

Simple Games

- Games: a mathematical formalism for rational interaction
- What is the best solution concept? (Nash Equilibrium)

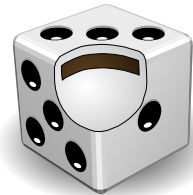
Types of Uncertainty

Alleatory



Markov Decision Process

Epistemic (Static)



Reinforcement Learning

Epistemic (Dynamic)



POMDP

Interaction



Game

Normal Form Games

- Alice and Bob are working on a homework assignment.
- They can either **share** or **withhold** their knowledge.
- If one player shares knowledge, the other benefits greatly, but the sharer also benefits by getting to test their knowledge

Alice's Payoffs

		B	
		S	W
A	S	4	2
	W	3	1

Bob's Payoffs

		B	
		S	W
A	S	4	3
	W	2	1

Bob

Alice

		S	W
Alice	S	4, 4	2, 3
	W	3, 2	1, 1

Called a **Normal Form, Simple**, or **Bimatrix** Game

Question for today: What **solution concept** should we use for games?

Dominant Strategies

		Bob	
		S	W
Alice	S	4, 4	2, 3
	W	3, 2	1, 1

- **Dominant (Pure) Strategy:** Action a is a dominant strategy if it is a best response to every action taken by the other player.
- **Dominant Strategy Equilibrium:** Every player plays a dominant strategy

Definitions

- Action $a^i \in A^i$
- Joint Action $a = (a^1, \dots, a^k)$
- All Other Actions
 $a^{-i} = (a^1, \dots, a^{i-1}, a^{i+1}, \dots, a^k)$
- Reward $R^i(a)$
- Joint Reward $R(a) = (R^1(a), \dots, R^k(a))$

Deterministic Best Response:

Action a^i is a deterministic best response to a^{-i} if

$$R^i(a^i, a^{-i}) \geq R^i(a^{i'}, a^{-i}) \quad \forall a^{i'}$$

Is the dominant strategy equilibrium always the best outcome for the players?

A more surprising example:

The Prisoner's Dilemma

- 2 criminals are captured
- Each can either keep silent or testify
 - other keeps silent -> minor conviction (1 year)
 - other testifies -> major conviction: 4 years
 - testify -> 1 year removed from sentence

Player 1

Player 2

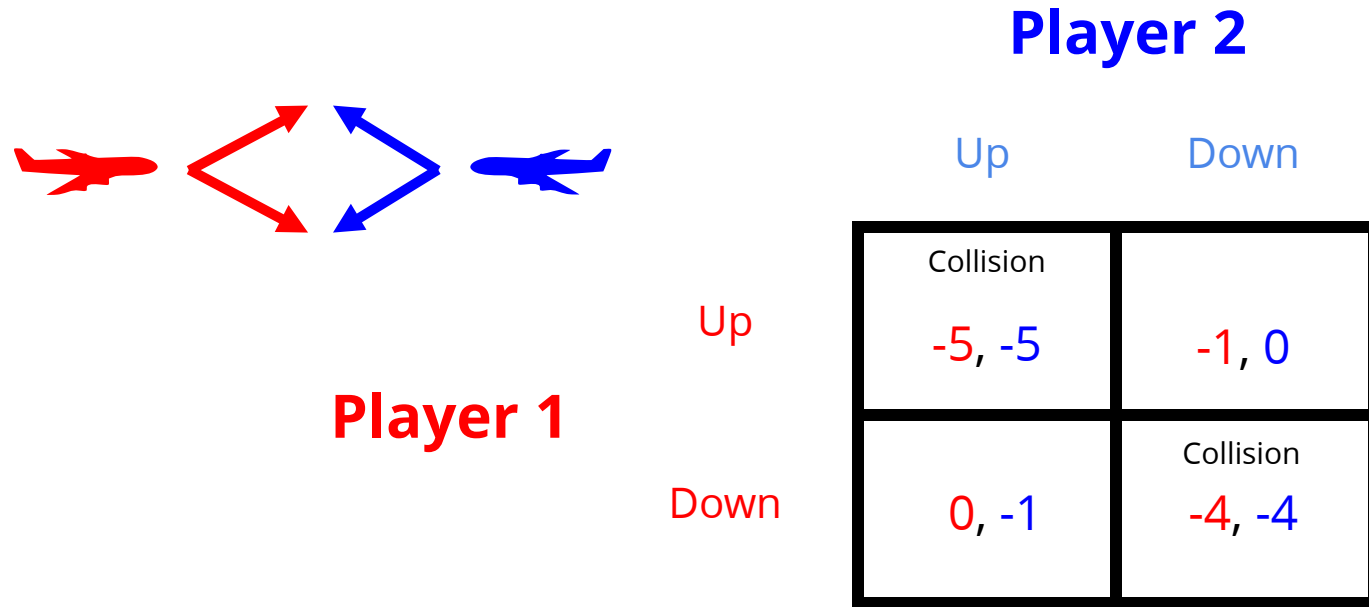
	S	T
S	-1, -1	-4, 0
T	0, -4	-3, -3

- Dominant strategy for both players is to testify
- Dominant strategy equilibrium is a very bad social result (for the criminals)

Do all simple games have a dominant strategy equilibrium?

Collision Avoidance Game

Example: Airborne Collision Avoidance



Pure Nash Equilibrium: All players play a deterministic best response.

Which equilibrium is better?

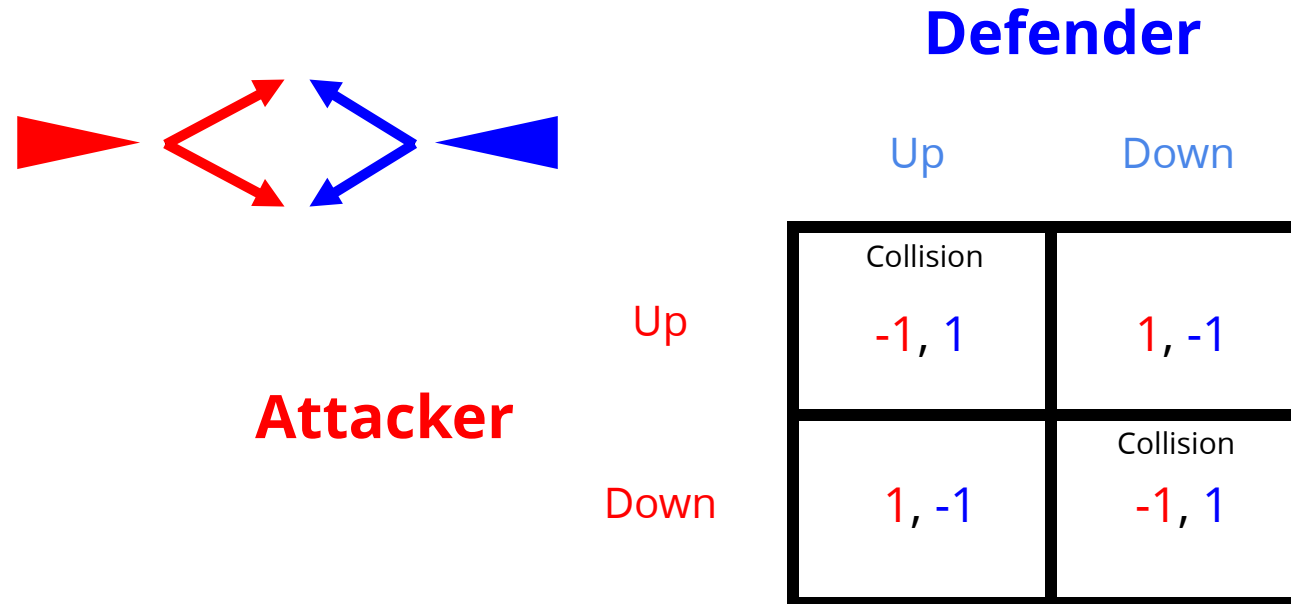
Do all simple games have a pure Nash equilibrium?

Practice: Find Pure Nash Equilibria

		Player 2		
		a	b	c
Player 1	a	4,4	2,5	0,0
	b	5,2	3,3	0,0
	c	0,0	0,0	10,10

Missile Defense Game

Missile Defense (simplified)



No Pure Nash Equilibrium!

Need a broader solution concept: Mixed Nash equilibrium.

Vocabulary and Notation for Mixed Strategies

	Single Player	Joint
• Action	$a^i \in A^i$	$a \in A$
• Policy (strategy)	$\pi^i(a^i)$	$\pi(a) = \prod_i \pi^i(a^i)$
• Reward	$R^i(a)$	$R(a)$
• Utility	$U^i(\pi) = \sum_a R^i(a)\pi(a)$	$U(\pi) = \sum_a R(a)\pi(a)$

Two Player Zero Sum:

$$R^1(a) + R^2(a) = 0 \quad \forall a$$

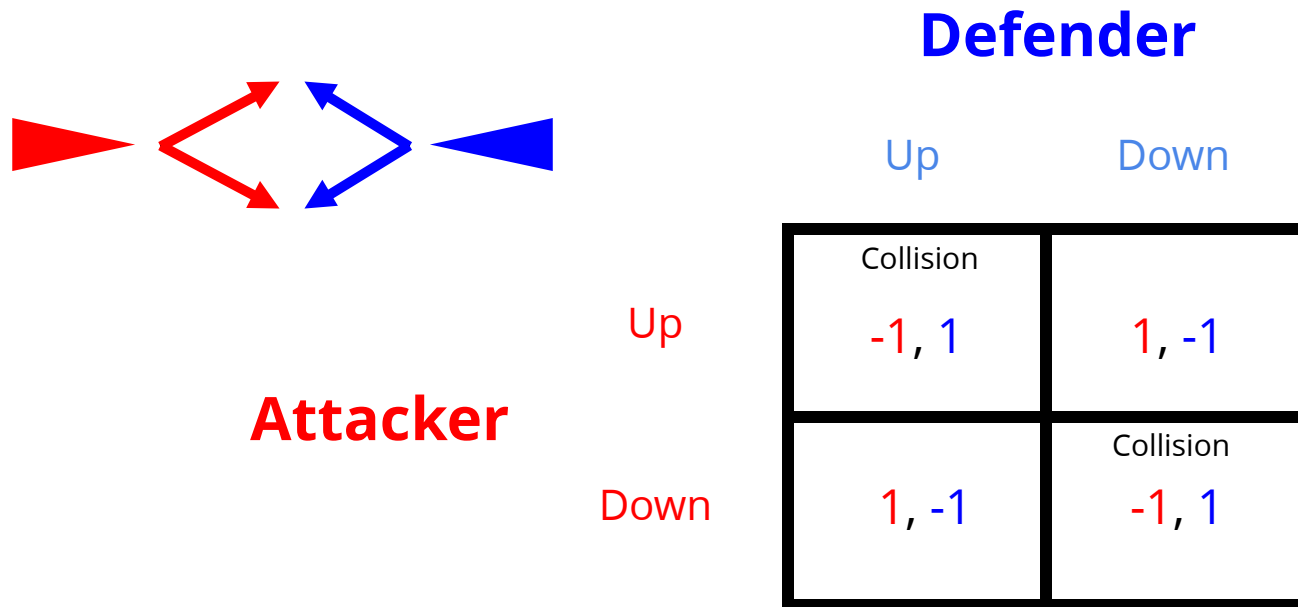
Best Response: Given a joint policy of all other agents, π^{-i} , a best response is a policy π^i that satisfies

$$U^i(\pi^i, \pi^{-i}) \geq U^i(\pi^{i'}, \pi^{-i})$$

for all other $\pi^{i'}$.

Missile Defense Game

Missile Defense (simplified)



- A ***Nash equilibrium*** is a joint policy in which all agents are following a best response

Rock-paper scissors

1. Guess the Nash Equilibrium argument
2. Make a qualitative argument that this is an NE based on best responses

		agent 2		
		rock	paper	scissors
agent 1	rock	0,0	-1,1	1,-1
	paper	1,-1	0,0	-1,1
	scissors	-1,1	1,-1	0,0

Do all simple games have at least one Nash equilibrium?

Yes!! (might be mixed) ¹¹

Every finite game has a Nash Equilibrium

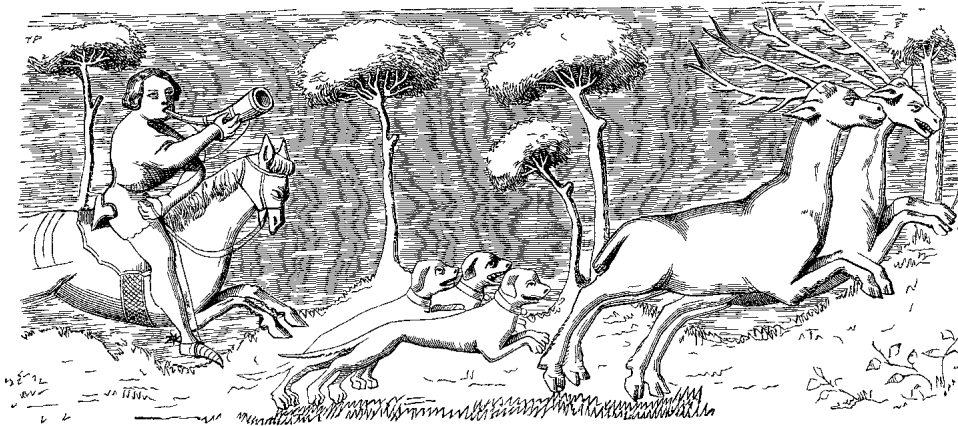
Kakutani's fixed-point theorem

A correspondence $f: X \rightarrow X$ has a fixed point (i.e., $\mathbf{x} \in f(\mathbf{x})$ for some $\mathbf{x} \in X$) if all of the following conditions hold.

- (1) X is a non-empty, closed, bounded, and convex set.
- (2) $f(\mathbf{x})$ is non-empty for any \mathbf{x} .
- (3) $f(\mathbf{x})$ is convex for any \mathbf{x} .
- (4) The set $\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in f(\mathbf{x}) \}$ is closed.

- Let x be a strategy profile, π .
- Let f be BR , that is, the best response operator
- A fixed point of BR is a Nash Equilibrium
- The BR operator and policy space for finite games meet the conditions above
- BR has a fixed point for every finite game, i.e. every finite game has a Nash Equilibrium

Calculating Mixed Nash



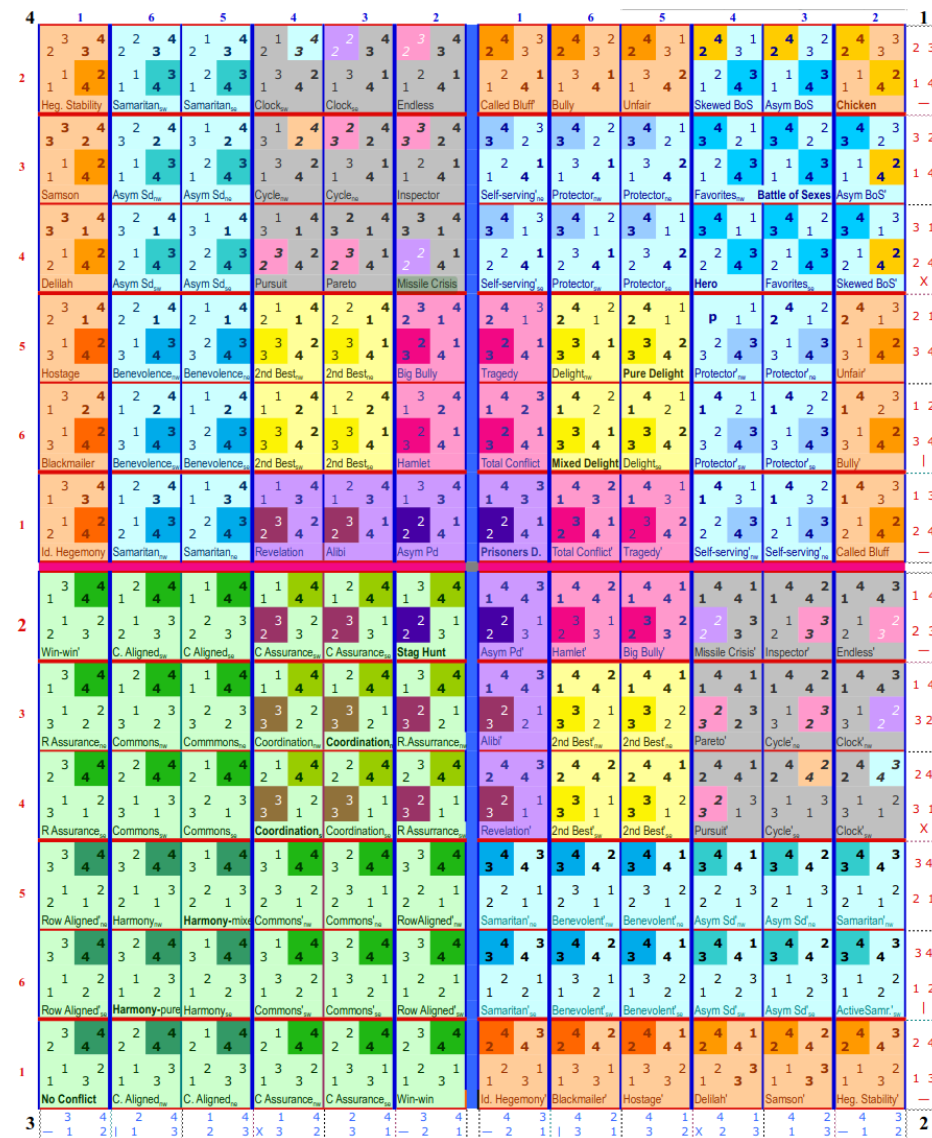
	Stag	Hare
Stag	4, 4	1, 3
Hare	3, 1	2, 2

- In a Mixed Nash Equilibrium, players must be *indifferent* between two or more actions
- (In large games, finding the support of the mixed strategies is the hard part)

General approach to find Nash Equilibria

$$\begin{aligned}
 &\underset{\pi, U}{\text{minimize}} && \sum_i \left(U^i - U^i(\pi) \right) \\
 &\text{subject to} && U^i \geq U^i(a^i, \pi^{-i}) \text{ for all } i, a^i \\
 &&& \sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i \\
 &&& \pi^i(a^i) \geq 0 \text{ for all } i, a^i
 \end{aligned}$$

Topology of bimatrix games:



Algorithms that use best response

Iterated Best Response: randomly cycle between agents who play the best response for the current policy (converges to Nash for certain narrow classes of games)

Fictitious Play:

1. Estimate maximum likelihood policies for opponents:

$$\pi^j(a^j) \propto N(j, a^j)$$

2. Play best response to estimated policy

(converges to Nash for wider class of games, notably zero-sum)

~~Battle of the Sexes~~

Bach or Stravinsky

- Two people want to go to a concert
- P1 prefers Bach, P2 Stravinsky

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

Correlated Equilibrium

- A *correlated joint policy* is a single distribution over the joint actions of all agents.
- A *correlated equilibrium* is a correlated joint policy where no agent i can increase their expected utility by deviating from their current action to another.
- Easier to find than Nash equilibrium (Linear Program)

Recap

- Games provide a mathematical framework for analyzing interaction between rational agents
- Games may not have a single "optimal" solution; instead there are equilibria
- If every player is playing a best response, that joint policy is a Nash Equilibrium
- Every finite game has at least one Nash Equilibrium (pure or mixed)
- Mixed Nash equilibria occur when players are indifferent between two outcomes