Solutions to Homework #4

1. In both parts of this problem A, B and C are sets.

(a) Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) Assume that $A \times B \subseteq A \times C$ and $A \neq \emptyset$. Prove that $B \subseteq C$. Explain why the conclusion may be false if $A = \emptyset$ and clearly state where you use that $A \neq \emptyset$ in your proof.

Solution: (a) We will prove inclusions in both directions:

"$\subseteq$" Take any $x \in A \times (B \cap C)$. By definition $x = (a, d)$ where $a \in A$ and $d \in B \cap C$. Since $d \in B \cap C$, we have $d \in B$ and $d \in C$. Thus, $x = (a, d)$ lies in both $A \times B$ and $A \times C$ and hence $x \in (A \times B) \cap (A \times C)$. Thus we proved the inclusion $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

"$\supseteq$" Now take any $x \in (A \times B) \cap (A \times C)$, so $x \in A \times B$ and $x \in A \times C$. The first condition implies that $x = (a, b)$ for some $a \in A$ and $b \in B$. Plugging (a, b) for x in the second condition, we get $(a, b) \in A \times C$, which means $a \in A$ (which we already know is true) and $b \in C$. Thus, $b \in B$ and $b \in C$, so $b \in B \cap C$ and hence $x = (a, b) \in A \times (B \cap C)$. Thus we proved the opposite inclusion $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

(b) First observe that $\emptyset \times D = \emptyset$ for any set D. Thus, if $A = \emptyset$, the inclusion $A \times B \subseteq A \times C$ holds for any sets B and C and cannot possibly imply any relation between B and C.

Now let us assume that $A \neq \emptyset$ and $A \times B \subseteq A \times C$. We want to prove that $B \subseteq C$.

So take any $b \in B$; our goal is to show that $b \in C$. Since A is non-empty, it has it least element, so we can pick some $a \in A$. Then $(a, b) \in A \times B$ by definition of the Cartesian product. Since $A \times B \subseteq A \times C$, it follows that $(a, b) \in A \times C$ which, in turn, implies that $b \in C$. Thus the implication $(b \in B \Rightarrow b \in C)$ is (identically) true and we proved that $B \subseteq C$.

2. Prove that there are no integers a and b such that $a^2 = 4b + 2$ (equivalently, if we divide the square of any integer by 4 with remainder, the remainder is never equal to 2).

Solution: By way of contradiction, assume that $a^2 = 4b + 2$ for some $a, b \in \mathbb{Z}$. Rewriting the equation as $a^2 = 2(2b + 1)$, we see that a^2 is even, so a is even by a theorem from Lecture 6. Thus, $a = 2k$ for some $k \in \mathbb{Z}$. Substituting $2k$ for a in the above equality, we get $4k^2 = 2(2b + 1)$, and dividing by 2, we get $2k^2 = 2b+1$. The obtained equality is impossible since the left-hand side is even and the right-hand side is odd. Thus, we reached the desired contradiction.

3. Let x and y be real numbers, at least one of which is irrational. Prove that at least one of the numbers $x+y$ and $x-y$ is irrational.

Solution: First observe that the statement of the problem can be rephrased as follows. Let $x, y \in \mathbb{R}$. Prove the implication

$$(x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}) \Rightarrow (x+y \notin \mathbb{Q} \text{ or } x-y \notin \mathbb{Q}).$$

We will prove this implication by contrapositive, that is, we will prove the implication

$$(x+y \in \mathbb{Q} \text{ and } x-y \in \mathbb{Q}) \Rightarrow (x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}) \qquad (***)$$

So, let us take any $x, y \in \mathbb{R}$ such that $x+y \in \mathbb{Q}$ and $x-y \in \mathbb{Q}$. This means that $x+y = \frac{m}{n}$ and $x-y = \frac{s}{t}$ for some $m, n, s, t \in \mathbb{Z}$ with $s, t \neq 0$. Taking the sum and the difference of these equalities and dividing by 2, we get $x = \frac{1}{2}\left(\left(\frac{m}{n} + \frac{s}{t}\right)\right) = \frac{mt+ns}{2nt}$ and $y = \frac{mt-ns}{2nt}$. Since $mt \pm ns \in \mathbb{Z}, 2nt \in \mathbb{Z}$ and $2nt \neq 0$ (as $2, n, t \neq 0$), we deduce that $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, which completes the proof.

4. Problem 10 in 2.1 from the BOOK.

Solution: we will proceed cyclically: "(a)$\Rightarrow$(b)","(b)$\Rightarrow$(c)" and finally "(c)$\Rightarrow$(a)".

"(a)$\Rightarrow$(b)" Assume that a is odd. This means that $a = 2k+1$ for some $k \in \mathbb{Z}$. Then $3a+1 = 6k+4 = 2(3k+2)$. Since $3k+2$ is an integer, $3a+1$ is even.

"(b)$\Rightarrow$(c)" Now assume that $3a+1$ is even, so $3a+1 = 2m$ for some $m \in \mathbb{Z}$. Then $a+1 = 2m-2a = 2(m-a)$, so $\frac{a+1}{2} = m-a \in \mathbb{Z}$.

"(c)$\Rightarrow$(a)" Finally, assume that $\frac{a+1}{2} \in \mathbb{Z}$. Thus $\frac{a+1}{2} = n$ for some $n \in \mathbb{Z}$. Then $a = 2n-1$, so a is odd.

Remark: Since the proofs of the three implications above are independent of each other, it would have been perfectly fine to use the same letter k in all three proofs (instead of using k in the first case, m in the second case and n in the third case).

5. Problem 19 in 2.1 from the BOOK.

Solution: Consider the real number $\sqrt{2}^{\sqrt{2}}$. It is either rational or irrational, so we consider two cases:

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. In this case we can simply set $a = b = \sqrt{2}$ (as we already proved that $\sqrt{2}$ is irrational).

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. In this case we take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. We know that b is irrational and a is irrational by the extra assumption in this case; on the other hand, by the exponent laws $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational.

Remark: One can show that the number $\sqrt{2}^{\sqrt{2}}$ is actually irrational, so case 2 occurs; however, this requires much more work and more advanced tools.

Note that if we were asked to find explicit irrational a and b such that a^b is rational, the above argument would not have been sufficient; however, we were only asked to prove the existence of such numbers a and b, which is accomplished above.

6. Prove by induction that the following equalities hold for any $n \in \mathbb{N}$:

(a) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
(b) $a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$ where $a, r \in \mathbb{R}$ and $r \neq 1$

Solution: (a) Let S_n be the statement $1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. We prove that S_n is true for all $n \geq 1$ by induction on n.

Induction base: S_1 is true since $1^2 = 1 = \frac{1\cdot 2\cdot 3}{6}$.

Induction step: Here it will be technically more convenient to do the induction step in the form $S_{n-1} \Rightarrow S_n$ for $n \geq 2$.

Assume S_{n-1} is true, that is, $1^2 + \ldots + (n-1)^2 = \frac{(n-1)n(2n-1)}{6}$. Then

$$1^2 + \ldots + n^2 = (1^2 + \ldots + (n-1)^2) + n^2 = \frac{(n-1)n(2n-1)}{6} + n^2 = \frac{2n^3 - 3n^2 + n}{6} + n^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}.$$

Thus, S_n is true as well.

(b) Let S_n be the statement $a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$

Induction base: LHS of S_1 is a, and RHS of S_1 is $a\frac{1-r}{1-r} = a$, so S_1 is true.

Induction step: "$S_n \Rightarrow S_{n+1}$". Assume that S_n is true for some n. Then $a + ar + ar^2 + \ldots + ar^n = (a + ar + ar^2 + \ldots + ar^{n-1}) + ar^n = a\frac{1-r^n}{1-r} + ar^n = a\frac{1-r^n+r^n(1-r)}{1-r} = a\frac{1-r^{n+1}}{1-r}$, so S_{n+1} is true.

7. In Lecture 7 we proved that for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \mathbb{Z}$ such that $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$. Moreover, it is shown that such a_n and b_n satisfy the following recursive relations: $a_1 = b_1 = 1$ and $a_{n+1} = a_n + 2b_n$, $b_{n+1} = a_n + b_n$ for all $n \in \mathbb{N}$.

(a) Use the above recursive formulas and mathematical induction to prove that $a_n^2 - 2b_n^2 = (-1)^n$ for all $n \in \mathbb{N}$.

(b) Prove that for all $n \in \mathbb{N}$ there exist $c_n, d_n \in \mathbb{Z}$ such that $(1+\sqrt{3})^n = c_n + d_n\sqrt{3}$.

(c) (bonus) Find a simple formula relating c_n and d_n (similar to the one in (a)) and prove it.

Solution:(a) *Base case:* By definition, $a_1 = b_1 = 1$, so $a_1^2 - 2b_1^2 = 1 - 2 = -1 = (-1)^1$.

Induction step: Assume that $a_n^2 - 2b_n^2 = (-1)^n$ for some $n \geq 1$. Since $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$ by construction, we have $a_{n+1}^2 - 2b_{n+1}^2 = (a_n+2b_n)^2 - 2(a_n+b_n)^2 = a_n^2 + 4a_nb_n + 4b_n^2 - 2a_n^2 - 4a_nb_n - 2b_n^2 = 2b_n^2 - a_n^2 = -(a_n^2 - 2b_n^2) = -(-1)^n = (-1)^{n+1}$, as desired.

(b) This is completely analogous to the corresponding example from Lecture 7. The recursive formulas for c_n and d_n are given by $c_1 = d_1 = 1$ and $c_{n+1} = c_n + 3d_n$ and $d_{n+1} = c_n + d_n$.

(c) Computing $c_n^2 - 3d_n^2$ for a few small values of n, we conjecture that $c_n^2 - 3d_n^2 = (-2)^n$. The proof of this formula is analogous to part (a).

8. Use induction to solve Problem 7 from 2.2 in the BOOK.

Solution: The inequality we asked to prove is clearly equivalent to

$$\frac{(n^2)!}{(n!)^2} \geq 1.$$

We will prove the inequality in this equivalent form by induction on n.

Base case: If $n = 1$, the LHS is $\frac{1!}{(1!)^2} = 1$. Since $1 \geq 1$, the inequality is true for $n = 1$.

Induction step: Now fix $n \geq 1$, and assume that $\frac{(n^2)!}{(n!)^2} \geq 1$. We want to show that $\frac{((n+1)^2)!}{((n+1)!)^2} \geq 1$. To do this let us first relate $(n+1)^2)!$ and $((n+1)!)^2$ to $(n^2)!$ and $(n!)^2$, respectively.

Since $(n+1)! = n!(n+1)$, we have $((n+1)!)^2 = (n!)^2(n+1)^2$. The number $((n+1)^2)! = (n^2+2n+1)!$ is equal to the product of all natural numbers from 1 to n^2+2n+1, while $(n^2)!$ is the product of all natural numbers from 1 to n^2. Thus, $((n+1)^2)!$ is equal to $(n^2)!$ times the product of all natural numbers from n^2+1 to n^2+2n+1:

$$((n+1)^2)! = (n^2)! \prod_{k=n^2+1}^{n^2+2n+1} k.$$

Substituting the obtained expressions for $((n+1)^2)!$ and $((n+1)!)^2$, we get

$$\frac{((n+1)^2)!}{((n+1)!)^2} = \frac{(n^2)! \prod\limits_{k=n^2+1}^{n^2+2n+1} k}{(n!)^2(n+1)^2} = \frac{(n^2)!}{(n!)^2} \cdot \frac{\prod\limits_{k=n^2+1}^{n^2+2n+1} k}{(n+1)^2}$$

We claim that each fraction in the last expression is ≥ 1. Indeed, $\frac{(n^2)!}{(n!)^2} \geq 1$ by the induction hypothesis. The numerator of the second fraction is the product of $2n+1$ integers, each of which is at least $n^2+1 \geq n+1$, so the second fraction is at least $\frac{(n+1)^{2n+1}}{(n+1)^2} = (n+1)^{2n-1} \geq n+1 > 1$ (since $n \geq 1$).

Thus, $\frac{\prod_{k=n^2+1}^{n^2+2n+1} k}{(n+1)^2}$ is ≥ 1 being a product of two real numbers ≥ 1, and we proved the desired inequality $\frac{((n+1)^2)!}{((n+1)!)^2} \geq 1$.