Homework #8. Due on Thursday, October 30th, 11:59pm on Canvas

Note: Lecture 14 on Monday, Oct 20th covered almost the same content as Lecture 14 from Fall 2018. Lecture 15 on Wednesday, Oct 22nd covered subsection 15.1 in Lecture 15 from Fall 2018 and Theorem 17.2 in Lecture 17 from Fall 2018.

Plan for next week: Monday, Oct 27th: Arzela-Ascoli Theorem, equicontinuity and compactness in function spaces (4.3 in Pugh, 7.6 in Rudin and online Lecture 18 from Fall 2018). Wednesday, Oct 29th: start talking about the Stone-Weierstrass Theorem (4.4 in Pugh, 7.7 in Rudin and online Lecture 19 from Fall 2018).

Problems:

Note on hints: Most hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

1. This problem describes a fancy way to show that closed bounded intervals in $\mathbb{R}$ are connected. A metric space (X, d) is called *chain-connected* if for any $x, y \in X$ and $\delta > 0$ there exists a finite sequence $x_0, x_1, \ldots, x_n$ of points of X such that $x_0 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \delta$ for all i.

(a)* Let X be metric space which is compact and chain-connected. Prove that X is connected.

(b) Prove that a closed bounded interval $[a, b] \subseteq \mathbb{R}$ is chain-connected and deduce from (a) that $[a, b]$ is connected.

2. Let X be a set and let $(f_n)_{n=1}^{\infty}, f$ be functions from X to $\mathbb{R}$. Suppose that $f_n \rightrightarrows f$ on X.

(i) Prove that if each f_n is bounded, then f is bounded.

(ii) Assume that f is bounded. Prove that there exists $M \in \mathbb{N}$ and $C \in \mathbb{R}$ such that $|f_n(x)| \leq C$ for all $n \geq M$ and $x \in X$. In other words, prove that the sequence (f_n) becomes uniformly bounded after we remove the first few terms at the beginning.

(iii) Give examples showing that both (i) and (ii) become false if we only assume that $f_n \to f$ pointwise on X.

3. Let X be a metric space and $(f_n), f$ functions from X to $\mathbb{R}$. Suppose that $f_n \rightrightarrows f$ on X and each f_n is uniformly continuous. Prove that f is uniformly continuous. **Hint:** Imitate the proof of Theorem 14.3.

4. Problem 5 on p. 263 in Pugh (see Exercise 3.36 for the definition of jump and removable discontinuities). A clarification on the statement: in each part of the problem you are given some property (P) of functions; the question is the following: if each f_n has property (P), is it always true that the limiting function f also has (P). In part (e) countable should mean 'infinite countable'.

5. Problem 7.3:15 from Bergman's supplement to Rudin (page 79), see

`http://math.berkeley.edu/~gbergman/ug.hndts/m104_Rudin_exs.pdf`

You can assume that the functions are real-valued (not complex-valued); also J denotes the natural numbers.

6. Let X be a set and let $\Omega = Func(X, \mathbb{R})$ be the set of all functions $f : X \to \mathbb{R}$. Prove that uniform convergence on Ω is metrizable, that is, there exists a metric D on Ω such that a sequence (f_n) in Ω converges uniformly on X to some $f \in \Omega \iff f_n \to f$ in the metric space (Ω, D). Recall that in Lecture 15 we proved that the uniform convergence is metrizable on the space $B(X)$ of all bounded functions $f : X \to \mathbb{R}$, and that in this case one can define metric $d_{unif} : B(X) \times B(X) \to \mathbb{R}_{\geq 0}$ by

$$d_{unif}(f, g) = \sup_{x \in X} |f(x) - g(x)|. \qquad (***)$$

To solve this problem think how to modify the above formula, so that $D(f, g)$ is defined for any f and g and the restriction of D to $B(X)$ gives a metric topologically equivalent to d_{unif}.

7. Let $a, b \in \mathbb{R}$ with $a < b$, and let (f_n) be a sequence of differentiable functions from $[a, b]$ to $\mathbb{R}$. Suppose that the sequence (f'_n) is uniformly bounded. Prove that the sequence (f_n) is equicontinuous. Deduce that if (f_n) is also uniformly bounded, then it has a uniformly convergent subsequence.

Hint for 1(a): Assume that X is disconnected, and use Problem 2 from HW#7 and uniform continuity to reach a contradiction.

Hint for 6: Look for D which coincides with d_{unif} on pairs of functions (f, g) such that $|f(x)| \leq 1$ for all x and $|g(x)| \leq 1$ for all x. (Here 1 does not play any special role; it can be replaced by any positive real number). Alternatively, Problem 116 on p.138 in Pugh implicitly provides a hint.