Solutions Homework #3

1. Let $\{B_i \mid i \in I\}$ be an indexed collection of subsets of a set U, and let A be a subset of U. Prove the generalized distributivity laws (a) and (b) as instructed below:

(a) $A \cap (\bigcup_{i\in I} B_i) = \bigcup_{i\in I}(A \cap B_i)$

(b) $A \cup (\bigcap_{i\in I} B_i) = \bigcap_{i\in I}(A \cup B_i)$

For (a) give a proof similar to the proof of de Morgan laws given in Lecture 4 (or § 1.5 of the book). Then deduce (b) from (a) and de Morgan laws.

Solution: (a) Let $x \in U$. We will prove the equality $A \cap (\bigcup_{i\in I} B_i) = \bigcup_{i\in I}(A \cap B_i)$ by showing that the statements $x \in A \cap (\bigcup_{i\in I} B_i)$ and $x \in \bigcup_{i\in I}(A \cap B_i)$ are equivalent.

We have $x \in A \cap (\bigcup_{i\in I} B_i) \iff (x \in A \text{ and } x \in \bigcup_{i\in I} B_i) \iff (x \in A$ and $x \in B_i$ for some $i \in I) \iff x \in A \cap B_i$ for some $i \in I \iff x \in \bigcup_{i\in I}(A \cap B_i)$.

In the above chain of equivalences we used

(i) the definition of intersection for the first and third $\iff$

(ii) the definition of union for the second and fourth $\iff$

Remark: A more common way to prove that two sets C and D are equal is to show that they are contained in each other: $C \subseteq D$ and $D \subseteq C$ or, equivalently, to prove the implications "$x \in C \Rightarrow x \in D$" and "$x \in D \Rightarrow x \in C$". Here we managed to prove both implications simultaneously, but this is not always possible.

(b) We can prove that two subsets of the universal set U are equal by showing that their complements are equal. So let us consider the complement of each of the two sets in (b) and use part (a) and de Morgan laws to show that these complements are equal.

Using de Morgan laws twice, the complement of the left-hand side of (b) is $(A \cup (\bigcap_{i\in I} B_i))^c = A^c \cap (\bigcap_{i\in I} B_i)^c = A^c \cap (\bigcup_{i\in I} B_i^c)$. Now applying the result of (a) with A^c playing the role of (a) and B_i^c playing the role of B_i, we get $A^c \cap (\bigcup_{i\in I} B_i^c) = \bigcup_{i\in I}(A^c \cap B_i^c)$. Finally, using de Morgan laws again $\bigcup_{i\in I}(A^c \cap B_i^c) = \bigcup_{i\in I}(A \cup B_i)^c$.

On the other hand, again by de Morgan laws the complement of the right-hand side of (b) is $(\bigcap_{i\in I}(A \cup B_i))^c = \bigcup_{i\in I}(A \cup B_i)^c$ which, by the previous paragraph, equals the complement of the left-hand side. This completes the proof.

2. Problem 2(a)(b)(d)(e) in 1.5 from the BOOK. Make sure to explain why the statement is true/false.

Solution: (a) This statement asserts that for any real number x we can find another real number a which is larger than $|x|$. This is true – no matter which x we start with, if we take $a = |x| + 1$, then $a \in \mathbb{R}$ and $|x| < a$. Using the formal negation rules, the negation of this statement is

$\exists x \in \mathbb{R}$ such that $\forall a \in \mathbb{R}$ we have $|x| \geq a$.

(b) This statement asserts that there exists a real number a which is less than any natural number. This is true – for instance, $a = 0$ has this property (in fact, any $a < 1$ would work). The negation of this statement is

$\forall a \in \mathbb{R}\ \ \exists x \in \mathbb{N}$ such that $a \geq x$.

(d) Note that for any $a, b \in \mathbb{R}$ we have $|a - b| \leq 100 \iff a \in [b - 100, b + 100]$ (here $[c, d]$ is the closed interval $\{x \in \mathbb{R} \mid c \leq x \leq d\}$). Thus the statement can be rephrased as follows: there exists a real number b such that every natural number a lies in the interval $[b - 100, b + 100]$. This is clearly false as the set of natural numbers is not bounded from above. The negation is

$\forall b \in \mathbb{R}\ \ \exists a \in \mathbb{N}$ such that $|a - b| > 100$.

(e) This statement is false. For instance, let $a = -1$. Then $\sqrt{a^2} = \sqrt{1} = 1 \neq -a$. The negation is

$\exists a \in \mathbb{R}$ such that $\sqrt{a^2} \neq a$.

3. Problem 4(b)(c)(e) in 1.5 from the BOOK. Make sure to explain why the statement is true/false.

Solution: (b) This statement asserts that there exists a subset S of $\mathbb{R}$ which contains exactly one element from each open interval $(k - 1, k)$ with $k \in \mathbb{N}$. This statement is true; for instance, the set $S = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\} = \{k - \frac{1}{2} \mid k \in \mathbb{N}\}$ has this property. The negation is

$\forall S \subseteq \mathbb{R}\ \ \exists k \in \mathbb{N}$ such that $|(k - 1, k) \cap S| \neq 1$.

(c) This statement asserts that for every $k \in \mathbb{N}$ there exists a non-empty subset S of $\{1, 2, \ldots, k\}$ such that the difference between any two elements of S is even. This statement is true; for instance, for any $k \in \mathbb{N}$ we can take $S = \{1\}$. The negation is

$\exists k \in \mathbb{N}$ such that $\forall S \in \mathcal{P}(\{1, 2, \ldots, k\})$ we have $S = \emptyset$ or $\exists\ x, y \in S$ such that $x - y$ is odd.

(e) This statement asserts that for any set S, the empty set is a PROPER subset of the power set $\mathcal{P}(S)$. This statement is true. Indeed, the empty set $\emptyset$ is a subset of any set A; in addition, if A is non-empty, then $\emptyset$ is a proper subset of A. Thus, the statement is equivalent to saying that for any set S the power set $\mathcal{P}(S)$ is non-empty, which is true since $\mathcal{P}(S)$ always contains the empty set. The negation is

$\exists$ a set S such that $\emptyset \not\subset \mathcal{P}(S)$.

4. Express the following statements in symbolic form using quantifiers:

(a) Distinct real numbers have distinct cubes.

(b) There exists the smallest natural number.

Solution: In both parts we transform the statement into symbolic form in several steps:

(a) Distinct real numbers have distinct cubes.

Any two distinct real numbers have distinct cubes.

For any two distinct real numbers, their cubes are distinct.

$\forall x, y \in \mathbb{R}$ such that $x \neq y$ we have $x^3 \neq y^3$

This is already an acceptable answer; however, it is not entirely satisfactory in the sense that it is not of the form $\forall x, y \in \mathbb{R}\ S(x, y)$ for some quantifier-free statement $S(x, y)$. One way to transform our answer into the latter form is as follows:

$\forall x, y \in \mathbb{R}$ we have $x = y$ or $x^3 \neq y^3$

(b) There exists the smallest natural number.

There exists a natural number which is less than or equal to any natural number.

$\exists n \in \mathbb{N}$ such that $\forall m \in \mathbb{N}$ we have $n \leq m$.

6. Problem 7 in 1.6 from the BOOK. In all parts of this problem we assume that a'priori m stands for an arbitrary integer.

Solution: (a) For instance, "m is divisible by 4" is such a condition. This condition is sufficient for $\frac{m}{2} \in \mathbb{Z}$ (if m is divisible by 4, it is also divisible by 2 and hence $\frac{m}{2} \in \mathbb{Z}$), but not necessary – for instance, $m = 2$ is not divisible by 4, but $\frac{2}{2} \in \mathbb{Z}$.

(b) For instance, "m is not relatively prime to 6" is such a condition. This condition is necessary for $\frac{m}{2} \in \mathbb{Z}$ (if $\frac{m}{2} \in \mathbb{Z}$, then m is even, so $gcd(m, 6) \geq 2$), but not sufficient – for instance, $m = 3$ is not relatively prime to 6, but $\frac{3}{2} \notin \mathbb{Z}$.

(c) For instance, "$m + 1$ is odd" is such a condition.

7. Let P, Q, R and S be statements. If U is any statement obtained from P, Q, R and S using logical connectives $\wedge$, $\vee$, $\sim$ and $\implies$, the *complexity of* U is the number of logical connectives that it involves. For instance, the statement $(P \vee Q) \implies (P \vee S)$ has complexity 3 (we used $\implies$ once and $\vee$ twice).

In each of the following examples find a statement V which is equivalent to the given statement U such that either V has smaller complexity than U or V has the same complexity as U, but involves fewer occurrences of $\implies$. Prove that your V is indeed equivalent to U.

(a) U is $P \implies (Q \implies R)$

(b) U is $(P \implies Q) \vee (P \implies R)$

(c) U is $P \wedge (\sim Q) \wedge (\sim R) \wedge (\sim S)$

Hint: There are several ways to figure out the answer. First, you can rewrite U using English sentences (and think of a different, but equivalent formulation). Second, you can try to use the laws we already established. Finally, you can write down the truth table for U, although this will not automatically tell you how to find V.

Solution: (a) The statement says 'If P is true, then the truth of Q implies the truth of R'. Equivalently, if both P and Q are true, then R is true. Thus, we can take V to be the statement $(P \wedge Q) \implies R$ (both U and V have complexity 2, but V has fewer occurrences of $\implies$). One can use truth tables to check that U and V are indeed equivalent.

(b) The statement says 'The truth of P implies the truth of Q or the truth of P implies the truth of R'. Equivalently, 'if P is true, then Q is true or R is true', so we can take V to be the statement $P \implies (Q \vee R)$ (it has complexity 2 while U has complexity 3). The last transition is probably less convincing than the corresponding transition in the proof of (a), so we should formally verify that $U \equiv V$. Perhaps the quickest way do this is to compute the negations of U and V.

Recall that for any statements S and T we have $\sim (S \implies T)$ is equivalent to $S \wedge (\sim T)$. Thus,

$$\begin{aligned} \sim U &\equiv (\sim (P \implies Q)) \wedge (\sim (P \implies R)) \\ &\equiv (P \wedge (\sim Q)) \wedge (P \wedge (\sim R)) \equiv P \wedge (\sim Q) \wedge (\sim R). \end{aligned}$$

On the other hand,

$$\sim V \equiv P \wedge (\sim (Q \vee R)) \equiv P \wedge ((\sim Q) \wedge (\sim R)) \equiv P \wedge (\sim Q) \wedge (\sim R).$$

Thus, we prove that $\sim U \equiv \sim V$ and hence $U \equiv V$.

(c) We can take V to be the statement $\sim ((\sim P) \vee Q \vee R \vee S)$. It is clear from the basic negation rules that $\sim U \equiv \sim V$ and hence $U \equiv V$.

8.

(a) Let P, Q and R be arbitrary statements. Prove that the following statements are equivalent: $(P \implies R) \vee (Q \implies R)$ and $(P \wedge Q) \implies R$.

- (b) Locate a logical mistake in the following argument:

  Consider the following statements.

  - P : n is divisible by 2
  - Q : n is divisible by 3
  - R : n is divisible by 6

  In each statement we assume that $n \in \mathbb{Z}$ and consider n as a free variable. The implication $P \implies R$ is false since there exist integers which are divisible by 2, but not divisible by 6, e.g. $n = 2$. Likewise, the implication $Q \implies R$ is also false since $n = 3$ is divisible by 3, but not divisible by 6. Hence $(P \implies R) \vee (Q \implies R)$ is also false (being the disjunction of two false statements).

  On the other hand, the implication $(P \wedge Q) \implies R$ asserts that if an integer n is divisible by both 2 and 3, then it is divisible by 6. This is a true implication (as we will prove in a few weeks).

  Thus, in the above example $(P \implies R) \vee (Q \implies R)$ is false while $(P \wedge Q) \implies R$ is true. This contradicts (a).

Solution: (a) This can be verified using truth tables or arguing similarly to 7(b).

(b) The mistake occurred when we said "Hence $(P \implies R) \vee (Q \implies R)$ is also false". The issue here is the following. The statements $P \implies R$ and $Q \implies R$ depend on a free variable n. By saying that these statements are false we mean that they are 'generally false' (that is, each of them is false for at least one value of n). It is NOT true that the disjunction of two generally false statements is generally false (we could only claim this if we knew that we could make both $P \implies R$ and $Q \implies R$ false by using the same value of n).