} =&\{\mathrm{Id}_{2n} + (A)_{i,\bar{j}} - (A^{\mathrm{tr}})_{j,\bar{i}}:\ A \in \mathrm{Mat}_{k_i\times k_j}(\mathbb{F}_q)\}\\ X_{2e_i} =&\{\mathrm{Id}_{2n} + (A)_{i,\bar{i}}:\ A \in \mathrm{Asym}_0(k_i, q)\}\end{aligned}$$

The negative root subgroups are given by $X_{-\gamma} = (X_\gamma)^{\mathrm{tr}}$.

This grading is the coarsened grading corresponding to the reduction $\eta : D_n \to C_4$ given by $\eta(e_i) = e_{\mathrm{block(i)}}$ for $1 \leq i \leq n$.

Let $R = \mathrm{Mat}_k(\mathbb{F}_q)$. As in the previous cases, let $Z = \mathrm{Id}_l \in R$ and recall that $\mathrm{Mat}_{k\times l}(\mathbb{F}_q) = RZ$ and $\mathrm{Mat}_{l\times k}(\mathbb{F}_q) = ZR$. Let $I = \mathrm{Asym}_0(k, q)$, and observe that

$$I = \mathrm{Asym}^{\min}(\mathrm{Mat}_k(\mathbb{F}_q), \mathrm{tr}) \quad \text{and} \quad ZIZ = \mathrm{Asym}^{\min}(ZRZ, \mathrm{tr}).$$

Now let $\widehat{R} = R_{main}$, let $*$ be the involution of $\widehat{R}$ that fixes x, y, z and w, and let $\widehat{I} = \mathrm{Asym}^{\min}(\widehat{R}, *)$. Define subsets $\{R_\gamma\}_{\gamma\in C_4}$ precisely as in Case 2 (but with the new meaning of $\widehat{R}, \widehat{I}$ and $*$), and define the group G_D in terms of $\{R_\gamma\}$ as G_C was defined in Case 2.

Choose symmetric matrices X and Y which generate R as a ring, and let $\pi : \widehat{R} \to R$ be the epimorphism given by $\pi(x) = X, \pi(y) = Y$, $\pi(z) = Z$ and $\pi(w) = W$. It is clear that π is involution preserving and $\pi(\widehat{I}) = I$. Thus, π induces an epimorphism $G_D \to \Omega^+_{2n}(\mathbb{F}_q)$. On the other hand, G_D is a quotient of the group $\mathrm{St}^1_{C_4}(\widehat{R}, *, \widehat{I}, \widehat{R}z)$, which has property (T) by Proposition 8.24.

Case 4: $\mathbf{\Phi} = {}^2A_{odd}$. The group G_D (constructed in case 3) maps onto $\mathrm{SU}_{2n}(\mathbb{F}_q)$ for $n \geq 18$.

Again write $n = 3k + l$, where $l \leq k \leq 2l$, and let

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_l & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_l & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{F}_{q^2}).$$

Let $x \mapsto \overline{x}$ be the automorphism of order 2 of $\mathbb{F}_{q^2}$, and given $A \in \mathrm{Mat}_{2n}(\mathbb{F}_{q^2})$, we let $\overline{A}$ be the matrix obtained from A by applying this automorphism to each entry. We realize the group $\mathrm{SU}_{2n}(\mathbb{F}_q)$ as follows:

$$\mathrm{SU}_{2n}(\mathbb{F}_q) = \{A \in \mathrm{SL}_{2n}(\mathbb{F}_{q^2}) \ : \overline{A}^{\mathrm{tr}} J A = J\}.$$

The group $\mathrm{SU}_{2n}(\mathbb{F}_q)$ admits the following C_4-grading $\{X_\gamma\}$. As in the previous case, below $1 \leq i < j \leq 4$, $\bar{i} = 9 - i$, $\bar{j} = 9 - j$, $k_1 = k_2 = k_3 = k$ and $k_4 = l$. We put

$$\begin{aligned} X_{e_i - e_j} &= \{\mathrm{Id}_{2n} + (A)_{i,j} - (\overline{A}^{\mathrm{tr}})_{\bar{j},\bar{i}} : \ A \in \mathrm{Mat}_{k_i \times k_j}(\mathbb{F}_{q^2})\} \\ X_{e_i + e_j} &= \{\mathrm{Id}_{2n} + (A)_{i,\bar{j}} - (\overline{A}^{\mathrm{tr}})_{j,\bar{i}} : \ A \in \mathrm{Mat}_{k_i \times k_j}(\mathbb{F}_{q^2})\} \\ X_{2e_i} &= \{\mathrm{Id}_{2n} + (A)_{i,\bar{i}} : \ A = -\overline{A}^{\mathrm{tr}} \in \mathrm{Mat}_{k_i \times k_i}(\mathbb{F}_{q^2})\} \end{aligned}$$

The negative root subgroups can be obtained by the formulas $X_{-\gamma} = (X_\gamma)^{\mathrm{tr}}$.

This grading is the coarsened grading corresponding to the reduction $\eta : C_n \to C_4$ given by $\eta(e_i) = e_{\mathrm{block(i)}}$ for $1 \leq i \leq n$.

Let $R = \mathrm{Mat}_k(\mathbb{F}_{q^2})$, $Z = \mathrm{Id}_l \in R$, let $\tau : R \to R$ be the involution given by $\tau(A) = \overline{A}^{\mathrm{tr}}$, and let

$$I = \{A \in R : \ A = -\tau(A)\}.$$

By Hilbert's theorem 90, any $\alpha \in \mathbb{F}_{q^2}$ satisfying $\overline{\alpha} = -\alpha$ is equal to $\overline{\beta} - \beta$ for some $\beta \in \mathbb{F}_{q^2}$, which implies that $I = \mathrm{Asym}^{\min}(R, \tau)$.

Let the ring $\widehat{R}$, the involution $*$, the form parameter $\widehat{I}$ and the group G_D be defined as in Case 3. Since $k \geq 3$ be assumption, R can be generated by two hermitian (that is, τ-invariant) matrices. Hence there exists an involution preserving epimorphism $\pi : \widehat{R} \to R$. By construction, $\pi(\widehat{I}) = I$, whence as in the previous case $\mathrm{SU}_{2n}(\mathbb{F}_q)$ is a quotient of G_D.

Case 5: $\mathbf{\Phi} = B$, *q is odd,* and **Case 6: $\mathbf{\Phi} = {}^2A_{even}$.** There exists a group G_B with property (T) which maps onto $\Omega_{2n+1}(\mathbb{F}_q)$ for $n \geq 36$ and q odd and onto $\mathrm{SU}_{2n+1}(\mathbb{F}_q)$ for $n \geq 36$.

We treat these two cases simultaneously because the arguments are almost identical. The following notations will have different meanings in cases 5 and 6. Let q be a prime power, which we assume to be odd in case 5. In case 5 we let $F = \mathbb{F}_q$ and $x \mapsto \bar{x}$ be the identity map on F, and in case 6 we let $F = \mathbb{F}_{q^2}$ and $x \mapsto \overline{x}$ the automorphism of F of order 2. In both cases, given a matrix A with entries in F, we denote by $\overline{A}$ the matrix obtained from A by applying the map $x \mapsto \overline{x}$ to each entry and we put $A^\tau = (\overline{A})^{\mathrm{tr}}$. Finally, we let $G(n,q) = \Omega_{2n+1}(\mathbb{F}_q)$ in case 5 and $G(n,q) = \mathrm{SU}_{2n+1}(\mathbb{F}_q)$ in case 6.

Since $n \geq 36$, it is easy to see that $n = 3k + \frac{l-1}{2}$ where $l \leq k \leq 2l$. We shall consider $(2n+1) \times (2n+1)$ matrices as 7×7 block matrices of type $(k_1, k_2, k_3, k_4, k_5, k_6, k_7) = (k, k, k, l, k, k, k)$. For $1 \leq i \leq 7$ we put $\bar{i} = 8 - i$.

Let

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_l & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_k & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \sum_{i=1}^{7} (\mathrm{Id}_{k_i})_{i,\bar{i}} \in \mathrm{Mat}_{2n+1}(\mathbb{F}_q).$$

Then (in both cases 5 and 6) the group $G(n,q)$ has the following realization:

$$G(n,q) = \{A \in SL_{2n+1}(\mathbb{F}_q) \ : A^\tau J A = J\}.$$

Define a BC_3-grading $\{X_\gamma\}_{\gamma \in BC_3}$ of the group $G(n,q)$ as follows. Let $R = \mathrm{Mat}_k(F)$, $Z = \mathrm{Id}_l \in R$, and recall that $RZ = \mathrm{Mat}_{k \times l}(F)$. Define the set $P(R, \tau, RZ)$ as in Example 2 of § 8. The positive root subgroups are given by

$$\begin{aligned} X_{e_i - e_j} =& \{\mathrm{Id}_{2n+1} + (A)_{i,j} - (A^\tau)_{\bar{j},\bar{i}} : \ A \in R\} \\ X_{e_i + e_j} =& \{\mathrm{Id}_{2n+1} + (A)_{i,\bar{j}} - (A^\tau)_{j,\bar{i}} : \ A \in R\} \\ X_{e_i} =& \{\mathrm{Id}_{2n+1} + (A)_{i,4} - (A^\tau)_{4,\bar{i}} + (B)_{i,\bar{i}} : \ (A, B) \in P(R, \tau, RZ)\} \\ X_{2e_i} =& \{\mathrm{Id}_{2n+1} + (B)_{i,\bar{i}} : \ B = -B^\tau \in R\}, \end{aligned}$$

where $1 \leq i < j \leq 3$ and $\bar{x} = 8 - x$. The negative root subgroups are given by $X_{-\gamma} = X_\gamma^{\mathrm{tr}}$.

This grading is the coarsened grading corresponding to the reduction $\eta : B_n \to BC_3$ in Case 5 and $\eta : B_n \to BC_3$ in Case 6 given by $\eta(e_i) = e_{\mathrm{block(i)}}$ for $1 \leq i \leq 3k$ and $\eta(e_i) = 0$ for $3k + 1 \leq i \leq 3k + \frac{l-1}{2} = n$.

Now let $\widehat{R} = R_{main}[a]$, and let $*$ be the involution of $\widehat{R}$ which fixes the canonical generators of R_{main} and sends a to $1-a$. Let G_B be the subgroup of $\mathrm{EL}_8(\widehat{R})$ generated by the subgroups $\{\widehat{X}_\gamma\}_{\gamma \in BC_3}$ which are defined as the corresponding subgroups $\{X_\gamma\}_{\gamma \in BC_3}$ with R, τ and $P(R, \tau, RZ)$ replaced by $\widehat{R}$, $*$ and $P(\widehat{R}, *, \widehat{R}z)$, respectively.

It is clear from the definition that the group G_B is a quotient of the Steinberg group $\mathrm{St}_{BC_3}(\widehat{R}, *, \widehat{R}z)$ and thus has property (T) by Proposition 8.10.

Thus it remains to prove that $G(n,q)$ is a quotient of G_B, for which it suffices to find an involution preserving epimorphism $\pi : \widehat{R} \to R$ such that

(i) $\pi(\widehat{R}z) = RZ$
(ii) $\pi(\mathrm{Asym}(\widehat{R}, *)) = \mathrm{Asym}(R, \tau)$
(iii) $\pi(P(\widehat{R}, *, \widehat{R}z)) = P(R, \tau, RZ)$ where $\pi((a,b)) = (\pi(a), \pi(b))$.

Choose τ-invariant matrices X and Y which generate R, and choose $\alpha \in F$ such that $\alpha + \overline{\alpha} = 1$ and $\alpha \notin \mathbb{F}_q$ in Case 6. In case 5 we simply set $\alpha = 1/2$, and in case 6 such α exists since the trace map $\mathbb{F}_{q^2} \to \mathbb{F}_q$ is surjective. Now define the epimorphism $\pi : \widehat{R} \to R$ by setting $\pi(x) = X$, $\pi(y) = Y$, $\pi(z) = Z$, $\pi(w) = W$ and $\pi(a) = \alpha$.

By construction, π is involution preserving and satisfies (i). It is also clear that $\pi(\mathrm{Asym}^{\min}(\widehat{R}, *)) = \mathrm{Asym}^{\min}(R, \tau)$. On the other hand, $\mathrm{Asym}^{\min}(\widehat{R}, *) = \mathrm{Asym}(\widehat{R}, *)$

and $\mathrm{Asym}^{\min}(R,\tau) = \mathrm{Asym}(R,\tau)$ by Lemma 8.9(1), which implies (ii). Finally, (iii) follows from (i),(ii) and Lemma 8.9(2).

Case 7: $\mathbf{\Phi} = {}^2D_n$. There exists a group $G_{^2D}$ with property (T) which maps onto $\Omega_{2n}^-(\mathbb{F}_q)$ for $n \geq 19$.

Since $n \geq 19$, we can write $n = 3k + l + 1$, where $l \leq k \leq 2l$. We will define $\mathrm{O}_{2n}^-(\mathbb{F}_q)$ as a subgroup of $\mathrm{O}_{2n}^+(\mathbb{F}_{q^2})$. Let $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $x \mapsto \overline{x}$ the automorphism of $\mathbb{F}_{q^2}$ of order 2. Let

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{F}_q)$$

and

$$T = \begin{pmatrix} \mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M + M^{\mathrm{tr}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{Id}_k \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{F}_q).$$

Recall that $\mathrm{O}_{2n}^+(\mathbb{F}_{q^2})$ is realized as the group of matrices $A \in \mathrm{GL}_{2n}(\mathbb{F}_{q^2})$ that fix the quadratic form $q(u,u) = u^{\mathrm{tr}} S u$, where $u \in \mathrm{Mat}_{2n\times 1}(\mathbb{F}_{q^2}) \cong \mathbb{F}_{q^2}^{2n}$. The group $\mathrm{O}_{2n}^-(\mathbb{F}_q)$ can be defined as follows:

$$O_{2n}^-(\mathbb{F}_q) = \{B \in O_{2n}^+(\mathbb{F}_{q^2}) : \ BT = T\overline{B}\}.$$

We shall consider $2n \times 2n$ matrices as 9×9 block matrices of type

$$(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9) = (k, k, k, l, 2, l, k, k, k).$$

Recall the following notation introduced in Example 3 of § 8 just before Proposition 8.25: If R is a ring with 1, $*$ an involution on R and σ an automorphism of R of order ≤ 2 commuting with $*$, then for any additive subgroups $S, I, J \subseteq R$ we put

$$Q(S, *, \sigma, I, J) = \{(r,t) : r \in I, t \in J \text{ and } t - r\sigma(r^*) \in S^\sigma\}$$

Now define

$$\Omega(R, *, \sigma, I, J) = \left\{(a,b,c) : \ a = (v, \sigma(v)),\ c = \begin{pmatrix} -\sigma(v^*) \\ -v^* \end{pmatrix}, (v, b + v\sigma(v^*)) \in Q(R, *, \sigma, I, J)\right\}$$

As in Case 3, for $m \in \mathbb{N}$ we denote by $\mathrm{Asym}_0(m, \mathbb{F}_q)$ the set of antisymmetric matrices in $\mathrm{Mat}_m(\mathbb{F}_q)$ with zeroes on the diagonal, and we put

$$\Omega_m = \left\{ (A,B,C) : \begin{array}{c} A = (V, \overline{V}), C = \begin{pmatrix} -\overline{V}^{\mathrm{tr}} \\ -V^{\mathrm{tr}} \end{pmatrix}, V \in \mathrm{Mat}_{m\times 1}(\mathbb{F}_{q^2}), \\ B \in \mathrm{Mat}_m(\mathbb{F}_q), B + V\overline{V}^{\mathrm{tr}} \in \mathrm{Asym}_0(m, \mathbb{F}_{q^2}) \end{array} \right\}.$$

The group $\Omega_{2n}^-(\mathbb{F}_q)$ has the following BC_4-grading $\{X_\gamma\}$. The positive root subgroups are described by

$$\begin{aligned}
X_{e_i-e_j} =&\{I+(A)_{i,j}-(A^{\mathrm{tr}})_{\bar{j},\bar{i}}:\ A\in \mathrm{Mat}_{k_i\times k_j}(\mathbb{F}_q)\}\\
X_{e_i+e_j} =&\{I+(A)_{i,\bar{j}}-(A^{\mathrm{tr}})_{j,\bar{i}}:\ A\in \mathrm{Mat}_{k_i\times k_j}(\mathbb{F}_q)\}\\
X_{e_i} =&\{I+(A)_{i,5}+(B)_{i,\bar{i}}+(C)_{5,\bar{i}}:\ (A,B,C)\in\Omega_{k_i})\}\\
X_{2e_i} =&\{I+(B)_{i,\bar{i}}:\ B\in \mathrm{Asym}_0(k_i,q)\}
\end{aligned}$$

where $1\le i<j\le 4$, $\bar{x}=10-x$. The negative root subgroups are given by $X_{-\gamma}=X_\gamma^{\mathrm{tr}}$.

This grading is the coarsened grading corresponding to the reduction $\eta: BC_n\to BC_4$ given by $\eta(e_i)=e_{\mathrm{block(i)}}$ for $1\le i\le n-1$ and $\eta(e_n)=0$.

Now let $R=\mathrm{Mat}_k(\mathbb{F}_{q^2})$. Note that the conjugation map conj : $R\to R$ given by $\mathrm{conj}(A)=\overline{A}$ is an automorphism of order 2 which commutes with the involution tr : $R\to R$. Let $U=E_{11}\in R$ and $Z=\mathrm{Id}_l\in R$. Observe that

$$\begin{aligned}
\mathrm{Mat}_{k\times 1}(\mathbb{F}_{q^2}) =&RU, & \mathrm{Mat}_{k\times l}(\mathbb{F}_{q^2}) =&RZ,\\
\mathrm{Mat}_{l\times k}(\mathbb{F}_{q^2}) =&ZR, & \mathrm{Mat}_{l\times l}(\mathbb{F}_{q^2}) =&ZRZ,\\
\mathrm{Mat}_{k\times k}(\mathbb{F}_{q}) =&R^{\mathrm{conj}}, & \mathrm{Mat}_{k\times l}(\mathbb{F}_{q}) =&R^{\mathrm{conj}}Z,\\
\mathrm{Mat}_{l\times k}(\mathbb{F}_{q}) =&ZR^{\mathrm{conj}}, & \mathrm{Mat}_{l\times l}(\mathbb{F}_{q}) =&ZR^{\mathrm{conj}}Z
\end{aligned}$$

$$\begin{aligned}
\mathrm{Asym}_0(k,q^2) =& \mathrm{Asym}^{\min}(R,\mathrm{tr})\\
\mathrm{Asym}_0(k,q) =&(\mathrm{Asym}^{\min}(R,\mathrm{tr}))^{\mathrm{conj}}\\
\mathrm{Asym}_0(l,q^2) =& \mathrm{Asym}^{\min}(R,\mathrm{tr})\cap ZRZ\\
\mathrm{Asym}_0(l,q) =&(\mathrm{Asym}^{\min}(R,\mathrm{tr})\cap ZRZ)^{\mathrm{conj}}\\
\Omega_k =&\Omega(R,\mathrm{tr},\mathrm{conj},RU,\mathrm{Asym}^{\min}(R,\mathrm{tr}))\\
\Omega_l =&\Omega(ZRZ,\mathrm{tr},\mathrm{conj},ZRU,\mathrm{Asym}^{\min}(R,\mathrm{tr}))
\end{aligned}$$

Therefore, the above definition of the positive root subgroups can be rewritten as follows:

$$\begin{aligned}
X_{e_i-e_j} =&\{I+(A)_{i,j}-(A^{\mathrm{tr}})_{\bar{j},\bar{i}}:\ A\in R^{\mathrm{conj}}\}\text{ for } 1\le i<j\le 3\\
X_{e_i-e_4} =&\{I+(A)_{i,4}-(A^{\mathrm{tr}})_{6,\bar{i}}:\ A\in R^{\mathrm{conj}}Z\}\text{ for } 1\le i\le 3\\
X_{e_i+e_j} =&\{I+(A)_{i,\bar{j}}-(A^{\mathrm{tr}})_{j,\bar{i}}:\ A\in R^{\mathrm{conj}}\}\text{ for } 1\le i<j\le 3\\
X_{e_i+e_4} =&\{I+(A)_{i,6}-(A^{\mathrm{tr}})_{4,\bar{i}}:\ A\in R^{\mathrm{conj}}Z\}\text{ for } 1\le i\le 3\\
X_{e_i} =&\{I+(A)_{i,5}+(B)_{i,\bar{i}}+(C)_{5,\bar{i}}:\ (A,B,C)\in\Omega(R,\mathrm{tr},\mathrm{conj},RU,\mathrm{Asym}^{\min}(R,\mathrm{tr}))\}\\
&\text{for } 1\le i\le 3\\
X_{2e_i} =&\{I+(B)_{i,\bar{i}}:\ B\in(\mathrm{Asym}^{\min}(R,\mathrm{tr}))^{\mathrm{conj}}\}\text{ for } 1\le i\le 3\\
X_{e_4} =&\{I+(A)_{4,5}+(B)_{4,6}+(C)_{5,6}:\ (A,B,C)\in\Omega(ZRZ,\mathrm{tr},\mathrm{conj},ZRU,\mathrm{Asym}^{\min}(R,\mathrm{tr}))\}\\
X_{2e_4} =&\{I+(B)_{4,6}:\ B\in(\mathrm{Asym}^{\min}(R,\mathrm{tr})\cap ZRZ)^{\mathrm{conj}}\}
\end{aligned}$$

Now, we will define the covering group $G_{^2D}$. Let $\widehat{R}_0$ be the quotient of the ring $R_{main}\langle u\rangle$ by the ideal generated by $zu-uz$,$zu-u$ and z^2-z, and let $\widehat{R}=\widehat{R}_0[a]$. Let $*$ be the involution of $\widehat{R}$ that fixes all 6 variables x,y,z,w,u and a, and let σ be the automorphism of $\widehat{R}$ of order 2 which fixes x,y,z,w,u and sends a to $1-a$. It is easy to show that

$$\widehat{R}^{\sigma}=\widehat{R}_0[a(1-a)].$$

Define $G_{^2D}$ to be the subgroup of $\mathrm{EL}_{10}(\widehat{R})$ generated by the subgroups $\{\widehat{X}_\gamma\}_{\gamma\in BC_4}$ which are defined in the same way as $\{X_\gamma\}$ (in their second description) with R, U, Z, conj and tr replaced by $\widehat{R}$, u, z, σ and $*$, respectively. It is straightforward to check that $G_{{}^2D}$ is a quotient of the Steinberg group $\mathrm{St}^1_{BC_4}(\widehat{R},*,\sigma,I,J;M)$ described in Proposition 8.25 where $I=\widehat{R}u$, $M=\widehat{R}z$ and $J=\mathrm{Asym}^{\min}(\widehat{R},*)$. Let us check that conditions (i)-(viii) of Proposition 8.25 hold. Condition (ii) holds by construction, and (i) and (iii) are clear from the above description of $\widehat{R}^{\sigma}$. Condition (viii) holds by Observation 8.26 thanks to the relation $z^2=z$. The latter also implies that $M^\sigma = M^\sigma z \subseteq \widehat{R}^{\sigma} z \subseteq (\widehat{R}z)^\sigma = M^\sigma$, whence $M^\sigma=\widehat{R}^{\sigma}z$, so (v) is satisfied. Condition (vi) holds as in all previous cases, and (vii) is automatic since M^σ is a principal ideal of $\widehat{R}^{\sigma}$, as we just verified.

It remains to check (iv), for which it will suffice to prove that $(\mathrm{Asym}^{\min}(\widehat{R},*))^\sigma = \mathrm{Asym}^{\min}(\widehat{R}^{\sigma},*)$. The inclusion $\mathrm{Asym}^{\min}(\widehat{R}^{\sigma},*)\subseteq(\mathrm{Asym}^{\min}(\widehat{R},*))^\sigma$ is clear. For the reverse inclusion we will use the following fact which follows easily from the above description of $\widehat{R}^{\sigma}$:

(*) Every $r\in\widehat{R}$ can be uniquely written as $r=r_1+ar_2$ with $r_1,r_2\in\widehat{R}^{\sigma}$.

Now take any $r\in(\mathrm{Asym}^{\min}(\widehat{R},*))^\sigma$. Thus, $r\in\widehat{R}^{\sigma}$ and $r=t^*-t$ for some $t\in\widehat{R}$. Write $t=c+ad$ with $c,d\in\widehat{R}^{\sigma}$. Then we have $r=c^*-c+a(d^*-d)$. Since c^*-c, d^*-d and r all lie in $\widehat{R}^{\sigma}$, by (*) we must have $d^*=d$, whence $r=c^*-c\in\mathrm{Asym}^{\min}(\widehat{R}^{\sigma},*)$.

Thus, we verified conditions (i)-(viii), so $G_{{}^2D}$ has property (T).

It remains to show that $\mathrm{O}^-_{2n}(\mathbb{F}_q)$ is a quotient of $G_{{}^2D}$ for which it suffices to construct an epimorphism $\pi:\widehat{R}\to R$ satisfying the following conditions:

(1) $\pi(r^*)=\pi(r)^{\mathrm{tr}}$ for all $r\in\widehat{R}$
(2) $\pi(\sigma(r))=\overline{\pi(r)}$ for all $r\in\widehat{R}$
(3) $\pi(\widehat{R}^{\sigma})=R^{\mathrm{conj}}$
(4) $\pi(z)=Z$ and $\pi(u)=U$
(5) $\pi((\mathrm{Asym}^{\min}(\widehat{R},*))^\sigma)=(\mathrm{Asym}^{\min}(R,\mathrm{tr}))^{\mathrm{conj}}$
(6) $\pi((\mathrm{Asym}^{\min}(\widehat{R},*)\cap z\widehat{R}z)^\sigma)=(\mathrm{Asym}^{\min}(R,\mathrm{tr})\cap ZRZ)^{\mathrm{conj}}$

Choose symmetric matrices $X,Y\in\mathrm{Mat}_k(\mathbb{F}_q)$ which generate $\mathrm{Mat}_k(\mathbb{F}_q)$ as a ring, and choose an element $\alpha\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$ such that $\alpha+\overline{\alpha}=1$. Let $\pi:\widehat{R}\to R$ be the homomorphism given by $\pi(x)=X$, $\pi(y)=Y$, $\pi(z)=Z$, $\pi(u)=U$, $\pi(w)=W$ and $\pi(a)=\alpha$. Since $R=\mathrm{Mat}_k(\mathbb{F}_q)+\alpha\,\mathrm{Mat}_k(\mathbb{F}_q)$, π is surjective, and conditions (1)-(4) trivially hold. Condition (5) follows from the equality $(\mathrm{Asym}^{\min}(\widehat{R},*))^\sigma=\mathrm{Asym}^{\min}(\widehat{R}^{\sigma},*)$ and the analogous equality $(\mathrm{Asym}^{\min}(R,\mathrm{tr}))^{\mathrm{conj}}=\mathrm{Asym}^{\min}(R^{\mathrm{conj}},\mathrm{tr})$, which is checked by direct calculation, with both sides equal to $\mathrm{Asym}_0(k,q)$. Finally, to prove (6) first note that by (1)-(4), we have the inclusions

$$\begin{aligned}\mathrm{Asym}^{\min}(ZR^{\mathrm{conj}}Z,\mathrm{tr}) &= \pi((\mathrm{Asym}^{\min}(z\widehat{R}^{\sigma}z),*))\\ &\subseteq \pi((\mathrm{Asym}^{\min}(\widehat{R},*)\cap z\widehat{R}z)^\sigma)\subseteq(\mathrm{Asym}^{\min}(R,\mathrm{tr})\cap ZRZ)^{\mathrm{conj}},\end{aligned}$$

so it remains to check that $(\mathrm{Asym}^{\min}(R,\mathrm{tr})\cap ZRZ)^{\mathrm{conj}}=\mathrm{Asym}^{\min}(ZR^{\mathrm{conj}}Z,\mathrm{tr})$. Again by direct verification both sets are equal to $\mathrm{Asym}_0(l,q)$. This completes the proof.

Proof of Lemma 9.6. Let a be a generator of $F^\times$, the multiplicative group of F. An easy computation shows that the matrices aE_{11} and $\sum_{i=1}^{k-1}(E_{i,i+1}+E_{i+1,i})$ generate $\mathrm{Mat}_k(F)$.

Now assume that $F=\mathbb{F}_{q^2}$. We will show that $A=aE_{1,2}+\sigma(a)E_{2,1}$ and $B=\sum_{i=1}^{k-1}E_{i+1,i}+E_{i,i+1}$ generate $\mathrm{Mat}_k(F)$. Let R be the subring generated by A and B. Since a generates $\mathbb{F}_{q^2}^\times$, $a\sigma(a)$ generates $\mathbb{F}_q^\times$ and so the subring generated by the matrix A^2 contains $\alpha(E_{1,1}+E_{2,2})$ for every $\alpha\in\mathbb{F}_q$. Thus

$$E_{1,1}=2(E_{1,1}+E_{2,2})-(E_{1,1}+E_{2,2})B^2(E_{1,1}+E_{2,2})\in R \text{ and } E_{2,2}\in R.$$

Hence for any $\alpha \in \mathbb{F}_q$,

$$\alpha E_{1,2} = \alpha(E_{1,1} + E_{2,2})E_{1,1}BE_{2,2} \in R \text{ and } \ \alpha a E_{1,2} = \alpha(E_{1,1} + E_{2,2})AE_{2,2} \in R.$$

Hence $\gamma E_{1,2} \in R$ for every $\gamma \in \mathbb{F}_{q^2}$. Similarly $\gamma E_{2,1} \in R$ for every $\gamma \in \mathbb{F}_{q^2}$. Therefore $\gamma E_{1,1}$ and $\gamma E_{2,2} \in R$ for every $\gamma \in \mathbb{F}_{q^2}$.

Now, by induction on k we show that $E_{i,k}, E_{k,i} \in R$ for any $i \leq k \leq n$. This clearly will finish the proof. The base of induction $k \leq 2$ is already established and the inductive step follows from the following equalities.

$$E_{k-1,k} = E_{k-1,k-1}B - E_{k-1,k-2}, \ E_{k,k-1} = BE_{k-1,k-1} - E_{k-2,k-1}.$$

□

9.5. **Groups of type A_1.** We finish this section by proving that any infinite family of groups of the form $\mathrm{PSL}_2(q)$ cannot have a mother group with property (T):

Theorem 9.7. *Let G be group which maps onto $\mathrm{PSL}_2(q)$ for infinitely many q. Then G does not have property (T).*

Proof. By way of contradiction, assume that G has property (T). Write G as F/K, where F is a finitely generated free group and let $\widetilde{G} = F/K^2[K, F]$. Then by Theorem 10.5 $\widetilde{G}$ also has property (T). Since G maps on infinitely many $\mathrm{PSL}_2(q)$ and $\mathrm{SL}_2(q)$ has no subgroups of index 2, it is easy to see that $\widetilde{G}$ maps on infinitely many $\mathrm{SL}_2(q)$. Let $\widetilde{G} = \langle X|R\rangle$ be a presentation of $\widetilde{G}$ with $|X| = n$ finite. Put $X_i = \begin{pmatrix} x^i_{11} & x^i_{12} \\ x^i_{21} & x^i_{22} \end{pmatrix}$ ($i = 1, \dots, n$) and let $B = \mathbb{Z}[x^i_{kl}\ : k, l = 1, 2;\ l = 1, \dots, n]$ and $A = B/I$ where I is the ideal of B generated by

$$x^i_{11}x^i_{22} - x^i_{12}x^i_{21} - 1 \ (i = 1, \dots, n)$$

and the entries of the following matrices

$$r(X_1, \dots, X_n) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for each $r \in R$.

Let $A_i = X_i(\bmod I) \in \mathrm{SL}_2(A)$, and let H be the group generated by $\{A_i\}_{i=1}^n$. It is easy to check that H is a quotient on $\widetilde{G}$, but on the other hand H maps onto any quotient of $\widetilde{G}$ of the form $\mathrm{SL}_2(R)$, with R commutative. In particular, H maps onto infinitely many $\mathrm{SL}_2(q)$.

Let N be the nilradical of A. We claim that the image of H in $\mathrm{SL}_2(A/N)$ is still infinite. Indeed, since A is Noetherian, N is nilpotent, so the kernel of the map $\mathrm{SL}_2(A) \to \mathrm{SL}_2(A/N)$ is also nilpotent. Thus, if the image of H in $\mathrm{SL}_2(A/N)$ was finite, H would be virtually nilpotent and could not map onto infinitely many $\mathrm{SL}_2(q)$.

Again since A is Noetherian, it has finitely many minimal prime ideals (whose intersection is N), so there exists a minimal prime ideal P of A such that the image of H in $\mathrm{SL}_2(A/P)$ is infinite. Now, we can finish the proof using, for instance [GHW, Theorem 4], which asserts that an infinite subgroup of $\mathrm{SL}_2(D)$, with D a commutative domain, has the Haagerup property and so cannot have property (T). The obtained contradiction finishes the proof. □

9.6. **Alternating Groups.** In this subsection we will show that there exists a mother group satisfying property (τ) for the family of alternating groups. The existence of a such group is established by combining ideas from [Ka2] and [KN2].

We start the proof with the following generalization of [Ka1, Lemma 4.2].

Proposition 9.8. *There exists a finitely generated dense subring R in*

$$\prod_{k=3}^{\infty} \mathrm{Mat}_k(\mathbb{F}_2)^{\times 2^{33k}}$$

containing $\bigoplus_{k=3}^{\infty} \mathrm{Mat}_k(\mathbb{F}_2)^{\times 2^{33k}}$.

Proof. The proof is a combination of the ideas from the proof of [Ka2, Lemma 2.1] and [Ka1, Lemma 4.2]. It is clear that it is enough to construct such a subring inside $\prod_{k=33}^{\infty} \mathrm{Mat}_k(\mathbb{F}_2)^{\times 2^{33k}}$. Denote by $R_{k,i}$ $(1 \le i \le 2^{33k})$ the i^{th} copy of the ring $\mathrm{Mat}_k(\mathbb{F}_2)$. Let R be a subring of $\prod_{k\ge 33, 1\le i\le 2^{33k}} R_{k,i}$ generated by the following 5 elements $\mathbf{a} = (a_{k,i})$, $\bar{\mathbf{a}} = ((a_{k,i})^{-1})$, $\mathbf{b} = (b_{k,i})$, $\mathbf{c} = (c_{k,i})$ and $\mathbf{x} = (x_{k,i})$, where

$$a_{k,i} = E_{1,2} + E_{2,3} + \ldots + E_{k,1},\ b_{k,i} = E_{1,2},\ c_{k,i} = E_{2,1}$$

and $\{x_{k,i} :\ 1 \le i \le 2^{33k}\}$ are different elements of $\mathrm{Mat}_k(\mathbb{F}_2)$ (this is possible because $k \ge 33$). Now as in the proof of [Ka1, Lemma 4.2], R is dense in $\prod_{k\ge 33, 1\le i\le 2^{33k}} R_{k,i}$ and contains $\bigoplus_{k\ge 33, 1\le i\le 2^{33k}} R_{k,i}$. □

Corollary 9.9. *There exists a finitely generated dense subgroup G_0 of*

$$\prod_{k\ge 3} \mathrm{SL}_{3k}(\mathbb{F}_2)^{\times 2^{33k}},$$

which contains $\bigoplus_{k\ge 3} \mathrm{SL}_{3k}(\mathbb{F}_2)^{\times 2^{33k}}$ *and has property (τ) with respect to the family of open subgroups.*

Proof. Let R be as in Proposition 9.8. Then from [EJ] we know that $G_0 = \mathrm{EL}_3(R)$ has property (T). This group is clearly dense in

$$\mathrm{EL}_3\left(\prod_{k=3}^{\infty} \mathrm{M}_k(\mathbb{F}_2)^{\times 2^{33k}}\right) \cong \prod_{k\ge 3} \mathrm{SL}_{3k}(\mathbb{F}_2)^{\times 2^{33k}}$$

and contains $\bigoplus_{k\ge 3} \mathrm{SL}_{3k}(\mathbb{F}_2)^{\times 2^{33k}}$. □

Proposition 9.10. *There exists a finitely generated dense subgroup G of $\prod_{n\ge 5} \mathrm{Alt}(n)$ which has property (τ) with respect to the family of open subgroups.*

Proof. Theorem 2 from [Ka2] states:

Theorem 9.11. *For every $n \ge 5$ there exists a generating set X_n of the alternating group $\mathrm{Alt}(n)$ such that: (a) $|X_n| = K$ and (b) $\kappa(\mathrm{Alt}(n); X_n) \ge \varepsilon_0 > 0$, where K and $\varepsilon_0 > 0$ are some explicit constants.*

The proposition cannot be derived from Theorem 9.11, but it follows relatively easily from its proof. Here we will only outline the main points.

The proof of Theorem 9.11 goes as follows. Let $n \ge 10^6$. We choose k_n such that

$$(2^{3k_n} - 1)^6 \le n < (2^{3(k_n+1)} - 1)^6$$

and we put $l_n = (2^{3k_n} - 1)^5$. Then it is shown in [Ka2] that there exists an embedding

$$\varphi_n : G_n = \mathrm{SL}_{3k_n}(\mathbb{F}_2)^{\times l_n} \to \mathrm{Alt}(n).$$

and the elements $\{g_{1,n}, \ldots, g_{C,n}\}$ of $\mathrm{Alt}(n)$ such that the Kazhdan constant

$$\kappa(\mathrm{Alt}(n), B_n) > \varepsilon_1 \tag{9.3}$$

for some $\varepsilon_1 > 0$, where $B_n = \bigcup_i (\varphi_n(G_n))^{g_{i,n}}$. Note that in particular B_n generates $\mathrm{Alt}(n)$. It is essential that the number of conjugates C and ε_1 are independent of n.

Let G_0 be as in Corollary 9.9. Note that the number of different n with the same k_n is at most 2^{18k_n}. Since $2^{18k_n} \cdot l_n \leq 2^{33k_n}$, we can construct a homomorphism $\varphi : G_0 \to \prod_{n\geq 10^6} \mathrm{Alt}(n)$ such that $\varphi(G_0) \cap \mathrm{Alt}(n) = \varphi_n(G_n)$.

Let S_0 be a finite generating set of G_0. Denote by $g_i \in \prod_{n\geq 10^6} \mathrm{Alt}(n)$ $(i = 1, \ldots, C)$ the element whose projection into $\mathrm{Alt}(n)$ is equal to $g_{i,n}$. Let $S = \cup_{i=1}^C \varphi(S_0)^{g_i}$ and let $G = \langle S \rangle$. We want to show that G satisfies the conclusion of Proposition 9.10 in the product $\prod_{n\geq 10^6} \mathrm{Alt}(n)$. This will clearly imply the proposition.

Let $B = \cup_{i=1}^C \varphi(G_0)^{g_i}$. It is clear that $\kappa_r(G, B; S) \geq \kappa_r(G_0, G_0, S_0)$ and $\kappa_r(G_0, G_0, S_0) \geq \frac{1}{2}\kappa(G_0, S_0)$ by Observation 2.2(ii). Since G_0 has (T), we deduce that $\kappa_r(G, B; S) > 0$.

Next observe that since $\varphi(G_0)$ contains $\varphi_n(G_n)$ and $\bigcup_i (\varphi_n(G_n))^{g_{i,n}}$ generates $\mathrm{Alt}(n)$, the group $G = \langle S \rangle$ contains $\bigoplus\limits_{n\geq 10^6} \mathrm{Alt}(n)$. In particular, G is dense in $\prod\limits_{n\geq 10^6} \mathrm{Alt}(n)$.

Let $l \geq 10^6$. Denote by U_l the subgroup $\prod_{n\geq l} \mathrm{Alt}(n)$. In order to finish the proof of the proposition we have to show that Kazhdan constants

$$\kappa(GU_l/U_l, SU_l/U_l)$$

are uniformly bounded from zero. This will follow if we show that the quantities $\kappa(G, S, V)$ are uniformly bounded from zero as V ranges over non-trivial irreducible unitary representations of GU_l/U_l for different l. Fix such a representation V – it is isomorphic to a tensor product $\bigotimes\limits_{i=10^6}^{l-1} V_i$ where each V_i is an irreducible representation of $\mathrm{Alt}(i)$ (which we can also view as a representation of G). Since V is non-trivial, V_i is non-trivial for some i. Since V is isomorphic to a direct sum of several copies of V_i as a representation of $\mathrm{Alt}(i)$, it follows that V does not contain nonzero $\mathrm{Alt}(i)$-invariant vectors. Take any $0 \neq v \in V$. By construction $B \supseteq B_i$, so there exists $g \in B$ such that $\|gv - v\| \geq \varepsilon_1 \|v\|$, where ε_1 is defined by (9.3). Thus there exists $s \in S$ such that $\|sv - v\| \geq \varepsilon_1\varepsilon_2\|v\|$ where $\varepsilon_2 = \kappa_r(G, B; S)$. Since $\varepsilon_2 > 0$ as shown above, we are done. □

Most likely, the group G does not have property (τ), because G might have finite quotients which are not visible via the embedding of G into the product of alternating groups. This complication can be bypassed using the ideas from [KN2]. First we introduce the following important definition.

Definition. Let $\mathcal{S} = \prod_{n=1}^\infty S_n$ be a Cartesian product of finite groups. A finitely generated subgroup G of $\mathcal{S}$ is *a frame for* $\mathcal{S}$ if the following hold:

(a) G contains $\bigoplus_{n=1}^\infty S_n$.

(b) The natural surjection $\widehat{G} \to \mathcal{S}$ is an isomorphism

The following property of frame subgroups was shown in [KN2]

Lemma 9.12 ([KN2, Lemma 8]). *Let A_n, $B_n \leq C_n$ $(n \in N)$ be finite groups with $C_n = \langle A_n, B_n \rangle$. Suppose that A (resp. B) is a frame for $\mathcal{A} = \prod\limits_{n=1}^\infty A_n$ (resp. $\mathcal{B} = \prod\limits_{n=1}^\infty B_n$). Each of A and B can be considered as a subgroup of $\mathcal{C} = \prod\limits_{n=1}^\infty C_n$ in the natural way. Then the group $\langle A, B \rangle$ is a frame for $\mathcal{C}$.*

The following result is a consequence of Proposition 13 of [KN2].

Proposition 9.13. *There exists a finitely generated frame subgroup H for $\prod\limits_{n\geq 5} \mathrm{Alt}(n)$.*

We will need a slight improvement of this proposition. Its proof will use the following lemma.

Lemma 9.14. *Let X be a set with k elements which generates a dense subgroup of $\prod\limits_{n\geq 5}\mathrm{Alt}(n)$. Then every $\mathbf{b}=(b_n)\in\prod_{n\geq 5}\mathrm{Alt}(n)$ is a product of $16k^2$ conjugates (in $\prod_{n\geq 5}\mathrm{Sym}(n)$) of elements from X.*

Proof. Let $X=\{\mathbf{x}_i=(x_{i,n}):\ i=1,\dots,k\}$. In order to prove the lemma it is enough to show that the following equations have solutions in $\{z_{i,j,n}: 1\leq i\leq k, 1\leq j\leq 16k, n\geq 5\}$:

$$x_{1,n}^{z_{1,1,n}}\cdot\ldots\cdot x_{1,n}^{z_{1,16k,n}}\cdot\ldots\cdot x_{k,n}^{z_{k,1,n}}\cdot\ldots\cdot x_{k,n}^{z_{k,16k,n}}=b_n.$$

Since $\{x_{i,n}:\ 1\leq i\leq |X|\}$ is a generating set of $\mathrm{Alt}(n)$ at least one of the elements $x_{i,n}$ moves at least $\frac{n}{k}$ points. Thus, the last claim follows from

Claim 9.15. *Let $n\geq 5$ and let $h\in\mathrm{Alt}(n)$ be a permutation that moves at least $\frac{n}{k}$ points. Then every $g\in\mathrm{Alt}(n)$ is a product of exactly $16k$ conjugates of h.*

Proof. Let $l=\max\{5,|\mathrm{supp}\ h|\}$. Without loss of generality we may assume that the support of h is contained in $\{1,\dots,l\}$. Let $\mathcal{K}$ be the conjugacy class of h. In [Br, Theorem 3.05] it is shown that $\mathrm{Alt}(l)\subseteq\mathcal{K}^4$. Hence, since $l\geq\frac{n}{k}$, the set $\mathcal{K}^{4k}$ contains an element without fixed points. Applying [Br, Theorem 3.05] again, we obtain that $\mathrm{Alt}(n)=\mathcal{K}^{16k}$. □

Corollary 9.16. *For any finite set $B\subset\prod\limits_{n\geq 5}\mathrm{Alt}(n)$ there exist a finitely generated frame subgroup T for $\prod\limits_{n\geq 5}\mathrm{Alt}(n)$ containing B.*

Proof. Let X be the generating set of the group H from Proposition 9.13. By Lemma 9.14 one can find finitely many elements $g_t\in\prod_{n\geq 5}\mathrm{Sym}(n)$ such that every element of the set B can be expressed as a product of at most C conjugates of elements from X by the elements g_t (where C is an absolute constant). Let T denote the subgroup of $\prod\mathrm{Alt}(n)$ generated by the subgroups $\{H^{g_t}\}$. Applying (possibly several times) Lemma 9.12, we obtain that T is also a frame. □

Using this corollary, we can finally prove Theorem 9.4 restated below:

Theorem 9.17. *There exists a group Γ with property (τ) which surjects onto all alternating groups.*

Proof. It suffices to apply Corollary 9.16 to the generating set of the group G from Proposition 9.10. The resulting group Γ contains G and therefore has property (τ) with respect to the family of open subgroups. However, since Γ is a frame, every finite index subgroup of Γ is open, that is, Γ has property (τ) with respect to all finite index subgroups. □

10. Estimating relative Kazhdan constants

The goal of this section is to prove Theorems 2.7 and 2.8 from § 2. For convenience we shall use the following terminology and notations:

- A unitary representation V of a group G will be referred to as a *G-space.*
- If U is a subspace of V, by P_U we denote the operator of orthogonal projection onto U. For any nonzero $v\in V$ we set $P_v=P_{\mathbb{C}v}$.

10.1. **Hilbert-Schmidt scalar product.** Consider the space $HS(V)$ of Hilbert-Schmidt operators on V, i.e., linear operators $A : V \to V$ such that $\sum_i \|A(e_i)\|^2$ is finite where $\{e_i\}$ is an orthonormal basis of V. The space $HS(V)$ is endowed with the *Hilbert-Schmidt scalar product* given by

$$\langle A, B\rangle = \sum_i \langle A(e_i), B(e_i)\rangle.$$

By a standard argument this definition does not depend on the choice of $\{e_i\}$. The associated norm on $HS(V)$ will be called the *Hilbert-Schmidt norm.*

If V is a unitary representation of a group G then $HS(V)$ is also a unitary representation of G – the action of an element $g \in G$ on an operator $A \in HS(V)$ is defined by $(gA)(v) = gA(g^{-1}v)$. If the element g acts by a scalar on V, for instance if g is in the center of G and V is an irreducible representation, then g acts trivially on $HS(V)$.

For any unit vector $v \in V$ the projection $P_v : V \to V$ is an element in $HS(V)$ of norm 1. The map $v \to P_v$ does not preserve the scalar product. However, we have the following explicit formula for $\langle P_u, P_v\rangle$.

Lemma 10.1. *If u and v are unit vectors in a Hilbert space V, then*

$$\langle P_u, P_v\rangle = |\langle u, v\rangle|^2 \quad \text{and therefore} \quad \|P_u - P_v\| \le \sqrt{2}\|u - v\|$$

Proof. Choose any orthonormal basis $\{e_i\}$ such that $e_1 = u$. Then

$$\langle P_u, P_v\rangle = \sum_i \langle P_u(e_i), P_v(e_i)\rangle = \langle u, \langle u, v\rangle v\rangle = |\langle u, v\rangle|^2.$$

Therefore,

$$\begin{aligned}\|P_u - P_v\|^2 = 2(1 - |\langle u, v\rangle|^2) &\le 2(1 + |\langle u, v\rangle|)(1 - \mathrm{Re}\langle u, v\rangle)\\ &= (1 + |\langle u, v\rangle|)\|u - v\|^2 \le 2\|u - v\|^2. \quad \square\end{aligned}$$

One can define a non-linear, norm preserving, map $\iota : V \to HS(V)$ by

$$\iota(v) = \|v\|P_v$$

The following lemma imposes a restriction on the change of codistance between vectors under the map ι.

Lemma 10.2. *Let $v_1, \dots, v_k$ be vectors in V. Then*

$$2\,\mathrm{codist}(v_1, v_2, \dots, v_k) - 1 \le \mathrm{codist}\,(\iota(v_1), \iota(v_2), \dots, \iota(v_k))$$

where $\mathrm{codist}(u_1, \dots, u_k)$ *denotes the ratio* $\dfrac{\|\sum u_i\|^2}{k\sum \|u_i\|^2}$.

Proof. The inequality $\cos^2\varphi \ge 2\cos\varphi - 1$ implies that

$$\langle \iota(v), \iota(w)\rangle = \|v\|\|w\| \left|\left\langle \frac{v}{\|v\|}, \frac{w}{\|w\|}\right\rangle\right|^2 \ge 2\langle v, w\rangle - \|v\|\|w\|$$

Therefore

$$\left\|\sum \iota(v_i)\right\|^2 = \sum_{i,j}\langle \iota(v_i), \iota(v_j)\rangle \ge$$

$$\sum_{i,j}(2\langle v_i, v_j\rangle - \|v_i\|\|v_j\|) = 2\left\|\sum v_i\right\|^2 - \left(\sum \|v_i\|\right)^2.$$

Since $\|\iota(v_i)\| = \|v_i\|$, we get

$$\begin{aligned}\mathrm{codist}\,(\iota(v_1), \iota(v_2), \dots, \iota(v_k)) &= \frac{\|\sum \iota(v_i)\|^2}{k\sum\|v_i\|^2}\\ &\ge 2\frac{\|\sum v_i\|^2}{k\sum\|v_i\|^2} - \frac{(\sum\|v_i\|)^2}{k\sum\|v_i\|^2} \ge 2\,\mathrm{codist}(v_1, \dots, v_k) - 1.\end{aligned}$$

which translates into the stated inequality between codistances. □

Let $\{(U_i, \langle\ ,\ \rangle_i)\}_{i\in I}$ be a family of Hilbert spaces. Recall that the Hilbert direct sum of U_i's denoted by $\oplus_{i\in I} U_i$ is the Hilbert space consisting of all families $(u_i)_{i\in I}$ with $u_i \in U_i$ such that $\sum_i \langle u_i, u_i\rangle_i < \infty$ with inner product

$$\langle (u_i), (w_i)\rangle = \sum_i \langle u_i, w_i\rangle_i.$$

Let V be a unitary representation of G and let N be a subgroup of G. Denote by $(\hat{N})_f$ the set of equivalence classes of irreducible finite dimensional representations of N. Let $\pi \in (\hat{N})_f$. Denote by $V(\pi)$ the N-subspace of V g spanned by all irreducible N-subspaces of V isomorphic to π. By Zorn' s Lemma, $V(\pi)$ is isomorphic to a Hilbert direct sum of N-spaces isomorphic to π. We may also decompose V as a Hilbert direct sum $V = V_\infty \oplus (\oplus_{\pi\in(\hat{N})_f} V(\pi))$, where V_∞ is the orthogonal complement of $\oplus_{\pi\in(\hat{N})_f} V(\pi)$ in V.

Lemma 10.3. *Let $v \in V$ be a unit vector and $v = v_\infty + \sum_{\pi\in(\hat{N})_f} v_\pi$ the decomposition of v such that $v_\infty \in V_\infty$ and $v_\pi \in V(\pi)$. Then*

$$\|P_{HS(V)^N}(P_v)\|^2 \le \sum_{\pi\in(\hat{N})_f} \frac{\|v_\pi\|^4}{\dim \pi}$$

(where the norm on the left-hand side is the Hilbert-Schmidt norm). Moreover, if V is an irreducible N-space, then $\|P_{HS(V)^N}(P_v)\|^2 = \frac{1}{\dim V}$.

Proof. Let $T \in HS(V)^N$. Then T preserves the decomposition $V = V_\infty \oplus (\oplus_{\pi\in(\hat{N})_f} V(\pi))$. Moreover, by Proposition A.1.12 of [BHV] T maps V_∞ to zero. Hence we have a decomposition

$$HS(V)^N = \oplus_{\pi\in(\hat{N})_f} HS(V)^{N,\pi}$$

where $HS(V)^{N,\pi}$ is the subspace of operators from $HS(V)^N$ which map the orthogonal complement of $V(\pi)$ to zero. Thus, we may write $T = \sum_{\pi\in(\hat{N})_f} T_\pi$ where $T_\pi \in HS(V)^{N,\pi}$, and T_{π_1} and T_{π_2} are orthogonal for non-isomorphic π_1 and π_2. Note also that

$$\|P_{HS(V)^N}(P_v)\|^2 = \sum_{\pi\in(\hat{N})_f} \|P_{HS(V)^{N,\pi}}(P_v)\|^2. \tag{10.1}$$

Now fix $\pi \in (\hat{N})_f$, and decompose $V(\pi)$ as a Hilbert direct sum $\oplus_{i\in I} U_i$ of (pairwise orthogonal) N-spaces $\{U_i\}$ each of which is isomorphic to π. Note that

$$HS(V)^{N,\pi} = \oplus_{i,j\in I} HS(V)^{N,\pi}_{i,j}$$

where $HS(V)^{N,\pi}_{i,j}$ is the subspace of operators from $HS(V)^{N,\pi}$ which map U_i onto U_j and map the orthogonal complement of U_i to zero. A standard application of Schur's lemma shows that each subspace $HS(V)^{N,\pi}_{i,j}$ is one-dimensional. Thus, if for each $i,j \in I$ we choose an element $T_{i,j} \in HS(V)^{N,\pi}_{i,j}$ with $\|T_{i,j}\| = 1$, then $\{T_{i,j}\}$ form an orthonormal basis of $HS(V)^{N,\pi}$. Therefore,

$$\|P_{HS(V)^{N,\pi}}(P_v)\|^2 = \sum_{i,j} |\langle P_v, T_{i,j}\rangle|^2 \tag{10.2}$$

Decompose v_π as $v_\pi = \sum_{i\in I} u_i$, where $u_i \in U_i$. Since there exists an orthonormal basis of V containing v, we have

$$\langle P_v, T_{i,j}\rangle = \langle v, T_{i,j}(v)\rangle = \langle u_j, T_{i,j}(u_i)\rangle. \tag{10.3}$$

Next note that $T_{i,j}^*T_{i,j}$ is an element of $HS(V)_{i,i}^{N,\pi}$ and thus by an earlier remark must act as multiplication by some scalar λ_i on U_i. Moreover, $\lambda_i = \frac{1}{\dim \pi}$ because if $f_1, \ldots, f_k$ is an orthonormal basis for U_i, then

$$\lambda_i \dim \pi = \sum_{l=1}^{k} \langle T_{i,j}^* T_{i,j} f_l, f_l \rangle = \langle T_{i,j} f_l, T_{i,j} f_l \rangle = \|T_{i,j}\|^2 = 1.$$

Hence $\|T_{i,j}u_i\|^2 = |\langle T_{i,j}^* T_{i,j} u_i, u_i\rangle| = \frac{\|u_i\|^2}{\dim \pi}$, whence $|\langle P_v, T_{i,j}\rangle|^2 \le \frac{\|u_i\|^2\|u_j\|^2}{\dim \pi}$ by (10.3), and (10.2) yields

$$\|P_{HS(V)^{N,\pi}}(P_v)\|^2 \le \sum_{i,j\in I} \frac{\|u_i\|^2\|u_j\|^2}{\dim \pi} = \frac{\|v_\pi\|^4}{\dim \pi}.$$

Combining this result with (10.1), we deduce the first assertion of the lemma.

Now we prove the second assertion. Assume that V is an irreducible N-space. As above, if V is infinite dimensional, then $HS(V)^N = 0$, so $\|P_{HS(V)^N}(P_v)\| = 0$. If V is finite-dimensional, then $HS(V)^N = HS(V)^G$ is one-dimensional consisting of scalar operators. The operator of multiplication by λ has Hilbert-Schmidt norm $|\lambda|\sqrt{\dim \pi}$, so $T_{1,1}$, being an element of Hilbert-Schmidt norm 1, must act as multiplication by some λ with $|\lambda| = \frac{1}{\sqrt{\dim \pi}}$. Therefore,

$$\|P_{HS(V)^N}(P_v)\|^2 = |\langle P_v, T_{1,1}\rangle|^2 = |\langle v, T_{1,1}v\rangle|^2 = \frac{1}{\dim \pi}. \quad \square$$

10.2. **Relative property** (T) **for group extensions.** We start with a simple result which reduces verification of relative property (T) to the case of irreducible representations.

Lemma 10.4. *Let G be a countable group, N a normal subgroup of G and S a finite subset of G. Assume that there exists a set of positive numbers $\{\varepsilon_s : s \in S\}$ such that for any irreducible G-space U without nonzero N-invariant vectors and any $0 \ne u \in U$ there exists $s \in S$ with $\|su - u\| \ge \varepsilon_s\|u\|$. Then*

$$\kappa(G, N; S) \ge \frac{1}{\sqrt{\sum_{s\in S} \frac{1}{\varepsilon_s^2}}}.$$

Remark. Since N is normal in G, for any irreducible G-space U, either U has no nonzero N-invariant vectors or N acts trivially on U.

Proof. Let V be a G-space without nonzero N-invariant vectors. We need to show that for any $0 \ne v \in V$ there exists $g \in S$ such that $\|gv - v\| \ge \frac{\|v\|}{\sqrt{\sum_{s\in S}\frac{1}{\varepsilon_s^2}}}$.

By the remark following the definition of a relative Kazhdan constant, we can assume that V is a cyclic G-space. Since G is countable, this implies that V is separable (that is, the ambient Hilbert space is separable). In this case, by [BHV, Theorem F.5.3], V is (unitarily equivalent to) the direct integral $\int_Z^{\oplus} V(z)d\mu(z)$ of a measurable field of irreducible G-spaces $V(z)$ over a measure space (Z, μ), where Z is a standard Borel space and $\mu(Z) < \infty$. We refer the reader to [BHV, § F.5] for the background on direct integrals.

Now take any $0 \ne v \in V$, and write it as $v = \int_Z v(z)$ with $v(z) \in V(z)$ for all z. For every $s \in S$ we put

$$Z_s = \{z \in Z : \|sv(z) - v(z)\| \ge \varepsilon_s\|v(z)\|\},$$

and let

$$Z_0 = \{z \in Z : N \text{ acts trivially on } V(z)\}.$$

By assumption, $(\cup_{s\in S} Z_s) \cup Z_0 = Z$; moreover, $\mu(Z_0) = 0$ since otherwise V would have a nonzero N-invariant vector. Hence

$$\sum_{s\in S}\int_{Z_s} \|v(z)\|^2 d\mu(z) \ge \int_Z \|v(z)\|^2 d\mu(z) = \|v\|^2,$$

and therefore there exists $g \in S$ such that

$$\int_{Z_g} \|v(z)\|^2 d\mu(z) \geq \frac{\|v\|^2}{\varepsilon_g^2 \sum_{s\in S} \frac{1}{\varepsilon_s^2}}.$$

Thus,

$$\|gv - v\|^2 \geq \int_{Z_g} \|gv(z) - v(z)\|^2 d\mu(z) > \int_{Z_g} \varepsilon_g^2 \|v(z)\|^2 d\mu(z) \geq \frac{\|v\|^2}{\sum_{s\in S} \frac{1}{\varepsilon_s^2}}. \quad \square$$

We are now ready to prove Theorem 2.7 whose statement (in fact, an extended version of it) is recalled below.

Theorem 10.5. *Let G be a group, N a normal subgroup of G and $Z \subseteq Z(G) \cap N$. Put $H = Z \cap [N, G]$. Let A, B and C be subsets of G satisfying the following conditions*

(1) *A and N generate G,*
(2) *$\kappa(G/Z, N/Z; B) \geq \varepsilon$,*
(3) *$\kappa(G/H, Z/H; C) \geq \delta$.*

Then the following hold:

(a)
$$\kappa(G, H; A \cup B) \geq \frac{12\varepsilon}{5\sqrt{72\varepsilon^2|A| + 25|B|}}.$$

(b)
$$\kappa(G, N; A \cup B \cup C) \geq \frac{1}{\sqrt{3}} \min\{\frac{12\varepsilon}{5\sqrt{72\varepsilon^2|A| + 25|B|}}, \delta\}.$$

Proof. (a) Using Lemma 10.4 we are reduced to proving the following claim:

Claim. *Let V be a non-trivial irreducible G-space without nonzero H-invariant vectors. Then there is no unit vector $v \in V$ such that*

$$\|sv - v\| \leq \frac{\sqrt{2}}{5} \text{ for any } s \in A \text{ and } \|sv - v\| \leq \frac{12\varepsilon}{25} \text{ for any } s \in B.$$

Let us assume the contrary, and let $v \in V$ be a unit vector satisfying the above conditions. First we shall show that

$$\|P_{HS(V)^N}(P_v)\|^2 \leq \frac{337}{625} \tag{10.4}$$

Case 1: V has an N-eigenvector. In this case V is spanned by N-eigenvectors, and thus we may write $v = \sum_i v_i$, where v_i are N-eigenvectors corresponding to distinct characters.

Assume that $\|v_j\| > \frac{4}{5}$ for some j. Since N is normal in G, any $g \in G$ sends the vector v_j to some eigenvector for N. Consider the subgroup

$$K = \{g \in G : g^{-1}ngv_j = nv_j \text{ for any } n \in N\}$$

consisting of elements fixing the character corresponding to v_j. Note that v_j is $[K, N]$-invariant. Since V has no nonzero H-invariant vectors and $H \subseteq [G, N]$, K is a proper subgroup of G. Thus, since $N \subseteq K$, there should exist $s \in A$ which is not in K. In particular $\langle sv_j, v_j\rangle = 0$ as sv_j and v_j are both N-eigenvectors corresponding to distinct characters. Hence

$$\|sv - v\|^2 \geq \left|\frac{\langle sv - v, v_j\rangle}{\|v_j\|}\right|^2 + \left|\frac{\langle sv - v, sv_j\rangle}{\|sv_j\|}\right|^2 = \left|\frac{\langle sv - v, v_j\rangle}{\|v_j\|}\right|^2 + \left|\frac{\langle s^{-1}v - v, v_j\rangle}{\|v_j\|}\right|^2 \tag{10.5}$$

Since $\langle sv_j, v_j\rangle = \langle v - v_j, v_j\rangle = 0$, we have $\langle sv - v, v_j\rangle = \langle s(v - v_j), v_j\rangle - \|v_j\|^2$. Since $\|v_j\| \geq \frac{4}{5}$ and hence $\|v - v_j\| \leq \frac{3}{5}$, we get

$$\left|\frac{\langle sv - v, v_j\rangle}{\|v_j\|}\right| \geq \|v_j\| - \left|\frac{\langle s(v - v_j), v_j\rangle}{\|v_j\|}\right| \geq \frac{4}{5} - \frac{3}{5} = \frac{1}{5}.$$

Thus, $\left|\frac{\langle sv-v,v_j\rangle}{\|v_j\|}\right|^2 \geq \frac{1}{25}$ and similarly $\left|\frac{\langle s^{-1}v-v,v_j\rangle}{\|v_j\|}\right|^2 \geq \frac{1}{25}$, so (10.5) yields $\|sv-v\|^2 > \frac{2}{25}$, which contradicts our assumptions on v.

Hence $\|v_i\| \leq \frac{4}{5}$ for all i, and Lemma 10.3 easily implies that $\|P_{HS(V)^N}(P_v)\|^2 \leq (4/5)^4 + (3/5)^4 = \frac{337}{625}$ (where the equality is achieved if after reindexing $\|v_1\| = 4/5$, $\|v_2\| = 3/5$ and $v_i = 0$ for $i \neq 1, 2$).

Case 2: V has no N-eigenvectors. Then we get directly from Lemma 10.3 that $\|P_{HS(V)^N}(P_v)\|^2 \leq \frac{1}{2} < \frac{337}{625}$.

Thus, we have established (10.4) in both cases. Let $Q = P_{(HS(V)^N)^\perp}(P_v)$. Then $\|Q\| \geq \sqrt{1 - \frac{337}{625}} = \frac{12\sqrt{2}}{25}$, so Lemma 10.1 yields

$$\|sQ - Q\| = \|sP_v - P_v\| = \|P_{sv} - P_v\| \leq \sqrt{2}\|sv - v\| \leq \frac{12\sqrt{2}\varepsilon}{25} \leq \varepsilon\|Q\| \tag{10.6}$$

for every $s \in B$.

Since V is an irreducible G-space, the elements of Z act as scalars on V, so Z acts trivially on $HS(V)$. Thus, $(HS(V)^N)^\perp$ is a G/Z-space without nonzero N/Z-invariant vectors, so (10.6) violates the assumption $\kappa(G/Z, N/Z; B) \geq \varepsilon$. This contradiction proves the claim and hence also part (a).

(b) Let V be a G-space without non-trivial N-invariant vectors and $0 \neq v \in V$. Let U be the orthogonal complement of V^Z in V and W the orthogonal complement of U^H in U. Then $V = V^Z \oplus U^H \oplus W$, so the projection of v onto at least one of the three subspaces V^Z, U^H and W has norm at least $\frac{\|v\|}{\sqrt{3}}$.

Case 1: $\|P_{V^Z}(v)\| \geq \frac{\|v\|}{\sqrt{3}}$. Since V^Z is a G/Z-space without nonzero N/Z-invariant vectors, by condition (2) there exists $s \in B$ such that $\|sP_{V^Z}(v) - P_{V^Z}(v)\| \geq \varepsilon\|P_{V^Z}(v)\|$. Therefore,

$$\|sv - v\| \geq \|sP_{V^Z}(v) - P_{V^Z}(v)\| > \frac{\varepsilon\|v\|}{\sqrt{3}} > \frac{12\varepsilon}{5\sqrt{72\varepsilon^2|A| + 25|B|}}\frac{\|v\|}{\sqrt{3}}.$$

Case 2: $\|P_{U^H}(v)\| \geq \frac{\|v\|}{\sqrt{3}}$. Similarly, since U^H is a G/H-space without nonzero Z/H-invariant vectors, by condition (3) there exists $s \in C$ such that

$$\|sv - v\| \geq \|sP_{U^H}(v) - P_{U^H}(v)\| > \frac{\delta\|v\|}{\sqrt{3}}.$$

Case 3: $\|P_W(v)\| \geq \frac{\|v\|}{\sqrt{3}}$. In this case, since W is a G-space without H-invariant vectors, we can apply part (a) to deduce that there exists $s \in A \cup B$ such that

$$\|sv - v\| \geq \|sP_W(v) - P_W(v)\| > \frac{12\varepsilon}{5\sqrt{72\varepsilon^2|A| + 25|B|}}\frac{\|v\|}{\sqrt{3}}.$$

$\square$

Remark. Theorem 10.5 generalizes a similar result due to Serre in the case $G = N$ (see, e.g., [BHV, Theorem 1.7.11] or [Ha, Theorem 1.8]). The case of a pair of subgroups (G, N) is also considered in [NPS, Lemma 1.1].

10.3. **Codistance bounds in nilpotent groups.** Let G be a nilpotent group generated by k subgroups $X_1, \ldots, X_k$. In this subsection we prove Theorem 2.8 which gives a bound for the codistance codist$(\{X_i\})$. The case when $k = 2$ and G is of nilpotency class 2 was considered in § 4 of [EJ]. Here we strengthen and generalize those results.

We will use the following auxiliary result.

Lemma 10.6. *Let (Z,μ) be a measure space and $z \to V(z)$ a measurable field of Hilbert spaces over Z. Let $A(z)$ and $B(z)$ be subspaces of $V(z)$. Put $A = \int_Z^{\oplus} A(z)$ and $B = \int_Z^{\oplus} B(z)$. Then for any measurable subset Z_1 of Z such that $\mu(Z \setminus Z_1) = 0$,*

$$\mathrm{orth}(A,B) \leq \sup_{z\in Z_1} \mathrm{orth}(A(z),B(z)).$$

Proof. Let $a = a(z) \in A(z)$ and $b = b(z) \in B(z)$ be two vectors. Then

$$\begin{aligned} |\langle a,b\rangle| &= \textstyle\int_Z |\langle a(z),b(z)\rangle|\, d\mu(z) = \int_{Z_1} |\langle a(z),b(z)\rangle|\, d\mu(z) \\ &\leq \textstyle\int_{Z_1} \mathrm{orth}(A(z),B(z))\|a(z)\|\|b(z)\| d\mu(z) \\ &\leq \sup_{z\in Z_1} \mathrm{orth}(A(z),B(z))\|a\|\|b\|. \end{aligned}$$

□

Corollary 10.7. *Let G be a countable group generated by subgroups $X_1,\ldots,X_k$. Then $\mathrm{codist}(X_1,\ldots,X_k)$ is equal to the supremum of the quantities $\mathrm{codist}(V^{X_1},\ldots,V^{X_k})$, where V runs over all non-trivial irreducible unitary representations of G.*

Proof. Let V be a unitary representation of G without G-invariant vectors. By the same argument as in Lemma 10.4, $V \cong \int_Z V(z)d\mu(z)$ for some measurable field of irreducible G-spaces $V(z)$ over a measure space (Z,μ). Put

$$Z_0 = \{z \in Z : \ V(z) \text{ is a trivial } G\text{-space}\}.$$

Since V has no nonzero G-invariant vectors, $\mu(Z_0) = 0$, whence by Lemma 10.6,

$$\begin{aligned} \mathrm{codist}(V^{X_1},\ldots,V^{X_k}) &= (\mathrm{orth}(V^{X_1}\times\cdots\times V^{X_k}, \mathrm{diag}\, V))^2 \\ &\leq \sup_{z\in Z\setminus Z_0} (\mathrm{orth}(V(z)^{X_1}\times\cdots\times V(z)^{X_k}, \mathrm{diag}\, V(z)))^2 \\ &= \sup_{z\in Z\setminus Z_0} \mathrm{codist}(V(z)^{X_1},\ldots,V(z)^{X_k}). \end{aligned}$$

□

We are now ready to prove the main result of this subsection.

Theorem 10.8. *Let G be a countable group generated by subgroups $X_1,\ldots,X_k$. Let H be a subgroup of $Z(G)$, and let m be the minimal dimension of an irreducible representation of G which is not trivial on H. Denote by $\bar{X}_i$ the image of X_i in G/H, and let $\varepsilon = 1-\mathrm{codist}(\bar{X}_1,\ldots,\bar{X}_k)$. Then $\mathrm{codist}(X_1,\ldots,X_k) \leq 1-\frac{(m-1)\varepsilon}{2m}$.*

Proof. By Corollary 10.7 we only have to consider non-trivial irreducible G-spaces. Let V be a non-trivial irreducible G-space, and let $n = \dim V \in \mathbb{N}\cup\{\infty\}$. If H acts trivially on V, there is nothing to prove since $\varepsilon > \frac{(m-1)\varepsilon}{2m}$. Thus, we can assume that H acts non-trivially, so $n \geq m$.

Now take any vectors $v_i \in V^{X_i}$ $(i = 1,\ldots,k)$. It is sufficient to show that

$$\mathrm{codist}(v_1,\ldots,v_k) \leq 1-\frac{(n-1)\varepsilon}{2n}. \tag{10.7}$$

Recall that for $v \in V$ we put $\iota(v) = \|v\|P_v \in HS(V)$ and that $\|\iota(v)\| = \|v\|$. Lemma 10.3 implies that

$$\|P_{HS(V)^G}(\iota(v_i))\|^2 = \frac{1}{n}\|v_i\|^2 = \frac{1}{n}\|\iota(v_i)\|^2 \tag{10.8}$$

and so

$$\|P_{HS(V)^G}(\sum_{i=1}^k \iota(v_i))\|^2 \leq k\sum_{i=1}^k \|P_{HS(V)^G}(\iota(v_i))\|^2 = \frac{k}{n}\sum_{i=1}^k \|\iota(v_i)\|^2.$$

On the other hand, since $Z(G)$ acts trivially on $HS(V)$, the action factors through G/H, which means that $(HS(V)^G)^\perp$ is a G/H-space without invariant vectors. Since $v_i \in V^{X_i}$, we have $\iota(v_i) \in HS(V)^{\bar{X}_i}$ and $P_{(HS(V)^G)^\perp}(\iota(v_i)) \in ((HS(V)^G)^\perp)^{\bar{X}_i}$. Hence by the definition of codistance

$$\begin{aligned}\|P_{(HS(V)^G)^\perp}(\sum_{i=1}^k \iota(v_i))\|^2 &\leq k \operatorname{codist}(\bar{X}_1,\dots,\bar{X}_k)\sum_{i=1}^k \|P_{(HS(V)^G)^\perp}(\iota(v_i))\|^2 \\ &= \tfrac{k(n-1)}{n}\operatorname{codist}(\bar{X}_1,\dots,\bar{X}_k)\sum_{i=1}^k \|\iota(v_i)\|^2.\end{aligned},$$

where the last equality holds by (10.8).

Combining these inequalities, we conclude that $\left\|\sum_{i=1}^k \iota(v_i)\right\|^2$ is bounded above by

$$k\left(\frac{1}{n}+\operatorname{codist}(\bar{X}_1,\dots,\bar{X}_k)\frac{n-1}{n}\right)\sum_{i=1}^k \|\iota(v_i)\|^2$$

Therefore

$$\operatorname{codist}(\iota(v_1),\dots,\iota(v_k)) \leq \frac{1}{n}+\operatorname{codist}(\bar{X}_1,\dots,\bar{X}_k)\frac{n-1}{n},$$

which, combined with Lemma 10.2, gives the following inequality equivalent to (10.7):

$$2\operatorname{codist}(v_1,\dots,v_k)-1 \leq \frac{1}{n}+\operatorname{codist}(\bar{X}_1,\dots,\bar{X}_k)\frac{n-1}{n} = \frac{1}{n}+(1-\varepsilon)\frac{n-1}{n} = 1-\frac{(n-1)\varepsilon}{n}.$$

□

We are now ready to prove Theorem 2.8 (whose statement is recalled below).

Theorem 10.9. *Let G be a countable nilpotent group of class c generated by subgroups $X_1,\dots,X_k$. Then*

$$\operatorname{codist}(X_1,\dots,X_k) \leq 1-\frac{1}{4^{c-1}k}.$$

Proof. We prove the theorem by induction on the nilpotency class c. The induction step follows from Theorem 10.8. To establish the base case $c=1$, in which case G is abelian, we use a separate induction on k. The case $k=2$ holds by [EJ, Lemma 3.4]. We now do the induction step on k. Let V be a G-space without G-invariant vectors, and let $v_i \in V^{X_i}$. Then using the induction hypothesis in the fourth line, we obtain

$$\begin{aligned}\|\sum_{i=1}^k v_i\|^2 &= \|\sum_{i=1}^{k-1} P_{(V^{X_k})^\perp}(v_i)\|^2 + \|\sum_{i=1}^k P_{V^{X_k}}(v_i)\|^2 \\ &\leq (k-1)\sum_{i=1}^{k-1}\|P_{(V^{X_k})^\perp}(v_i)\|^2 + \|(k-2)\frac{\sum_{i=1}^{k-1} P_{V^{X_k}}(v_i)}{k-2}+v_k\|^2 \\ &\leq (k-1)\sum_{i=1}^{k-1}\|P_{(V^{X_k})^\perp}(v_i)\|^2 + \frac{k-1}{k-2}\|\sum_{i=1}^{k-1} P_{V^{X_k}}(v_i)\|^2 + (k-1)\|v_k\|^2 \\ &\leq (k-1)\sum_{i=1}^{k-1}\|P_{V^{(X_k)^\perp}}(v_i)\|^2 + (k-1)\sum_{i=1}^{k-1}\|P_{V^{X_k}}(v_i)\|^2 + (k-1)\|v_k\|^2 \\ &= (k-1)\sum_{i=1}^k \|v_i\|^2.\end{aligned}$$

□

11. Appendix

This section contains proofs of Theorem 2.3 and 7.10. Both results are virtually identical to [Ka1, Cor 1.8] and [Ka1, Cor 1.10], respectively (the only difference is that EL_n is replaced by St_n, which does not affect the argument), but we have chosen to provide proofs of both results to make this paper essentially self-contained.

Let R be a finitely generated associative ring (with 1). Recall that we defined the Steinberg group $\mathrm{St}_2(R)$ to be the free product $R \star R$. Given $r \in R$, the element of $\mathrm{St}_2(R)$ corresponding to r from the first (resp. second) copy of R will be denoted by $x_{12}(r)$ (resp. $x_{21}(r)$).

Let $\{r_1, \dots r_k\}$ be a finite generating set of R, and let $M = \langle m_1, \dots, m_k\rangle$ be a finitely generated left R-module. Consider the action of the group $\mathrm{St}_2(R) = R \star R$ on M^2 where $x_{12}(r)$ acts by the matrix $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ and $x_{21}(r)$ acts by the matrix $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$. Define $F_1 \subset \mathrm{St}_2(R)$ and $F_2 \subset M^2$ by

$$F_1 = \{x_{12}(\pm r_i), x_{12}(\pm 1), x_{21}(\pm r_i), x_{21}(\pm 1)\} \quad \text{and} \quad F_2 = \{(\pm m_i, 0), (0, \pm m_i)\}.$$

Note that $|F_1| \le 4(k+1)$ and $|F_2| \le 4d$.

Theorem 11.1. *Let $\langle F_1\rangle$ be the subgroup of $\mathrm{St}_2(R)$ generated by F_1, let $\Gamma = \langle F_1\rangle \ltimes M^2$, and let $F = F_1 \cup F_2$. Then the pair (Γ, M^2) has relative property (T), and moreover*

$$\kappa_r(\Gamma, M^2; F) > \frac{1}{2K(k,d)},$$

where $K(k,d) = 12(\sqrt{k} + \sqrt{d} + 1)$. [3]

Remark. Theorem 11.1 remains true for the semi-direct product $M^2 \rtimes \langle F_1\rangle$ where M is a right R-module and elements of F_1 act on M^2 by right multiplication. This is because such semi-direct product $M^2 \rtimes \langle F_1\rangle$ is isomorphic to $\langle (F_1)^{\mathrm{op}}\rangle \ltimes (M^2)^{\mathrm{op}}$ where $(M^2)^{\mathrm{op}} = M^2$ considered as a left module over the opposite ring R^{op} and $(F_1)^{\mathrm{op}}$ is the image of F_1^{op} in the Steinberg group $\mathrm{St}_2(R^{\mathrm{op}})$ under the canonical isomorphism $\mathrm{St}_2(R) \to \mathrm{St}_2(R^{\mathrm{op}})$.

Proof. Let V be a unitary representation of Γ, and let $v \in V$ be (F, ε)-invariant for some $\varepsilon > 0$. By definition of the Kazhdan ratio $\kappa_r(\Gamma, M^2; F)$ we need to prove that v is $(M^2, 2K(k,d)\varepsilon)$-invariant. Without loss of generality we will assume that v is a unit vector.

Let $\mathcal{P}$ be the projection valued measure on the dual $(M^2)^* = {M^*}^2$, coming from the restriction of the representation V to M^2. Here M^* stands for the Pontryagin dual of $(M,+)$, the additive group of M, that is, $M^* = \mathrm{Hom}\big((M,+), S^1\big)$, where S^1 is the unit circle in the complex plane. The set M^* is an abelian group with natural topology. Let μ_v be the probability measure on ${M^*}^2$, defined by $\mu_v(B) = \langle \mathcal{P}(B)v, v\rangle$.

The group $\mathrm{St}_2(R)$ has a natural right action on ${M^*}^2$, which is dual to the standard left action of $\mathrm{St}_2(R)$ on M^2 and is given by

$$\chi^g((m,n)) = \chi((gm, gn)) \text{ for all } g \in G, \chi \in {M^*}^2 \text{ and } (m,n) \in M^2.$$

The following lemma shows that the measure μ_v is almost F_1-invariant with respect to this action.

Lemma 11.2. *For every measurable set $B \subset {M^*}^2$ and every $g \in F_1$ we have*

$$|\mu_v(B^g) - \mu_v(B)| \le 2\varepsilon\sqrt{\mu_v(B)} + \varepsilon^2 \quad \text{or, equivalently,} \quad |\sqrt{\mu_v(B^g)} - \sqrt{\mu_v(B)}| \le \varepsilon.$$

[3] This bound can be improved slightly.

Proof. Using properties of projection valued measures, it is easy to show that $\mu_v(B^g) = \langle \pi(g^{-1})\mathcal{P}(B)\pi(g)v, v\rangle$. Therefore,

$$\begin{aligned}
|\mu_v(B^g) - \mu_v(B)| &= |\langle \pi(g^{-1})\mathcal{P}(B)\pi(g)v, v\rangle - \langle \mathcal{P}(B)v, v\rangle| \\
&\le |\langle \pi(g^{-1})\mathcal{P}(B)(\pi(g)v - v), v\rangle| + |\langle \mathcal{P}(B)v, (\pi(g)v - v\rangle)| \\
&\le 2|\langle \pi(g)v - v, \mathcal{P}(B)v\rangle| + \langle \mathcal{P}(B)(\pi(g)v - v), \pi(g)v - v\rangle \\
&\le 2\varepsilon\sqrt{\mu_v(B)} + \varepsilon^2,
\end{aligned}$$

where the final inequality follows from the facts that v is (F,ε) invariant vector and $||\mathcal{P}(B)v||^2 = \mu_v(B)$. □

We will need the following consequence of the above lemma:

Lemma 11.3. *Let A and B be measurable sets in ${M^*}^2$. Suppose that A decomposes as a disjoint union of the sets A_i for $1 \le i \le s$ and there exist elements $g_i \in F_1$ such that the sets $B_i = (A_i)^{g_i}$ are disjoint subsets of B. Then*

$$\sqrt{\mu_v(A)} \le \sqrt{\mu_v(B)} + \sqrt{s}\varepsilon.$$

Proof. Applying lemma 11.2 to the sets A_i yields

$$\begin{aligned}
\mu_v(A) = \sum_{i=1}^s \mu_v(A_i) &\le \sum_{i=1}^s \left[\mu_v(B_i) + 2\varepsilon\sqrt{\mu_v(B_i)} + \varepsilon^2\right] \le \\
&\le \sum \mu_v(B_i) + 2\varepsilon\sqrt{s\sum \mu_v(B_i)} + s\varepsilon^2 \le \\
&\le \mu_v(B) + 2\varepsilon\sqrt{s\mu_v(B)} + s\varepsilon^2 = \left(\sqrt{\mu_v(B)} + \sqrt{s}\varepsilon\right)^2,
\end{aligned}$$

where we have used that for nonnegative numbers a_i the following inequality holds:

$$\sqrt{\sum_{i=1}^s a_i} \le \sum_{i=1}^s \sqrt{a_i} \le \sqrt{s\sum_{i=1}^s a_i}.$$

□

For an element $\chi \in {M^*}^2$ we will write $\chi = (\chi_1, \chi_2)$, where $\chi_1, \chi_2 \in M^*$.

Lemma 11.4. *For $1 \le s \le d$ and $i = 1, 2$ let*

$$P_{i,s} = \{\chi \in {M^*}^2 : \operatorname{Re}\chi_i(m_s) > 0\}.$$

Then $\mu_v(P_{i,s}) \ge 1 - \varepsilon^2/2$. (Recall that $m_1, \dots, m_d$ are the given generators of M).

Proof. For $1 \le i \le s$ let $g_{1,s} = (m_s, 0) \in M^2$ and $g_{2,s} = (0, m_s) \in M^2$, so that by construction $g_{i,s} \in F_2$. By the definition of the measure μ_v, we have

$$||g_{i,s}v - v||^2 = \int_{{M^*}^2} |\chi_i(m_s) - 1|^2 \, d\mu_v.$$

By assumption v is (F_2,ε)-invariant, whence $||g_{i,s}v - v|| \le \varepsilon$. If we break the integral into two integrals over $P_{i,s}$ and its complement, we get

$$\varepsilon^2 \ge \int_{{M^*}^2\setminus P_{i,s}} |\chi_i(m_s) - 1|^2 \, d\mu_v + \int_{P_{i,s}} |\chi_i(m_s) - 1|^2 \, d\mu_v \ge \int_{{M^*}^2\setminus P_{i,s}} 2 \, d\mu_v = 2(1 - \mu_v(P_{i,s})),$$

which yields the desired inequality. □

The rest of the proof shows that any probability measure on ${M^*}^2$ satisfying the above lemmas must be very close to the Dirac measure at the origin, which implies that the vector v must be almost M^2-invariant.

First we consider the special case $R = M = \mathbb{Z}$, so that we can identify M^* with S^1 and ${M^*}^2$ with the two-dimensional torus $\mathbb{T}^2$. The action of $\mathrm{St}_2(\mathbb{Z})$ on $\mathbb{T}^2$ factors through the action of $\mathrm{SL}_2(\mathbb{Z})$. It is well known that the only $\mathrm{SL}_2(\mathbb{Z})$-invariant measures on the torus $\mathbb{T}^2$ are multiples of the Dirac measure at the origin. The following lemma is a quantitative version of this fact.

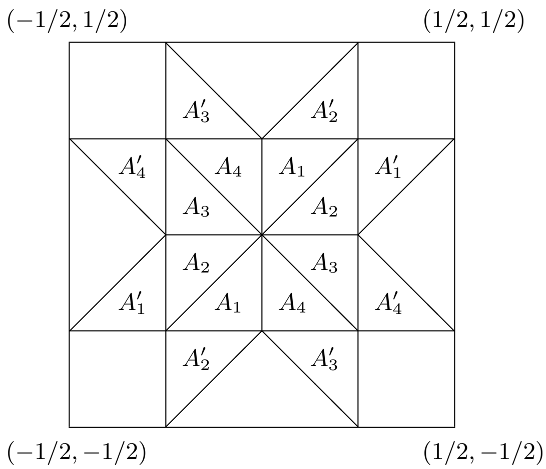
Lemma 11.5. *Let μ be a finitely additive measure on $\mathbb{T}^2$ such that*

(1) $\mu(\{(x,y) : \mathrm{Re}\, x < 0\}) \le \varepsilon^2/2$ *and* $\mu(\{(x,y) : \mathrm{Re}\, y < 0\}) \le \varepsilon^2/2$,
(2) $|\mu(B^g) - \mu(B)| \le 2\varepsilon\sqrt{\mu(B)} + \varepsilon^2$ *for any Borel set* $B \subseteq \mathbb{T}^2$ *and any elementary matrix* $g \in \mathrm{SL}_2(\mathbb{Z})$ *with* ± 1 *off the diagonal.*

Then the measure μ satisfies

$$\mu(\mathbb{T}^2 \setminus \{(0,0)\}) \le (2+\sqrt{10})^2\varepsilon^2.$$

Proof. Define the Borel subsets A_i and A_i' of $\mathbb{T}^2$ using the picture below – here we identify the torus $\mathbb{T}^2$ with the square $(-1/2, 1/2] \times (-1/2, 1/2]$:



Each set A_i or A_i' consists of the interiors of two triangles and part of their boundary (not including the vertices). The sets A_i do not contain the side which is part of the small square, they also do not contain their clockwise boundary but contain the counter-clockwise one. Each set A_i' includes only the part of its boundary which lies on the small square.

It can be seen, from the picture, that the elementary matrices $e_{ij}(\pm 1) \in \mathrm{SL}_2(\mathbb{Z})$, act on the sets A_i as follows:

$$\begin{array}{ll} (A_3 \cup A_4')^{e_{21}(1)} = A_3 \cup A_4 & (A_3' \cup A_4)^{e_{12}(1)} = A_3 \cup A_4 \\ (A_1' \cup A_2)^{e_{21}(-1)} = A_1 \cup A_2 & (A_1 \cup A_2')^{e_{12}(-1)} = A_1 \cup A_2. \end{array}$$

In view of hypothesis (2) of the lemma, these equalities yield the following inequalities:

$$\begin{aligned} \mu(A_1) + \mu(A_2) &\le \mu(A_1') + \mu(A_2) + \varepsilon^2 + 2\varepsilon\sqrt{\mu(A_1') + \mu(A_2)} \\ \mu(A_1) + \mu(A_2) &\le \mu(A_1) + \mu(A_2') + \varepsilon^2 + 2\varepsilon\sqrt{\mu(A_1) + \mu(A_2')} \\ \mu(A_3) + \mu(A_4) &\le \mu(A_3') + \mu(A_4) + \varepsilon^2 + 2\varepsilon\sqrt{\mu(A_3') + \mu(A_4)} \\ \mu(A_3) + \mu(A_4) &\le \mu(A_3) + \mu(A_4') + \varepsilon^2 + 2\varepsilon\sqrt{\mu(A_3) + \mu(A_4')}. \end{aligned}$$

Adding these inequalities and noticing that

$$\begin{aligned} &\mu(A_1') + \mu(A_4') \le \mu(\{|x| \ge 1/4\}) \le \varepsilon^2/2 \quad \text{and} \\ &\mu(A_2') + \mu(A_3') \le \mu(\{|y| \ge 1/4\}) \le \varepsilon^2/2 \end{aligned}$$

we obtain

$$\sum \mu(A_i) \le 4\varepsilon^2 + \sum \mu(A_i') + 2\varepsilon\sqrt{4\left(\sum \mu(A_i) + \sum \mu(A_i')\right)} \le$$

$$\leq 5\varepsilon^2 + 4\varepsilon\sqrt{\sum \mu(A_i) + \varepsilon^2}.$$

This yields $\sum \mu(A_i) \leq (13 + 4\sqrt{10})\varepsilon^2$, and therefore

$$\mu(\mathbb{T}^2 \setminus \{(0,0)\}) \leq \sum \mu(A_i) + \mu(\{|x| \geq 1/4\}) + \mu(\{|y| \geq 1/4\}) \leq$$
$$\leq (14 + 4\sqrt{10})\varepsilon^2 = (2 + \sqrt{10})^2\varepsilon^2,$$

which completes the proof. □

We are now ready to prove the analogue of Lemma 11.5 dealing with arbitrary ring R and R-module M.

Lemma 11.6. *Let μ be a finitely additive measure on ${M^*}^2$ such that*

(1) $\mu(\{\chi : \operatorname{Re}\chi_i(m_s) < 0\}) \leq \varepsilon^2/2$ *for all $i = 1,2$ and $s = 1,\dots,d$.*
(2) $|\mu(B^g) - \mu(B)| \leq 2\varepsilon\sqrt{\mu(B)} + \varepsilon^2$ *for any Borel set $B \subseteq {M^*}^2$ and any $g \in F_1 \subset \mathrm{St}_2(R)$.*

Then we have

$$\mu({M^*}^2 \setminus \{(0,0)\}) \leq K(k,d)^2\varepsilon^2,$$

where the constant $K(k,d)$ is defined in Theorem 11.1.

Proof. Define the following increasing filtration (as abelian group) of the module M : Let $M^{(0)} = \mathrm{span}_{\mathbb{Z}}\{m_1,\dots,m_d\}$ and $M^{(i+1)} = M^{(i)} + r_1M^{(i)} + \cdots + r_kM^{(i)}$, that is, $M^{(i)}$ is the span of all elements in M which can be obtained from the module generators using at most i multiplications by the ring generators. One has $M = \bigcup_i M^{(i)}$, since R and M are generated by $\{r_i\}$ and $\{m_j\}$ respectively.

This filtration induces a decreasing filtration of the dual

$$M^*_{(i)} = \left\{\chi \in M^* \mid \chi(m) = 1,\, \forall m \in M^{(i)}\right\}$$

such that $\{0\} = \cap M^*_{(i)}$. This filtration yields a valuation $\nu : M^* \to \mathbb{N} \cup \{\infty\}$ by setting $\nu(\chi) = i$ if $\chi \in M^*_{(i)} \setminus M^*_{(i+1)}$ for some i and $\nu(0) = \infty$.

Let us define the following subsets of ${M^*}^2 \setminus \{0,0\}$:

$$\begin{aligned}
A &= \{(\chi_1,\chi_2) \mid \nu(\chi_1) > \nu(\chi_2) > 0\}\\
B &= \{(\chi_1,\chi_2) \mid \nu(\chi_1) = \nu(\chi_2) > 0\}\\
C &= \{(\chi_1,\chi_2) \mid \nu(\chi_2) > \nu(\chi_1) > 0\}\\
D &= \{(\chi_1,\chi_2) \mid \nu(\chi_1) = 0 \text{ or } \nu(\chi_2) = 0\}\,.
\end{aligned}$$

Consider the action of $x_{12}(r), x_{21}(r) \in F_1 \subset \mathrm{St}_2(R)$ on the element $(\chi_1,\chi_2) \in {M^*}^2$. By the definition of this action we have

$$\left((\chi_1,\chi_2)^{x_{12}(r)}\right)\begin{pmatrix} m_1\\ m_2\end{pmatrix} = (\chi_1,\chi_2)\left(x_{12}(r)\begin{pmatrix} m_1\\ m_2\end{pmatrix}\right) = (\chi_1,\chi_2)\begin{pmatrix} m_1 + r\cdot m_2\\ m_2\end{pmatrix}$$

and

$$\left((\chi_1,\chi_2)^{x_{21}(r)}\right)\begin{pmatrix} m_1\\ m_2\end{pmatrix} = (\chi_1,\chi_2)\left(x_{21}(r)\begin{pmatrix} m_1\\ m_2\end{pmatrix}\right) = (\chi_1,\chi_2)\begin{pmatrix} m_1\\ m_2 + r\cdot m_1\end{pmatrix}$$

This shows that

$$A^{x_{21}(1)} \subseteq B \quad\text{and}\quad C^{x_{12}(1)} \subseteq B,$$

and by hypothesis (2) of Lemma 11.6 we get

$$\sqrt{\mu(A)} \leq \sqrt{\mu(B)} + \varepsilon \quad\text{and}\quad \sqrt{\mu(C)} \leq \sqrt{\mu(B)} + \varepsilon. \tag{11.1}$$

Claim 11.7. *The union $A \cup B$ can be decomposed as a disjoint union of k sets A_i such that the sets $\widetilde{A_i} = (A_i)^{x_{21}(r_i)}$ are disjoint and lie in $C \cup D$. Similarly, $C \cup B$ can be written as $\bigcup C_i$ such that $\widetilde{C_i} = (C_i)^{x_{12}(r_i)}$ are disjoint subsets of $A \cup D$.*

Proof. Let $(\chi_1,\chi_2)\in A\cup B$. Define $\chi_{2,j}\in M^*$ by

$$\chi_{2,j}(f)=\chi_2(r_jf).$$

Using the definition of the subsets $M^{(i)}$ one can see that there exists some j between 1 and k such that $\nu(\chi_{2,j})=\nu(\chi_2)-1$. Let $s(\chi_2)$ be the smallest index j such that $\chi_{2,j}$ satisfies this condition. Define the sets A_i by

$$A_i=\{(\chi_1,\chi_2)\in A\cup B : s(\chi_2)=i\}.$$

It is easy to check that $A\cup B$ is a disjoint union of A_i and that $\widetilde{A_i}=(A_i)^{x_{21}(r_i)}$ are disjoint subsets of $C\cup D$.

The second part of the claim is proved by the same argument applied to the first component χ_1 instead of the second one χ_2. □

We are ready to complete the proof of Lemma 11.6. Applying Lemma 11.3 to the sets $A\cup B$ and $C\cup B$ yields

$$\sqrt{\mu(A\cup B)}\le\sqrt{\mu(C\cup D)}+\sqrt{k}\varepsilon$$
$$\sqrt{\mu(C\cup B)}\le\sqrt{\mu(A\cup D)}+\sqrt{k}\varepsilon.$$

Squaring and adding the the above inequalities and taking (11.1) into account, we get

$$\begin{aligned}2\mu(B)&\le 2\mu(D)+2\varepsilon\sqrt{k(\mu(C)+\mu(D))}+2\varepsilon\sqrt{k(\mu(A)+\mu(D))}+2k\varepsilon^2\le\\&\le 2\mu(D)+4\varepsilon\sqrt{k\mu(D)}+2\varepsilon\sqrt{k\mu(A)}+2\varepsilon\sqrt{k\mu(C)}+2k\varepsilon^2\le\\&\le 2\mu(D)+4\varepsilon\sqrt{k\mu(D)}+4\varepsilon\sqrt{k\mu(B)}+(2k+4\sqrt{k})\varepsilon^2\end{aligned}$$

The last inequality can be rewritten as

$$(\sqrt{\mu(B)}-\sqrt{k}\varepsilon)^2\le(\sqrt{\mu(D)}+\sqrt{k}\varepsilon)^2+(k+2\sqrt{k})\varepsilon^2. \tag{11.2}$$

Note that (11.1) and (11.2) yield an upper bound for $\mu(A),\mu(B)$ and $\mu(C)$ in terms of $\mu(D)$, so to finish the argument we just need to find a suitable bound for $\mu(D)$.

Fix $s\in\{1,\dots,d\}$, and let M_s be the subgroup of M^2 generated by $(m_s,0)$ and $(0,m_s)$. Restricting the functionals χ from M to M_s, we obtain a map $M^{*\,2}\to(M_s^*)^2$. Let $\pi_s : M^{*\,2}\to\mathbb{T}^2$ be the composition of this map with the natural embedding $(M_s^*)^2\to\mathbb{T}^2$ (which is an isomorphism if m_s has infinite additive order). Let $\mu_s=\mu\circ\pi_s^{-1}$ be the measure on $\mathbb{T}^2$ obtained from μ by pullback via π_s.

Since M_s is invariant under the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on M^2, the kernel of the map π_s is invariant under the dual (right) action of $\mathrm{SL}_2(\mathbb{Z})$ on $M^{*\,2}$, and therefore we have a well-defined right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$ given by

$$f^g=\pi_s((\pi_s^{-1}(f))^g)\text{ for all } f\in\mathbb{T}^2 \text{ and } g\in SL_2(\mathbb{Z}).$$

It is easy to see that the measure μ_s satisfies the hypotheses of Lemma 11.5 with respect to this action, so $\mu_s(\mathbb{T}^2\setminus(0,0))\le(2+\sqrt{10})^2\varepsilon^2$.

Finally, observe that the set D is the same as $\bigcup_s\pi_s^{-1}(\mathbb{T}^2\setminus(0,0))$, and therefore

$$\mu(D)\le d(2+\sqrt{10})^2\varepsilon^2. \tag{11.3}$$

The desired inequality $\mu(M^{*\,2}\setminus(0,0))\le K(k,d)^2\varepsilon^2$ now follows from (11.1),(11.2) and (11.3) by a straightforward computation. □

Theorem 11.1 now follows easily. For any $g\in M^2$ we have

$$||gv-v||^2=\int_{M^{*\,2}}|\chi(g)-1|^2\,d\mu_v\le\int_{M^{*\,2}\setminus(0,0)}4\,d\mu_v=4\mu_v(M^{*\,2}\setminus(0,0))\le 4K(k,d)^2\varepsilon^2,$$

where the last inequality holds by Lemma 11.6. Hence v is $(F,2K(k,d)\varepsilon)$-invariant, as desired.

Alternatively, we can finish the argument as follows. Lemma 11.6 implies that for $\varepsilon < \frac{1}{K(k,d)}$ we have $\mu_v(M^{*\,2} \setminus (0,0)) < 1$. Hence the projection operator $\mathcal{P}(\{(0,0\})$ is nonzero, that is, V has a nonzero M^2-invariant vector. Thus, by definition, $\kappa(\Gamma, M^2; F) \geq \frac{1}{K(k,d)}$, whence $\kappa_r(\Gamma, M^2; F) \geq \frac{1}{2K(k,d)}$ by Observation 2.2(ii). $\square$

Using a similar method we can prove the higher-dimensional analog of Theorem 11.1:

Theorem 11.8. *Let R and M be as in Theorem 11.1, with generating set $\{r_1, \ldots, r_k\}$ and $\{m_1, \ldots, m_d\}$, respectively. Let $\Gamma = \mathrm{St}_p(R) \ltimes M^p$ for some $p \geq 3$, let F_1 be the generating set of $\mathrm{St}_p(R)$ consisting of the elements of root subgroups of the form $x_{ij}(\pm r_l)$ and $x_{ij}(\pm 1)$, and let F_2 be the set of standard generators of M^p and their inverses (so that $|F_1| \leq 2p(p-1)(k+1)$ and $|F_2| \leq 2pd$), and let $F = F_1 \cup F_2$. Then the pair (Γ, M^p) has relative property (T) and*

$$\kappa_r(\Gamma, M^p; F) > \frac{1}{2K(k,d,p)},$$

where $K(k,d,p) = 20(\sqrt{k} + \sqrt{d} + \sqrt{p})$.

Remark. In the above theorem one can replace the group $\mathrm{St}_p(R)$ by the free product of $p(p-1)$ copies of the additive group of R. Also note that $St_p(R) = \langle F_1 \rangle$ since $p \geq 3$.

Proof. The proof is similar to that of Theorem 11.1. We start with a unitary representation V of $\Gamma = \mathrm{St}_p(R) \ltimes M^p$ and an (F, ε)-invariant unit vector $v \in V$. Let μ_v be the measure on $M^{*\,p}$ coming from the restriction of the representation to the abelian group M^p. As in the case $p = 2$, the measure μ_v is almost invariant under the action of the generators of $\mathrm{St}_p(R)$.

We will write an element $\chi \in M^{*\,p}$ as $(\chi_1, \ldots, \chi_p)$, where $\chi_i \in M^*$. Define the Borel subsets B_i, C_i of $M_k^{*\,p}$ for $2 \leq i \leq p$ by

$$B_i = \{\chi \in M^{*\,p} : \chi_j = 0 \text{ for } j \leq i\} \text{ and}$$
$$C_i = \{\chi \in M^{*\,p} : \chi_1 = \chi_i \neq 0, \chi_j = 0 \text{ for } 1 < j < i\}.$$

Using the restriction $M^{*\,p} \to M^{*\,2}$ coming form the inclusion $M^2 \subset M^p$ and Lemma 11.6, it is easy to see that

$$\mu_v(\{\chi : \chi_1 \neq 0 \text{ or } \chi_2 \neq 0\}) \leq K(k,d)\varepsilon^2, \text{ that is,}$$

$$\mu_v(M^{*\,p} \setminus B_2) \leq K(k,d)^2\varepsilon^2. \tag{11.4}$$

On the other hand, the elementary matrix $x_{1i}(1) \in \mathrm{St}_p(R)$ sends $B_{i-1} \setminus B_i$ into C_i for any $i \geq 3$. Now notice that the sets C_i are disjoint for $i = 2, \ldots, p$ and their union lies in the set

$$C := \{\chi : \chi_1 \neq 0, \chi_2 = 0\} \subseteq M^{*\,p} \setminus B_2. \tag{11.5}$$

Applying Lemma 11.3 to the action of $x_{1i}(1)$ on the set $B_{i-1} \setminus B_i$ for $3 \leq i \leq p$, after an easy computation we get

$$\sqrt{\mu_v(B_2 \setminus B_p)} \leq \sqrt{\mu_v(C)} + \sqrt{(p-2)}\,\varepsilon. \tag{11.6}$$

Combining (11.4), (11.5) and (11.6) and observing that $B_p = \{(0, \cdots, 0)\}$, we conclude that

$$\mu_v(M^{*\,p} \setminus \{(0,\ldots,0)\}) = \mu_v(M^{*\,p} \setminus B_2) + \mu_v(B_2 \setminus B_p) \leq$$
$$\leq K(k,d)^2\varepsilon^2 + \Big(K(k,d)\varepsilon + \sqrt{(p-2)}\,\varepsilon\Big)^2 \leq K(k,d,p)^2\varepsilon^2.$$

The assertion of Theorem 11.8 follows from this inequality by the same argument as in Theorem 11.1. $\square$

The above argument can be generalized even further. Let N be an R-bimodule generated (as a bi-module) by d elements. Let Γ denote the semidirect product $(\mathrm{St}_p(R) \times \mathrm{St}_q(R)) \ltimes N^{pq}$, $p, q \geq 2$, where we identify N^{pq} with the abelian group $\mathrm{Mat}_{p\times q}(N)$ of $p \times q$ matrices over N and let $\mathrm{St}_p(R)$ (resp. $\mathrm{St}_q(R)$) act by left multiplication (resp. right multiplication). Somewhat informally, we can think of Γ as the "block upper-triangular" group

$$\begin{pmatrix} \mathrm{St}_p(R) & \mathrm{Mat}_{p\times q}(N) \\ 0 & \mathrm{St}_q(R) \end{pmatrix}.$$

This group has a natural generating set F consisting of the elementary matrices with the generators of R and N off the diagonal.

Theorem 11.9. *In the above setting, the pair* (Γ, N^{pq}) *has relative property* (T) *and*

$$\kappa_r(\Gamma, N^{pq}; F) > \frac{1}{2K(k,d,p,q)},$$

where $K(k,d,p) = 50(\sqrt{k} + \sqrt{d} + \sqrt{p} + \sqrt{q})$.

Index

References

[Bak1] Anthony Bak, *The stable structure of quadratic modules,* Thesis, Columbia University, 1969.

[Bak2] Anthony Bak, *K-theory of forms.* Annals of Mathematics Studies, 98. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981. viii+268 pp.

[BGM] Catalin Badea, Sophie Grivaux and Vladimir Muller, *The rate of convergence in the method of alternating projections,* Algebra i Analiz 23 (2011), no. 3, 1–30; translation in St. Petersburg Math. J. 23 (2012), no. 3, 413-434.

[BHV] Bachir Bekka, Pierre de la Harpe and Alain Valette, *Kazhdan's property* (T). New Math. Monogr. 11, Cambridge Univ. Press, Cambridge, 2008.

[Bou] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4-6.* Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. xii+300

[Br] J. L. Brenner, Covering theorems for FINASIGs. VIII. Almost all conjugacy classes in Alt$n$ have exponent ≤ 4. J. Austral. Math. Soc. Ser. A 25 (1978), no. 2, 210–214.

[BGT] Emmanuel Breuillard, Ben Green and Terence Tao, *Suzuki groups as expanders,* Groups Geom. Dyn. 5 (2011), no. 2, 281–299.

[Bu] Marc Burger, *Kazhdan constants for* $SL(3,\mathbb{Z})$*,* J. Reine Angew. Math. 413 (1991), 36–67.

[Ca] Roger W. Carter, *Simple groups of Lie type.* Reprint of the 1972 original. John Wiley & Sons, Inc., New York, 1989. x+335 pp.

[Co] Yves de Cornulier, *Relative Kazhdan property,* Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 2, 301–333.

[DJ] Jan Dymara and Tadeusz Januszkiewicz *Cohomology of buildings and their automorphism groups.* Invent. Math. 150 (2002), no. 3, 579–627

[EJ] Mikhail Ershov and Andrei Jaikin-Zapirain, *Property* (T) *for noncommutative universal lattices,* Invent. Math., 179 (2010), no. 2, 303–347.

[Fa] John R. Faulkner, *Barbilian planes*, Geom. Dedicata 30 (1989), no. 2, 125-181.

[Fa2] John R. Faulkner, *Groups with Steinberg relations and coordinatization of polygonal geometries.* Mem. Amer. Math. Soc. 10 (1977), no. 185, iii+135 pp.

[GHW] Erik Guentner, Nigel, Higson, Shmuel Weinberger, *The Novikov conjecture for linear groups*, Publ. Math. Inst. Hautes Etudes Sci. (2005), no 101, 243–268.

[GMV] Fritz Grunewald, Jens Mennicke and Leonid Vaserstein, *On symplectic groups over polynomial rings.* Math. Z. 206 (1991), no. 1, 35–56.

[Ha] Uzy Hadad, *Kazhdan constants of group extensions*, Internat. J. Algebra Comput. 20 (2010), no. 5, 671-688.

[HO] Alexander J. Hahn and O. Timothy O'Meara, *The classical groups and K-theory.* Springer-Verlag, Berlin, New York, 1989, 565 pp.

[HLW] Shlomo Hoory, Nathan Linial and Avi Wigderson, *Expander graphs and their applications,* Bull. Amer. Math. Soc. (N.S.) 43 (2006), no. 4, 439–561.

[Hu1] James E. Humphreys, *Introduction to Lie algebras and representation theory.* Second printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1978. xii+171 pp.

[Hu2] James E. Humphreys, *Linear algebraic groups.* Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975. xiv+247 pp.

[Ka1] Martin Kassabov, *Universal lattices and unbounded rank expanders,* Invent. Math. 170 (2007), no. 2, 297–326.

[Ka2] Martin Kassabov, *Symmetric groups and expander graphs.* Invent. Math. 170 (2007), no. 2, 327-354.

[Ka3] Martin Kassabov, *Subspace arrangements and property* (T)*,* Groups Geom. Dyn. 5 (2011), no. 2, 445-477.

[KLN] Martin Kassabov, Alexander Lubotzky and Nikolay Nikolov, *Finite simple groups as expanders,* Proc. Natl. Acad. Sci. USA 103 (2006), no. 16, 6116–6119.

[KN1] Martin Kassabov and Nikolay Nikolov, *Universal lattices and property* (τ)*,* Invent. Math. 165 (2006), no. 1, 209–224.

[KN2] Martin Kassabov and Nikolay Nikolov, *Cartesian products as profinite completions.* Int. Math. Res. Not. 2006, Art. ID 72947, 17 pp.

[Kazh] David Kazhdan, *On the connection of the dual space of a group with the structure of its closed subgroups. (Russian)* Funkcional. Anal. i Priložen. 1 (1967), 71–74

[KL] Peter Kleidman and Martin Liebeck, *The subgroup structure if the finite classical groups,* Cambridge University Press 1990.

[LN] Ottmar Loos and Erhard Neher, *Steinberg groups for Jordan pairs,* Oberwolfach preprints 2011-29.

[LZ] Alexander Lubotzky and Andrzej Zuk, *On property* (τ)*,* book in preparation, available at http://www.ma.huji.ac.il/ alexlub/

[NPS] Remus Nicoara, Sorin Popa and Roman Sasyk, *On* II_1 *factors arising from 2-cocycles of* w*-rigid groups.* J. Funct. Anal. 242 (2007), no. 1, 230–246.

[Sh1] Yehuda Shalom, *Bounded generation and Kazhdan's property* (T). Inst. Hautes Études Sci. Publ. Math. No. 90 (1999), 145–168.

[Sh2] Y. Shalom, *The algebraization of Kazhdan's property* (T). International Congress of Mathematicians. Vol. II, 1283–1310, Eur. Math. Soc., Zürich, 2006.

[Shi] Z. Shi, *Groups graded by finite root systems.* Tohoku Math. J. (2) 45 (1993), no. 1, 89-108.

[St] Robert Steinberg, *Lectures on Chevalley groups.* Yale University, 1968.

[Su] Andrei A. Suslin, *The structure of the special linear group over rings of polynomials. (Russian)* Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 2, 235–252.

[Ti] Jacques Tits, *Moufang octagons and the Ree groups of type* ${}^2F_4$. Amer. J. Math. 105 (1983), no. 2, 539–594.

[Va] Leonid Vaserstein, *Groups having the property* T*. (Russian)* Funkcional. Anal. i Prilozen. 2 (1968), no. 2, 86.

[Va2] L. Vaserstein, *Bounded reduction of invertible matrices over polynomial rings by addition operations,* preprint, 2007.

[Zh] Z. Zhang, *Nonassociative algebras and groups with property* (T)*,* preprint (2013), arXiv:1311.4089.

[Zh2] Z. Zhang, *work in preparation*

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