Math 8851. Homework #4. To be completed by 6pm on Thu, Mar 27

1. Let $G = \langle S \rangle$ be a group and $H = \langle T \rangle$ a subgroup of G, with S, T finite. The *distortion* function of H in G is $D_H : \mathbb{N} \to \mathbb{N}$ defined as follows: $D_H(n)$ is the maximal word length $\|h\|_T$ where h ranges over all elements of H with $\|h\|_S \leq n$.

While D_H depends on the choice of generating sets S and T, its equivalence class (with respect to standard notion of equivalence of functions) does not. In particular, one can talk about D_H being polynomial of fixed degree or exponential.

(a) Prove that if H is infinite, there exists $C > 0$ such that $D_H(n) \geq Cn$ for all sufficiently large n.
(b) Now prove that H is undistorted (as defined in class) if and only if there exist $K > 0$ such that $D_H(n) \leq Kn$ for all n.
(c) Let $G = BS(1, 2) = \langle a, b | bab^{-1} = a^2 \rangle$ and $H = \langle a \rangle$. Prove that the distortion function is exponential. By definition this means there exist constants $\lambda, \mu > 1$ such that $\lambda^n \leq D_H(n) \leq \mu^n$ for all sufficiently large n. **Note:** Unlike word growth, $D_H(n)$ can grow super-exponentially.
(d) Let G be the Heisenberg group over $\mathbb{Z}$. It can be defined as the group of 3×3 upper unitriangular matrices over $\mathbb{Z}$ (upper unitriangular means that the entries below the diagonal are 0 and the entries on the diagonal are 1) and has the presentation $G = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 1 \rangle$ where $x = E_{12}(1), y = E_{23}(1)$ and $z = E_{13}(1)$ (by $E_{ij}(\lambda)$ we denote the matrix which has 1's on the diagonal, λ in the (i, j)-position and 0's everywhere else). Prove that the subgroups $\langle x \rangle$ and $\langle y \rangle$ are undistorted while for $H = \langle z \rangle$ the distortion is quadratic.

2. Let $G = \mathbb{Z}^2$. Prove that for any non-trivial subgroup H of G there exist generating sets S and T of G such that H is quasi-convex with respect to S, but not quasi-convex with respect to T. **Hint:** First prove this for H generated by $(n, 0)$ for some $n \in \mathbb{N}$ and then use a suitable theorem to deduce the general case.

3. Prove that an undistorted (equivalently, quasi-convex) subgroup of a hyperbolic group is hyperbolic (Corollary 19.3 from class).

4. Let G be a group, S a finite generating set for G and H a subgroup of G which is quasi-convex with respect to S. Use the Švarč-Milnor lemma to prove that H is finitely generated and undistorted in G. **Hint:** Apply the Švarč-Milnor lemma to the left-multiplication action of H on a suitable subspace X of $Cay(G, S)$. Note that you cannot take $X = Cay(G, S)$ since in that case the action need not be cobounded (in fact, will not be cobounded unless H has finite index in G). Also keep in mind that X needs to be geodesic.

5. This problem deals with quasi-convex subsets (not necessary subgroups) in hyperbolic groups. Let G be a hyperbolic group and S a finite generating set for G.

(a) Let X be an arbitrary subset of G. Prove that the following equivalent:
   (i) X is quasi-convex when considered as a subset of $Cay(G, S)$
   (ii) There exists τ such that for any $x \in X$ and any geodesic $[1, x]$ in $Cay(G, S)$ we have $[1, x] \subseteq X^{+\tau}$. **Note:** We are not assuming that $1 \in X$. Here 1 could be replaced by any fixed element of $Cay(G, S)$.

(b) Use (a) to show that if X and Y are quasi-convex subsets of X, then the sets $X \cup Y$ and $XY = \{xy : x \in X, y \in Y\}$ are also quasi-convex.

Hint: Directly apply the slim triangle condition to relate (i) and (ii) in (a).

6. In 1921 Nielsen proved that finitely generated subgroups of free groups are free. This problem relates Nielsen's proof to the fact that finitely generated subgroups of free groups are undistorted.

Let X be a finite set, $F = F(X)$, the free group on X and $U = \{u_1, \ldots, u_n\}$ a finite tuple of elements of F (the order here does not really play a role; we are talking about tuples rather than sets to allow repetitions). Then U is called *Nielsen reduced* if

(i) $u_i \neq 1$ for all i, all u_i are distinct and $U \cap U^{-1} = \emptyset$
(ii) for all $v, w \in U \cup U^{-1}$ with $w \neq v^{-1}$ we have $\|vw\| \geq \max(\|v\|, \|w\|)$.
(iii) for all $u, v, w \in U \cup U^{-1}$ with $v \neq u^{-1}, w^{-1}$ we have

$$\|uvw\| > \|u\| - \|v\| + \|w\|.$$

Nielsen proved that if we are given any $U = \{u_1, \ldots, u_n\}$ such that $u_i \neq 1$ for some i, one can obtain a Nielsen-reduced set from U using the following operations:

(a) replace u_i by u_i^{-1} for some i;

(b) replace u_i by $u_i u_j^{\pm 1}$ or by $u_j^{\pm 1} u_i$ for some $i \neq j$;
(c) if $u_i = 1$ for some i, remove u_i.

Since none of operations (a), (b) and (c) changes the subgroup generated by the tuple, it follows that any non-trivial finitely generated subgroup of F has a Nielsen-reduced generating set.

Suppose now that $H \neq \{1\}$ is a finitely generated subgroup and $U = \{u_1, \ldots, u_n\}$ a Nielsen reduced generating set for H. Consider another alphabet with n elements $Y = \{y_1, \ldots, y_n\}$, and let $\phi : F(Y) \to F(X)$ be the unique homomorphism such that $\phi(y_i) = u_i$ for all i. In other words, ϕ takes a reduced word in $Y \cup Y^{-1}$ and replaces each y_i by $u_i \in F(X)$. Note that the image of ϕ is precisely H.

Now we arrive at the actual statement of the problem:

(1) Prove that for any $1 \neq w \in F(X)$ we have $\|\phi(w)\|_X \geq \|w\|_Y$. **Hint:** Show that for any subword of length 3 in w, say $y_1 y_2 y_3$ (with $y_i \in Y^{\pm 1}$), the middle word $\phi(y_2)$ of the corresponding product $\phi(y_1)\phi(y_2)\phi(y_3)$ cannot completely cancel in $F(X)$.
(2) Property (1) immediately implies that ϕ has trivial kernel which means that H is free and U is a free generating set for H. Show that (1) also implies that H is undistorted in $F(X)$.

7. In 1926 Schreier proved that arbitrary subgroups of free groups are free. This problem relates Schreier's proof to the fact that finitely generated subgroups of free groups are quasi-convex.

We start with Schreier's method for computing a generating set for a subgroup of an arbitrary group for which a generating set is given (part (a) below). So let G be any group, X a generating set for G, H a subgroup of G and T a right transversal for H (a subset of G containing exactly one element from each right coset Hg). As in HW#3.6, for each $g \in G$ we denote by $\overline{g}$ the unique element of T such that $H\overline{g} = Hg$. Let us also assume that $1 \in T$ (this condition is not really necessary, but slightly simplifies the argument). Let Y be the set of all elements of the form $tx(\overline{tx})^{-1}$ with $t \in T$ and $x \in X$.

(a) Prove that Y is a generating set for H. Moreover prove that for any $h \in H$ we have $\|h\|_Y \leq \|h\|_X$ and describe an algorithm with starts with a word of length n in X representing some $h \in H$ and produces a word of length n in Y representing the same $h \in H$.
(b) Now assume that G is free, H is finitely generated and T is a Schreier transversal as defined in HW#3.6. As stated in

HW#3.6, by Schreier's theorem non-identity element of Y are all distinct and freely generate Y. Since H is finitely generated, it follows that Y contains only finitely many non-identity elements. Use this fact to show that H is quasi-convex in G (with respect to X).

Hint for (a): It is probably more convenient to prove the following more general statement: any $g \in G$ with $\|g\|_X = n$ can be written as $g = y_1 \dots y_n t$ where $y_i \in Y^{\pm 1}$ and $t \in T$ (note that if $g \in H$, we are forced to have $t = 1$ since $y_i \in H$ for each i). To prove this statement by induction on n it suffices to show that for any $t_0 \in Y$ and x one can write each of the words $t_0 x$ and $t_0 x^{-1}$ in the form yt for some $y \in Y^{\pm 1}$ and $t \in T$. The first case is quite straightforward. For the second case, first show that if $t' = \overline{t_0 x^{-1}}$, then $\overline{t' x} = t_0$.