Homework # 3, to be submitted on Canvas by 11:59pm on Fri, Feb 6th.

Plan for the next 2 classes (Feb 3 and 5): Canonical Forms of Linear Transformations (12.2-12.3 in DF; Lectures 6-8 from Spring 21 and Lecture 10-12 from Spring 10).

Note on hints: Some hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which a hint is available at the end are marked with *.

Problem 1. Let $R = \mathbb{R}[x]$, $F = R^3$ (the standard 3-dimensional R-module) and N the R-submodule of F generated by $(1-x, 1, 0)$, $(-2, 4-x, 0)$ and $(1, -5, -x)$.

(a) Find compatible bases for F and N, that is, bases satisfying the conclusion of the compatible bases theorem (AKA submodule structure theorem). **Note:** an algorithm for computing such bases is given in Lecture 8 from Spring 2010.

(b) Describe the quotient module F/N in IF and ED forms.

Problem 2. Let R be a PID. For an R-module M denote by $d(M)$ the minimal number of generators of M.

(a) Prove that if M is a finitely generated R-module and N is a submodule of M, then $d(N) \leq d(M)$

(b) Let $a \in R$ be a nonzero non-unit. Find (with proof) the number of submodules of R/aR in terms of the prime decomposition of a

Problem 3:

(a)* Let R be a PID, M be a finitely generated R-module and $R/a_1R \oplus \ldots \oplus R/a_mR \oplus R^s$ its invariant factor decomposition, that is, $a_1, \ldots, a_m$ are nonzero non-units and $a_1 \mid a_2 \mid \ldots \mid a_m$. Prove that

$$d(M) = m + s.$$

Warning: It is not true in general that $d(P \oplus Q) = d(P) + d(Q)$.

(b) Again let R be a PID. Let F be a free R-module of rank n with basis $e_1, \ldots, e_n$, let N be the submodule of F generated by some elements $v_1, \ldots, v_n \in F$, and let $A \in Mat_n(F)$ be the matrix such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

Find a simple condition on the entries of A which holds if and only if $d(F/N) = n$.

Problem 4. DF, Problem 2, page 469. By definition, the rank of an R-module is the smallest number of linearly independent elements.

Problem 5: Let R be a PID, M a finitely generated free R-module and N a submodule of M. Prove that the following are equivalent:

(1) any basis of N can be extended to a basis of M;
(2) some basis of N can be extended to a basis of M;
(3) M/N is free;
(4) M/N is torsion-free.

Problem 6: Given a commutative ring R with 1 and R-module V and W, define $\mathrm{Hom}_R(V, W)$ to be the set of R-module homomorphisms from V to W. This is an R-module with addition defined by $(f+g)(v) = f(v)+g(v)$ for all $f, g \in \mathrm{Hom}_R(V, W)$ and $v \in V$ and scalar action defined by $(rf)(v) = f(rv)$ for all $f \in \mathrm{Hom}(V, W)$, $v \in V$ and $r \in R$. The module $V^* = \mathrm{Hom}_R(V, R)$ is called the dual module of V.

(1) Let V and W be as above. Show that there is a natural homomorphism $\varphi : V^* \otimes_R W \to \mathrm{Hom}_R(V, W)$ such that $(\varphi(f \otimes w))(v) = f(v)w$ for all $f \in V^*$, $v \in V$ and $w \in W$.
(2) Assume that W is a finitely generated free R-module. Prove that φ is an isomorphism. **Hint:** Problem 2 from HW#1 is relevant.
(3) Give examples showing that φ need not be surjective if either W is not free or W is free but not finitely generated.
(4) (bonus)Now give an example where φ is not injective.

Hint for 3(a): Let p be a prime dividing a_1. How is M related to $M' = (R/pR)^{m+s}$ and what is $d(M')$ (and why)?