Homework #3. Solutions to selected problems.

1. Let V be a finite-dimensional vector space over a field F of characteristic 2 and H a symmetric (=skew-symmetric since $\mathrm{char} F = 2$) bilinear form on V. Prove that there exist subspaces V_1 and V_2 of V such that

(a) $V = V_1 \oplus V_2$ and $V_1 \perp V_2$ (that is, $H(v, w) = 0$ for all $v \in V_1$ and $w \in V_2$).

(b) $H_{|V_1}$ is diagonalizable (that is, $[H_{|V_1}]_{\beta_1}$ is diagonal for some basis β_1 of V_1)

(c) $H_{|V_2}$ is alternating and non-degenerate (such a form is called symplectic).

Solution: First we clarify the statement. In order for the assertion to be true in all cases, we need to allow the possibility that $V_1 = 0$. In this case we consider condition (b) as being vacuous (the formal meaning of (b) in this case may be unclear since it talks about 0×0 matrix being diagonal).

We argue by induction on n. If $\dim V = 1$, then $[H]_\beta$ is diagonal for any basis β, so we simply set $V_1 = V$ and $V_2 = 0$.

Now fix $n > 1$, and assume that the statement of the problem holds for all spaces of dimension less than n. We consider two cases.

Case 1: There exists $x \in V$ with $H(x, x) \neq 0$. In this case we imitate the induction step from the proof of Theorem 3.4. More precisely, let $W = Fx = Span(x)$. Then $H_{|W}$ is non-degenerate, whence $V = W \oplus W^\perp$. Applying the induction hypotheses to H restricted to $W^\perp$, we conclude that there exist subspaces W_1 and W_2 of $W^\perp$ such that $W^\perp = W_1 \oplus W_2$, $W_1 \perp W_2$, $[H_{|W_1}]_{\gamma_1}$ is diagonal for some basis γ_1 of W_1 and $H_{|W_2}$ is alternating and non-degenerate.

If we now define $\beta_1 = \gamma_1 \cup x$ and $V_1 = Span(\beta)$, then $[H_{|V_1}]_{\beta_1}$ is still diagonal. It is also true that $V = V_1 \oplus W_2$ and $V_1 \perp W_2$ (since W_2 is orthogonal to both x and V_1).

Case 2: $H(x, x) = 0$ for all $x \in V$. In this case H is alternating (by definition), whence by Theorem 5.1 there exists a basis β of V such that $[H]_\beta$ is block-diagonal with the first k diagonal blocks equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (note that $1 = -1$ since $\mathrm{char}(F) = 2$ although this is not important for the proof) and the last diagonal block is the $l \times l$ zero matrix (here k and l satisfy $2k + l = n$). If we now let V_1 be the span of the first $2k$ elements of β and let V_2 be the span of the last l elements of β, it is clear that conditions (a),(b) and (c) are all satisfied.

2. Let H be a bilinear form on a vector space V.

(a) Assume that V is finite-dimensional. Prove that H is left-nondegenerate if and only if H is right-nondegenerate.

(b) (bonus) Construct an example of an infinite-dimensional vector space V and a bilinear form H on V which is left-nondenerate but not right-nondegenerate.

Solution: (a) Fix any basis β of V. We shall use the result of Problem 6(b) in HW#1 in the following form: A bilinear form H on a finite-dimensional vector space V is left-nondegenerate $\iff$ the matrix $[H]_\beta$ is invertible.

Given a bilinear form H, define $H^T : V \times V \to F$ by $H^T(x, y) = H(y, x)$. It is clear that

(i) H^T is also bilinear

(ii) H^T is right-nondegenerate $\iff$ H is left-nondegenerate

(iii) $[H^T]_\beta = [H_\beta]^T$ (the matrix of H^T is the transpose of the matrix of H)

Since a matrix is invertible if and only if its transpose is invertible, we have the following chain of equivalences:

H is left-nondegenerate $\iff$ $[H]_\beta$ is invertible $\iff$ $[H^T]_\beta = [H_\beta]^T$ is invertible $\iff$ H^T is left-nondegenerate $\iff$ H is right-nondegenerate.

(b) Let $V = F^0_\infty$ be the vector space defined in Problem 5 of HW#2, and define the bilinear form H on V by $H(v, w) = \sum\limits_{i=1}^{\infty} v_{i+1} w_i$ (where v_k and w_k are the k^{th} coordinates of v and w respectively). Note that $H(v, w)$ is well defined since by definition elements of V only have finitely many nonzero coordinates. It is clear that H is bilinear.

The form H is left-degenerate since v_1 does not appear in the formula for $H(v, w)$ and thus $H(e_1, w) = 0$ for all w. On the other hand, H is right-nondegenerate. Indeed, take any nonzero $w \in V$ and choose any i with $w_i \neq 0$. Then $H(e_{i+1}, w) = w_{i+1} \neq 0$.

In general, it is not hard to show that a bilinear form H on a vector space V is left-nondegenerate if and only if the columns of the matrix $[H]_\beta$ (where β is any basis of V) are linearly independent and H is right-nondegenerate if and only if the rows of the matrix $[H]_\beta$ are linearly independent. For a finite square matrix linear independence of columns is equivalent to linear independence of rows, which is why in the finite-dimensional case being left-nondegenerate is equivalent to being right-nondegenerate.

The matrix of the form H in the above example (with respect to the standard basis) is obtained by placing the (infinite size) identity matrix to the right of a column of zeroes. The presence of a zero column forces columns to be linearly dependent; on the other hand, the rows are $e_2, e_3, \ldots$ and hence are linearly independent.

3. Let V be an inner product space.

(a) Prove the parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in V$.

(b) Show that $\langle x, y\rangle$ can be expressed as a linear combination of squares of norms. In Lecture 6 we discussed how to do this for the real inner product spaces.

Solution to 3(b): As we showed in class $\operatorname{Re}\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$. In order to find a similar expression for $\langle x, y\rangle$, we first express the imaginary part $\operatorname{Im}\langle x, y\rangle$ as the real part of the inner product of some vectors.

For all $x, y \in V$ we have $\langle x, y\rangle = \operatorname{Re}\langle x, y\rangle + i\operatorname{Im}\langle x, y\rangle$ and hence also $\langle x, iy\rangle = \operatorname{Re}\langle x, iy\rangle + i\operatorname{Im}\langle x, iy\rangle = i\operatorname{Re}\langle x, y\rangle + i^2\operatorname{Im}\langle x, y\rangle = -\operatorname{Im}\langle x, y\rangle + i\operatorname{Re}\langle x, y\rangle$.

Thus, $\operatorname{Im}\langle ix, y\rangle = -\operatorname{Re}(\langle ix, y\rangle) = \frac{1}{4}(\|x-iy\|^2 - \|x+iy\|^2$. Hence $\langle x, y\rangle = \operatorname{Re}\langle x, y\rangle + i\operatorname{Im}\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2) = \frac{1}{4}\sum_{k=0}^{3}(-i)^k\|x+i^k y\|^2$.

4. Let V be a finite-dimensional complex inner product space and $A \in \mathcal{L}(V)$. Prove that $\operatorname{Im}(A^*) = \operatorname{Ker}(A)^\perp$ (where the orthogonal complement is with respect to the inner product on V).

Solution: We will prove the equality by showing that inclusions in both directions hold:

1. Why $\operatorname{Im}(A^*) \subseteq \operatorname{Ker}(A)^\perp$. Take any $x \in \operatorname{Im}(A^*)$. Thus, $x = A^*v$ for some $v \in V$. By definition of adjoint, for all $y \in \operatorname{Ker}(A)$ we have $\langle y, x\rangle = \langle y, A^*v\rangle = \langle Ay, v\rangle = 0$, so $x \in \operatorname{Ker}(A)^\perp$.

2. Why $\operatorname{Ker}(A)^{\perp} \subseteq \operatorname{Im}(A^*)$. This will require a bit more work.

First we claim that $\operatorname{Im}(A^*)^{\perp} \subseteq \operatorname{Ker}(A)$. Indeed, take any $x \in \operatorname{Im}(A^*)^{\perp}$. This means that $\langle x, A^* v\rangle = 0$ for all $v \in V$. Equivalently, $\langle Ax, v\rangle = 0$ for all $v \in V$. Since the inner product is non-degenerate, this forces $Ax = 0$, that is, $x \in \operatorname{Ker}(A)$.

Next we will use the following two facts which follow easily from the definition of orthogonal complements:

(i) If Y, Z are subspaces of V, with $Y \subseteq Z$, then $Z^{\perp} \subseteq Y^{\perp}$

(ii) For any subspace Y of V we have $Y \subseteq (Y^{\perp})^{\perp}$.

Note that (i) holds for the orthogonal complements with respect to any form (bilinear or sesquilinear). For (ii) to be true we only need the form H (with respect to which the complements are taken) to be reflexive, which by definition means $H(x, y) = 0 \iff H(y, x) = 0$ (in particular, H could be symmetric, skew-symmetric, Hermitian or skew-Hermitian).

Finally, we claim that if H is a reflexive non-degenerate form on a finite-dimensional vector space V (which is the case in our problem), then

(iii) $Y = (Y^{\perp})^{\perp}$ for any subspace Y.

Indeed, since H is non-degenerate, by Problem 4 in HW#2 we have $\dim Z^{\perp} = \dim V - \dim Z$ for any subspace Z. Thus, $\dim(Y^{\perp})^{\perp} = \dim V - \dim Y^{\perp} = \dim V - (\dim V - \dim Y) = \dim Y$. Combined with the inclusion $Y \subseteq (Y^{\perp})^{\perp}$, this implies (iii)

Following this digression, we can now finish the proof. Applying (i) to the inclusion $\operatorname{Im}(A^*)^{\perp} \subseteq \operatorname{Ker}(A)$, we get $\operatorname{Ker}(A)^{\perp} \subseteq (\operatorname{Im}(A^*)^{\perp})^{\perp}$. By (iii) we have $(\operatorname{Im}(A^*)^{\perp})^{\perp} = \operatorname{Im}(A^*)$, and therefore $\operatorname{Ker}(A)^{\perp} \subseteq \operatorname{Im}(A^*)$, as desired.

5. Let V be an inner product space where $\dim V$ is finite or countable, β an orthonormal basis of V and $A \in \mathcal{L}(V)$.

(a) Prove that if $A^* \in \mathcal{L}(V)$ is any operator such that $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all $x, y \in V$, then $[A^*]_\beta = [A]_\beta^*$ (where $[A]_\beta^*$ is the conjugate transpose of A). In particular, this shows that the adjoint operator is unique (if exists).

(b) As we proved in class, the adjoint A^* always exists if $\dim V$ is finite. Now use (a) and a result from earlier homeworks to show that if V is countably-dimensional, then the adjoint A^* may not exist.

Solution: (a) Let $v_1, v_2, \ldots$ be the elements of β. Since β is orthonormal, we have $\langle Av_i, v_j\rangle = \overline{[Av_i]_\beta}^T [v_j]_\beta = \overline{[A]_\beta [v_i]_\beta}^T [v_j]_\beta = \overline{[v_i]_\beta}^T \overline{[A]_\beta}^T [v_j]_\beta = e_i^T [A]_\beta^* e_j = (i,j)$ entry of $[A]_\beta^*$.

On the other hand, $\langle Av_i, v_j\rangle = \langle v_i, A^* v_j\rangle = \overline{[v_i]_\beta}^T [A^* v_j]_\beta = \overline{[v_i]_\beta}^T [A^*]_\beta [v_j]_\beta = e_i^T [A^*]_\beta e_j = (i,j)$ entry of $[A^*]_\beta$

Thus, each entry of $[A^*]_\beta$ is equal to the respective entry of $[A]_\beta^*$, whence $[A^*]_\beta = [A]_\beta^*$.

(b) As in 2(b), let $V = F_0^\infty$, and let $A \in \mathcal{L}(V)$ be the unique operator such that $A(e_i) = e_1$ for all i. Note that the matrix of A (with respect to the standard basis) has 1 everywhere in the first row and 0 everywhere else. If this A had the adjoint A^*, then by (a) the matrix of A^* would have had 1 everywhere in the first column. But this matrix has infinitely many nonzero entires in the same column and hence does not correspond to any linear map by Problem 5 in HW#2.