Homework # 4, to be submitted on Canvas by 6pm on Sat, Feb 14th.

Plan for the next 2 classes (Feb 10 and 12): Jordan Canonical Form (12.3 in DF; Lecture 8 from Spring 21 and Lecture 12+ from Spring 10). Start talking about Representation Theory of Groups.

Problem 1: Let F be a field, $a(x) = x^n + \sum_{k=0}^{n-1} a_k x^k \in F[x]$ a non-constant monic polynomial, and let $A = C_{a(x)}$ be its companion matrix. Prove by direct computation that $\mathrm{SNF}(xI - A) = diag(\underbrace{1, \ldots, 1}_{n-1 \text{ times}}, a(x))$.

Problem 2: DF, Problem 6, page 488.

Problem 3: Determine the number of possible RCFs of 8×8 matrices A over $\mathbb{Q}$ with $\chi_A(x) = x^8 - x^4$. Explain your argument in detail.

Problem 4: Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Give an example showing that this does not hold for 4×4 matrices.

Problem 5: Find the number of distinct conjugacy classes in the group $\mathrm{GL}_3(\mathbb{F}_2)$ (where $\mathbb{F}_2$ is the field with 2 elements) and specify one element in each conjugacy class.

Problem 6: Prove that there is no matrix $A \in Mat_{10}(\mathbb{Q})$ satisfying $A^4 = -I$ (where I is the identity matrix).

Problem 7: DF, Problem 15 on page 500. **Hint:** This problem does not require long and tedious computations.

Problem 8: DF, Problem 20 on page 501. Also find an explicit $P \in \mathrm{GL}_n(F)$ such that $A = PBP^{-1}$ where A and B are the two matrices in the form (you may choose which one is A and which one is B).

Problem 9: Let V be an n-dimensional vector space over some field F, and let $T : V \to V$ be a **nilpotent** F-linear map. Prove that $T^n = 0$ in two different ways:

(a) using JCF

(b) without using JCF or RCF, but instead looking at the sequence of kernels $\{\mathrm{Ker}\,(T^k)\}_{k=1}^{\infty}$.