

**Homework #10. Due on Monday, November 24th, 11:59pm on Canvas**

**Plan for the next 3 classes (Mon, Nov 17, Wed, Nov 19 and Mon, Nov 24):** Measurable sets and measurable functions – I plan to follow online Lectures from 2018 pretty closely (the corresponding lectures are 22–24). Alternative references: Kolmogorov-Fomin (Sections 25 and 28), Pugh (6.1, 6.2 and 6.6) – very interesting approach, but apart from 6.1, very different from what we will do in class, Rudin (11.2 and 11.4) and Tao II (Chapter 7).

**Problems:**

**Note on hints:** Most hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with \*.

**1. (Density of trigonometric polynomials).** Let  $\lambda \in \mathbb{R}$ . A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called  $\lambda$ -periodic if  $f(x + \lambda) = f(x)$  for all  $x \in \mathbb{R}$ . Denote by  $Func_\lambda(\mathbb{R}, \mathbb{R})$  the set of all  $\lambda$ -periodic functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and by  $C_\lambda(\mathbb{R})$  the set of all continuous  $\lambda$ -periodic functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

Now let  $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  be the unit circle in  $\mathbb{R}^2$ , and define the function  $\Phi : Func(S^1, \mathbb{R}) \rightarrow Func(\mathbb{R}, \mathbb{R})$  (where  $Func(X, Y)$  is the set of all functions from  $X$  to  $Y$ ) by

$$(\Phi(f))(t) = f(\cos(t), \sin(t)) \text{ for all } f \in Func(S^1, \mathbb{R}) \text{ and } t \in \mathbb{R}.$$

- (a) (practice) Prove that  $\Phi$  is injective and  $\text{Im}(\Phi) = Func_{2\pi}(\mathbb{R})$ , the set of all  $2\pi$ -periodic functions.
- (b) \* (practice) Prove that  $\Phi$  maps  $C(S^1)$  onto  $C_{2\pi}(\mathbb{R})$ . Moreover, show that  $\Phi$  restricted to  $C(S^1)$  is both an isometry from  $C(S^1)$  to  $C_{2\pi}(\mathbb{R})$  (with respect to the uniform metrics) and an isomorphism of algebras (the latter means that  $\Phi$  preserves addition, multiplication and scalar multiplication).
- (c) Use (b) to derive the following variation of the Stone-Weierstrass Theorem (in the case  $X = \mathbb{R}$ ). Let  $A \subseteq C_{2\pi}(\mathbb{R})$  be an algebra. Suppose that  $A$  vanishes nowhere and separates points on  $[0, 2\pi)$ . Then  $A$  is dense in  $C_{2\pi}(\mathbb{R})$ .
- (d) For each  $k \in \mathbb{N}$  define  $f_k, g_k : \mathbb{R} \rightarrow \mathbb{R}$  by  $f_k(x) = \cos(kx)$  and  $g_k(x) = \sin(kx)$ . Note that  $f_k, g_k \in C_{2\pi}(\mathbb{R})$ . Let

$$\Omega = \{1, f_k, g_k : k \in \mathbb{N}\}$$

(here  $\mathbf{1}$  is the constant function equal to 1 at any point), and let  $A = \text{Span}(\Omega)$ , the set of (finite) linear combinations of functions from  $\Omega$  with real coefficients (such functions are called **trigonometric polynomials**). Use (c) to prove that  $A$  is dense in  $C_{2\pi}(\mathbb{R})$  (you will likely need to use some trigonometric identities).

**Remark:** An alternative formulation of (d) is that for any  $f \in C_{2\pi}(\mathbb{R})$ , there exists a sequence of trigonometric polynomials uniformly converging to  $f$  on  $\mathbb{R}$ . If you are familiar with Fourier series, you may be tempted to think that the partial sums of the Fourier series of  $f$  provide a sequence of such trigonometric polynomials. However, this is not true in general. If  $f$  is merely assumed to be continuous, the Fourier series does not even have to converge pointwise. Requiring that  $f$  is differentiable is sufficient to deduce that the Fourier series converges to  $f$  pointwise, but is not sufficient for uniform convergence.

**2. (Newton's method)** This is a well-known method for approximating roots (zeroes) of a function. The goal of this problem is to provide justification for this method using the proof of the Contraction Mapping Theorem given in Lecture 19 (on Monday, Nov 10); the same proof is given in Pugh, pp.240-241.

The setup for Newton's method is as follows. Let  $[a, b] \subseteq \mathbb{R}$  be a closed bounded interval, and let  $f$  be a real-valued function which is defined and twice differentiable on some open interval containing  $[a, b]$ . Suppose in addition that

- (i)  $f(a) < 0$  and  $f(b) > 0$ ;
- (ii) there exists  $C > 0$  such that  $f'(x) \geq C$  for all  $x \in [a, b]$ , so in particular  $f$  is increasing on  $[a, b]$ ;
- (iii) there exists  $M > 0$  such that  $|f''(x)| \leq M$  for all  $x \in [a, b]$ .

Conditions (i) and (ii) and the Intermediate Value Theorem imply that  $f$  has a unique root on  $[a, b]$ , that is, there exists a unique  $\tau \in [a, b]$  such that  $f(\tau) = 0$ .

Now define the function  $g : [a, b] \rightarrow \mathbb{R}$  by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Geometrically,  $g(x)$  is the  $x$ -intercept of the tangent line to the graph of  $f$  at the point  $(x, f(x))$  (see, e.g., the wikipedia page on Newton's method for a picture). Note that  $g(x) = x \iff f(x) = 0$ , so the unique root of  $f$  (which we denoted above by  $\tau$ ) is equal to the unique fixed point of  $g$ .

Now we get to the actual problem. Given  $\varepsilon > 0$ , let  $I_\varepsilon = [\tau - \varepsilon, \tau + \varepsilon]$ . Prove that if  $\varepsilon$  is sufficiently small, then  $g$  is defined on  $I_\varepsilon$  and moreover

- (1) there exists  $q < 1$  such that  $|g(x) - g(y)| \leq q|x - y|$  for all  $x, y \in I_\varepsilon$ ;
- (2)  $g(I_\varepsilon) \subseteq I_\varepsilon$ .

(It is more convenient to prove things in this order). Note that (i) and (ii) imply that  $g$  is a contraction from  $I_\varepsilon$  to  $I_\varepsilon$ .

Hence the proof of the Contraction Mapping Theorem given in class implies that if we pick any  $x_0 \in I_\varepsilon$  (with  $\varepsilon$  satisfying (1) and (2)) and define the sequence  $(x_n)$  by  $x_n = g(x_{n-1})$  for all  $n \in \mathbb{N}$ , then  $(x_n)$  converges to  $\tau$ . This is the Newton's method.

**3\*.** The goal of this problem is to show that in the statement of the Contraction Mapping Theorem, one cannot replace the condition ' $f : X \rightarrow X$  is a contraction' by  $d(f(x), f(y)) < d(x, y)$  for all distinct  $x, y \in X$ . Thus, you are asked to give an example (with proof) of a complete metric space  $X$  and a map  $f : X \rightarrow X$  such that  $d(f(x), f(y)) < d(x, y)$  for all distinct  $x, y \in X$ , but  $f$  has no fixed point.

**4.** (practice) Let  $A_1, A_2, B_1$  and  $B_2$  be subsets of the same set. Prove that

- (a)  $(A_1 \cup A_2) \triangle (B_1 \cup B_2) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$
- (b)  $(A_1 \cap A_2) \triangle (B_1 \cap B_2) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$

As in Lecture 22 from Fall 2018,  $A \triangle B$  is the symmetric difference of  $A$  and  $B$  defined by

$$A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

Recall that both formulas were used in verification of properties of Lebesgue measure.

**5.** In all parts of this problem  $X = \mathbb{R}$  or  $\mathbb{R}^2$ , and  $m$  denotes the Lebesgue measure on  $X$ .

- (a)\* Prove that every open subset of  $X$  is measurable. Deduce that every closed subset of  $X$  is measurable.

Now let  $\Omega_0, \Omega_1, \Omega_2, \dots$  be the following collections of subsets of  $X$ . First define  $\Omega_0$  to be the set of all subsets of  $X$  which are either open or closed. For each  $k \geq 1$  define  $\Omega_k$  to be the set of all subsets which can be represented either as a countable union or a countable intersection of subsets from  $\Omega_{k-1}$ .

- (b) Deduce from (a) that each set in each  $\Omega_k$  is measurable.
- (c) Assume that  $X = \mathbb{R}$  and  $S = \mathbb{Q}$ . Does there exist  $k \in \mathbb{N}$  such that  $S \in \Omega_k$ ? If yes, what is the smallest such  $k$ ?
- (d) Same question as (c) for  $S = \mathbb{R} \setminus \mathbb{Q}$ .

6.

- (a) Let  $A$  be a countable subset of  $\mathbb{R}$ . Prove that  $A$  has measure zero (that is,  $A$  is measurable and  $m(A) = 0$ ).
- (b) Prove that the (standard) Cantor set  $C$  has measure 0 (see p.105 in Pugh for the definition of the standard Cantor set).

7. The goal of this problem is to construct a non-measurable subset of  $\mathbb{R}$ .

- (a) Define a relation  $\sim$  on  $\mathbb{R}$  by  $x \sim y \iff y - x \in \mathbb{Q}$ . Prove that  $\sim$  is an equivalence relation.
- (b) Explain why each equivalence class with respect to  $\sim$  contains some a real number in  $[0, 1]$ . Thus, there exists a subset  $V$  of  $[0, 1]$  which contains exactly one element from each equivalence (this step requires axiom of choice).
- (c) Prove that the sets  $\{q + V : q \in \mathbb{Q} \cap [-1, 1]\}$  are pairwise disjoint. Next prove that if  $W = \sqcup_{q \in \mathbb{Q} \cap [-1, 1]} (q + V)$ , then  $[0, 1] \subseteq W \subseteq [-1, 2]$ .
- (d) Now use (c) and basic properties of the Lebesgue measure to prove that  $V$  is not measurable. **Hint:** Argue by contradiction and consider the cases  $m(V) = 0$  and  $m(V) > 0$  separately.

8.

- (a) Let  $A, B$  and  $C$  be subsets of the same set. Prove that

$$A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$$

- (b) Now let  $X = [0, 1]$  or  $[0, 1]^2$ . Let  $A$  be a subset of  $X$ , and suppose that for every  $\varepsilon > 0$  there exists a measurable subset  $B \subseteq X$  such that  $m^*(A \triangle B) < \varepsilon$ . Prove that  $A$  is measurable.

**Hint for 1(b):** The inclusion  $\Phi(C(S^1)) \subseteq C_{2\pi}(\mathbb{R})$  is straightforward. To prove the equality use the fact that the map  $\psi : t \mapsto (\cos(t), \sin(t))$  from  $\mathbb{R}$  to  $S^1$  is locally injective (any point  $t \in \mathbb{R}$  has a neighborhood on which  $\psi$  is injective) and show that if  $I \subseteq \mathbb{R}$  is a closed interval on which  $\psi$  is injective, then the inverse map  $\psi^{-1} : \psi(I) \rightarrow I$  is also continuous – this can be deduced immediately from one of HW#7 problems.

**Hint for 3:** You can find such an example where  $X = [a, \infty)$  for some  $a \in \mathbb{R}$  (the value of  $a$  is not essential here). First find a natural sufficient condition for a differentiable function  $f : X \rightarrow X$  to satisfy  $|f(x) - f(y)| < |x - y|$  for all distinct  $x, y \in X$ . Then determine what it means geometrically for  $f$  to have no fixed points. Now try to draw a graph of a function satisfying both conditions and then find an explicit formula for such function (and prove that it has desired properties).

**Hint for 5(a):** Show that any open subset of  $\mathbb{R}^2$  can be written as a union of squares whose endpoints have rational coordinates.