Homework # 1, to be submitted on Canvas by 11:59pm on Fri, Jan 23rd.

Plan for the next 2 classes (Jan 20 and 22): Tensor products of modules (continued) and tensor products of algebras (10.4). Start nodules over PIDs (12.1).

Note: All rings are assumed to have 1. DF = Dummit and Foote.

Problem 1. Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists nonzero $r \in R$ such that $rm = 0$. An R-module M is called *divisible* if for any nonzero $r \in R$ we have $rM = M$. In other words, M is divisible if for any $m \in M$ and nonzero $r \in R$ there exists $x \in M$ such that $rx = m$.

(a) Suppose that M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.
(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a $\mathbb{Z}$-module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

Problem 2. Let R be a commutative ring, $\{N_\alpha\}$ a collection of R-modules and M another R-module.

(a) ([DF, problem 14, p.376]) Prove that $M \otimes (\oplus N_\alpha) \cong \oplus(M \otimes N_\alpha)$ as R-modules (tensor products are over R).
(b) ([DF, problem 15, p. 376]) Show by example that $M \otimes (\prod N_\alpha)$ need not be isomorphic to $\prod(M \otimes N_\alpha)$. **Hint:** Use the result of one of the previous problems on p. 376.

Problem 3. Let R be a commutative domain, and let M be a free R-module with basis $e_1, \ldots, e_k$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1 \in M \otimes M$ is not representable as a simple tensor $m \otimes n$ for some $m, n \in M$.

Problem 4. Problem 16 on p.376 of DF.

Problem 5. Problem 17 on pp.376-377 of DF.

Problem 6. Problem 21 on p.377 of DF.

Problem 7.

(a) Let V be a finite-dimensional vector space over $\mathbb{C}$ (complex numbers). Note that V can also be considered as a vector space over $\mathbb{R}$, but $\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{C}}(V)$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as vector spaces over $\mathbb{R}$ and compute their dimensions over $\mathbb{R}$.
(b) Let R be a commutative integral domain and F its field of fractions. Prove that $F \otimes_R F \cong F \otimes_F F \cong F$ as F-modules.

Note: As we will explain in Lecture 3, if T and S are rings, M is a left T-module and N is an (S,T)-bimodule, then $N \otimes_T M$ is a left S-module where the action of S on simple tensors is given by $s(n \otimes m) = (sn) \otimes m$ for all $s \in S, m \in M, n \in N$. The F-module structures on $F \otimes_R F$ and $F \otimes_F F$ are given by this construction (in both cases $S = M = N = F$ and $T = R$ in the first case and $T = F$ in the second case).

Problem 8. Problem 25 on p.377 of DF.