Math 3000. Solutions to the Second Midterm.

1. For each of the following functions determine whether it is injective and whether it is surjective. Give a **detailed** argument. Recall that $\mathbb{N}$ denotes the set of natural numbers.

(a) (5 pts) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 1$

(b) (5 pts) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) =$ the sum of all digits of x (e.g. $f(236) = 2 + 3 + 6 = 11$; if x is a single-digit number, then $f(x) = x$).

Solution: (a) The function is injective. Indeed, if $f(a) = f(b)$ for some $a, b \in \mathbb{N}$, then $a+1 = b+1$, and subtracting 1 from both sides, we get $a = b$.

The function is not surjective since 1 lies in the codomain, but there there is no $a \in \mathbb{N}$ such that $f(a) = 1$ (since if $a \in \mathbb{N}$, then $a \geq 1$ and hence $f(a) = a + 1 \geq 2 > 1$).

(b) The function is not injective. For instance, $f(11) = f(2) = 2$. The function is surjective. Indeed, take any $b \in \mathbb{N}$, and let n_b be the number with b digits all of which are equal to 1 (that is, $n_1 = 1, n_2 = 11, n_3 = 111$ etc.). Then $n_b \in \mathbb{N}$ and $f(n_b) = \underbrace{1 + \ldots + 1}_{b \text{ times}} = b$.

2. Let $A = \mathbb{R} \times \mathbb{R}$, and consider the relation $\sim$ on A given by

$$(x, y) \sim (z, w) \iff \max\{|x|, |y|\} = \max\{|z|, |w|\}$$

(a) (4 pts) Prove that $\sim$ is an equivalence relation

(b) (4 pts) Compute the equivalence class of $(1, 2)$. You should give a completely formal argument. Do not skip steps. The answer should be given in the form

$$[(1, 2)] = \{(x, y) \in \mathbb{R} \times \mathbb{R} \text{ s.t. } ***\}$$

where *** is a certain condition on x and y which does NOT involve the *max* function.

(c) (2 pts) Now draw $[(1, 2)]$ as a subset of $\mathbb{R} \times \mathbb{R}$.

Solution: (a) $\sim$ is reflexive since $\max\{|x|, |y|\} = \max\{|x|, |y|\}$ for any $(x, y) \in A$.

Symmetry: take any $(x, y), (z, w) \in A$ such that $(x, y) \sim (z, w)$. By definition this means that $\max\{|x|, |y|\} = \max\{|z|, |w|\}$ which is equivalent to $\max\{|z|, |w|\} = \max\{|x|, |y|\}$, and the latter (by definition) means that $(z, w) \sim (x, y)$.

Transitivity: take any $(x,y),(z,w),(u,v) \in A$ such that $(x,y) \sim (z,w)$ and $(z,w) \sim (u,v)$. By definition this means that $\max\{|x|,|y|\} = \max\{|z|,|w|\}$ and $\max\{|z|,|w|\} = \max\{|u|,|v|\}$. By transitivity of equality of real numbers we get $\max\{|x|,|y|\} = \max\{|u|,|v|\}$, so $(x,y) \sim (u,v)$.

(b) By the definition of an equivalence class we have

$$[(1,2)] = \{(x,y) \in \mathbb{R}\times\mathbb{R} \text{ s.t. } [(x,y)] \sim [(1,2)]\}$$
$$= \{(x,y) \in \mathbb{R}\times\mathbb{R} \text{ s.t. } \max\{|x|,|y|\} = \max\{|1|,|2|\}$$
$$= \{(x,y) \in \mathbb{R}\times\mathbb{R} \text{ s.t. } \max\{|x|,|y|\} = 2\}$$
$$= \{(x,y) \in \mathbb{R}\times\mathbb{R} \text{ s.t. } (|x| = 2 \text{ and } |y| \le 2) \text{ or } (|x| \le 2 \text{ and } |y| = 2)\}$$
$$\{(x,y) \in \mathbb{R}\times\mathbb{R} \text{ s.t. } (x = \pm 2 \text{ and } -2 \le y \le 2) \text{ or } (-2 \le x \le 2 \text{ and } y = \pm 2)\}.$$

(c) Geometrically, $[(1,2)]$ is the square with vertices $(-2,-2),(-2,2),(2,-2),(2,2)$. By a square here we mean just the boundary (perimeter), not the interior.

3. (5 pts) A midterm in a UVA math class had 3 problems. You know that

- 10 students solved problem 1 (we are counting all students who solved problem 1, not just students who solved problem 1 and did not solve anything else)
- 8 students solved problem 2
- 6 students solved problem 3
- 6 students solved both problems 1 and 2
- 5 students solved both problems 1 and 3
- 4 students solved both problems 2 and 3
- 3 students solved all 3 problems
- every student solved at least one problem

How many students took the midterm? Make sure to prove your answer.

Solution: For $i = 1,2,3$, let A_i be the set of students who solved problem i. Then the given information can be reformulated as follows: $|A_1| = 10$, $|A_2| = 8$, $|A_3| = 6$, $|A_1 \cap A_2| = 6$, $|A_1 \cap A_3| = 5$, $|A_2 \cap A_3| = 4$ and $|A_1 \cap A_2 \cap A_3| = 3$. Hence by the inclusion-exclusion principle we get

$$|A_1 \cup A_2 \cup A_3| = 10 + 8 + 6 - 6 - 5 - 4 + 3 = 12.$$

Finally, since every student solved at least one problem, $A_1 \cap A_2 \cap A_3$ is precisely the set of students who took the midterm. Thus, 12 students took the midterm.

4. (5 pts) Let $A = \{1, 2, 3, 4\}$. Find the total number of relations R on A such that R is **reflexive** and $|R| = 6$. Make sure to prove your answer.

Solution: By definition R is reflexive $\iff$ R contains each of the pairs $(1, 1), (2, 2), (3, 3), (4, 4)$. Thus, to construct a reflexive relation R with $|R| = 6$ we need to choose 2 of the 12 non-diagonal pairs in the set $A \times A$. The number of ways to do so is $\binom{12}{2} = \frac{12 \cdot 11}{2} = 66$.

5. Let $n \in \mathbb{N}$, and write $n = p_1^{a_1} \dots p_k^{a_k}$ where $p_1, \dots, p_k$ are distinct primes and each $a_i \in \mathbb{Z}_{\geq 0}$. Recall that for each prime p we define $\mathrm{ord}_p(n) = a_i$ if $p = p_i$ for some $1 \leq i \leq k$ and $\mathrm{ord}_p(n) = 0$ if $p \notin \{p_1, \dots, p_k\}$. The following three properties were established in HW#6:

(i) $\mathrm{ord}_p(mn) = \mathrm{ord}_p(m) + \mathrm{ord}_p(n)$ for all $m, n \in \mathbb{N}$ and all primes p

(ii) Let $n \in \mathbb{N}$. Then n is a perfect square $\iff$ $\mathrm{ord}_p(n)$ is even for each prime p

(iii) Let $n, m \in \mathbb{N}$. Then n and m are relatively prime $\iff$ for every prime p we have $\mathrm{ord}_p(m) = 0$ or $\mathrm{ord}_p(n) = 0$.

(a) (5 pts) Choose one of the three properties above and prove it. Do NOT prove more than one, but give a detailed proof for the one you chose.

(b) (5 pts) Use properties (i)-(iii) above to prove the following theorem. Let $m, n \in \mathbb{N}$. Suppose that m and n are relatively prime and mn is a perfect square. Prove that m and n are both perfect squares.

Solution: (a) see solutions to HW#6 and page 3 of Lecture 12 (for the backwards direction of (ii))

(b) By the backwards direction of (ii), to prove that m and n are perfect squares, it suffices to show that $\mathrm{ord}_p(m)$ and $\mathrm{ord}_p(n)$ are both even for all primes p. We will prove this separately for each prime p.

So fix a prime p. Since m and n are relatively prime, by (iii) $\mathrm{ord}_p(m) = 0$ or $\mathrm{ord}_p(n) = 0$. We consider two case accordingly.

Case 1: $\mathrm{ord}_p(m) = 0$. This already tells us that $\mathrm{ord}_p(m)$ is even. Also, by (i) we have $\mathrm{ord}_p(mn) = \mathrm{ord}_p(m) + \mathrm{ord}_p(n) = 0 + \mathrm{ord}_p(n)$, so $\mathrm{ord}_p(n) = \mathrm{ord}_p(mn)$. Since mn is a perfect square by assumption, the forward direction of (ii) implies that $\mathrm{ord}_p(mn)$ is even, so $\mathrm{ord}_p(n)$ is even. Thus, we showed that $\mathrm{ord}_p(m)$ and $\mathrm{ord}_p(n)$ are both even, as desired.

Case 2: $\mathrm{ord}_p(n) = 0$. The argument in this case is identical to case 1 (with roles of m and n switched).

Bonus. (5 pts) Given $n \in \mathbb{N}$, let $d(n)$ denote the number of positive divisors of n. In homework #7 we used the formula for $d(n)$ in terms of the prime factorization of n to prove the following result:

$$d(n) \text{ is odd } \iff n \text{ is a perfect square.}$$

Give another proof of this result which does not use the formula for $d(n)$.

Solution: The idea is that most positive divisors of n come in pairs. Indeed, if $d \mid n$, then $n = de$ for some $e \in \mathbb{Z}$, so $e \mid n$ as well; further, if $d > 0$, then $e > 0$ as well. Thus, if $e \neq d$, we get an (unordered) pair of positive divisors $\{d, e\}$ with $de = n$. Note that distinct pairs do not overlap since if $d_1 e_1 = n$ and $d_2 e_2 = n$ and, say, $d_1 = d_2$, then $e_1 = \frac{n}{d_1} = \frac{n}{d_2} = e_2$.

The only way we can have $d = e$ in the above situation is when n is a perfect square and $d = \sqrt{n}$. Thus, if n is not perfect square, all of its positive divisors can be split into disjoint pairs, so the total number of positive divisors is twice the number of pairs and hence even. On the other hand, if n is a perfect square, by the same logic, the number of positive divisors of n different from $\sqrt{n}$ is even, and hence the total number of positive divisors (with $\sqrt{n}$ included) is odd.