Homework # 5, to be submitted on Canvas by 11:59pm on Fri, Feb 27th.

Plan for the next 2 classes (Feb 24 and 26): Unitary and unitarizable representations. Proof of Maschke's theorem over $\mathbb{C}$. Schur's Lemma and representations of abelian groups. Start talking about characters and character tables (time permitting)

Problem 1: Let F be a field. Use JCF to prove that any square matrix $A \in Mat_n(F)$ is similar to its transpose A^T. You can assume without proof that any field can be embedded into an algebraically closed field (we will prove this later this semester).

Problem 2: Let G be a group, F a field and V a vector space over F. Prove (including all the details) that there is a natural bijection between linear representations of G of the form (ρ, V) and FG-module structures on V which extend the given F-vector space structure on V.

Problem 3: (Schur's Lemma) This problem collects several (related) results (parts (a),(b) and (d) below), each of which may be referred to as Schur's Lemma.

(a) Let R be a ring, M and N irreducible (left) R-modules and $f : M \to N$ a homomorphism of R-modules. Prove that f is either an isomorphism or the zero map.

(b) Let R be a ring and M an irreducible R-module. Prove that $\mathrm{End}_R(M)$, the ring of endomorphisms of M as an R-module, is a division ring (a division ring is defined in the same way as a field except that multiplication need not be commutative).

(c) Let G be a group, $g \in Z(G)$ an element of the center of G, and (ρ, V) a linear representation of G over some field F. Prove that for any $\lambda \in F$, the map $\rho(g) - \lambda I : V \to V$ lies in $\mathrm{End}_{FG}(V)$ (where V is a considered as an FG-module via the correspondence from Problem 2).

(d) In the setting of (c), assume that F is algebraically closed and V is finite-dimensional and irreducible. Use (b) and (c) to prove that $\rho(g) = \lambda I$ for some $\lambda \in F$. In other words, if we are given a finite-dimensional irreducible representation over an algebraically closed field, then any central element must act as a scalar operator.

Problem 4: (Lemma 11.2 from class) Let R be a ring and M an R-module. We will say that M has *the complement property* if for every submodule N of M there exists a submodule P such that $M = N \oplus P$. By Theorem 11.1 from class, M has the complement property if and only if M is completely reducible. However, the proof of Theorem 11.1 relies on Lemma 11.2, and the main goal of this problem is to prove the latter, so we need this temporary terminology for this problem.

(a) Suppose that $M = P \oplus Q$ for some submodules P and Q. Prove that if N is any submodule containing P, then $N = P \oplus (N \cap Q)$.
(b) Deduce from (a) that if M has the complement property, then so does any submodule of M.
(c) Now prove Lemma 11.2 from class which asserts that if M has the complement property, then any nonzero submodule L of M contains an irreducible submodule.

Hint for (c): First use (b) to reduce (c) to the case $L = M = Rx$ for some nonzero $x \in M$. Use Zorn's Lemma to show that M contains a maximal submodule N not containing x. Now show that M/N is irreducible and then use the complement property of M to deduce that it contains an irreducible submodule.

Problem 5: Let $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$ and S_n the symmetric group on $[n]$. Let F be any field and (ρ, V) the permutation representation of S_n over F corresponding to the defining action of S_n on $[n]$. Up to equivalence, we can think of V simply as F^n, in which case $\rho : S_n \to \mathrm{GL}(F^n)$ is given by $(\rho(\sigma))(e_i) = e_{\sigma(i)}$ where $\{e_1, \ldots, e_n\}$ is the standard basis of F^n.

(a) Let $Z = F(e_1 + \ldots + e_n)$ and $W = \{(x_1, \ldots, x_n) \in F^n : \sum x_i = 0\}$. Prove that Z and W are both subrepresentations of V.
(b) Prove that $V = W \oplus Z$ if and only if $\mathrm{char}(F)$ does not divide n.
(c) * Assume that $\mathrm{char}(F)$ does not divide $n!$. Prove that W is an irreducible representation of S_n (see a hint at the end of the assignment).

Problem 6: Let R be a commutative ring with 1, let M_1, M_2, N_1 and N_2 be R-modules, and let $\varphi : M_1 \to M_2$ and $\psi : N_1 \to N_2$ be homomorphisms of R-modules.

(a) Prove that there exists a unique linear map $\varphi \otimes \psi : M_1 \otimes N_1 \to M_2 \otimes N_2$ such that $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(m)$ for all $m \in M_1$ and $n \in N_1$.
(b) Now assume that R a field, M_1, M_2, N_1, N_2 are finite-dimensional (as vector spaces over R), and choose bases α_i for M_i and β_i for N_i for $i = 1, 2$. Note that $\gamma_i = \alpha_i \otimes \beta_i = \{x_i \otimes y_i : x_i \in \alpha_i, y_i \in \beta_i\}$ is a basis for $M_i \otimes N_i$. Consider the matrices $A = [\varphi]_{\alpha_1}^{\alpha_2}$, $B = [\psi]_{\beta_1}^{\beta_2}$ and $C = [\varphi \otimes \psi]_{\gamma_1}^{\gamma_2}$. Prove that if γ_1 and γ_2 are ordered in a suitable way, then C is the Kronecker product of A and B as defined below.

(c) In the setting of (b), assume that $M_1 = M_2$ and $N_1 = N_2$. Prove that

$$\mathrm{tr}(\varphi \otimes \psi) = \mathrm{tr}(\varphi)\mathrm{tr}(\psi).$$

Note: If V is a finite-dimensional vector space and $T : V \to V$ a linear map, we defines $\mathrm{tr}(T) = \mathrm{tr}([T]_\beta)$ where β is any basis of T. The right-hand side does not depend on β since similar matrices have the same trace.

Kronecker product: Let R be a commutative ring, and let $A = (a_{ij}) \in Mat_{k\times l}(R)$ and $B \in Mat_{m\times n}(R)$. The Kronecker product of A and B is a $km \times ln$-matrix over R defined as a block matrix:

$$\begin{pmatrix} a_1 1B & \dots & a_{1l}B \\ \vdots & \ddots & \vdots \\ a_k 1B & \dots & a_{kl}B. \end{pmatrix}$$

Hint for 6(c): Given $v \in F^n$, denote by $\mathrm{wt}(w)$ (the *weight* of w) the number of nonzero coordinates of w. Now show that if $w \in W$ and $\mathrm{wt}(w) > 2$, then there exists $g \in G$ such that $w' = \sigma(g)w - w$ is nonzero and $\mathrm{wt}(w') < \mathrm{wt}(w)$. Deduce that every nonzero S_n-invariant subspace of W contains a vector of weight 2. It remains to show that if $w \in W$ and $\mathrm{wt}(w) = 2$, then the smallest G-invariant subspace of W containing w is the entire W, which can be proved by a direct computation.