Homework #7. Due on Thursday, October 23rd, 11:59pm on Canvas

Plan for next week:

Below Pugh stands for 'Real Mathematical Analysis' by Charles Pugh, Rudin for 'Principles of Mathematical Analysis' by Walter Rudin, Tao I for 'Real Analysis I' by Terrence Tao and Tao II for 'Real Analysis II' by Terrence Tao. Pugh, Tao I and Tao II are freely available via UVA subscription (use Springer Link).

Plan for next week: Uniform Convergence (4.1 in Pugh, 7.1-7.3 in Rudin and 3.1-3.3 in Tao II) and Banach's Contraction Mapping Theorem (4.5 in Pugh, 9.3 in Rudin and 6.6 in Tao II).

Problems:

Note on hints: Most hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

1. (practice) Let X be a metric space and Y a subset of X. Prove that if X is complete and Y is closed in X, then Y is complete. **Note:** In class we proved an analogous result with 'complete' replaced by 'compact' (Theorem 8.7). We also proved a related result involving completeness: if Y is complete, then Y is closed in X (this is part of Theorem 10.1).

2. The goal of this problem is to fill in the details of the construction of the completion of a metric space discussed in Lecture 13.

We start by recalling the notations introduced in class. Let (X, d) be a metric space. Let $\Omega = \Omega(X)$ be the set of all Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X$ for each n. Define the relation $\sim$ on Ω by setting

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

(a) Prove that $\sim$ is an equivalence relation.

Now let $\widehat{X} = \Omega/\sim$, the set of equivalence classes with respect to $\sim$. The equivalence class of a sequence (x_n) will be denoted by $[x_n]$. Given an element $x \in X$, we will denote by $[x] \in \widehat{X}$ the equivalence class of the constant sequence all of whose elements are equal to x.

Now define the function $D : \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0}$ by setting

$$D([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n) \qquad (***)$$

(b)* Prove that the limit on the right-hand side of (***) always exists and that the function D is well-defined (that is, if $[x_n] = [x'_n]$ and $[y_n] = [y'_n]$, then $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$).

(c) Prove that $(\widehat{X}, D)$ is a metric space.

3. (Distance from a point to a subset). Let (X, d) be a metric space. Given a point $x \in X$ and a subset Z of X, we define $d(x, Z)$ (the distance from x to Z) by

$$d(x, Z) = \inf\{d(x, z) : z \in Z\}.$$

Note that one indeed has to take infimum, not minimum. There may be no $z \in Z$ which is closest to x.

(a) Prove that $d(x, Z) = 0 \iff x \in \overline{Z}$.

(b*) Prove that $d(x, Z) \geq d(y, Z) - d(x, y)$ for all $x, y \in X$ and $Z \subseteq X$

(c) Now fix $Z \subseteq X$, and define $d_Z : X \to \mathbb{R}$ by $d_Z(x) = d(x, Z)$. Use (b) to prove that d_Z is continuous.

4*. Again let (X, d) be a metric space. Let $a \in X$ and K a compact subset of X. Prove that there exists $k \in K$ such that $d(a, k) = d(a, K)$ (this is equivalent to saying that the set $\{d(a, z) : z \in K\}$ has the minimal element).

5.

(a) Theorem 40 in Pugh states the following: Let X, Y are metric metric spaces, assume that X is a compact, and assume that $f : X \to Y$ is continuous and bijective. Then $f^{-1} : Y \to X$ is also continuous. Give a short proof of this theorem by combining Corollary 7.4, Theorem 8.2, Theorem 8.7 and Theorem 11.1 from class (note that in Theorems 8.2 and 8.7 we can now replace sequential compactness and covering compactness by just 'compactness' as we already proved the equivalence of the two versions of compactness). The respective references in Pugh are Theorem 11, equivalence of (i) and (iii), Theorem 26, Theorem 32 and Theorem 36.

(b) Now give an example of metric spaces X and Y and a function $f : X \to Y$ such that f is continuous and bijective, but $f^{-1} : Y \to X$ is not continuous (so that the assumption that X is compact in (a) is essential).

Before solving Probelms 6 and 7 below read the subsection on uniform continuity at the end of Lecture 11.

6. (practice) Read (and understand) the proof of Theorem 42 in Pugh which asserts that if $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

7*. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

(a) Assume that f' is bounded, that is, there exists $M \in \mathbb{R}$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.

(b) Now assume that $f'(x) \to \infty$ as $x \to \infty$. Prove that f is not uniformly continuous.

8. Consider functions $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{nx+1}$. Let $0 \leq a \leq b$ be real numbers. Prove that $\{f_n\}$ converges uniformly on $[a, b] \iff a > 0$ or $a = b = 0$. Include all the details!

Hint for 2(b): For the existence of the limit prove that the sequence $(d(x_n, y_n))$ is Cauchy using the inequality $d(x, w) \leq d(x, y)+d(y, z)+d(z, w)$.

Hint for 3(b): If S is a subset of $\mathbb{R}$ bounded below and $\alpha \in \mathbb{R}$, how to show that $\inf(S) \geq \alpha$?

Hint for 4: Apply Corollary 11.2 to a suitable function.

Hint for 7: Use the Mean Value Theorem.