Homework #3. Due on Thursday, September 18th, 11:59pm on Canvas

Plan for next week:

Below Pugh stands for 'Real Mathematical Analysis' by Charles Pugh, Rudin for 'Principles of Mathematical Analysis' by Walter Rudin, Tao I for 'Real Analysis I' by Terrence Tao and Tao II for 'Real Analysis II' by Terrence Tao. Pugh, Tao I and Tao II are freely available via UVA subscription (use Springer Link).

The main topics next week (Mon, Sep 15 + Wed, Sep 17) will be convergence in metric spaces, continuous functions between metric spaces, complete metric spaces and completions of metric spaces. This is the order in which I covered these topics in 2018, but this time I may talk about complete spaces and completions before continuity. References: for convergence see 2.1 in Pugh, 3.1 in Rudin and 1.1 in Tao II; for continuity see 2.2 and 2.3.1 in Pugh, 4.1 and 4.2 in Rudin and 2.1 and 2.2 in Tao II; for complete metric spaces see 2.3.4 in Pugh and 1.4 in Tao II (there is also a brief description of completions at the end of 1.4 in Tao II).

Problems:

Note on hints and practice problems: Most hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *. Parts of problems marked as 'practice' should be completed, but need not included in your submission.

1. Let $f : A \to B$ be a function. Give a detailed proof of the following properties:

(a) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ for all $U, V \subseteq B$

(b) $f(f^{-1}(D)) \subseteq D$ for all $D \subseteq B$. Give an example showing that the inclusion may be strict.

(c) $f^{-1}(f(C)) \supseteq C$ for all $C \subseteq A$. Give an example showing that the inclusion may be strict.

Note: Once we start talking about continuous functions, preimages will play a key role, and the goal of this problem is to give you extra practice with preimages.

2. This is an extended version of Problem 3(d) from HW#2. The statement of 3(d) itself is part (d) below; earlier parts establish auxiliary results which should help with the proof of part (d).

(a) Prove that $\mathbb{R}$ (constructed as in Lecture 2) has the Archimedean property, that is, for every $\alpha \in \mathbb{R}$ there exists $M \in \mathbb{N}$ such that $\alpha < M$. Recall that by definition α is the equivalence class of some Cauchy sequence (a_n) with $a_n \in \mathbb{Q}$, $M \in \mathbb{N}$ is viewed as the class of the constant sequence $M, M, \ldots$ and the definition of inequalities between real numbers is given on page 5 of online Lecture 2 (after Lemma 2.5).

(b) Let $a, b \in \mathbb{R}$. Use the Archimedean property to prove that the following are equivalent:
  (i) $a \leq b$ (by definition this means $a < b$ or $a = b$);
  (ii) $a < b + \frac{1}{n}$ for all $n \in \mathbb{N}$.

(c) As in HW#2.3, define $Z_n = \{z \in \mathbb{Q} : z = \frac{m}{2^n} \text{ for some } m \in \mathbb{Z}\}$. Use the Archimedean property and HW#2.3(b) to prove that for all $a, b \in \mathbb{R}$ with $a < b$ there exists z such that $a < z < b$ and $z \in Z_n$ for some $n \in \mathbb{N}$.

(d) Now let $A \subseteq \mathbb{R}$ be a non-empty bounded above subset. Define the Cauchy sequence (x_n) as in HW#2.3, and let $x = [x_n]$. Prove that $x = \sup(A)$.

  Hint: By definition of supremum, we first need to show that x is an upper bound for A and then show that $x \leq y$ for any upper bound y for A. For the first part, start by showing that for any fixed k we have $x_k < x + \frac{1}{2^{k-1}}$ (this should follow from your calculation in the solution to HW#2.3(c)) and then use (b) and definition of x_k. For the second part, argue by contradiction – assume that A has an upper bound $y < x$ and deduce from (c) that A has an upper bound $z < x$ such that $z \in Z_m$ for some $m \in \mathbb{N}$. Finally, show that this inequality contradicts the definition of the sequence (x_n).

3. Let (X, d) be a metric space and S is a subset of X. Prove that S is open $\iff$ S is the union of some collection of open balls (which could be centered at different points).

4.

(a)* Use Problem 2 to prove Theorem 5.8 from online notes: Let X be a metric space, and suppose that $S \subseteq Y \subseteq X$. Then S is open in Y $\iff$ $\exists$ an open subset U of X such that $S = U \cap Y$.

(b) Now use (a) to prove a direct analogue of (a) for closed sets: Let $Z \subseteq Y$. Then Z is closed as a subset of Y $\iff$ $Z = Y \cap K$ for some closed subset K of X. Deduce that if Z is closed in X, then Z is closed in Y. **Note:** this should be essentially a set-theoretic proof

– you just need the statement of (a), definiton of a closed set and some basic set-theoretic identities.

5. Given a metric space (X,d), a point $x \in X$ and $\varepsilon > 0$, define $B_\varepsilon(x) = \{y \in X : d(y,x) \le \varepsilon\}$, called the *closed ball of radius ε centered at x*.

(a) Prove that $B_\varepsilon(x)$ is always a closed subset of X.
(b) Deduce from (a) that $\overline{N_\varepsilon(x)} \subseteq B_\varepsilon(x)$, that is, the closure of the open ball of radius ε centered at x is contained in the respective closed ball.
(c) Is it always true that $\overline{N_\varepsilon(x)} = B_\varepsilon(x)$? Prove or give a counterexample.

6. Let (X,d) be a metric space and S a subset of X. Prove that the following three conditions are equivalent. The set S is called *bounded* if it satisfies either of those conditions:

(i) There exists $x \in X$ and $R \in \mathbb{R}$ such that $S \subseteq N_R(x)$.
(ii) For any $x \in X$ there exists $R \in \mathbb{R}$ such that $S \subseteq N_R(x)$.
(iii) The set $\{d(s,t) : s,t \in S\}$ is bounded above as a subset of $\mathbb{R}$.

Definition: Let (X,d) be a metric space and $\varepsilon > 0$. A subset S of X is called an *ε-net* if for any $x \in X$ there exists $s \in S$ such that $d(x,s) < \varepsilon$. In other words, S is an ε-net if X is the union of open balls of radius ε centered at elements of S.

7. Let S be a subset of a metric space (X,d). Prove that the following are equivalent:

(i) The closure of S is the entire X;
(ii) $U \cap S \neq \emptyset$ for any non-empty open subset U of X;
(iii) S is an ε-net for every $\varepsilon > 0$.

The subset S is called *dense* (in X) if it satisfies these equivalent conditions.

8. A metric space (X,d) is called **ultrametric** if for any $x,y,z \in X$ the following inequality holds:

$$d(x,z) \le \max\{d(x,y), d(y,z)\}.$$

(Note that this inequality is much stronger than the triangle inequality). If X is any set and d is the discrete metric on X, then clearly (X,d) is ultrametric. A more interesting example of an ultrametric space will be given in the next homework.

Prove that properties (i) and (ii) below hold in any ultrametric space (X,d) (note that both properties are counter-intuitive since they are very far from being true in $\mathbb{R}$).

(i) Take any $x \in X$, $\varepsilon > 0$ and take any $y \in N_\varepsilon(x)$. Then $N_\varepsilon(y) = N_\varepsilon(x)$. This means that if we take an open ball of fixed radius around some point x, then for any other point y from that open ball, the open ball of the same radius, but now centered at y, coincides with the original ball. In other words, any point of an open ball happens to be its center.

(ii) Prove that a sequence $\{x_n\}$ in X is Cauchy $\iff$ for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $d(x_{n+1}, x_n) < \varepsilon$ for all $n \geq M$. **Note:** The forward implication holds in any metric space.

Hint for #3(a): Let $x \in Y$ and $\varepsilon > 0$. Express $N_\varepsilon^Y(x)$ in terms of $N_\varepsilon^X(x)$ and Y.