Homework # 2, to be submitted on Canvas by 11:59pm on Fri, Jan 30th.

Plan for the next 2 classes (Jan 27 and 29): Modules over PIDs, continued (12.1 in DF; Lectures 3-5 from Spring 21 and Lecture 7-9 from Spring 10). Possibly start talking about the rational canonical form (12.2 in DF; Lecture 6-7 from Spring 21 and Lectures 10-11 from Spring 10).

Note on hints: Some hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which a hint is available at the end are marked with *.

Problem 1. Let R be a Euclidean domain and $n \geq 2$ an integer.

(a) Use the proof of the Smith Normal Form theorem from class to show that every matrix $A \in \mathrm{GL}_n(R)$ can be written as a product of elementary matrices $E_{ij}(\lambda)$, flip matrices F_{ij} and a diagonal matrix D.

(b) Now show that the flip matrices can be eliminated from the product in (a), and one can assume that $D = diag(d, 1, \ldots, 1)$, that is, all diagonal entries of D except possibly the $(1, 1)$-entry are equal to 1.

(c) Deduce from (b) that $\mathrm{SL}_n(R)$ (the subgroup of matrices of determinant 1) is generated by the elementary matrices $E_{ij}(\lambda)$.

Problem 2. Let R be a Euclidean domain, let $k, n \in \mathbb{N}$ and $i \leq \min\{k, n\}$. Given a matrix $A \in Mat_{k\times n}(R)$, define $d_i(A)$ to be the gcd of all $i \times i$ minors of A. Prove that $d_i(PAQ) = d_i(A)$ for all $P \in \mathrm{GL}_k(R)$ and $Q \in \mathrm{GL}_n(R)$. Recall that this was a key fact in the proof of uniqueness of the Smith Normal Form. **Hint:** Use Problem 1.

Problem 3: Let R be a commutative ring (with 1).

(a) Let C be an R-algebra and let A and B be R-subalgebras of C which commute with each other, that is, $ab = ba$ for any $a \in A, b \in B$ (note that A and B themselves do not have to be commutative). Prove that there is an $\underline{R\text{-algebra homomorphism}}$ $\varphi : A \otimes_R B \to C$ such that $\varphi(a \otimes b) = ab$ for each $a \in A$ and $b \in B$.

(b) Prove that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$ as rings (as usual $\mathbb{R}$ is real numbers and $\mathbb{C}$ are complex numbers).

(c) Now assume that R is a field, and let A be a finite-dimensional R-algebra. Prove that the algebra $A \otimes_R A$ cannot be a field unless $\dim_R A = 1$. **Hint:** use (a).

Problem 4.

(a) Prove that $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $B = \mathbb{C} \times \mathbb{C}$ are isomorphic as $\mathbb{C}$-algebras. Here $\mathbb{C}$ acts on $\mathbb{C} \times \mathbb{C}$ in the usual way, that is, $\lambda(a, b) = (\lambda a, \lambda b)$ and on $\mathbb{C}\otimes\mathbb{C}$ by $\lambda(a\otimes b) = (\lambda a\otimes b)$ (so we apply the extension of scalars construction where the second copy of $\mathbb{C}$ is viewed as the original $\mathbb{R}$-algebra and the first copy of $\mathbb{C}$ is used for the scalar extension).

Hint: You may use without proof the following generalization of HW#1.8: if R is a subring of a commutative ring S with $1_R = 1_S$, then for any polynomial $f(x) \in R[x]$ we have

$$S \otimes (R[x]/(f(x)) \cong S[x]/(f(x))$$

as S-algebras.

(b) Explain why $\{1\otimes 1, 1\otimes i\}$ is a basis for A over $\mathbb{C}$. Now compute $\varphi(1\otimes 1)$ and $\varphi(1\otimes i)$ where $\varphi : A \to B$ is your isomorphism from (a) (note that φ is completely determined by its values on a $\mathbb{C}$-basis of A).

(c)* Prove that there exist precisely 2 $\mathbb{C}$-algebra isomorphisms from A to B. **Hint:** First prove ≥ 2 and then ≤ 2.

Problem 5. Let V and W be finite dimensional vector spaces over a field F, let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{w_1, \ldots, w_m\}$ a basis of W.
Let $\varphi : V \otimes_F W \to Mat_{n\times m}(F)$ be the F-linear transformation such that $\varphi(v_i \otimes w_j) = e_{ij}$ where e_{ij} is the matrix whose (i, j)-entry is equal to 1 and all other entries are equal to 0 (note that such transformation exists and is unique because $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes_F W$; furthermore, φ is an isomorphism since matrices $\{e_{ij}\}$ form a basis of $Mat_{n\times m}(F)$).

(a) Prove that for a matrix $A \in Mat_{n\times m}(F)$ the following are equivalent:
   (i) $A = \varphi(v \otimes w)$ for some $v \in V, w \in W$ (note: v and w need not be elements of the above bases)
   (ii) $rk(A) \leq 1$.

(b) Let $A \in Mat_{n\times m}(F)$. Prove that $\mathrm{rk}(A)$ is the smallest d such that $\varphi^{-1}(A)$ can be written as a sum of d simple tensors.

Problem 6*. Let R be a ring (with 1), let M be a left R-module and N its submodule. Prove that M is Noetherian $\iff$ N and M/N are both Noetherian. **Note:** we will discuss Noetherian modules at the beginning of Lecture 5.

Hint for 4:

- To prove that there are at least 2 $\mathbb{C}$-algebra isomorphisms from A to B show that either A or B has a non-trivial $\mathbb{C}$-algebra automorphism (the assertion is true for both A and B, but you only need to prove it for one of them).
- To prove ≤ 2 find enough restrictions on $\psi(1 \otimes 1)$ and $\psi(1 \otimes i)$ where $\psi : A \to B$ is a $\mathbb{C}$-algebra isomorphism to deduce that there are at most 2 choices for the pair $(\psi(1 \otimes 1), \psi(1 \otimes i))$.

Hint for 6: The forward direction is easy. For the backwards direction, observe that if $\{P_i\}$ is an ascending chain of submodules of M, then $\{P_i \cap N\}$ is an ascending chain of submodules of N and $\{(P_i + N)/N\}$ is an ascending chain of submodules of M/N.