

Formalisation of Ground Resolution and CDCL in Isabelle/HOL

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theory <i>DPLL-W</i>		
imports		
<i>Entailment-Definition.Partial-Herbrand-Interpretation</i>		
<i>Entailment-Definition.Partial-Annotated-Herbrand-Interpretation</i>		
<i>Weidenbach-Book-Base.Wellfounded-More</i>		
begin		

0.1 Weidenbach's DPLL

0.1.1 Rules

type-synonym *'a dpll_W-ann-lit* = (*'a, unit*) *ann-lit*
type-synonym *'a dpll_W-ann-lits* = (*'a, unit*) *ann-lits*
type-synonym *'v dpll_W-state* = *'v dpll_W-ann-lits* × *'v clauses*

abbreviation *trail* :: *'v dpll_W-state* ⇒ *'v dpll_W-ann-lits* **where**
trail ≡ *fst*

abbreviation *clauses* :: *'v dpll_W-state* ⇒ *'v clauses* **where**
clauses ≡ *snd*

inductive *dpll_W* :: *'v dpll_W-state* ⇒ *'v dpll_W-state* ⇒ *bool* **where**
propagate: *add-mset L C ∈ # clauses S* ⇒ *trail S ⊨_{as} CNot C* ⇒ *undefined-lit (trail S) L*
 ⇒ *dpll_W S (Propagated L () # trail S, clauses S) |*
decided: *undefined-lit (trail S) L* ⇒ *atm-of L ∈ atms-of-mm (clauses S)*
 ⇒ *dpll_W S (Decided L # trail S, clauses S) |*
backtrack: *backtrack-split (trail S) = (M', L # M)* ⇒ *is-decided L* ⇒ *D ∈ # clauses S*
 ⇒ *trail S ⊨_{as} CNot D* ⇒ *dpll_W S (Propagated (− (lit-of L)) () # M, clauses S)*

0.1.2 Invariants

lemma *dpll_W-distinct-inv*:
 assumes *dpll_W S S'*
 and *no-dup (trail S)*
 shows *no-dup (trail S')*
 ⟨*proof*⟩

lemma *dpll_W-consistent-interp-inv*:
 assumes *dpll_W S S'*
 and *consistent-interp (lits-of-l (trail S))*
 and *no-dup (trail S)*

shows *consistent-interp* (*lits-of-l* (*trail S'*))
 ⟨*proof*⟩

lemma *dpll_W-vars-in-snd-inv*:

assumes *dpll_W* *S S'*
and *atm-of* ' (*lits-of-l* (*trail S*)) \subseteq *atms-of-mm* (*clauses S*)
shows *atm-of* ' (*lits-of-l* (*trail S'*)) \subseteq *atms-of-mm* (*clauses S'*)
 ⟨*proof*⟩

lemma *atms-of-ms-lit-of-atms-of*: *atms-of-ms* (*unmark* ' *c*) = *atm-of* ' *lit-of* ' *c*
 ⟨*proof*⟩

theorem 2.8.3 page 86 of Weidenbach's book

lemma *dpll_W-propagate-is-conclusion*:

assumes *dpll_W* *S S'*
and *all-decomposition-implies-m* (*clauses S*) (*get-all-ann-decomposition* (*trail S*))
and *atm-of* ' *lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*)
shows *all-decomposition-implies-m* (*clauses S'*) (*get-all-ann-decomposition* (*trail S'*))
 ⟨*proof*⟩

theorem 2.8.4 page 86 of Weidenbach's book

theorem *dpll_W-propagate-is-conclusion-of-decided*:

assumes *dpll_W* *S S'*
and *all-decomposition-implies-m* (*clauses S*) (*get-all-ann-decomposition* (*trail S*))
and *atm-of* ' *lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*)
shows *set-mset* (*clauses S'*) $\cup \{\{\#lit-of L\# \mid L. is-decided L \wedge L \in set (trail S')\}\}$
 $\models_{ps} unmark ' \bigcup (set ' snd ' set (get-all-ann-decomposition (trail S')))$
 ⟨*proof*⟩

theorem 2.8.5 page 86 of Weidenbach's book

lemma *only-propagated-vars-unsat*:

assumes *decided*: $\forall x \in set M. \neg is-decided x$
and *DN*: *D* $\in N$ **and** *D*: *M* $\models_{as} CNot D$
and *inv*: *all-decomposition-implies* *N* (*get-all-ann-decomposition* *M*)
and *atm-incl*: *atm-of* ' *lits-of-l* *M* \subseteq *atms-of-ms* *N*
shows *unsatisfiable* *N*
 ⟨*proof*⟩

lemma *dpll_W-same-clauses*:

assumes *dpll_W* *S S'*
shows *clauses S* = *clauses S'*
 ⟨*proof*⟩

lemma *rtrancp-dpll_W-inv*:

assumes *rtrancp* *dpll_W* *S S'*
and *inv*: *all-decomposition-implies-m* (*clauses S*) (*get-all-ann-decomposition* (*trail S*))
and *atm-incl*: *atm-of* ' *lits-of-l* (*trail S*) \subseteq *atms-of-mm* (*clauses S*)
and *consistent-interp* (*lits-of-l* (*trail S*))
and *no-dup* (*trail S*)
shows *all-decomposition-implies-m* (*clauses S'*) (*get-all-ann-decomposition* (*trail S'*))
and *atm-of* ' *lits-of-l* (*trail S'*) \subseteq *atms-of-mm* (*clauses S'*)
and *clauses S* = *clauses S'*
and *consistent-interp* (*lits-of-l* (*trail S'*))
and *no-dup* (*trail S'*)
 ⟨*proof*⟩

definition $dpll_W\text{-all-inv } S \equiv$
 $(all\text{-decomposition-implies-}m \text{ (clauses } S) \text{ (get-all-ann-decomposition (trail } S)))$
 $\wedge atm\text{-of ' lits-of-l (trail } S) \subseteq atms\text{-of-mm (clauses } S)$
 $\wedge consistent\text{-interp (lits-of-l (trail } S))$
 $\wedge no\text{-dup (trail } S))$

lemma $dpll_W\text{-all-inv-dest[dest]:}$
assumes $dpll_W\text{-all-inv } S$
shows $all\text{-decomposition-implies-}m \text{ (clauses } S) \text{ (get-all-ann-decomposition (trail } S))$
and $atm\text{-of ' lits-of-l (trail } S) \subseteq atms\text{-of-mm (clauses } S)$
and $consistent\text{-interp (lits-of-l (trail } S)) \wedge no\text{-dup (trail } S)$
 $\langle proof \rangle$

lemma $rtranclp\text{-}dpll_W\text{-all-inv:}$
assumes $rtranclp\text{-}dpll_W\text{-all-inv } S\ S'$
and $dpll_W\text{-all-inv } S$
shows $dpll_W\text{-all-inv } S'$
 $\langle proof \rangle$

lemma $dpll_W\text{-all-inv:}$
assumes $dpll_W\text{-all-inv } S\ S'$
and $dpll_W\text{-all-inv } S$
shows $dpll_W\text{-all-inv } S'$
 $\langle proof \rangle$

lemma $rtranclp\text{-}dpll_W\text{-inv-starting-from-0:}$
assumes $rtranclp\text{-}dpll_W\text{-all-inv } S\ S'$
and $inv: trail\ S = []$
shows $dpll_W\text{-all-inv } S'$
 $\langle proof \rangle$

lemma $dpll_W\text{-can-do-step:}$
assumes $consistent\text{-interp (set } M)$
and $distinct\ M$
and $atm\text{-of ' (set } M) \subseteq atms\text{-of-mm } N$
shows $rtranclp\text{-}dpll_W\text{-all-inv } ([], N) \text{ (map Decided } M, N)$
 $\langle proof \rangle$

definition $conclusive\text{-}dpll_W\text{-state } (S:: 'v\ dpll_W\text{-state}) \longleftrightarrow$
 $(trail\ S \models_{asm} clauses\ S \vee ((\forall L \in set\ (trail\ S). \neg is\text{-decided } L)$
 $\wedge (\exists C \in \# clauses\ S. trail\ S \models_{as} CNot\ C)))$

theorem 2.8.7 page 87 of Weidenbach's book

lemma $dpll_W\text{-strong-completeness:}$
assumes $set\ M \models_{sm} N$
and $consistent\text{-interp (set } M)$
and $distinct\ M$
and $atm\text{-of ' (set } M) \subseteq atms\text{-of-mm } N$
shows $dpll_W^{**}\text{-all-inv } ([], N) \text{ (map Decided } M, N)$
and $conclusive\text{-}dpll_W\text{-state (map Decided } M, N)$
 $\langle proof \rangle$

theorem 2.8.6 page 86 of Weidenbach's book

lemma $dpll_W\text{-sound:}$
assumes

$\text{rtrancpl } dpll_W ([], N) (M, N) \text{ and}$
 $\forall S. \neg dpll_W (M, N) S$
shows $M \models_{asm} N \longleftrightarrow \text{satisfiable } (\text{set-mset } N) \text{ (is } ?A \longleftrightarrow ?B)$
 $\langle \text{proof} \rangle$

0.1.3 Termination

definition $dpll_W\text{-mes } M n =$
 $\text{map } (\lambda l. \text{ if is-decided } l \text{ then } 2 \text{ else } (1::nat)) (\text{rev } M) @ \text{replicate } (n - \text{length } M) 3$

lemma $\text{length-}dpll_W\text{-mes}$:
assumes $\text{length } M \leq n$
shows $\text{length } (dpll_W\text{-mes } M n) = n$
 $\langle \text{proof} \rangle$

lemma $\text{distinctcard-atm-of-lit-of-eq-length}$:
assumes $\text{no-dup } S$
shows $\text{card } (\text{atm-of } ' \text{ lits-of-l } S) = \text{length } S$
 $\langle \text{proof} \rangle$

lemma Cons-lexn-iff :
shows $\langle (x \# xs, y \# ys) \in \text{lexn } R \ n \longleftrightarrow (\text{length } (x \# xs) = n \wedge \text{length } (y \# ys) = n \wedge ((x, y) \in R \vee (x = y \wedge (xs, ys) \in \text{lexn } R \ (n - 1)))) \rangle$
 $\langle \text{proof} \rangle$
declare $\text{append-same-lexn}[\text{simp}] \ \text{prepend-same-lexn}[\text{simp}] \ \text{Cons-lexn-iff}[\text{simp}]$
declare $\text{lexn.simps}(2)[\text{simp del}]$

lemma $dpll_W\text{-card-decrease}$:
assumes
 $dpll: dpll_W \ S \ S' \text{ and}$
 $[\text{simp}]: \text{length } (\text{trail } S') \leq \text{card vars} \text{ and}$
 $\text{length } (\text{trail } S) \leq \text{card vars}$
shows
 $(dpll_W\text{-mes } (\text{trail } S') (\text{card vars}), dpll_W\text{-mes } (\text{trail } S) (\text{card vars})) \in \text{lexn less-than } (\text{card vars})$
 $\langle \text{proof} \rangle$

theorem 2.8.8 page 87 of Weidenbach's book

lemma $dpll_W\text{-card-decrease'}$:
assumes $dpll: dpll_W \ S \ S'$
and $\text{atm-incl}: \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-mm } (\text{clauses } S)$
and $\text{no-dup}: \text{no-dup } (\text{trail } S)$
shows $(dpll_W\text{-mes } (\text{trail } S') (\text{card } (\text{atms-of-mm } (\text{clauses } S'))), dpll_W\text{-mes } (\text{trail } S) (\text{card } (\text{atms-of-mm } (\text{clauses } S)))) \in \text{lex less-than}$
 $\langle \text{proof} \rangle$

lemma wf-lexn : $\text{wf } (\text{lexn } \{(a, b). (a::nat) < b\} (\text{card } (\text{atms-of-mm } (\text{clauses } S))))$
 $\langle \text{proof} \rangle$

lemma $\text{wf-}dpll_W$:
 $\text{wf } \{(S', S). dpll_W\text{-all-inv } S \wedge dpll_W \ S \ S'\}$
 $\langle \text{proof} \rangle$

lemma $dpll_W\text{-trancpl-star-commute}$:
 $\{(S', S). dpll_W\text{-all-inv } S \wedge dpll_W \ S \ S'\}^+ = \{(S', S). dpll_W\text{-all-inv } S \wedge \text{trancpl } dpll_W \ S \ S'\}$
(is $?A = ?B$ **)**

⟨proof⟩

lemma *wf-dpll_W-transclp*: $wf \{(S', S). dpll_W\text{-all-inv } S \wedge dpll_W^{++} S S'\}$
 ⟨proof⟩

lemma *wf-dpll_W-plus*:
 $wf \{(S', ([], N)) \mid S'. dpll_W^{++} ([], N) S'\} \text{ (is } wf ?P)$
 ⟨proof⟩

0.1.4 Final States

Proposition 2.8.1: final states are the normal forms of $dpll_W$

lemma *dpll_W-no-more-step-is-a-conclusive-state*:

assumes $\forall S'. \neg dpll_W S S'$
shows *conclusive-dpll_W-state* S

⟨proof⟩

lemma *dpll_W-conclusive-state-correct*:

assumes $dpll_W^{**} ([], N) (M, N)$ **and** *conclusive-dpll_W-state* (M, N)
shows $M \models_{asm} N \longleftrightarrow \text{satisfiable } (set\text{-mset } N) \text{ (is } ?A \longleftrightarrow ?B)$

⟨proof⟩

end

theory *CDCL-W-Level*

imports

Entailment-Definition.Partial-Annotated-Herbrand-Interpretation

begin

Level of literals and clauses

Getting the level of a variable, implies that the list has to be reversed. Here is the function *after* reversing.

definition *count-decided* :: $('v, 'b, 'm) \text{ annotated-lit list} \Rightarrow nat$ **where**
count-decided $l = \text{length } (\text{filter is-decided } l)$

definition *get-level* :: $('v, 'm) \text{ ann-lits} \Rightarrow 'v \text{ literal} \Rightarrow nat$ **where**
get-level $S L = \text{length } (\text{filter is-decided } (\text{dropWhile } (\lambda S. \text{atm-of } (\text{lit-of } S) \neq \text{atm-of } L) S))$

lemma *get-level-uminus[simp]*: $\langle \text{get-level } M (-L) = \text{get-level } M L \rangle$
 ⟨proof⟩

lemma *get-level-Neg-Pos*: $\langle \text{get-level } M (\text{Neg } L) = \text{get-level } M (\text{Pos } L) \rangle$
 ⟨proof⟩

lemma *count-decided-0-iff*:

$\langle \text{count-decided } M = 0 \longleftrightarrow (\forall L \in \text{set } M. \neg \text{is-decided } L) \rangle$

⟨proof⟩

lemma

shows

count-decided-nil[simp]: $\langle \text{count-decided } [] = 0 \rangle$ **and**

count-decided-cons[simp]:

$\langle \text{count-decided } (a \# M) = (\text{if is-decided } a \text{ then } \text{Suc } (\text{count-decided } M) \text{ else } \text{count-decided } M) \rangle$ **and**

count-decided-append[simp]:

$\langle \text{count-decided } (M @ M') = \text{count-decided } M + \text{count-decided } M' \rangle$
 $\langle \text{proof} \rangle$

lemma *atm-of-notin-get-level-eq-0[simp]*:

assumes *undefined-lit* $M L$

shows $\text{get-level } M L = 0$

$\langle \text{proof} \rangle$

lemma *get-level-ge-0-atm-of-in*:

assumes $\text{get-level } M L > n$

shows $\text{atm-of } L \in \text{atm-of ' lits-of-l } M$

$\langle \text{proof} \rangle$

In *get-level* (resp. *get-level*), the beginning (resp. the end) can be skipped if the literal is not in the beginning (resp. the end).

lemma *get-level-skip[simp]*:

assumes *undefined-lit* $M L$

shows $\text{get-level } (M @ M') L = \text{get-level } M' L$

$\langle \text{proof} \rangle$

If the literal is at the beginning, then the end can be skipped

lemma *get-level-skip-end[simp]*:

assumes *defined-lit* $M L$

shows $\text{get-level } (M @ M') L = \text{get-level } M L + \text{count-decided } M'$

$\langle \text{proof} \rangle$

lemma *get-level-skip-beginning[simp]*:

assumes $\text{atm-of } L' \neq \text{atm-of } (\text{lit-of } K)$

shows $\text{get-level } (K \# M) L' = \text{get-level } M L'$

$\langle \text{proof} \rangle$

lemma *get-level-take-beginning[simp]*:

assumes $\text{atm-of } L' = \text{atm-of } (\text{lit-of } K)$

shows $\text{get-level } (K \# M) L' = \text{count-decided } (K \# M)$

$\langle \text{proof} \rangle$

lemma *get-level-cons-if*:

$\langle \text{get-level } (K \# M) L' =$

$(\text{if } \text{atm-of } L' = \text{atm-of } (\text{lit-of } K) \text{ then } \text{count-decided } (K \# M) \text{ else } \text{get-level } M L') \rangle$

$\langle \text{proof} \rangle$

lemma *get-level-skip-beginning-not-decided[simp]*:

assumes *undefined-lit* $S L$

and $\forall s \in \text{set } S. \neg \text{is-decided } s$

shows $\text{get-level } (M @ S) L = \text{get-level } M L$

$\langle \text{proof} \rangle$

lemma *get-level-skip-all-not-decided[simp]*:

fixes M

assumes $\forall m \in \text{set } M. \neg \text{is-decided } m$

shows $\text{get-level } M L = 0$

$\langle \text{proof} \rangle$

the $\{\#0::'a\# \}$ is there to ensure that the set is not empty.

definition *get-maximum-level* :: $('a, 'b) \text{ ann-lits} \Rightarrow 'a \text{ clause} \Rightarrow \text{nat}$

where

$get\text{-}maximum\text{-}level\ M\ D = Max\text{-}mset\ (\{\#0\#\} + image\text{-}mset\ (get\text{-}level\ M)\ D)$

lemma *get-maximum-level-ge-get-level*:

$L \in \# D \implies get\text{-}maximum\text{-}level\ M\ D \geq get\text{-}level\ M\ L$

$\langle proof \rangle$

lemma *get-maximum-level-empty[simp]*:

$get\text{-}maximum\text{-}level\ M\ \{\#\} = 0$

$\langle proof \rangle$

lemma *get-maximum-level-exists-lit-of-max-level*:

$D \neq \{\#\} \implies \exists L \in \# D. get\text{-}level\ M\ L = get\text{-}maximum\text{-}level\ M\ D$

$\langle proof \rangle$

lemma *get-maximum-level-empty-list[simp]*:

$get\text{-}maximum\text{-}level\ []\ D = 0$

$\langle proof \rangle$

lemma *get-maximum-level-add-mset*:

$get\text{-}maximum\text{-}level\ M\ (add\text{-}mset\ L\ D) = \max\ (get\text{-}level\ M\ L)\ (get\text{-}maximum\text{-}level\ M\ D)$

$\langle proof \rangle$

lemma *get-level-append-if*:

$\langle get\text{-}level\ (M\ @\ M')\ L = (if\ defined\text{-}lit\ M\ L\ then\ get\text{-}level\ M\ L + count\text{-}decided\ M'\ else\ get\text{-}level\ M'\ L) \rangle$

$\langle proof \rangle$

Do not activate as [simp] rules. It breaks everything.

lemma *get-maximum-level-single*:

$get\text{-}maximum\text{-}level\ M\ \{\#x\#\} = get\text{-}level\ M\ x$

$\langle proof \rangle$

lemma *get-maximum-level-plus*:

$get\text{-}maximum\text{-}level\ M\ (D + D') = \max\ (get\text{-}maximum\text{-}level\ M\ D)\ (get\text{-}maximum\text{-}level\ M\ D')$

$\langle proof \rangle$

lemma *get-maximum-level-cong*:

assumes $\forall L \in \# D. get\text{-}level\ M\ L = get\text{-}level\ M'\ L$

shows $get\text{-}maximum\text{-}level\ M\ D = get\text{-}maximum\text{-}level\ M'\ D$

$\langle proof \rangle$

lemma *get-maximum-level-exists-lit*:

assumes $n: n > 0$

and $max: get\text{-}maximum\text{-}level\ M\ D = n$

shows $\exists L \in \# D. get\text{-}level\ M\ L = n$

$\langle proof \rangle$

lemma *get-maximum-level-skip-first[simp]*:

assumes $atm\text{-}of\ (lit\text{-}of\ K) \notin atm\text{-}s\text{-}of\ D$

shows $get\text{-}maximum\text{-}level\ (K\ \#\ M)\ D = get\text{-}maximum\text{-}level\ M\ D$

$\langle proof \rangle$

lemma *get-maximum-level-skip-beginning*:

assumes $DH: \forall x \in \# D. undefined\text{-}lit\ c\ x$

shows $get\text{-}maximum\text{-}level\ (c\ @\ H)\ D = get\text{-}maximum\text{-}level\ H\ D$

$\langle \text{proof} \rangle$

lemma *get-maximum-level-D-single-propagated*:
get-maximum-level [Propagated x21 x22] D = 0
 $\langle \text{proof} \rangle$

lemma *get-maximum-level-union-mset*:
get-maximum-level M (A \cup B) = get-maximum-level M (A + B)
 $\langle \text{proof} \rangle$

lemma *count-decided-rev[simp]*:
count-decided (rev M) = count-decided M
 $\langle \text{proof} \rangle$

lemma *count-decided-ge-get-level*:
count-decided M \geq get-level M L
 $\langle \text{proof} \rangle$

lemma *count-decided-ge-get-maximum-level*:
count-decided M \geq get-maximum-level M D
 $\langle \text{proof} \rangle$

lemma *get-level-last-decided-ge*:
 $\langle \text{defined-lit } (c @ [\text{Decided } K]) L' \implies 0 < \text{get-level } (c @ [\text{Decided } K]) L' \rangle$
 $\langle \text{proof} \rangle$

lemma *get-maximum-level-mono*:
 $\langle D \subseteq \# D' \implies \text{get-maximum-level } M D \leq \text{get-maximum-level } M D' \rangle$
 $\langle \text{proof} \rangle$

fun *get-all-mark-of-propagated where*
get-all-mark-of-propagated [] = [] |
get-all-mark-of-propagated (Decided - # L) = get-all-mark-of-propagated L |
get-all-mark-of-propagated (Propagated - mark # L) = mark # get-all-mark-of-propagated L

lemma *get-all-mark-of-propagated-append[simp]*:
get-all-mark-of-propagated (A @ B) = get-all-mark-of-propagated A @ get-all-mark-of-propagated B
 $\langle \text{proof} \rangle$

lemma *get-all-mark-of-propagated-tl-proped*:
 $\langle M \neq [] \implies \text{is-proped } (\text{hd } M) \implies \text{get-all-mark-of-propagated } (\text{tl } M) = \text{tl } (\text{get-all-mark-of-propagated } M) \rangle$
 $\langle \text{proof} \rangle$

Properties about the levels

lemma *atm-lit-of-set-lits-of-l*:
($\lambda l. \text{atm-of } (\text{lit-of } l)$) ' set xs = atm-of ' lits-of-l xs
 $\langle \text{proof} \rangle$

Before I try yet another time to prove that I can remove the assumption *no-dup M*: It does not work. The problem is that *get-level M K = Suc i* peaks the first occurrence of the literal *K*. This is for example an issue for the trail *replicate n (Decided K)*. An explicit counter-example is below.

lemma *le-count-decided-decomp*:

```

assumes  $\langle \text{no-dup } M \rangle$ 
shows  $\langle i < \text{count-decided } M \longleftrightarrow (\exists c \ K \ c'. \ M = c @ \text{Decided } K \ \# \ c' \wedge \text{get-level } M \ K = \text{Suc } i) \rangle$ 
   $\langle \text{is } ?A \longleftrightarrow ?B \rangle$ 
 $\langle \text{proof} \rangle$ 

```

The counter-example if the assumption *no-dup* *M*.

```

lemma
  fixes K
  defines  $\langle M \equiv \text{replicate } 3 \ (\text{Decided } K) \rangle$ 
  defines  $\langle i \equiv 1 \rangle$ 
  assumes  $\langle i < \text{count-decided } M \longleftrightarrow (\exists c \ K \ c'. \ M = c @ \text{Decided } K \ \# \ c' \wedge \text{get-level } M \ K = \text{Suc } i) \rangle$ 
  shows False
 $\langle \text{proof} \rangle$ 

```

```

lemma Suc-count-decided-gt-get-level:
   $\langle \text{get-level } M \ L < \text{Suc } (\text{count-decided } M) \rangle$ 
 $\langle \text{proof} \rangle$ 

```

```

lemma get-level-neq-Suc-count-decided[simp]:
   $\langle \text{get-level } M \ L \neq \text{Suc } (\text{count-decided } M) \rangle$ 
 $\langle \text{proof} \rangle$ 

```

```

lemma length-get-all-ann-decomposition:  $\langle \text{length } (\text{get-all-ann-decomposition } M) = 1 + \text{count-decided } M \rangle$ 
 $\langle \text{proof} \rangle$ 

```

```

lemma get-maximum-level-remove-non-max-lvl:
   $\langle \text{get-level } M \ a < \text{get-maximum-level } M \ D \implies$ 
   $\text{get-maximum-level } M \ (\text{remove1-mset } a \ D) = \text{get-maximum-level } M \ D \rangle$ 
 $\langle \text{proof} \rangle$ 

```

```

lemma exists-lit-max-level-in-negate-ann-lits:
   $\langle \text{negate-ann-lits } M \neq \{\#\} \implies \exists L \in \#\text{negate-ann-lits } M. \text{get-level } M \ L = \text{count-decided } M \rangle$ 
 $\langle \text{proof} \rangle$ 

```

```

end
theory CDCL-W
  imports CDCL-W-Level Weidenbach-Book-Base.Wellfounded-More
begin

```

Chapter 1

Weidenbach's CDCL

The organisation of the development is the following:

- `CDCL_W.thy` contains the specification of the rules: the rules and the strategy are defined, and we proof the correctness of CDCL.
- `CDCL_W_Termination.thy` contains the proof of termination, based on the book.
- `CDCL_W_Merge.thy` contains a variant of the calculus: some rules of the raw calculus are always applied together (like the rules analysing the conflict and then backtracking). This is useful for the refinement from NOT.
- `CDCL_WNOT.thy` proves the inclusion of Weidenbach's version of CDCL in NOT's version. We use here the version defined in `CDCL_W_Merge.thy`. We need this, because NOT's backjump corresponds to multiple applications of three rules in Weidenbach's calculus. We show also the termination of the calculus without strategy. There are two different refinement: one from NOT's to Weidenbach's CDCL and another to W's CDCL with strategy.

We have some variants build on the top of Weidenbach's CDCL calculus:

- `CDCL_W_Incremental.thy` adds incrementality on the top of `CDCL_W.thy`. The way we are doing it is not compatible with `CDCL_W_Merge.thy`, because we add conflicts and the `CDCL_W_Merge.thy` cannot analyse conflicts added externally, since the conflict and analyse are merged.
- `CDCL_W_Restart.thy` adds restart and forget while restarting. It is built on the top of `CDCL_W_Merge.thy`.

1.1 Weidenbach's CDCL with Multisets

```
declare upt.simps(2)[simp del]
```

1.1.1 The State

We will abstract the representation of clause and clauses via two locales. We here use multisets, contrary to `CDCL_W_Abstract_State.thy` where we assume only the existence of a conversion to the state.

```

locale stateW-ops =
  fixes
    state :: 'st  $\Rightarrow$  ('v, 'v clause) ann-lits  $\times$  'v clauses  $\times$  'v clauses  $\times$  'v clause option  $\times$ 
      'b and
    trail :: 'st  $\Rightarrow$  ('v, 'v clause) ann-lits and
    init-clss :: 'st  $\Rightarrow$  'v clauses and
    learned-clss :: 'st  $\Rightarrow$  'v clauses and
    conflicting :: 'st  $\Rightarrow$  'v clause option and

    cons-trail :: ('v, 'v clause) ann-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
    tl-trail :: 'st  $\Rightarrow$  'st and

    add-learned-clss :: 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st and
    remove-clss :: 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st and
    update-conflicting :: 'v clause option  $\Rightarrow$  'st  $\Rightarrow$  'st and

    init-state :: 'v clauses  $\Rightarrow$  'st
  begin

  abbreviation hd-trail :: 'st  $\Rightarrow$  ('v, 'v clause) ann-lit where
    hd-trail S  $\equiv$  hd (trail S)

  definition clauses :: 'st  $\Rightarrow$  'v clauses where
    clauses S = init-clss S + learned-clss S

  abbreviation resolve-clss :: ('a literal  $\Rightarrow$  'a clause  $\Rightarrow$  'a clause  $\Rightarrow$  'a clause) where
    resolve-clss L D' E  $\equiv$  remove1-mset ( $-L$ ) D'  $\cup\#$  remove1-mset L E

  abbreviation state-butlast :: 'st  $\Rightarrow$  ('v, 'v clause) ann-lits  $\times$  'v clauses  $\times$  'v clauses
     $\times$  'v clause option where
    state-butlast S  $\equiv$  (trail S, init-clss S, learned-clss S, conflicting S)

  definition additional-info :: 'st  $\Rightarrow$  'b where
    additional-info S = ( $\lambda(-, -, -, -, D). D$ ) (state S)

  end

```

We are using an abstract state to abstract away the detail of the implementation: we do not need to know how the clauses are represented internally, we just need to know that they can be converted to multisets.

Weidenbach state is a five-tuple composed of:

1. the trail is a list of decided literals;
2. the initial set of clauses (that is not changed during the whole calculus);
3. the learned clauses (clauses can be added or remove);
4. the conflicting clause (if any has been found so far).

Contrary to the original version, we have removed the maximum level of the trail, since the information is redundant and required an additional invariant.

There are two different clause representation: one for the conflicting clause ('v clause, standing for conflicting clause) and one for the initial and learned clauses ('v clause, standing for clause).

The representation of the clauses annotating literals in the trail is slightly different: being able to convert it to *'v clause* is enough (needed for function *hd-trail* below).

There are several axioms to state the independance of the different fields of the state: for example, adding a clause to the learned clauses does not change the trail.

```

locale stateW-no-state =
  stateW-ops
  state
  — functions about the state:
    — getter:
  trail init-clss learned-clss conflicting
    — setter:
  cons-trail tl-trail add-learned-cls remove-cls
  update-conflicting

  — Some specific states:
  init-state
for
  state-eq :: 'st ⇒ 'st ⇒ bool (infix ~ 50) and
  state :: 'st ⇒ ('v, 'v clause) ann-lits × 'v clauses × 'v clauses × 'v clause option ×
    'b and
  trail :: 'st ⇒ ('v, 'v clause) ann-lits and
  init-clss :: 'st ⇒ 'v clauses and
  learned-clss :: 'st ⇒ 'v clauses and
  conflicting :: 'st ⇒ 'v clause option and

  cons-trail :: ('v, 'v clause) ann-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-learned-cls :: 'v clause ⇒ 'st ⇒ 'st and
  remove-cls :: 'v clause ⇒ 'st ⇒ 'st and
  update-conflicting :: 'v clause option ⇒ 'st ⇒ 'st and

  init-state :: 'v clauses ⇒ 'st +
assumes
  state-eq-ref[simp, intro]: ⟨S ~ S⟩ and
  state-eq-sym: ⟨S ~ T ⟷ T ~ S⟩ and
  state-eq-trans: ⟨S ~ T ⟹ T ~ U' ⟹ S ~ U'⟩ and
  state-eq-state: ⟨S ~ T ⟹ state S = state T⟩ and

  cons-trail:
    ∧S'. state st = (M, S') ⟹
      state (cons-trail L st) = (L # M, S') and

  tl-trail:
    ∧S'. state st = (M, S') ⟹ state (tl-trail st) = (tl M, S') and

  remove-cls:
    ∧S'. state st = (M, N, U, S') ⟹
      state (remove-cls C st) =
        (M, removeAll-mset C N, removeAll-mset C U, S') and

  add-learned-cls:
    ∧S'. state st = (M, N, U, S') ⟹
      state (add-learned-cls C st) = (M, N, {#C#} + U, S') and

  update-conflicting:

```

$\bigwedge S'. \text{state } st = (M, N, U, D, S') \implies$
 $\text{state } (\text{update-conflicting } E \text{ } st) = (M, N, U, E, S') \text{ and}$

init-state:
 $\text{state-butlast } (\text{init-state } N) = ([], N, \{\#\}, \text{None}) \text{ and}$

cons-trail-state-eq:
 $\langle S \sim S' \implies \text{cons-trail } L \text{ } S \sim \text{cons-trail } L \text{ } S' \rangle \text{ and}$

tl-trail-state-eq:
 $\langle S \sim S' \implies \text{tl-trail } S \sim \text{tl-trail } S' \rangle \text{ and}$

add-learned-cls-state-eq:
 $\langle S \sim S' \implies \text{add-learned-cls } C \text{ } S \sim \text{add-learned-cls } C \text{ } S' \rangle \text{ and}$

remove-cls-state-eq:
 $\langle S \sim S' \implies \text{remove-cls } C \text{ } S \sim \text{remove-cls } C \text{ } S' \rangle \text{ and}$

update-conflicting-state-eq:
 $\langle S \sim S' \implies \text{update-conflicting } D \text{ } S \sim \text{update-conflicting } D \text{ } S' \rangle \text{ and}$

tl-trail-add-learned-cls-commute:
 $\langle \text{tl-trail } (\text{add-learned-cls } C \text{ } T) \sim \text{add-learned-cls } C \text{ } (\text{tl-trail } T) \rangle \text{ and}$

tl-trail-update-conflicting:
 $\langle \text{tl-trail } (\text{update-conflicting } D \text{ } T) \sim \text{update-conflicting } D \text{ } (\text{tl-trail } T) \rangle \text{ and}$

update-conflicting-update-conflicting:
 $\langle \bigwedge D \text{ } D' \text{ } S \text{ } S'. S \sim S' \implies$
 $\text{update-conflicting } D \text{ } (\text{update-conflicting } D' \text{ } S) \sim \text{update-conflicting } D \text{ } S' \rangle \text{ and}$

update-conflicting-itself:
 $\langle \bigwedge D \text{ } S'. \text{conflicting } S' = D \implies \text{update-conflicting } D \text{ } S' \sim S' \rangle$

locale *state_W* =

state_W-no-state

state-eq state

— functions about the state:

— getter:

trail init-clss learned-clss conflicting

— setter:

cons-trail tl-trail add-learned-cls remove-cls

update-conflicting

— Some specific states:

init-state

for

state-eq :: 'st \Rightarrow 'st \Rightarrow bool (**infix** \sim 50) **and**

state :: 'st \Rightarrow ('v, 'v clause) ann-lits \times 'v clauses \times 'v clauses \times 'v clause option \times 'b **and**

trail :: 'st \Rightarrow ('v, 'v clause) ann-lits **and**

init-clss :: 'st \Rightarrow 'v clauses **and**

learned-clss :: 'st \Rightarrow 'v clauses **and**

conflicting :: 'st \Rightarrow 'v clause option **and**

cons-trail :: ('v, 'v clause) ann-lit \Rightarrow 'st \Rightarrow 'st **and**

tl-trail :: 'st \Rightarrow 'st **and**

add-learned-cls :: 'v clause \Rightarrow 'st \Rightarrow 'st **and**


```

remove-cls :: 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st and
update-conflicting :: 'v clause option  $\Rightarrow$  'st  $\Rightarrow$  'st and

init-state :: 'v clauses  $\Rightarrow$  'st +
assumes
  state-prop[simp]:
     $\langle \text{state } S = (\text{trail } S, \text{init-clss } S, \text{learned-clss } S, \text{conflicting } S, \text{additional-info } S) \rangle$ 
begin

lemma
  trail-cons-trail[simp]:
    trail (cons-trail L st) = L # trail st and
  trail-tl-trail[simp]: trail (tl-trail st) = tl (trail st) and
  trail-add-learned-cls[simp]:
    trail (add-learned-cls C st) = trail st and
  trail-remove-cls[simp]:
    trail (remove-cls C st) = trail st and
  trail-update-conflicting[simp]: trail (update-conflicting E st) = trail st and

  init-clss-cons-trail[simp]:
    init-clss (cons-trail M st) = init-clss st
    and
  init-clss-tl-trail[simp]:
    init-clss (tl-trail st) = init-clss st and
  init-clss-add-learned-cls[simp]:
    init-clss (add-learned-cls C st) = init-clss st and
  init-clss-remove-cls[simp]:
    init-clss (remove-cls C st) = removeAll-mset C (init-clss st) and
  init-clss-update-conflicting[simp]:
    init-clss (update-conflicting E st) = init-clss st and

  learned-clss-cons-trail[simp]:
    learned-clss (cons-trail M st) = learned-clss st and
  learned-clss-tl-trail[simp]:
    learned-clss (tl-trail st) = learned-clss st and
  learned-clss-add-learned-cls[simp]:
    learned-clss (add-learned-cls C st) =  $\{\#C\# \} + \text{learned-clss st}$  and
  learned-clss-remove-cls[simp]:
    learned-clss (remove-cls C st) = removeAll-mset C (learned-clss st) and
  learned-clss-update-conflicting[simp]:
    learned-clss (update-conflicting E st) = learned-clss st and

  conflicting-cons-trail[simp]:
    conflicting (cons-trail M st) = conflicting st and
  conflicting-tl-trail[simp]:
    conflicting (tl-trail st) = conflicting st and
  conflicting-add-learned-cls[simp]:
    conflicting (add-learned-cls C st) = conflicting st
    and
  conflicting-remove-cls[simp]:
    conflicting (remove-cls C st) = conflicting st and
  conflicting-update-conflicting[simp]:
    conflicting (update-conflicting E st) = E and

  init-state-trail[simp]: trail (init-state N) = [] and
  init-state-clss[simp]: init-clss (init-state N) = N and

```

init-state-learned-clss[simp]: learned-clss (init-state N) = {#} and
init-state-conflicting[simp]: conflicting (init-state N) = None
 ⟨proof⟩

lemma

shows

clauses-cons-trail[simp]:
clauses (cons-trail M S) = clauses S and

clss-tl-trail[simp]: clauses (tl-trail S) = clauses S and
clauses-add-learned-cls-unfolded:
clauses (add-learned-cls U S) = {#U#} + learned-clss S + init-clss S
and
clauses-update-conflicting[simp]: clauses (update-conflicting D S) = clauses S and
clauses-remove-cls[simp]:
clauses (remove-cls C S) = removeAll-mset C (clauses S) and
clauses-add-learned-cls[simp]:
clauses (add-learned-cls C S) = {#C#} + clauses S and
clauses-init-state[simp]: clauses (init-state N) = N
 ⟨proof⟩

lemma *state-eq-trans'*: $\langle S \sim S' \implies T \sim S' \implies T \sim S \rangle$

⟨proof⟩

abbreviation *backtrack-lvl* :: 'st \Rightarrow nat **where**

backtrack-lvl S \equiv count-decided (trail S)

named-theorems *state-simp* ⟨contains all theorems of the form $\textcircled{\{term\}} \langle S \sim T \implies P S = P T \rangle$.
 These theorems can cause a signifecant blow-up of the simp-space⟩

lemma

shows

state-eq-trail[state-simp]: $S \sim T \implies \text{trail } S = \text{trail } T$ and
state-eq-init-clss[state-simp]: $S \sim T \implies \text{init-clss } S = \text{init-clss } T$ and
state-eq-learned-clss[state-simp]: $S \sim T \implies \text{learned-clss } S = \text{learned-clss } T$ and
state-eq-conflicting[state-simp]: $S \sim T \implies \text{conflicting } S = \text{conflicting } T$ and
state-eq-clauses[state-simp]: $S \sim T \implies \text{clauses } S = \text{clauses } T$ and
state-eq-undefined-lit[state-simp]: $S \sim T \implies \text{undefined-lit (trail } S) L = \text{undefined-lit (trail } T) L$ and
state-eq-backtrack-lvl[state-simp]: $S \sim T \implies \text{backtrack-lvl } S = \text{backtrack-lvl } T$
 ⟨proof⟩

lemma *state-eq-conflicting-None:*

$S \sim T \implies \text{conflicting } T = \text{None} \implies \text{conflicting } S = \text{None}$

⟨proof⟩

We combine all simplification rules about (\sim) in a single list of theorems. While they are handy as simplification rule as long as we are working on the state, they also cause a *huge* slow-down in all other cases.

declare *state-simp[simp]*

function *reduce-trail-to* :: 'a list \Rightarrow 'st \Rightarrow 'st **where**

reduce-trail-to F S =

(if length (trail S) = length F \vee trail S = [] then S else reduce-trail-to F (tl-trail S))

⟨proof⟩

termination

$\langle \text{proof} \rangle$

declare *reduce-trail-to.simps*[*simp del*]

lemma *reduce-trail-to-induct*:

assumes

$\langle \bigwedge^F S. \text{length } (\text{trail } S) = \text{length } F \implies P \ F \ S \rangle$ **and**

$\langle \bigwedge^F S. \text{trail } S = [] \implies P \ F \ S \rangle$ **and**

$\langle \bigwedge^F S. \text{length } (\text{trail } S) \neq \text{length } F \implies \text{trail } S \neq [] \implies P \ F \ (\text{tl-trail } S) \implies P \ F \ S \rangle$

shows

$\langle P \ F \ S \rangle$

$\langle \text{proof} \rangle$

lemma

shows

reduce-trail-to-Nil[*simp*]: $\text{trail } S = [] \implies \text{reduce-trail-to } F \ S = S$ **and**

reduce-trail-to-eq-length[*simp*]: $\text{length } (\text{trail } S) = \text{length } F \implies \text{reduce-trail-to } F \ S = S$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-length-ne*:

$\text{length } (\text{trail } S) \neq \text{length } F \implies \text{trail } S \neq [] \implies$

$\text{reduce-trail-to } F \ S = \text{reduce-trail-to } F \ (\text{tl-trail } S)$

$\langle \text{proof} \rangle$

lemma *trail-reduce-trail-to-length-le*:

assumes $\text{length } F > \text{length } (\text{trail } S)$

shows $\text{trail } (\text{reduce-trail-to } F \ S) = []$

$\langle \text{proof} \rangle$

lemma *trail-reduce-trail-to-Nil*[*simp*]:

$\text{trail } (\text{reduce-trail-to } [] \ S) = []$

$\langle \text{proof} \rangle$

lemma *clauses-reduce-trail-to-Nil*:

$\text{clauses } (\text{reduce-trail-to } [] \ S) = \text{clauses } S$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-skip-beginning*:

assumes $\text{trail } S = F' @ F$

shows $\text{trail } (\text{reduce-trail-to } F \ S) = F$

$\langle \text{proof} \rangle$

lemma *clauses-reduce-trail-to*[*simp*]:

$\text{clauses } (\text{reduce-trail-to } F \ S) = \text{clauses } S$

$\langle \text{proof} \rangle$

lemma *conflicting-update-trail*[*simp*]:

$\text{conflicting } (\text{reduce-trail-to } F \ S) = \text{conflicting } S$

$\langle \text{proof} \rangle$

lemma *init-clss-update-trail*[*simp*]:

$\text{init-clss } (\text{reduce-trail-to } F \ S) = \text{init-clss } S$

$\langle \text{proof} \rangle$

lemma *learned-clss-update-trail*[*simp*]:

$\text{learned-clss } (\text{reduce-trail-to } F \ S) = \text{learned-clss } S$

$\langle \text{proof} \rangle$

lemma *conflicting-reduce-trail-to[simp]*:

$\text{conflicting } (\text{reduce-trail-to } F \ S) = \text{None} \longleftrightarrow \text{conflicting } S = \text{None}$

$\langle \text{proof} \rangle$

lemma *trail-eq-reduce-trail-to-eq*:

$\text{trail } S = \text{trail } T \implies \text{trail } (\text{reduce-trail-to } F \ S) = \text{trail } (\text{reduce-trail-to } F \ T)$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-trail-tl-trail-decomp[simp]*:

$\text{trail } S = F' \ @ \ \text{Decided } K \ \# \ F \implies \text{trail } (\text{reduce-trail-to } F \ S) = F$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-add-learned-cls[simp]*:

$\text{trail } (\text{reduce-trail-to } F \ (\text{add-learned-cls } C \ S)) = \text{trail } (\text{reduce-trail-to } F \ S)$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-remove-learned-cls[simp]*:

$\text{trail } (\text{reduce-trail-to } F \ (\text{remove-cls } C \ S)) = \text{trail } (\text{reduce-trail-to } F \ S)$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-update-conflicting[simp]*:

$\text{trail } (\text{reduce-trail-to } F \ (\text{update-conflicting } C \ S)) = \text{trail } (\text{reduce-trail-to } F \ S)$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-length*:

$\text{length } M = \text{length } M' \implies \text{reduce-trail-to } M \ S = \text{reduce-trail-to } M' \ S$

$\langle \text{proof} \rangle$

lemma *trail-reduce-trail-to-drop*:

$\text{trail } (\text{reduce-trail-to } F \ S) =$
 $(\text{if } \text{length } (\text{trail } S) \geq \text{length } F$
 $\text{then } \text{drop } (\text{length } (\text{trail } S) - \text{length } F) (\text{trail } S)$
 $\text{else } [])$

$\langle \text{proof} \rangle$

lemma *in-get-all-ann-decomposition-trail-update-trail[simp]*:

assumes $H: (L \ \# \ M1, M2) \in \text{set } (\text{get-all-ann-decomposition } (\text{trail } S))$

shows $\text{trail } (\text{reduce-trail-to } M1 \ S) = M1$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-state-eq*:

$\langle S \sim S' \implies \text{length } M = \text{length } M' \implies \text{reduce-trail-to } M \ S \sim \text{reduce-trail-to } M' \ S' \rangle$

$\langle \text{proof} \rangle$

lemma *conflicting-cons-trail-conflicting[iff]*:

$\text{conflicting } (\text{cons-trail } L \ S) = \text{None} \longleftrightarrow \text{conflicting } S = \text{None}$

$\langle \text{proof} \rangle$

lemma *conflicting-add-learned-cls-conflicting[iff]*:

$\text{conflicting } (\text{add-learned-cls } C \ S) = \text{None} \longleftrightarrow \text{conflicting } S = \text{None}$

$\langle \text{proof} \rangle$

lemma *reduce-trail-to-compow-tl-trail-le*:

assumes $\langle \text{length } M < \text{length } (\text{trail } M') \rangle$

shows $\langle \text{reduce-trail-to } M \ M' = (\text{tl-trail} \sim (\text{length } (\text{trail } M') - \text{length } M)) \ M' \rangle$
 $\langle \text{proof} \rangle$

lemma *reduce-trail-to-compow-tl-trail-eq*:

$\langle \text{length } M = \text{length } (\text{trail } M') \implies \text{reduce-trail-to } M \ M' = (\text{tl-trail} \sim (\text{length } (\text{trail } M') - \text{length } M)) \ M' \rangle$
 $\langle \text{proof} \rangle$

lemma *reduce-trail-to-compow-tl-trail*:

$\langle \text{length } M \leq \text{length } (\text{trail } M') \implies \text{reduce-trail-to } M \ M' = (\text{tl-trail} \sim (\text{length } (\text{trail } M') - \text{length } M)) \ M' \rangle$
 $\langle \text{proof} \rangle$

lemma *tl-trail-reduce-trail-to-cons*:

$\langle \text{length } (L \# M) < \text{length } (\text{trail } M') \implies \text{tl-trail } (\text{reduce-trail-to } (L \# M) \ M') = \text{reduce-trail-to } M \ M' \rangle$
 $\langle \text{proof} \rangle$

lemma *compow-tl-trail-add-learned-cls-swap*:

$\langle (\text{tl-trail} \sim n) (\text{add-learned-cls } D \ S) \sim \text{add-learned-cls } D ((\text{tl-trail} \sim n) \ S) \rangle$
 $\langle \text{proof} \rangle$

lemma *reduce-trail-to-add-learned-cls-state-eq*:

$\langle \text{length } M \leq \text{length } (\text{trail } S) \implies$
 $\text{reduce-trail-to } M (\text{add-learned-cls } D \ S) \sim \text{add-learned-cls } D (\text{reduce-trail-to } M \ S) \rangle$
 $\langle \text{proof} \rangle$

lemma *compow-tl-trail-update-conflicting-swap*:

$\langle (\text{tl-trail} \sim n) (\text{update-conflicting } D \ S) \sim \text{update-conflicting } D ((\text{tl-trail} \sim n) \ S) \rangle$
 $\langle \text{proof} \rangle$

lemma *reduce-trail-to-update-conflicting-state-eq*:

$\langle \text{length } M \leq \text{length } (\text{trail } S) \implies$
 $\text{reduce-trail-to } M (\text{update-conflicting } D \ S) \sim \text{update-conflicting } D (\text{reduce-trail-to } M \ S) \rangle$
 $\langle \text{proof} \rangle$

lemma

additional-info-cons-trail[simp]:
 $\langle \text{additional-info } (\text{cons-trail } L \ S) = \text{additional-info } S \rangle$ **and**
additional-info-tl-trail[simp]:
 $\text{additional-info } (\text{tl-trail } S) = \text{additional-info } S$ **and**
additional-info-add-learned-cls-unfolded:
 $\text{additional-info } (\text{add-learned-cls } U \ S) = \text{additional-info } S$ **and**
additional-info-update-conflicting[simp]:
 $\text{additional-info } (\text{update-conflicting } D \ S) = \text{additional-info } S$ **and**
additional-info-remove-cls[simp]:
 $\text{additional-info } (\text{remove-cls } C \ S) = \text{additional-info } S$ **and**
additional-info-add-learned-cls[simp]:
 $\text{additional-info } (\text{add-learned-cls } C \ S) = \text{additional-info } S$
 $\langle \text{proof} \rangle$

lemma *additional-info-reduce-trail-to[simp]*:

$\langle \text{additional-info } (\text{reduce-trail-to } F \ S) = \text{additional-info } S \rangle$
 $\langle \text{proof} \rangle$

lemma *reduce-trail-to*:

$\text{state } (\text{reduce-trail-to } F \ S) =$

((if length (trail S) ≥ length F
 then drop (length (trail S) − length F) (trail S)
 else []), init-clss S, learned-clss S, conflicting S, additional-info S)
 ⟨proof⟩

end — end of $state_W$ locale

1.1.2 CDCL Rules

Because of the strategy we will later use, we distinguish propagate, conflict from the other rules

locale *conflict-driven-clause-learning_W* =
state_W
state-eq
state
 — functions for the state:
 — access functions:
trail init-clss learned-clss conflicting
 — changing state:
cons-trail tl-trail add-learned-cls remove-cls
update-conflicting
 — get state:
init-state
for
state-eq :: 'st ⇒ 'st ⇒ bool (**infix** ~ 50) **and**
state :: 'st ⇒ ('v, 'v clause) ann-lits × 'v clauses × 'v clauses × 'v clause option ×
 'b **and**
trail :: 'st ⇒ ('v, 'v clause) ann-lits **and**
init-clss :: 'st ⇒ 'v clauses **and**
learned-clss :: 'st ⇒ 'v clauses **and**
conflicting :: 'st ⇒ 'v clause option **and**

cons-trail :: ('v, 'v clause) ann-lit ⇒ 'st ⇒ 'st **and**
tl-trail :: 'st ⇒ 'st **and**
add-learned-cls :: 'v clause ⇒ 'st ⇒ 'st **and**
remove-cls :: 'v clause ⇒ 'st ⇒ 'st **and**
update-conflicting :: 'v clause option ⇒ 'st ⇒ 'st **and**

init-state :: 'v clauses ⇒ 'st
begin
inductive *propagate* :: 'st ⇒ 'st ⇒ bool **for** *S* :: 'st **where**
propagate-rule: *conflicting S = None* ⇒
E ∈ # clauses S ⇒
L ∈ # E ⇒
trail S ⊨_{as} CNot (E − {#L#}) ⇒
undefined-lit (trail S) L ⇒
T ∼ cons-trail (Propagated L E) S ⇒
propagate S T

inductive-cases *propagateE*: *propagate S T*

inductive *conflict* :: 'st ⇒ 'st ⇒ bool **for** *S* :: 'st **where**
conflict-rule:
conflicting S = None ⇒

$D \in \# \text{ clauses } S \implies$
 $\text{trail } S \models_{as} CNot \ D \implies$
 $T \sim \text{update-conflicting } (Some \ D) \ S \implies$
 $\text{conflict } S \ T$

inductive-cases *conflictE*: *conflict S T*

inductive *backtrack* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

backtrack-rule:

$\text{conflicting } S = Some \ (add-mset \ L \ D) \implies$
 $(Decided \ K \ \# \ M1, \ M2) \in set \ (get-all-ann-decomposition \ (trail \ S)) \implies$
 $get-level \ (trail \ S) \ L = backtrack-lvl \ S \implies$
 $get-level \ (trail \ S) \ L = get-maximum-level \ (trail \ S) \ (add-mset \ L \ D') \implies$
 $get-maximum-level \ (trail \ S) \ D' \equiv i \implies$
 $get-level \ (trail \ S) \ K = i + 1 \implies$
 $D' \subseteq \# \ D \implies$
 $clauses \ S \models_{pm} add-mset \ L \ D' \implies$
 $T \sim cons-trail \ (Propagated \ L \ (add-mset \ L \ D'))$
 $\quad (reduce-trail-to \ M1$
 $\quad \quad (add-learned-cls \ (add-mset \ L \ D')$
 $\quad \quad \quad (update-conflicting \ None \ S))) \implies$
 $backtrack \ S \ T$

inductive-cases *backtrackE*: *backtrack S T*

Here is the normal backtrack rule from Weidenbach's book:

inductive *simple-backtrack* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

simple-backtrack-rule:

$\text{conflicting } S = Some \ (add-mset \ L \ D) \implies$
 $(Decided \ K \ \# \ M1, \ M2) \in set \ (get-all-ann-decomposition \ (trail \ S)) \implies$
 $get-level \ (trail \ S) \ L = backtrack-lvl \ S \implies$
 $get-level \ (trail \ S) \ L = get-maximum-level \ (trail \ S) \ (add-mset \ L \ D) \implies$
 $get-maximum-level \ (trail \ S) \ D \equiv i \implies$
 $get-level \ (trail \ S) \ K = i + 1 \implies$
 $T \sim cons-trail \ (Propagated \ L \ (add-mset \ L \ D))$
 $\quad (reduce-trail-to \ M1$
 $\quad \quad (add-learned-cls \ (add-mset \ L \ D)$
 $\quad \quad \quad (update-conflicting \ None \ S))) \implies$
 $simple-backtrack \ S \ T$

inductive-cases *simple-backtrackE*: *simple-backtrack S T*

This is a generalised version of backtrack: It is general enough to also include OCDCL's version.

inductive *backtrackg* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

backtrackg-rule:

$\text{conflicting } S = Some \ (add-mset \ L \ D) \implies$
 $(Decided \ K \ \# \ M1, \ M2) \in set \ (get-all-ann-decomposition \ (trail \ S)) \implies$
 $get-level \ (trail \ S) \ L = backtrack-lvl \ S \implies$
 $get-level \ (trail \ S) \ L = get-maximum-level \ (trail \ S) \ (add-mset \ L \ D') \implies$
 $get-maximum-level \ (trail \ S) \ D' \equiv i \implies$
 $get-level \ (trail \ S) \ K = i + 1 \implies$
 $D' \subseteq \# \ D \implies$
 $T \sim cons-trail \ (Propagated \ L \ (add-mset \ L \ D'))$
 $\quad (reduce-trail-to \ M1$
 $\quad \quad (add-learned-cls \ (add-mset \ L \ D')$
 $\quad \quad \quad (update-conflicting \ None \ S))) \implies$

backtrackg S T

inductive-cases *backtrackgE*: *backtrackg S T*

inductive *decide* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

decide-rule:

conflicting S = None \Rightarrow
undefined-lit (trail S) L \Rightarrow
atm-of L \in *atms-of-mm (init-clss S)* \Rightarrow
T \sim *cons-trail (Decided L) S* \Rightarrow
decide S T

inductive-cases *decideE*: *decide S T*

inductive *skip* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

skip-rule:

trail S = Propagated L C' # M \Rightarrow
conflicting S = Some E \Rightarrow
 $\neg L \in \# E$ \Rightarrow
 $E \neq \{\#\}$ \Rightarrow
T \sim *tl-trail S* \Rightarrow
skip S T

inductive-cases *skipE*: *skip S T*

get-maximum-level (Propagated L (C + $\{\#L\# \}$) # M) D = k \vee k = 0 (that was in a previous version of the book) is equivalent to *get-maximum-level (Propagated L (C + $\{\#L\# \}$) # M) D = k*, when the structural invariants holds.

inductive *resolve* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

resolve-rule: *trail S* $\neq []$ \Rightarrow

hd-trail S = Propagated L E \Rightarrow

$L \in \# E$ \Rightarrow

conflicting S = Some D' \Rightarrow

$\neg L \in \# D'$ \Rightarrow

get-maximum-level (trail S) ((remove1-mset ($\neg L$) D')) = backtrack-lvl S \Rightarrow

T \sim *update-conflicting (Some (resolve-cls L D' E))*

(tl-trail S) \Rightarrow

resolve S T

inductive-cases *resolveE*: *resolve S T*

Christoph's version restricts restarts to the the case where $\neg M \models N + U$. While it is possible to implement this (by watching a clause), This is an unnecessary restriction.

inductive *restart* :: 'st \Rightarrow 'st \Rightarrow bool **for** *S* :: 'st **where**

restart: *state S = (M, N, U, None, S')* \Rightarrow

$U' \subseteq \# U$ \Rightarrow

state T = ([], N, U', None, S') \Rightarrow

restart S T

inductive-cases *restartE*: *restart S T*

We add the condition $C \notin \# \text{init-clss } S$, to maintain consistency even without the strategy.

inductive *forget* :: 'st \Rightarrow 'st \Rightarrow bool **where**

forget-rule:

conflicting S = None \Rightarrow

$C \in \# \text{ learned-clss } S \implies$
 $\neg(\text{trail } S) \models_{asm} \text{ clauses } S \implies$
 $C \notin \text{ set } (\text{get-all-mark-of-propagated } (\text{trail } S)) \implies$
 $C \notin \# \text{ init-clss } S \implies$
 $\text{removeAll-mset } C (\text{clauses } S) \models_{pm} C \implies$
 $T \sim \text{remove-cls } C S \implies$
 $\text{forget } S T$

inductive-cases *forgetE*: *forget* $S T$

inductive *cdcl_W-rf* :: '*st* \Rightarrow '*st* \Rightarrow *bool* for $S ::$ '*st* where
restart: *restart* $S T \implies \text{cdcl}_W\text{-rf } S T \mid$
forget: *forget* $S T \implies \text{cdcl}_W\text{-rf } S T$

inductive *cdcl_W-bj* :: '*st* \Rightarrow '*st* \Rightarrow *bool* where
skip: *skip* $S S' \implies \text{cdcl}_W\text{-bj } S S' \mid$
resolve: *resolve* $S S' \implies \text{cdcl}_W\text{-bj } S S' \mid$
backtrack: *backtrack* $S S' \implies \text{cdcl}_W\text{-bj } S S'$

inductive-cases *cdcl_W-bjE*: *cdcl_W-bj* $S T$

inductive *cdcl_W-o* :: '*st* \Rightarrow '*st* \Rightarrow *bool* for $S ::$ '*st* where
decide: *decide* $S S' \implies \text{cdcl}_W\text{-o } S S' \mid$
bj: *cdcl_W-bj* $S S' \implies \text{cdcl}_W\text{-o } S S'$

inductive *cdcl_W-restart* :: '*st* \Rightarrow '*st* \Rightarrow *bool* for $S ::$ '*st* where
propagate: *propagate* $S S' \implies \text{cdcl}_W\text{-restart } S S' \mid$
conflict: *conflict* $S S' \implies \text{cdcl}_W\text{-restart } S S' \mid$
other: *cdcl_W-o* $S S' \implies \text{cdcl}_W\text{-restart } S S' \mid$
rf: *cdcl_W-rf* $S S' \implies \text{cdcl}_W\text{-restart } S S'$

lemma *rtrancplp-propagate-is-rtrancplp-cdcl_W-restart*:
 $\text{propagate}^{**} S S' \implies \text{cdcl}_W\text{-restart}^{**} S S'$
 $\langle \text{proof} \rangle$

inductive *cdcl_W* :: '*st* \Rightarrow '*st* \Rightarrow *bool* for $S ::$ '*st* where
W-propagate: *propagate* $S S' \implies \text{cdcl}_W S S' \mid$
W-conflict: *conflict* $S S' \implies \text{cdcl}_W S S' \mid$
W-other: *cdcl_W-o* $S S' \implies \text{cdcl}_W S S'$

lemma *cdcl_W-cdcl_W-restart*:
 $\text{cdcl}_W S T \implies \text{cdcl}_W\text{-restart } S T$
 $\langle \text{proof} \rangle$

lemma *rtrancplp-cdcl_W-cdcl_W-restart*:
 $\langle \text{cdcl}_W^{**} S T \implies \text{cdcl}_W\text{-restart}^{**} S T \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-restart-all-rules-induct*[*consumes* 1, *case-names* *propagate conflict forget restart decide skip resolve backtrack*]:
fixes $S ::$ '*st*
assumes
cdcl_W-restart: *cdcl_W-restart* $S S'$ **and**
propagate: $\bigwedge T. \text{propagate } S T \implies P S T$ **and**
conflict: $\bigwedge T. \text{conflict } S T \implies P S T$ **and**
forget: $\bigwedge T. \text{forget } S T \implies P S T$ **and**

restart: $\bigwedge T. \text{ restart } S \ T \implies P \ S \ T$ **and**
decide: $\bigwedge T. \text{ decide } S \ T \implies P \ S \ T$ **and**
skip: $\bigwedge T. \text{ skip } S \ T \implies P \ S \ T$ **and**
resolve: $\bigwedge T. \text{ resolve } S \ T \implies P \ S \ T$ **and**
backtrack: $\bigwedge T. \text{ backtrack } S \ T \implies P \ S \ T$
shows $P \ S \ S'$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-restart-all-induct*[consumes 1, case-names propagate conflict forget restart decide skip resolve backtrack]:

fixes $S :: 'st$

assumes

cdcl_W-restart: *cdcl_W-restart* $S \ S'$ **and**

propagateH: $\bigwedge C \ L \ T. \text{ conflicting } S = \text{None} \implies$

$C \in \# \text{ clauses } S \implies$

$L \in \# \ C \implies$

$\text{trail } S \models_{\text{as}} C \text{Not } (\text{remove1-mset } L \ C) \implies$

$\text{undefined-lit } (\text{trail } S) \ L \implies$

$T \sim \text{cons-trail } (\text{Propagated } L \ C) \ S \implies$

$P \ S \ T$ **and**

conflictH: $\bigwedge D \ T. \text{ conflicting } S = \text{None} \implies$

$D \in \# \text{ clauses } S \implies$

$\text{trail } S \models_{\text{as}} C \text{Not } D \implies$

$T \sim \text{update-conflicting } (\text{Some } D) \ S \implies$

$P \ S \ T$ **and**

forgetH: $\bigwedge C \ T. \text{ conflicting } S = \text{None} \implies$

$C \in \# \text{ learned-clss } S \implies$

$\neg(\text{trail } S) \models_{\text{asm}} \text{clauses } S \implies$

$C \notin \text{set } (\text{get-all-mark-of-propagated } (\text{trail } S)) \implies$

$C \notin \# \text{ init-clss } S \implies$

$\text{removeAll-mset } C \ (\text{clauses } S) \models_{\text{pm}} C \implies$

$T \sim \text{remove-clss } C \ S \implies$

$P \ S \ T$ **and**

restartH: $\bigwedge T \ U. \text{ conflicting } S = \text{None} \implies$

$\text{state } T = ([], \text{init-clss } S, U, \text{None}, \text{additional-info } S) \implies$

$U \subseteq \# \text{ learned-clss } S \implies$

$P \ S \ T$ **and**

decideH: $\bigwedge L \ T. \text{ conflicting } S = \text{None} \implies$

$\text{undefined-lit } (\text{trail } S) \ L \implies$

$\text{atm-of } L \in \text{atms-of-mm } (\text{init-clss } S) \implies$

$T \sim \text{cons-trail } (\text{Decided } L) \ S \implies$

$P \ S \ T$ **and**

skipH: $\bigwedge L \ C' \ M \ E \ T.$

$\text{trail } S = \text{Propagated } L \ C' \ \# \ M \implies$

$\text{conflicting } S = \text{Some } E \implies$

$\neg L \notin \# \ E \implies E \neq \{\#\} \implies$

$T \sim \text{tl-trail } S \implies$

$P \ S \ T$ **and**

resolveH: $\bigwedge L \ E \ M \ D \ T.$

$\text{trail } S = \text{Propagated } L \ E \ \# \ M \implies$

$L \in \# \ E \implies$

$\text{hd-trail } S = \text{Propagated } L \ E \implies$

$\text{conflicting } S = \text{Some } D \implies$

$\neg L \in \# \ D \implies$

$\text{get-maximum-level } (\text{trail } S) ((\text{remove1-mset } (\neg L) \ D)) = \text{backtrack-lvl } S \implies$

$T \sim \text{update-conflicting}$

$(\text{Some } (\text{resolve-cls } L \ D \ E)) \ (tl\text{-trail } S) \implies$
 $P \ S \ T$ and
 $\text{backtrackH}: \bigwedge L \ D \ K \ i \ M1 \ M2 \ T \ D'.$
 $\text{conflicting } S = \text{Some } (\text{add-mset } L \ D) \implies$
 $(\text{Decided } K \ \# \ M1, \ M2) \in \text{set } (\text{get-all-ann-decomposition } (\text{trail } S)) \implies$
 $\text{get-level } (\text{trail } S) \ L = \text{backtrack-lvl } S \implies$
 $\text{get-level } (\text{trail } S) \ L = \text{get-maximum-level } (\text{trail } S) \ (\text{add-mset } L \ D') \implies$
 $\text{get-maximum-level } (\text{trail } S) \ D' \equiv i \implies$
 $\text{get-level } (\text{trail } S) \ K = i+1 \implies$
 $D' \subseteq \# \ D \implies$
 $\text{clauses } S \models_{pm} \text{add-mset } L \ D' \implies$
 $T \sim \text{cons-trail } (\text{Propagated } L \ (\text{add-mset } L \ D'))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{add-mset } L \ D')$
 $(\text{update-conflicting } \text{None } S))) \implies$
 $P \ S \ T$
shows $P \ S \ S'$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-o-induct}[\text{consumes } 1, \text{case-names decide skip resolve backtrack}]:$

fixes $S :: 'st$
assumes $\text{cdcl}_W\text{-restart}: \text{cdcl}_W\text{-o } S \ T$ **and**
 $\text{decideH}: \bigwedge L \ T. \text{conflicting } S = \text{None} \implies \text{undefined-lit } (\text{trail } S) \ L$
 $\implies \text{atm-of } L \in \text{atms-of-mm } (\text{init-clss } S)$
 $\implies T \sim \text{cons-trail } (\text{Decided } L) \ S$
 $\implies P \ S \ T$ **and**
 $\text{skipH}: \bigwedge L \ C' \ M \ E \ T.$
 $\text{trail } S = \text{Propagated } L \ C' \ \# \ M \implies$
 $\text{conflicting } S = \text{Some } E \implies$
 $-L \notin \# \ E \implies E \neq \{\#\} \implies$
 $T \sim tl\text{-trail } S \implies$
 $P \ S \ T$ and
 $\text{resolveH}: \bigwedge L \ E \ M \ D \ T.$
 $\text{trail } S = \text{Propagated } L \ E \ \# \ M \implies$
 $L \in \# \ E \implies$
 $hd\text{-trail } S = \text{Propagated } L \ E \implies$
 $\text{conflicting } S = \text{Some } D \implies$
 $-L \in \# \ D \implies$
 $\text{get-maximum-level } (\text{trail } S) \ ((\text{remove1-mset } (-L) \ D)) = \text{backtrack-lvl } S \implies$
 $T \sim \text{update-conflicting}$
 $(\text{Some } (\text{resolve-cls } L \ D \ E)) \ (tl\text{-trail } S) \implies$
 $P \ S \ T$ and
 $\text{backtrackH}: \bigwedge L \ D \ K \ i \ M1 \ M2 \ T \ D'.$
 $\text{conflicting } S = \text{Some } (\text{add-mset } L \ D) \implies$
 $(\text{Decided } K \ \# \ M1, \ M2) \in \text{set } (\text{get-all-ann-decomposition } (\text{trail } S)) \implies$
 $\text{get-level } (\text{trail } S) \ L = \text{backtrack-lvl } S \implies$
 $\text{get-level } (\text{trail } S) \ L = \text{get-maximum-level } (\text{trail } S) \ (\text{add-mset } L \ D') \implies$
 $\text{get-maximum-level } (\text{trail } S) \ D' \equiv i \implies$
 $\text{get-level } (\text{trail } S) \ K = i+1 \implies$
 $D' \subseteq \# \ D \implies$
 $\text{clauses } S \models_{pm} \text{add-mset } L \ D' \implies$
 $T \sim \text{cons-trail } (\text{Propagated } L \ (\text{add-mset } L \ D'))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{add-mset } L \ D')$
 $(\text{update-conflicting } \text{None } S))) \implies$
 $P \ S \ T$

shows $P\ S\ T$

$\langle proof \rangle$

lemma *cdcl_W-o-all-rules-induct*[consumes 1, case-names decide backtrack skip resolve]:

fixes $S\ T :: 'st$

assumes

cdcl_W-o $S\ T$ **and**

$\bigwedge T. \text{decide } S\ T \implies P\ S\ T$ **and**

$\bigwedge T. \text{backtrack } S\ T \implies P\ S\ T$ **and**

$\bigwedge T. \text{skip } S\ T \implies P\ S\ T$ **and**

$\bigwedge T. \text{resolve } S\ T \implies P\ S\ T$

shows $P\ S\ T$

$\langle proof \rangle$

lemma *cdcl_W-o-rule-cases*[consumes 1, case-names decide backtrack skip resolve]:

fixes $S\ T :: 'st$

assumes

cdcl_W-o $S\ T$ **and**

$\text{decide } S\ T \implies P$ **and**

$\text{backtrack } S\ T \implies P$ **and**

$\text{skip } S\ T \implies P$ **and**

$\text{resolve } S\ T \implies P$

shows P

$\langle proof \rangle$

lemma *backtrack-backtrackg*:

$\langle \text{backtrack } S\ T \implies \text{backtrackg } S\ T \rangle$

$\langle proof \rangle$

lemma *simple-backtrack-backtrackg*:

$\langle \text{simple-backtrack } S\ T \implies \text{backtrackg } S\ T \rangle$

$\langle proof \rangle$

1.1.3 Structural Invariants

Properties of the trail

We here establish that:

- the consistency of the trail;
- the fact that there is no duplicate in the trail.

Nitpicking 0.1. *As one can see in the following proof, the properties of the levels are required to prove Item 1 page 94 of Weidenbach's book. However, this point is only mentioned later, and only in the proof of Item 7 page 95 of Weidenbach's book.*

lemma *backtrack-lit-skipped*:

assumes

L: get-level (*trail* S) $L = \text{backtrack-lvl } S$ **and**

M1: (Decided $K \# M1, M2) \in \text{set } (\text{get-all-ann-decomposition } (\text{trail } S))$ **and**

no-dup: no-dup (*trail* S) **and**

lev-K: get-level (*trail* S) $K = i + 1$

shows *undefined-lit M1 L*
 $\langle proof \rangle$

lemma *cdcl_W-restart-distinctinv-1*:

assumes
cdcl_W-restart S S' **and**
n-d: no-dup (trail S)
shows *no-dup (trail S')*
 $\langle proof \rangle$

Item 1 page 94 of Weidenbach's book

lemma *cdcl_W-restart-consistent-inv-2*:

assumes
cdcl_W-restart S S' **and**
no-dup (trail S)
shows *consistent-interp (lits-of-l (trail S'))*
 $\langle proof \rangle$

definition *cdcl_W-M-level-inv :: 'st \Rightarrow bool* **where**

cdcl_W-M-level-inv S \longleftrightarrow
consistent-interp (lits-of-l (trail S))
 \wedge *no-dup (trail S)*

lemma *cdcl_W-M-level-inv-decomp*:

assumes *cdcl_W-M-level-inv S*
shows
consistent-interp (lits-of-l (trail S)) **and**
no-dup (trail S)
 $\langle proof \rangle$

lemma *cdcl_W-restart-consistent-inv*:

fixes *S S' :: 'st*
assumes
cdcl_W-restart S S' **and**
cdcl_W-M-level-inv S
shows *cdcl_W-M-level-inv S'*
 $\langle proof \rangle$

lemma *rtrancp-cdcl_W-restart-consistent-inv*:

assumes
*cdcl_W-restart** S S'* **and**
cdcl_W-M-level-inv S
shows *cdcl_W-M-level-inv S'*
 $\langle proof \rangle$

lemma *trancp-cdcl_W-restart-consistent-inv*:

assumes
cdcl_W-restart++ S S' **and**
cdcl_W-M-level-inv S
shows *cdcl_W-M-level-inv S'*
 $\langle proof \rangle$

lemma *cdcl_W-M-level-inv-S0-cdcl_W-restart[simp]*:

cdcl_W-M-level-inv (init-state N)
 $\langle proof \rangle$

lemma *backtrack-ex-decomp*:

assumes

M-l: *no-dup* (*trail S*) **and**

i-S: *i* < *backtrack-lvl S*

shows $\exists K\ M1\ M2. (Decided\ K\ \# \ M1, M2) \in set\ (get-all-ann-decomposition\ (trail\ S)) \wedge$
 $get-level\ (trail\ S)\ K = Suc\ i$

<proof>

lemma *backtrack-lvl-backtrack-decrease*:

assumes *inv*: *cdcl_W-M-level-inv S* **and** *bt*: *backtrack S T*

shows *backtrack-lvl T* < *backtrack-lvl S*

<proof>

Compatibility with (\sim)

declare *state-eq-trans*[*trans*]

lemma *propagate-state-eq-compatible*:

assumes

propa: *propagate S T* **and**

SS': *S* \sim *S'* **and**

TT': *T* \sim *T'*

shows *propagate S' T'*

<proof>

lemma *conflict-state-eq-compatible*:

assumes

conf: *conflict S T* **and**

TT': *T* \sim *T'* **and**

SS': *S* \sim *S'*

shows *conflict S' T'*

<proof>

lemma *backtrack-state-eq-compatible*:

assumes

bt: *backtrack S T* **and**

SS': *S* \sim *S'* **and**

TT': *T* \sim *T'*

shows *backtrack S' T'*

<proof>

lemma *decide-state-eq-compatible*:

assumes

dec: *decide S T* **and**

SS': *S* \sim *S'* **and**

TT': *T* \sim *T'*

shows *decide S' T'*

<proof>

lemma *skip-state-eq-compatible*:

assumes

skip: *skip S T* **and**

SS': *S* \sim *S'* **and**

TT': *T* \sim *T'*

shows *skip S' T'*

<proof>

lemma *resolve-state-eq-compatible:*

assumes

res: resolve S T and

TT': T ~ T' and

SS': S ~ S'

shows *resolve S' T'*

<proof>

lemma *forget-state-eq-compatible:*

assumes

forget: forget S T and

SS': S ~ S' and

TT': T ~ T'

shows *forget S' T'*

<proof>

lemma *cdcl_W-restart-state-eq-compatible:*

assumes

cdcl_W-restart S T and \neg restart S T and

S ~ S'

T ~ T'

shows *cdcl_W-restart S' T'*

<proof>

lemma *cdcl_W-bj-state-eq-compatible:*

assumes

cdcl_W-bj S T

T ~ T'

shows *cdcl_W-bj S' T'*

<proof>

lemma *trancpl-cdcl_W-bj-state-eq-compatible:*

assumes

cdcl_W-bj⁺⁺ S T

S ~ S' and

T ~ T'

shows *cdcl_W-bj⁺⁺ S' T'*

<proof>

lemma *skip-unique:*

skip S T \implies skip S T' \implies T ~ T'

<proof>

lemma *resolve-unique:*

resolve S T \implies resolve S T' \implies T ~ T'

<proof>

The same holds for backtrack, but more invariants are needed.

Conservation of some Properties

lemma *cdcl_W-o-no-more-init-clss:*

assumes

cdcl_W-o S S' and

inv: cdcl_W-M-level-inv S

shows $\text{init-clss } S = \text{init-clss } S'$
 $\langle \text{proof} \rangle$

lemma $\text{trancpl-cdcl}_W\text{-o-no-more-init-clss}$:

assumes
 $\text{cdcl}_W\text{-o}^{++} S S'$ **and**
 $\text{inv: cdcl}_W\text{-M-level-inv } S$
shows $\text{init-clss } S = \text{init-clss } S'$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_W\text{-o-no-more-init-clss}$:

assumes
 $\text{cdcl}_W\text{-o}^{**} S S'$ **and**
 $\text{inv: cdcl}_W\text{-M-level-inv } S$
shows $\text{init-clss } S = \text{init-clss } S'$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-restart-init-clss}$:

assumes
 $\text{cdcl}_W\text{-restart } S T$
shows $\text{init-clss } S = \text{init-clss } T$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_W\text{-restart-init-clss}$:

$\text{cdcl}_W\text{-restart}^{**} S T \implies \text{init-clss } S = \text{init-clss } T$
 $\langle \text{proof} \rangle$

lemma $\text{trancpl-cdcl}_W\text{-restart-init-clss}$:

$\text{cdcl}_W\text{-restart}^{++} S T \implies \text{init-clss } S = \text{init-clss } T$
 $\langle \text{proof} \rangle$

Learned Clause

This invariant shows that:

- the learned clauses are entailed by the initial set of clauses.
- the conflicting clause is entailed by the initial set of clauses.
- the marks belong to the clauses. We could also restrict it to entailment by the clauses, to allow forgetting this clauses.

definition (**in** $\text{state}_W\text{-ops}$) $\text{reasons-in-clauses} :: \langle 'st \Rightarrow \text{bool} \rangle$ **where**

$\langle \text{reasons-in-clauses } (S :: 'st) \longleftrightarrow$
 $(\text{set } (\text{get-all-mark-of-propagated } (\text{trail } S)) \subseteq \text{set-mset } (\text{clauses } S)) \rangle$

definition (**in** $\text{state}_W\text{-ops}$) $\text{cdcl}_W\text{-learned-clause} :: \langle 'st \Rightarrow \text{bool} \rangle$ **where**

$\text{cdcl}_W\text{-learned-clause } (S :: 'st) \longleftrightarrow$
 $((\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{clauses } S \models_{\text{pm}} T)$
 $\wedge \text{reasons-in-clauses } S)$

lemma $\text{cdcl}_W\text{-learned-clause-alt-def}$:

$\langle \text{cdcl}_W\text{-learned-clause } (S :: 'st) \longleftrightarrow$
 $((\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{clauses } S \models_{\text{pm}} T)$
 $\wedge \text{set } (\text{get-all-mark-of-propagated } (\text{trail } S)) \subseteq \text{set-mset } (\text{clauses } S)) \rangle$

$\langle proof \rangle$

lemma *reasons-in-clauses-init-state*[simp]: $\langle reasons-in-clauses \ (init-state \ N) \rangle$
 $\langle proof \rangle$

Item 3 page 95 of Weidenbach's book for the initial state and some additional structural properties about the trail.

lemma *cdcl_W-learned-clause-S0-cdcl_W-restart*[simp]:
cdcl_W-learned-clause (init-state *N*)
 $\langle proof \rangle$

Item 4 page 95 of Weidenbach's book

lemma *cdcl_W-restart-learned-clss*:
assumes
cdcl_W-restart *S S'* **and**
learned: cdcl_W-learned-clause *S* **and**
lev-inv: cdcl_W-M-level-inv *S*
shows *cdcl_W-learned-clause* *S'*
 $\langle proof \rangle$

lemma *rtrancp-cdcl_W-restart-learned-clss*:
assumes
*cdcl_W-restart*** *S S'* **and**
cdcl_W-M-level-inv *S*
cdcl_W-learned-clause *S*
shows *cdcl_W-learned-clause* *S'*
 $\langle proof \rangle$

lemma *cdcl_W-restart-reasons-in-clauses*:
assumes
cdcl_W-restart *S S'* **and**
learned: reasons-in-clauses *S*
shows *reasons-in-clauses* *S'*
 $\langle proof \rangle$

lemma *rtrancp-cdcl_W-restart-reasons-in-clauses*:
assumes
*cdcl_W-restart*** *S S'* **and**
learned: reasons-in-clauses *S*
shows *reasons-in-clauses* *S'*
 $\langle proof \rangle$

No alien atom in the state

This invariant means that all the literals are in the set of clauses. These properties are implicit in Weidenbach's book.

definition *no-strange-atm* $S' \longleftrightarrow$
 $(\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S'))$
 $\wedge (\forall L \text{ mark}. \text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S') \longrightarrow \text{atms-of mark} \subseteq \text{atms-of-mm } (\text{init-clss } S'))$
 $\wedge \text{atms-of-mm } (\text{learned-clss } S') \subseteq \text{atms-of-mm } (\text{init-clss } S')$
 $\wedge \text{atm-of } ' (\text{lits-of-l } (\text{trail } S')) \subseteq \text{atms-of-mm } (\text{init-clss } S')$

lemma *no-strange-atm-decomp*:
assumes *no-strange-atm* *S*

shows *conflicting* $S = \text{Some } T \implies \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S)$
and $(\forall L \text{ mark. } \text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{atms-of mark} \subseteq \text{atms-of-mm } (\text{init-clss } S))$
and $\text{atms-of-mm } (\text{learned-clss } S) \subseteq \text{atms-of-mm } (\text{init-clss } S)$
and $\text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-mm } (\text{init-clss } S)$
 $\langle \text{proof} \rangle$

lemma *no-strange-atm-S0* [simp]: *no-strange-atm* (*init-state* N)
 $\langle \text{proof} \rangle$

lemma *propagate-no-strange-atm-inv*:

assumes
propagate S T **and**
alien: *no-strange-atm* S
shows *no-strange-atm* T
 $\langle \text{proof} \rangle$

lemma *atms-of-ms-learned-clss-restart-state-in-atms-of-ms-learned-clssI*:

$\text{atms-of-mm } (\text{learned-clss } S) \subseteq \text{atms-of-mm } (\text{init-clss } S) \implies$
 $x \in \text{atms-of-mm } (\text{learned-clss } T) \implies$
 $\text{learned-clss } T \subseteq \# \text{ learned-clss } S \implies$
 $x \in \text{atms-of-mm } (\text{init-clss } S)$
 $\langle \text{proof} \rangle$

lemma (**in** $-$) *atms-of-subset-mset-mono*: $\langle D' \subseteq \# D \implies \text{atms-of } D' \subseteq \text{atms-of } D \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-restart-no-strange-atm-explicit*:

assumes
cdcl_W-restart S S' **and**
lev: *cdcl_W-M-level-inv* S **and**
conf: $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S)$ **and**
decided: $\forall L \text{ mark. } \text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{atms-of mark} \subseteq \text{atms-of-mm } (\text{init-clss } S)$ **and**
learned: $\text{atms-of-mm } (\text{learned-clss } S) \subseteq \text{atms-of-mm } (\text{init-clss } S)$ **and**
trail: $\text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-mm } (\text{init-clss } S)$
shows
 $(\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{atms-of } T \subseteq \text{atms-of-mm } (\text{init-clss } S')) \wedge$
 $(\forall L \text{ mark. } \text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S') \longrightarrow \text{atms-of mark} \subseteq \text{atms-of-mm } (\text{init-clss } S')) \wedge$
 $\text{atms-of-mm } (\text{learned-clss } S') \subseteq \text{atms-of-mm } (\text{init-clss } S') \wedge$
 $\text{atm-of } ' (\text{lits-of-l } (\text{trail } S')) \subseteq \text{atms-of-mm } (\text{init-clss } S')$
 $(\text{is } ?C S' \wedge ?M S' \wedge ?U S' \wedge ?V S')$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-restart-no-strange-atm-inv*:

assumes *cdcl_W-restart* S S' **and** *no-strange-atm* S **and** *cdcl_W-M-level-inv* S
shows *no-strange-atm* S'
 $\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_W-restart-no-strange-atm-inv*:

assumes *cdcl_W-restart*** S S' **and** *no-strange-atm* S **and** *cdcl_W-M-level-inv* S
shows *no-strange-atm* S'
 $\langle \text{proof} \rangle$

No Duplicates all Around

This invariant shows that there is no duplicate (no literal appearing twice in the formula). The last part could be proven using the previous invariant also. Remark that we will show later that there cannot be duplicate *clause*.

definition *distinct-cdcl_W-state* ($S :: 'st$)
 $\longleftrightarrow ((\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{distinct-mset } T)$
 $\wedge \text{distinct-mset-mset } (\text{learned-clss } S)$
 $\wedge \text{distinct-mset-mset } (\text{init-clss } S)$
 $\wedge (\forall L \text{ mark. } (\text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{distinct-mset mark})))$

lemma *distinct-cdcl_W-state-decomp*:

assumes *distinct-cdcl_W-state* S

shows

$\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{distinct-mset } T$ **and**

distinct-mset-mset (*learned-clss* S) **and**

distinct-mset-mset (*init-clss* S) **and**

$\forall L \text{ mark. } (\text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{distinct-mset mark})$

$\langle \text{proof} \rangle$

lemma *distinct-cdcl_W-state-decomp-2*:

assumes *distinct-cdcl_W-state* S **and** *conflicting* $S = \text{Some } T$

shows *distinct-mset* T

$\langle \text{proof} \rangle$

lemma *distinct-cdcl_W-state-S0-cdcl_W-restart*[*simp*]:

distinct-mset-mset $N \implies \text{distinct-cdcl}_W\text{-state } (\text{init-state } N)$

$\langle \text{proof} \rangle$

lemma *distinct-cdcl_W-state-inv*:

assumes

cdcl_W-restart $S S'$ **and**

lev-inv: *cdcl_W-M-level-inv* S **and**

distinct-cdcl_W-state S

shows *distinct-cdcl_W-state* S'

$\langle \text{proof} \rangle$

lemma *rtanclp-distinct-cdcl_W-state-inv*:

assumes

*cdcl_W-restart*** $S S'$ **and**

cdcl_W-M-level-inv S **and**

distinct-cdcl_W-state S

shows *distinct-cdcl_W-state* S'

$\langle \text{proof} \rangle$

Conflicts and Annotations

This invariant shows that each mark contains a contradiction only related to the previously defined variable.

abbreviation *every-mark-is-a-conflict* :: $'st \Rightarrow \text{bool}$ **where**

every-mark-is-a-conflict $S \equiv$

$\forall L \text{ mark } a \ b. \ a \ @ \ \text{Propagated } L \text{ mark} \ \# \ b = (\text{trail } S)$

$\longrightarrow (b \models_{\text{as}} \text{CNot } (\text{mark} - \{\#L\}) \wedge L \in \# \text{ mark})$

definition *cdcl_W-conflicting* :: $'st \Rightarrow \text{bool}$ **where**

$cdcl_W\text{-conflicting } S \longleftrightarrow$
 $(\forall T. \text{ conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot \ T) \wedge \text{every-mark-is-a-conflict } S$

lemma *backtrack-atms-of-D-in-M1*:

fixes $S \ T :: 'st$ **and** $D \ D' :: \langle 'v \text{ clause} \rangle$ **and** $K \ L :: \langle 'v \text{ literal} \rangle$ **and** $i :: nat$ **and**

$M1 \ M2 :: \langle ('v, 'v \text{ clause}) \text{ ann-lits} \rangle$

assumes

inv : $no\text{-dup } (trail \ S)$ **and**

i : $get\text{-maximum-level } (trail \ S) \ D' \equiv i$ **and**

$decomp$: $(Decided \ K \ \# \ M1, \ M2)$

$\in \text{set } (get\text{-all-ann-decomposition } (trail \ S))$ **and**

$S\text{-lvl}$: $backtrack\text{-lvl } S = get\text{-maximum-level } (trail \ S) \ (add\text{-mset } L \ D')$ **and**

$S\text{-confl}$: $conflicting \ S = \text{Some } D$ **and**

$lev\text{-}K$: $get\text{-level } (trail \ S) \ K = Suc \ i$ **and**

T : $T \sim cons\text{-trail } K''$

$(reduce\text{-trail-to } M1$

$(add\text{-learned-cls } (add\text{-mset } L \ D')$

$(update\text{-conflicting } None \ S)))$ **and**

$confl$: $\forall T. \text{ conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot \ T$ **and**

$D\text{-}D'$: $\langle D' \subseteq \# \ D \rangle$

shows $atms\text{-of } D' \subseteq atm\text{-of } \langle lits\text{-of-l } (tl \ (trail \ T)) \rangle$

$\langle proof \rangle$

lemma *distinct-atms-of-incl-not-in-other*:

assumes

$a1$: $no\text{-dup } (M \ @ \ M')$ **and**

$a2$: $atms\text{-of } D \subseteq atm\text{-of } \langle lits\text{-of-l } M' \rangle$ **and**

$a3$: $x \in atms\text{-of } D$

shows $x \notin atm\text{-of } \langle lits\text{-of-l } M \rangle$

$\langle proof \rangle$

lemma *backtrack-M1-CNot-D'*:

fixes $S \ T :: 'st$ **and** $D \ D' :: \langle 'v \text{ clause} \rangle$ **and** $K \ L :: \langle 'v \text{ literal} \rangle$ **and** $i :: nat$ **and**

$M1 \ M2 :: \langle ('v, 'v \text{ clause}) \text{ ann-lits} \rangle$

assumes

inv : $no\text{-dup } (trail \ S)$ **and**

i : $get\text{-maximum-level } (trail \ S) \ D' \equiv i$ **and**

$decomp$: $(Decided \ K \ \# \ M1, \ M2)$

$\in \text{set } (get\text{-all-ann-decomposition } (trail \ S))$ **and**

$S\text{-lvl}$: $backtrack\text{-lvl } S = get\text{-maximum-level } (trail \ S) \ (add\text{-mset } L \ D')$ **and**

$S\text{-confl}$: $conflicting \ S = \text{Some } D$ **and**

$lev\text{-}K$: $get\text{-level } (trail \ S) \ K = Suc \ i$ **and**

T : $T \sim cons\text{-trail } K''$

$(reduce\text{-trail-to } M1$

$(add\text{-learned-cls } (add\text{-mset } L \ D')$

$(update\text{-conflicting } None \ S)))$ **and**

$confl$: $\forall T. \text{ conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot \ T$ **and**

$D\text{-}D'$: $\langle D' \subseteq \# \ D \rangle$

shows $M1 \models_{as} CNot \ D'$ **and**

$\langle atm\text{-of } (lit\text{-of } K'') = atm\text{-of } L \implies no\text{-dup } (trail \ T) \rangle$

$\langle proof \rangle$

Item 5 page 95 of Weidenbach's book

lemma *cdcl_W-restart-propagate-is-conclusion*:

assumes

$cdcl_W\text{-restart } S \ S'$ **and**

inv: $cdcl_W$ -M-level-inv S **and**
decomp: all-decomposition-implies-m (clauses S) (get-all-ann-decomposition (trail S)) **and**
learned: $cdcl_W$ -learned-clause S **and**
confl: $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot \ T$ **and**
alien: no-strange-atm S
shows all-decomposition-implies-m (clauses S') (get-all-ann-decomposition (trail S'))
 ⟨proof⟩

lemma $cdcl_W$ -restart-propagate-is-false:

assumes
cdcl_W-restart $S \ S'$ **and**
lev: $cdcl_W$ -M-level-inv S **and**
learned: $cdcl_W$ -learned-clause S **and**
decomp: all-decomposition-implies-m (clauses S) (get-all-ann-decomposition (trail S)) **and**
confl: $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot \ T$ **and**
alien: no-strange-atm S **and**
mark-confl: every-mark-is-a-conflict S
shows every-mark-is-a-conflict S'
 ⟨proof⟩

lemma $cdcl_W$ -conflicting-is-false:

assumes
cdcl_W-restart $S \ S'$ **and**
M-lev: $cdcl_W$ -M-level-inv S **and**
confl-inv: $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot \ T$ **and**
decided-confl: $\forall L \text{ mark } a \ b. a @ \text{Propagated } L \text{ mark } \# \ b = (\text{trail } S) \longrightarrow (b \models_{as} CNot (\text{mark} - \{\#L\}) \wedge L \in \# \text{ mark})$ **and**
dist: distinct- $cdcl_W$ -state S
shows $\forall T. \text{conflicting } S' = \text{Some } T \longrightarrow \text{trail } S' \models_{as} CNot \ T$
 ⟨proof⟩

lemma $cdcl_W$ -conflicting-decomp:

assumes $cdcl_W$ -conflicting S
shows
 $\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{trail } S \models_{as} CNot \ T$ **and**
 $\forall L \text{ mark } a \ b. a @ \text{Propagated } L \text{ mark } \# \ b = (\text{trail } S) \longrightarrow (b \models_{as} CNot (\text{mark} - \{\#L\}) \wedge L \in \# \text{ mark})$
 ⟨proof⟩

lemma $cdcl_W$ -conflicting-decomp2:

assumes $cdcl_W$ -conflicting S **and** conflicting $S = \text{Some } T$
shows trail $S \models_{as} CNot \ T$
 ⟨proof⟩

lemma $cdcl_W$ -conflicting-S0- $cdcl_W$ -restart[simp]:

$cdcl_W$ -conflicting (init-state N)
 ⟨proof⟩

definition $cdcl_W$ -learned-clauses-entailed-by-init **where**

$\langle cdcl_W$ -learned-clauses-entailed-by-init $S \longleftrightarrow \text{init-clss } S \models_{psm} \text{learned-clss } S \rangle$

lemma $cdcl_W$ -learned-clauses-entailed-init[simp]:

$\langle cdcl_W$ -learned-clauses-entailed-by-init (init-state N)
 ⟨proof⟩

lemma $cdcl_W$ -learned-clauses-entailed:

assumes
cdcl_W-restart: *cdcl_W-restart S S'* **and**
 2: *cdcl_W-learned-clause S* **and**
 9: *cdcl_W-learned-clauses-entailed-by-init S*
shows *cdcl_W-learned-clauses-entailed-by-init S'*
⟨proof⟩

lemma *rtrancp-cdcl_W-learned-clauses-entailed*:

assumes
cdcl_W-restart: *cdcl_W-restart** S S'* **and**
 2: *cdcl_W-learned-clause S* **and**
 4: *cdcl_W-M-level-inv S* **and**
 9: *cdcl_W-learned-clauses-entailed-by-init S*
shows *cdcl_W-learned-clauses-entailed-by-init S'*
⟨proof⟩

Putting all the Invariants Together

lemma *cdcl_W-restart-all-inv*:

assumes
cdcl_W-restart: *cdcl_W-restart S S'* **and**
 1: *all-decomposition-implies-m (clauses S) (get-all-ann-decomposition (trail S))* **and**
 2: *cdcl_W-learned-clause S* **and**
 4: *cdcl_W-M-level-inv S* **and**
 5: *no-strange-atm S* **and**
 7: *distinct-cdcl_W-state S* **and**
 8: *cdcl_W-conflicting S*
shows
all-decomposition-implies-m (clauses S') (get-all-ann-decomposition (trail S')) **and**
cdcl_W-learned-clause S' **and**
cdcl_W-M-level-inv S' **and**
no-strange-atm S' **and**
distinct-cdcl_W-state S' **and**
cdcl_W-conflicting S'
⟨proof⟩

lemma *rtrancp-cdcl_W-restart-all-inv*:

assumes
cdcl_W-restart: *rtrancp cdcl_W-restart S S'* **and**
 1: *all-decomposition-implies-m (clauses S) (get-all-ann-decomposition (trail S))* **and**
 2: *cdcl_W-learned-clause S* **and**
 4: *cdcl_W-M-level-inv S* **and**
 5: *no-strange-atm S* **and**
 7: *distinct-cdcl_W-state S* **and**
 8: *cdcl_W-conflicting S*
shows
all-decomposition-implies-m (clauses S') (get-all-ann-decomposition (trail S')) **and**
cdcl_W-learned-clause S' **and**
cdcl_W-M-level-inv S' **and**
no-strange-atm S' **and**
distinct-cdcl_W-state S' **and**
cdcl_W-conflicting S'
⟨proof⟩

lemma *all-invariant-S0-cdcl_W-restart*:

assumes *distinct-mset-mset N*

$$\begin{aligned}
& \text{all-decomposition-implies-m } (\text{init-clss } (\text{init-state } N)) \\
& \quad (\text{get-all-ann-decomposition } (\text{trail } (\text{init-state } N))) \text{ and} \\
& \text{cdcl}_W\text{-learned-clause } (\text{init-state } N) \text{ and} \\
& \forall T. \text{ conflicting } (\text{init-state } N) = \text{Some } T \longrightarrow (\text{trail } (\text{init-state } N)) \models_{\text{as}} \text{CNot } T \text{ and} \\
& \text{no-strange-atm } (\text{init-state } N) \text{ and} \\
& \text{consistent-interp } (\text{lits-of-l } (\text{trail } (\text{init-state } N))) \text{ and} \\
& \forall L \text{ mark } a \text{ b. } a @ \text{Propagated } L \text{ mark } \# \text{ b} = \text{trail } (\text{init-state } N) \longrightarrow \\
& \quad (b \models_{\text{as}} \text{CNot } (\text{mark} - \{\#L\}) \wedge L \in \# \text{ mark}) \text{ and} \\
& \text{distinct-cdcl}_W\text{-state } (\text{init-state } N) \\
& \langle \text{proof} \rangle
\end{aligned}$$

lemma *cdcl_W-only-propagated-vars-unsat*:

decided: $\forall x \in \text{set } M. \neg \text{is-decided } x$ **and**
DN: $D \in \# \text{ clauses } S$ **and**
D: $M \models_{as} CNot \ D$ **and**
inv: *all-decomposition-implies-m* $(N + U)$ (*get-all-ann-decomposition* M) **and**
state: *state* $S = (M, N, U, k, C)$ **and**
learned-cl: *cdcl_W-learned-clause* S **and**
atm-incl: *no-strange-atm* S
hows *unsatisfiable* (*set-mset* $(N + U)$)

We have actually a much stronger theorem, namely *all-decomposition-implies-propagated-lits-are-implied*, that show that the only choices we made are decided in the formula

assumes *all-decomposition-implies-m N (get-all-ann-decomposition M)*
and $\forall m \in \text{set } M. \neg \text{is-decided } m$
shows $\text{set-mset } N \models_{ps} \text{unmark-l } M$

lemma *conflict-with-false-implies-unsat*:

$cdcl_W$ -restart: $cdcl_W$ -restart S S' and
 lev: $cdcl_W$ -M-level-inv S and
 [simp]: conflicting $S' = \text{Some } \{\#\}$ and
 learned: $cdcl_W$ -learned-clause S and
 learned-entailed: $\langle cdcl_W$ -learned-clauses-entailed-by-init $S \rangle$
hows unsatisfiable (set-mset (clauses S))
 proof)

lemma *conflict-with-false-implies-terminated*:

assumes *cdcl_W-restart* S S' **and** *conflicting* $S = \text{Some } \{\#\}$
shows *False*
 $\langle \text{proof} \rangle$

This is a simple consequence of all we have shown previously. It is not strictly necessary, but helps finding a better bound on the number of learned clauses.

lemma *learned-clss-are-not-tautologies*:

assumes

cdcl_W-restart S S' **and**

lev: *cdcl_W-M-level-inv* S **and**

conflicting: *cdcl_W-conflicting* S **and**

no-tauto: $\forall s \in \#$ *learned-clss* S . \neg *tautology* s

shows $\forall s \in \#$ *learned-clss* S' . \neg *tautology* s

\langle *proof* \rangle

definition *final-cdcl_W-restart-state* ($S :: 'st$)

\longleftrightarrow (*trail* $S \models_{asm}$ *init-clss* S

$\vee ((\forall L \in \text{set } (\text{trail } S). \neg \text{is-decided } L) \wedge$

$(\exists C \in \#$ *init-clss* S . *trail* $S \models_{as}$ *CNot* $C)))$

definition *termination-cdcl_W-restart-state* ($S :: 'st$)

\longleftrightarrow (*trail* $S \models_{asm}$ *init-clss* S

$\vee ((\forall L \in \text{atms-of-mm } (\text{init-clss } S). L \in \text{atm-of ' lits-of-l } (\text{trail } S))$

$\wedge (\exists C \in \#$ *init-clss* S . *trail* $S \models_{as}$ *CNot* $C)))$

1.1.4 CDCL Strong Completeness

lemma *cdcl_W-restart-can-do-step*:

assumes

consistent-interp (*set* M) **and**

distinct M **and**

atm-of ' (set $M) \subseteq \text{atms-of-mm } N$

shows $\exists S$. *rtranclp cdcl_W-restart* (*init-state* N) S

$\wedge \text{state-butlast } S = (\text{map } (\lambda L. \text{Decided } L) M, N, \{\#\}, \text{None})$

\langle *proof* \rangle

theorem 2.9.11 page 98 of Weidenbach's book

lemma *cdcl_W-restart-strong-completeness*:

assumes

MN: *set* $M \models_{sm} N$ **and**

cons: *consistent-interp* (*set* M) **and**

dist: *distinct* M **and**

atm: *atm-of ' (set* $M) \subseteq \text{atms-of-mm } N$

obtains S **where**

state-butlast $S = (\text{map } (\lambda L. \text{Decided } L) M, N, \{\#\}, \text{None})$ **and**

rtranclp cdcl_W-restart (*init-state* N) S **and**

final-cdcl_W-restart-state S

\langle *proof* \rangle

1.1.5 Higher level strategy

The rules described previously do not necessary lead to a conclusive state. We have to add a strategy:

- do propagate and conflict when possible;
- otherwise, do another rule (except forget and restart).

Definition

lemma *tranclp-conflict*:

tranclp conflict S S' \implies conflict S S'
 $\langle \text{proof} \rangle$

lemma *no-chained-conflict*:

assumes *conflict S S' and conflict S' S''*
shows *False*
 $\langle \text{proof} \rangle$

lemma *tranclp-conflict-iff*:

full1 conflict S S' \iff conflict S S'
 $\langle \text{proof} \rangle$

lemma *no-conflict-after-conflict*:

conflict S T \implies \neg conflict T U
 $\langle \text{proof} \rangle$

lemma *no-propagate-after-conflict*:

conflict S T \implies \neg propagate T U
 $\langle \text{proof} \rangle$

inductive *cdcl_W-stgy* :: 'st \Rightarrow 'st \Rightarrow bool **for** S :: 'st **where**

conflict': *conflict S S' \implies cdcl_W-stgy S S' |*

propagate': *propagate S S' \implies cdcl_W-stgy S S' |*

other': *no-step conflict S \implies no-step propagate S \implies cdcl_W-o S S' \implies cdcl_W-stgy S S'*

lemma *cdcl_W-stgy-cdcl_W*: *cdcl_W-stgy S T \implies cdcl_W S T*

$\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-induct*[consumes 1, case-names *conflict propagate decide skip resolve backtrack*]:

assumes

$\langle \text{cdcl}_W\text{-stgy } S \ T \rangle$ **and**

$\langle \bigwedge T. \text{conflict } S \ T \implies P \ T \rangle$ **and**

$\langle \bigwedge T. \text{propagate } S \ T \implies P \ T \rangle$ **and**

$\langle \bigwedge T. \text{no-step conflict } S \implies \text{no-step propagate } S \implies \text{decide } S \ T \implies P \ T \rangle$ **and**

$\langle \bigwedge T. \text{no-step conflict } S \implies \text{no-step propagate } S \implies \text{skip } S \ T \implies P \ T \rangle$ **and**

$\langle \bigwedge T. \text{no-step conflict } S \implies \text{no-step propagate } S \implies \text{resolve } S \ T \implies P \ T \rangle$ **and**

$\langle \bigwedge T. \text{no-step conflict } S \implies \text{no-step propagate } S \implies \text{backtrack } S \ T \implies P \ T \rangle$

shows

$\langle P \ T \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-cases*[consumes 1, case-names *conflict propagate decide skip resolve backtrack*]:

assumes

$\langle \text{cdcl}_W\text{-stgy } S \ T \rangle$ **and**

$\langle \text{conflict } S \ T \implies P \rangle$ **and**

$\langle \text{propagate } S \ T \implies P \rangle$ **and**

$\langle \text{no-step conflict } S \implies \text{no-step propagate } S \implies \text{decide } S \ T \implies P \rangle$ **and**

$\langle \text{no-step conflict } S \implies \text{no-step propagate } S \implies \text{skip } S \ T \implies P \rangle$ **and**

$\langle \text{no-step conflict } S \implies \text{no-step propagate } S \implies \text{resolve } S \ T \implies P \rangle$ **and**

$\langle \text{no-step conflict } S \implies \text{no-step propagate } S \implies \text{backtrack } S \ T \implies P \rangle$

shows

$\langle P \rangle$

$\langle \text{proof} \rangle$

Invariants

lemma *cdcl_W-stgy-consistent-inv*:
assumes *cdcl_W-stgy* *S S'* **and** *cdcl_W-M-level-inv* *S*
shows *cdcl_W-M-level-inv* *S'*
 $\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_W-stgy-consistent-inv*:
assumes *cdcl_W-stgy*** *S S'* **and** *cdcl_W-M-level-inv* *S*
shows *cdcl_W-M-level-inv* *S'*
 $\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-no-more-init-clss*:
assumes *cdcl_W-stgy* *S S'*
shows *init-clss* *S* = *init-clss* *S'*
 $\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_W-stgy-no-more-init-clss*:
assumes *cdcl_W-stgy*** *S S'*
shows *init-clss* *S* = *init-clss* *S'*
 $\langle \text{proof} \rangle$

Literal of highest level in conflicting clauses

One important property of the *cdcl_W-restart* with strategy is that, whenever a conflict takes place, there is at least a literal of level *k* involved (except if we have derived the false clause). The reason is that we apply conflicts before a decision is taken.

definition *conflict-is-false-with-level* :: 'st \Rightarrow bool **where**
conflict-is-false-with-level *S* $\equiv \forall D. \text{conflicting } S = \text{Some } D \longrightarrow D \neq \{\#\}$
 $\longrightarrow (\exists L \in \# D. \text{get-level } (\text{trail } S) L = \text{backtrack-lvl } S)$

declare *conflict-is-false-with-level-def*[simp]

Literal of highest level in decided literals

definition *mark-is-false-with-level* :: 'st \Rightarrow bool **where**
mark-is-false-with-level *S'* \equiv
 $\forall D M1 M2 L. M1 @ \text{Propagated } L D \# M2 = \text{trail } S' \longrightarrow D - \{\#L\} \neq \{\#\}$
 $\longrightarrow (\exists L. L \in \# D \wedge \text{get-level } (\text{trail } S') L = \text{count-decided } M1)$

lemma *backtrack_W-rule*:
assumes
conf: $\langle \text{conflicting } S = \text{Some } (\text{add-mset } L D) \rangle$ **and**
decomp: $\langle (\text{Decided } K \# M1, M2) \in \text{set } (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$ **and**
lev-L: $\langle \text{get-level } (\text{trail } S) L = \text{backtrack-lvl } S \rangle$ **and**
max-lev: $\langle \text{get-level } (\text{trail } S) L = \text{get-maximum-level } (\text{trail } S) (\text{add-mset } L D) \rangle$ **and**
max-D: $\langle \text{get-maximum-level } (\text{trail } S) D \equiv i \rangle$ **and**
lev-K: $\langle \text{get-level } (\text{trail } S) K = i + 1 \rangle$ **and**
T: $\langle T \sim \text{cons-trail } (\text{Propagated } L (\text{add-mset } L D))$
 $(\text{reduce-trail-to } M1$
 $(\text{add-learned-cls } (\text{add-mset } L D)$
 $(\text{update-conflicting } \text{None } S))) \rangle$ **and**
lev-inv: *cdcl_W-M-level-inv* *S* **and**
conf: $\langle \text{cdcl}_W\text{-conflicting } S \rangle$ **and**
learned: $\langle \text{cdcl}_W\text{-learned-clause } S \rangle$
shows $\langle \text{backtrack } S T \rangle$

$\langle \text{proof} \rangle$

lemma *backtrack-no-decomp:*

assumes

S : *conflicting* $S = \text{Some } (\text{add-mset } L \ E)$ **and**

L : *get-level* $(\text{trail } S) \ L = \text{backtrack-lvl } S$ **and**

D : *get-maximum-level* $(\text{trail } S) \ E < \text{backtrack-lvl } S$ **and**

bt : *backtrack-lvl* $S = \text{get-maximum-level } (\text{trail } S) (\text{add-mset } L \ E)$ **and**

$lev\text{-}inv$: *cdcl_W-M-level-inv* S **and**

$conf$: *cdcl_W-conflicting* S **and**

$learned$: *cdcl_W-learned-clause* S

shows $\exists S'. \text{cdcl}_W\text{-o } S \ S' \exists S'. \text{backtrack } S \ S'$

$\langle \text{proof} \rangle$

lemma *no-analyse-backtrack-Ex-simple-backtrack:*

assumes

bt : *backtrack* $S \ T$ **and**

$lev\text{-}inv$: *cdcl_W-M-level-inv* S **and**

$conf$: *cdcl_W-conflicting* S **and**

$learned$: *cdcl_W-learned-clause* S **and**

$no\text{-}dup$: *distinct-cdcl_W-state* S **and**

$ns\text{-}s$: *no-step skip* S **and**

$ns\text{-}r$: *no-step resolve* S

shows $\langle \text{Ex}(\text{simple-backtrack } S) \rangle$

$\langle \text{proof} \rangle$

lemma *trail-begins-with-decided-conflicting-exists-backtrack:*

assumes

$conf\text{-}k$: *conflict-is-false-with-level* S **and**

$conf$: *cdcl_W-conflicting* S **and**

$level\text{-}inv$: *cdcl_W-M-level-inv* S **and**

$no\text{-}dup$: *distinct-cdcl_W-state* S **and**

$learned$: *cdcl_W-learned-clause* S **and**

$alien$: *no-strange-atm* S **and**

$tr\text{-}ne$: *trail* $S \neq []$ **and**

L' : *hd-trail* $S = \text{Decided } L'$ **and**

$nempty$: $C \neq \{\#\}$ **and**

$conf$: *conflicting* $S = \text{Some } C$

shows $\langle \text{Ex } (\text{backtrack } S) \rangle$ **and** $\langle \text{no-step skip } S \rangle$ **and** $\langle \text{no-step resolve } S \rangle$

$\langle \text{proof} \rangle$

lemma *conflicting-no-false-can-do-step:*

assumes

$conf$: *conflicting* $S = \text{Some } C$ **and**

$nempty$: $C \neq \{\#\}$ **and**

$conf\text{-}k$: *conflict-is-false-with-level* S **and**

$conf$: *cdcl_W-conflicting* S **and**

$level\text{-}inv$: *cdcl_W-M-level-inv* S **and**

$no\text{-}dup$: *distinct-cdcl_W-state* S **and**

$learned$: *cdcl_W-learned-clause* S **and**

$alien$: *no-strange-atm* S **and**

$termi$: *no-step cdcl_W-stgy* S

shows *False*

$\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-final-state-conclusive2:*

assumes

termi: *no-step cdcl_W-stgy S* **and**
decomp: *all-decomposition-implies-m (clauses S) (get-all-ann-decomposition (trail S))* **and**
learned: *cdcl_W-learned-clause S* **and**
level-inv: *cdcl_W-M-level-inv S* **and**
alien: *no-strange-atm S* **and**
no-dup: *distinct-cdcl_W-state S* **and**
conft: *cdcl_W-conflicting S* **and**
conft-k: *conflict-is-false-with-level S*

shows (*conflicting S = Some {#} \wedge unsatisfiable (set-mset (clauses S))*)
 \vee (*conflicting S = None \wedge trail S \models as set-mset (clauses S)*)

<proof>

lemma *cdcl_W-stgy-final-state-conclusive*:

assumes

termi: *no-step cdcl_W-stgy S* **and**
decomp: *all-decomposition-implies-m (clauses S) (get-all-ann-decomposition (trail S))* **and**
learned: *cdcl_W-learned-clause S* **and**
level-inv: *cdcl_W-M-level-inv S* **and**
alien: *no-strange-atm S* **and**
no-dup: *distinct-cdcl_W-state S* **and**
conft: *cdcl_W-conflicting S* **and**
conft-k: *conflict-is-false-with-level S* **and**
learned-entailed: *(cdcl_W-learned-clauses-entailed-by-init S)*

shows (*conflicting S = Some {#} \wedge unsatisfiable (set-mset (init-clss S))*)
 \vee (*conflicting S = None \wedge trail S \models as set-mset (init-clss S)*)

<proof>

lemma *cdcl_W-stgy-tranclp-cdcl_W-restart*:

cdcl_W-stgy S S' \implies cdcl_W-restart⁺⁺ S S'
<proof>

lemma *tranclp-cdcl_W-stgy-tranclp-cdcl_W-restart*:

cdcl_W-stgy⁺⁺ S S' \implies cdcl_W-restart⁺⁺ S S'
<proof>

lemma *rtranclp-cdcl_W-stgy-rtranclp-cdcl_W-restart*:

*cdcl_W-stgy^{**} S S' \implies cdcl_W-restart^{**} S S'*
<proof>

lemma *cdcl_W-o-conflict-is-false-with-level-inv*:

assumes

cdcl_W-o S S' **and**
lev: *cdcl_W-M-level-inv S* **and**
conft-inv: *conflict-is-false-with-level S* **and**
n-d: *distinct-cdcl_W-state S* **and**
conflicting: *cdcl_W-conflicting S*

shows *conflict-is-false-with-level S'*

<proof>

Strong completeness

lemma *propagate-high-levelE*:

assumes *propagate S T*

obtains *M' N' U L C* **where**

state-butlast $S = (M', N', U, \text{None})$ **and**
state-butlast $T = (\text{Propagated } L \ (C + \{\#L\}) \ \# \ M', N', U, \text{None})$ **and**
 $C + \{\#L\} \in \# \text{ local.clauses } S$ **and**
 $M' \models_{as} C \text{Not } C$ **and**
undefined-lit (*trail* S) L
 <proof>

lemma *cdcl_W-propagate-conflict-completeness:*

assumes

MN: $\text{set } M \models_s \text{set-mset } N$ **and**
cons: *consistent-interp* (*set* M) **and**
tot: *total-over-m* (*set* M) (*set-mset* N) **and**
lits-of-l (*trail* S) $\subseteq \text{set } M$ **and**
init-clss $S = N$ **and**
*propagate*** $S \ S'$ **and**
learned-clss $S = \{\#\}$

shows $\text{length } (\text{trail } S) \leq \text{length } (\text{trail } S') \wedge \text{lits-of-l } (\text{trail } S') \subseteq \text{set } M$

<proof>

lemma

assumes *propagate*** $S \ X$

shows

rtrancpl-propagate-init-clss: *init-clss* $X = \text{init-clss } S$ **and**
rtrancpl-propagate-learned-clss: *learned-clss* $X = \text{learned-clss } S$

<proof>

lemma *cdcl_W-stgy-strong-completeness-n:*

assumes

MN: $\text{set } M \models_s \text{set-mset } N$ **and**
cons: *consistent-interp* (*set* M) **and**
tot: *total-over-m* (*set* M) (*set-mset* N) **and**
atm-incl: *atm-of* ' (*set* M) $\subseteq \text{atms-of-mm } N$ **and**
distM: *distinct* M **and**
length: $n \leq \text{length } M$

shows

$\exists M' \ S. \text{length } M' \geq n \wedge$
 $\text{lits-of-l } M' \subseteq \text{set } M \wedge$
 $\text{no-dup } M' \wedge$
 $\text{state-butlast } S = (M', N, \{\#\}, \text{None}) \wedge$
 $\text{cdcl}_W\text{-stgy}^{**} (\text{init-state } N) \ S$

<proof>

lemma *cdcl_W-stgy-strong-completeness':*

assumes

MN: $\text{set } M \models_s \text{set-mset } N$ **and**
cons: *consistent-interp* (*set* M) **and**
tot: *total-over-m* (*set* M) (*set-mset* N) **and**
atm-incl: *atm-of* ' (*set* M) $\subseteq \text{atms-of-mm } N$ **and**
distM: *distinct* M

shows

$\exists M' \ S. \text{lits-of-l } M' = \text{set } M \wedge$
 $\text{state-butlast } S = (M', N, \{\#\}, \text{None}) \wedge$
 $\text{cdcl}_W\text{-stgy}^{**} (\text{init-state } N) \ S$

<proof>

theorem 2.9.11 page 98 of Weidenbach's book (with strategy)

lemma *cdcl_W-stgy-strong-completeness*:
assumes
MN: *set M* \models_s *set-mset N* **and**
cons: *consistent-interp* (*set M*) **and**
tot: *total-over-m* (*set M*) (*set-mset N*) **and**
atm-incl: *atm-of* ' (*set M*) \subseteq *atms-of-mm N* **and**
distM: *distinct M*
shows
 $\exists M' k S.$
lits-of-l M' = *set M* \wedge
state-butlast S = (*M'*, *N*, {#}, *None*) \wedge
*cdcl_W-stgy*** (*init-state N*) *S* \wedge
final-cdcl_W-restart-state S
 $\langle \text{proof} \rangle$

No conflict with only variables of level less than backtrack level

This invariant is stronger than the previous argument in the sense that it is a property about all possible conflicts.

definition *no-smaller-conf* (*S* :: 'st) \equiv
 $(\forall M K M' D. \text{trail } S = M' @ \text{Decided } K \# M \longrightarrow D \in \# \text{ clauses } S \longrightarrow \neg M \models_{as} CNot D)$

lemma *no-smaller-conf-init-sate[simp]*:
no-smaller-conf (*init-state N*) $\langle \text{proof} \rangle$

lemma *cdcl_W-o-no-smaller-conf-inv*:
fixes *S S'* :: 'st
assumes
cdcl_W-o S S' **and**
n-s: *no-step conflict S* **and**
lev: *cdcl_W-M-level-inv S* **and**
max-lev: *conflict-is-false-with-level S* **and**
smaller: *no-smaller-conf S*
shows *no-smaller-conf S'*
 $\langle \text{proof} \rangle$

lemma *conflict-no-smaller-conf-inv*:
assumes *conflict S S'*
and *no-smaller-conf S*
shows *no-smaller-conf S'*
 $\langle \text{proof} \rangle$

lemma *propagate-no-smaller-conf-inv*:
assumes *propagate: propagate S S'*
and *n-l*: *no-smaller-conf S*
shows *no-smaller-conf S'*
 $\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-no-smaller-conf*:
assumes *cdcl_W-stgy S S'*
and *n-l*: *no-smaller-conf S*
and *conflict-is-false-with-level S*
and *cdcl_W-M-level-inv S*
shows *no-smaller-conf S'*
 $\langle \text{proof} \rangle$

lemma *conflict-conflict-is-false-with-level:*

assumes

conflict: conflict S T and

smaller: no-smaller-confl S and

M-lev: cdcl_W-M-level-inv S

shows *conflict-is-false-with-level T*

<proof>

lemma *cdcl_W-stgy-ex-lit-of-max-level:*

assumes

cdcl_W-stgy S S' and

n-l: no-smaller-confl S and

conflict-is-false-with-level S and

cdcl_W-M-level-inv S and

distinct-cdcl_W-state S and

cdcl_W-conflicting S

shows *conflict-is-false-with-level S'*

<proof>

lemma *rtrancpl-cdcl_W-stgy-no-smaller-confl-inv:*

assumes

*cdcl_W-stgy** S S' and*

n-l: no-smaller-confl S and

cls-false: conflict-is-false-with-level S and

lev: cdcl_W-M-level-inv S and

dist: distinct-cdcl_W-state S and

conflicting: cdcl_W-conflicting S and

decomp: all-decomposition-implies-m (clauses S) (get-all-ann-decomposition (trail S)) and

learned: cdcl_W-learned-clause S and

alien: no-strange-atm S

shows *no-smaller-confl S' \wedge conflict-is-false-with-level S'*

<proof>

Final States are Conclusive

theorem 2.9.9 page 97 of Weidenbach's book

lemma *full-cdcl_W-stgy-final-state-conclusive:*

fixes *S' :: 'st*

assumes *full: full cdcl_W-stgy (init-state N) S'*

and *no-d: distinct-mset-mset N*

shows *(conflicting S' = Some {#} \wedge unsatisfiable (set-mset (init-clss S')))*

\vee (conflicting S' = None \wedge trail S' \models_{asm} init-clss S')

<proof>

lemma *cdcl_W-o-fst-empty-conflicting-false:*

assumes

cdcl_W-o S S' and

trail S = [] and

conflicting S \neq None

shows *False*

<proof>

lemma *cdcl_W-stgy-fst-empty-conflicting-false:*

assumes

$cdcl_W\text{-stgy } S \ S'$ **and**
 $trail \ S = []$ **and**
 $conflicting \ S \neq None$
shows $False$
 $\langle proof \rangle$

lemma $cdcl_W\text{-o-conflicting-is-false}$:
 $cdcl_W\text{-o } S \ S' \implies conflicting \ S = Some \ \{\#\} \implies False$
 $\langle proof \rangle$

lemma $cdcl_W\text{-stgy-conflicting-is-false}$:
 $cdcl_W\text{-stgy } S \ S' \implies conflicting \ S = Some \ \{\#\} \implies False$
 $\langle proof \rangle$

lemma $rtrancp\text{-}cdcl_W\text{-stgy-conflicting-is-false}$:
 $cdcl_W\text{-stgy}^{**} \ S \ S' \implies conflicting \ S = Some \ \{\#\} \implies S' = S$
 $\langle proof \rangle$

definition $conflict\text{-or-propagate} :: 'st \Rightarrow 'st \Rightarrow bool$ **where**
 $conflict\text{-or-propagate} \ S \ T \longleftrightarrow conflict \ S \ T \vee propagate \ S \ T$

declare $conflict\text{-or-propagate-def}[simp]$

lemma $conflict\text{-or-propagate-intros}$:
 $conflict \ S \ T \implies conflict\text{-or-propagate} \ S \ T$
 $propagate \ S \ T \implies conflict\text{-or-propagate} \ S \ T$
 $\langle proof \rangle$

theorem 2.9.9 page 97 of Weidenbach's book

lemma $full\text{-}cdcl_W\text{-stgy-final-state-conclusive-from-init-state}$:
fixes $S' :: 'st$
assumes $full$: $full \ cdcl_W\text{-stgy} \ (init\text{-state} \ N) \ S'$
and $no\text{-d}$: $distinct\text{-mset-mset} \ N$
shows $(conflicting \ S' = Some \ \{\#\} \wedge unsatisfiable \ (set\text{-mset} \ N))$
 $\vee (conflicting \ S' = None \wedge trail \ S' \models_{asm} N \wedge satisfiable \ (set\text{-mset} \ N))$
 $\langle proof \rangle$

1.1.6 Structural Invariant

The condition that no learned clause is a tautology is overkill for the termination (in the sense that the no-duplicate condition is enough), but it allows to reuse *simple-clss*.

The invariant contains all the structural invariants that holds,

definition $cdcl_W\text{-all-struct-inv}$ **where**
 $cdcl_W\text{-all-struct-inv} \ S \longleftrightarrow$
 $no\text{-strange-atm} \ S \wedge$
 $cdcl_W\text{-M-level-inv} \ S \wedge$
 $(\forall s \in \# \text{ learned-clss } S. \neg tautology \ s) \wedge$
 $distinct\text{-}cdcl_W\text{-state} \ S \wedge$
 $cdcl_W\text{-conflicting} \ S \wedge$
 $all\text{-decomposition-implies-m} \ (clauses \ S) \ (get\text{-all-ann-decomposition} \ (trail \ S)) \wedge$
 $cdcl_W\text{-learned-clause} \ S$

lemma $cdcl_W\text{-all-struct-inv-inv}$:
assumes $cdcl_W\text{-restart} \ S \ S'$ **and** $cdcl_W\text{-all-struct-inv} \ S$
shows $cdcl_W\text{-all-struct-inv} \ S'$

$\langle proof \rangle$

lemma *rtrancp-cdcl_W-all-struct-inv-inv*:
assumes *cdcl_W-restart** S S'* **and** *cdcl_W-all-struct-inv S*
shows *cdcl_W-all-struct-inv S'*
 $\langle proof \rangle$

lemma *cdcl_W-stgy-cdcl_W-all-struct-inv*:
cdcl_W-stgy S T \implies cdcl_W-all-struct-inv S \implies cdcl_W-all-struct-inv T
 $\langle proof \rangle$

lemma *rtrancp-cdcl_W-stgy-cdcl_W-all-struct-inv*:
*cdcl_W-stgy** S T \implies cdcl_W-all-struct-inv S \implies cdcl_W-all-struct-inv T*
 $\langle proof \rangle$

lemma *beginning-not-decided-invert*:
assumes *A: M @ A = M' @ Decided K # H* **and**
nm: $\forall m \in \text{set } M. \neg \text{is-decided } m$
shows $\exists M. A = M @ \text{Decided } K \# H$
 $\langle proof \rangle$

1.1.7 Strategy-Specific Invariant

definition *cdcl_W-stgy-invariant* **where**
cdcl_W-stgy-invariant S \longleftrightarrow
conflict-is-false-with-level S
 \wedge *no-smaller-confl S*

lemma *cdcl_W-stgy-cdcl_W-stgy-invariant*:
assumes
cdcl_W-restart: cdcl_W-stgy S T **and**
inv-s: cdcl_W-stgy-invariant S **and**
inv: cdcl_W-all-struct-inv S
shows
cdcl_W-stgy-invariant T
 $\langle proof \rangle$

lemma *rtrancp-cdcl_W-stgy-cdcl_W-stgy-invariant*:
assumes
*cdcl_W-restart: cdcl_W-stgy** S T* **and**
inv-s: cdcl_W-stgy-invariant S **and**
inv: cdcl_W-all-struct-inv S
shows
cdcl_W-stgy-invariant T
 $\langle proof \rangle$

lemma *full-cdcl_W-stgy-inv-normal-form*:
assumes
full: full cdcl_W-stgy S T **and**
inv-s: cdcl_W-stgy-invariant S **and**
inv: cdcl_W-all-struct-inv S **and**
learned-entailed: $\langle \text{cdcl}_W\text{-learned-clauses-entailed-by-init } S \rangle$
shows *conflicting T = Some {#} \wedge unsatisfiable (set-mset (init-cls S))*
 \vee *conflicting T = None \wedge trail T \models_{asm} init-cls S \wedge satisfiable (set-mset (init-cls S))*
 $\langle proof \rangle$

lemma *full-cdcl_W-stgy-inv-normal-form2*:

assumes

full: *full cdcl_W-stgy S T* **and**

inv-s: *cdcl_W-stgy-invariant S* **and**

inv: *cdcl_W-all-struct-inv S*

shows *conflicting T = Some {#} \wedge unsatisfiable (set-mset (clauses T))*

\vee *conflicting T = None \wedge satisfiable (set-mset (clauses T))*

<proof>

1.1.8 Additional Invariant: No Smaller Propagation

definition *no-smaller-propa* :: *<'st \Rightarrow bool>* **where**

no-smaller-propa (S :: 'st) \equiv

$(\forall M K M' D L. \text{trail } S = M' @ \text{Decided } K \# M \longrightarrow D + \{\#L\} \in \# \text{ clauses } S \longrightarrow \text{undefined-lit } M$
 $L \longrightarrow \neg M \models_{as} CNot D)$

lemma *propagated-cons-eq-append-decide-cons*:

Propagated L E # Ms = M' @ Decided K # M \longleftrightarrow

M' $\neq [] \wedge Ms = tl M' @ Decided K \# M \wedge hd M' = Propagated L E$

<proof>

lemma *in-get-all-mark-of-propagated-in-trail*:

<C \in set (get-all-mark-of-propagated M) $\longleftrightarrow (\exists L. Propagated L C \in \text{set } M)$ >

<proof>

lemma *no-smaller-propa-tl*:

assumes

<no-smaller-propa S> **and**

<trail S $\neq []$ > **and**

< \neg is-decided(hd-trail S)> **and**

<trail U = tl (trail S)> **and**

<clauses U = clauses S>

shows

<no-smaller-propa U>

<proof>

lemmas *rulesE* =

skipE resolveE backtrackE propagateE conflictE decideE restartE forgetE backtrackgE

lemma *decide-no-smaller-step*:

assumes *dec*: *<decide S T>* **and** *smaller-propa*: *<no-smaller-propa S>* **and**

n-s: *<no-step propagate S>*

shows *<no-smaller-propa T>*

<proof>

lemma *no-smaller-propa-reduce-trail-to*:

<no-smaller-propa S \implies no-smaller-propa (reduce-trail-to M1 S)>

<proof>

lemma *backtrackg-no-smaller-propa*:

assumes *o*: *<backtrackg S T>* **and** *smaller-propa*: *<no-smaller-propa S>* **and**

n-d: *<no-dup (trail S)>* **and**

n-s: *<no-step propagate S>* **and**

tr-CNot: *<trail S $\models_{as} CNot$ (the (conflicting S))>*

shows $\langle \text{no-smaller-propa } T \rangle$
 $\langle \text{proof} \rangle$

lemmas $\text{backtrack-no-smaller-propa} = \text{backtrackg-no-smaller-propa}[OF \text{ backtrack-backtrackg}]$

lemma $\text{cdcl}_W\text{-stgy-no-smaller-propa}$:
assumes
 $\text{cdcl}: \langle \text{cdcl}_W\text{-stgy } S \ T \rangle$ **and**
 $\text{smaller-propa}: \langle \text{no-smaller-propa } S \rangle$ **and**
 $\text{inv}: \langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$
shows $\langle \text{no-smaller-propa } T \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_W\text{-stgy-no-smaller-propa}$:
assumes
 $\text{cdcl}: \langle \text{cdcl}_W\text{-stgy}^{**} S \ T \rangle$ **and**
 $\text{smaller-propa}: \langle \text{no-smaller-propa } S \rangle$ **and**
 $\text{inv}: \langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$
shows $\langle \text{no-smaller-propa } T \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{hd-trail-level-ge-1-length-gt-1}$:
fixes $S :: 'st$
defines $M[\text{symmetric}, \text{simp}]: \langle M \equiv \text{trail } S \rangle$
defines $L[\text{symmetric}, \text{simp}]: \langle L \equiv \text{hd } M \rangle$
assumes
 $\text{smaller}: \langle \text{no-smaller-propa } S \rangle$ **and**
 $\text{struct}: \langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**
 $\text{dec}: \langle \text{count-decided } M \geq 1 \rangle$ **and**
 $\text{proped}: \langle \text{is-proped } L \rangle$
shows $\langle \text{size } (\text{mark-of } L) > 1 \rangle$
 $\langle \text{proof} \rangle$

1.1.9 More Invariants: Conflict is False if no decision

If the level is higher than 0, then the conflict is not empty.

definition $\text{conflict-non-zero-unless-level-0} :: \langle 'st \Rightarrow \text{bool} \rangle$ **where**
 $\langle \text{conflict-non-zero-unless-level-0 } S \longleftrightarrow$
 $(\text{conflicting } S = \text{Some } \{\#\} \longrightarrow \text{count-decided } (\text{trail } S) = 0) \rangle$

definition $\text{no-false-clause} :: \langle 'st \Rightarrow \text{bool} \rangle$ **where**
 $\langle \text{no-false-clause } S \longleftrightarrow (\forall C \in \# \text{ clauses } S. C \neq \{\#\}) \rangle$

lemma $\text{cdcl}_W\text{-restart-no-false-clause}$:
assumes
 $\langle \text{cdcl}_W\text{-restart } S \ T \rangle$
 $\langle \text{no-false-clause } S \rangle$
shows $\langle \text{no-false-clause } T \rangle$
 $\langle \text{proof} \rangle$

The proofs work smoothly thanks to the side-conditions about levels of the rule *resolve*.

lemma $\text{cdcl}_W\text{-restart-conflict-non-zero-unless-level-0}$:
assumes
 $\langle \text{cdcl}_W\text{-restart } S \ T \rangle$

$\langle \text{no-false-clause } S \rangle$ **and**
 $\langle \text{conflict-non-zero-unless-level-0 } S \rangle$
shows $\langle \text{conflict-non-zero-unless-level-0 } T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_W-restart-no-false-clause:*

assumes
 $\langle \text{cdcl}_W\text{-restart}^{**} S T \rangle$
 $\langle \text{no-false-clause } S \rangle$
shows $\langle \text{no-false-clause } T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_W-restart-conflict-non-zero-unless-level-0:*

assumes
 $\langle \text{cdcl}_W\text{-restart}^{**} S T \rangle$
 $\langle \text{no-false-clause } S \rangle$ **and**
 $\langle \text{conflict-non-zero-unless-level-0 } S \rangle$
shows $\langle \text{conflict-non-zero-unless-level-0 } T \rangle$
 $\langle \text{proof} \rangle$

definition *propagated-clauses-clauses* :: 'st \Rightarrow bool **where**

$\langle \text{propagated-clauses-clauses } S \equiv \forall L K. \text{Propagated } L K \in \text{set } (\text{trail } S) \longrightarrow K \in \# \text{ clauses } S \rangle$

lemma *propagate-single-literal-clause-get-level-is-0:*

assumes
smaller: $\langle \text{no-smaller-propa } S \rangle$ **and**
propa-tr: $\langle \text{Propagated } L \{ \#L\# \} \in \text{set } (\text{trail } S) \rangle$ **and**
n-d: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**
propa: $\langle \text{propagated-clauses-clauses } S \rangle$
shows $\langle \text{get-level } (\text{trail } S) L = 0 \rangle$
 $\langle \text{proof} \rangle$

Conflict Minimisation

Remove Literals of Level 0 **lemma** *conflict-minimisation-level-0:*

fixes $S :: \text{'st}$
defines $D[\text{simp}]$: $\langle D \equiv \text{the } (\text{conflicting } S) \rangle$
defines $[\text{simp}]$: $\langle M \equiv \text{trail } S \rangle$
defines $\langle D' \equiv \text{filter-mset } (\lambda L. \text{get-level } M L > 0) D \rangle$
assumes
ns-s: $\langle \text{no-step skip } S \rangle$ **and**
ns-r: $\langle \text{no-step resolve } S \rangle$ **and**
inv-s: $\text{cdcl}_W\text{-stgy-invariant } S$ **and**
inv: $\text{cdcl}_W\text{-all-struct-inv } S$ **and**
conf: $\langle \text{conflicting } S \neq \text{None} \rangle \langle \text{conflicting } S \neq \text{Some } \{ \# \} \rangle$ **and**
M-nempty: $\langle M \sim = [] \rangle$
shows
 $\text{clauses } S \models_{\text{pm}} D'$ **and**
 $\langle \text{lit-of } (\text{hd } M) \in \# D' \rangle$
 $\langle \text{proof} \rangle$

lemma *literals-of-level0-entailed:*

assumes
struct-invs: $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**
in-trail: $\langle L \in \text{lits-of-l } (\text{trail } S) \rangle$ **and**
lev: $\langle \text{get-level } (\text{trail } S) L = 0 \rangle$

shows
 $\langle \text{clauses } S \models_{pm} \{\#L\# \} \rangle$
 $\langle \text{proof} \rangle$

1.1.10 Some higher level use on the invariants

In later refinement we mostly use the group invariants and don't try to be as specific as above. The corresponding theorems are collected here.

lemma *conflict-conflict-is-false-with-level-all-inv:*

$\langle \text{conflict } S \ T \implies$
 $\text{no-smaller-confl } S \implies$
 $\text{cdcl}_W\text{-all-struct-inv } S \implies$
 $\text{conflict-is-false-with-level } T \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-ex-lit-of-max-level-all-inv:*

assumes
 $\text{cdcl}_W\text{-stgy } S \ S'$ **and**
 $n\text{-l: no-smaller-confl } S$ **and**
 $\text{conflict-is-false-with-level } S$ **and**
 $\text{cdcl}_W\text{-all-struct-inv } S$
shows $\text{conflict-is-false-with-level } S'$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-o-conflict-is-false-with-level-inv-all-inv:*

assumes
 $\langle \text{cdcl}_W\text{-o } S \ T \rangle$
 $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$
 $\langle \text{conflict-is-false-with-level } S \rangle$
shows $\langle \text{conflict-is-false-with-level } T \rangle$
 $\langle \text{proof} \rangle$

lemma *no-step-cdcl_W-total:*

assumes
 $\langle \text{no-step cdcl}_W \ S \rangle$
 $\langle \text{conflicting } S = \text{None} \rangle$
 $\langle \text{no-strange-atm } S \rangle$
shows $\langle \text{total-over-m (lits-of-l (trail } S)) (set-mset (clauses } S)) \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-Ex-cdcl_W-stgy:*

assumes
 $\langle \text{cdcl}_W \ S \ T \rangle$
shows $\langle \text{Ex}(\text{cdcl}_W\text{-stgy } S) \rangle$
 $\langle \text{proof} \rangle$

lemma *no-step-skip-hd-in-conflicting:*

assumes
 $\text{inv-s: } \langle \text{cdcl}_W\text{-stgy-invariant } S \rangle$ **and**
 $\text{inv: } \langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**
 $\text{ns: } \langle \text{no-step skip } S \rangle$ **and**
 $\text{confl: } \langle \text{conflicting } S \neq \text{None} \rangle \langle \text{conflicting } S \neq \text{Some } \{\#\} \rangle$

```

shows  $\langle \text{lit-of } (\text{hd } (\text{trail } S)) \in \# \text{ the } (\text{conflicting } S) \rangle$ 
 $\langle \text{proof} \rangle$ 

lemma
fixes  $S$ 
assumes
   $nss: \langle \text{no-step skip } S \rangle$  and
   $nsr: \langle \text{no-step resolve } S \rangle$  and
   $invs: \langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$  and
   $stgy: \langle \text{cdcl}_W\text{-stgy-invariant } S \rangle$  and
   $confl: \langle \text{conflicting } S \neq \text{None} \rangle$  and
   $confl': \langle \text{conflicting } S \neq \text{Some } \{\#\} \rangle$ 
shows  $\text{no-skip-no-resolve-single-highest-level:}$ 
 $\langle \text{the } (\text{conflicting } S) =$ 
   $\text{add-mset } (\neg(\text{lit-of } (\text{hd } (\text{trail } S)))) \{\#L \in \# \text{ the } (\text{conflicting } S).$ 
   $\text{get-level } (\text{trail } S) L < \text{local.backtrack-lvl } S \# \} \rangle$  (is ?A) and
   $\text{no-skip-no-resolve-level-lvl-nonzero:}$ 
 $\langle 0 < \text{backtrack-lvl } S \rangle$  (is ?B) and
   $\text{no-skip-no-resolve-level-get-maximum-lvl-le:}$ 
 $\langle \text{get-maximum-level } (\text{trail } S) (\text{remove1-mset } (\neg(\text{lit-of } (\text{hd } (\text{trail } S)))) (\text{the } (\text{conflicting } S)))$ 
   $< \text{backtrack-lvl } S \rangle$  (is ?C)
 $\langle \text{proof} \rangle$ 

```

end

end

theory *CDCL-W-Termination*

imports *CDCL-W*

begin

context *conflict-driven-clause-learning_W*

begin

1.1.11 Termination

No Relearning of a clause

Because of the conflict minimisation, this version is less clear than the version without: instead of extracting the clause from the conflicting clause, we must take it from the clause used to backjump; i.e., the annotation of the first literal of the trail.

We also prove below that no learned clause is subsumed by a (smaller) clause in the clause set.

lemma *cdcl_W-stgy-no-relearned-clause:*

assumes

$cdcl: \langle \text{backtrack } S T \rangle$ **and**

$inv: \langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**

$smaller: \langle \text{no-smaller-propa } S \rangle$

shows

$\langle \text{mark-of } (\text{hd-trail } T) \notin \# \text{ clauses } S \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-no-relearned-larger-clause:*

assumes

$cdcl: \langle \text{backtrack } S T \rangle$ **and**

$inv: \langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**

$smaller: \langle \text{no-smaller-propa } S \rangle$ **and**

smaller-conf: $\langle \text{no-smaller-conf } S \rangle$ **and**
E-subset: $\langle E \subset \# \text{ mark-of } (\text{hd-trail } T) \rangle$
shows $\langle E \not\subset \# \text{ clauses } S \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-no-relearned-highest-subres-clause*:

assumes
cdcl: $\langle \text{backtrack } S \ T \rangle$ **and**
inv: $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**
smaller: $\langle \text{no-smaller-propa } S \rangle$ **and**
smaller-conf: $\langle \text{no-smaller-conf } S \rangle$ **and**
E-subset: $\langle \text{mark-of } (\text{hd-trail } T) = \text{add-mset } (\text{lit-of } (\text{hd-trail } T)) \ E \rangle$
shows $\langle \text{add-mset } (- \text{ lit-of } (\text{hd-trail } T)) \ E \not\subset \# \text{ clauses } S \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-distinct-mset*:

assumes
cdcl: $\langle \text{cdcl}_W\text{-stgy } S \ T \rangle$ **and**
inv: $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**
smaller: $\langle \text{no-smaller-propa } S \rangle$ **and**
dist: $\langle \text{distinct-mset } (\text{clauses } S) \rangle$
shows
 $\langle \text{distinct-mset } (\text{clauses } T) \rangle$
 $\langle \text{proof} \rangle$

This is a more restrictive version of the previous theorem, but is a better bound for an implementation that does not do duplication removal (esp. as part of preprocessing).

lemma *cdcl_W-stgy-learned-distinct-mset*:

assumes
cdcl: $\langle \text{cdcl}_W\text{-stgy } S \ T \rangle$ **and**
inv: $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**
smaller: $\langle \text{no-smaller-propa } S \rangle$ **and**
dist: $\langle \text{distinct-mset } (\text{learned-clss } S + \text{remdups-mset } (\text{init-clss } S)) \rangle$
shows
 $\langle \text{distinct-mset } (\text{learned-clss } T + \text{remdups-mset } (\text{init-clss } T)) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_W-stgy-distinct-mset-clauses*:

assumes
st: $\text{cdcl}_W\text{-stgy}^{**} \ R \ S$ **and**
invR: $\langle \text{cdcl}_W\text{-all-struct-inv } R \rangle$ **and**
dist: $\langle \text{distinct-mset } (\text{clauses } R) \rangle$ **and**
no-smaller: $\langle \text{no-smaller-propa } R \rangle$
shows $\langle \text{distinct-mset } (\text{clauses } S) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_W-stgy-distinct-mset-learned-clauses*:

assumes
st: $\text{cdcl}_W\text{-stgy}^{**} \ R \ S$ **and**
invR: $\langle \text{cdcl}_W\text{-all-struct-inv } R \rangle$ **and**
dist: $\langle \text{distinct-mset } (\text{learned-clss } R + \text{remdups-mset } (\text{init-clss } R)) \rangle$ **and**
no-smaller: $\langle \text{no-smaller-propa } R \rangle$
shows $\langle \text{distinct-mset } (\text{learned-clss } S + \text{remdups-mset } (\text{init-clss } S)) \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-distinct-mset-clauses*:
assumes
st: *cdcl_W-stgy*** (*init-state N*) *S* **and**
no-duplicate-clause: *distinct-mset N* **and**
no-duplicate-in-clause: *distinct-mset-mset N*
shows *distinct-mset (clauses S)*
 ⟨*proof*⟩

lemma *cdcl_W-stgy-learned-distinct-mset-new*:
assumes
cdcl: ⟨*cdcl_W-stgy S T*⟩ **and**
inv: *cdcl_W-all-struct-inv S* **and**
smaller: ⟨*no-smaller-propa S*⟩ **and**
dist: ⟨*distinct-mset (learned-clss S - A)*⟩
shows ⟨*distinct-mset (learned-clss T - A)*⟩
 ⟨*proof*⟩

lemma *rtrancp-cdcl_W-stgy-distinct-mset-clauses-new-abs*:
assumes
st: *cdcl_W-stgy*** *R S* **and**
invR: *cdcl_W-all-struct-inv R* **and**
no-smaller: ⟨*no-smaller-propa R*⟩ **and**
 ⟨*distinct-mset (learned-clss R - A)*⟩
shows *distinct-mset (learned-clss S - A)*
 ⟨*proof*⟩

lemma *rtrancp-cdcl_W-stgy-distinct-mset-clauses-new*:
assumes
st: *cdcl_W-stgy*** *R S* **and**
invR: *cdcl_W-all-struct-inv R* **and**
no-smaller: ⟨*no-smaller-propa R*⟩
shows *distinct-mset (learned-clss S - learned-clss R)*
 ⟨*proof*⟩

Decrease of a Measure

fun *cdcl_W-restart-measure* **where**
cdcl_W-restart-measure S =
 [(3::nat) ^ (card (atms-of-mm (init-clss S))) - card (set-mset (learned-clss S)),
 if conflicting *S* = None then 1 else 0,
 if conflicting *S* = None then card (atms-of-mm (init-clss S)) - length (trail S)
 else length (trail S)
]

lemma *length-model-le-vars*:
assumes
no-strange-atm S **and**
no-d: *no-dup (trail S)* **and**
finite (atms-of-mm (init-clss S))
shows *length (trail S) ≤ card (atms-of-mm (init-clss S))*
 ⟨*proof*⟩

lemma *length-model-le-vars-all-inv*:
assumes *cdcl_W-all-struct-inv S*
shows *length (trail S) ≤ card (atms-of-mm (init-clss S))*

$\langle \text{proof} \rangle$

lemma *learned-clss-less-upper-bound:*

fixes $S :: 'st$

assumes

distinct-cdcl_W-state S **and**

$\forall s \in \# \text{ learned-clss } S. \neg \text{tautology } s$

shows $\text{card}(\text{set-mset } (\text{learned-clss } S)) \leq 3 \wedge \text{card } (\text{atms-of-mm } (\text{learned-clss } S))$

$\langle \text{proof} \rangle$

lemma *cdcl_W-restart-measure-decreasing:*

fixes $S :: 'st$

assumes

cdcl_W-restart $S S'$ **and**

no-restart:

$\neg(\text{learned-clss } S \subseteq \# \text{ learned-clss } S' \wedge [] = \text{trail } S' \wedge \text{conflicting } S' = \text{None})$

and

no-forget: $\text{learned-clss } S \subseteq \# \text{ learned-clss } S'$ **and**

no-relearn: $\bigwedge S'. \text{backtrack } S S' \implies \text{mark-of } (\text{hd-trail } S') \notin \# \text{ learned-clss } S$

and

alien: *no-strange-atm* S **and**

M-level: *cdcl_W-M-level-inv* S **and**

no-taut: $\forall s \in \# \text{ learned-clss } S. \neg \text{tautology } s$ **and**

no-dup: *distinct-cdcl_W-state* S **and**

confl: *cdcl_W-conflicting* S

shows $(\text{cdcl}_W\text{-restart-measure } S', \text{cdcl}_W\text{-restart-measure } S) \in \text{lexn less-than } 3$

$\langle \text{proof} \rangle$

lemma *cdcl_W-stgy-step-decreasing:*

fixes $S T :: 'st$

assumes

cdcl: $\langle \text{cdcl}_W\text{-stgy } S T \rangle$ **and**

struct-inv: $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$ **and**

smaller: $\langle \text{no-smaller-propa } S \rangle$

shows $(\text{cdcl}_W\text{-restart-measure } T, \text{cdcl}_W\text{-restart-measure } S) \in \text{lexn less-than } 3$

$\langle \text{proof} \rangle$

lemma *empty-trail-no-smaller-propa:* $\langle \text{trail } R = [] \implies \text{no-smaller-propa } R \rangle$

$\langle \text{proof} \rangle$

Roughly corresponds to theorem 2.9.15 page 100 of Weidenbach's book but using a different bound (the bound is below)

lemma *tranclp-cdcl_W-stgy-decreasing:*

fixes $R S T :: 'st$

assumes $\text{cdcl}_W\text{-stgy}^{++} R S$ **and**

tr: $\text{trail } R = []$ **and**

cdcl_W-all-struct-inv R

shows $(\text{cdcl}_W\text{-restart-measure } S, \text{cdcl}_W\text{-restart-measure } R) \in \text{lexn less-than } 3$

$\langle \text{proof} \rangle$

lemma *tranclp-cdcl_W-stgy-S0-decreasing:*

fixes $R S T :: 'st$

assumes

pl: $\text{cdcl}_W\text{-stgy}^{++} (\text{init-state } N) S$ **and**

no-dup: *distinct-mset-mset* N

shows $(\text{cdcl}_W\text{-restart-measure } S, \text{cdcl}_W\text{-restart-measure } (\text{init-state } N)) \in \text{learn less-than } 3$
 $\langle \text{proof} \rangle$

lemma $\text{wf-tranclp-cdcl}_W\text{-stgy}$:

$\text{wf } \{(S::'\text{st}, \text{init-state } N) \mid S \ N. \text{distinct-mset-mset } N \wedge \text{cdcl}_W\text{-stgy}^{++} (\text{init-state } N) \ S\}$
 $\langle \text{proof} \rangle$

The following theorems is deeply linked with the strategy: It shows that a decision alone cannot lead to a conflict. This is obvious but I expect this to be a major part of the proof that the number of learnt clause cannot be larger than $(2::'a)^n$.

lemma $\text{no-conflict-after-decide}$:

assumes

$\text{dec}: \langle \text{decide } S \ T \rangle$ **and**

$\text{inv}: \langle \text{cdcl}_W\text{-all-struct-inv } T \rangle$ **and**

$\text{smaller}: \langle \text{no-smaller-propa } T \rangle$ **and**

$\text{smaller-conf}: \langle \text{no-smaller-conf } T \rangle$

shows $\langle \neg \text{conflict } T \ U \rangle$

$\langle \text{proof} \rangle$

abbreviation $\text{list-weight-propa-trail} :: \langle ('v \text{ literal}, 'v \text{ literal}, 'v \text{ literal multiset}) \text{ annotated-lit list} \Rightarrow \text{bool list} \rangle$ **where**

$\langle \text{list-weight-propa-trail } M \equiv \text{map is-proped } M \rangle$

definition $\text{comp-list-weight-propa-trail} :: \langle \text{nat} \Rightarrow ('v \text{ literal}, 'v \text{ literal}, 'v \text{ literal multiset}) \text{ annotated-lit list} \Rightarrow \text{bool list} \rangle$ **where**

$\langle \text{comp-list-weight-propa-trail } b \ M \equiv \text{replicate } (b - \text{length } M) \ \text{False} \ @ \ \text{list-weight-propa-trail } M \rangle$

lemma $\text{comp-list-weight-propa-trail-append[simp]}$:

$\langle \text{comp-list-weight-propa-trail } b \ (M \ @ \ M') =$

$\text{comp-list-weight-propa-trail } (b - \text{length } M') \ M \ @ \ \text{list-weight-propa-trail } M' \rangle$

$\langle \text{proof} \rangle$

lemma $\text{comp-list-weight-propa-trail-append-single[simp]}$:

$\langle \text{comp-list-weight-propa-trail } b \ (M \ @ \ [K]) =$

$\text{comp-list-weight-propa-trail } (b - 1) \ M \ @ \ [\text{is-proped } K] \rangle$

$\langle \text{proof} \rangle$

lemma $\text{comp-list-weight-propa-trail-cons[simp]}$:

$\langle \text{comp-list-weight-propa-trail } b \ (K \ \# \ M') =$

$\text{comp-list-weight-propa-trail } (b - \text{Suc } (\text{length } M')) \ [] \ @ \ \text{is-proped } K \ \# \ \text{list-weight-propa-trail } M' \rangle$

$\langle \text{proof} \rangle$

fun $\text{of-list-weight} :: \langle \text{bool list} \Rightarrow \text{nat} \rangle$ **where**

$\langle \text{of-list-weight } [] = 0 \rangle$

$| \langle \text{of-list-weight } (b \ \# \ xs) = (\text{if } b \text{ then } 1 \text{ else } 0) + 2 * \text{of-list-weight } xs \rangle$

lemma $\text{of-list-weight-append[simp]}$:

$\langle \text{of-list-weight } (a \ @ \ b) = \text{of-list-weight } a + 2^{\text{length } a} * \text{of-list-weight } b \rangle$

$\langle \text{proof} \rangle$

lemma $\text{of-list-weight-append-single[simp]}$:

$\langle \text{of-list-weight } (a \ @ \ [b]) = \text{of-list-weight } a + 2^{\text{length } a} * (\text{if } b \text{ then } 1 \text{ else } 0) \rangle$

$\langle \text{proof} \rangle$

lemma $\text{of-list-weight-replicate-False[simp]}$: $\langle \text{of-list-weight } (\text{replicate } n \ \text{False}) = 0 \rangle$

$\langle \text{proof} \rangle$

lemma *of-list-weight-replicate-True[simp]*: $\langle \text{of-list-weight } (\text{replicate } n \text{ True}) = 2^n - 1 \rangle$
 $\langle \text{proof} \rangle$

lemma *of-list-weight-le*: $\langle \text{of-list-weight } xs \leq 2^{\text{length } xs} - 1 \rangle$
 $\langle \text{proof} \rangle$

lemma *of-list-weight-lt*: $\langle \text{of-list-weight } xs < 2^{\text{length } xs} \rangle$
 $\langle \text{proof} \rangle$

lemma [simp]: $\langle \text{of-list-weight } (\text{comp-list-weight-propa-trail } n []) = 0 \rangle$
 $\langle \text{proof} \rangle$

abbreviation *propa-weight*

$:: \langle \text{nat} \Rightarrow ('v \text{ literal}, 'v \text{ literal}, 'v \text{ literal multiset}) \text{ annotated-lit list} \Rightarrow \text{nat} \rangle$

where

$\langle \text{propa-weight } n \ M \equiv \text{of-list-weight } (\text{comp-list-weight-propa-trail } n \ M) \rangle$

lemma *length-comp-list-weight-propa-trail[simp]*: $\langle \text{length } (\text{comp-list-weight-propa-trail } a \ M) = \max (\text{length } M) \ a \rangle$
 $\langle \text{proof} \rangle$

lemma (in *pow2-times-n*):

$\langle \text{Suc } a \leq n \implies 2 * 2^{(n - \text{Suc } a)} = (2::\text{nat})^{(n - a)} \rangle$
 $\langle \text{Suc } a \leq n \implies 2^{(n - \text{Suc } a)} * 2 = (2::\text{nat})^{(n - a)} \rangle$
 $\langle \text{Suc } a \leq n \implies 2^{(n - \text{Suc } a)} * b * 2 = (2::\text{nat})^{(n - a)} * b \rangle$
 $\langle \text{Suc } a \leq n \implies 2^{(n - \text{Suc } a)} * (b * 2) = (2::\text{nat})^{(n - a)} * b \rangle$
 $\langle \text{Suc } a \leq n \implies 2^{(n - \text{Suc } a)} * (2 * b) = (2::\text{nat})^{(n - a)} * b \rangle$
 $\langle \text{Suc } a \leq n \implies 2 * b * 2^{(n - \text{Suc } a)} = (2::\text{nat})^{(n - a)} * b \rangle$
 $\langle \text{Suc } a \leq n \implies 2 * (b * 2^{(n - \text{Suc } a)}) = (2::\text{nat})^{(n - a)} * b \rangle$
 $\langle \text{proof} \rangle$

lemma *decide-propa-weight*:

$\langle \text{decide } S \ T \implies n \geq \text{length } (\text{trail } T) \implies \text{propa-weight } n \ (\text{trail } S) \leq \text{propa-weight } n \ (\text{trail } T) \rangle$
 $\langle \text{proof} \rangle$

lemma *propagate-propa-weight*:

$\langle \text{propagate } S \ T \implies n \geq \text{length } (\text{trail } T) \implies \text{propa-weight } n \ (\text{trail } S) < \text{propa-weight } n \ (\text{trail } T) \rangle$
 $\langle \text{proof} \rangle$

The theorem below corresponds the bound of theorem 2.9.15 page 100 of Weidenbach's book. In the current version there is no proof of the bound.

The following proof contains an immense amount of stupid bookkeeping. The proof itself is rather easy and Isabelle makes it extra-complicated.

Let's consider the sequence $S \rightarrow \dots \rightarrow T$. The bookkeeping part:

1. We decompose it into its components $f \ 0 \rightarrow f \ 1 \rightarrow \dots \rightarrow f \ n$.
2. Then we extract the backjumps out of it, which are at position $\text{nth-nj } 0, \text{nth-nj } 1, \dots$
3. Then we extract the conflicts out of it, which are at position $\text{nth-confl } 0, \text{nth-confl } 1, \dots$

Then the simple part:

1. each backtrack increases *propa-weight*
2. but *propa-weight* is bounded by $(2::'a)^{\text{card } (\text{atms-of-mm } (\text{init-clss } S))}$ Therefore, we get the bound.

Comments on the proof:

- The main problem of the proof is the number of inductions in the bookkeeping part.
- The proof is actually by contradiction to make sure that enough backtrack step exists. This could probably be avoided, but without change in the proof.

Comments on the bound:

- The proof is very very crude: Any propagation also decreases the bound. The lemma $\llbracket \text{decide } ?S \text{ } ?T; \text{cdcl}_W\text{-all-struct-inv } ?T; \text{no-smaller-propa } ?T; \text{no-smaller-confl } ?T \rrbracket \implies \neg \text{conflict } ?T \text{ } ?U$ above shows that a decision cannot lead immediately to a conflict.
- TODO: can a backtrack could be immediately followed by another conflict (if there are several conflicts for the initial backtrack)? If not the bound can be divided by two.

lemma *cdcl-pow2-n-learned-clauses:*

assumes

cdcl: $\langle \text{cdcl}_W^{**} S T \rangle$ **and**

confl: $\langle \text{conflicting } S = \text{None} \rangle$ **and**

inv: $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$

shows $\langle \text{size } (\text{learned-clss } T) \leq \text{size } (\text{learned-clss } S) + 2^{\wedge} (\text{card } (\text{atms-of-mm } (\text{init-clss } S))) \rangle$
(is $\langle - \leq - + ?b \rangle$)

$\langle \text{proof} \rangle$

Application of the previous theorem to an initial state:

corollary *cdcl-pow2-n-learned-clauses2:*

assumes

cdcl: $\langle \text{cdcl}_W^{**} (\text{init-state } N) T \rangle$ **and**

inv: $\langle \text{cdcl}_W\text{-all-struct-inv } (\text{init-state } N) \rangle$

shows $\langle \text{size } (\text{learned-clss } T) \leq 2^{\wedge} (\text{card } (\text{atms-of-mm } N)) \rangle$

$\langle \text{proof} \rangle$

end

end

1.2 Merging backjump rules

theory *CDCL-W-Merge*

imports *CDCL-W*

begin

Before showing that Weidenbach's CDCL is included in NOT's CDCL, we need to work on a variant of Weidenbach's calculus: NOT's backjump assumes the existence of a clause that is suitable to backjump. This clause is obtained in W's CDCL by applying:

1. *conflict-driven-clause-learning_W.conflict* to find the conflict

2. the conflict is analysed by repetitive application of *conflict-driven-clause-learning_W.resolve* and *conflict-driven-clause-learning_W.skip*,
3. finally *conflict-driven-clause-learning_W.backtrack* is used to backtrack.

We show that this new calculus has the same final states than Weidenbach's CDCL if the calculus starts in a state such that the invariant holds and no conflict has been found yet. The latter condition holds for initial states.

1.2.1 Inclusion of the States

context *conflict-driven-clause-learning_W*
begin

declare *cdcl_W-restart.intros[intro]* *cdcl_W-bj.intros[intro]* *cdcl_W-o.intros[intro]*
state-prop [simp del]

lemma *backtrack-no-cdcl_W-bj*:
assumes *cdcl*: *cdcl_W-bj T U*
shows $\neg \text{backtrack } V T$
 $\langle \text{proof} \rangle$

skip-or-resolve corresponds to the *analyze* function in the code of MiniSAT.

inductive *skip-or-resolve* :: '*st* \Rightarrow '*st* \Rightarrow bool **where**
s-or-r-skip[intro]: *skip S T* \Longrightarrow *skip-or-resolve S T* |
s-or-r-resolve[intro]: *resolve S T* \Longrightarrow *skip-or-resolve S T*

lemma *rtrancp-cdcl_W-bj-skip-or-resolve-backtrack*:
assumes *cdcl_W-bj** S U*
shows *skip-or-resolve** S U* $\vee (\exists T. \text{skip-or-resolve** } S T \wedge \text{backtrack } T U)$
 $\langle \text{proof} \rangle$

lemma *rtrancp-skip-or-resolve-rtrancp-cdcl_W-restart*:
*skip-or-resolve** S T* \Longrightarrow *cdcl_W-restart** S T*
 $\langle \text{proof} \rangle$

definition *backjump-l-cond* :: '*v clause* \Rightarrow '*v clause* \Rightarrow '*v literal* \Rightarrow '*st* \Rightarrow '*st* \Rightarrow bool **where**
backjump-l-cond $\equiv \lambda C C' L S T. \text{True}$

lemma *wf-skip-or-resolve*:
wf $\{(T, S). \text{skip-or-resolve } S T\}$
 $\langle \text{proof} \rangle$

definition *inv_{NOT}* :: '*st* \Rightarrow bool **where**
inv_{NOT} $\equiv \lambda S. \text{no-dup } (\text{trail } S)$

declare *inv_{NOT}-def[simp]*
end

context *conflict-driven-clause-learning_W*
begin

1.2.2 More lemmas about Conflict, Propagate and Backjumping

Termination

lemma *cdcl_W-bj-measure*:

assumes *cdcl_W-bj* S T

shows $\text{length } (\text{trail } S) + (\text{if conflicting } S = \text{None then } 0 \text{ else } 1)$
 $> \text{length } (\text{trail } T) + (\text{if conflicting } T = \text{None then } 0 \text{ else } 1)$

$\langle \text{proof} \rangle$

lemma *wf-cdcl_W-bj*:

wf $\{(b, a). \text{cdcl}_W\text{-bj } a \ b\}$

$\langle \text{proof} \rangle$

lemma *cdcl_W-bj-exists-normal-form*:

shows $\exists T. \text{full } \text{cdcl}_W\text{-bj } S \ T$

$\langle \text{proof} \rangle$

lemma *rtrancpl-skip-state-decomp*:

assumes *skip*** S T

shows

$\exists M. \text{trail } S = M @ \text{trail } T \wedge (\forall m \in \text{set } M. \neg \text{is-decided } m)$

init-clss $S = \text{init-clss } T$

learned-clss $S = \text{learned-clss } T$

backtrack-lvl $S = \text{backtrack-lvl } T$

conflicting $S = \text{conflicting } T$

$\langle \text{proof} \rangle$

Analysing is confluent

lemma *backtrack-reduce-trail-to-state-eq*:

assumes

V-T: $\langle V \sim \text{tl-trail } T \rangle$ **and**

decomp: $\langle (\text{Decided } K \# M1, M2) \in \text{set } (\text{get-all-ann-decomposition } (\text{trail } V)) \rangle$

shows $\langle \text{reduce-trail-to } M1 \ (\text{add-learned-clss } E \ (\text{update-conflicting } \text{None } V))$

$\sim \text{reduce-trail-to } M1 \ (\text{add-learned-clss } E \ (\text{update-conflicting } \text{None } T)) \rangle$

$\langle \text{proof} \rangle$

lemma *rtrancpl-skip-backtrack-reduce-trail-to-state-eq*:

assumes

V-T: $\langle \text{skip** } T \ V \rangle$ **and**

decomp: $\langle (\text{Decided } K \# M1, M2) \in \text{set } (\text{get-all-ann-decomposition } (\text{trail } V)) \rangle$

shows $\langle \text{reduce-trail-to } M1 \ (\text{add-learned-clss } E \ (\text{update-conflicting } \text{None } T))$

$\sim \text{reduce-trail-to } M1 \ (\text{add-learned-clss } E \ (\text{update-conflicting } \text{None } V)) \rangle$

$\langle \text{proof} \rangle$

Backjumping after skipping or jump directly **lemma** *rtrancpl-skip-backtrack-backtrack*:

assumes

*skip*** S T **and**

backtrack T W **and**

cdcl_W-all-struct-inv S

shows *backtrack* S W

$\langle \text{proof} \rangle$

See also theorem *rtrancpl-skip-backtrack-backtrack*

lemma *rtrancpl-skip-backtrack-backtrack-end*:

assumes
skip: $\text{skip}^{**} S T$ **and**
bt: $\text{backtrack} S W$ **and**
inv: $\text{cdcl}_W\text{-all-struct-inv } S$
shows $\text{backtrack} T W$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-bj-decomp-resolve-skip-and-bj}$:
assumes $\text{cdcl}_W\text{-bj}^{**} S T$
shows $(\text{skip-or-resolve}^{**} S T \vee (\exists U. \text{skip-or-resolve}^{**} S U \wedge \text{backtrack } U T))$
 $\langle \text{proof} \rangle$

1.2.3 CDCL with Merging

inductive $\text{cdcl}_W\text{-merge-restart} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$ **where**
fw-r-propagate: $\text{propagate } S S' \Longrightarrow \text{cdcl}_W\text{-merge-restart } S S' \mid$
fw-r-conflict: $\text{conflict } S T \Longrightarrow \text{full } \text{cdcl}_W\text{-bj } T U \Longrightarrow \text{cdcl}_W\text{-merge-restart } S U \mid$
fw-r-decide: $\text{decide } S S' \Longrightarrow \text{cdcl}_W\text{-merge-restart } S S' \mid$
fw-r-rf: $\text{cdcl}_W\text{-rf } S S' \Longrightarrow \text{cdcl}_W\text{-merge-restart } S S'$

lemma $\text{rtrancpl-cdcl}_W\text{-bj-rtrancpl-cdcl}_W\text{-restart}$:
 $\text{cdcl}_W\text{-bj}^{**} S T \Longrightarrow \text{cdcl}_W\text{-restart}^{**} S T$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-merge-restart-cdcl}_W\text{-restart}$:
assumes $\text{cdcl}_W\text{-merge-restart } S T$
shows $\text{cdcl}_W\text{-restart}^{**} S T$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-merge-restart-conflicting-true-or-no-step}$:
assumes $\text{cdcl}_W\text{-merge-restart } S T$
shows $\text{conflicting } T = \text{None} \vee \text{no-step } \text{cdcl}_W\text{-restart } T$
 $\langle \text{proof} \rangle$

inductive $\text{cdcl}_W\text{-merge} :: 'st \Rightarrow 'st \Rightarrow \text{bool}$ **where**
fw-propagate: $\text{propagate } S S' \Longrightarrow \text{cdcl}_W\text{-merge } S S' \mid$
fw-conflict: $\text{conflict } S T \Longrightarrow \text{full } \text{cdcl}_W\text{-bj } T U \Longrightarrow \text{cdcl}_W\text{-merge } S U \mid$
fw-decide: $\text{decide } S S' \Longrightarrow \text{cdcl}_W\text{-merge } S S' \mid$
fw-forget: $\text{forget } S S' \Longrightarrow \text{cdcl}_W\text{-merge } S S'$

lemma $\text{cdcl}_W\text{-merge-cdcl}_W\text{-merge-restart}$:
 $\text{cdcl}_W\text{-merge } S T \Longrightarrow \text{cdcl}_W\text{-merge-restart } S T$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_W\text{-merge-trancpl-cdcl}_W\text{-merge-restart}$:
 $\text{cdcl}_W\text{-merge}^{**} S T \Longrightarrow \text{cdcl}_W\text{-merge-restart}^{**} S T$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-merge-rtrancpl-cdcl}_W\text{-restart}$:
 $\text{cdcl}_W\text{-merge } S T \Longrightarrow \text{cdcl}_W\text{-restart}^{**} S T$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_W\text{-merge-rtrancpl-cdcl}_W\text{-restart}$:
 $\text{cdcl}_W\text{-merge}^{**} S T \Longrightarrow \text{cdcl}_W\text{-restart}^{**} S T$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-all-struct-inv-tranclp-cdcl_W-merge-tranclp-cdcl_W-merge-cdcl_W-all-struct-inv*:

assumes

inv: *cdcl_W-all-struct-inv* *b*

cdcl_W-merge⁺⁺ *b a*

shows $(\lambda S T. \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{cdcl}_W\text{-merge } S T)^{++} b a$

<proof>

lemma *backtrack-is-full1-cdcl_W-bj*:

assumes *bt*: *backtrack* *S T*

shows *full1 cdcl_W-bj* *S T*

<proof>

lemma *rtrancl-cdcl_W-conflicting-true-cdcl_W-merge-restart*:

assumes *cdcl_W-restart*^{**} *S V* **and** *inv*: *cdcl_W-M-level-inv* *S* **and** *conflicting* *S = None*

shows $(\text{cdcl}_W\text{-merge-restart}^{**} S V \wedge \text{conflicting } V = \text{None})$

$\vee (\exists T U. \text{cdcl}_W\text{-merge-restart}^{**} S T \wedge \text{conflicting } V \neq \text{None} \wedge \text{conflict } T U \wedge \text{cdcl}_W\text{-bj}^{**} U V)$

<proof>

lemma *no-step-cdcl_W-restart-no-step-cdcl_W-merge-restart*:

no-step cdcl_W-restart *S* \implies *no-step cdcl_W-merge-restart* *S*

<proof>

lemma *no-step-cdcl_W-merge-restart-no-step-cdcl_W-restart*:

assumes

conflicting *S = None* **and**

cdcl_W-M-level-inv *S* **and**

no-step cdcl_W-merge-restart *S*

shows *no-step cdcl_W-restart* *S*

<proof>

lemma *cdcl_W-merge-restart-no-step-cdcl_W-bj*:

assumes

cdcl_W-merge-restart *S T*

shows *no-step cdcl_W-bj* *T*

<proof>

lemma *rtranclp-cdcl_W-merge-restart-no-step-cdcl_W-bj*:

assumes

cdcl_W-merge-restart^{**} *S T* **and**

conflicting *S = None*

shows *no-step cdcl_W-bj* *T*

<proof>

If *conflicting* *S* \neq *None*, we cannot say anything.

Remark that this theorem does not say anything about well-foundedness: even if you know that one relation is well-founded, it only states that the normal forms are shared.

lemma *conflicting-true-full-cdcl_W-restart-iff-full-cdcl_W-merge*:

assumes *conf*: *conflicting* *S = None* **and** *lev*: *cdcl_W-M-level-inv* *S*

shows *full cdcl_W-restart* *S V* \longleftrightarrow *full cdcl_W-merge-restart* *S V*

<proof>

lemma *init-state-true-full-cdcl_W-restart-iff-full-cdcl_W-merge*:

shows *full cdcl_W-restart* (*init-state* *N*) *V* \longleftrightarrow *full cdcl_W-merge-restart* (*init-state* *N*) *V*

<proof>

1.2.4 CDCL with Merge and Strategy

The intermediate step

inductive $cdcl_W-s' :: 'st \Rightarrow 'st \Rightarrow \text{bool}$ **for** $S :: 'st$ **where**

conflict': $\text{conflict } S \ S' \Longrightarrow cdcl_W-s' \ S \ S' \mid$

propagate': $\text{propagate } S \ S' \Longrightarrow cdcl_W-s' \ S \ S' \mid$

decide': $\text{no-step conflict } S \Longrightarrow \text{no-step propagate } S \Longrightarrow \text{decide } S \ S' \Longrightarrow cdcl_W-s' \ S \ S' \mid$

bj': $\text{full1 } cdcl_W-bj \ S \ S' \Longrightarrow cdcl_W-s' \ S \ S'$

inductive-cases $cdcl_W-s'E$: $cdcl_W-s' \ S \ T$

lemma *rtrancpl-cdcl_W-bj-full1-cdclp-cdcl_W-stgy*:

$cdcl_W-bj^{**} \ S \ S' \Longrightarrow cdcl_W-stgy^{**} \ S \ S'$

<proof>

lemma *cdcl_W-s'-is-rtrancpl-cdcl_W-stgy*:

$cdcl_W-s' \ S \ T \Longrightarrow cdcl_W-stgy^{**} \ S \ T$

<proof>

lemma *cdcl_W-stgy-cdcl_W-s'-no-step*:

assumes $cdcl_W-stgy \ S \ U$ **and** $cdcl_W\text{-all-struct-inv } S$ **and** $\text{no-step } cdcl_W-bj \ U$

shows $cdcl_W-s' \ S \ U$

<proof>

lemma *rtrancpl-cdcl_W-stgy-connected-to-rtrancpl-cdcl_W-s'*:

assumes $cdcl_W-stgy^{**} \ S \ U$ **and** $\text{inv: } cdcl_W\text{-M-level-inv } S$

shows $cdcl_W-s'^{**} \ S \ U \vee (\exists T. cdcl_W-s'^{**} \ S \ T \wedge cdcl_W-bj^{++} \ T \ U \wedge \text{conflicting } U \neq \text{None})$

<proof>

lemma *n-step-cdcl_W-stgy-iff-no-step-cdcl_W-restart-cl-cdcl_W-o*:

assumes $\text{inv: } cdcl_W\text{-all-struct-inv } S$

shows $\text{no-step } cdcl_W-s' \ S \longleftrightarrow \text{no-step } cdcl_W-stgy \ S \ (\text{is } ?S' \ S \longleftrightarrow ?C \ S)$

<proof>

lemma *cdcl_W-s'-trancpl-cdcl_W-restart*:

assumes $cdcl_W-s' \ S \ S'$ **shows** $cdcl_W\text{-restart}^{++} \ S \ S'$

<proof>

lemma *trancpl-cdcl_W-s'-trancpl-cdcl_W-restart*:

$cdcl_W-s'^{++} \ S \ S' \Longrightarrow cdcl_W\text{-restart}^{++} \ S \ S'$

<proof>

lemma *rtrancpl-cdcl_W-s'-rtrancpl-cdcl_W-restart*:

$cdcl_W-s'^{**} \ S \ S' \Longrightarrow cdcl_W\text{-restart}^{**} \ S \ S'$

<proof>

lemma *full-cdcl_W-stgy-iff-full-cdcl_W-s'*:

assumes $\text{inv: } cdcl_W\text{-all-struct-inv } S$

shows $\text{full } cdcl_W-stgy \ S \ T \longleftrightarrow \text{full } cdcl_W-s' \ S \ T \ (\text{is } ?S \longleftrightarrow ?S')$

<proof>

end

end

Chapter 2

NOT's CDCL and DPLL

```
theory CDCL-WNOT-Measure
imports Weidenbach-Book-Base.WB-List-More
begin
```

The organisation of the development is the following:

- `CDCL_WNOT_Measure.thy` contains the measure used to show the termination the core of CDCL.
- `CDCL_NOT.thy` contains the specification of the rules: the rules are defined, and we proof the correctness and termination for some strategies CDCL.
- `DPLL_NOT.thy` contains the DPLL calculus based on the CDCL version.
- `DPLL_W.thy` contains Weidenbach's version of DPLL and the proof of equivalence between the two DPLL versions.

2.1 Measure

This measure show the termination of the core of CDCL: each step improves the number of literals we know for sure.

This measure can also be seen as the increasing lexicographic order: it is an order on bounded sequences, when each element is bounded. The proof involves a measure like the one defined here (the same?).

definition $\mu_C :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat list} \Rightarrow \text{nat}$ **where**
 $\mu_C s b M \equiv (\sum i=0..<\text{length } M. M!i * b^{\wedge} (s + i - \text{length } M))$

lemma $\mu_C\text{-Nil}[simp]$:
 $\mu_C s b [] = 0$
<proof>

lemma $\mu_C\text{-single}[simp]$:
 $\mu_C s b [L] = L * b^{\wedge} (s - \text{Suc } 0)$
<proof>

lemma $\text{set-sum-atLeastLessThan-add}$:
 $(\sum i=k..<k+(b::\text{nat}). f i) = (\sum i=0..<b. f (k + i))$
<proof>

lemma *set-sum-atLeastLessThan-Suc*:

$(\sum i=1..<Suc\ j. f\ i) = (\sum i=0..<j. f\ (Suc\ i))$
 $\langle proof \rangle$

lemma μ_C -cons:

$\mu_C\ s\ b\ (L\ \# \ M) = L * b^{\wedge} (s - 1 - length\ M) + \mu_C\ s\ b\ M$
 $\langle proof \rangle$

lemma μ_C -append:

assumes $s \geq length\ (M@M')$
shows $\mu_C\ s\ b\ (M@M') = \mu_C\ (s - length\ M')\ b\ M + \mu_C\ s\ b\ M'$
 $\langle proof \rangle$

lemma μ_C -cons-non-empty-inf:

assumes M -ge-1: $\forall i \in set\ M. i \geq 1$ **and** $M: M \neq []$
shows $\mu_C\ s\ b\ M \geq b^{\wedge} (s - length\ M)$
 $\langle proof \rangle$

Copy of `~~/src/HOL/ex/NatSum.thy` (but generalized to $0 \leq k$)

lemma *sum-of-powers*: $0 \leq k \implies (k - 1) * (\sum i=0..<n. k^i) = k^n - (1::nat)$
 $\langle proof \rangle$

In the degenerated cases, we only have the large inequality holds. In the other cases, the following strict inequality holds:

lemma μ_C -bounded-non-degenerated:

fixes $b :: nat$
assumes
 $b > 0$ **and**
 $M \neq []$ **and**
 M -le: $\forall i < length\ M. M[i] < b$ **and**
 $s \geq length\ M$
shows $\mu_C\ s\ b\ M < b^{\wedge} s$
 $\langle proof \rangle$

In the degenerate case $b = (0::'a)$, the list M is empty (since the list cannot contain any element).

lemma μ_C -bounded:

fixes $b :: nat$
assumes
 M -le: $\forall i < length\ M. M[i] < b$ **and**
 $s \geq length\ M$
 $b > 0$
shows $\mu_C\ s\ b\ M < b^{\wedge} s$
 $\langle proof \rangle$

When $b = 0$, we cannot show that the measure is empty, since $0^0 = 1$.

lemma μ_C -base-0:

assumes $length\ M \leq s$
shows $\mu_C\ s\ 0\ M \leq M!0$
 $\langle proof \rangle$

lemma *finite-bounded-pair-list*:

fixes $b :: nat$
shows *finite* $\{(ys, xs). length\ xs < s \wedge length\ ys < s \wedge$

$(\forall i < \text{length } xs. xs ! i < b) \wedge (\forall i < \text{length } ys. ys ! i < b)\}$
 $\langle \text{proof} \rangle$

definition $\nu NOT :: nat \Rightarrow nat \Rightarrow (nat\ list \times nat\ list)\ set$ **where**
 $\nu NOT\ s\ base = \{(ys, xs). \text{length } xs < s \wedge \text{length } ys < s \wedge$
 $(\forall i < \text{length } xs. xs ! i < base) \wedge (\forall i < \text{length } ys. ys ! i < base) \wedge$
 $(ys, xs) \in \text{lenlex less-than}\}$

lemma $\text{finite-}\nu NOT[\text{simp}]$:
 $\text{finite } (\nu NOT\ s\ base)$
 $\langle \text{proof} \rangle$

lemma $\text{acyclic-}\nu NOT$: $\text{acyclic } (\nu NOT\ s\ base)$
 $\langle \text{proof} \rangle$

lemma $\text{wf-}\nu NOT$: $\text{wf } (\nu NOT\ s\ base)$
 $\langle \text{proof} \rangle$

end
theory $CDCL\text{-}NOT$
imports
 $Weidenbach\text{-}Book\text{-}Base.WB\text{-}List\text{-}More$
 $Weidenbach\text{-}Book\text{-}Base.Wellfounded\text{-}More$
 $Entailment\text{-}Definition.Partial\text{-}Annotated\text{-}Herbrand\text{-}Interpretation$
 $CDCL\text{-}WNOT\text{-}Measure$
begin

2.2 NOT's CDCL

2.2.1 Auxiliary Lemmas and Measure

We define here some more simplification rules, or rules that have been useful as help for some tactic

lemma $\text{atms-of-uminus-lit-atm-of-lit-of}$:
 $\langle \text{atms-of } \{\# \text{ -lit-of } x. x \in \# A \# \} = \text{atm-of } ' (\text{lit-of } ' (\text{set-mset } A)) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{atms-of-ms-single-image-atm-of-lit-of}$:
 $\langle \text{atms-of-ms } (\text{unmark-s } A) = \text{atm-of } ' (\text{lit-of } ' A) \rangle$
 $\langle \text{proof} \rangle$

2.2.2 Initial Definitions

The State

We define here an abstraction over operation on the state we are manipulating.

locale $\text{dpll-state-ops} =$
fixes
 $\text{trail} :: \langle 'st \Rightarrow ('v, \text{unit}) \text{ ann-lits} \rangle$ **and**
 $\text{clauses}_{NOT} :: \langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**
 $\text{prepend-trail} :: \langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
 $\text{tl-trail} :: \langle 'st \Rightarrow 'st \rangle$ **and**
 $\text{add-cl}_{NOT} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
 $\text{remove-cl}_{NOT} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$

begin
abbreviation $state_{NOT} :: \langle 'st \Rightarrow ('v, unit) \text{ ann-lit list} \times 'v \text{ clauses} \rangle$ **where**
 $\langle state_{NOT} S \equiv (trail\ S, clauses_{NOT}\ S) \rangle$
end

NOT's state is basically a pair composed of the trail (i.e. the candidate model) and the set of clauses. We abstract this state to convert this state to other states. like Weidenbach's five-tuple.

locale $dpll\text{-}state =$
 $dpll\text{-}state\text{-}ops$
 $trail\ clauses_{NOT}\ prepend\text{-}trail\ tl\text{-}trail\ add\text{-}cls_{NOT}\ remove\text{-}cls_{NOT}$ — related to the state
for
 $trail :: \langle 'st \Rightarrow ('v, unit) \text{ ann-lits} \rangle$ **and**
 $clauses_{NOT} :: \langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**
 $prepend\text{-}trail :: \langle ('v, unit) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
 $tl\text{-}trail :: \langle 'st \Rightarrow 'st \rangle$ **and**
 $add\text{-}cls_{NOT} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
 $remove\text{-}cls_{NOT} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle +$
assumes
 $prepend\text{-}trail_{NOT}:$
 $\langle state_{NOT} (prepend\text{-}trail\ L\ st) = (L \# trail\ st, clauses_{NOT}\ st) \rangle$ **and**
 $tl\text{-}trail_{NOT}:$
 $\langle state_{NOT} (tl\text{-}trail\ st) = (tl\ (trail\ st), clauses_{NOT}\ st) \rangle$ **and**
 $add\text{-}cls_{NOT}:$
 $\langle state_{NOT} (add\text{-}cls_{NOT}\ C\ st) = (trail\ st, add\text{-}mset\ C\ (clauses_{NOT}\ st)) \rangle$ **and**
 $remove\text{-}cls_{NOT}:$
 $\langle state_{NOT} (remove\text{-}cls_{NOT}\ C\ st) = (trail\ st, removeAll\text{-}mset\ C\ (clauses_{NOT}\ st)) \rangle$
begin
lemma
 $trail\text{-}prepend\text{-}trail[simp]:$
 $\langle trail\ (prepend\text{-}trail\ L\ st) = L \# trail\ st \rangle$
and
 $trail\text{-}tl\text{-}trail_{NOT}[simp]: \langle trail\ (tl\text{-}trail\ st) = tl\ (trail\ st) \rangle$ **and**
 $trail\text{-}add\text{-}cls_{NOT}[simp]: \langle trail\ (add\text{-}cls_{NOT}\ C\ st) = trail\ st \rangle$ **and**
 $trail\text{-}remove\text{-}cls_{NOT}[simp]: \langle trail\ (remove\text{-}cls_{NOT}\ C\ st) = trail\ st \rangle$ **and**

 $clauses\text{-}prepend\text{-}trail[simp]:$
 $\langle clauses_{NOT}\ (prepend\text{-}trail\ L\ st) = clauses_{NOT}\ st \rangle$
and
 $clauses\text{-}tl\text{-}trail[simp]: \langle clauses_{NOT}\ (tl\text{-}trail\ st) = clauses_{NOT}\ st \rangle$ **and**
 $clauses\text{-}add\text{-}cls_{NOT}[simp]:$
 $\langle clauses_{NOT}\ (add\text{-}cls_{NOT}\ C\ st) = add\text{-}mset\ C\ (clauses_{NOT}\ st) \rangle$ **and**
 $clauses\text{-}remove\text{-}cls_{NOT}[simp]:$
 $\langle clauses_{NOT}\ (remove\text{-}cls_{NOT}\ C\ st) = removeAll\text{-}mset\ C\ (clauses_{NOT}\ st) \rangle$
 $\langle proof \rangle$

We define the following function doing the backtrack in the trail:

function $reduce\text{-}trail\text{-}to_{NOT} :: \langle 'a \text{ list} \Rightarrow 'st \Rightarrow 'st \rangle$ **where**
 $\langle reduce\text{-}trail\text{-}to_{NOT}\ F\ S =$
 $(if\ length\ (trail\ S) = length\ F \vee trail\ S = []\ then\ S\ else\ reduce\text{-}trail\text{-}to_{NOT}\ F\ (tl\text{-}trail\ S)) \rangle$
 $\langle proof \rangle$
termination $\langle proof \rangle$

declare $reduce\text{-}trail\text{-}to_{NOT}.simps[simp\ del]$

Then we need several lemmas about the $reduce\text{-}trail\text{-}to_{NOT}$.

lemma

shows

$\text{reduce-trail-to}_{NOT}\text{-Nil}[simp]: \langle \text{trail } S = [] \implies \text{reduce-trail-to}_{NOT} F S = S \rangle$ and
 $\text{reduce-trail-to}_{NOT}\text{-eq-length}[simp]: \langle \text{length } (\text{trail } S) = \text{length } F \implies \text{reduce-trail-to}_{NOT} F S = S \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{reduce-trail-to}_{NOT}\text{-length-ne}[simp]:$

$\langle \text{length } (\text{trail } S) \neq \text{length } F \implies \text{trail } S \neq [] \implies$
 $\text{reduce-trail-to}_{NOT} F S = \text{reduce-trail-to}_{NOT} F (\text{tl-trail } S) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{trail-reduce-trail-to}_{NOT}\text{-length-le}:$

assumes $\langle \text{length } F > \text{length } (\text{trail } S) \rangle$

shows $\langle \text{trail } (\text{reduce-trail-to}_{NOT} F S) = [] \rangle$

$\langle \text{proof} \rangle$

lemma $\text{trail-reduce-trail-to}_{NOT}\text{-Nil}[simp]:$

$\langle \text{trail } (\text{reduce-trail-to}_{NOT} [] S) = [] \rangle$

$\langle \text{proof} \rangle$

lemma $\text{clauses-reduce-trail-to}_{NOT}\text{-Nil}:$

$\langle \text{clauses}_{NOT} (\text{reduce-trail-to}_{NOT} [] S) = \text{clauses}_{NOT} S \rangle$

$\langle \text{proof} \rangle$

lemma $\text{trail-reduce-trail-to}_{NOT}\text{-drop}:$

$\langle \text{trail } (\text{reduce-trail-to}_{NOT} F S) =$
 $(\text{if } \text{length } (\text{trail } S) \geq \text{length } F$
 $\text{then } \text{drop } (\text{length } (\text{trail } S) - \text{length } F) (\text{trail } S)$
 $\text{else } []) \rangle$

$\langle \text{proof} \rangle$

lemma $\text{reduce-trail-to}_{NOT}\text{-skip-beginning}:$

assumes $\langle \text{trail } S = F' @ F \rangle$

shows $\langle \text{trail } (\text{reduce-trail-to}_{NOT} F S) = F \rangle$

$\langle \text{proof} \rangle$

lemma $\text{reduce-trail-to}_{NOT}\text{-clauses}[simp]:$

$\langle \text{clauses}_{NOT} (\text{reduce-trail-to}_{NOT} F S) = \text{clauses}_{NOT} S \rangle$

$\langle \text{proof} \rangle$

lemma $\text{trail-eq-reduce-trail-to}_{NOT}\text{-eq}:$

$\langle \text{trail } S = \text{trail } T \implies \text{trail } (\text{reduce-trail-to}_{NOT} F S) = \text{trail } (\text{reduce-trail-to}_{NOT} F T) \rangle$

$\langle \text{proof} \rangle$

lemma $\text{trail-reduce-trail-to}_{NOT}\text{-add-cl}_{NOT}[simp]:$

$\langle \text{no-dup } (\text{trail } S) \implies$

$\text{trail } (\text{reduce-trail-to}_{NOT} F (\text{add-cl}_{NOT} C S)) = \text{trail } (\text{reduce-trail-to}_{NOT} F S) \rangle$

$\langle \text{proof} \rangle$

lemma $\text{reduce-trail-to}_{NOT}\text{-trail-tl-trail-decomp}[simp]:$

$\langle \text{trail } S = F' @ \text{Decided } K \# F \implies$

$\text{trail } (\text{reduce-trail-to}_{NOT} F (\text{tl-trail } S)) = F \rangle$

$\langle \text{proof} \rangle$

lemma $\text{reduce-trail-to}_{NOT}\text{-length}:$

$\langle \text{length } M = \text{length } M' \implies \text{reduce-trail-to}_{NOT} M S = \text{reduce-trail-to}_{NOT} M' S \rangle$

$\langle \text{proof} \rangle$

abbreviation *trail-weight* **where**

$\langle \text{trail-weight } S \equiv \text{map } ((\lambda l. 1 + \text{length } l) \circ \text{snd}) (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$

As we are defining abstract states, the Isabelle equality about them is too strong: we want the weaker equivalence stating that two states are equal if they cannot be distinguished, i.e. given the getter *trail* and *clauses_{NOT}* do not distinguish them.

definition *state-eq_{NOT}* :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ (**infix** ~ 50) **where**

$\langle S \sim T \iff \text{trail } S = \text{trail } T \wedge \text{clauses}_{\text{NOT}} S = \text{clauses}_{\text{NOT}} T \rangle$

lemma *state-eq_{NOT}-ref*[*intro*, *simp*]:

$\langle S \sim S \rangle$

$\langle \text{proof} \rangle$

lemma *state-eq_{NOT}-sym*:

$\langle S \sim T \iff T \sim S \rangle$

$\langle \text{proof} \rangle$

lemma *state-eq_{NOT}-trans*:

$\langle S \sim T \implies T \sim U \implies S \sim U \rangle$

$\langle \text{proof} \rangle$

lemma

shows

state-eq_{NOT}-trail: $\langle S \sim T \implies \text{trail } S = \text{trail } T \rangle$ **and**

state-eq_{NOT}-clauses: $\langle S \sim T \implies \text{clauses}_{\text{NOT}} S = \text{clauses}_{\text{NOT}} T \rangle$

$\langle \text{proof} \rangle$

lemmas *state-simp_{NOT}*[*simp*] = *state-eq_{NOT}-trail* *state-eq_{NOT}-clauses*

lemma *reduce-trail-to_{NOT}-state-eq_{NOT}-compatible*:

assumes *ST*: $\langle S \sim T \rangle$

shows $\langle \text{reduce-trail-to}_{\text{NOT}} F S \sim \text{reduce-trail-to}_{\text{NOT}} F T \rangle$

$\langle \text{proof} \rangle$

end — End on locale *dpll-state*.

Definition of the Transitions

Each possible is in its own locale.

locale *propagate-ops* =

dpll-state *trail* *clauses_{NOT}* *prepend-trail* *tl-trail* *add-cl_s_{NOT}* *remove-cl_s_{NOT}*

for

trail :: $\langle 'st \Rightarrow ('v, \text{unit}) \text{ ann-lits} \rangle$ **and**

clauses_{NOT} :: $\langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**

prepend-trail :: $\langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

tl-trail :: $\langle 'st \Rightarrow 'st \rangle$ **and**

add-cl_s_{NOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

remove-cl_s_{NOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ +

fixes

propagate-conds :: $\langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$

begin

inductive *propagate_{NOT}* :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **where**

propagate_{NOT}[*intro*]: $\langle \text{add-mset } L \ C \in \# \text{ clauses}_{\text{NOT}} S \implies \text{trail } S \models_{\text{as}} C \text{Not } C \rangle$


```

     $\Rightarrow$  undefined-lit (trail S) L
     $\Rightarrow$  propagate-conds (Propagated L ()) S T
     $\Rightarrow$  T  $\sim$  prepend-trail (Propagated L ()) S
     $\Rightarrow$  propagateNOT S T
inductive-cases propagateNOTE[elim]:  $\langle$ propagateNOT S T $\rangle$ 

end

locale decide-ops =
  dpll-state trail clausesNOT prepend-trail tl-trail add-clsNOT remove-clsNOT
for
  trail ::  $\langle$ 'st  $\Rightarrow$  ('v, unit) ann-lits $\rangle$  and
  clausesNOT ::  $\langle$ 'st  $\Rightarrow$  'v clauses $\rangle$  and
  prepend-trail ::  $\langle$ ('v, unit) ann-lit  $\Rightarrow$  'st  $\Rightarrow$  'st $\rangle$  and
  tl-trail ::  $\langle$ 'st  $\Rightarrow$  'st $\rangle$  and
  add-clsNOT ::  $\langle$ 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st $\rangle$  and
  remove-clsNOT ::  $\langle$ 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st +
fixes
  decide-conds ::  $\langle$ 'st  $\Rightarrow$  'st  $\Rightarrow$  bool $\rangle$ 
begin
inductive decideNOT ::  $\langle$ 'st  $\Rightarrow$  'st  $\Rightarrow$  bool $\rangle$  where
decideNOT[intro]:
   $\langle$ undefined-lit (trail S) L  $\Rightarrow$ 
  atm-of L  $\in$  atms-of-mm (clausesNOT S)  $\Rightarrow$ 
  T  $\sim$  prepend-trail (Decided L) S  $\Rightarrow$ 
  decide-conds S T  $\Rightarrow$ 
  decideNOT S T $\rangle$ 

inductive-cases decideNOTE[elim]:  $\langle$ decideNOT S S $\rangle$ 
end

locale backjumping-ops =
  dpll-state trail clausesNOT prepend-trail tl-trail add-clsNOT remove-clsNOT
for
  trail ::  $\langle$ 'st  $\Rightarrow$  ('v, unit) ann-lits $\rangle$  and
  clausesNOT ::  $\langle$ 'st  $\Rightarrow$  'v clauses $\rangle$  and
  prepend-trail ::  $\langle$ ('v, unit) ann-lit  $\Rightarrow$  'st  $\Rightarrow$  'st $\rangle$  and
  tl-trail ::  $\langle$ 'st  $\Rightarrow$  'st $\rangle$  and
  add-clsNOT ::  $\langle$ 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st $\rangle$  and
  remove-clsNOT ::  $\langle$ 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st +
fixes
  backjump-conds ::  $\langle$ 'v clause  $\Rightarrow$  'v clause  $\Rightarrow$  'v literal  $\Rightarrow$  'st  $\Rightarrow$  'st  $\Rightarrow$  bool $\rangle$ 
begin

inductive backjump where
 $\langle$ trail S = F' @ Decided K # F
 $\Rightarrow$  T  $\sim$  prepend-trail (Propagated L ()) (reduce-trail-toNOT F S)
 $\Rightarrow$  C  $\in$  # clausesNOT S
 $\Rightarrow$  trail S  $\models_{as}$  CNot C
 $\Rightarrow$  undefined-lit F L
 $\Rightarrow$  atm-of L  $\in$  atms-of-mm (clausesNOT S)  $\cup$  atm-of ' (lits-of-l (trail S))
 $\Rightarrow$  clausesNOT S  $\models_{pm}$  add-mset L C'
 $\Rightarrow$  F  $\models_{as}$  CNot C'
 $\Rightarrow$  backjump-conds C C' L S T
 $\Rightarrow$  backjump S T $\rangle$ 
inductive-cases backjumpE:  $\langle$ backjump S T $\rangle$ 

```

The condition $atm\text{-}of\ L \in atm\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \cup atm\text{-}of\ ' \textit{lits-of-l}\ (trail\ S)$ is not implied by the condition $clauses_{NOT}\ S \models_{pm} add\text{-}mset\ L\ C'$ (no negation).

end

2.2.3 DPLL with Backjumping

locale *dpll-with-backjumping-ops* =

propagate-ops *trail* *clauses*_{NOT} *prepend-trail* *tl-trail* *add-cl*_{NOT} *remove-cl*_{NOT} *propagate-con*_{ds} +
decide-ops *trail* *clauses*_{NOT} *prepend-trail* *tl-trail* *add-cl*_{NOT} *remove-cl*_{NOT} *decide-con*_{ds} +
backjumping-ops *trail* *clauses*_{NOT} *prepend-trail* *tl-trail* *add-cl*_{NOT} *remove-cl*_{NOT} *backjump-con*_{ds}

for

trail :: $\langle 'st \Rightarrow ('v, unit)\ ann\text{-}lits \rangle$ **and**
*clauses*_{NOT} :: $\langle 'st \Rightarrow 'v\ clauses \rangle$ **and**
prepend-trail :: $\langle ('v, unit)\ ann\text{-}lit \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
tl-trail :: $\langle 'st \Rightarrow 'st \rangle$ **and**
*add-cl*_{NOT} :: $\langle 'v\ clause \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
*remove-cl*_{NOT} :: $\langle 'v\ clause \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
inv :: $\langle 'st \Rightarrow bool \rangle$ **and**
*decide-con*_{ds} :: $\langle 'st \Rightarrow 'st \Rightarrow bool \rangle$ **and**
*backjump-con*_{ds} :: $\langle 'v\ clause \Rightarrow 'v\ clause \Rightarrow 'v\ literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool \rangle$ **and**
*propagate-con*_{ds} :: $\langle ('v, unit)\ ann\text{-}lit \Rightarrow 'st \Rightarrow 'st \Rightarrow bool \rangle$ +

assumes

bj-can-jump:

$\langle \bigwedge S\ C\ F'\ K\ F\ L.$
 $inv\ S \Rightarrow$
 $trail\ S = F' @ Decided\ K \# F \Rightarrow$
 $C \in \# clauses_{NOT}\ S \Rightarrow$
 $trail\ S \models_{as}\ CNot\ C \Rightarrow$
 $undefined\text{-}lit\ F\ L \Rightarrow$
 $atm\text{-}of\ L \in atm\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \cup atm\text{-}of\ ' \textit{lits-of-l}\ (F' @ Decided\ K \# F) \Rightarrow$
 $clauses_{NOT}\ S \models_{pm}\ add\text{-}mset\ L\ C' \Rightarrow$
 $F \models_{as}\ CNot\ C' \Rightarrow$
 $\neg no\text{-}step\ backjump\ S \rangle$ **and**

can-propagate-or-decide-or-backjump:

$\langle atm\text{-}of\ L \in atm\text{-}of\text{-}mm\ (clauses_{NOT}\ S) \Rightarrow$
 $undefined\text{-}lit\ (trail\ S)\ L \Rightarrow$
 $satisfiable\ (set\text{-}mset\ (clauses_{NOT}\ S)) \Rightarrow$
 $inv\ S \Rightarrow$
 $no\text{-}dup\ (trail\ S) \Rightarrow$
 $\exists T. decide_{NOT}\ S\ T \vee propagate_{NOT}\ S\ T \vee backjump\ S\ T \rangle$

begin

We cannot add a like condition $atms\text{-}of\ C' \subseteq atm\text{-}of\text{-}ms\ N$ to ensure that we can backjump even if the last decision variable has disappeared from the set of clauses.

The part of the condition $atm\text{-}of\ L \in atm\text{-}of\ ' \textit{lits-of-l}\ (F' @ Decided\ K \# F)$ is important, otherwise you are not sure that you can backtrack.

Definition

We define *dpll* with backjumping:

inductive *dpll-bj* :: $\langle 'st \Rightarrow 'st \Rightarrow bool \rangle$ **for** *S* :: $'st$ **where**

*bj-decide*_{NOT}: $\langle decide_{NOT}\ S\ S' \Rightarrow dpll\text{-}bj\ S\ S' \rangle$ |

*bj-propagate*_{NOT}: $\langle propagate_{NOT}\ S\ S' \Rightarrow dpll\text{-}bj\ S\ S' \rangle$ |

bj-backjump: $\langle backjump\ S\ S' \Rightarrow dpll\text{-}bj\ S\ S' \rangle$

lemmas *dpll-bj-induct* = *dpll-bj.induct*[*split-format*(*complete*)]
thm *dpll-bj-induct*[*OF dpll-with-backjumping-ops-axioms*]
lemma *dpll-bj-all-induct*[*consumes 2, case-names decide_{NOT} propagate_{NOT} backjump*]:
fixes *S T* :: *<'st>*
assumes
<dpll-bj S T> **and**
<inv S>
<∧ L T. undefined-lit (trail S) L ⇒ atm-of L ∈ atms-of-mm (clauses_{NOT} S)
⇒ T ∼ prepend-trail (Decided L) S
⇒ P S T> **and**
<∧ C L T. add-mset L C ∈ # clauses_{NOT} S ⇒ trail S ⊨_{as} CNot C ⇒ undefined-lit (trail S) L
⇒ T ∼ prepend-trail (Propagated L ()) S
⇒ P S T> **and**
<∧ C F' K F L C' T. C ∈ # clauses_{NOT} S ⇒ F' @ Decided K # F ⊨_{as} CNot C
⇒ trail S = F' @ Decided K # F
⇒ undefined-lit F L
⇒ atm-of L ∈ atms-of-mm (clauses_{NOT} S) ∪ atm-of ' (lits-of-l (F' @ Decided K # F))
⇒ clauses_{NOT} S ⊨_{pm} add-mset L C'
⇒ F ⊨_{as} CNot C'
⇒ T ∼ prepend-trail (Propagated L ()) (reduce-trail-to_{NOT} F S)
⇒ P S T>
shows *<P S T>*
<proof>

Basic properties

First, some better suited induction principle **lemma** *dpll-bj-clauses*:

assumes *<dpll-bj S T>* **and** *<inv S>*
shows *<clauses_{NOT} S = clauses_{NOT} T>*
<proof>

No duplicates in the trail **lemma** *dpll-bj-no-dup*:

assumes *<dpll-bj S T>* **and** *<inv S>*
and *<no-dup (trail S)>*
shows *<no-dup (trail T)>*
<proof>

Valuations **lemma** *dpll-bj-sat-iff*:

assumes *<dpll-bj S T>* **and** *<inv S>*
shows *<I ⊨_{sm} clauses_{NOT} S ⇔ I ⊨_{sm} clauses_{NOT} T>*
<proof>

Clauses **lemma** *dpll-bj-atms-of-ms-clauses-inv*:

assumes
<dpll-bj S T> **and**
<inv S>
shows *<atms-of-mm (clauses_{NOT} S) = atms-of-mm (clauses_{NOT} T)>*
<proof>

lemma *dpll-bj-atms-in-trail*:

assumes
<dpll-bj S T> **and**
<inv S> **and**
<atm-of ' (lits-of-l (trail S)) ⊆ atms-of-mm (clauses_{NOT} S)>

shows $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S) \rangle$
 $\langle \text{proof} \rangle$

lemma *dpll-bj-atms-in-trail-in-set:*

assumes $\langle \text{dpll-bj } S \ T \rangle$ **and**
 $\langle \text{inv } S \rangle$ **and**
 $\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**
 $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$
shows $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq A \rangle$
 $\langle \text{proof} \rangle$

lemma *dpll-bj-all-decomposition-implies-inv:*

assumes
 $\langle \text{dpll-bj } S \ T \rangle$ **and**
 $\text{inv: } \langle \text{inv } S \rangle$ **and**
 $\text{decomp: } \langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} S) (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$
shows $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} T) (\text{get-all-ann-decomposition } (\text{trail } T)) \rangle$
 $\langle \text{proof} \rangle$

Termination

Using a proper measure lemma *length-get-all-ann-decomposition-append-Decided:*

$\langle \text{length } (\text{get-all-ann-decomposition } (F' @ \text{Decided } K \# F)) =$
 $\text{length } (\text{get-all-ann-decomposition } F')$
 $+ \text{length } (\text{get-all-ann-decomposition } (\text{Decided } K \# F))$
 $- 1 \rangle$
 $\langle \text{proof} \rangle$

lemma *take-length-get-all-ann-decomposition-decided-sandwich:*

$\langle \text{take } (\text{length } (\text{get-all-ann-decomposition } F))$
 $(\text{map } (f \circ \text{snd}) (\text{rev } (\text{get-all-ann-decomposition } (F' @ \text{Decided } K \# F))))$
 $=$
 $\text{map } (f \circ \text{snd}) (\text{rev } (\text{get-all-ann-decomposition } F))$
 \rangle
 $\langle \text{proof} \rangle$

lemma *length-get-all-ann-decomposition-length:*

$\langle \text{length } (\text{get-all-ann-decomposition } M) \leq 1 + \text{length } M \rangle$
 $\langle \text{proof} \rangle$

lemma *length-in-get-all-ann-decomposition-bounded:*

assumes $i: \langle i \in \text{set } (\text{trail-weight } S) \rangle$
shows $\langle i \leq \text{Suc } (\text{length } (\text{trail } S)) \rangle$
 $\langle \text{proof} \rangle$

Well-foundedness The bounds are the following:

- $1 + \text{card } (\text{atms-of-ms } A)$: $\text{card } (\text{atms-of-ms } A)$ is an upper bound on the length of the list. As *get-all-ann-decomposition* appends an possibly empty couple at the end, adding one is needed.
- $2 + \text{card } (\text{atms-of-ms } A)$: $\text{card } (\text{atms-of-ms } A)$ is an upper bound on the number of elements, where adding one is necessary for the same reason as for the bound on the list, and one is needed to have a strict bound.

abbreviation *unassigned-lit* :: $\langle 'b \text{ clause set} \Rightarrow 'a \text{ list} \Rightarrow \text{nat} \rangle$ **where**
 $\langle \text{unassigned-lit } N \ M \equiv \text{card } (\text{atms-of-ms } N) - \text{length } M \rangle$

lemma *dpll-bj-trail-mes-increasing-prop*:

fixes $M :: \langle ('v, \text{unit}) \text{ ann-lits} \rangle$ **and** $N :: \langle 'v \text{ clauses} \rangle$

assumes

$\langle \text{dpll-bj } S \ T \rangle$ **and**

$\langle \text{inv } S \rangle$ **and**

$N\text{-}A: \langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**

$M\text{-}A: \langle \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**

$n\text{-}d: \langle \text{no-dup } (\text{trail } S) \rangle$ **and**

$\text{finite}: \langle \text{finite } A \rangle$

shows $\mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$
 $> \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S)$

$\langle \text{proof} \rangle$

lemma *dpll-bj-trail-mes-decreasing-prop*:

assumes *dpll*: $\langle \text{dpll-bj } S \ T \rangle$ **and** *inv*: $\langle \text{inv } S \rangle$ **and**

$N\text{-}A: \langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**

$M\text{-}A: \langle \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**

$nd: \langle \text{no-dup } (\text{trail } S) \rangle$ **and**

$\text{fin}\text{-}A: \langle \text{finite } A \rangle$

shows $\langle (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$
 $< (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$
 $- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S) \rangle$

$\langle \text{proof} \rangle$

lemma *wf-dpll-bj*:

assumes *fin*: $\langle \text{finite } A \rangle$

shows $\langle \text{wf } \{(T, S). \text{dpll-bj } S \ T$

$\wedge \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$

$\wedge \text{no-dup } (\text{trail } S) \wedge \text{inv } S \}$

$\langle \text{is } \langle \text{wf } ?A \rangle \rangle$

$\langle \text{proof} \rangle$

Alternative termination proof **abbreviation** *DPLL-mes_W* **where**

$\langle \text{DPLL-mes}_W \ A \ M \equiv$

$\text{map } (\lambda L. \text{if is-decided } L \text{ then } 2::\text{nat} \text{ else } 1) (\text{rev } M) \ @ \ \text{replicate } (\text{card } A - \text{length } M) \ 3 \rangle$

lemma *distinctcard-atm-of-lit-of-eq-length*:

assumes *no-dup* S

shows $\text{card } (\text{atm-of } ' \text{ lits-of-l } S) = \text{length } S$

$\langle \text{proof} \rangle$

lemma *dpll-bj-trail-mes-decreasing-less-than*:

assumes *dpll*: $\langle \text{dpll-bj } S \ T \rangle$ **and** *inv*: $\langle \text{inv } S \rangle$ **and**

$N\text{-}A: \langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**

$M\text{-}A: \langle \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**

$nd: \langle \text{no-dup } (\text{trail } S) \rangle$ **and**

$\text{fin}\text{-}A: \langle \text{finite } A \rangle$

shows $\langle (\text{DPLL-mes}_W (\text{atms-of-ms } A) (\text{trail } T), \text{DPLL-mes}_W (\text{atms-of-ms } A) (\text{trail } S)) \in$
 $\text{lexn less-than } (\text{card } ((\text{atms-of-ms } A))) \rangle$

$\langle \text{proof} \rangle$

lemma

assumes $\text{fin}[\text{simp}] : \langle \text{finite } A \rangle$
shows $\langle \text{wf } \{(T, S). \text{dpll-bj } S \text{ } T$
 $\wedge \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } \text{' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{no-dup } (\text{trail } S) \wedge \text{inv } S \rangle$
 $(\text{is } \langle \text{wf } ?A \rangle)$
 $\langle \text{proof} \rangle$

Normal Forms

We prove that given a normal form of DPLL, with some structural invariants, then either N is satisfiable and the built valuation M is a model; or N is unsatisfiable.

Idea of the proof: We have to prove that $\text{satisfiable } N, \neg M \models_{as} N$ and there is no remaining step is incompatible.

1. The *decide* rule tells us that every variable in N has a value.
2. The assumption $\neg M \models_{as} N$ implies that there is conflict.
3. There is at least one decision in the trail (otherwise, M would be a model of the set of clauses N).
4. Now if we build the clause with all the decision literals of the trail, we can apply the *backjump* rule.

The assumption are saying that we have a finite upper bound A for the literals, that we cannot do any step $\forall S'. \neg \text{dpll-bj } S S'$

theorem *dpll-backjump-final-state:*

fixes $A :: \langle 'v \text{ clause set} \rangle$ **and** $S \text{ } T :: \langle 'st \rangle$

assumes

$\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{atm-of } \text{' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**
 $\langle \text{finite } A \rangle$ **and**
 $\text{inv} : \langle \text{inv } S \rangle$ **and**
 $\text{n-d} : \langle \text{no-dup } (\text{trail } S) \rangle$ **and**
 $\text{n-s} : \langle \text{no-step dpll-bj } S \rangle$ **and**
 $\text{decomp} : \langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} S) (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$

shows $\langle \text{unsatisfiable } (\text{set-mset } (\text{clauses}_{NOT} S))$

$\vee (\text{trail } S \models_{asm} \text{clauses}_{NOT} S \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} S))) \rangle$

$\langle \text{proof} \rangle$

end — End of the locale *dpll-with-backjumping-ops*.

locale *dpll-with-backjumping* =

dpll-with-backjumping-ops $\text{trail clauses}_{NOT} \text{prepend-trail tl-trail add-cl}_s_{NOT} \text{remove-cl}_s_{NOT} \text{inv}$
 $\text{decide-conds backjump-conds propagate-conds}$

for

$\text{trail} :: \langle 'st \Rightarrow ('v, \text{unit}) \text{ann-lits} \rangle$ **and**
 $\text{clauses}_{NOT} :: \langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**
 $\text{prepend-trail} :: \langle ('v, \text{unit}) \text{ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
 $\text{tl-trail} :: \langle 'st \Rightarrow 'st \rangle$ **and**
 $\text{add-cl}_s_{NOT} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
 $\text{remove-cl}_s_{NOT} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
 $\text{inv} :: \langle 'st \Rightarrow \text{bool} \rangle$ **and**

$decide-conds :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$ **and**
 $backjump-conds :: \langle 'v \text{ clause} \Rightarrow 'v \text{ clause} \Rightarrow 'v \text{ literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow bool \rangle$ **and**
 $propagate-conds :: \langle ('v, unit) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \Rightarrow bool \rangle$
 $+$
assumes $dpll-bj-inv: \langle \bigwedge S \ T. \ dpll-bj \ S \ T \Longrightarrow inv \ S \Longrightarrow inv \ T \rangle$
begin

lemma $rtranclp-dpll-bj-inv$:
assumes $\langle dpll-bj^{**} \ S \ T \rangle$ **and** $\langle inv \ S \rangle$
shows $\langle inv \ T \rangle$
 $\langle proof \rangle$

lemma $rtranclp-dpll-bj-no-dup$:
assumes $\langle dpll-bj^{**} \ S \ T \rangle$ **and** $\langle inv \ S \rangle$
and $\langle no-dup \ (trail \ S) \rangle$
shows $\langle no-dup \ (trail \ T) \rangle$
 $\langle proof \rangle$

lemma $rtranclp-dpll-bj-atms-of-ms-clauses-inv$:
assumes
 $\langle dpll-bj^{**} \ S \ T \rangle$ **and** $\langle inv \ S \rangle$
shows $\langle atms-of-mm \ (clauses_{NOT} \ S) = atms-of-mm \ (clauses_{NOT} \ T) \rangle$
 $\langle proof \rangle$

lemma $rtranclp-dpll-bj-atms-in-trail$:
assumes
 $\langle dpll-bj^{**} \ S \ T \rangle$ **and**
 $\langle inv \ S \rangle$ **and**
 $\langle atm-of \ ' \ (lits-of-l \ (trail \ S)) \subseteq atms-of-mm \ (clauses_{NOT} \ S) \rangle$
shows $\langle atm-of \ ' \ (lits-of-l \ (trail \ T)) \subseteq atms-of-mm \ (clauses_{NOT} \ T) \rangle$
 $\langle proof \rangle$

lemma $rtranclp-dpll-bj-sat-iff$:
assumes $\langle dpll-bj^{**} \ S \ T \rangle$ **and** $\langle inv \ S \rangle$
shows $\langle I \models_{sm} clauses_{NOT} \ S \longleftrightarrow I \models_{sm} clauses_{NOT} \ T \rangle$
 $\langle proof \rangle$

lemma $rtranclp-dpll-bj-atms-in-trail-in-set$:
assumes
 $\langle dpll-bj^{**} \ S \ T \rangle$ **and**
 $\langle inv \ S \rangle$
 $\langle atms-of-mm \ (clauses_{NOT} \ S) \subseteq A \rangle$ **and**
 $\langle atm-of \ ' \ (lits-of-l \ (trail \ S)) \subseteq A \rangle$
shows $\langle atm-of \ ' \ (lits-of-l \ (trail \ T)) \subseteq A \rangle$
 $\langle proof \rangle$

lemma $rtranclp-dpll-bj-all-decomposition-implies-inv$:
assumes
 $\langle dpll-bj^{**} \ S \ T \rangle$ **and**
 $\langle inv \ S \rangle$
 $\langle all-decomposition-implies-m \ (clauses_{NOT} \ S) \ (get-all-ann-decomposition \ (trail \ S)) \rangle$
shows $\langle all-decomposition-implies-m \ (clauses_{NOT} \ T) \ (get-all-ann-decomposition \ (trail \ T)) \rangle$
 $\langle proof \rangle$

lemma $rtranclp-dpll-bj-inv-incl-dpll-bj-inv-trancl$:
 $\langle \{(T, S). \ dpll-bj^{++} \ S \ T$

$\wedge \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{no-dup } (\text{trail } S) \wedge \text{inv } S\}$
 $\subseteq \{(T, S). \text{dpll-bj } S T \wedge \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \wedge \text{no-dup } (\text{trail } S) \wedge \text{inv } S\}^+\}$
(is $\langle ?A \subseteq ?B^+ \rangle$)
 $\langle \text{proof} \rangle$

lemma *wf-tranclp-dpll-bj*:

assumes *fin*: $\langle \text{finite } A \rangle$

shows $\langle \text{wf } \{(T, S). \text{dpll-bj}^{++} S T$

$\wedge \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$

$\wedge \text{no-dup } (\text{trail } S) \wedge \text{inv } S\}$

\rangle
 $\langle \text{proof} \rangle$

lemma *dpll-bj-sat-ext-iff*:

$\langle \text{dpll-bj } S T \implies \text{inv } S \implies I \models_{\text{sextm}} \text{clauses}_{NOT} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} T \rangle$

$\langle \text{proof} \rangle$

lemma *rtranclp-dpll-bj-sat-ext-iff*:

$\langle \text{dpll-bj}^{**} S T \implies \text{inv } S \implies I \models_{\text{sextm}} \text{clauses}_{NOT} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} T \rangle$

$\langle \text{proof} \rangle$

theorem *full-dpll-backjump-final-state*:

fixes *A* :: $\langle 'v \text{ clause set} \rangle$ **and** *S T* :: $\langle 'st \rangle$

assumes

full: $\langle \text{full dpll-bj } S T \rangle$ **and**

atms-S: $\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**

atms-trail: $\langle \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**

n-d: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**

$\langle \text{finite } A \rangle$ **and**

inv: $\langle \text{inv } S \rangle$ **and**

decomp: $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} S) (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$

shows $\langle \text{unsatisfiable } (\text{set-mset } (\text{clauses}_{NOT} S))$

$\vee (\text{trail } T \models_{\text{asm}} \text{clauses}_{NOT} S \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} S))) \rangle$

$\langle \text{proof} \rangle$

corollary *full-dpll-backjump-final-state-from-init-state*:

fixes *A* :: $\langle 'v \text{ clause set} \rangle$ **and** *S T* :: $\langle 'st \rangle$

assumes

full: $\langle \text{full dpll-bj } S T \rangle$ **and**

$\langle \text{trail } S = [] \rangle$ **and**

$\langle \text{clauses}_{NOT} S = N \rangle$ **and**

$\langle \text{inv } S \rangle$

shows $\langle \text{unsatisfiable } (\text{set-mset } N) \vee (\text{trail } T \models_{\text{asm}} N \wedge \text{satisfiable } (\text{set-mset } N)) \rangle$

$\langle \text{proof} \rangle$

lemma *tranclp-dpll-bj-trail-mes-decreasing-prop*:

assumes *dpll*: $\langle \text{dpll-bj}^{++} S T \rangle$ **and** *inv*: $\langle \text{inv } S \rangle$ **and**

N-A: $\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**

M-A: $\langle \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**

n-d: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**

fin-A: $\langle \text{finite } A \rangle$

shows $\langle (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$

$- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } T)$

$< (2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))$

$- \mu_C (1 + \text{card } (\text{atms-of-ms } A)) (2 + \text{card } (\text{atms-of-ms } A)) (\text{trail-weight } S) \rangle$

$\langle proof \rangle$

end — End of the locale *dpll-with-backjumping*.

2.2.4 CDCL

In this section we will now define the conflict driven clause learning above DPLL: we first introduce the rules learn and forget, and then add these rules to the DPLL calculus.

Learn and Forget

Learning adds a new clause where all the literals are already included in the clauses.

```

locale learn-ops =
  dpll-state trail clausesNOT prepend-trail tl-trail add-clNOT remove-clNOT
for
  trail ::  $\langle 'st \Rightarrow ('v, unit) \text{ ann-lits} \rangle$  and
  clausesNOT ::  $\langle 'st \Rightarrow 'v \text{ clauses} \rangle$  and
  prepend-trail ::  $\langle ('v, unit) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$  and
  tl-trail ::  $\langle 'st \Rightarrow 'st \rangle$  and
  add-clNOT ::  $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$  and
  remove-clNOT ::  $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle +$ 
fixes
  learn-conds ::  $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow bool \rangle$ 
begin

inductive learn ::  $\langle 'st \Rightarrow 'st \Rightarrow bool \rangle$  where
  learnNOT-rule:  $\langle clauses_{NOT} \ S \models_{pm} C \Rightarrow$ 
    atms-of  $C \subseteq \text{atms-of-mm } (clauses_{NOT} \ S) \cup \text{atm-of } ' (lits-of-l (trail \ S)) \Rightarrow$ 
    learn-conds  $C \ S \Rightarrow$ 
     $T \sim \text{add-cl}_{NOT} \ C \ S \Rightarrow$ 
    learn  $S \ T \rangle$ 
inductive-cases learnNOTE:  $\langle learn \ S \ T \rangle$ 

lemma learn- $\mu_C$ -stable:
  assumes  $\langle learn \ S \ T \rangle$  and  $\langle no\text{-dup } (trail \ S) \rangle$ 
shows  $\langle \mu_C \ A \ B \ (trail\text{-weight } S) = \mu_C \ A \ B \ (trail\text{-weight } T) \rangle$ 
 $\langle proof \rangle$ 
end

```

Forget removes an information that can be deduced from the context (e.g. redundant clauses, tautologies)

```

locale forget-ops =
  dpll-state trail clausesNOT prepend-trail tl-trail add-clNOT remove-clNOT
for
  trail ::  $\langle 'st \Rightarrow ('v, unit) \text{ ann-lits} \rangle$  and
  clausesNOT ::  $\langle 'st \Rightarrow 'v \text{ clauses} \rangle$  and
  prepend-trail ::  $\langle ('v, unit) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$  and
  tl-trail ::  $\langle 'st \Rightarrow 'st \rangle$  and
  add-clNOT ::  $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$  and
  remove-clNOT ::  $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle +$ 
fixes
  forget-conds ::  $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow bool \rangle$ 
begin

```

inductive $\text{forget}_{NOT} :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$ **where**

forget_{NOT} :

$\langle \text{removeAll-mset } C(\text{clauses}_{NOT} S) \models_{pm} C \Rightarrow$

$\text{forget-conds } C S \Rightarrow$

$C \in \# \text{ clauses}_{NOT} S \Rightarrow$

$T \sim \text{remove-cl}_S C S \Rightarrow$

$\text{forget}_{NOT} S T \rangle$

inductive-cases $\text{forget}_{NOT} E$: $\langle \text{forget}_{NOT} S T \rangle$

lemma $\text{forget-}\mu_C\text{-stable}$:

assumes $\langle \text{forget}_{NOT} S T \rangle$

shows $\langle \mu_C A B (\text{trail-weight } S) = \mu_C A B (\text{trail-weight } T) \rangle$

$\langle \text{proof} \rangle$

end

locale $\text{learn-and-forget}_{NOT} =$

$\text{learn-ops trail clauses}_{NOT} \text{prepend-trail tl-trail add-cl}_S \text{remove-cl}_S \text{learn-conds} +$

$\text{forget-ops trail clauses}_{NOT} \text{prepend-trail tl-trail add-cl}_S \text{remove-cl}_S \text{forget-conds}$

for

$\text{trail} :: \langle 'st \Rightarrow ('v, \text{unit}) \text{ann-lits} \rangle$ **and**

$\text{clauses}_{NOT} :: \langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**

$\text{prepend-trail} :: \langle ('v, \text{unit}) \text{ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{tl-trail} :: \langle 'st \Rightarrow 'st \rangle$ **and**

$\text{add-cl}_S :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{remove-cl}_S :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{learn-conds forget-conds} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow bool \rangle$

begin

inductive $\text{learn-and-forget}_{NOT} :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$

where

$\text{lf-learn} :: \langle \text{learn } S T \Rightarrow \text{learn-and-forget}_{NOT} S T \rangle \mid$

$\text{lf-forget} :: \langle \text{forget}_{NOT} S T \Rightarrow \text{learn-and-forget}_{NOT} S T \rangle$

end

Definition of CDCL

locale $\text{conflict-driven-clause-learning-ops} =$

$\text{dpll-with-backjumping-ops trail clauses}_{NOT} \text{prepend-trail tl-trail add-cl}_S \text{remove-cl}_S$

$\text{inv decide-conds backjump-conds propagate-conds} +$

$\text{learn-and-forget}_{NOT} \text{trail clauses}_{NOT} \text{prepend-trail tl-trail add-cl}_S \text{remove-cl}_S \text{learn-conds}$

forget-conds

for

$\text{trail} :: \langle 'st \Rightarrow ('v, \text{unit}) \text{ann-lits} \rangle$ **and**

$\text{clauses}_{NOT} :: \langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**

$\text{prepend-trail} :: \langle ('v, \text{unit}) \text{ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{tl-trail} :: \langle 'st \Rightarrow 'st \rangle$ **and**

$\text{add-cl}_S :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{remove-cl}_S :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{inv} :: \langle 'st \Rightarrow bool \rangle$ **and**

$\text{decide-conds} :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$ **and**

$\text{backjump-conds} :: \langle 'v \text{ clause} \Rightarrow 'v \text{ clause} \Rightarrow 'v \text{ literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow bool \rangle$ **and**

$\text{propagate-conds} :: \langle ('v, \text{unit}) \text{ann-lit} \Rightarrow 'st \Rightarrow 'st \Rightarrow bool \rangle$ **and**

$\text{learn-conds forget-conds} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow bool \rangle$

begin

inductive $\text{cdcl}_{NOT} :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$ **for** $S :: 'st$ **where**

$c\text{-dpll-bj}: \langle \text{dpll-bj } S S' \Rightarrow \text{cdcl}_{NOT} S S' \rangle \mid$
 $c\text{-learn}: \langle \text{learn } S S' \Rightarrow \text{cdcl}_{NOT} S S' \rangle \mid$
 $c\text{-forget}_{NOT}: \langle \text{forget}_{NOT} S S' \Rightarrow \text{cdcl}_{NOT} S S' \rangle$

lemma $\text{cdcl}_{NOT}\text{-all-induct}[\text{consumes } 1, \text{ case-names dpll-bj learn forget}_{NOT}]$:

fixes $S T :: \langle 'st \rangle$

assumes $\langle \text{cdcl}_{NOT} S T \rangle$ **and**

$\text{dpll}: \langle \bigwedge T. \text{dpll-bj } S T \Rightarrow P S T \rangle$ **and**

learning:

$\langle \bigwedge C T. \text{clauses}_{NOT} S \models_{pm} C \Rightarrow$

$\text{atms-of } C \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \Rightarrow$

$T \sim \text{add-cl}_{NOT} C S \Rightarrow$

$P S T \rangle$ **and**

forgetting: $\langle \bigwedge C T. \text{removeAll-mset } C (\text{clauses}_{NOT} S) \models_{pm} C \Rightarrow$

$C \in \# \text{clauses}_{NOT} S \Rightarrow$

$T \sim \text{remove-cl}_{NOT} C S \Rightarrow$

$P S T \rangle$

shows $\langle P S T \rangle$

$\langle \text{proof} \rangle$

lemma $\text{cdcl}_{NOT}\text{-no-dup}$:

assumes

$\langle \text{cdcl}_{NOT} S T \rangle$ **and**

$\langle \text{inv } S \rangle$ **and**

$\langle \text{no-dup } (\text{trail } S) \rangle$

shows $\langle \text{no-dup } (\text{trail } T) \rangle$

$\langle \text{proof} \rangle$

Consistency of the trail **lemma** $\text{cdcl}_{NOT}\text{-consistent}$:

assumes

$\langle \text{cdcl}_{NOT} S T \rangle$ **and**

$\langle \text{inv } S \rangle$ **and**

$\langle \text{no-dup } (\text{trail } S) \rangle$

shows $\langle \text{consistent-interp } (\text{lits-of-l } (\text{trail } T)) \rangle$

$\langle \text{proof} \rangle$

The subtle problem here is that tautologies can be removed, meaning that some variable can disappear of the problem. It is also means that some variable of the trail might not be present in the clauses anymore.

lemma $\text{cdcl}_{NOT}\text{-atms-of-ms-clauses-decreasing}$:

assumes $\langle \text{cdcl}_{NOT} S T \rangle$ **and** $\langle \text{inv } S \rangle$

shows $\langle \text{atms-of-mm } (\text{clauses}_{NOT} T) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \rangle$

$\langle \text{proof} \rangle$

lemma $\text{cdcl}_{NOT}\text{-atms-in-trail}$:

assumes $\langle \text{cdcl}_{NOT} S T \rangle$ **and** $\langle \text{inv } S \rangle$

and $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S) \rangle$

shows $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S) \rangle$

$\langle \text{proof} \rangle$

lemma $\text{cdcl}_{NOT}\text{-atms-in-trail-in-set}$:

assumes

$\langle \text{cdcl}_{NOT} S T \rangle$ **and** $\langle \text{inv } S \rangle$ **and**

$\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

$\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$

shows $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq A \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-all-decomposition-implies*:

assumes $\langle \text{cdcl}_{NOT} S T \rangle$ **and** $\langle \text{inv } S \rangle$ **and**
 $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} S) (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$
shows
 $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} T) (\text{get-all-ann-decomposition } (\text{trail } T)) \rangle$
 $\langle \text{proof} \rangle$

Extension of models lemma *cdcl_{NOT}-bj-sat-ext-iff*:

assumes $\langle \text{cdcl}_{NOT} S T \rangle$ **and** $\langle \text{inv } S \rangle$
shows $\langle I \models_{\text{sextm}} \text{clauses}_{NOT} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} T \rangle$
 $\langle \text{proof} \rangle$

end — End of the locale *conflict-driven-clause-learning-ops*.

CDCL with invariant

locale *conflict-driven-clause-learning* =
 $\text{conflict-driven-clause-learning-ops} +$
assumes *cdcl_{NOT}-inv*: $\langle \bigwedge S T. \text{cdcl}_{NOT} S T \implies \text{inv } S \implies \text{inv } T \rangle$
begin
sublocale *dpll-with-backjumping*
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-inv*:

$\langle \text{cdcl}_{NOT}^{**} S T \implies \text{inv } S \implies \text{inv } T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-no-dup*:

assumes $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and** $\langle \text{inv } S \rangle$
and $\langle \text{no-dup } (\text{trail } S) \rangle$
shows $\langle \text{no-dup } (\text{trail } T) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-trail-clauses-bound*:

assumes
cdcl: $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and**
inv: $\langle \text{inv } S \rangle$ **and**
atms-clauses-S: $\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**
atms-trail-S: $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$
shows $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq A \wedge \text{atms-of-mm } (\text{clauses}_{NOT} T) \subseteq A \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-all-decomposition-implies*:

assumes $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and** $\langle \text{inv } S \rangle$ **and** $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**
 $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} S) (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$
shows
 $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} T) (\text{get-all-ann-decomposition } (\text{trail } T)) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-bj-sat-ext-iff*:

assumes $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and** $\langle \text{inv } S \rangle$
shows $\langle I \models_{\text{sextm}} \text{clauses}_{NOT} S \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} T \rangle$
 $\langle \text{proof} \rangle$

definition $cdcl_{NOT-}NOT-all-inv$ **where**

$\langle cdcl_{NOT-}NOT-all-inv A S \longleftrightarrow (finite A \wedge inv S \wedge atms-of-mm (clauses_{NOT} S) \subseteq atms-of-ms A \wedge atm-of \text{ ' } lits-of-l (trail S) \subseteq atms-of-ms A \wedge no-dup (trail S)) \rangle$

lemma $cdcl_{NOT-}NOT-all-inv$:

assumes $\langle cdcl_{NOT}^{**} S T \rangle$ **and** $\langle cdcl_{NOT-}NOT-all-inv A S \rangle$

shows $\langle cdcl_{NOT-}NOT-all-inv A T \rangle$

$\langle proof \rangle$

abbreviation $learn-or-forget$ **where**

$\langle learn-or-forget S T \equiv learn S T \vee forget_{NOT} S T \rangle$

lemma $rtranclp-learn-or-forget-cdcl_{NOT}$:

$\langle learn-or-forget^{**} S T \implies cdcl_{NOT}^{**} S T \rangle$

$\langle proof \rangle$

lemma $learn-or-forget-dpll-\mu_C$:

assumes

$l-f$: $\langle learn-or-forget^{**} S T \rangle$ **and**

$dpll$: $\langle dpll-bj T U \rangle$ **and**

inv : $\langle cdcl_{NOT-}NOT-all-inv A S \rangle$

shows $\langle (2 + card (atms-of-ms A)) \wedge (1 + card (atms-of-ms A))$
 $- \mu_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight U)$
 $< (2 + card (atms-of-ms A)) \wedge (1 + card (atms-of-ms A))$
 $- \mu_C (1 + card (atms-of-ms A)) (2 + card (atms-of-ms A)) (trail-weight S) \rangle$
(is $\langle ?\mu U < ?\mu S \rangle$)

$\langle proof \rangle$

lemma $infinite-cdcl_{NOT-}exists-learn-and-forget-infinite-chain$:

assumes

$\langle \bigwedge i. cdcl_{NOT} (f i) (f (Suc i)) \rangle$ **and**

inv : $\langle cdcl_{NOT-}NOT-all-inv A (f 0) \rangle$

shows $\langle \exists j. \forall i \geq j. learn-or-forget (f i) (f (Suc i)) \rangle$

$\langle proof \rangle$

lemma $wf-cdcl_{NOT-}no-learn-and-forget-infinite-chain$:

assumes

$no-infinite-lf$: $\langle \bigwedge j. \neg (\forall i \geq j. learn-or-forget (f i) (f (Suc i))) \rangle$

shows $\langle wf \{ (T, S). cdcl_{NOT} S T \wedge cdcl_{NOT-}NOT-all-inv A S \} \rangle$

(is $\langle wf \{ (T, S). cdcl_{NOT} S T \wedge ?inv S \} \rangle$)

$\langle proof \rangle$

lemma $inv-and-tranclp-cdcl_{NOT-}tranclp-cdcl_{NOT-}and-inv$:

$\langle cdcl_{NOT}^{++} S T \wedge cdcl_{NOT-}NOT-all-inv A S \longleftrightarrow (\lambda S T. cdcl_{NOT} S T \wedge cdcl_{NOT-}NOT-all-inv A S)^{++} S T \rangle$

(is $\langle ?A \wedge ?I \longleftrightarrow ?B \rangle$)

$\langle proof \rangle$

lemma $wf-tranclp-cdcl_{NOT-}no-learn-and-forget-infinite-chain$:

assumes

$no-infinite-lf$: $\langle \bigwedge j. \neg (\forall i \geq j. learn-or-forget (f i) (f (Suc i))) \rangle$

shows $\langle wf \{ (T, S). cdcl_{NOT}^{++} S T \wedge cdcl_{NOT-}NOT-all-inv A S \} \rangle$

$\langle proof \rangle$

lemma *cdcl_{NOT}-final-state*:

assumes

n-s: $\langle \text{no-step } \text{cdcl}_{NOT} S \rangle$ **and**

inv: $\langle \text{cdcl}_{NOT}\text{-NOT-all-inv } A S \rangle$ **and**

decomp: $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} S) (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$

shows $\langle \text{unsatisfiable } (\text{set-mset } (\text{clauses}_{NOT} S))$

$\vee (\text{trail } S \models_{asm} \text{clauses}_{NOT} S \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} S))) \rangle$

$\langle \text{proof} \rangle$

lemma *full-cdcl_{NOT}-final-state*:

assumes

full: $\langle \text{full } \text{cdcl}_{NOT} S T \rangle$ **and**

inv: $\langle \text{cdcl}_{NOT}\text{-NOT-all-inv } A S \rangle$ **and**

n-d: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**

decomp: $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} S) (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$

shows $\langle \text{unsatisfiable } (\text{set-mset } (\text{clauses}_{NOT} T))$

$\vee (\text{trail } T \models_{asm} \text{clauses}_{NOT} T \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} T))) \rangle$

$\langle \text{proof} \rangle$

end — End of the locale *conflict-driven-clause-learning*.

Termination

To prove termination we need to restrict learn and forget. Otherwise we could forget and relearn the exact same clause over and over. A first idea is to forbid removing clauses that can be used to backjump. This does not change the rules of the calculus. A second idea is to “merge” backjump and learn: that way, though closer to implementation, needs a change of the rules, since the backjump-rule learns the clause used to backjump.

Restricting learn and forget

locale *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learn* =

dp_{ll}-state trail clauses_{NOT} prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT} +

conflict-driven-clause-learning trail clauses_{NOT} prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}

inv decide-conds backjump-conds propagate-conds

$\langle \lambda C S. \text{distinct-mset } C \wedge \neg \text{tautology } C \wedge \text{learn-restrictions } C S \wedge$

$(\exists F K d F' C' L. \text{trail } S = F' @ \text{Decided } K \# F \wedge C = \text{add-mset } L C' \wedge F \models_{as} C \text{Not } C'$
 $\wedge \text{add-mset } L C' \notin \# \text{clauses}_{NOT} S) \rangle$

$\langle \lambda C S. \neg (\exists F' F K d L. \text{trail } S = F' @ \text{Decided } K \# F \wedge F \models_{as} C \text{Not } (\text{remove1-mset } L C))$
 $\wedge \text{forget-restrictions } C S \rangle$

for

trail :: $\langle 'st \Rightarrow ('v, \text{unit}) \text{ann-lits} \rangle$ **and**

clauses_{NOT} :: $\langle 'st \Rightarrow 'v \text{clauses} \rangle$ **and**

prepend-trail :: $\langle ('v, \text{unit}) \text{ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

tl-trail :: $\langle 'st \Rightarrow 'st \rangle$ **and**

add-cl_{NOT} :: $\langle 'v \text{clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

remove-cl_{NOT} :: $\langle 'v \text{clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

inv :: $\langle 'st \Rightarrow \text{bool} \rangle$ **and**

decide-conds :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**

backjump-conds :: $\langle 'v \text{clause} \Rightarrow 'v \text{clause} \Rightarrow 'v \text{literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**

propagate-conds :: $\langle ('v, \text{unit}) \text{ann-lit} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**

learn-restrictions *forget-restrictions* :: $\langle 'v \text{clause} \Rightarrow 'st \Rightarrow \text{bool} \rangle$

begin

lemma *cdcl_{NOT}-learn-all-induct*[*consumes 1, case-names dp_{ll}-bj learn forget_{NOT}*]:

fixes $S T :: \langle 'st \rangle$

assumes $\langle cdcl_{NOT} S T \rangle$ **and**

$dpll: \langle \bigwedge T. dpll\text{-}bj S T \implies P S T \rangle$ **and**

learning:

$\langle \bigwedge C F K F' C' L T. clauses_{NOT} S \models_{pm} C \implies$
 $atms\text{-}of C \subseteq atms\text{-}of\text{-}mm (clauses_{NOT} S) \cup atm\text{-}of ' (lits\text{-}of\text{-}l (trail S)) \implies$
 $distinct\text{-}mset C \implies$

$\neg tautology C \implies$

$learn\text{-}restrictions C S \implies$

$trail S = F' @ Decided K \# F \implies$

$C = add\text{-}mset L C' \implies$

$F \models_{as} CNot C' \implies$

$add\text{-}mset L C' \notin clauses_{NOT} S \implies$

$T \sim add\text{-}cls_{NOT} C S \implies$

$P S T \rangle$ **and**

forgetting: $\langle \bigwedge C T. removeAll\text{-}mset C (clauses_{NOT} S) \models_{pm} C \implies$

$C \in \# clauses_{NOT} S \implies$

$\neg (\exists F' F K L. trail S = F' @ Decided K \# F \wedge F \models_{as} CNot (C - \{\#L\# \})) \implies$

$T \sim remove\text{-}cls_{NOT} C S \implies$

$forget\text{-}restrictions C S \implies$

$P S T \rangle$

shows $\langle P S T \rangle$

$\langle proof \rangle$

lemma $rtranclp\text{-}cdcl_{NOT}\text{-}inv$:

$\langle cdcl_{NOT}^{**} S T \implies inv S \implies inv T \rangle$

$\langle proof \rangle$

lemma $learn\text{-}always\text{-}simple\text{-}clauses$:

assumes

$learn: \langle learn S T \rangle$ **and**

$n\text{-}d: \langle no\text{-}dup (trail S) \rangle$

shows $\langle set\text{-}mset (clauses_{NOT} T - clauses_{NOT} S)$

$\subseteq simple\text{-}clss (atms\text{-}of\text{-}mm (clauses_{NOT} S) \cup atm\text{-}of ' lits\text{-}of\text{-}l (trail S)) \rangle$

$\langle proof \rangle$

definition $\langle conflicting\text{-}bj\text{-}clss S \equiv$

$\{C + \{\#L\#\} \mid C L. C + \{\#L\#\} \in \# clauses_{NOT} S \wedge distinct\text{-}mset (C + \{\#L\#\})$

$\wedge \neg tautology (C + \{\#L\#\})$

$\wedge (\exists F' K F. trail S = F' @ Decided K \# F \wedge F \models_{as} CNot C) \rangle$

lemma $conflicting\text{-}bj\text{-}clss\text{-}remove\text{-}cls_{NOT}[simp]$:

$\langle conflicting\text{-}bj\text{-}clss (remove\text{-}cls_{NOT} C S) = conflicting\text{-}bj\text{-}clss S - \{C\} \rangle$

$\langle proof \rangle$

lemma $conflicting\text{-}bj\text{-}clss\text{-}remove\text{-}cls_{NOT}'[simp]$:

$\langle T \sim remove\text{-}cls_{NOT} C S \implies conflicting\text{-}bj\text{-}clss T = conflicting\text{-}bj\text{-}clss S - \{C\} \rangle$

$\langle proof \rangle$

lemma $conflicting\text{-}bj\text{-}clss\text{-}add\text{-}cls_{NOT}\text{-}state\text{-}eq$:

assumes

$T: \langle T \sim add\text{-}cls_{NOT} C' S \rangle$ **and**

$n\text{-}d: \langle no\text{-}dup (trail S) \rangle$

shows $\langle conflicting\text{-}bj\text{-}clss T$

$= conflicting\text{-}bj\text{-}clss S$

$\cup (if \exists C L. C' = add\text{-}mset L C \wedge distinct\text{-}mset (add\text{-}mset L C) \wedge \neg tautology (add\text{-}mset L C))$

$\wedge (\exists F' K d F. \text{trail } S = F' @ \text{Decided } K \# F \wedge F \models_{as} C \text{Not } C)$
 then $\{C'\}$ else $\{\}$

<proof>

lemma *conflicting-bj-clss-add-clss_{NOT}*:

$\langle \text{no-dup } (\text{trail } S) \implies$
 $\text{conflicting-bj-clss } (\text{add-clss}_{NOT} C' S)$
 $= \text{conflicting-bj-clss } S$
 $\cup (\text{if } \exists C L. C' = C + \{\#L\} \wedge \text{distinct-mset } (C + \{\#L\}) \wedge \neg \text{tautology } (C + \{\#L\})$
 $\wedge (\exists F' K d F. \text{trail } S = F' @ \text{Decided } K \# F \wedge F \models_{as} C \text{Not } C)$
 then $\{C'\}$ else $\{\}$

<proof>

lemma *conflicting-bj-clss-incl-clauses*:

$\langle \text{conflicting-bj-clss } S \subseteq \text{set-mset } (\text{clauses}_{NOT} S) \rangle$
<proof>

lemma *finite-conflicting-bj-clss[simp]*:

$\langle \text{finite } (\text{conflicting-bj-clss } S) \rangle$
<proof>

lemma *learn-conflicting-increasing*:

$\langle \text{no-dup } (\text{trail } S) \implies \text{learn } S T \implies \text{conflicting-bj-clss } S \subseteq \text{conflicting-bj-clss } T \rangle$
<proof>

abbreviation $\langle \text{conflicting-bj-clss-yet } b S \equiv$

$3 \wedge b - \text{card } (\text{conflicting-bj-clss } S) \rangle$

abbreviation $\mu_L :: \langle \text{nat} \Rightarrow 'st \Rightarrow \text{nat} \times \text{nat} \rangle$ **where**

$\langle \mu_L b S \equiv (\text{conflicting-bj-clss-yet } b S, \text{card } (\text{set-mset } (\text{clauses}_{NOT} S))) \rangle$

lemma *do-not-forget-before-backtrack-rule-clause-learned-clause-untouched*:

assumes $\langle \text{forget}_{NOT} S T \rangle$
shows $\langle \text{conflicting-bj-clss } S = \text{conflicting-bj-clss } T \rangle$
<proof>

lemma *forget- μ_L -decrease*:

assumes $\text{forget}_{NOT}: \langle \text{forget}_{NOT} S T \rangle$
shows $\langle (\mu_L b T, \mu_L b S) \in \text{less-than } <*\text{lex}*> \text{less-than} \rangle$
<proof>

lemma *set-condition-or-split*:

$\langle \{a. (a = b \vee Q a) \wedge S a\} = (\text{if } S b \text{ then } \{b\} \text{ else } \{\}) \cup \{a. Q a \wedge S a\} \rangle$
<proof>

lemma *set-insert-neq*:

$\langle A \neq \text{insert } a A \longleftrightarrow a \notin A \rangle$
<proof>

lemma *learn- μ_L -decrease*:

assumes $\text{learnST}: \langle \text{learn } S T \rangle$ **and** $n\text{-d}: \langle \text{no-dup } (\text{trail } S) \rangle$ **and**
 $A: \langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq A \rangle$ **and**
 $\text{fin-A}: \langle \text{finite } A \rangle$
shows $\langle (\mu_L (\text{card } A) T, \mu_L (\text{card } A) S) \in \text{less-than } <*\text{lex}*> \text{less-than} \rangle$
<proof>

We have to assume the following:

- *inv S*: the invariant holds in the initial state.
- *A* is a (finite *finite A*) superset of the literals in the trail *atm-of* ‘ *lits-of-l (trail S)* \subseteq *atms-of-ms A* and in the clauses *atms-of-mm (clauses_{NOT} S)* \subseteq *atms-of-ms A*. This can be the set of all the literals in the starting set of clauses.
- *no-dup (trail S)*: no duplicate in the trail. This is invariant along the path.

definition μ_{CDCL} **where**

$$\langle \mu_{CDCL} A T \equiv ((2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A)) \\ - \mu_C (1 + \text{card} (\text{atms-of-ms } A)) (2 + \text{card} (\text{atms-of-ms } A)) (\text{trail-weight } T), \\ \text{conflicting-bj-clss-yet} (\text{card} (\text{atms-of-ms } A)) T, \text{card} (\text{set-mset} (\text{clauses}_{NOT} T))) \rangle$$

lemma *cdcl_{NOT}-decreasing-measure*:

assumes

$\langle \text{cdcl}_{NOT} S T \rangle$ **and**

inv: $\langle \text{inv } S \rangle$ **and**

atm-clss: $\langle \text{atms-of-mm} (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**

atm-lits: $\langle \text{atm-of} \text{ ‘ } \text{lits-of-l} (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**

n-d: $\langle \text{no-dup} (\text{trail } S) \rangle$ **and**

fin-A: $\langle \text{finite } A \rangle$

shows $\langle \mu_{CDCL} A T, \mu_{CDCL} A S \rangle$

$\in \text{less-than} < * \text{lex} * > (\text{less-than} < * \text{lex} * > \text{less-than})$

$\langle \text{proof} \rangle$

lemma *wf-cdcl_{NOT}-restricted-learning*:

assumes $\langle \text{finite } A \rangle$

shows $\langle \text{wf } \{(T, S)\}.$

$(\text{atms-of-mm} (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of} \text{ ‘ } \text{lits-of-l} (\text{trail } S) \subseteq \text{atms-of-ms } A$

$\wedge \text{no-dup} (\text{trail } S)$

$\wedge \text{inv } S)$

$\wedge \text{cdcl}_{NOT} S T \rangle$

$\langle \text{proof} \rangle$

definition $\mu_C' :: \langle 'v \text{ clause set} \Rightarrow 'st \Rightarrow \text{nat} \rangle$ **where**

$$\langle \mu_C' A T \equiv \mu_C (1 + \text{card} (\text{atms-of-ms } A)) (2 + \text{card} (\text{atms-of-ms } A)) (\text{trail-weight } T) \rangle$$

definition $\mu_{CDCL}' :: \langle 'v \text{ clause set} \Rightarrow 'st \Rightarrow \text{nat} \rangle$ **where**

$$\langle \mu_{CDCL}' A T \equiv$$

$$((2 + \text{card} (\text{atms-of-ms } A)) \wedge (1 + \text{card} (\text{atms-of-ms } A)) - \mu_C' A T) * (1 + 3^{\text{card} (\text{atms-of-ms } A)}) * 2$$

$$+ \text{conflicting-bj-clss-yet} (\text{card} (\text{atms-of-ms } A)) T * 2$$

$$+ \text{card} (\text{set-mset} (\text{clauses}_{NOT} T)) \rangle$$

lemma *cdcl_{NOT}-decreasing-measure'*:

assumes

$\langle \text{cdcl}_{NOT} S T \rangle$ **and**

inv: $\langle \text{inv } S \rangle$ **and**

atms-clss: $\langle \text{atms-of-mm} (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**

atms-trail: $\langle \text{atm-of} \text{ ‘ } \text{lits-of-l} (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**

n-d: $\langle \text{no-dup} (\text{trail } S) \rangle$ **and**

fin-A: $\langle \text{finite } A \rangle$

shows $\langle \mu_{CDCL}' A T < \mu_{CDCL}' A S \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-clauses-bound*:

assumes

$\langle \text{cdcl}_{NOT} S T \rangle$ **and**

$\langle \text{inv } S \rangle$ **and**

$\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

$\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$ **and**

$n\text{-d}$: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**

$\text{fin-A}[\text{simp}]$: $\langle \text{finite } A \rangle$

shows $\langle \text{set-mset } (\text{clauses}_{NOT} T) \subseteq \text{set-mset } (\text{clauses}_{NOT} S) \cup \text{simple-clss } A \rangle$

$\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-clauses-bound*:

assumes

$\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and**

$\langle \text{inv } S \rangle$ **and**

$\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

$\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$ **and**

$n\text{-d}$: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**

finite : $\langle \text{finite } A \rangle$

shows $\langle \text{set-mset } (\text{clauses}_{NOT} T) \subseteq \text{set-mset } (\text{clauses}_{NOT} S) \cup \text{simple-clss } A \rangle$

$\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-card-clauses-bound*:

assumes

$\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and**

$\langle \text{inv } S \rangle$ **and**

$\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

$\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$ **and**

$n\text{-d}$: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**

finite : $\langle \text{finite } A \rangle$

shows $\langle \text{card } (\text{set-mset } (\text{clauses}_{NOT} T)) \leq \text{card } (\text{set-mset } (\text{clauses}_{NOT} S)) + 3 \wedge (\text{card } A) \rangle$

$\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-card-clauses-bound'*:

assumes

$\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and**

$\langle \text{inv } S \rangle$ **and**

$\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

$\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$ **and**

$n\text{-d}$: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**

finite : $\langle \text{finite } A \rangle$

shows $\langle \text{card } \{C \mid C. C \in \# \text{ clauses}_{NOT} T \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} \leq \text{card } \{C \mid C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} + 3 \wedge (\text{card } A) \rangle$

$\leq \text{card } \{C \mid C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} + 3 \wedge (\text{card } A) \rangle$

$(\text{is } \langle \text{card } ?T \leq \text{card } ?S + - \rangle)$

$\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-card-simple-clauses-bound*:

assumes

$\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and**

$\langle \text{inv } S \rangle$ **and**

NA : $\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

MA : $\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$ **and**

$n\text{-d}$: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**

finite : $\langle \text{finite } A \rangle$

shows $\langle \text{card } (\text{set-mset } (\text{clauses}_{NOT} T)) \leq \text{card } (\text{set-mset } (\text{clauses}_{NOT} S)) + 3 \wedge (\text{card } A) \rangle$

$\leq \text{card } \{C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} + 3 \wedge (\text{card } A)$
 $\langle \text{is } \langle \text{card } ?T \leq \text{card } ?S + \neg \rangle \rangle$
 $\langle \text{proof} \rangle$

definition $\mu_{CDCL}'\text{-bound} :: \langle 'v \text{ clause set} \Rightarrow 'st \Rightarrow nat \rangle$ **where**

$\langle \mu_{CDCL}'\text{-bound } A \ S =$
 $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))) * (1 + 3 \wedge \text{card } (\text{atms-of-ms } A)) * 2$
 $+ 2 * 3 \wedge (\text{card } (\text{atms-of-ms } A))$
 $+ \text{card } \{C. C \in \# \text{ clauses}_{NOT} S \wedge (\text{tautology } C \vee \neg \text{distinct-mset } C)\} + 3 \wedge (\text{card } (\text{atms-of-ms } A)) \rangle$

lemma $\mu_{CDCL}'\text{-bound-reduce-trail-to}_{NOT}[\text{simp}]$:

$\langle \mu_{CDCL}'\text{-bound } A \ (\text{reduce-trail-to}_{NOT} \ M \ S) = \mu_{CDCL}'\text{-bound } A \ S \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_{NOT}\text{-}\mu_{CDCL}'\text{-bound-reduce-trail-to}_{NOT}$:

assumes

$\langle \text{cdcl}_{NOT}^{**} \ S \ T \rangle$ **and**
 $\langle \text{inv } S \rangle$ **and**
 $\langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $n\text{-d}: \langle \text{no-dup } (\text{trail } S) \rangle$ **and**
 $\text{finite}: \langle \text{finite } (\text{atms-of-ms } A) \rangle$ **and**
 $U: \langle U \sim \text{reduce-trail-to}_{NOT} \ M \ T \rangle$

shows $\langle \mu_{CDCL}' \ A \ U \leq \mu_{CDCL}'\text{-bound } A \ S \rangle$

$\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_{NOT}\text{-}\mu_{CDCL}'\text{-bound}$:

assumes

$\langle \text{cdcl}_{NOT}^{**} \ S \ T \rangle$ **and**
 $\langle \text{inv } S \rangle$ **and**
 $\langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $n\text{-d}: \langle \text{no-dup } (\text{trail } S) \rangle$ **and**
 $\text{finite}: \langle \text{finite } (\text{atms-of-ms } A) \rangle$

shows $\langle \mu_{CDCL}' \ A \ T \leq \mu_{CDCL}'\text{-bound } A \ S \rangle$

$\langle \text{proof} \rangle$

lemma $\text{rtrancpl-}\mu_{CDCL}'\text{-bound-decreasing}$:

assumes

$\langle \text{cdcl}_{NOT}^{**} \ S \ T \rangle$ **and**
 $\langle \text{inv } S \rangle$ **and**
 $\langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $n\text{-d}: \langle \text{no-dup } (\text{trail } S) \rangle$ **and**
 $\text{finite}[\text{simp}]: \langle \text{finite } (\text{atms-of-ms } A) \rangle$

shows $\langle \mu_{CDCL}'\text{-bound } A \ T \leq \mu_{CDCL}'\text{-bound } A \ S \rangle$

$\langle \text{proof} \rangle$

end — End of the locale *conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt*.

2.2.5 CDCL with Restarts

Definition

locale *restart-ops* =

```

fixes
   $cdcl_{NOT} :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$  and
   $restart :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$ 
begin
inductive  $cdcl_{NOT}\text{-raw-restart} :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$  where
 $\langle cdcl_{NOT} S T \Rightarrow cdcl_{NOT}\text{-raw-restart} S T \rangle \mid$ 
 $\langle restart S T \Rightarrow cdcl_{NOT}\text{-raw-restart} S T \rangle$ 

end

locale  $conflict\text{-driven-clause-learning-with-restarts} =$ 
   $conflict\text{-driven-clause-learning trail clauses}_{NOT} prepend\text{-trail tl-trail add-cl}_{NOT} remove\text{-cl}_{NOT}$ 
   $inv decide\text{-conds backjump\text{-conds propagate\text{-conds learn\text{-conds forget\text{-conds}}$ 
for
   $trail :: \langle 'st \Rightarrow ('v, unit) ann\text{-lits} \rangle$  and
   $clauses_{NOT} :: \langle 'st \Rightarrow 'v clauses \rangle$  and
   $prepend\text{-trail} :: \langle ('v, unit) ann\text{-lit} \Rightarrow 'st \Rightarrow 'st \rangle$  and
   $tl\text{-trail} :: \langle 'st \Rightarrow 'st \rangle$  and
   $add\text{-cl}_{NOT} :: \langle 'v clause \Rightarrow 'st \Rightarrow 'st \rangle$  and
   $remove\text{-cl}_{NOT} :: \langle 'v clause \Rightarrow 'st \Rightarrow 'st \rangle$  and
   $inv :: \langle 'st \Rightarrow bool \rangle$  and
   $decide\text{-conds} :: \langle 'st \Rightarrow 'st \Rightarrow bool \rangle$  and
   $backjump\text{-conds} :: \langle 'v clause \Rightarrow 'v clause \Rightarrow 'v literal \Rightarrow 'st \Rightarrow 'st \Rightarrow bool \rangle$  and
   $propagate\text{-conds} :: \langle ('v, unit) ann\text{-lit} \Rightarrow 'st \Rightarrow 'st \Rightarrow bool \rangle$  and
   $learn\text{-conds forget\text{-conds} :: \langle 'v clause \Rightarrow 'st \Rightarrow bool \rangle$ 
begin

lemma  $cdcl_{NOT}\text{-iff-cdcl}_{NOT}\text{-raw-restart-no-restarts}$ :
   $\langle cdcl_{NOT} S T \longleftrightarrow restart\text{-ops.}cdcl_{NOT}\text{-raw-restart } cdcl_{NOT} (\lambda\text{-} \cdot False) S T \rangle$ 
  (is  $\langle ?C S T \longleftrightarrow ?R S T \rangle$ )
  <proof>

lemma  $cdcl_{NOT}\text{-cdcl}_{NOT}\text{-raw-restart}$ :
   $\langle cdcl_{NOT} S T \Rightarrow restart\text{-ops.}cdcl_{NOT}\text{-raw-restart } cdcl_{NOT} restart S T \rangle$ 
  <proof>
end

```

Increasing restarts

Definition We define our increasing restart very abstractly: the predicate (called $cdcl_{NOT}$) does not have to be a CDCL calculus. We just need some assumptions to prove termination:

- a function f that is strictly monotonic. The first step is actually only used as a restart to clean the state (e.g. to ensure that the trail is empty). Then we assume that $(1::'a) \leq f n$ for $(1::'a) \leq n$: it means that between two consecutive restarts, at least one step will be done. This is necessary to avoid sequence. like: full – restart – full – ...
- a measure μ : it should decrease under the assumptions $bound\text{-}inv$, whenever a $cdcl_{NOT}$ or a $restart$ is done. A parameter is given to μ : for conflict- driven clause learning, it is an upper-bound of the clauses. We are assuming that such a bound can be found after a restart whenever the invariant holds.
- we also assume that the measure decrease after any $cdcl_{NOT}$ step.
- an invariant on the states $cdcl_{NOT}\text{-}inv$ that also holds after restarts.

- it is *not required* that the measure decrease with respect to restarts, but the measure has to be bound by some function μ -bound taking the same parameter as μ and the initial state of the considered $cdcl_{NOT}$ chain.

```

locale  $cdcl_{NOT}$ -increasing-restarts-ops =
  restart-ops  $cdcl_{NOT}$  restart for
    restart ::  $\langle 'st \Rightarrow 'st \Rightarrow bool \rangle$  and
     $cdcl_{NOT}$  ::  $\langle 'st \Rightarrow 'st \Rightarrow bool \rangle +$ 
fixes
  f ::  $\langle nat \Rightarrow nat \rangle$  and
  bound-inv ::  $\langle 'bound \Rightarrow 'st \Rightarrow bool \rangle$  and
   $\mu$  ::  $\langle 'bound \Rightarrow 'st \Rightarrow nat \rangle$  and
   $cdcl_{NOT}$ -inv ::  $\langle 'st \Rightarrow bool \rangle$  and
   $\mu$ -bound ::  $\langle 'bound \Rightarrow 'st \Rightarrow nat \rangle$ 
assumes
  f:  $\langle unbounded f \rangle$  and
  f-ge-1:  $\langle \bigwedge n. n \geq 1 \implies f n \neq 0 \rangle$  and
  bound-inv:  $\langle \bigwedge A S T. cdcl_{NOT}$ -inv  $S \implies bound$ -inv  $A S \implies cdcl_{NOT} S T \implies bound$ -inv  $A T \rangle$  and
   $cdcl_{NOT}$ -measure:  $\langle \bigwedge A S T. cdcl_{NOT}$ -inv  $S \implies bound$ -inv  $A S \implies cdcl_{NOT} S T \implies \mu A T < \mu$ 
   $A S \rangle$  and
  measure-bound2:  $\langle \bigwedge A T U. cdcl_{NOT}$ -inv  $T \implies bound$ -inv  $A T \implies cdcl_{NOT}^{**} T U$ 
     $\implies \mu A U \leq \mu$ -bound  $A T \rangle$  and
  measure-bound4:  $\langle \bigwedge A T U. cdcl_{NOT}$ -inv  $T \implies bound$ -inv  $A T \implies cdcl_{NOT}^{**} T U$ 
     $\implies \mu$ -bound  $A U \leq \mu$ -bound  $A T \rangle$  and
   $cdcl_{NOT}$ -restart-inv:  $\langle \bigwedge A U V. cdcl_{NOT}$ -inv  $U \implies restart U V \implies bound$ -inv  $A U \implies bound$ -inv
   $A V \rangle$ 
and
  exists-bound:  $\langle \bigwedge R S. cdcl_{NOT}$ -inv  $R \implies restart R S \implies \exists A. bound$ -inv  $A S \rangle$  and
   $cdcl_{NOT}$ -inv:  $\langle \bigwedge S T. cdcl_{NOT}$ -inv  $S \implies cdcl_{NOT} S T \implies cdcl_{NOT}$ -inv  $T \rangle$  and
   $cdcl_{NOT}$ -inv-restart:  $\langle \bigwedge S T. cdcl_{NOT}$ -inv  $S \implies restart S T \implies cdcl_{NOT}$ -inv  $T \rangle$ 
begin

```

lemma $cdcl_{NOT}$ - $cdcl_{NOT}$ -inv:

```

assumes
   $\langle (cdcl_{NOT} \sim n) S T \rangle$  and
   $\langle cdcl_{NOT}$ -inv  $S \rangle$ 
shows  $\langle cdcl_{NOT}$ -inv  $T \rangle$ 
 $\langle proof \rangle$ 

```

lemma $cdcl_{NOT}$ -bound-inv:

```

assumes
   $\langle (cdcl_{NOT} \sim n) S T \rangle$  and
   $\langle cdcl_{NOT}$ -inv  $S \rangle$ 
   $\langle bound$ -inv  $A S \rangle$ 
shows  $\langle bound$ -inv  $A T \rangle$ 
 $\langle proof \rangle$ 

```

lemma $rtrancp$ - $cdcl_{NOT}$ - $cdcl_{NOT}$ -inv:

```

assumes
   $\langle cdcl_{NOT}^{**} S T \rangle$  and
   $\langle cdcl_{NOT}$ -inv  $S \rangle$ 
shows  $\langle cdcl_{NOT}$ -inv  $T \rangle$ 
 $\langle proof \rangle$ 

```

lemma $rtrancp$ - $cdcl_{NOT}$ -bound-inv:

assumes
 $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and**
 $\langle \text{bound-inv } A S \rangle$ **and**
 $\langle \text{cdcl}_{NOT-inv} S \rangle$
shows $\langle \text{bound-inv } A T \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_{NOT-comp-n-le}$:

assumes
 $\langle (\text{cdcl}_{NOT} \sim (Suc\ n)) S T \rangle$ **and**
 $\langle \text{bound-inv } A S \rangle$
 $\langle \text{cdcl}_{NOT-inv} S \rangle$
shows $\langle \mu A T < \mu A S - n \rangle$
 $\langle \text{proof} \rangle$

lemma wf-cdcl_{NOT} :

$\langle \text{wf } \{(T, S). \text{cdcl}_{NOT} S T \wedge \text{cdcl}_{NOT-inv} S \wedge \text{bound-inv } A S\} \rangle$ **(is** $\langle \text{wf } ?A \rangle$ **)**
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_{NOT-measure}$:

assumes
 $\langle \text{cdcl}_{NOT}^{**} S T \rangle$ **and**
 $\langle \text{bound-inv } A S \rangle$ **and**
 $\langle \text{cdcl}_{NOT-inv} S \rangle$
shows $\langle \mu A T \leq \mu A S \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_{NOT-comp-bounded}$:

assumes
 $\langle \text{bound-inv } A S \rangle$ **and** $\langle \text{cdcl}_{NOT-inv} S \rangle$ **and** $\langle m \geq 1 + \mu A S \rangle$
shows $\langle \neg(\text{cdcl}_{NOT} \sim m) S T \rangle$
 $\langle \text{proof} \rangle$

- $f\ n < m$ ensures that at least one step has been done.

inductive $\text{cdcl}_{NOT-restart}$ **where**

$\text{restart-step: } \langle (\text{cdcl}_{NOT} \sim m) S T \implies m \geq f\ n \implies \text{restart } T U$

$\implies \text{cdcl}_{NOT-restart} (S, n) (U, Suc\ n) \rangle$ |

$\text{restart-full: } \langle \text{full1 } \text{cdcl}_{NOT} S T \implies \text{cdcl}_{NOT-restart} (S, n) (T, Suc\ n) \rangle$

lemmas $\text{cdcl}_{NOT-with-restart-induct} = \text{cdcl}_{NOT-restart.induct}[\text{split-format}(\text{complete}),$
 $\text{OF } \text{cdcl}_{NOT-increasing-restarts-ops-axioms}]$

lemma $\text{cdcl}_{NOT-restart-cdcl}_{NOT-raw-restart}$:

$\langle \text{cdcl}_{NOT-restart} S T \implies \text{cdcl}_{NOT-raw-restart}^{**} (fst\ S) (fst\ T) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_{NOT-with-restart-bound-inv}$:

assumes
 $\langle \text{cdcl}_{NOT-restart} S T \rangle$ **and**
 $\langle \text{bound-inv } A (fst\ S) \rangle$ **and**
 $\langle \text{cdcl}_{NOT-inv} (fst\ S) \rangle$
shows $\langle \text{bound-inv } A (fst\ T) \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-with-restart-cdcl_{NOT}-inv*:

assumes
 $\langle \text{cdcl}_{\text{NOT-restart}} S T \rangle$ **and**
 $\langle \text{cdcl}_{\text{NOT-inv}} (\text{fst } S) \rangle$
shows $\langle \text{cdcl}_{\text{NOT-inv}} (\text{fst } T) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-with-restart-cdcl_{NOT}-inv*:

assumes
 $\langle \text{cdcl}_{\text{NOT-restart}^{**}} S T \rangle$ **and**
 $\langle \text{cdcl}_{\text{NOT-inv}} (\text{fst } S) \rangle$
shows $\langle \text{cdcl}_{\text{NOT-inv}} (\text{fst } T) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-with-restart-bound-inv*:

assumes
 $\langle \text{cdcl}_{\text{NOT-restart}^{**}} S T \rangle$ **and**
 $\langle \text{cdcl}_{\text{NOT-inv}} (\text{fst } S) \rangle$ **and**
 $\langle \text{bound-inv } A (\text{fst } S) \rangle$
shows $\langle \text{bound-inv } A (\text{fst } T) \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-with-restart-increasing-number*:

$\langle \text{cdcl}_{\text{NOT-restart}} S T \implies \text{snd } T = 1 + \text{snd } S \rangle$
 $\langle \text{proof} \rangle$

end

locale *cdcl_{NOT}-increasing-restarts* =

cdcl_{NOT}-increasing-restarts-ops restart *cdcl_{NOT}* *f* *bound-inv* μ *cdcl_{NOT}-inv* μ -*bound* +
dpll-state *trail* *clauses_{NOT}* *prepend-trail* *tl-trail* *add-cl_{sNOT}* *remove-cl_{sNOT}*

for

trail :: $\langle 'st \Rightarrow ('v, \text{unit}) \text{ ann-lits} \rangle$ **and**
clauses_{NOT} :: $\langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**
prepend-trail :: $\langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
tl-trail :: $\langle 'st \Rightarrow 'st \rangle$ **and**
add-cl_{sNOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
remove-cl_{sNOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
f :: $\langle \text{nat} \Rightarrow \text{nat} \rangle$ **and**
restart :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
bound-inv :: $\langle 'bound \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
 μ :: $\langle 'bound \Rightarrow 'st \Rightarrow \text{nat} \rangle$ **and**
cdcl_{NOT} :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
cdcl_{NOT}-inv :: $\langle 'st \Rightarrow \text{bool} \rangle$ **and**
 μ -*bound* :: $\langle 'bound \Rightarrow 'st \Rightarrow \text{nat} \rangle$ +

assumes

measure-bound: $\langle \bigwedge A T V n. \text{cdcl}_{\text{NOT-inv}} T \implies \text{bound-inv } A T \implies \text{cdcl}_{\text{NOT-restart}} (T, n) (V, \text{Suc } n) \implies \mu A V \leq \mu\text{-bound } A T \rangle$ **and**
cdcl_{NOT}-raw-restart- μ -bound:
 $\langle \text{cdcl}_{\text{NOT-restart}} (T, a) (V, b) \implies \text{cdcl}_{\text{NOT-inv}} T \implies \text{bound-inv } A T \implies \mu\text{-bound } A V \leq \mu\text{-bound } A T \rangle$

begin

lemma *rtrancpl-cdcl_{NOT}-raw-restart- μ -bound*:

$\langle \text{cdcl}_{\text{NOT-restart}^{**}} (T, a) (V, b) \implies \text{cdcl}_{\text{NOT-inv}} T \implies \text{bound-inv } A T \implies \mu\text{-bound } A V \leq \mu\text{-bound } A T \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-raw-restart-measure-bound*:

$\langle \text{cdcl}_{NOT}\text{-restart } (T, a) (V, b) \Rightarrow \text{cdcl}_{NOT}\text{-inv } T \Rightarrow \text{bound-inv } A \ T \Rightarrow \mu \ A \ V \leq \mu\text{-bound } A \ T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-raw-restart-measure-bound*:

$\langle \text{cdcl}_{NOT}\text{-restart}^{**} (T, a) (V, b) \Rightarrow \text{cdcl}_{NOT}\text{-inv } T \Rightarrow \text{bound-inv } A \ T \Rightarrow \mu \ A \ V \leq \mu\text{-bound } A \ T \rangle$
 $\langle \text{proof} \rangle$

lemma *wf-cdcl_{NOT}-restart*:

$\langle \text{wf } \{(T, S). \text{cdcl}_{NOT}\text{-restart } S \ T \wedge \text{cdcl}_{NOT}\text{-inv } (fst \ S)\} \rangle \text{ (is } \langle \text{wf } ?A \rangle)$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-restart-steps-bigger-than-bound*:

assumes
 $\langle \text{cdcl}_{NOT}\text{-restart } S \ T \rangle$ **and**
 $\langle \text{bound-inv } A \ (fst \ S) \rangle$ **and**
 $\langle \text{cdcl}_{NOT}\text{-inv } (fst \ S) \rangle$ **and**
 $\langle f \ (snd \ S) > \mu\text{-bound } A \ (fst \ S) \rangle$
shows $\langle \text{full1 } \text{cdcl}_{NOT} \ (fst \ S) \ (fst \ T) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-with-inv-inv-rtrancpl-cdcl_{NOT}*:

assumes
 $\text{inv}: \langle \text{cdcl}_{NOT}\text{-inv } S \rangle$ **and**
 $\text{binv}: \langle \text{bound-inv } A \ S \rangle$
shows $\langle (\lambda S \ T. \text{cdcl}_{NOT} \ S \ T \wedge \text{cdcl}_{NOT}\text{-inv } S \wedge \text{bound-inv } A \ S)^{**} \ S \ T \longleftrightarrow \text{cdcl}_{NOT}^{**} \ S \ T \rangle$
 $\text{(is } \langle ?A^{**} \ S \ T \longleftrightarrow ?B^{**} \ S \ T \rangle)$
 $\langle \text{proof} \rangle$

lemma *no-step-cdcl_{NOT}-restart-no-step-cdcl_{NOT}*:

assumes
 $\text{n-s}: \langle \text{no-step } \text{cdcl}_{NOT}\text{-restart } S \rangle$ **and**
 $\text{inv}: \langle \text{cdcl}_{NOT}\text{-inv } (fst \ S) \rangle$ **and**
 $\text{binv}: \langle \text{bound-inv } A \ (fst \ S) \rangle$
shows $\langle \text{no-step } \text{cdcl}_{NOT} \ (fst \ S) \rangle$
 $\langle \text{proof} \rangle$

end

2.2.6 Merging backjump and learning

locale *cdcl_{NOT}-merge-bj-learn-ops* =

decide-ops *trail* *clauses_{NOT}* *prepend-trail* *tl-trail* *add-cls_{NOT}* *remove-cls_{NOT}* *decide-conds* +
forget-ops *trail* *clauses_{NOT}* *prepend-trail* *tl-trail* *add-cls_{NOT}* *remove-cls_{NOT}* *forget-conds* +
propagate-ops *trail* *clauses_{NOT}* *prepend-trail* *tl-trail* *add-cls_{NOT}* *remove-cls_{NOT}* *propagate-conds*
for

trail :: $\langle 'st \Rightarrow ('v, \text{unit}) \text{ ann-lits} \rangle$ **and**
clauses_{NOT} :: $\langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**
prepend-trail :: $\langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
tl-trail :: $\langle 'st \Rightarrow 'st \rangle$ **and**
add-cls_{NOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
remove-cls_{NOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
decide-conds :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**


```

    propagate-conds :: ⟨('v, unit) ann-lit ⇒ 'st ⇒ 'st ⇒ bool⟩ and
    forget-conds :: ⟨'v clause ⇒ 'st ⇒ bool⟩ +
    fixes backjump-l-cond :: ⟨'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool⟩
begin

```

We have a new backjump that combines the backjumping on the trail and the learning of the used clause (called C'' below)

```

inductive backjump-l where
backjump-l: ⟨trail S = F' @ Decided K # F
  ⇒ T ~ prepend-trail (Propagated L ()) (reduce-trail-toNOT F (add-clNOT C'' S))
  ⇒ C ∈# clausesNOT S
  ⇒ trail S ⊨as CNot C
  ⇒ undefined-lit F L
  ⇒ atm-of L ∈ atms-of-mm (clausesNOT S) ∪ atm-of ' (lits-of-l (trail S))
  ⇒ clausesNOT S ⊨pm add-mset L C'
  ⇒ C'' = add-mset L C'
  ⇒ F ⊨as CNot C'
  ⇒ backjump-l-cond C C' L S T
  ⇒ backjump-l S T⟩

```

Avoid (meaningless) simplification in the theorem generated by *inductive-cases*:

```

declare reduce-trail-toNOT-length-ne[simp del] Set.Un-iff[simp del] Set.insert-iff[simp del]
inductive-cases backjump-lE: ⟨backjump-l S T⟩
thm backjump-lE
declare reduce-trail-toNOT-length-ne[simp] Set.Un-iff[simp] Set.insert-iff[simp]

```

```

inductive cdclNOT-merged-bj-learn :: ⟨'st ⇒ 'st ⇒ bool⟩ for S :: 'st where
cdclNOT-merged-bj-learn-decideNOT: ⟨decideNOT S S' ⇒ cdclNOT-merged-bj-learn S S'⟩ |
cdclNOT-merged-bj-learn-propagateNOT: ⟨propagateNOT S S' ⇒ cdclNOT-merged-bj-learn S S'⟩ |
cdclNOT-merged-bj-learn-backjump-l: ⟨backjump-l S S' ⇒ cdclNOT-merged-bj-learn S S'⟩ |
cdclNOT-merged-bj-learn-forgetNOT: ⟨forgetNOT S S' ⇒ cdclNOT-merged-bj-learn S S'⟩

```

```

lemma cdclNOT-merged-bj-learn-no-dup-inv:
  ⟨cdclNOT-merged-bj-learn S T ⇒ no-dup (trail S) ⇒ no-dup (trail T)⟩
  ⟨proof⟩
end

```

```

locale cdclNOT-merge-bj-learn-proxy =
  cdclNOT-merge-bj-learn-ops trail clausesNOT prepend-trail tl-trail add-clNOT remove-clNOT
  decide-conds propagate-conds forget-conds
  ⟨λC C' L' S T. backjump-l-cond C C' L' S T
  ∧ distinct-mset C' ∧ L' ∉# C' ∧ ¬tautology (add-mset L' C')⟩
for
  trail :: ⟨'st ⇒ ('v, unit) ann-lits⟩ and
  clausesNOT :: ⟨'st ⇒ 'v clauses⟩ and
  prepend-trail :: ⟨('v, unit) ann-lit ⇒ 'st ⇒ 'st⟩ and
  tl-trail :: ⟨'st ⇒ 'st⟩ and
  add-clNOT :: ⟨'v clause ⇒ 'st ⇒ 'st⟩ and
  remove-clNOT :: ⟨'v clause ⇒ 'st ⇒ 'st⟩ and
  decide-conds :: ⟨'st ⇒ 'st ⇒ bool⟩ and
  propagate-conds :: ⟨('v, unit) ann-lit ⇒ 'st ⇒ 'st ⇒ bool⟩ and
  forget-conds :: ⟨'v clause ⇒ 'st ⇒ bool⟩ and
  backjump-l-cond :: ⟨'v clause ⇒ 'v clause ⇒ 'v literal ⇒ 'st ⇒ 'st ⇒ bool⟩ +
fixes
  inv :: ⟨'st ⇒ bool⟩

```

begin

abbreviation *backjump-conds* :: $\langle 'v \text{ clause} \Rightarrow 'v \text{ clause} \Rightarrow 'v \text{ literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$

where

$\langle \text{backjump-conds} \equiv \lambda C \ C' \ L' \ S \ T. \text{distinct-mset } C' \wedge L' \notin \# \ C' \wedge \neg \text{tautology } (\text{add-mset } L' \ C') \rangle$

sublocale *backjumping-ops* *trail clauses_{NOT} prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}*
backjump-conds
 $\langle \text{proof} \rangle$

end

locale *cdcl_{NOT}-merge-bj-learn* =

cdcl_{NOT}-merge-bj-learn-proxy *trail clauses_{NOT} prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}*
decide-conds propagate-conds forget-conds backjump-l-cond inv

for

trail :: $\langle 'st \Rightarrow ('v, \text{unit}) \text{ ann-lits} \rangle$ **and**
clauses_{NOT} :: $\langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**
prepend-trail :: $\langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
tl-trail :: $\langle 'st \Rightarrow 'st \rangle$ **and**
add-cl_{NOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
remove-cl_{NOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
decide-conds :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
propagate-conds :: $\langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
forget-conds :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
backjump-l-cond :: $\langle 'v \text{ clause} \Rightarrow 'v \text{ clause} \Rightarrow 'v \text{ literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
inv :: $\langle 'st \Rightarrow \text{bool} \rangle$ +

assumes

bj-merge-can-jump:

$\langle \bigwedge S \ C \ F' \ K \ F \ L.$

inv S

$\implies \text{trail } S = F' @ \text{Decided } K \ \# \ F$

$\implies C \in \# \text{ clauses}_{\text{NOT}} S$

$\implies \text{trail } S \models_{\text{as}} C \text{Not } C$

$\implies \text{undefined-lit } F \ L$

$\implies \text{atm-of } L \in \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \cup \text{atm-of } ' (\text{lits-of-l } (F' @ \text{Decided } K \ \# \ F))$

$\implies \text{clauses}_{\text{NOT}} S \models_{\text{pm}} \text{add-mset } L \ C'$

$\implies F \models_{\text{as}} C \text{Not } C'$

$\implies \neg \text{no-step backjump-l } S \rangle$ **and**

cdcl-merged-inv: $\langle \bigwedge S \ T. \text{cdcl}_{\text{NOT}}\text{-merged-bj-learn } S \ T \implies \text{inv } S \implies \text{inv } T \rangle$ **and**

can-propagate-or-decide-or-backjump-l:

$\langle \text{atm-of } L \in \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \implies$

$\text{undefined-lit } (\text{trail } S) \ L \implies$

$\text{inv } S \implies$

$\text{satisfiable } (\text{set-mset } (\text{clauses}_{\text{NOT}} S)) \implies$

$\exists T. \text{decide}_{\text{NOT}} S \ T \vee \text{propagate}_{\text{NOT}} S \ T \vee \text{backjump-l } S \ T \rangle$

begin

lemma *backjump-no-step-backjump-l*:

$\langle \text{backjump } S \ T \implies \text{inv } S \implies \neg \text{no-step backjump-l } S \rangle$

$\langle \text{proof} \rangle$

lemma *tautology-single-add*:

$\langle \text{tautology } (L + \{\#a\}) \longleftrightarrow \text{tautology } L \vee \neg a \in \# \ L \rangle$

$\langle \text{proof} \rangle$

lemma *backjump-l-implies-exists-backjump*:
assumes bj : $\langle \text{backjump-l } S \ T \rangle$ **and** $\langle \text{inv } S \rangle$ **and** $n\text{-d}$: $\langle \text{no-dup } (\text{trail } S) \rangle$
shows $\exists U. \text{backjump } S \ U$
 $\langle \text{proof} \rangle$

Without additional knowledge on *backjump-l-cond*, it is impossible to have the same invariant.

sublocale *dpll-with-backjumping-ops* $\text{trail clauses}_{NOT} \text{prepend-trail tl-trail add-cl}_{NOT} \text{remove-cl}_{NOT}$
 $\text{inv decide-conds backjump-conds propagate-conds}$
 $\langle \text{proof} \rangle$

sublocale *conflict-driven-clause-learning-ops* $\text{trail clauses}_{NOT} \text{prepend-trail tl-trail add-cl}_{NOT}$
 $\text{remove-cl}_{NOT} \text{inv decide-conds backjump-conds propagate-conds}$
 $\langle \lambda C -. \text{distinct-mset } C \wedge \neg \text{tautology } C \rangle$
 forget-conds
 $\langle \text{proof} \rangle$

lemma *backjump-l-learn-backjump*:
assumes bt : $\langle \text{backjump-l } S \ T \rangle$ **and** inv : $\langle \text{inv } S \rangle$
shows $\exists C' L D. \text{learn } S \ (\text{add-cl}_{NOT} D \ S)$
 $\wedge D = \text{add-mset } L \ C'$
 $\wedge \text{backjump } (\text{add-cl}_{NOT} D \ S) \ T$
 $\wedge \text{atms-of } (\text{add-mset } L \ C') \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S))$
 $\langle \text{proof} \rangle$

lemma *backjump-l-backjump-learn*:
assumes bt : $\langle \text{backjump-l } S \ T \rangle$ **and** inv : $\langle \text{inv } S \rangle$
shows $\exists C' L D S'. \text{backjump } S \ S'$
 $\wedge \text{learn } S' \ T$
 $\wedge D = (\text{add-mset } L \ C')$
 $\wedge T \sim \text{add-cl}_{NOT} D \ S'$
 $\wedge \text{atms-of } (\text{add-mset } L \ C') \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S))$
 $\wedge \text{clauses}_{NOT} \ S \models_{pm} D$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-merged-bj-learn-is-tranclp-cdcl_{NOT}*:
 $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn } S \ T \implies \text{inv } S \implies \text{cdcl}_{NOT}^{++} \ S \ T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT}-and-inv*:
 $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} \ S \ T \implies \text{inv } S \implies \text{cdcl}_{NOT}^{**} \ S \ T \wedge \text{inv } T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtranclp-cdcl_{NOT}-merged-bj-learn-is-rtranclp-cdcl_{NOT}*:
 $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} \ S \ T \implies \text{inv } S \implies \text{cdcl}_{NOT}^{**} \ S \ T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtranclp-cdcl_{NOT}-merged-bj-learn-inv*:
 $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} \ S \ T \implies \text{inv } S \implies \text{inv } T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtranclp-cdcl_{NOT}-merged-bj-learn-no-dup-inv*:
 $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} \ S \ T \implies \text{no-dup } (\text{trail } S) \implies \text{no-dup } (\text{trail } T) \rangle$
 $\langle \text{proof} \rangle$

definition $\mu_C' :: \langle 'v \text{ clause set} \Rightarrow 'st \Rightarrow \text{nat} \rangle$ **where**
 $\mu_C' \ A \ T \equiv \mu_C \ (1 + \text{card } (\text{atms-of-ms } A)) \ (2 + \text{card } (\text{atms-of-ms } A)) \ (\text{trail-weight } T)$

definition $\mu_{CDCL}'\text{-merged} :: \langle 'v \text{ clause set} \Rightarrow 'st \Rightarrow nat \rangle$ **where**
 $\langle \mu_{CDCL}'\text{-merged } A \ T \equiv$
 $((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A)) - \mu_C' A \ T) * 2 + \text{card } (\text{set-mset } (\text{clauses}_{NOT} T)) \rangle$

lemma $cdcl_{NOT}\text{-decreasing-measure}'$:

assumes

$\langle cdcl_{NOT}\text{-merged-bj-learn } S \ T \rangle$ **and**

$\langle inv: \langle inv \ S \rangle$ **and**

$\text{atm-clss}: \langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**

$\text{atm-trail}: \langle \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**

$n\text{-d}: \langle \text{no-dup } (\text{trail } S) \rangle$ **and**

$\text{fin-A}: \langle \text{finite } A \rangle$

shows $\langle \mu_{CDCL}'\text{-merged } A \ T < \mu_{CDCL}'\text{-merged } A \ S \rangle$

$\langle \text{proof} \rangle$

lemma $wf\text{-}cdcl_{NOT}\text{-merged-bj-learn}$:

assumes

$\text{fin-A}: \langle \text{finite } A \rangle$

shows $\langle wf \ \{ (T, S). \}$

$(inv \ S \wedge \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{no-dup } (\text{trail } S))$

$\wedge cdcl_{NOT}\text{-merged-bj-learn } S \ T \rangle$

$\langle \text{proof} \rangle$

lemma $\text{in-atms-neg-defined}$: $\langle x \in \text{atms-of } C' \implies F \models_{as} CNot \ C' \implies x \in \text{atm-of } ' \text{ lits-of-l } F \rangle$

$\langle \text{proof} \rangle$

lemma $cdcl_{NOT}\text{-merged-bj-learn-atms-of-ms-clauses-decreasing}$:

assumes $\langle cdcl_{NOT}\text{-merged-bj-learn } S \ T \rangle$ **and** $\langle inv \ S \rangle$

shows $\langle \text{atms-of-mm } (\text{clauses}_{NOT} T) \subseteq \text{atms-of-mm } (\text{clauses}_{NOT} S) \cup \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \rangle$

$\langle \text{proof} \rangle$

lemma $cdcl_{NOT}\text{-merged-bj-learn-atms-in-trail-in-set}$:

assumes

$\langle cdcl_{NOT}\text{-merged-bj-learn } S \ T \rangle$ **and** $\langle inv \ S \rangle$ **and**

$\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

$\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$

shows $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq A \rangle$

$\langle \text{proof} \rangle$

lemma $rtrancpl\text{-}cdcl_{NOT}\text{-merged-bj-learn-trail-clauses-bound}$:

assumes

$cdcl: \langle cdcl_{NOT}\text{-merged-bj-learn}^{**} S \ T \rangle$ **and**

$inv: \langle inv \ S \rangle$ **and**

$\text{atms-clauses-S}: \langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

$\text{atms-trail-S}: \langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } S)) \subseteq A \rangle$

shows $\langle \text{atm-of } ' (\text{lits-of-l } (\text{trail } T)) \subseteq A \wedge \text{atms-of-mm } (\text{clauses}_{NOT} T) \subseteq A \rangle$

$\langle \text{proof} \rangle$

lemma $cdcl_{NOT}\text{-merged-bj-learn-trail-clauses-bound}$:

assumes

$cdcl: \langle cdcl_{NOT}\text{-merged-bj-learn } S \ T \rangle$ **and**

$inv: \langle inv \ S \rangle$ **and**

$\text{atms-clauses-S}: \langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq A \rangle$ **and**

atms-trail-S: $\langle \text{atm-of } (lits\text{-of-l } (trail\ S)) \subseteq A \rangle$
shows $\langle \text{atm-of } (lits\text{-of-l } (trail\ T)) \subseteq A \wedge \text{atms-of-mm } (clauses_{NOT}\ T) \subseteq A \rangle$
 $\langle \text{proof} \rangle$

lemma *trancpl-cdcl_{NOT}-cdcl_{NOT}-trancpl*:

assumes

$\langle cdcl_{NOT}\text{-merged-bj-learn}^{++}\ S\ T \rangle$ **and**

inv: $\langle inv\ S \rangle$ **and**

atm-cls: $\langle \text{atms-of-mm } (clauses_{NOT}\ S) \subseteq \text{atms-of-ms } A \rangle$ **and**

atm-trail: $\langle \text{atm-of } (lits\text{-of-l } (trail\ S)) \subseteq \text{atms-of-ms } A \rangle$ **and**

n-d: $\langle \text{no-dup } (trail\ S) \rangle$ **and**

fin-A[simp]: $\langle \text{finite } A \rangle$

shows $\langle (T, S) \in \{(T, S)\}.$

$(inv\ S \wedge \text{atms-of-mm } (clauses_{NOT}\ S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } (lits\text{-of-l } (trail\ S)) \subseteq \text{atms-of-ms } A$
 $\wedge \text{no-dup } (trail\ S))$

$\wedge cdcl_{NOT}\text{-merged-bj-learn } S\ T \rangle^+ \text{ (is } \langle \cdot \in ?P^+ \rangle)$

$\langle \text{proof} \rangle$

lemma *wf-trancpl-cdcl_{NOT}-merged-bj-learn*:

assumes $\langle \text{finite } A \rangle$

shows $\langle wf\ \{(T, S)\}.$

$(inv\ S \wedge \text{atms-of-mm } (clauses_{NOT}\ S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } (lits\text{-of-l } (trail\ S)) \subseteq \text{atms-of-ms } A$
 $\wedge \text{no-dup } (trail\ S))$

$\wedge cdcl_{NOT}\text{-merged-bj-learn}^{++}\ S\ T \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-merged-bj-learn-final-state*:

fixes $A :: \langle 'v\ \text{clause set} \rangle$ **and** $S\ T :: \langle 'st \rangle$

assumes

n-s: $\langle \text{no-step } cdcl_{NOT}\text{-merged-bj-learn } S \rangle$ **and**

atms-S: $\langle \text{atms-of-mm } (clauses_{NOT}\ S) \subseteq \text{atms-of-ms } A \rangle$ **and**

atms-trail: $\langle \text{atm-of } (lits\text{-of-l } (trail\ S)) \subseteq \text{atms-of-ms } A \rangle$ **and**

n-d: $\langle \text{no-dup } (trail\ S) \rangle$ **and**

$\langle \text{finite } A \rangle$ **and**

inv: $\langle inv\ S \rangle$ **and**

decomp: $\langle \text{all-decomposition-implies-m } (clauses_{NOT}\ S) (\text{get-all-ann-decomposition } (trail\ S)) \rangle$

shows $\langle \text{unsatisfiable } (\text{set-mset } (clauses_{NOT}\ S))$

$\vee (trail\ S \models_{asm} clauses_{NOT}\ S \wedge \text{satisfiable } (\text{set-mset } (clauses_{NOT}\ S))) \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-merged-bj-learn-all-decomposition-implies*:

assumes $\langle cdcl_{NOT}\text{-merged-bj-learn } S\ T \rangle$ **and** *inv*: $\langle inv\ S \rangle$

$\langle \text{all-decomposition-implies-m } (clauses_{NOT}\ S) (\text{get-all-ann-decomposition } (trail\ S)) \rangle$

shows

$\langle \text{all-decomposition-implies-m } (clauses_{NOT}\ T) (\text{get-all-ann-decomposition } (trail\ T)) \rangle$

$\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-merged-bj-learn-all-decomposition-implies*:

assumes $\langle cdcl_{NOT}\text{-merged-bj-learn}^{**}\ S\ T \rangle$ **and** *inv*: $\langle inv\ S \rangle$

$\langle \text{all-decomposition-implies-m } (clauses_{NOT}\ S) (\text{get-all-ann-decomposition } (trail\ S)) \rangle$

shows

$\langle \text{all-decomposition-implies-m } (clauses_{NOT}\ T) (\text{get-all-ann-decomposition } (trail\ T)) \rangle$

$\langle \text{proof} \rangle$

lemma *full-cdcl_{NOT}-merged-bj-learn-final-state*:

fixes $A :: \langle 'v\ \text{clause set} \rangle$ **and** $S\ T :: \langle 'st \rangle$

assumes
full: $\langle \text{full cdcl}_{NOT}\text{-merged-bj-learn } S \ T \rangle$ **and**
atms-S: $\langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**
atms-trail: $\langle \text{atm-of } \text{ ' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**
n-d: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**
 $\langle \text{finite } A \rangle$ **and**
inv: $\langle \text{inv } S \rangle$ **and**
decomp: $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \ S) \ (\text{get-all-ann-decomposition } (\text{trail } S)) \rangle$
shows $\langle \text{unsatisfiable } (\text{set-mset } (\text{clauses}_{NOT} \ T)) \vee (\text{trail } T \models_{asm} \text{clauses}_{NOT} \ T \wedge \text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} \ T))) \rangle$
 $\langle \text{proof} \rangle$
end

2.2.7 Instantiations

In this section, we instantiate the previous locales to ensure that the assumption are not contradictory.

locale *cdcl_{NOT}-with-backtrack-and-restarts* =
conflict-driven-clause-learning-learning-before-backjump-only-distinct-learnt
trail clauses_{NOT} prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}
inv decide-conds backjump-conds propagate-conds learn-restrictions forget-restrictions
for
trail :: $\langle 'st \Rightarrow ('v, \text{unit}) \text{ ann-lits} \rangle$ **and**
clauses_{NOT} :: $\langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**
prepend-trail :: $\langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
tl-trail :: $\langle 'st \Rightarrow 'st \rangle$ **and**
add-cl_{NOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
remove-cl_{NOT} :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
inv :: $\langle 'st \Rightarrow \text{bool} \rangle$ **and**
decide-conds :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
backjump-conds :: $\langle 'v \text{ clause} \Rightarrow 'v \text{ clause} \Rightarrow 'v \text{ literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
propagate-conds :: $\langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
learn-restrictions forget-restrictions :: $\langle 'v \text{ clause} \Rightarrow 'st \Rightarrow \text{bool} \rangle$
+
fixes *f* :: $\langle \text{nat} \Rightarrow \text{nat} \rangle$
assumes
unbounded: $\langle \text{unbounded } f \rangle$ **and** *f-ge-1*: $\langle \bigwedge n. n \geq 1 \Rightarrow f \ n \geq 1 \rangle$ **and**
inv-restart: $\langle \bigwedge S \ T. \text{inv } S \Rightarrow T \sim \text{reduce-trail-to}_{NOT} \ ([::'a \text{ list}) \ S \Rightarrow \text{inv } T \rangle$
begin

lemma *bound-inv-inv*:

assumes
 $\langle \text{inv } S \rangle$ **and**
n-d: $\langle \text{no-dup } (\text{trail } S) \rangle$ **and**
atms-clss-S-A: $\langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**
atms-trail-S-A: $\langle \text{atm-of } \text{ ' lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{finite } A \rangle$ **and**
cdcl_{NOT}: $\langle \text{cdcl}_{NOT} \ S \ T \rangle$
shows
 $\langle \text{atms-of-mm } (\text{clauses}_{NOT} \ T) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{atm-of } \text{ ' lits-of-l } (\text{trail } T) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{finite } A \rangle$
 $\langle \text{proof} \rangle$

sublocale *cdcl_{NOT}-increasing-restarts-ops* $\langle \lambda S T. T \sim \text{reduce-trail-to}_{NOT} ([\cdot]::'a \text{ list}) S \rangle$ *cdcl_{NOT} f*
 $\langle \lambda A S. \text{atms-of-mm} (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \wedge \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \wedge$
finite A
 $\mu_{CDCL}' \langle \lambda S. \text{inv } S \wedge \text{no-dup } (\text{trail } S) \rangle$
 $\mu_{CDCL}'\text{-bound}$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-with-restart- μ_{CDCL}' -le- μ_{CDCL}' -bound:*

assumes

cdcl_{NOT}: $\langle \text{cdcl}_{NOT}\text{-restart } (T, a) (V, b) \rangle$ **and**

cdcl_{NOT}-inv:

$\langle \text{inv } T \rangle$

$\langle \text{no-dup } (\text{trail } T) \rangle$ **and**

bound-inv:

$\langle \text{atms-of-mm} (\text{clauses}_{NOT} T) \subseteq \text{atms-of-ms } A \rangle$

$\langle \text{atm-of } ' \text{ lits-of-l } (\text{trail } T) \subseteq \text{atms-of-ms } A \rangle$

$\langle \text{finite } A \rangle$

shows $\langle \mu_{CDCL}' A V \leq \mu_{CDCL}'\text{-bound } A T \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-with-restart- μ_{CDCL}' -bound-le- μ_{CDCL}' -bound:*

assumes

cdcl_{NOT}: $\langle \text{cdcl}_{NOT}\text{-restart } (T, a) (V, b) \rangle$ **and**

cdcl_{NOT}-inv:

$\langle \text{inv } T \rangle$

$\langle \text{no-dup } (\text{trail } T) \rangle$ **and**

bound-inv:

$\langle \text{atms-of-mm} (\text{clauses}_{NOT} T) \subseteq \text{atms-of-ms } A \rangle$

$\langle \text{atm-of } ' \text{ lits-of-l } (\text{trail } T) \subseteq \text{atms-of-ms } A \rangle$

$\langle \text{finite } A \rangle$

shows $\langle \mu_{CDCL}'\text{-bound } A V \leq \mu_{CDCL}'\text{-bound } A T \rangle$

$\langle \text{proof} \rangle$

sublocale *cdcl_{NOT}-increasing-restarts - - - - -*

f

$\langle \lambda S T. T \sim \text{reduce-trail-to}_{NOT} ([\cdot]::'a \text{ list}) S \rangle$

$\langle \lambda A S. \text{atms-of-mm} (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A$

$\wedge \text{atm-of } ' \text{ lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \wedge \text{finite } A \rangle$

$\mu_{CDCL}' \text{ cdcl}_{NOT}$

$\langle \lambda S. \text{inv } S \wedge \text{no-dup } (\text{trail } S) \rangle$

$\mu_{CDCL}'\text{-bound}$

$\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-restart-all-decomposition-implies:*

assumes $\langle \text{cdcl}_{NOT}\text{-restart } S T \rangle$ **and**

$\langle \text{inv } (\text{fst } S) \rangle$ **and**

$\langle \text{no-dup } (\text{trail } (\text{fst } S)) \rangle$

$\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} (\text{fst } S)) (\text{get-all-ann-decomposition } (\text{trail } (\text{fst } S))) \rangle$

shows

$\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} (\text{fst } T)) (\text{get-all-ann-decomposition } (\text{trail } (\text{fst } T))) \rangle$

$\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_{NOT}-restart-all-decomposition-implies:*

assumes $\langle \text{cdcl}_{NOT}\text{-restart}^{**} S T \rangle$ **and**

inv: $\langle \text{inv } (\text{fst } S) \rangle$ **and**

n-d: $\langle \text{no-dup } (\text{trail } (\text{fst } S)) \rangle$ **and**

decomp:
 $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \text{ (fst } S)) \text{ (get-all-ann-decomposition (trail (fst } S))) \rangle$
shows
 $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \text{ (fst } T)) \text{ (get-all-ann-decomposition (trail (fst } T))) \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-restart-sat-ext-iff:*
assumes
st: $\langle \text{cdcl}_{NOT}\text{-restart } S \ T \rangle$ **and**
n-d: $\langle \text{no-dup (trail (fst } S)) \rangle$ **and**
inv: $\langle \text{inv (fst } S) \rangle$
shows $\langle I \models_{\text{sextm}} \text{clauses}_{NOT} \text{ (fst } S) \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} \text{ (fst } T) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancp-cdcl_{NOT}-restart-sat-ext-iff:*
fixes $S \ T :: \langle 'st \times \text{nat} \rangle$
assumes
st: $\langle \text{cdcl}_{NOT}\text{-restart}^{**} S \ T \rangle$ **and**
n-d: $\langle \text{no-dup (trail (fst } S)) \rangle$ **and**
inv: $\langle \text{inv (fst } S) \rangle$
shows $\langle I \models_{\text{sextm}} \text{clauses}_{NOT} \text{ (fst } S) \longleftrightarrow I \models_{\text{sextm}} \text{clauses}_{NOT} \text{ (fst } T) \rangle$
 $\langle \text{proof} \rangle$

theorem *full-cdcl_{NOT}-restart-backjump-final-state:*
fixes $A :: \langle 'v \text{ clause set} \rangle$ **and** $S \ T :: \langle 'st \rangle$
assumes
full: $\langle \text{full cdcl}_{NOT}\text{-restart } (S, n) \ (T, m) \rangle$ **and**
atms-S: $\langle \text{atms-of-mm } (\text{clauses}_{NOT} \ S) \subseteq \text{atms-of-ms } A \rangle$ **and**
atms-trail: $\langle \text{atm-of ' lits-of-l (trail } S) \subseteq \text{atms-of-ms } A \rangle$ **and**
n-d: $\langle \text{no-dup (trail } S) \rangle$ **and**
fin-A[simp]: $\langle \text{finite } A \rangle$ **and**
inv: $\langle \text{inv } S \rangle$ **and**
decomp: $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \ S) \text{ (get-all-ann-decomposition (trail } S)) \rangle$
shows $\langle \text{unsatisfiable (set-mset (clauses}_{NOT} \ S)) \vee (\text{lits-of-l (trail } T) \models_{\text{sextm}} \text{clauses}_{NOT} \ S \wedge \text{satisfiable (set-mset (clauses}_{NOT} \ S))) \rangle$
 $\langle \text{proof} \rangle$

end — End of the locale *cdcl_{NOT}-with-backtrack-and-restarts*.

The restart does only reset the trail, contrary to Weidenbach's version where forget and restart are always combined. But there is a forget rule.

locale *cdcl_{NOT}-merge-bj-learn-with-backtrack-restarts =*
cdcl_{NOT}-merge-bj-learn trail clauses_{NOT} prepend-trail tl-trail add-cl_{NOT} remove-cl_{NOT}
decide-conds propagate-conds forget-conds
 $\langle \lambda C \ C' \ L' \ S \ T. \text{distinct-mset } C' \wedge L' \notin \# \ C' \wedge \text{backjump-l-cond } C \ C' \ L' \ S \ T \rangle \text{ inv}$
for
trail $:: \langle 'st \Rightarrow ('v, \text{unit}) \text{ ann-lits} \rangle$ **and**
clauses_{NOT} $:: \langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**
prepend-trail $:: \langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
tl-trail $:: \langle 'st \Rightarrow 'st \rangle$ **and**
add-cl_{NOT} $:: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
remove-cl_{NOT} $:: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**
decide-conds $:: \langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
propagate-conds $:: \langle ('v, \text{unit}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
inv $:: \langle 'st \Rightarrow \text{bool} \rangle$ **and**
forget-conds $:: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow \text{bool} \rangle$ **and**
backjump-l-cond $:: \langle 'v \text{ clause} \Rightarrow 'v \text{ clause} \Rightarrow 'v \text{ literal} \Rightarrow 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$

$+$
fixes $f :: \langle \text{nat} \Rightarrow \text{nat} \rangle$
assumes
 $\text{unbounded}: \langle \text{unbounded } f \rangle$ **and** $f\text{-ge-1}: \langle \bigwedge n. n \geq 1 \implies f\ n \geq 1 \rangle$ **and**
 $\text{inv-restart}: \langle \bigwedge S\ T. \text{inv } S \implies T \sim \text{reduce-trail-to}_{NOT} []\ S \implies \text{inv } T \rangle$
begin

definition $\text{not-simplified-clcs} :: \langle 'b \text{ clause multiset} \Rightarrow 'b \text{ clauses} \rangle$
where
 $\langle \text{not-simplified-clcs } A \equiv \{ \#C \in \# A. C \notin \text{simple-clss } (\text{atms-of-mm } A) \# \} \rangle$

lemma $\text{not-simplified-clcs-tautology-distinct-mset}$:
 $\langle \text{not-simplified-clcs } A = \{ \#C \in \# A. \text{tautology } C \vee \neg \text{distinct-mset } C \# \} \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{simple-clss-or-not-simplified-clcs}$:
assumes $\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle x \in \# \text{clauses}_{NOT} S \rangle$ **and** $\langle \text{finite } A \rangle$
shows $\langle x \in \text{simple-clss } (\text{atms-of-ms } A) \vee x \in \# \text{not-simplified-clcs } (\text{clauses}_{NOT} S) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_{NOT}\text{-merged-bj-learn-clauses-bound}$:
assumes
 $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn } S\ T \rangle$ **and**
 $\text{inv}: \langle \text{inv } S \rangle$ **and**
 $\text{atms-clss}: \langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\text{atms-trail}: \langle \text{atm-of } (\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\text{fin-A[simp]}: \langle \text{finite } A \rangle$
shows $\langle \text{set-mset } (\text{clauses}_{NOT} T) \subseteq \text{set-mset } (\text{not-simplified-clcs } (\text{clauses}_{NOT} S)) \cup \text{simple-clss } (\text{atms-of-ms } A) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_{NOT}\text{-merged-bj-learn-not-simplified-decreasing}$:
assumes $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn } S\ T \rangle$
shows $\langle \text{not-simplified-clcs } (\text{clauses}_{NOT} T) \subseteq \# \text{not-simplified-clcs } (\text{clauses}_{NOT} S) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_{NOT}\text{-merged-bj-learn-not-simplified-decreasing}$:
assumes $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} S\ T \rangle$
shows $\langle \text{not-simplified-clcs } (\text{clauses}_{NOT} T) \subseteq \# \text{not-simplified-clcs } (\text{clauses}_{NOT} S) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancpl-cdcl}_{NOT}\text{-merged-bj-learn-clauses-bound}$:
assumes
 $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} S\ T \rangle$ **and**
 $\langle \text{inv } S \rangle$ **and**
 $\langle \text{atms-of-mm } (\text{clauses}_{NOT} S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{atm-of } (\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\text{finite[simp]}: \langle \text{finite } A \rangle$
shows $\langle \text{set-mset } (\text{clauses}_{NOT} T) \subseteq \text{set-mset } (\text{not-simplified-clcs } (\text{clauses}_{NOT} S)) \cup \text{simple-clss } (\text{atms-of-ms } A) \rangle$
 $\langle \text{proof} \rangle$

abbreviation $\mu_{CDCL}'\text{-bound}$ **where**
 $\langle \mu_{CDCL}'\text{-bound } A\ T \equiv ((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))) * 2 + \text{card } (\text{set-mset } (\text{not-simplified-clcs } (\text{clauses}_{NOT} T))) \rangle$

+ $3 \wedge \text{card} (\text{atms-of-ms } A)$

lemma *rtranchp-cdcl_{NOT}-merged-bj-learn-clauses-bound-card:*

assumes

$\langle \text{cdcl}_{\text{NOT}}\text{-merged-bj-learn}^{**} S T \rangle$ **and**
 $\langle \text{inv } S \rangle$ **and**
 $\langle \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{atm-of } '(\text{lits-of-l } (\text{trail } S)) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\text{finite: } \langle \text{finite } A \rangle$

shows $\langle \mu_{\text{CDCL}}'\text{-merged } A T \leq \mu_{\text{CDCL}}'\text{-bound } A S \rangle$

$\langle \text{proof} \rangle$

sublocale *cdcl_{NOT}-increasing-restarts-ops* $\langle \lambda S T. T \sim \text{reduce-trail-to}_{\text{NOT}} ([::'a \text{ list}) S \rangle$

cdcl_{NOT}-merged-bj-learn *f*

$\langle \lambda A S. \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{atm-of } ' \text{lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \wedge \text{finite } A \rangle$

$\mu_{\text{CDCL}}'\text{-merged}$

$\langle \lambda S. \text{inv } S \wedge \text{no-dup } (\text{trail } S) \rangle$

$\mu_{\text{CDCL}}'\text{-bound}$

$\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-restart- $\mu_{\text{CDCL}}'\text{-merged-le-}\mu_{\text{CDCL}}'\text{-bound}$:*

assumes

$\langle \text{cdcl}_{\text{NOT}}\text{-restart } T V \rangle$
 $\langle \text{inv } (\text{fst } T) \rangle$ **and**
 $\langle \text{no-dup } (\text{trail } (\text{fst } T)) \rangle$ **and**
 $\langle \text{atms-of-mm } (\text{clauses}_{\text{NOT}} (\text{fst } T)) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{atm-of } ' \text{lits-of-l } (\text{trail } (\text{fst } T)) \subseteq \text{atms-of-ms } A \rangle$ **and**
 $\langle \text{finite } A \rangle$

shows $\langle \mu_{\text{CDCL}}'\text{-merged } A (\text{fst } V) \leq \mu_{\text{CDCL}}'\text{-bound } A (\text{fst } T) \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-restart- $\mu_{\text{CDCL}}'\text{-bound-le-}\mu_{\text{CDCL}}'\text{-bound}$:*

assumes

$\langle \text{cdcl}_{\text{NOT}}\text{-restart } T V \rangle$ **and**
 $\langle \text{no-dup } (\text{trail } (\text{fst } T)) \rangle$ **and**
 $\langle \text{inv } (\text{fst } T) \rangle$ **and**
 $\text{fin: } \langle \text{finite } A \rangle$

shows $\langle \mu_{\text{CDCL}}'\text{-bound } A (\text{fst } V) \leq \mu_{\text{CDCL}}'\text{-bound } A (\text{fst } T) \rangle$

$\langle \text{proof} \rangle$

sublocale *cdcl_{NOT}-increasing-restarts - - - - f*

$\langle \lambda S T. T \sim \text{reduce-trail-to}_{\text{NOT}} ([::'a \text{ list}) S \rangle$

$\langle \lambda A S. \text{atms-of-mm } (\text{clauses}_{\text{NOT}} S) \subseteq \text{atms-of-ms } A$
 $\wedge \text{atm-of } ' \text{lits-of-l } (\text{trail } S) \subseteq \text{atms-of-ms } A \wedge \text{finite } A \rangle$

$\mu_{\text{CDCL}}'\text{-merged}$ *cdcl_{NOT}-merged-bj-learn*

$\langle \lambda S. \text{inv } S \wedge \text{no-dup } (\text{trail } S) \rangle$

$\langle \lambda A T. ((2 + \text{card } (\text{atms-of-ms } A)) \wedge (1 + \text{card } (\text{atms-of-ms } A))) * 2$
 $+ \text{card } (\text{set-mset } (\text{not-simplified-cls}(\text{clauses}_{\text{NOT}} T)))$
 $+ 3 \wedge \text{card } (\text{atms-of-ms } A) \rangle$

$\langle \text{proof} \rangle$

lemma *true-clss-ext-decrease-right-insert:* $\langle I \models_{\text{sext}} \text{insert } C (\text{set-mset } M) \implies I \models_{\text{sextm}} M \rangle$

$\langle \text{proof} \rangle$

lemma *true-clss-ext-decrease-add-implied:*

assumes $\langle M \models_{pm} C \rangle$
shows $\langle I \models_{sext} \text{insert } C \text{ (set-mset } M) \longleftrightarrow I \models_{sextm} M \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-merged-bj-learn-bj-sat-ext-iff:*

assumes $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn } S \ T \rangle$ **and** $\text{inv: } \langle \text{inv } S \rangle$
shows $\langle I \models_{sextm} \text{clauses}_{NOT} \ S \longleftrightarrow I \models_{sextm} \text{clauses}_{NOT} \ T \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-merged-bj-learn-bj-sat-ext-iff:*

assumes $\langle \text{cdcl}_{NOT}\text{-merged-bj-learn}^{**} \ S \ T \rangle$ **and** $\langle \text{inv } S \rangle$
shows $\langle I \models_{sextm} \text{clauses}_{NOT} \ S \longleftrightarrow I \models_{sextm} \text{clauses}_{NOT} \ T \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-restart-eq-sat-iff:*

assumes
 $\langle \text{cdcl}_{NOT}\text{-restart } S \ T \rangle$ **and**
 $\text{inv: } \langle \text{inv } (\text{fst } S) \rangle$
shows $\langle I \models_{sextm} \text{clauses}_{NOT} \ (\text{fst } S) \longleftrightarrow I \models_{sextm} \text{clauses}_{NOT} \ (\text{fst } T) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-restart-eq-sat-iff:*

assumes
 $\langle \text{cdcl}_{NOT}\text{-restart}^{**} \ S \ T \rangle$ **and**
 $\text{inv: } \langle \text{inv } (\text{fst } S) \rangle$ **and** $n\text{-d: } \langle \text{no-dup}(\text{trail } (\text{fst } S)) \rangle$
shows $\langle I \models_{sextm} \text{clauses}_{NOT} \ (\text{fst } S) \longleftrightarrow I \models_{sextm} \text{clauses}_{NOT} \ (\text{fst } T) \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_{NOT}-restart-all-decomposition-implies-m:*

assumes
 $\langle \text{cdcl}_{NOT}\text{-restart } S \ T \rangle$ **and**
 $\text{inv: } \langle \text{inv } (\text{fst } S) \rangle$ **and** $n\text{-d: } \langle \text{no-dup}(\text{trail } (\text{fst } S)) \rangle$ **and**
 $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \ (\text{fst } S))$
 $\text{ (get-all-ann-decomposition } (\text{trail } (\text{fst } S))) \rangle$
shows $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \ (\text{fst } T))$
 $\text{ (get-all-ann-decomposition } (\text{trail } (\text{fst } T))) \rangle$
 $\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_{NOT}-restart-all-decomposition-implies-m:*

assumes
 $\langle \text{cdcl}_{NOT}\text{-restart}^{**} \ S \ T \rangle$ **and**
 $\text{inv: } \langle \text{inv } (\text{fst } S) \rangle$ **and** $n\text{-d: } \langle \text{no-dup}(\text{trail } (\text{fst } S)) \rangle$ **and**
 $\text{decomp: } \langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \ (\text{fst } S))$
 $\text{ (get-all-ann-decomposition } (\text{trail } (\text{fst } S))) \rangle$
shows $\langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \ (\text{fst } T))$
 $\text{ (get-all-ann-decomposition } (\text{trail } (\text{fst } T))) \rangle$
 $\langle \text{proof} \rangle$

lemma *full-cdcl_{NOT}-restart-normal-form:*

assumes
 $\text{full: } \langle \text{full } \text{cdcl}_{NOT}\text{-restart } S \ T \rangle$ **and**
 $\text{inv: } \langle \text{inv } (\text{fst } S) \rangle$ **and** $n\text{-d: } \langle \text{no-dup}(\text{trail } (\text{fst } S)) \rangle$ **and**
 $\text{decomp: } \langle \text{all-decomposition-implies-m } (\text{clauses}_{NOT} \ (\text{fst } S))$
 $\text{ (get-all-ann-decomposition } (\text{trail } (\text{fst } S))) \rangle$ **and**
 $\text{atms-cls: } \langle \text{atms-of-mm } (\text{clauses}_{NOT} \ (\text{fst } S)) \subseteq \text{atms-of-ms } A \rangle$ **and**

atms-trail: $\langle \text{atm-of } \text{ lits-of-l } (\text{trail } (\text{fst } S)) \subseteq \text{atms-of-ms } A \rangle$ **and**
fin: $\langle \text{finite } A \rangle$
shows $\langle \text{unsatisfiable } (\text{set-mset } (\text{clauses}_{NOT} (\text{fst } S)))$
 $\vee \text{ lits-of-l } (\text{trail } (\text{fst } T)) \models_{\text{sextm}} \text{clauses}_{NOT} (\text{fst } S) \wedge$
 $\text{satisfiable } (\text{set-mset } (\text{clauses}_{NOT} (\text{fst } S))) \rangle$
 $\langle \text{proof} \rangle$

corollary *full-cdcl_{NOT}-restart-normal-form-init-state*:

assumes

init-state: $\langle \text{trail } S = [] \rangle \langle \text{clauses}_{NOT} S = N \rangle$ **and**

full: $\langle \text{full cdcl}_{NOT}\text{-restart } (S, 0) T \rangle$ **and**

inv: $\langle \text{inv } S \rangle$

shows $\langle \text{unsatisfiable } (\text{set-mset } N)$

$\vee \text{ lits-of-l } (\text{trail } (\text{fst } T)) \models_{\text{sextm}} N \wedge \text{satisfiable } (\text{set-mset } N) \rangle$

$\langle \text{proof} \rangle$

end — End of locale *cdcl_{NOT}-merge-bj-learn-with-backtrack-restarts*.

end

theory *CDCL-WNOT*

imports *CDCL-NOT CDCL-W-Merge*

begin

2.3 Link between Weidenbach's and NOT's CDCL

2.3.1 Inclusion of the states

declare *upt.simps*(2)[*simp del*]

fun *convert-ann-lit-from-W* **where**

convert-ann-lit-from-W (*Propagated L* -) = *Propagated L* () |

convert-ann-lit-from-W (*Decided L*) = *Decided L*

abbreviation *convert-trail-from-W* ::

$(\text{'v}, \text{'mark}) \text{ ann-lits}$

$\Rightarrow (\text{'v}, \text{unit}) \text{ ann-lits}$ **where**

convert-trail-from-W $\equiv \text{map } \text{convert-ann-lit-from-W}$

lemma *lits-of-l-convert-trail-from-W*[*simp*]:

lits-of-l (*convert-trail-from-W M*) = *lits-of-l M*

$\langle \text{proof} \rangle$

lemma *lit-of-convert-trail-from-W*[*simp*]:

lit-of (*convert-ann-lit-from-W L*) = *lit-of L*

$\langle \text{proof} \rangle$

lemma *no-dup-convert-from-W*[*simp*]:

no-dup (*convert-trail-from-W M*) $\longleftrightarrow \text{no-dup } M$

$\langle \text{proof} \rangle$

lemma *convert-trail-from-W-true-annots*[*simp*]:

convert-trail-from-W M $\models_{\text{as}} C \longleftrightarrow M \models_{\text{as}} C$

$\langle \text{proof} \rangle$

lemma *defined-lit-convert-trail-from-W*[*simp*]:

defined-lit (*convert-trail-from-W S*) = *defined-lit S*
 ⟨proof⟩

lemma *is-decided-convert-trail-from-W*[simp]:
 ⟨*is-decided* (*convert-ann-lit-from-W L*) = *is-decided L*⟩
 ⟨proof⟩

lemma *count-decided-conver-Trail-from-W*[simp]:
 ⟨*count-decided* (*convert-trail-from-W M*) = *count-decided M*⟩
 ⟨proof⟩

The values 0 and {#} are dummy values.

consts *dummy-cl*s :: 'cls
fun *convert-ann-lit-from-NOT*
 :: ('v, 'mark) *ann-lit* ⇒ ('v, 'cls) *ann-lit* **where**
convert-ann-lit-from-NOT (*Propagated L* -) = *Propagated L dummy-cl*s |
convert-ann-lit-from-NOT (*Decided L*) = *Decided L*

abbreviation *convert-trail-from-NOT* **where**
convert-trail-from-NOT ≡ *map convert-ann-lit-from-NOT*

lemma *undefined-lit-convert-trail-from-NOT*[simp]:
undefined-lit (*convert-trail-from-NOT F*) *L* ↔ *undefined-lit F L*
 ⟨proof⟩

lemma *lits-of-l-convert-trail-from-NOT*:
lits-of-l (*convert-trail-from-NOT F*) = *lits-of-l F*
 ⟨proof⟩

lemma *convert-trail-from-W-from-NOT*[simp]:
convert-trail-from-W (*convert-trail-from-NOT M*) = *M*
 ⟨proof⟩

lemma *convert-trail-from-W-convert-lit-from-NOT*[simp]:
convert-ann-lit-from-W (*convert-ann-lit-from-NOT L*) = *L*
 ⟨proof⟩

abbreviation *trail_{NOT}* **where**
trail_{NOT} S ≡ *convert-trail-from-W (fst S)*

lemma *undefined-lit-convert-trail-from-W*[iff]:
undefined-lit (*convert-trail-from-W M*) *L* ↔ *undefined-lit M L*
 ⟨proof⟩

lemma *lit-of-convert-ann-lit-from-NOT*[iff]:
lit-of (*convert-ann-lit-from-NOT L*) = *lit-of L*
 ⟨proof⟩

sublocale *state_W* ⊆ *dpll-state-ops* **where**
trail = λ*S*. *convert-trail-from-W (trail S)* **and**
clauses_{NOT} = *clauses* **and**
prepend-trail = λ*L S*. *cons-trail (convert-ann-lit-from-NOT L) S* **and**
tl-trail = λ*S*. *tl-trail S* **and**
add-cl_{NOT} = λ*C S*. *add-learned-cl_s C S* **and**
remove-cl_{NOT} = λ*C S*. *remove-cl_s C S*
 ⟨proof⟩

sublocale $state_W \subseteq dpll\text{-}state$ **where**
trail = $\lambda S. \text{convert-trail-from-}W \text{ (trail } S) \text{ and}$
*clauses*_{NOT} = *clauses* **and**
prepend-trail = $\lambda L S. \text{cons-trail (convert-ann-lit-from-NOT } L) S \text{ and}$
tl-trail = $\lambda S. \text{tl-trail } S \text{ and}$
*add-cls*_{NOT} = $\lambda C S. \text{add-learned-cls } C S \text{ and}$
*remove-cls*_{NOT} = $\lambda C S. \text{remove-cls } C S$
 $\langle \text{proof} \rangle$

context $state_W$
begin
declare $state\text{-}simp_{NOT}[simp \text{ del}]$
end

2.3.2 Inclusion of Weidendenbch's CDCL without Strategy

sublocale $conflict\text{-}driven\text{-}clause\text{-}learning_W \subseteq cdcl_{NOT}\text{-}merge\text{-}bj\text{-}learn\text{-}ops$ **where**
trail = $\lambda S. \text{convert-trail-from-}W \text{ (trail } S) \text{ and}$
*clauses*_{NOT} = *clauses* **and**
prepend-trail = $\lambda L S. \text{cons-trail (convert-ann-lit-from-NOT } L) S \text{ and}$
tl-trail = $\lambda S. \text{tl-trail } S \text{ and}$
*add-cls*_{NOT} = $\lambda C S. \text{add-learned-cls } C S \text{ and}$
*remove-cls*_{NOT} = $\lambda C S. \text{remove-cls } C S \text{ and}$
decide-conds = $\lambda - -. \text{True and}$
propagate-conds = $\lambda - -. \text{True and}$
forget-conds = $\lambda - S. \text{conflicting } S = \text{None and}$
backjump-l-cond = $\lambda C C' L' S T. \text{backjump-l-cond } C C' L' S T$
 $\wedge \text{distinct-mset } C' \wedge L' \notin \# C' \wedge \neg \text{tautology (add-mset } L' C')$
 $\langle \text{proof} \rangle$

sublocale $conflict\text{-}driven\text{-}clause\text{-}learning_W \subseteq cdcl_{NOT}\text{-}merge\text{-}bj\text{-}learn\text{-}proxy$ **where**
trail = $\lambda S. \text{convert-trail-from-}W \text{ (trail } S) \text{ and}$
*clauses*_{NOT} = *clauses* **and**
prepend-trail = $\lambda L S. \text{cons-trail (convert-ann-lit-from-NOT } L) S \text{ and}$
tl-trail = $\lambda S. \text{tl-trail } S \text{ and}$
*add-cls*_{NOT} = $\lambda C S. \text{add-learned-cls } C S \text{ and}$
*remove-cls*_{NOT} = $\lambda C S. \text{remove-cls } C S \text{ and}$
decide-conds = $\lambda - -. \text{True and}$
propagate-conds = $\lambda - -. \text{True and}$
forget-conds = $\lambda - S. \text{conflicting } S = \text{None and}$
backjump-l-cond = *backjump-l-cond* **and**
inv = *inv*_{NOT}
 $\langle \text{proof} \rangle$

sublocale $conflict\text{-}driven\text{-}clause\text{-}learning_W \subseteq cdcl_{NOT}\text{-}merge\text{-}bj\text{-}learn$ **where**
trail = $\lambda S. \text{convert-trail-from-}W \text{ (trail } S) \text{ and}$
*clauses*_{NOT} = *clauses* **and**
prepend-trail = $\lambda L S. \text{cons-trail (convert-ann-lit-from-NOT } L) S \text{ and}$
tl-trail = $\lambda S. \text{tl-trail } S \text{ and}$
*add-cls*_{NOT} = $\lambda C S. \text{add-learned-cls } C S \text{ and}$
*remove-cls*_{NOT} = $\lambda C S. \text{remove-cls } C S \text{ and}$
decide-conds = $\lambda - -. \text{True and}$
propagate-conds = $\lambda - -. \text{True and}$
forget-conds = $\lambda - S. \text{conflicting } S = \text{None and}$
backjump-l-cond = *backjump-l-cond* **and**

$inv = inv_{NOT}$
 $\langle proof \rangle$

context *conflict-driven-clause-learning_W*
begin

Notations are lost while proving locale inclusion:

notation *state-eq_{NOT}* (**infix** \sim_{NOT} 50)

2.3.3 Additional Lemmas between NOT and W states

lemma *trail_W-eq-reduce-trail-to_{NOT}-eq*:
 $trail\ S = trail\ T \implies trail\ (reduce-trail-to_{NOT}\ F\ S) = trail\ (reduce-trail-to_{NOT}\ F\ T)$
 $\langle proof \rangle$

lemma *trail-reduce-trail-to_{NOT}-add-learned-cls*:
 $no-dup\ (trail\ S) \implies$
 $trail\ (reduce-trail-to_{NOT}\ M\ (add-learned-cls\ D\ S)) = trail\ (reduce-trail-to_{NOT}\ M\ S)$
 $\langle proof \rangle$

lemma *reduce-trail-to_{NOT}-reduce-trail-convert*:
 $reduce-trail-to_{NOT}\ C\ S = reduce-trail-to\ (convert-trail-from-NOT\ C)\ S$
 $\langle proof \rangle$

lemma *reduce-trail-to-map[simp]*:
 $reduce-trail-to\ (map\ f\ M)\ S = reduce-trail-to\ M\ S$
 $\langle proof \rangle$

lemma *reduce-trail-to_{NOT}-map[simp]*:
 $reduce-trail-to_{NOT}\ (map\ f\ M)\ S = reduce-trail-to_{NOT}\ M\ S$
 $\langle proof \rangle$

lemma *skip-or-resolve-state-change*:
assumes *skip-or-resolve*** $S\ T$
shows
 $\exists M. trail\ S = M @ trail\ T \wedge (\forall m \in set\ M. \neg is-decided\ m)$
 $clauses\ S = clauses\ T$
 $backtrack-lvl\ S = backtrack-lvl\ T$
 $init-clss\ S = init-clss\ T$
 $learned-clss\ S = learned-clss\ T$
 $\langle proof \rangle$

2.3.4 Inclusion of Weidenbach's CDCL in NOT's CDCL

This lemma shows the inclusion of Weidenbach's CDCL *cdcl_W-merge* (with merging) in NOT's *cdcl_{NOT}-merged-bj-learn*.

lemma *cdcl_W-merge-is-cdcl_{NOT}-merged-bj-learn*:
assumes
 $inv: cdcl_W-all-struct-inv\ S$ **and**
 $cdcl_W-restart: cdcl_W-merge\ S\ T$
shows *cdcl_{NOT}-merged-bj-learn* $S\ T$
 $\vee (no-step\ cdcl_W-merge\ T \wedge conflicting\ T \neq None)$
 $\langle proof \rangle$

abbreviation $cdcl_{NOT}\text{-restart}$ **where**

$cdcl_{NOT}\text{-restart} \equiv restart\text{-ops}.cdcl_{NOT}\text{-raw-restart } cdcl_{NOT} \text{ restart}$

lemma $cdcl_W\text{-merge-restart-is-}cdcl_{NOT}\text{-merged-bj-learn-restart-no-step}$:

assumes

inv : $cdcl_W\text{-all-struct-inv } S$ **and**

$cdcl_W\text{-restart}:cdcl_W\text{-merge-restart } S \ T$

shows $cdcl_{NOT}\text{-restart}^{**} S \ T \vee (no\text{-step } cdcl_W\text{-merge } T \wedge conflicting \ T \neq None)$

$\langle proof \rangle$

abbreviation $\mu_{FW} :: 'st \Rightarrow nat$ **where**

$\mu_{FW} \ S \equiv (if \ no\text{-step } cdcl_W\text{-merge } S \text{ then } 0 \text{ else } 1 + \mu_{CDCL}'\text{-merged } (set\text{-mset } (init\text{-class } S)) \ S)$

lemma $cdcl_W\text{-merge-}\mu_{FW}\text{-decreasing}$:

assumes

inv : $cdcl_W\text{-all-struct-inv } S$ **and**

fw : $cdcl_W\text{-merge } S \ T$

shows $\mu_{FW} \ T < \mu_{FW} \ S$

$\langle proof \rangle$

lemma $wf\text{-}cdcl_W\text{-merge}$: $wf \ \{(T, S). \ cdcl_W\text{-all-struct-inv } S \wedge cdcl_W\text{-merge } S \ T\}$

$\langle proof \rangle$

lemma $transclp\text{-}cdcl_W\text{-merge-}cdcl_W\text{-merge-transcl}$:

$\{(T, S). \ cdcl_W\text{-all-struct-inv } S \wedge cdcl_W\text{-merge}^{++} S \ T\}$

$\subseteq \{(T, S). \ cdcl_W\text{-all-struct-inv } S \wedge cdcl_W\text{-merge } S \ T\}^+$

$\langle proof \rangle$

lemma $wf\text{-}transclp\text{-}cdcl_W\text{-merge}$: $wf \ \{(T, S). \ cdcl_W\text{-all-struct-inv } S \wedge cdcl_W\text{-merge}^{++} S \ T\}$

$\langle proof \rangle$

lemma $wf\text{-}cdcl_W\text{-bj-all-struct}$: $wf \ \{(T, S). \ cdcl_W\text{-all-struct-inv } S \wedge cdcl_W\text{-bj } S \ T\}$

$\langle proof \rangle$

lemma $cdcl_W\text{-conflicting-true-}cdcl_W\text{-merge-restart}$:

assumes $cdcl_W \ S \ V$ **and** $conf$: $conflicting \ S = None$

shows $(cdcl_W\text{-merge } S \ V \wedge conflicting \ V = None) \vee (conflicting \ V \neq None \wedge conflict \ S \ V)$

$\langle proof \rangle$

lemma $transcl\text{-}cdcl_W\text{-conflicting-true-}cdcl_W\text{-merge-restart}$:

assumes $cdcl_W^{++} S \ V$ **and** inv : $cdcl_W\text{-M-level-inv } S$ **and** $conflicting \ S = None$

shows $(cdcl_W\text{-merge}^{++} S \ V \wedge conflicting \ V = None)$

$\vee (\exists T \ U. \ cdcl_W\text{-merge}^{**} S \ T \wedge conflicting \ V \neq None \wedge conflict \ T \ U \wedge cdcl_W\text{-bj}^{**} U \ V)$

$\langle proof \rangle$

lemma $wf\text{-}cdcl_W$: $wf \ \{(T, S). \ cdcl_W\text{-all-struct-inv } S \wedge cdcl_W \ S \ T\}$

$\langle proof \rangle$

lemma $wf\text{-}cdcl_W\text{-stgy}$:

$\langle wf \ \{(T, S). \ cdcl_W\text{-all-struct-inv } S \wedge cdcl_W\text{-stgy } S \ T\} \rangle$

$\langle proof \rangle$

end

2.3.5 Inclusion of Weidendenbch's CDCL with Strategy

context *conflict-driven-clause-learning_W*

begin

abbreviation *propagate-conds* **where**

propagate-conds $\equiv \lambda -. \text{propagate}$

abbreviation (*input*) *decide-conds* **where**

decide-conds $S\ T \equiv \text{decide } S\ T \wedge \text{no-step conflict } S \wedge \text{no-step propagate } S$

abbreviation *backjump-l-conds-stgy* :: '*v* clause \Rightarrow '*v* clause \Rightarrow '*v* literal \Rightarrow '*st* \Rightarrow '*st* \Rightarrow bool **where**

backjump-l-conds-stgy $C\ C'\ L\ S\ V \equiv$

$(\exists\ T\ U. \text{conflict } S\ T \wedge \text{full skip-or-resolve } T\ U \wedge \text{conflicting } T = \text{Some } C \wedge$
 $\text{mark-of } (\text{hd-trail } V) = \text{add-mset } L\ C' \wedge \text{backtrack } U\ V)$

abbreviation *inv_{NOT}-stgy* **where**

inv_{NOT}-stgy $S \equiv \text{conflicting } S = \text{None} \wedge \text{cdcl}_W\text{-all-struct-inv } S \wedge \text{no-smaller-propa } S \wedge$
 $\text{cdcl}_W\text{-stgy-invariant } S \wedge \text{propagated-clauses-clauses } S$

interpretation *cdcl_W-with-strategy*: *cdcl_{NOT}-merge-bj-learn-ops* **where**

trail $= \lambda S. \text{convert-trail-from-}W\ (\text{trail } S)$ **and**

clauses_{NOT} $= \text{clauses}$ **and**

prepend-trail $= \lambda L\ S. \text{cons-trail } (\text{convert-ann-lit-from-NOT } L)\ S$ **and**

tl-trail $= \lambda S. \text{tl-trail } S$ **and**

add-cl_{NOT} $= \lambda C\ S. \text{add-learned-cls } C\ S$ **and**

remove-cl_{NOT} $= \lambda C\ S. \text{remove-cls } C\ S$ **and**

decide-conds $= \text{decide-conds}$ **and**

propagate-conds $= \text{propagate-conds}$ **and**

forget-conds $= \lambda -. \text{False}$ **and**

backjump-l-cond $= \lambda C\ C'\ L'\ S\ T. \text{backjump-l-conds-stgy } C\ C'\ L'\ S\ T$
 $\wedge \text{distinct-mset } C' \wedge L' \notin \# C' \wedge \neg \text{tautology } (\text{add-mset } L'\ C')$

$\langle \text{proof} \rangle$

interpretation *cdcl_W-with-strategy*: *cdcl_{NOT}-merge-bj-learn-proxy* **where**

trail $= \lambda S. \text{convert-trail-from-}W\ (\text{trail } S)$ **and**

clauses_{NOT} $= \text{clauses}$ **and**

prepend-trail $= \lambda L\ S. \text{cons-trail } (\text{convert-ann-lit-from-NOT } L)\ S$ **and**

tl-trail $= \lambda S. \text{tl-trail } S$ **and**

add-cl_{NOT} $= \lambda C\ S. \text{add-learned-cls } C\ S$ **and**

remove-cl_{NOT} $= \lambda C\ S. \text{remove-cls } C\ S$ **and**

decide-conds $= \text{decide-conds}$ **and**

propagate-conds $= \text{propagate-conds}$ **and**

forget-conds $= \lambda -. \text{False}$ **and**

backjump-l-cond $= \text{backjump-l-conds-stgy}$ **and**

inv $= \text{inv}_{\text{NOT-stgy}}$

$\langle \text{proof} \rangle$

lemma *cdcl_W-with-strategy-cdcl_{NOT}-merged-bj-learn-conflict*:

assumes

cdcl_W-with-strategy.cdcl_{NOT}-merged-bj-learn $S\ T$

conflicting $S = \text{None}$

shows

conflicting $T = \text{None}$

$\langle \text{proof} \rangle$

lemma *cdcl_W-with-strategy-no-forget_{NOT}[iff]*: *cdcl_W-with-strategy.forget_{NOT}* $S\ T \longleftrightarrow \text{False}$

$\langle \text{proof} \rangle$

lemma *cdcl_W-with-strategy-cdcl_{NOT}-merged-bj-learn-cdcl_W-stgy*:

assumes

cdcl_W-with-strategy.cdcl_{NOT}-merged-bj-learn S V

shows

*cdcl_W-stgy** S V*

$\langle \text{proof} \rangle$

lemma *rtrancpl-transition-function*:

$\langle R^{**} a b \implies \exists f j. (\forall i < j. R (f i) (f (Suc i))) \wedge f 0 = a \wedge f j = b \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_W-bj-cdcl_W-stgy*: $\langle \text{cdcl}_W\text{-bj } S T \implies \text{cdcl}_W\text{-stgy } S T \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_W-restart-propagated-clauses-clauses*:

$\langle \text{cdcl}_W\text{-restart } S T \implies \text{propagated-clauses-clauses } S \implies \text{propagated-clauses-clauses } T \rangle$

$\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_W-restart-propagated-clauses-clauses*:

$\langle \text{cdcl}_W\text{-restart}^{**} S T \implies \text{propagated-clauses-clauses } S \implies \text{propagated-clauses-clauses } T \rangle$

$\langle \text{proof} \rangle$

lemma *rtrancpl-cdcl_W-stgy-propagated-clauses-clauses*:

$\langle \text{cdcl}_W\text{-stgy}^{**} S T \implies \text{propagated-clauses-clauses } S \implies \text{propagated-clauses-clauses } T \rangle$

$\langle \text{proof} \rangle$

lemma *conflicting-clause-bt-lvl-gt-0-backjump*:

assumes

inv: $\langle \text{inv}_{NOT}\text{-stgy } S \rangle$ and

C: $\langle C \in \# \text{ clauses } S \rangle$ and

tr-C: $\langle \text{trail } S \models_{as} C \text{Not } C \rangle$ and

bt: $\langle \text{backtrack-lvl } S > 0 \rangle$

shows $\langle \exists T U V. \text{conflict } S T \wedge \text{full skip-or-resolve } T U \wedge \text{backtrack } U V \rangle$

$\langle \text{proof} \rangle$

lemma *conflict-full-skip-or-resolve-backtrack-backjump-l*:

assumes

conf: $\langle \text{conflict } S T \rangle$ and

full: $\langle \text{full skip-or-resolve } T U \rangle$ and

bt: $\langle \text{backtrack } U V \rangle$ and

inv: $\langle \text{cdcl}_W\text{-all-struct-inv } S \rangle$

shows $\langle \text{cdcl}_W\text{-with-strategy.backjump-l } S V \rangle$

$\langle \text{proof} \rangle$

lemma *is-decided-o-convert-ann-lit-from-W[simp]*:

$\langle \text{is-decided o convert-ann-lit-from-W} = \text{is-decided} \rangle$

$\langle \text{proof} \rangle$

lemma *cdcl_W-with-strategy-propagate_{NOT}-propagate-iff[iff]*:

$\langle \text{cdcl}_W\text{-with-strategy.propagate}_{NOT} S T \longleftrightarrow \text{propagate } S T \rangle$ (is ?NOT \longleftrightarrow ?W)

$\langle \text{proof} \rangle$

interpretation *cdcl_W-with-strategy: cdcl_{NOT}-merge-bj-learn where*

```

trail =  $\lambda S.$  convert-trail-from-W (trail S) and
clausesNOT = clauses and
prepend-trail =  $\lambda L S.$  cons-trail (convert-ann-lit-from-NOT L) S and
tl-trail =  $\lambda S.$  tl-trail S and
add-clsNOT =  $\lambda C S.$  add-learned-cls C S and
remove-clsNOT =  $\lambda C S.$  remove-cls C S and
decide-conds = decide-conds and
propagate-conds = propagate-conds and
forget-conds =  $\lambda -.$  False and
backjump-l-cond = backjump-l-conds-stgy and
inv = invNOT-stgy
⟨proof⟩

```

thm *cdcl_W-with-strategy.full-cdcl_{NOT}-merged-bj-learn-final-state*

end

end

theory *CDCL-W-Full*

imports *CDCL-W-Termination CDCL-WNOT*

begin

context *conflict-driven-clause-learning_W*

begin

lemma *rtrancpl-cdcl_W-merge-stgy-distinct-mset-clauses:*

assumes

invR: *cdcl_W-all-struct-inv* *R* **and**

st: *cdcl_W-s^{'**}* *R* *S* **and**

smaller: ⟨*no-smaller-propa* *R*⟩ **and**

dist: *distinct-mset* (*clauses* *R*)

shows *distinct-mset* (*clauses* *S*)

⟨*proof*⟩

end

end

theory *CDCL-W-Restart*

imports *CDCL-W-Full*

begin

Chapter 3

Extensions on Weidenbach's CDCL

We here extend our calculus.

3.1 Restarts

context *conflict-driven-clause-learning_W*
begin

This is an unrestricted version.

inductive *cdcl_W-restart-stgy* **for** $S\ T :: \langle 'st \times nat \rangle$ **where**
 $\langle cdcl_W\text{-stgy}\ (fst\ S)\ (fst\ T) \implies snd\ S = snd\ T \implies cdcl_W\text{-restart-stgy}\ S\ T \rangle \mid$
 $\langle restart\ (fst\ S)\ (fst\ T) \implies snd\ T = Suc\ (snd\ S) \implies cdcl_W\text{-restart-stgy}\ S\ T \rangle$

lemma *cdcl_W-stgy-cdcl_W-restart*: $\langle cdcl_W\text{-stgy}\ S\ S' \implies cdcl_W\text{-restart}\ S\ S' \rangle$
<proof>

lemma *cdcl_W-restart-stgy-cdcl_W-restart*:
 $\langle cdcl_W\text{-restart-stgy}\ S\ T \implies cdcl_W\text{-restart}\ (fst\ S)\ (fst\ T) \rangle$
<proof>

lemma *rtrancp-cdcl_W-restart-stgy-cdcl_W-restart*:
 $\langle cdcl_W\text{-restart-stgy}^{**}\ S\ T \implies cdcl_W\text{-restart}^{**}\ (fst\ S)\ (fst\ T) \rangle$
<proof>

lemma *cdcl_W-stgy-cdcl_W-restart-stgy*:
 $\langle cdcl_W\text{-stgy}\ S\ T \implies cdcl_W\text{-restart-stgy}\ (S,\ n)\ (T,\ n) \rangle$
<proof>

lemma *rtrancp-cdcl_W-stgy-cdcl_W-restart-stgy*:
 $\langle cdcl_W\text{-stgy}^{**}\ S\ T \implies cdcl_W\text{-restart-stgy}^{**}\ (S,\ n)\ (T,\ n) \rangle$
<proof>

lemma *cdcl_W-restart-dcl_W-all-struct-inv*:
 $\langle cdcl_W\text{-restart-stgy}\ S\ T \implies cdcl_W\text{-all-struct-inv}\ (fst\ S) \implies cdcl_W\text{-all-struct-inv}\ (fst\ T) \rangle$
<proof>

lemma *rtrancp-cdcl_W-restart-dcl_W-all-struct-inv*:
 $\langle cdcl_W\text{-restart-stgy}^{**}\ S\ T \implies cdcl_W\text{-all-struct-inv}\ (fst\ S) \implies cdcl_W\text{-all-struct-inv}\ (fst\ T) \rangle$
<proof>

lemma *restart-cdcl_W-stgy-invariant*:

$\langle \text{restart } S \ T \implies \text{cdcl}_W\text{-stgy-invariant } T \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-restart-dcl}_W\text{-stgy-invariant}$:

$\langle \text{cdcl}_W\text{-restart-stgy } S \ T \implies \text{cdcl}_W\text{-all-struct-inv } (\text{fst } S) \implies \text{cdcl}_W\text{-stgy-invariant } (\text{fst } S) \implies$
 $\text{cdcl}_W\text{-stgy-invariant } (\text{fst } T) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{rtrancp-cdcl}_W\text{-restart-dcl}_W\text{-stgy-invariant}$:

$\langle \text{cdcl}_W\text{-restart-stgy}^{**} S \ T \implies \text{cdcl}_W\text{-all-struct-inv } (\text{fst } S) \implies \text{cdcl}_W\text{-stgy-invariant } (\text{fst } S) \implies$
 $\text{cdcl}_W\text{-stgy-invariant } (\text{fst } T) \rangle$
 $\langle \text{proof} \rangle$

end

locale $\text{cdcl}_W\text{-restart-restart-ops} =$

$\text{conflict-driven-clause-learning}_W$

state-eq

state

— functions for the state:

— access functions:

$\text{trail init-clss learned-clss conflicting}$

— changing state:

$\text{cons-trail tl-trail add-learned-cls remove-cls}$

$\text{update-conflicting}$

— get state:

init-state

for

$\text{state-eq} :: \langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ (**infix** ~ 50) **and**

$\text{state} :: \langle 'st \Rightarrow ('v, 'v \text{ clause}) \text{ ann-lits} \times 'v \text{ clauses} \times 'v \text{ clauses} \times 'v \text{ clause option} \times$
 $'b \rangle$ **and**

$\text{trail} :: \langle 'st \Rightarrow ('v, 'v \text{ clause}) \text{ ann-lits} \rangle$ **and**

$\text{init-clss} :: \langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**

$\text{learned-clss} :: \langle 'st \Rightarrow 'v \text{ clauses} \rangle$ **and**

$\text{conflicting} :: \langle 'st \Rightarrow 'v \text{ clause option} \rangle$ **and**

$\text{cons-trail} :: \langle ('v, 'v \text{ clause}) \text{ ann-lit} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{tl-trail} :: \langle 'st \Rightarrow 'st \rangle$ **and**

$\text{add-learned-cls} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{remove-cls} :: \langle 'v \text{ clause} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{update-conflicting} :: \langle 'v \text{ clause option} \Rightarrow 'st \Rightarrow 'st \rangle$ **and**

$\text{init-state} :: \langle 'v \text{ clauses} \Rightarrow 'st \rangle +$

fixes

$f :: \langle \text{nat} \Rightarrow \text{nat} \rangle$

locale $\text{cdcl}_W\text{-restart-restart} =$

$\text{cdcl}_W\text{-restart-restart-ops} +$

assumes

$f: \langle \text{unbounded } f \rangle$

The condition of the differences of cardinality has to be strict. Otherwise, you could be in a strange state, where nothing remains to do, but a restart is done. See the proof of well-foundedness. The same applies for the $\text{cdcl}_W\text{-stgy}^{+\downarrow} S \ T$: With a $\text{cdcl}_W\text{-stgy}^\downarrow S \ T$, this rules could be applied one after the other, doing nothing each time.

context $cdcl_W$ -restart-restart-ops

begin

inductive $cdcl_W$ -merge-with-restart **where**

restart-step:

$\langle (cdcl_W\text{-stgy} \sim (card (set\text{-mset} (learned\text{-clss} T)) - card (set\text{-mset} (learned\text{-clss} S)))) S T$
 $\implies card (set\text{-mset} (learned\text{-clss} T)) - card (set\text{-mset} (learned\text{-clss} S)) > f n$
 $\implies restart T U \implies cdcl_W\text{-merge-with-restart} (S, n) (U, Suc n) \mid$

restart-full: $\langle full1\ cdcl_W\text{-stgy} S T \implies cdcl_W\text{-merge-with-restart} (S, n) (T, Suc n) \rangle$

lemma $cdcl_W$ -merge-with-restart-rtrancpl- $cdcl_W$ -restart:

$\langle cdcl_W\text{-merge-with-restart} S T \implies cdcl_W\text{-restart}^{**} (fst S) (fst T) \rangle$
 $\langle proof \rangle$

lemma $cdcl_W$ -merge-with-restart-increasing-number:

$\langle cdcl_W\text{-merge-with-restart} S T \implies snd T = 1 + snd S \rangle$
 $\langle proof \rangle$

lemma $\langle full1\ cdcl_W\text{-stgy} S T \implies cdcl_W\text{-merge-with-restart} (S, n) (T, Suc n) \rangle$

$\langle proof \rangle$

lemma $cdcl_W$ -all-struct-inv-learned-clss-bound:

assumes inv : $\langle cdcl_W\text{-all-struct-inv} S \rangle$
shows $\langle set\text{-mset} (learned\text{-clss} S) \subseteq simple\text{-clss} (atms\text{-of-mm} (init\text{-clss} S)) \rangle$
 $\langle proof \rangle$

lemma $cdcl_W$ -merge-with-restart-init-clss:

$\langle cdcl_W\text{-merge-with-restart} S T \implies cdcl_W\text{-M-level-inv} (fst S) \implies$
 $init\text{-clss} (fst S) = init\text{-clss} (fst T) \rangle$
 $\langle proof \rangle$

lemma (in $cdcl_W$ -restart-restart)

$\langle wf \{ (T, S). cdcl_W\text{-all-struct-inv} (fst S) \wedge cdcl_W\text{-merge-with-restart} S T \} \rangle$
 $\langle proof \rangle$

lemma $cdcl_W$ -merge-with-restart-distinct-mset-clauses:

assumes $invR$: $\langle cdcl_W\text{-all-struct-inv} (fst R) \rangle$ **and**
 st : $\langle cdcl_W\text{-merge-with-restart} R S \rangle$ **and**
 $dist$: $\langle distinct\text{-mset} (clauses (fst R)) \rangle$ **and**
 R : $\langle no\text{-smaller-propa} (fst R) \rangle$
shows $\langle distinct\text{-mset} (clauses (fst S)) \rangle$
 $\langle proof \rangle$

inductive $cdcl_W$ -restart-with-restart **where**

restart-step:

$\langle cdcl_W\text{-stgy}^{**} S T \implies$
 $card (set\text{-mset} (learned\text{-clss} T)) - card (set\text{-mset} (learned\text{-clss} S)) > f n \implies$
 $restart T U \implies$
 $cdcl_W\text{-restart-with-restart} (S, n) (U, Suc n) \mid$

restart-full: $\langle full1\ cdcl_W\text{-stgy} S T \implies cdcl_W\text{-restart-with-restart} (S, n) (T, Suc n) \rangle$

lemma $cdcl_W$ -restart-with-restart-rtrancpl- $cdcl_W$ -restart:

$\langle cdcl_W\text{-restart-with-restart} S T \implies cdcl_W\text{-restart}^{**} (fst S) (fst T) \rangle$
 $\langle proof \rangle$

lemma $cdcl_W$ -restart-with-restart-increasing-number:

$\langle cdcl_W\text{-restart-with-restart} S T \implies snd T = 1 + snd S \rangle$

$\langle \text{proof} \rangle$

lemma $\langle \text{full1 } \text{cdcl}_W\text{-stgy } S \ T \implies \text{cdcl}_W\text{-restart-with-restart } (S, n) \ (T, \text{Suc } n) \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-restart-with-restart-init-clss}$:
 $\langle \text{cdcl}_W\text{-restart-with-restart } S \ T \implies \text{cdcl}_W\text{-M-level-inv } (\text{fst } S) \implies$
 $\text{init-clss } (\text{fst } S) = \text{init-clss } (\text{fst } T) \rangle$
 $\langle \text{proof} \rangle$

theorem (in $\text{cdcl}_W\text{-restart-restart}$)
 $\langle \text{wf } \{(T, S). \text{cdcl}_W\text{-all-struct-inv } (\text{fst } S) \wedge \text{cdcl}_W\text{-restart-with-restart } S \ T\} \rangle$
 $\langle \text{proof} \rangle$

lemma $\text{cdcl}_W\text{-restart-with-restart-distinct-mset-clauses}$:
assumes invR : $\langle \text{cdcl}_W\text{-all-struct-inv } (\text{fst } R) \rangle$ **and**
 st : $\langle \text{cdcl}_W\text{-restart-with-restart } R \ S \rangle$ **and**
 dist : $\langle \text{distinct-mset } (\text{clauses } (\text{fst } R)) \rangle$ **and**
 R : $\langle \text{no-smaller-propa } (\text{fst } R) \rangle$
shows $\langle \text{distinct-mset } (\text{clauses } (\text{fst } S)) \rangle$
 $\langle \text{proof} \rangle$

end

locale $\text{luby-sequence} =$
fixes $ur :: \text{nat}$
assumes $\langle ur > 0 \rangle$
begin

lemma $\text{exists-luby-decomp}$:
fixes $i :: \text{nat}$
shows $\langle \exists k :: \text{nat}. (2 \wedge (k - 1) \leq i \wedge i < 2 \wedge k - 1) \vee i = 2 \wedge k - 1 \rangle$
 $\langle \text{proof} \rangle$

Luby sequences are defined by:

- $2^k - 1$, if $i = (2 :: 'a)^k - (1 :: 'a)$
- $\text{luby-sequence-core } (i - 2^{k-1} + 1)$, if $(2 :: 'a)^{k-1} \leq i$ and $i \leq (2 :: 'a)^k - (1 :: 'a)$

Then the sequence is then scaled by a constant unit run (called ur here), strictly positive.

function $\text{luby-sequence-core} :: \langle \text{nat} \Rightarrow \text{nat} \rangle$ **where**
 $\langle \text{luby-sequence-core } i =$
 $(\text{if } \exists k. i = 2^k - 1$
 $\text{then } 2^{\wedge}((\text{SOME } k. i = 2^k - 1) - 1)$
 $\text{else } \text{luby-sequence-core } (i - 2^{\wedge}((\text{SOME } k. 2^{\wedge}(k-1) \leq i \wedge i < 2^k - 1) - 1) + 1)) \rangle$
 $\langle \text{proof} \rangle$
termination
 $\langle \text{proof} \rangle$

declare $\text{luby-sequence-core.simps[simp del]}$

lemma $\text{two-pover-n-eq-two-power-n'-eq}$:
assumes H : $\langle (2 :: \text{nat}) \wedge (k :: \text{nat}) - 1 = 2 \wedge k' - 1 \rangle$
shows $\langle k' = k \rangle$

$\langle \text{proof} \rangle$

lemma *luby-sequence-core-two-power-minus-one:*

$\langle \text{luby-sequence-core } (2^k - 1) = 2^{(k-1)} \rangle$ (is $\langle ?L = ?K \rangle$)

$\langle \text{proof} \rangle$

lemma *different-luby-decomposition-false:*

assumes

$H: \langle 2^k \wedge (k - \text{Suc } 0) \leq i \rangle$ **and**

$k': \langle i < 2^{k'} - \text{Suc } 0 \rangle$ **and**

$k-k': \langle k > k' \rangle$

shows $\langle \text{False} \rangle$

$\langle \text{proof} \rangle$

lemma *luby-sequence-core-not-two-power-minus-one:*

assumes

$k-i: \langle 2^k \wedge (k - 1) \leq i \rangle$ **and**

$i-k: \langle i < 2^k - 1 \rangle$

shows $\langle \text{luby-sequence-core } i = \text{luby-sequence-core } (i - 2^{(k-1)} + 1) \rangle$

$\langle \text{proof} \rangle$

lemma *unbounded-luby-sequence-core:* $\langle \text{unbounded luby-sequence-core} \rangle$

$\langle \text{proof} \rangle$

abbreviation *luby-sequence* :: $\langle \text{nat} \Rightarrow \text{nat} \rangle$ **where**

$\langle \text{luby-sequence } n \equiv \text{ur} * \text{luby-sequence-core } n \rangle$

lemma *bounded-luby-sequence:* $\langle \text{unbounded luby-sequence} \rangle$

$\langle \text{proof} \rangle$

lemma *luby-sequence-core-0:* $\langle \text{luby-sequence-core } 0 = 1 \rangle$

$\langle \text{proof} \rangle$

lemma $\langle \text{luby-sequence-core } n \geq 1 \rangle$

$\langle \text{proof} \rangle$

end

locale *luby-sequence-restart* =

luby-sequence ur +

conflict-driven-clause-learning_W

— functions for the state:

state-eq state

— access functions:

trail init-clss learned-clss conflicting

— changing state:

cons-trail tl-trail add-learned-cls remove-cls

update-conflicting

— get state:

init-state

for

ur :: *nat* **and**

state-eq :: $\langle 'st \Rightarrow 'st \Rightarrow \text{bool} \rangle$ (**infix** ~ 50) **and**

state :: $\langle 'st \Rightarrow ('v, 'v \text{ clause}) \text{ ann-lits} \times 'v \text{ clauses} \times 'v \text{ clauses} \times 'v \text{ clause option} \times 'b \rangle$ **and**

trail :: $\langle 'st \Rightarrow ('v, 'v \text{ clause}) \text{ ann-lits} \rangle$ **and**

```

hd-trail :: ⟨'st ⇒ ('v, 'v clause) ann-lit⟩ and
init-clss :: ⟨'st ⇒ 'v clauses⟩ and
learned-clss :: ⟨'st ⇒ 'v clauses⟩ and
conflicting :: ⟨'st ⇒ 'v clause option⟩ and

cons-trail :: ⟨('v, 'v clause) ann-lit ⇒ 'st ⇒ 'st⟩ and
tl-trail :: ⟨'st ⇒ 'st⟩ and
add-learned-clss :: ⟨'v clause ⇒ 'st ⇒ 'st⟩ and
remove-clss :: ⟨'v clause ⇒ 'st ⇒ 'st⟩ and
update-conflicting :: ⟨'v clause option ⇒ 'st ⇒ 'st⟩ and

init-state :: ⟨'v clauses ⇒ 'st⟩
begin

sublocale cdclW-restart-restart where
  f = luby-sequence
  ⟨proof⟩

end

end
theory CDCL-W-Incremental
imports CDCL-W-Full
begin

```

3.2 Incremental SAT solving

```

locale stateW-adding-init-clause-no-state =
  stateW-no-state
  state-eq
  state
  — functions about the state:
    — getter:
  trail init-clss learned-clss conflicting
    — setter:
  cons-trail tl-trail add-learned-clss remove-clss
  update-conflicting

  — Some specific states:
  init-state
for
  state-eq :: 'st ⇒ 'st ⇒ bool (infix ~ 50) and
  state :: 'st ⇒ ('v, 'v clause) ann-lits × 'v clauses × 'v clauses × 'v clause option ×
    'b and
  trail :: 'st ⇒ ('v, 'v clause) ann-lits and
  init-clss :: 'st ⇒ 'v clauses and
  learned-clss :: 'st ⇒ 'v clauses and
  conflicting :: 'st ⇒ 'v clause option and

  cons-trail :: ('v, 'v clause) ann-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-learned-clss :: 'v clause ⇒ 'st ⇒ 'st and
  remove-clss :: 'v clause ⇒ 'st ⇒ 'st and
  update-conflicting :: 'v clause option ⇒ 'st ⇒ 'st and

```

```

  init-state :: 'v clauses  $\Rightarrow$  'st +
fixes
  add-init-cls :: 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st
assumes
  add-init-cls:
    state st = (M, N, U, S')  $\Rightarrow$ 
    state (add-init-cls C st) = (M, {#C#} + N, U, S')

locale stateW-adding-init-clause-ops =
  stateW-adding-init-clause-no-state
  state-eq
  state
  — functions about the state:
    — getter:
  trail init-clss learned-clss conflicting
    — setter:
  cons-trail tl-trail add-learned-cls remove-cls update-conflicting

  — Some specific states:
  init-state
  add-init-cls
for
  state-eq :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool (infix  $\sim$  50) and
  state :: 'st  $\Rightarrow$  ('v, 'v clause) ann-lits  $\times$  'v clauses  $\times$  'v clauses  $\times$  'v clause option  $\times$ 
    'b and
  trail :: 'st  $\Rightarrow$  ('v, 'v clause) ann-lits and
  init-clss :: 'st  $\Rightarrow$  'v clauses and
  learned-clss :: 'st  $\Rightarrow$  'v clauses and
  conflicting :: 'st  $\Rightarrow$  'v clause option and

  cons-trail :: ('v, 'v clause) ann-lit  $\Rightarrow$  'st  $\Rightarrow$  'st and
  tl-trail :: 'st  $\Rightarrow$  'st and
  add-learned-cls :: 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st and
  remove-cls :: 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st and
  update-conflicting :: 'v clause option  $\Rightarrow$  'st  $\Rightarrow$  'st and

  init-state :: 'v clauses  $\Rightarrow$  'st and
  add-init-cls :: 'v clause  $\Rightarrow$  'st  $\Rightarrow$  'st +
assumes
  state-prop[simp]:
     $\langle$ state S = (trail S, init-clss S, learned-clss S, conflicting S, additional-info S) $\rangle$ 

locale stateW-adding-init-clause =
  stateW-adding-init-clause-ops
  state-eq
  state
  — functions about the state:
    — getter:
  trail init-clss learned-clss conflicting
    — setter:
  cons-trail tl-trail add-learned-cls remove-cls update-conflicting

  — Some specific states:
  init-state add-init-cls
for
  state-eq :: 'st  $\Rightarrow$  'st  $\Rightarrow$  bool (infix  $\sim$  50) and

```

```

state :: 'st ⇒ ('v, 'v clause) ann-lits × 'v clauses × 'v clauses × 'v clause option ×
  'b and
trail :: 'st ⇒ ('v, 'v clause) ann-lits and
init-clss :: 'st ⇒ 'v clauses and
learned-clss :: 'st ⇒ 'v clauses and
conflicting :: 'st ⇒ 'v clause option and

cons-trail :: ('v, 'v clause) ann-lit ⇒ 'st ⇒ 'st and
tl-trail :: 'st ⇒ 'st and
add-learned-clss :: 'v clause ⇒ 'st ⇒ 'st and
remove-clss :: 'v clause ⇒ 'st ⇒ 'st and
update-conflicting :: 'v clause option ⇒ 'st ⇒ 'st and

init-state :: 'v clauses ⇒ 'st and
add-init-clss :: 'v clause ⇒ 'st ⇒ 'st
begin

sublocale stateW
  ⟨proof⟩

lemma
  trail-add-init-clss[simp]:
    trail (add-init-clss C st) = trail st and
  init-clss-add-init-clss[simp]:
    init-clss (add-init-clss C st) = {#C#} + init-clss st
  and
  learned-clss-add-init-clss[simp]:
    learned-clss (add-init-clss C st) = learned-clss st and
  conflicting-add-init-clss[simp]:
    conflicting (add-init-clss C st) = conflicting st
  ⟨proof⟩

lemma clauses-add-init-clss[simp]:
  clauses (add-init-clss N S) = {#N#} + init-clss S + learned-clss S
  ⟨proof⟩

lemma reduce-trail-to-add-init-clss[simp]:
  trail (reduce-trail-to F (add-init-clss C S)) = trail (reduce-trail-to F S)
  ⟨proof⟩

lemma conflicting-add-init-clss-iff-conflicting[simp]:
  conflicting (add-init-clss C S) = None ⟷ conflicting S = None
  ⟨proof⟩
end

locale conflict-driven-clause-learning-with-adding-init-clauseW =
  stateW-adding-init-clause
  state-eq
  state
  — functions for the state:
  — access functions:
  trail init-clss learned-clss conflicting
  — changing state:
  cons-trail tl-trail add-learned-clss remove-clss update-conflicting

  — get state:

```

```

init-state
  — Adding a clause:
add-init-cls
for
  state-eq :: 'st ⇒ 'st ⇒ bool (infix ~ 50) and
  state :: 'st ⇒ ('v, 'v clause) ann-lits × 'v clauses × 'v clauses × 'v clause option ×
    'b and
  trail :: 'st ⇒ ('v, 'v clause) ann-lits and
  init-clss :: 'st ⇒ 'v clauses and
  learned-clss :: 'st ⇒ 'v clauses and
  conflicting :: 'st ⇒ 'v clause option and

  cons-trail :: ('v, 'v clause) ann-lit ⇒ 'st ⇒ 'st and
  tl-trail :: 'st ⇒ 'st and
  add-learned-cls :: 'v clause ⇒ 'st ⇒ 'st and
  remove-cls :: 'v clause ⇒ 'st ⇒ 'st and
  update-conflicting :: 'v clause option ⇒ 'st ⇒ 'st and

  init-state :: 'v clauses ⇒ 'st and
  add-init-cls :: 'v clause ⇒ 'st ⇒ 'st
begin

sublocale conflict-driven-clause-learningW
  ⟨proof⟩

```

This invariant holds all the invariant related to the strategy. See the structural invariant in *cdcl_W-all-struct-inv*

When we add a new clause, we reduce the trail until we get to the first literal included in C. Then we can mark the conflict.

```

fun cut-trail-wrt-clause where
  cut-trail-wrt-clause C [] S = S |
  cut-trail-wrt-clause C (Decided L # M) S =
    (if -L ∈# C then S
     else cut-trail-wrt-clause C M (tl-trail S)) |
  cut-trail-wrt-clause C (Propagated L - # M) S =
    (if -L ∈# C then S
     else cut-trail-wrt-clause C M (tl-trail S))

```

```

definition add-new-clause-and-update :: 'v clause ⇒ 'st ⇒ 'st where
  add-new-clause-and-update C S =
    (if trail S ⊨as CNot C
     then update-conflicting (Some C) (add-init-cls C
       (cut-trail-wrt-clause C (trail S) S))
     else add-init-cls C S)

```

```

lemma init-clss-cut-trail-wrt-clause[simp]:
  init-clss (cut-trail-wrt-clause C M S) = init-clss S
  ⟨proof⟩

```

```

lemma learned-clss-cut-trail-wrt-clause[simp]:
  learned-clss (cut-trail-wrt-clause C M S) = learned-clss S
  ⟨proof⟩

```

```

lemma conflicting-clss-cut-trail-wrt-clause[simp]:
  conflicting (cut-trail-wrt-clause C M S) = conflicting S

```

$\langle \text{proof} \rangle$

lemma *trail-cut-trail-wrt-clause*:

$\exists M. \text{ trail } S = M @ \text{ trail } (\text{cut-trail-wrt-clause } C (\text{trail } S) S)$

$\langle \text{proof} \rangle$

lemma *n-dup-no-dup-trail-cut-trail-wrt-clause[simp]*:

assumes *n-d*: *no-dup* (*trail T*)

shows *no-dup* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))

$\langle \text{proof} \rangle$

lemma *cut-trail-wrt-clause-backtrack-lvl-length-decided*:

assumes

backtrack-lvl T = *count-decided* (*trail T*)

shows

backtrack-lvl (*cut-trail-wrt-clause C* (*trail T*) *T*) =
count-decided (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))

$\langle \text{proof} \rangle$

lemma *cut-trail-wrt-clause-CNot-trail*:

assumes *trail T* \models_{as} *CNot C*

shows

(*trail* ((*cut-trail-wrt-clause C* (*trail T*) *T*))) \models_{as} *CNot C*

$\langle \text{proof} \rangle$

lemma *cut-trail-wrt-clause-hd-trail-in-or-empty-trail*:

(($\forall L \in \#C. -L \notin \text{ lits-of-l } (\text{trail } T)$) \wedge *trail* (*cut-trail-wrt-clause C* (*trail T*) *T*) = [])
 \vee ($-\text{lit-of}$ (*hd* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*))) $\in \# C$
 \wedge *length* (*trail* (*cut-trail-wrt-clause C* (*trail T*) *T*)) ≥ 1)

$\langle \text{proof} \rangle$

We can fully run *cdcl_W-restart-s* or add a clause. Remark that we use *cdcl_W-restart-s* to avoid an explicit *skip*, *resolve*, and *backtrack* normalisation to get rid of the conflict *C* if possible.

inductive *incremental-cdcl_W* :: '*st* \Rightarrow '*st* \Rightarrow *bool* **for** *S* **where**

add-conf:

trail S \models_{asm} *init-clss S* \Rightarrow *distinct-mset C* \Rightarrow *conflicting S* = *None* \Rightarrow

trail S \models_{as} *CNot C* \Rightarrow

full cdcl_W-stgy

(*update-conflicting* (*Some C*)

(*add-init-cls C* (*cut-trail-wrt-clause C* (*trail S*) *S*))) *T* \Rightarrow

incremental-cdcl_W S T |

add-no-conf:

trail S \models_{asm} *init-clss S* \Rightarrow *distinct-mset C* \Rightarrow *conflicting S* = *None* \Rightarrow

\neg *trail S* \models_{as} *CNot C* \Rightarrow

full cdcl_W-stgy (*add-init-cls C S*) *T* \Rightarrow

incremental-cdcl_W S T

lemma *cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-all-struct-inv*:

assumes

inv-T: *cdcl_W-all-struct-inv T* **and**

tr-T-N[simp]: *trail T* \models_{asm} *N* **and**

tr-C[simp]: *trail T* \models_{as} *CNot C* **and**

[*simp*]: *distinct-mset C*

shows *cdcl_W-all-struct-inv* (*add-new-clause-and-update C T*) (**is** *cdcl_W-all-struct-inv ?T'*)

$\langle \text{proof} \rangle$

lemma *cdcl_W-all-struct-inv-add-new-clause-and-update-cdcl_W-stgy-inv*:

assumes

inv-s: *cdcl_W-stgy-invariant* *T* **and**
inv: *cdcl_W-all-struct-inv* *T* **and**
tr-T-N[simp]: *trail* *T* \models_{asm} *N* **and**
tr-C[simp]: *trail* *T* \models_{as} *CNot C* **and**
[simp]: *distinct-mset* *C*

shows *cdcl_W-stgy-invariant* (*add-new-clause-and-update* *C T*)
 (**is** *cdcl_W-stgy-invariant* ?*T'*)

<proof>

lemma *incremental-cdcl_W-inv*:

assumes

inc: *incremental-cdcl_W* *S T* **and**
inv: *cdcl_W-all-struct-inv* *S* **and**
s-inv: *cdcl_W-stgy-invariant* *S* **and**
learned-entailed: *<cdcl_W-learned-clauses-entailed-by-init* *S>*

shows

cdcl_W-all-struct-inv *T* **and**
cdcl_W-stgy-invariant *T* **and**
learned-entailed: *<cdcl_W-learned-clauses-entailed-by-init* *T>*

<proof>

lemma *rtrancpl-incremental-cdcl_W-inv*:

assumes

inc: *incremental-cdcl_W*** *S T* **and**
inv: *cdcl_W-all-struct-inv* *S* **and**
s-inv: *cdcl_W-stgy-invariant* *S* **and**
learned-entailed: *<cdcl_W-learned-clauses-entailed-by-init* *S>*

shows

cdcl_W-all-struct-inv *T* **and**
cdcl_W-stgy-invariant *T* **and**
<cdcl_W-learned-clauses-entailed-by-init *T>*
<proof>

lemma *incremental-conclusive-state*:

assumes

inc: *incremental-cdcl_W* *S T* **and**
inv: *cdcl_W-all-struct-inv* *S* **and**
s-inv: *cdcl_W-stgy-invariant* *S* **and**
learned-entailed: *<cdcl_W-learned-clauses-entailed-by-init* *S>*

shows *conflicting* *T = Some {#} \wedge unsatisfiable (set-mset (init-clss T))*

\vee *conflicting* *T = None \wedge trail T \models_{asm} init-clss T \wedge satisfiable (set-mset (init-clss T))*

<proof>

lemma *trancpl-incremental-correct*:

assumes

inc: *incremental-cdcl_W⁺⁺* *S T* **and**
inv: *cdcl_W-all-struct-inv* *S* **and**
s-inv: *cdcl_W-stgy-invariant* *S* **and**
learned-entailed: *<cdcl_W-learned-clauses-entailed-by-init* *S>*

shows *conflicting* *T = Some {#} \wedge unsatisfiable (set-mset (init-clss T))*

\vee *conflicting* *T = None \wedge trail T \models_{asm} init-clss T \wedge satisfiable (set-mset (init-clss T))*

<proof>

end

```
end  
theory DPLL-CDCL-W-Implementation  
imports  
  Entailment-Definition.Partial-Annotated-Herbrand-Interpretation  
  CDCL-W-Level  
begin
```


Chapter 4

List-based Implementation of DPLL and CDCL

We can now reuse all the theorems to go towards an implementation using 2-watched literals:

- `CDCL_W_Abstract_State.thy` defines a better-suited state: the operation operating on it are more constrained, allowing simpler proofs and less edge cases later.

4.1 Simple List-Based Implementation of the DPLL and CDCL

The idea of the list-based implementation is to test the stack: the theories about the calculi, adapting the theorems to a simple implementation and the code exportation. The implementation are very simple and simply iterate over-and-over on lists.

4.1.1 Common Rules

Propagation

The following theorem holds:

lemma *lits-of-l-unfold*:

$$(\forall c \in \text{set } C. -c \in \text{lits-of-l } Ms) \longleftrightarrow Ms \models_{as} CNot (\text{mset } C)$$

<proof>

The right-hand version is written at a high-level, but only the left-hand side is executable.

definition *is-unit-clause* :: 'a literal list \Rightarrow ('a, 'b) ann-lits \Rightarrow 'a literal option

where

is-unit-clause l M =

(case List.filter ($\lambda a. \text{atm-of } a \notin \text{atm-of ' lits-of-l } M$) l of
a # [] \Rightarrow if M \models_{as} CNot (mset l - {#a#}) then Some a else None
| - \Rightarrow None)

definition *is-unit-clause-code* :: 'a literal list \Rightarrow ('a, 'b) ann-lits

\Rightarrow 'a literal option **where**

is-unit-clause-code l M =

(case List.filter ($\lambda a. \text{atm-of } a \notin \text{atm-of ' lits-of-l } M$) l of
a # [] \Rightarrow if ($\forall c \in \text{set } (\text{remove1 } a \text{ l}). -c \in \text{lits-of-l } M$) then Some a else None
| - \Rightarrow None)

lemma *is-unit-clause-is-unit-clause-code*[code]:
is-unit-clause *l* *M* = *is-unit-clause-code* *l* *M*
 ⟨proof⟩

lemma *is-unit-clause-some-undef*:
assumes *is-unit-clause* *l* *M* = *Some a*
shows *undefined-lit* *M* *a*
 ⟨proof⟩

lemma *is-unit-clause-some-CNot*: *is-unit-clause* *l* *M* = *Some a* \implies *M* \models_{as} *CNot* (*mset* *l* - {#a#})
 ⟨proof⟩

lemma *is-unit-clause-some-in*: *is-unit-clause* *l* *M* = *Some a* \implies *a* \in *set l*
 ⟨proof⟩

lemma *is-unit-clause-Nil*[simp]: *is-unit-clause* [] *M* = *None*
 ⟨proof⟩

Unit propagation for all clauses

Finding the first clause to propagate

fun *find-first-unit-clause* :: 'a literal list list \Rightarrow ('a, 'b) ann-lits
 \Rightarrow ('a literal \times 'a literal list) option **where**
find-first-unit-clause (*a* # *l*) *M* =
 (case *is-unit-clause* *a* *M* of
 None \Rightarrow *find-first-unit-clause* *l* *M*
 | *Some L* \Rightarrow *Some* (*L*, *a*)) |
find-first-unit-clause [] - = *None*

lemma *find-first-unit-clause-some*:
find-first-unit-clause *l* *M* = *Some* (*a*, *c*)
 \implies *c* \in *set l* \wedge *M* \models_{as} *CNot* (*mset* *c* - {#a#}) \wedge *undefined-lit* *M* *a* \wedge *a* \in *set c*
 ⟨proof⟩

lemma *propagate-is-unit-clause-not-None*:
assumes
M: *M* \models_{as} *CNot* (*mset* *c* - {#a#}) **and**
undef: *undefined-lit* *M* *a* **and**
ac: *a* \in *set c*
shows *is-unit-clause* *c* *M* \neq *None*
 ⟨proof⟩

lemma *find-first-unit-clause-none*:
c \in *set l* \implies *M* \models_{as} *CNot* (*mset* *c* - {#a#}) \implies *undefined-lit* *M* *a* \implies *a* \in *set c*
 \implies *find-first-unit-clause* *l* *M* \neq *None*
 ⟨proof⟩

Decide

fun *find-first-unused-var* :: 'a literal list list \Rightarrow 'a literal set \Rightarrow 'a literal option **where**
find-first-unused-var (*a* # *l*) *M* =
 (case *List.find* ($\lambda lit. lit \notin M \wedge \neg lit \in M$) *a* of
 None \Rightarrow *find-first-unused-var* *l* *M*
 | *Some a* \Rightarrow *Some a*) |

find-first-unused-var [] - = None

lemma *find-none*[*iff*]:

List.find ($\lambda lit. lit \notin M \wedge \neg lit \notin M$) *a* = None \longleftrightarrow *atm-of* ' *set a* \subseteq *atm-of* ' *M*
 ⟨*proof*⟩

lemma *find-some*: *List.find* ($\lambda lit. lit \notin M \wedge \neg lit \notin M$) *a* = *Some b* $\implies b \in set\ a \wedge b \notin M \wedge \neg b \notin M$
 ⟨*proof*⟩

lemma *find-first-unused-var-None*[*iff*]:

find-first-unused-var l M = None $\longleftrightarrow (\forall a \in set\ l. atm-of\ ' set\ a \subseteq atm-of\ ' M)$
 ⟨*proof*⟩

lemma *find-first-unused-var-Some-not-all-incl*:

assumes *find-first-unused-var l M* = *Some c*
shows $\neg(\forall a \in set\ l. atm-of\ ' set\ a \subseteq atm-of\ ' M)$
 ⟨*proof*⟩

lemma *find-first-unused-var-Some*:

find-first-unused-var l M = *Some a* $\implies (\exists m \in set\ l. a \in set\ m \wedge a \notin M \wedge \neg a \notin M)$
 ⟨*proof*⟩

lemma *find-first-unused-var-undefined*:

find-first-unused-var l (lits-of-l Ms) = *Some a* $\implies undefined-lit\ Ms\ a$
 ⟨*proof*⟩

4.1.2 CDCL specific functions

Level

fun *maximum-level-code*:: 'a literal list \Rightarrow ('a, 'b) ann-lits \Rightarrow nat

where

maximum-level-code [] - = 0 |

maximum-level-code (*L* # *Ls*) *M* = max (*get-level M L*) (*maximum-level-code Ls M*)

lemma *maximum-level-code-eq-get-maximum-level*[*simp*]:

maximum-level-code D M = *get-maximum-level M (mset D)*
 ⟨*proof*⟩

lemma [*code*]:

fixes *M* :: ('a, 'b) ann-lits

shows *get-maximum-level M (mset D)* = *maximum-level-code D M*

⟨*proof*⟩

Backjumping

fun *find-level-decomp* **where**

find-level-decomp M [] *D k* = None |

find-level-decomp M (*L* # *Ls*) *D k* =

(case (*get-level M L*, *maximum-level-code (D @ Ls) M*) of
 (*i*, *j*) \Rightarrow if *i* = *k* \wedge *j* < *i* then *Some (L, j)* else *find-level-decomp M Ls (L#D) k*
)

lemma *find-level-decomp-some*:

assumes *find-level-decomp M Ls D k* = *Some (L, j)*

shows $L \in set\ Ls \wedge get-maximum-level\ M\ (mset\ (remove1\ L\ (Ls\ @\ D))) = j \wedge get-level\ M\ L = k$

$\langle proof \rangle$

lemma *find-level-decomp-none*:

assumes *find-level-decomp* $M \ Ls \ E \ k = None$ **and** $mset \ (L \# D) = mset \ (Ls \ @ \ E)$

shows $\neg(L \in set \ Ls \wedge get_maximum_level \ M \ (mset \ D) < k \wedge k = get_level \ M \ L)$

$\langle proof \rangle$

fun *bt-cut* **where**

bt-cut $i \ (Propagated \ - \ \# \ Ls) = bt_cut \ i \ Ls \ |$

bt-cut $i \ (Decided \ K \ \# \ Ls) = (if \ count_decided \ Ls = i \ then \ Some \ (Decided \ K \ \# \ Ls) \ else \ bt_cut \ i \ Ls) \ |$

bt-cut $i \ [] = None$

lemma *bt-cut-some-decomp*:

assumes *no-dup* M **and** *bt-cut* $i \ M = Some \ M'$

shows $\exists K \ M2 \ M1. \ M = M2 \ @ \ M' \wedge M' = Decided \ K \ \# \ M1 \wedge get_level \ M \ K = (i+1)$

$\langle proof \rangle$

lemma *bt-cut-not-none*:

assumes *no-dup* M **and** $M = M2 \ @ \ Decided \ K \ \# \ M'$ **and** $get_level \ M \ K = (i+1)$

shows *bt-cut* $i \ M \neq None$

$\langle proof \rangle$

lemma *get-all-ann-decomposition-ex*:

$\exists N. \ (Decided \ K \ \# \ M', \ N) \in set \ (get_all_ann_decomposition \ (M2 @ Decided \ K \ \# \ M'))$

$\langle proof \rangle$

lemma *bt-cut-in-get-all-ann-decomposition*:

assumes *no-dup* M **and** *bt-cut* $i \ M = Some \ M'$

shows $\exists M2. \ (M', \ M2) \in set \ (get_all_ann_decomposition \ M)$

$\langle proof \rangle$

fun *do-backtrack-step* **where**

do-backtrack-step $(M, \ N, \ U, \ Some \ D) =$

$(case \ find_level_decomp \ M \ D \ [] \ (count_decided \ M) \ of$

$None \Rightarrow (M, \ N, \ U, \ Some \ D)$

$| \ Some \ (L, \ j) \Rightarrow$

$(case \ bt_cut \ j \ M \ of$

$Some \ (Decided \ - \ \# \ Ls) \Rightarrow (Propagated \ L \ D \ \# \ Ls, \ N, \ D \ \# \ U, \ None)$

$| \ - \Rightarrow (M, \ N, \ U, \ Some \ D))$

$) \ |$

do-backtrack-step $S = S$

end

theory *DPLL-W-Implementation*

imports *DPLL-CDCL-W-Implementation DPLL-W HOL-Library.Code-Target-Numeral*

begin

4.1.3 Simple Implementation of DPLL

Combining the propagate and decide: a DPLL step

definition *DPLL-step* $:: int \ dpll_W_ann_lits \times int \ literal \ list \ list$

$\Rightarrow int \ dpll_W_ann_lits \times int \ literal \ list \ list$ **where**

DPLL-step $= (\lambda(Ms, \ N).$

$(case \ find_first_unit_clause \ N \ Ms \ of$

$Some \ (L, \ -) \Rightarrow (Propagated \ L \ () \ \# \ Ms, \ N)$

```

| - ⇒
  if ∃ C ∈ set N. (∀ c ∈ set C. -c ∈ lits-of-l Ms)
  then
    (case backtrack-split Ms of
      (-, L # M) ⇒ (Propagated (- (lit-of L)) () # M, N)
    | (-, -) ⇒ (Ms, N)
    )
  else
    (case find-first-unused-var N (lits-of-l Ms) of
      Some a ⇒ (Decided a # Ms, N)
    | None ⇒ (Ms, N)))

```

Example of propagation:

```

value DPLL-step ([Decided (Neg 1)], [[Pos (1::int), Neg 2]])

```

We define the conversion function between the states as defined in *Prop-DPLL* (with multisets) and here (with lists).

```

abbreviation toS ≡ λ(Ms::(int, unit) ann-lits)
  (N:: int literal list list). (Ms, mset (map mset N))
abbreviation toS' ≡ λ(Ms::(int, unit) ann-lits,
  N:: int literal list list). (Ms, mset (map mset N))

```

Proof of correctness of *DPLL-step*

```

lemma DPLL-step-is-a-dpllW-step:
  assumes step: (Ms', N') = DPLL-step (Ms, N)
  and neg: (Ms, N) ≠ (Ms', N')
  shows dpllW (toS Ms N) (toS Ms' N')
<proof>

```

```

lemma DPLL-step-stuck-final-state:
  assumes step: (Ms, N) = DPLL-step (Ms, N)
  shows conclusive-dpllW-state (toS Ms N)
<proof>

```

Adding invariants

Invariant tested in the function **function** *DPLL-ci* :: int dpll_W-ann-lits ⇒ int literal list list ⇒ int dpll_W-ann-lits × int literal list list **where**

```

DPLL-ci Ms N =
  (if ¬dpllW-all-inv (Ms, mset (map mset N))
  then (Ms, N)
  else
    let (Ms', N') = DPLL-step (Ms, N) in
    if (Ms', N') = (Ms, N) then (Ms, N) else DPLL-ci Ms' N)
<proof>

```

termination

<proof>

No invariant tested **function** (*domintros*) *DPLL-part* :: int dpll_W-ann-lits ⇒ int literal list list ⇒ int dpll_W-ann-lits × int literal list list **where**

```

DPLL-part Ms N =
  (let (Ms', N') = DPLL-step (Ms, N) in
  if (Ms', N') = (Ms, N) then (Ms, N) else DPLL-part Ms' N)
<proof>

```

lemma *snd-DPLL-step[simp]*:
snd (DPLL-step (Ms, N)) = N
<proof>

lemma *dpll_W-all-inv-implies-2-eq3-and-dom*:
assumes *dpll_W-all-inv (Ms, mset (map mset N))*
shows *DPLL-ci Ms N = DPLL-part Ms N ∧ DPLL-part-dom (Ms, N)*
<proof>

lemma *DPLL-ci-dpll_W-rtrancp*:
assumes *DPLL-ci Ms N = (Ms', N')*
shows *dpll_W** (toS Ms N) (toS Ms' N)*
<proof>

lemma *dpll_W-all-inv-dpll_W-trancp-irrefl*:
assumes *dpll_W-all-inv (Ms, N)*
and *dpll_W⁺⁺ (Ms, N) (Ms, N)*
shows *False*
<proof>

lemma *DPLL-ci-final-state*:
assumes *step: DPLL-ci Ms N = (Ms, N)*
and *inv: dpll_W-all-inv (toS Ms N)*
shows *conclusive-dpll_W-state (toS Ms N)*
<proof>

lemma *DPLL-step-obtains*:
obtains *Ms' where (Ms', N) = DPLL-step (Ms, N)*
<proof>

lemma *DPLL-ci-obtains*:
obtains *Ms' where (Ms', N) = DPLL-ci Ms N*
<proof>

lemma *DPLL-ci-no-more-step*:
assumes *step: DPLL-ci Ms N = (Ms', N')*
shows *DPLL-ci Ms' N' = (Ms', N')*
<proof>

lemma *DPLL-part-dpll_W-all-inv-final*:
fixes *M Ms':: (int, unit) ann-lits and*
N :: int literal list list
assumes *inv: dpll_W-all-inv (Ms, mset (map mset N))*
and *MsN: DPLL-part Ms N = (Ms', N)*
shows *conclusive-dpll_W-state (toS Ms' N) ∧ dpll_W** (toS Ms N) (toS Ms' N)*
<proof>

Embedding the invariant into the type

Defining the type **typedef** *dpll_W-state =*
{(M::(int, unit) ann-lits, N::int literal list list).
dpll_W-all-inv (toS M N))}
morphisms *rough-state-of state-of*

$\langle proof \rangle$

lemma

DPLL-part-dom (\square , N)

$\langle proof \rangle$

Some type classes instantiation *dpll_W-state* :: *equal*

begin

definition *equal-dpll_W-state* :: *dpll_W-state* \Rightarrow *dpll_W-state* \Rightarrow *bool* **where**

equal-dpll_W-state $S S' = (\text{rough-state-of } S = \text{rough-state-of } S')$

instance

$\langle proof \rangle$

end

DPLL definition *DPLL-step'* :: *dpll_W-state* \Rightarrow *dpll_W-state* **where**

DPLL-step' $S = \text{state-of } (\text{DPLL-step } (\text{rough-state-of } S))$

declare *rough-state-of-inverse*[*simp*]

lemma *DPLL-step-dpll_W-conc-inv*:

DPLL-step (*rough-state-of* S) $\in \{(M, N). \text{dpll}_W\text{-all-inv } (toS\ M\ N)\}$

$\langle proof \rangle$

lemma *rough-state-of-DPLL-step'-DPLL-step*[*simp*]:

rough-state-of (*DPLL-step'* S) = *DPLL-step* (*rough-state-of* S)

$\langle proof \rangle$

function *DPLL-tot*:: *dpll_W-state* \Rightarrow *dpll_W-state* **where**

DPLL-tot $S =$

(*let* $S' = \text{DPLL-step}'\ S$ *in*

if $S' = S$ *then* S *else* *DPLL-tot* S')

$\langle proof \rangle$

termination

$\langle proof \rangle$

lemma [*code*]:

DPLL-tot $S =$

(*let* $S' = \text{DPLL-step}'\ S$ *in*

if $S' = S$ *then* S *else* *DPLL-tot* S') $\langle proof \rangle$

lemma *DPLL-tot-DPLL-step-DPLL-tot*[*simp*]: *DPLL-tot* (*DPLL-step'* S) = *DPLL-tot* S

$\langle proof \rangle$

lemma *DOPLL-step'-DPLL-tot*[*simp*]:

DPLL-step' (*DPLL-tot* S) = *DPLL-tot* S

$\langle proof \rangle$

lemma *DPLL-tot-final-state*:

assumes *DPLL-tot* $S = S$

shows *conclusive-dpll_W-state* (*toS'* (*rough-state-of* S))

$\langle proof \rangle$

lemma *DPLL-tot-star*:

assumes *rough-state-of* (*DPLL-tot* *S*) = *S'*
shows *dpll_W^{**}* (*toS'* (*rough-state-of* *S*)) (*toS'* *S'*)
 ⟨*proof*⟩

lemma *rough-state-of-rough-state-of-Nil*[*simp*]:
rough-state-of (*state-of* ([], *N*)) = ([], *N*)
 ⟨*proof*⟩

Theorem of correctness

lemma *DPLL-tot-correct*:
assumes *rough-state-of* (*DPLL-tot* (*state-of* ([], *N*)))) = (*M*, *N'*)
and (*M'*, *N''*) = *toS'* (*M*, *N'*)
shows *M'* ⊨*asm* *N''* \longleftrightarrow *satisfiable* (*set-mset* *N''*)
 ⟨*proof*⟩

Code export

A conversion to DPLL-W-Implementation. *dpll_W-state* **definition** *Con* :: (*int*, *unit*) *ann-lits* × *int literal list list*

⇒ *dpll_W-state* **where**

Con *xs* = *state-of* (*if* *dpll_W-all-inv* (*toS* (*fst* *xs*) (*snd* *xs*)) *then* *xs* *else* ([], []))

lemma [*code abstype*]:
Con (*rough-state-of* *S*) = *S*
 ⟨*proof*⟩

declare *rough-state-of-DPLL-step'-DPLL-step*[*code abstract*]

lemma *Con-DPLL-step-rough-state-of-state-of*[*simp*]:
Con (*DPLL-step* (*rough-state-of* *s*)) = *state-of* (*DPLL-step* (*rough-state-of* *s*))
 ⟨*proof*⟩

A slightly different version of *DPLL-tot* where the returned boolean indicates the result.

definition *DPLL-tot-rep* **where**

DPLL-tot-rep *S* =

(*let* (*M*, *N*) = (*rough-state-of* (*DPLL-tot* *S*)) *in* ($\forall A \in \text{set } N. (\exists a \in \text{set } A. a \in \text{lits-of-l } M), M$))

One version of the generated SML code is here, but not included in the generated document.
 The only differences are:

- export '*a literal* from the SML Module *Clausal-Logic*;
- export the constructor *Con* from *DPLL-W-Implementation*;
- export the *int* constructor from *Arith*.

All these allows to test on the code on some examples.

end

theory *CDCL-W-Implementation*

imports *DPLL-CDCL-W-Implementation* *CDCL-W-Termination*
HOL-Library.Code-Target-Numeral

begin

4.1.4 List-based CDCL Implementation

We here have a very simple implementation of Weidenbach's CDCL, based on the same principle as the implementation of DPLL: iterating over-and-over on lists. We do not use any fancy data-structure (see the two-watched literals for a better suited data-structure).

The goal was (as for DPLL) to test the infrastructure and see if an important lemma was missing to prove the correctness and the termination of a simple implementation.

Types and Instantiation

notation *image-mset* (infixr '# 90)

type-synonym *'a cdcl_W-restart-mark* = *'a clause*

type-synonym *'v cdcl_W-restart-ann-lit* = (*'v*, *'v cdcl_W-restart-mark*) *ann-lit*

type-synonym *'v cdcl_W-restart-ann-lits* = (*'v*, *'v cdcl_W-restart-mark*) *ann-lits*

type-synonym *'v cdcl_W-restart-state* =
'v cdcl_W-restart-ann-lits × *'v clauses* × *'v clauses* × *'v clause option*

abbreviation *raw-trail* :: *'a* × *'b* × *'c* × *'d* ⇒ *'a* **where**

raw-trail ≡ (λ(*M*, -). *M*)

abbreviation *raw-cons-trail* :: *'a* ⇒ *'a list* × *'b* × *'c* × *'d* ⇒ *'a list* × *'b* × *'c* × *'d*
where

raw-cons-trail ≡ (λ*L* (*M*, *S*). (*L*#*M*, *S*))

abbreviation *raw-tl-trail* :: *'a list* × *'b* × *'c* × *'d* ⇒ *'a list* × *'b* × *'c* × *'d* **where**

raw-tl-trail ≡ (λ(*M*, *S*). (*tl M*, *S*))

abbreviation *raw-init-clss* :: *'a* × *'b* × *'c* × *'d* ⇒ *'b* **where**

raw-init-clss ≡ λ(*M*, *N*, -). *N*

abbreviation *raw-learned-clss* :: *'a* × *'b* × *'c* × *'d* ⇒ *'c* **where**

raw-learned-clss ≡ λ(*M*, *N*, *U*, -). *U*

abbreviation *raw-conflicting* :: *'a* × *'b* × *'c* × *'d* ⇒ *'d* **where**

raw-conflicting ≡ λ(*M*, *N*, *U*, *D*). *D*

abbreviation *raw-update-conflicting* :: *'d* ⇒ *'a* × *'b* × *'c* × *'d* ⇒ *'a* × *'b* × *'c* × *'d*
where

raw-update-conflicting ≡ λ*S* (*M*, *N*, *U*, -). (*M*, *N*, *U*, *S*)

abbreviation *S0-cdcl_W-restart* *N* ≡ (([], *N*, {#}, *None*)):: *'v cdcl_W-restart-state*

abbreviation *raw-add-learned-clss* **where**

raw-add-learned-clss ≡ λ*C* (*M*, *N*, *U*, *S*). (*M*, *N*, {#*C*#} + *U*, *S*)

abbreviation *raw-remove-cls* **where**

raw-remove-cls ≡ λ*C* (*M*, *N*, *U*, *S*). (*M*, *removeAll-mset C N*, *removeAll-mset C U*, *S*)

lemma *raw-trail-conv*: *raw-trail* (*M*, *N*, *U*, *D*) = *M* **and**

clauses-conv: *raw-init-clss* (*M*, *N*, *U*, *D*) = *N* **and**

raw-learned-clss-conv: *raw-learned-clss* (*M*, *N*, *U*, *D*) = *U* **and**

raw-conflicting-conv: *raw-conflicting* (*M*, *N*, *U*, *D*) = *D*

⟨*proof*⟩

lemma *state-conv*:

$S = (\text{raw-trail } S, \text{raw-init-clss } S, \text{raw-learned-clss } S, \text{raw-conflicting } S)$
 $\langle \text{proof} \rangle$

definition *state where*

$\langle \text{state } S = (\text{raw-trail } S, \text{raw-init-clss } S, \text{raw-learned-clss } S, \text{raw-conflicting } S, ()) \rangle$

interpretation *state_W*

$(=)$
state
raw-trail raw-init-clss raw-learned-clss raw-conflicting
 $\lambda L (M, S). (L \# M, S)$
 $\lambda (M, S). (tl \ M, S)$
 $\lambda C (M, N, U, S). (M, N, \text{add-mset } C \ U, S)$
 $\lambda C (M, N, U, S). (M, \text{removeAll-mset } C \ N, \text{removeAll-mset } C \ U, S)$
 $\lambda D (M, N, U, -). (M, N, U, D)$
 $\lambda N. ([], N, \{\#\}, \text{None})$
 $\langle \text{proof} \rangle$

declare *state-simp*[*simp del*]

interpretation *conflict-driven-clause-learning_W*

$(=)$ *state*
raw-trail raw-init-clss raw-learned-clss
raw-conflicting
 $\lambda L (M, S). (L \# M, S)$
 $\lambda (M, S). (tl \ M, S)$
 $\lambda C (M, N, U, S). (M, N, \text{add-mset } C \ U, S)$
 $\lambda C (M, N, U, S). (M, \text{removeAll-mset } C \ N, \text{removeAll-mset } C \ U, S)$
 $\lambda D (M, N, U, -). (M, N, U, D)$
 $\lambda N. ([], N, \{\#\}, \text{None})$
 $\langle \text{proof} \rangle$

declare *clauses-def*[*simp*]

lemma *reduce-trail-to-empty-trail*[*simp*]:

$\text{reduce-trail-to } F \ ([], aa, ab, b) = ([], aa, ab, b)$
 $\langle \text{proof} \rangle$

lemma *reduce-trail-to'*:

$\text{reduce-trail-to } F \ S =$
 $((\text{if } \text{length } (\text{raw-trail } S) \geq \text{length } F$
 $\text{then drop } (\text{length } (\text{raw-trail } S) - \text{length } F) (\text{raw-trail } S)$
 $\text{else } []), \text{raw-init-clss } S, \text{raw-learned-clss } S, \text{raw-conflicting } S)$
 $(\text{is } ?S = -)$
 $\langle \text{proof} \rangle$

Definition of the rules

Types **lemma** *true-raw-init-clss-remdups*[*simp*]:

$I \models s \ (\text{mset} \circ \text{remdups}) \ 'N \longleftrightarrow I \models s \ \text{mset} \ 'N$
 $\langle \text{proof} \rangle$

lemma *true-clss-raw-remdups-mset-mset*[*simp*]:

$\langle I \models s \ (\lambda L. \text{remdups-mset } (\text{mset } L)) \ 'N' \longleftrightarrow I \models s \ \text{mset} \ 'N' \rangle$

$\langle \text{proof} \rangle$

declare *satisfiable-carac*[*iff del*]

lemma *satisfiable-mset-remdups*[*simp*]:

satisfiable ((*mset* \circ *remdups*) ' *N*) \longleftrightarrow *satisfiable* (*mset* ' *N*)

satisfiable (($\lambda L.$ *remdups-mset* (*mset* *L*)) ' *N'*) \longleftrightarrow *satisfiable* (*mset* ' *N'*)

$\langle \text{proof} \rangle$

type-synonym 'v *cdcl_W-restart-state-inv-st* = ('v, 'v *literal list*) *ann-lit list* \times
'v *literal list list* \times 'v *literal list list* \times 'v *literal list option*

We need some functions to convert between our abstract state 'v *cdcl_W-restart-state* and the concrete state 'v *cdcl_W-restart-state-inv-st*.

fun *convert* :: ('a, 'c *list*) *ann-lit* \Rightarrow ('a, 'c *mset*) *ann-lit* **where**

convert (*Propagated* *L* *C*) = *Propagated* *L* (*mset* *C*) |

convert (*Decided* *K*) = *Decided* *K*

abbreviation *convertC* :: 'a *list option* \Rightarrow 'a *mset option* **where**

convertC \equiv *map-option* *mset*

lemma *convert-Propagated*[*elim!*]:

convert *z* = *Propagated* *L* *C* \implies (\exists *C'*. *z* = *Propagated* *L* *C'* \wedge *C* = *mset* *C'*)

$\langle \text{proof} \rangle$

lemma *is-decided-convert*[*simp*]: *is-decided* (*convert* *x*) = *is-decided* *x*

$\langle \text{proof} \rangle$

lemma *is-decided-convert-is-decided*[*simp*]: $\langle (\text{is-decided} \circ \text{convert}) = (\text{is-decided}) \rangle$

$\langle \text{proof} \rangle$

lemma *get-level-map-convert*[*simp*]:

get-level (*map convert* *M*) *x* = *get-level* *M* *x*

$\langle \text{proof} \rangle$

lemma *get-maximum-level-map-convert*[*simp*]:

get-maximum-level (*map convert* *M*) *D* = *get-maximum-level* *M* *D*

$\langle \text{proof} \rangle$

lemma *count-decided-convert*[*simp*]:

$\langle \text{count-decided} (\text{map convert } M) = \text{count-decided } M \rangle$

$\langle \text{proof} \rangle$

lemma *atm-lit-of-convert*[*simp*]:

lit-of (*convert* *x*) = *lit-of* *x*

$\langle \text{proof} \rangle$

lemma *no-dup-convert*[*simp*]:

$\langle \text{no-dup} (\text{map convert } M) = \text{no-dup } M \rangle$

$\langle \text{proof} \rangle$

Conversion function

fun *toS* :: 'v *cdcl_W-restart-state-inv-st* \Rightarrow 'v *cdcl_W-restart-state* **where**

toS (*M*, *N*, *U*, *C*) = (*map convert* *M*, *mset* (*map mset* *N*), *mset* (*map mset* *U*), *convertC* *C*)

Definition an abstract type

typedef *'v cdcl_W-restart-state-inv* = {*S* :: *'v cdcl_W-restart-state-inv-st*. *cdcl_W-all-struct-inv* (*toS S*)}
morphisms *rough-state-of state-of*
<proof>

instantiation *cdcl_W-restart-state-inv* :: (*type*) *equal*

begin

definition *equal-cdcl_W-restart-state-inv* :: *'v cdcl_W-restart-state-inv* \Rightarrow
'v cdcl_W-restart-state-inv \Rightarrow *bool* **where**
equal-cdcl_W-restart-state-inv S S' = (*rough-state-of S* = *rough-state-of S'*)

instance

<proof>

end

lemma *lits-of-map-convert[simp]*: *lits-of-l* (*map convert M*) = *lits-of-l M*

<proof>

lemma *undefined-lit-map-convert[iff]*:

undefined-lit (*map convert M*) *L* \longleftrightarrow *undefined-lit M L*

<proof>

lemma *true-annot-map-convert[simp]*: *map convert M* $\models_a N \longleftrightarrow M \models_a N$

<proof>

lemma *true-annots-map-convert[simp]*: *map convert M* $\models_{as} N \longleftrightarrow M \models_{as} N$

<proof>

lemmas *propagateE*

lemma *find-first-unit-clause-some-is-propagate*:

assumes *H*: *find-first-unit-clause* (*N @ U*) *M* = *Some (L, C)*

shows *propagate* (*toS (M, N, U, None)*) (*toS (Propagated L C # M, N, U, None)*)

<proof>

The Transitions

Propagate **definition** *do-propagate-step* :: *'v cdcl_W-restart-state-inv-st* \Rightarrow *'v cdcl_W-restart-state-inv-st*
where

do-propagate-step S =

(*case S of*

(*M, N, U, None*) \Rightarrow

(*case find-first-unit-clause* (*N @ U*) *M of*

Some (L, C) \Rightarrow (*Propagated L C # M, N, U, None*)

| *None* \Rightarrow (*M, N, U, None*))

| *S* \Rightarrow *S*)

lemma *do-propagate-step*:

do-propagate-step S $\neq S \implies$ *propagate* (*toS S*) (*toS (do-propagate-step S)*)

<proof>

lemma *do-propagate-step-option[simp]*:

raw-conflicting S \neq *None* \implies *do-propagate-step S* = *S*

<proof>

lemma *do-propagate-step-no-step*:

assumes *prop-step*: *do-propagate-step S* = *S*

shows *no-step propagate* (*toS S*)

$\langle \text{proof} \rangle$

Conflict fun *find-conflict* where

find-conflict $M \ [] = \text{None} \mid$

find-conflict $M \ (N \ \# \ Ns) = (\text{if } (\forall c \in \text{set } N. \neg c \in \text{ lits-of-}l \ M) \text{ then } \text{Some } N \text{ else } \text{find-conflict } M \ Ns)$

lemma *find-conflict-Some*:

find-conflict $M \ Ns = \text{Some } N \implies N \in \text{set } Ns \wedge M \models_{\text{as}} \text{CNot } (\text{mset } N)$

$\langle \text{proof} \rangle$

lemma *find-conflict-None*:

find-conflict $M \ Ns = \text{None} \iff (\forall N \in \text{set } Ns. \neg M \models_{\text{as}} \text{CNot } (\text{mset } N))$

$\langle \text{proof} \rangle$

lemma *find-conflict-None-no-conflict*:

find-conflict $M \ (N @ U) = \text{None} \iff \text{no-step conflict } (\text{toS } (M, N, U, \text{None}))$

$\langle \text{proof} \rangle$

definition *do-conflict-step* :: $'v \text{ cdcl}_W\text{-restart-state-inv-st} \Rightarrow 'v \text{ cdcl}_W\text{-restart-state-inv-st}$ where

do-conflict-step $S =$

(case S of

$(M, N, U, \text{None}) \Rightarrow$

(case *find-conflict* $M \ (N @ U)$ of

$\text{Some } a \Rightarrow (M, N, U, \text{Some } a)$

$\mid \text{None} \Rightarrow (M, N, U, \text{None}))$

$\mid S \Rightarrow S)$

lemma *do-conflict-step*:

do-conflict-step $S \neq S \implies \text{conflict } (\text{toS } S) (\text{toS } (\text{do-conflict-step } S))$

$\langle \text{proof} \rangle$

lemma *do-conflict-step-no-step*:

do-conflict-step $S = S \implies \text{no-step conflict } (\text{toS } S)$

$\langle \text{proof} \rangle$

lemma *do-conflict-step-option[simp]*:

raw-conflicting $S \neq \text{None} \implies \text{do-conflict-step } S = S$

$\langle \text{proof} \rangle$

lemma *do-conflict-step-raw-conflicting[dest]*:

do-conflict-step $S \neq S \implies \text{raw-conflicting } (\text{do-conflict-step } S) \neq \text{None}$

$\langle \text{proof} \rangle$

definition *do-cp-step* where

do-cp-step $S =$

(*do-propagate-step* o *do-conflict-step*) S

lemma *cdcl_W-all-struct-inv-rough-state[simp]*: *cdcl_W-all-struct-inv* (*toS* (*rough-state-of* S))

$\langle \text{proof} \rangle$

lemma [simp]: *cdcl_W-all-struct-inv* (*toS* S) $\implies \text{rough-state-of } (\text{state-of } S) = S$

$\langle \text{proof} \rangle$

Skip fun *do-skip-step* :: $'v \text{ cdcl}_W\text{-restart-state-inv-st} \Rightarrow 'v \text{ cdcl}_W\text{-restart-state-inv-st}$ where

do-skip-step (*Propagated* $L \ C \ \# \ Ls, N, U, \text{Some } D$) =

(if $-L \notin \text{set } D \wedge D \neq []$
 then $(Ls, N, U, \text{Some } D)$
 else $(\text{Propagated } L \ C \ \# Ls, N, U, \text{Some } D)) \mid$
 do-skip-step $S = S$

lemma *do-skip-step*:

do-skip-step $S \neq S \implies \text{skip } (\text{toS } S) (\text{toS } (\text{do-skip-step } S))$
 <proof>

lemma *do-skip-step-no*:

do-skip-step $S = S \implies \text{no-step skip } (\text{toS } S)$
 <proof>

lemma *do-skip-step-raw-trail-is-None*[iff]:

do-skip-step $S = (a, b, c, \text{None}) \longleftrightarrow S = (a, b, c, \text{None})$
 <proof>

Resolve **fun** *maximum-level-code*:: 'a literal list \Rightarrow ('a, 'a literal list) ann-lit list \Rightarrow nat
where

maximum-level-code $[] = 0 \mid$
maximum-level-code $(L \ \# \ Ls) \ M = \max (\text{get-level } M \ L) (\text{maximum-level-code } Ls \ M)$

lemma *maximum-level-code-eq-get-maximum-level*[code, simp]:

maximum-level-code $D \ M = \text{get-maximum-level } M \ (\text{mset } D)$
 <proof>

fun *do-resolve-step* :: 'v cdcl_W-restart-state-inv-st \Rightarrow 'v cdcl_W-restart-state-inv-st **where**

do-resolve-step $(\text{Propagated } L \ C \ \# \ Ls, N, U, \text{Some } D) =$
 (if $-L \in \text{set } D \wedge \text{maximum-level-code } (\text{remove1 } (-L) \ D) (\text{Propagated } L \ C \ \# \ Ls) = \text{count-decided } Ls$
 then $(Ls, N, U, \text{Some } (\text{remdups } (\text{remove1 } L \ C \ @ \ \text{remove1 } (-L) \ D)))$
 else $(\text{Propagated } L \ C \ \# \ Ls, N, U, \text{Some } D)) \mid$
do-resolve-step $S = S$

lemma *do-resolve-step*:

cdcl_W-all-struct-inv $(\text{toS } S) \implies \text{do-resolve-step } S \neq S$
 $\implies \text{resolve } (\text{toS } S) (\text{toS } (\text{do-resolve-step } S))$
 <proof>

lemma *do-resolve-step-no*:

do-resolve-step $S = S \implies \text{no-step resolve } (\text{toS } S)$
 <proof>

lemma *rough-state-of-state-of-resolve*[simp]:

cdcl_W-all-struct-inv $(\text{toS } S) \implies$
 rough-state-of $(\text{state-of } (\text{do-resolve-step } S)) = \text{do-resolve-step } S$
 <proof>

lemma *do-resolve-step-raw-trail-is-None*[iff]:

do-resolve-step $S = (a, b, c, \text{None}) \longleftrightarrow S = (a, b, c, \text{None})$
 <proof>

Backjumping **lemma** *get-all-ann-decomposition-map-convert*:

(get-all-ann-decomposition $(\text{map convert } M)) =$
 map $(\lambda(a, b). (\text{map convert } a, \text{map convert } b)) (\text{get-all-ann-decomposition } M)$
 <proof>

lemma *do-backtrack-step*:

assumes

db: *do-backtrack-step* $S \neq S$ **and**

inv: *cdcl_W-all-struct-inv* (*toS* S)

shows *backtrack* (*toS* S) (*toS* (*do-backtrack-step* S))

<proof>

lemma *map-eq-list-length*:

map f $L = L' \implies \text{length } L = \text{length } L'$

<proof>

lemma *map-mmset-of-mlit-eq-cons*:

assumes *map convert* $M = a @ c$

obtains $a' c'$ **where**

$M = a' @ c'$ **and**

$a = \text{map convert } a'$ **and**

$c = \text{map convert } c'$

<proof>

lemma *Decided-convert-iff*:

Decided $K = \text{convert } za \longleftrightarrow za = \text{Decided } K$

<proof>

declare *conflict-is-false-with-level-def*[*simp del*]

lemma *do-backtrack-step-no*:

assumes

db: *do-backtrack-step* $S = S$ **and**

inv: *cdcl_W-all-struct-inv* (*toS* S) **and**

ns: *<no-step skip* (*toS* S) *<no-step resolve* (*toS* S)

shows *no-step backtrack* (*toS* S)

<proof>

lemma *rough-state-of-state-of-backtrack*[*simp*]:

assumes *inv*: *cdcl_W-all-struct-inv* (*toS* S)

shows *rough-state-of* (*state-of* (*do-backtrack-step* S)) = *do-backtrack-step* S

<proof>

Decide fun *do-decide-step* **where**

do-decide-step (M, N, U, None) =

(*case find-first-unused-var* N (*lits-of-l* M) *of*

$\text{None} \Rightarrow (M, N, U, \text{None})$

| $\text{Some } L \Rightarrow (\text{Decided } L \# M, N, U, \text{None}))$ |

do-decide-step $S = S$

lemma *do-decide-step*:

do-decide-step $S \neq S \implies \text{decide}$ (*toS* S) (*toS* (*do-decide-step* S))

<proof>

lemma *do-decide-step-no*:

do-decide-step $S = S \implies \text{no-step decide}$ (*toS* S)

<proof>

lemma *rough-state-of-state-of-do-decide-step*[*simp*]:

$cdcl_W\text{-all-struct-inv } (toS\ S) \implies \text{rough-state-of } (state\text{-of } (do\text{-decide-step } S)) = do\text{-decide-step } S$
 $\langle proof \rangle$

lemma *rough-state-of-state-of-do-skip-step*[simp]:

$cdcl_W\text{-all-struct-inv } (toS\ S) \implies \text{rough-state-of } (state\text{-of } (do\text{-skip-step } S)) = do\text{-skip-step } S$
 $\langle proof \rangle$

Code generation

Type definition There are two invariants: one while applying conflict and propagate and one for the other rules

declare *rough-state-of-inverse*[simp add]

definition *Con* **where**

$Con\ xs = state\text{-of } (if\ cdcl_W\text{-all-struct-inv } (toS\ (fst\ xs,\ snd\ xs))\ then\ xs$
 $else\ ([], [], [], None))$

lemma [code abstype]:

$Con\ (rough\text{-state-of } S) = S$
 $\langle proof \rangle$

definition *do-cp-step'* **where**

$do\text{-cp-step}'\ S = state\text{-of } (do\text{-cp-step } (rough\text{-state-of } S))$

typedef *'v cdcl_W-restart-state-inv-from-init-state* =

$\{S :: 'v\ cdcl_W\text{-restart-state-inv-st. } cdcl_W\text{-all-struct-inv } (toS\ S)$
 $\wedge\ cdcl_W\text{-stgy}^{**}\ (S0\text{-cdcl}_W\text{-restart } (raw\text{-init-clss } (toS\ S)))\ (toS\ S)\}$

morphisms *rough-state-from-init-state-of* *state-from-init-state-of*

$\langle proof \rangle$

instantiation *cdcl_W-restart-state-inv-from-init-state* :: (type) equal

begin

definition *equal-cdcl_W-restart-state-inv-from-init-state* :: *'v cdcl_W-restart-state-inv-from-init-state* \Rightarrow

'v cdcl_W-restart-state-inv-from-init-state \Rightarrow bool **where**

equal-cdcl_W-restart-state-inv-from-init-state *S S'* \longleftrightarrow

$(rough\text{-state-from-init-state-of } S = rough\text{-state-from-init-state-of } S')$

instance

$\langle proof \rangle$

end

definition *ConI* **where**

$ConI\ S = state\text{-from-init-state-of } (if\ cdcl_W\text{-all-struct-inv } (toS\ (fst\ S,\ snd\ S))$
 $\wedge\ cdcl_W\text{-stgy}^{**}\ (S0\text{-cdcl}_W\text{-restart } (raw\text{-init-clss } (toS\ S)))\ (toS\ S)\ then\ S\ else\ ([], [], [], None))$

lemma [code abstype]:

$ConI\ (rough\text{-state-from-init-state-of } S) = S$
 $\langle proof \rangle$

definition *id-of-I-to*:: *'v cdcl_W-restart-state-inv-from-init-state* \Rightarrow *'v cdcl_W-restart-state-inv* **where**

id-of-I-to *S* = *state-of* (*rough-state-from-init-state-of* *S*)

lemma [code abstract]:

$rough\text{-state-of } (id\text{-of-I-to } S) = rough\text{-state-from-init-state-of } S$
 $\langle proof \rangle$

lemma *in-clauses-rough-state-of-is-distinct*:

$c \in \text{set } (\text{raw-init-clss } (\text{rough-state-of } S) @ \text{raw-learned-clss } (\text{rough-state-of } S)) \implies \text{distinct } c$
 $\langle \text{proof} \rangle$

The other rules **fun** *do-if-not-equal* **where**

do-if-not-equal [] $S = S$ |
do-if-not-equal ($f \# fs$) $S =$
 (let $T = f S$ in
 if $T \neq S$ then T else *do-if-not-equal* $fs S$)

fun *do-cdcl-step* **where**

do-cdcl-step $S =$
do-if-not-equal [*do-conflict-step*, *do-propagate-step*, *do-skip-step*, *do-resolve-step*,
do-backtrack-step, *do-decide-step*] S

lemma *do-cdcl-step*:

assumes *inv*: *cdcl_W-all-struct-inv* (*toS* S) **and**
st: *do-cdcl-step* $S \neq S$
shows *cdcl_W-stgy* (*toS* S) (*toS* (*do-cdcl-step* S))
 $\langle \text{proof} \rangle$

lemma *do-cdcl-step-no*:

assumes *inv*: *cdcl_W-all-struct-inv* (*toS* S) **and**
st: *do-cdcl-step* $S = S$
shows *no-step cdcl_W* (*toS* S)
 $\langle \text{proof} \rangle$

lemma *rough-state-of-state-of-do-cdcl-step[simp]*:

rough-state-of (*state-of* (*do-cdcl-step* (*rough-state-of* S))) = *do-cdcl-step* (*rough-state-of* S)
 $\langle \text{proof} \rangle$

definition *do-cdcl_W-stgy-step* :: '*v cdcl_W-restart-state-inv* \Rightarrow '*v cdcl_W-restart-state-inv* **where**

do-cdcl_W-stgy-step $S =$
state-of (*do-cdcl-step* (*rough-state-of* S))

lemma *rough-state-of-do-cdcl_W-stgy-step[code abstract]*:

rough-state-of (*do-cdcl_W-stgy-step* S) = *do-cdcl-step* (*rough-state-of* S)
 $\langle \text{proof} \rangle$

definition *do-cdcl_W-stgy-step'* **where**

do-cdcl_W-stgy-step' $S = \text{state-from-init-state-of } (\text{rough-state-of } (\text{do-cdcl_W-stgy-step } (\text{id-of-I-to } S)))$

Correction of the transformation **lemma** *do-cdcl_W-stgy-step*:

assumes *do-cdcl_W-stgy-step* $S \neq S$
shows *cdcl_W-stgy* (*toS* (*rough-state-of* S)) (*toS* (*rough-state-of* (*do-cdcl_W-stgy-step* S)))
 $\langle \text{proof} \rangle$

lemma *length-raw-trail-toS[simp]*:

length (*raw-trail* (*toS* S)) = *length* (*raw-trail* S)
 $\langle \text{proof} \rangle$

lemma *raw-conflicting-noTrue-iff-toS[simp]*:

raw-conflicting (*toS* S) $\neq \text{None} \longleftrightarrow \text{raw-conflicting } S \neq \text{None}$
 $\langle \text{proof} \rangle$

lemma *raw-trail-toS-neq-imp-raw-trail-neq*:

$\text{raw-trail } (\text{toS } S) \neq \text{raw-trail } (\text{toS } S') \implies \text{raw-trail } S \neq \text{raw-trail } S'$
 $\langle \text{proof} \rangle$

lemma *do-cp-step-neq-raw-trail-increase*:

$\exists c. \text{raw-trail } (\text{do-cp-step } S) = c @ \text{raw-trail } S \wedge (\forall m \in \text{set } c. \neg \text{is-decided } m)$
 $\langle \text{proof} \rangle$

lemma *do-cp-step-raw-conflicting*:

$\text{raw-conflicting } (\text{rough-state-of } S) \neq \text{None} \implies \text{do-cp-step}' S = S$
 $\langle \text{proof} \rangle$

lemma *do-decide-step-not-raw-conflicting-one-more-decide*:

assumes

$\text{raw-conflicting } S = \text{None}$ **and**

$\text{do-decide-step } S \neq S$

shows $\text{Suc } (\text{length } (\text{filter is-decided } (\text{raw-trail } S)))$
 $= \text{length } (\text{filter is-decided } (\text{raw-trail } (\text{do-decide-step } S)))$
 $\langle \text{proof} \rangle$

lemma *do-decide-step-not-raw-conflicting-one-more-decide-bt*:

assumes $\text{raw-conflicting } S \neq \text{None}$ **and**

$\text{do-decide-step } S \neq S$

shows $\text{length } (\text{filter is-decided } (\text{raw-trail } S)) < \text{length } (\text{filter is-decided } (\text{raw-trail } (\text{do-decide-step } S)))$
 $\langle \text{proof} \rangle$

lemma *count-decided-raw-trail-toS*:

$\text{count-decided } (\text{raw-trail } (\text{toS } S)) = \text{count-decided } (\text{raw-trail } S)$
 $\langle \text{proof} \rangle$

lemma *rough-state-of-state-of-do-skip-step-rough-state-of[simp]*:

$\text{rough-state-of } (\text{state-of } (\text{do-skip-step } (\text{rough-state-of } S))) = \text{do-skip-step } (\text{rough-state-of } S)$
 $\langle \text{proof} \rangle$

lemma *raw-conflicting-do-resolve-step-iff[iff]*:

$\text{raw-conflicting } (\text{do-resolve-step } S) = \text{None} \longleftrightarrow \text{raw-conflicting } S = \text{None}$
 $\langle \text{proof} \rangle$

lemma *raw-conflicting-do-skip-step-iff[iff]*:

$\text{raw-conflicting } (\text{do-skip-step } S) = \text{None} \longleftrightarrow \text{raw-conflicting } S = \text{None}$
 $\langle \text{proof} \rangle$

lemma *raw-conflicting-do-decide-step-iff[iff]*:

$\text{raw-conflicting } (\text{do-decide-step } S) = \text{None} \longleftrightarrow \text{raw-conflicting } S = \text{None}$
 $\langle \text{proof} \rangle$

lemma *raw-conflicting-do-backtrack-step-imp[simp]*:

$\text{do-backtrack-step } S \neq S \implies \text{raw-conflicting } (\text{do-backtrack-step } S) = \text{None}$
 $\langle \text{proof} \rangle$

lemma *do-skip-step-eq-iff-raw-trail-eq*:

$\text{do-skip-step } S = S \longleftrightarrow \text{raw-trail } (\text{do-skip-step } S) = \text{raw-trail } S$
 $\langle \text{proof} \rangle$

lemma *do-decide-step-eq-iff-raw-trail-eq*:

$\text{do-decide-step } S = S \longleftrightarrow \text{raw-trail } (\text{do-decide-step } S) = \text{raw-trail } S$
 $\langle \text{proof} \rangle$

lemma *do-backtrack-step-eq-iff-raw-trail-eq*:
assumes *no-dup* (*raw-trail* *S*)
shows *do-backtrack-step* *S* = *S* \longleftrightarrow *raw-trail* (*do-backtrack-step* *S*) = *raw-trail* *S*
 \langle *proof* \rangle

lemma *do-resolve-step-eq-iff-raw-trail-eq*:
do-resolve-step *S* = *S* \longleftrightarrow *raw-trail* (*do-resolve-step* *S*) = *raw-trail* *S*
 \langle *proof* \rangle

lemma *do-cdcl_W-stgy-step-no*:
assumes *S*: *do-cdcl_W-stgy-step* *S* = *S*
shows *no-step* *cdcl_W-stgy* (*toS* (*rough-state-of* *S*))
 \langle *proof* \rangle

lemma *toS-rough-state-of-state-of-rough-state-from-init-state-of*[*simp*]:
toS (*rough-state-of* (*state-of* (*rough-state-from-init-state-of* *S*)))
= *toS* (*rough-state-from-init-state-of* *S*)
 \langle *proof* \rangle

lemma *cdcl_W-stgy-is-rtrancp-cdcl_W-restart*:
cdcl_W-stgy *S* *T* \implies *cdcl_W-restart*** *S* *T*
 \langle *proof* \rangle

lemma *cdcl_W-stgy-init-raw-init-clss*:
cdcl_W-stgy *S* *T* \implies *cdcl_W-M-level-inv* *S* \implies *raw-init-clss* *S* = *raw-init-clss* *T*
 \langle *proof* \rangle

lemma *clauses-toS-rough-state-of-do-cdcl_W-stgy-step*[*simp*]:
raw-init-clss (*toS* (*rough-state-of* (*do-cdcl_W-stgy-step* (*state-of* (*rough-state-from-init-state-of* *S*))))))
= *raw-init-clss* (*toS* (*rough-state-from-init-state-of* *S*)) (**is** - = *raw-init-clss* (*toS* ?*S*))
 \langle *proof* \rangle

lemma *rough-state-from-init-state-of-do-cdcl_W-stgy-step'*[*code abstract*]:
rough-state-from-init-state-of (*do-cdcl_W-stgy-step'* *S*) =
rough-state-of (*do-cdcl_W-stgy-step* (*id-of-I-to* *S*))
 \langle *proof* \rangle

All rules together **function** *do-all-cdcl_W-stgy* **where**

do-all-cdcl_W-stgy *S* =
(*let* *T* = *do-cdcl_W-stgy-step'* *S* *in*
if *T* = *S* *then* *S* *else* *do-all-cdcl_W-stgy* *T*)
 \langle *proof* \rangle

termination
 \langle *proof* \rangle

thm *do-all-cdcl_W-stgy.induct*

lemma *do-all-cdcl_W-stgy-induct*:
(\bigwedge *S*. (*do-cdcl_W-stgy-step'* *S* \neq *S* \implies *P* (*do-cdcl_W-stgy-step'* *S*)) \implies *P* *S*) \implies *P* *a0*
 \langle *proof* \rangle

lemma *no-step-cdcl_W-stgy-cdcl_W-restart-all*:
fixes *S* :: 'a *cdcl_W-restart-state-inv-from-init-state*
shows *no-step* *cdcl_W-stgy* (*toS* (*rough-state-from-init-state-of* (*do-all-cdcl_W-stgy* *S*)))
 \langle *proof* \rangle

```

lemma do-all-cdclW-stgy-is-rtrancp-cdclW-stgy:
  cdclW-stgy** (toS (rough-state-from-init-state-of S))
    (toS (rough-state-from-init-state-of (do-all-cdclW-stgy S)))
  <proof>

```

Final theorem:

```

lemma DPLL-tot-correct:
  assumes
    r: rough-state-from-init-state-of (do-all-cdclW-stgy (state-from-init-state-of
      (([], map remdups N, [], None)))) = S and
    S: (M', N', U', E) = toS S
  shows (E ≠ Some {#} ∧ satisfiable (set (map mset N)))
    ∨ (E = Some {#} ∧ unsatisfiable (set (map mset N)))
  <proof>

```

The Code The SML code is skipped in the documentation, but stays to ensure that some version of the exported code is working. The only difference between the generated code and the one used here is the export of the constructor `ConI`.

```

<proof>
theory CDCL-Abstract-Clause-Representation
imports Entailment-Definition.Partial-Herbrand-Interpretation
begin

```

```

type-synonym 'v clause = 'v literal multiset
type-synonym 'v clauses = 'v clause multiset

```

4.1.5 Abstract Clause Representation

We will abstract the representation of clause and clauses via two locales. We expect our representation to behave like multiset, but the internal representation can be done using list or whatever other representation.

We assume the following:

- there is an equivalent to adding and removing a literal and to taking the union of clauses.

```

locale raw-cls =
  fixes
    mset-cls :: 'cls ⇒ 'v clause
begin
end

```

The two following locales are the *exact same* locale, but we need two different locales. Otherwise, instantiating *raw-cls* would lead to duplicate constants.

```

locale abstract-with-index =
  fixes
    get-lit :: 'a ⇒ 'it ⇒ 'conc option and
    convert-to-mset :: 'a ⇒ 'conc multiset
  assumes
    in-clss-mset-cls[dest]:
      get-lit Cs a = Some e ⇒ e ∈ # convert-to-mset Cs and
    in-mset-cls-exists-preimage:
      b ∈ # convert-to-mset Cs ⇒ ∃ b'. get-lit Cs b' = Some b

```

locale *abstract-with-index2* =
fixes
get-lit :: 'a \Rightarrow 'it \Rightarrow 'conc option **and**
convert-to-mset :: 'a \Rightarrow 'conc multiset
assumes
in-clss-mset-clss[*dest*]:
get-lit *Cs* *a* = *Some* *e* \implies *e* \in # *convert-to-mset* *Cs* **and**
in-mset-clss-exists-preimage:
b \in # *convert-to-mset* *Cs* \implies \exists *b'*. *get-lit* *Cs* *b'* = *Some* *b*

locale *raw-clss* =
abstract-with-index *get-lit* *mset-cl* +
abstract-with-index2 *get-cl* *mset-clss*
for
get-lit :: 'cls \Rightarrow 'lit \Rightarrow 'v literal option **and**
mset-cl :: 'cls \Rightarrow 'v clause **and**

get-cl :: 'clss \Rightarrow 'cls-it \Rightarrow 'cls option **and**
mset-clss:: 'clss \Rightarrow 'cls multiset

begin

definition *cls-lit* :: 'cls \Rightarrow 'lit \Rightarrow 'v literal (**infix** \downarrow 49) **where**
C \downarrow *a* \equiv the (*get-lit* *C* *a*)

definition *clss-cl* :: 'clss \Rightarrow 'cls-it \Rightarrow 'cls (**infix** \Downarrow 49) **where**
C \Downarrow *a* \equiv the (*get-cl* *C* *a*)

definition *in-cl* :: 'lit \Rightarrow 'cls \Rightarrow bool (**infix** $\in\downarrow$ 49) **where**
a $\in\downarrow$ *Cs* \equiv *get-lit* *Cs* *a* \neq None

definition *in-clss* :: 'cls-it \Rightarrow 'clss \Rightarrow bool (**infix** $\in\Downarrow$ 49) **where**
a $\in\Downarrow$ *Cs* \equiv *get-cl* *Cs* *a* \neq None

definition *raw-clss* **where**
raw-clss *S* \equiv *image-mset* *mset-cl* (*mset-clss* *S*)

end

experiment

begin

fun *safe-nth* **where**
safe-nth (*x* # -) 0 = *Some* *x* |
safe-nth (- # *xs*) (*Suc* *n*) = *safe-nth* *xs* *n* |
safe-nth [] - = None

lemma *safe-nth-nth*: *n* < *length* *l* \implies *safe-nth* *l* *n* = *Some* (*nth* *l* *n*)
 <proof>

lemma *safe-nth-None*: *n* \geq *length* *l* \implies *safe-nth* *l* *n* = None
 <proof>

lemma *safe-nth-Some-iff*: *safe-nth* *l* *n* = *Some* *m* \longleftrightarrow *n* < *length* *l* \wedge *m* = *nth* *l* *n*
 <proof>

lemma *safe-nth-None-iff*: *safe-nth* *l* *n* = None \longleftrightarrow *n* \geq *length* *l*

```

    <proof>

interpretation abstract-with-index
  safe-nth
  mset
  <proof>

interpretation abstract-with-index2
  safe-nth
  mset
  <proof>

interpretation list-clss: raw-clss
  safe-nth mset
  safe-nth mset
  <proof>
end

end
theory CDCL-W-Abstract-State
imports CDCL-W-Full CDCL-W-Restart

begin

```

4.2 Instantiation of Weidenbach's CDCL by Multisets

We first instantiate the locale of Weidenbach's locale. Then we refine it to a 2-WL program.

```

type-synonym 'v cdclW-restart-mset = ('v, 'v clause) ann-lit list ×
  'v clauses ×
  'v clauses ×
  'v clause option

```

We use definition, otherwise we could not use the simplification theorems we have already shown.

```

fun trail :: 'v cdclW-restart-mset ⇒ ('v, 'v clause) ann-lit list where
trail (M, -) = M

```

```

fun init-clss :: 'v cdclW-restart-mset ⇒ 'v clauses where
init-clss (-, N, -) = N

```

```

fun learned-clss :: 'v cdclW-restart-mset ⇒ 'v clauses where
learned-clss (-, -, U, -) = U

```

```

fun conflicting :: 'v cdclW-restart-mset ⇒ 'v clause option where
conflicting (-, -, -, C) = C

```

```

fun cons-trail :: ('v, 'v clause) ann-lit ⇒ 'v cdclW-restart-mset ⇒ 'v cdclW-restart-mset where
cons-trail L (M, R) = (L # M, R)

```

```

fun tl-trail where
tl-trail (M, R) = (tl M, R)

```

```

fun add-learned-clss where
add-learned-clss C (M, N, U, R) = (M, N, {#C#} + U, R)

```

fun *remove-cls* **where**
remove-cls *C* (*M*, *N*, *U*, *R*) = (*M*, *removeAll-mset* *C* *N*, *removeAll-mset* *C* *U*, *R*)

fun *update-conflicting* **where**
update-conflicting *D* (*M*, *N*, *U*, *-*) = (*M*, *N*, *U*, *D*)

fun *init-state* **where**
init-state *N* = ([], *N*, {#}, *None*)

declare *trail.simps*[*simp del*] *cons-trail.simps*[*simp del*] *tl-trail.simps*[*simp del*]
add-learned-cls.simps[*simp del*] *remove-cls.simps*[*simp del*]
update-conflicting.simps[*simp del*] *init-clss.simps*[*simp del*] *learned-clss.simps*[*simp del*]
conflicting.simps[*simp del*] *init-state.simps*[*simp del*]

lemmas *cdcl_W-restart-mset-state* = *trail.simps* *cons-trail.simps* *tl-trail.simps* *add-learned-cls.simps*
remove-cls.simps *update-conflicting.simps* *init-clss.simps* *learned-clss.simps*
conflicting.simps *init-state.simps*

definition *state* **where**
 $\langle \text{state } S = (\text{trail } S, \text{init-clss } S, \text{learned-clss } S, \text{conflicting } S, ()) \rangle$

interpretation *cdcl_W-restart-mset: state_W-ops* **where**

state = *state* **and**
trail = *trail* **and**
init-clss = *init-clss* **and**
learned-clss = *learned-clss* **and**
conflicting = *conflicting* **and**

cons-trail = *cons-trail* **and**
tl-trail = *tl-trail* **and**
add-learned-cls = *add-learned-cls* **and**
remove-cls = *remove-cls* **and**
update-conflicting = *update-conflicting* **and**
init-state = *init-state*
 $\langle \text{proof} \rangle$

definition *state-eq* :: '*v* *cdcl_W-restart-mset* \Rightarrow '*v* *cdcl_W-restart-mset* \Rightarrow bool (**infix** \sim_m 50) **where**
 $\langle S \sim_m T \longleftrightarrow \text{state } S = \text{state } T \rangle$

interpretation *cdcl_W-restart-mset: state_W* **where**

state = *state* **and**
trail = *trail* **and**
init-clss = *init-clss* **and**
learned-clss = *learned-clss* **and**
conflicting = *conflicting* **and**
state-eq = *state-eq* **and**
cons-trail = *cons-trail* **and**
tl-trail = *tl-trail* **and**
add-learned-cls = *add-learned-cls* **and**
remove-cls = *remove-cls* **and**
update-conflicting = *update-conflicting* **and**
init-state = *init-state*
 $\langle \text{proof} \rangle$

abbreviation *backtrack-lvl* :: '*v* *cdcl_W-restart-mset* \Rightarrow nat **where**

backtrack-lvl \equiv *cdcl_W-restart-mset.backtrack-lvl*

interpretation *cdcl_W-restart-mset*: *conflict-driven-clause-learning_W* **where**

state = *state* **and**
trail = *trail* **and**
init-clss = *init-clss* **and**
learned-clss = *learned-clss* **and**
conflicting = *conflicting* **and**

state-eq = *state-eq* **and**
cons-trail = *cons-trail* **and**
tl-trail = *tl-trail* **and**
add-learned-cls = *add-learned-cls* **and**
remove-cls = *remove-cls* **and**
update-conflicting = *update-conflicting* **and**
init-state = *init-state*
 $\langle \text{proof} \rangle$

lemma *cdcl_W-restart-mset-state-eq-eq*: *state-eq* = (=)
 $\langle \text{proof} \rangle$

lemma *clauses-def*: $\langle \text{cdcl}_W\text{-restart-mset.clauses } (M, N, U, C) = N + U \rangle$
 $\langle \text{proof} \rangle$

lemma *cdcl_W-restart-mset-reduce-trail-to*:

cdcl_W-restart-mset.reduce-trail-to *F S* =
 ((if *length* (*trail S*) \geq *length F*
 then *drop* (*length* (*trail S*) - *length F*) (*trail S*)
 else []), *init-clss S*, *learned-clss S*, *conflicting S*)
 (is ?*S* = -)

$\langle \text{proof} \rangle$

lemma *full-cdcl_W-init-state*:

$\langle \text{full cdcl}_W\text{-restart-mset.cdcl}_W\text{-stgy } (\text{init-state } \{\#\}) S \longleftrightarrow S = \text{init-state } \{\#\} \rangle$
 $\langle \text{proof} \rangle$

locale *twl-restart-ops* =

fixes
f :: $\langle \text{nat} \Rightarrow \text{nat} \rangle$

begin

interpretation *cdcl_W-restart-mset*: *cdcl_W-restart-restart-ops* **where**

state = *state* **and**
trail = *trail* **and**
init-clss = *init-clss* **and**
learned-clss = *learned-clss* **and**
conflicting = *conflicting* **and**

state-eq = *state-eq* **and**
cons-trail = *cons-trail* **and**
tl-trail = *tl-trail* **and**
add-learned-cls = *add-learned-cls* **and**
remove-cls = *remove-cls* **and**
update-conflicting = *update-conflicting* **and**


```

    init-state = init-state and
    f = f
     $\langle \text{proof} \rangle$ 

end

locale twl-restart =
  twl-restart-ops f for f ::  $\langle \text{nat} \Rightarrow \text{nat} \rangle$  +
  assumes
    f:  $\langle \text{unbounded } f \rangle$ 
begin

interpretation cdclW-restart-mset: cdclW-restart-restart where
  state = state and
  trail = trail and
  init-clss = init-clss and
  learned-clss = learned-clss and
  conflicting = conflicting and

  state-eq = state-eq and
  cons-trail = cons-trail and
  tl-trail = tl-trail and
  add-learned-cls = add-learned-cls and
  remove-cls = remove-cls and
  update-conflicting = update-conflicting and
  init-state = init-state and
  f = f
   $\langle \text{proof} \rangle$ 

end

context conflict-driven-clause-learningW
begin

lemma distinct-cdclW-state-alt-def:
   $\langle \text{distinct-cdcl}_W\text{-state } S =$ 
     $((\forall T. \text{conflicting } S = \text{Some } T \longrightarrow \text{distinct-mset } T) \wedge$ 
     $\text{distinct-mset-mset } (\text{clauses } S) \wedge$ 
     $(\forall L \text{ mark. } \text{Propagated } L \text{ mark} \in \text{set } (\text{trail } S) \longrightarrow \text{distinct-mset mark})) \rangle$ 
   $\langle \text{proof} \rangle$ 
end

lemma cdclW-stgy-cdclW-init-state-empty-no-step:
   $\langle \text{cdcl}_W\text{-restart-mset.cdcl}_W\text{-stgy } (\text{init-state } \{\#\}) S \longleftrightarrow \text{False} \rangle$ 
   $\langle \text{proof} \rangle$ 

lemma cdclW-stgy-cdclW-init-state:
   $\langle \text{cdcl}_W\text{-restart-mset.cdcl}_W\text{-stgy}^{**} (\text{init-state } \{\#\}) S \longleftrightarrow S = \text{init-state } \{\#\} \rangle$ 
   $\langle \text{proof} \rangle$ 

end

```