Bilinear Forms and Group Representations
Fall 2017. Midterm #2 (take-home part).
Due on Thursday, November 9th, in class

Directions: Provide complete arguments (do not skip steps). State clearly and FULLY any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given. If you are unable to solve a problem (or a part of a problem), you may still use its result to solve a later part of the same problem or a later problem in the exam.

Rules: You are NOT allowed to discuss midterm problems with anyone else except me. You may ask me any questions about the problems (e.g. if the formulation is unclear), but as a rule I will only provide minor hints. You may freely use the following resources:

(i) your class notes
(ii) homework solutions (both your solutions and solutions posted on the course webpage)
(iii) Ben Webster's 4657 notes
(iv) the book 'Representation Theory of Finite Groups' by Benjamin Steinberg

The use of other books or online sources is NOT allowed.

1. (8 pts) Let G be a group and H a subgroup of G. Let (ρ, V) be a representation of G, and let $V^H = \{v \in V : \rho(h)v = v \text{ for all } h \in H\}$ be the subspace of H-invariant vectors.

(i) Assume that H is normal in G. Prove that the subspace V^H is G-invariant.
(ii) Give an example showing that if H is not normal, then V^H need not be G-invariant.

2. (6 pts) Let (α, V) and (β, W) be representations of a group G. Let (α^*, V^*) be the dual of (α, V), and let $(\alpha^* \otimes \beta, V^* \otimes W)$ be the tensor product of the representations (α^*, V^*) and (β, W). Define the representation $(\gamma, \mathrm{Hom}(V, W))$ of G by

$$(\gamma(g))(f) = \beta(g) \circ f \circ \alpha(g)^{-1}$$

for all $f \in \mathrm{Hom}(V, W)$. Prove that

$$(\gamma, \mathrm{Hom}(V, W)) \cong (\alpha^* \otimes \beta, V^* \otimes W)$$

as representations of G. Make sure to provide all the details.

3. (4 pts) Let (ρ, V) and (ρ', V) be complex representations of a finite group G (the vector space V is the same for both representations). Suppose that $\rho'(g)$ is conjugate to $\rho(g)$ in $\mathrm{GL}(V)$ for every $g \in G$. Prove that the representations (ρ, V) and (ρ', V) are equivalent. **Note:** The result is not automatic since the matrix which conjugates $\rho'(g)$ to $\rho(g)$ may depend on g.

4. (8 pts) Let G be a finite group, and suppose you are given the character table of G. Give a simple algorithm to determine which elements of G lie in $[G, G]$ based on the character table. Make sure to prove that your algorithm works.

5. (6 pts) Let $G = D_{2n}$, the dihedral group of order $2n$. Let F be an algebraically closed field. Prove that every irreducible representation of G over F has dimension 1 or 2. **Note:** Partial credit for the case $F = \mathbb{C}$ will be given.

6. (8 pts) Let G be the group of order 20 from HW#8.6. Explicitly construct a 4-dimensional irreducible complex representation of G (and prove that your representation has the required property). Two bonus points for two substantially different constructions.

Hint for #6: S_5 or Heisenberg.