

**Homework # 2, to be submitted on Canvas by 11:59pm on Fri, Jan 30th.**

**Plan for the next 2 classes (Jan 27 and 29):** Modules over PIDs, continued (12.1 in DF; Lectures 3-5 from Spring 21 and Lecture 7-9 from Spring 10). Possibly start talking about the rational canonical form (12.2 in DF; Lecture 6-7 from Spring 21 and Lectures 10-11 from Spring 10).

**Note on hints:** Some hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which a hint is available at the end are marked with \*.

**Problem 1.** Let  $R$  be a Euclidean domain and  $n \geq 2$  an integer.

- (a) Use the proof of the Smith Normal Form theorem from class to show that every matrix  $A \in \mathrm{GL}_n(R)$  can be written as a product of elementary matrices  $E_{ij}(\lambda)$ , flip matrices  $F_{ij}$  and a diagonal matrix  $D$ .
- (b) Now show that the flip matrices can be eliminated from the product in (a), and one can assume that  $D = \mathrm{diag}(d, 1, \dots, 1)$ , that is, all diagonal entries of  $D$  except possibly the  $(1, 1)$ -entry are equal to 1.
- (c) Deduce from (b) that  $\mathrm{SL}_n(R)$  (the subgroup of matrices of determinant 1) is generated by the elementary matrices  $E_{ij}(\lambda)$ .

**Problem 2.** Let  $R$  be a Euclidean domain, let  $k, n \in \mathbb{N}$  and  $i \leq \min\{k, n\}$ . Given a matrix  $A \in \mathrm{Mat}_{k \times n}(R)$ , define  $d_i(A)$  to be the gcd of all  $i \times i$  minors of  $A$ . Prove that  $d_i(PAQ) = d_i(A)$  for all  $P \in \mathrm{GL}_k(R)$  and  $Q \in \mathrm{GL}_n(R)$ . Recall that this was a key fact in the proof of uniqueness of the Smith Normal Form. **Hint:** Use Problem 1.

**Problem 3:** Let  $R$  be a commutative ring (with 1).

- (a) Let  $C$  be an  $R$ -algebra and let  $A$  and  $B$  be  $R$ -subalgebras of  $C$  which commute with each other, that is,  $ab = ba$  for any  $a \in A, b \in B$  (note that  $A$  and  $B$  themselves do not have to be commutative). Prove that there is an  $R$ -algebra homomorphism  $\varphi : A \otimes_R B \rightarrow C$  such that  $\varphi(a \otimes b) = ab$  for each  $a \in A$  and  $b \in B$ .
- (b) Prove that  $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[i] \cong \mathbb{C}$  as rings (as usual  $\mathbb{R}$  is real numbers and  $\mathbb{C}$  are complex numbers).
- (c) Now assume that  $R$  is a field, and let  $A$  be a finite-dimensional  $R$ -algebra. Prove that the algebra  $A \otimes_R A$  cannot be a field unless  $\dim_R A = 1$ . **Hint:** use (a).

**Problem 4.**

- (a) Prove that  $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$  and  $B = \mathbb{C} \times \mathbb{C}$  are isomorphic as  $\mathbb{C}$ -algebras. Here  $\mathbb{C}$  acts on  $\mathbb{C} \times \mathbb{C}$  in the usual way, that is,  $\lambda(a, b) = (\lambda a, \lambda b)$  and

on  $\mathbb{C} \otimes \mathbb{C}$  by  $\lambda(a \otimes b) = (\lambda a \otimes b)$  (so we apply the extension of scalars construction where the second copy of  $\mathbb{C}$  is viewed as the original  $\mathbb{R}$ -algebra and the first copy of  $\mathbb{C}$  is used for the scalar extension).

**Hint:** You may use without proof the following generalization of HW#1.8: if  $R$  is a subring of a commutative ring  $S$  with  $1_R = 1_S$ , then for any polynomial  $f(x) \in R[x]$  we have

$$S \times (R[x]/(f(x))) \cong S[x]/(f(x))$$

as  $S$ -algebras.

- (b) Explain why  $\{1 \otimes 1, 1 \otimes i\}$  is a basis for  $A$  over  $\mathbb{C}$ . Now compute  $\varphi(1 \otimes 1)$  and  $\varphi(1 \otimes i)$  where  $\varphi : A \rightarrow B$  is your isomorphism from (a) (note that  $\varphi$  is completely determined by its values on a  $\mathbb{C}$ -basis of  $A$ ).
- (c)\* Prove that there exist precisely 2  $\mathbb{C}$ -algebra isomorphisms from  $A$  to  $B$ . **Hint:** First prove  $\geq 2$  and then  $\leq 2$ .

**Problem 5.** Let  $V$  and  $W$  be finite dimensional vector spaces over a field  $F$ , let  $\{v_1, \dots, v_n\}$  be a basis of  $V$  and  $\{w_1, \dots, w_m\}$  a basis of  $W$ .

Let  $\varphi : V \otimes_F W \rightarrow \text{Mat}_{n \times m}(F)$  be the  $F$ -linear transformation such that  $\varphi(v_i \otimes w_j) = e_{ij}$  where  $e_{ij}$  is the matrix whose  $(i, j)$ -entry is equal to 1 and all other entries are equal to 0 (note that such transformation exists and is unique because  $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$  is a basis for  $V \otimes_F W$ ; furthermore,  $\varphi$  is an isomorphism since matrices  $\{e_{ij}\}$  form a basis of  $\text{Mat}_{n \times m}(F)$ ).

- (a) Prove that for a matrix  $A \in \text{Mat}_{n \times m}(F)$  the following are equivalent:
  - (i)  $A = \varphi(v \otimes w)$  for some  $v \in V, w \in W$  (note:  $v$  and  $w$  need not be elements of the above bases)
  - (ii)  $\text{rk}(A) \leq 1$ .
- (b) Let  $A \in \text{Mat}_{n \times m}(F)$ . Prove that  $\text{rk}(A)$  is the smallest  $d$  such that  $\varphi^{-1}(A)$  can be written as a sum of  $d$  simple tensors.

**Problem 6\*.** Let  $R$  be a ring (with 1), let  $M$  be a left  $R$ -module and  $N$  its submodule. Prove that  $M$  is Noetherian  $\iff N$  and  $M/N$  are both Noetherian. **Note:** we will discuss Noetherian modules at the beginning of Lecture 5.

**Hint for 4:**

- To prove that there are at least 2  $\mathbb{C}$ -algebra isomorphisms from  $A$  to  $B$  show that either  $A$  or  $B$  has a non-trivial  $\mathbb{C}$ -algebra automorphism (the assertion is true for both  $A$  and  $B$ , but you only need to prove it for one of them).
- To prove  $\leq 2$  find enough restrictions on  $\psi(1 \otimes 1)$  and  $\psi(1 \otimes i)$  where  $\psi : A \rightarrow B$  is a  $\mathbb{C}$ -algebra isomorphism to deduce that there are at most 2 choices for the pair  $(\psi(1 \otimes 1), \psi(1 \otimes i))$ .

**Hint for 6:** The forward direction is easy. For the backwards direction, observe that if  $\{P_i\}$  is an ascending chain of submodules of  $M$ , then  $\{P_i \cap N\}$  is an ascending chain of submodules of  $N$  and  $\{(P_i + N)/N\}$  is an ascending chain of submodules of  $M/N$ .