Homework #5. Due on Thursday, October 2nd, 11:59pm on Canvas

Plan for next week:

Below Pugh stands for 'Real Mathematical Analysis' by Charles Pugh, Rudin for 'Principles of Mathematical Analysis' by Walter Rudin, Tao I for 'Real Analysis I' by Terrence Tao and Tao II for 'Real Analysis II' by Terrence Tao. Pugh, Tao I and Tao II are freely available via UVA subscription (use Springer Link).

Next week (Mon, Sep 29 + Wed, Oct 1) will continue talking about compactness and its relation with other properties of metric space. The main topic on Monday will be 'compactness vs completeness' – see Lectures 10 and 11 from Fall 18 (this material is not discussed in detail in any of our main books). The main topic on Wednesday will be compactness and continuity – see 2.4.3-2.4.6 in Pugh, 4.3 in Rudin and 2.3 in Tao II. If time allows, we will also start talking about connectedness – see 2.5 in Pugh, 4.4 in Rudin and 2.4 in Tao II.

Problems:

Note on hints: Most hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

1. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Prove that K is covering compact in two different ways:

(i) by showing that K is closed and bounded as a subset of $\mathbb{R}$.

(ii)* directly from definition of covering compactness.

2. Let (X, d) be a metric space and Y a subset of X.

(a) Suppose that X is sequentially compact and Y is a closed subset of X. Prove that Y is also sequentially compact.

(b)* Now assume that Y is covering compact. Prove that Y is closed in X (we are not assuming anything about X).

Note: Since sequential compactness is equivalent to compactness for metric spaces, the results of parts (a) and (b) of this problem are equivalent to Lemma 8.7 (which proved the analogous result for covering compactness) and Theorem 8.2 (which proved the analogous result for sequential compactness). The point of this exercise is to prove the results working directly with the specified version of compactness.

3. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $X = X_1 \times X_2$, and define the function $d : X \times X \to \mathbb{R}_{\geq 0}$ by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

(a) (practice) Prove that (X, d) is a metric space.

(b)* Assume that (X_1, d_1) and (X_2, d_2) are both sequentially compact. Prove that (X, d) is also sequentially compact.

4*. Let X be a metric space. Prove that X is covering compact $\iff$ X satisfies the following property called the *finite intersection property*:

> Let $\{K_\alpha\}$ be any collection of closed subsets of X such that for any finite subcollection $K_{\alpha_1}, \ldots, K_{\alpha_n}$, the intersection $K_{\alpha_1} \cap \ldots \cap K_{\alpha_n}$ is non-empty. Then the intersection of all sets in $\{K_\alpha\}$ is non-empty.

5*. Let $X = C[a, b]$, considered as a metric space with uniform metric d_{unif} (as defined in Problem 7 in HW#2). Prove that the set $B_1(\mathbf{0})$, the closed ball of radius 1 centered at $\mathbf{0}$ in X, is not sequentially compact. Here $\mathbf{0}$ is the function which is identically 0.

6. Let X be a set, and let d_1 and d_2 be two different metrics on X. We will say that d_1 and d_2 are *topologically equivalent* if a subset S of X is open with respect to $d_1 \iff$ it is open with respect to d_2.

(a)* Assume that the notions of convergence in the metric spaces (X, d_1) and (X, d_2) coincide, that is, if (x_n) is any sequence in X, then (x_n) converges in (X, d_1) if and only if (x_n) converges in (X, d_2). Prove that d_1 and d_2 are topologically equivalent.

(b) Suppose now that there exist real numbers $A, B > 0$ such that $d_1(x, y) \leq A d_2(x, y)$ and $d_2(x, y) \leq B d_1(x, y)$ for all $x, y \in X$. Use (a) to prove that d_1 and d_2 are topologically equivalent.

(c) Now use (b) to prove that the Euclidean and Manhattan metrics on $\mathbb{R}^n$ are topologically equivalent.

Hint for #1(ii): Let $\{U_\alpha\}$ be any open cover of K. One of the U_α must contain 0. What can you say about that U_α?

Hint for 2(b): Argue by contradiction, that is, prove that if Y is not closed in X, then Y is not covering compact. In class we proved that the half-open interval $[0, 1)$ is not covering compact by constructing an explicit open cover without a finite subcover. Try to generalize that argument to the case where Y is an arbitrary non-closed subset of some metric space X.

Hint for 3(b): Take any sequence in X – it has the form $((a_n, b_n))$ for some $a_n \in X_1$ and $b_n \in X_2$. Since X_1 is sequentially compact, (a_n) has a convergent subsequence (a_{n_k}). Now look at the sequence (b_{n_k}). Since X_2 is sequentially compact, it has a convergent subsequence $(b_{n_{k_s}})_{s\in\mathbb{N}}$. Now prove that the subsequence $((a_{n_{k_s}}, b_{n_{k_s}}))$ of $((a_n, b_n))$ converges.

Hint for #4: There is a natural bijection between open covers of X and collections of closed subsets with empty intersection (here we consider open covers in the special case $S = X$).

Hint for #5: Use a result from one of the previous homeworks.

Hint for #6(a): It may be easier to prove that a subset S of X is closed with respect to d_1 if and only if it is closed with respect to d_2. How can you express the condition 'S is closed' in terms of convergence of sequences?