Percolation games: percolation structures in non-oriented games on $\mathbb{Z}^2$ and random transitions in oriented games

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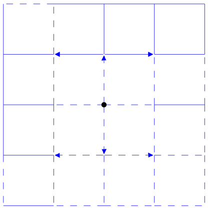
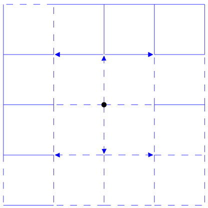
1. Explores two examples of percolation games on $\mathbb{Z}^2$:
    - Characterize lattice structures that provide strategic advantages to the players.
    - Analyze the critical probabilities of their occurrence.
    - Examine their strategic implications.
    - Use computational simulations for further insights.
2. Extends the model:
    - Incorporate a stochastic process into the transitions.
    - Prove similar results as known for percolation games.

Example of a zero-sum stochastic game on $\mathbb{Z}^2$

- The state space is $\mathbb{Z}^2$.
- For $p \in [0,1]$:
  - Random costs on the edges $\{c(e)\}_{e \in \mathbb{E}^2}$, with $c(e) \sim$ Bernoulli(p) i.i.d.
  - We consider $(\Omega = \{0,1\}^{\mathbb{E}^2}, \mathcal{F}, \mathbb{P}_p)$.





Let p be fixed, $\omega \in \Omega$ and a token is placed at $z \in \mathbb{Z}^2$.
The players are informed of ω and the game proceed as follows:

At each stage $m \in \mathbb{N}_+$:

- Player 1 moves the token up or down.
- Then, player 2 moves it left or right.
- Player 1 receives the costs of the corresponding traversed edges, and player 2 receives the opposite amount.

For the n-stage game, the total payoff and value are:

$$
\begin{aligned}
\gamma_n(z,\sigma,\tau) &= \frac{1}{2n}\sum_{m=1}^{2n} c(e_m), \\
v_n(z) &= \max_{\sigma\in\Sigma}\min_{\tau\in T}\gamma_n(z,\sigma,\tau) = \min_{\tau\in T}\max_{\sigma\in\Sigma}\gamma_n(z,\sigma,\tau).
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For the ∞-stage game,

$$\begin{aligned}
\gamma(z,\sigma,\tau) &= \limsup_{n\to\infty}\frac{1}{2n}\sum_{m=1}^{2n} c(e_m), \\
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\end{aligned}$$

Does $v_p(z)$ depends on z? Is v_p random?

Theorem

$v_p(z)$ is independent of z.

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v_p is almost surely deterministic.

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Theorem

v_p *is almost surely deterministic.*

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What should we focus on to answer this question?

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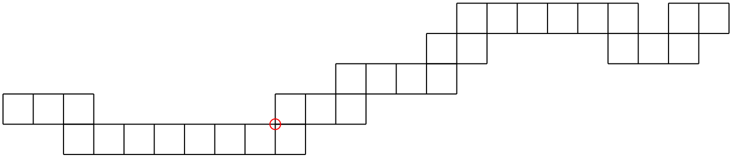
v_p is almost surely deterministic.

A natural question is: What is its value?

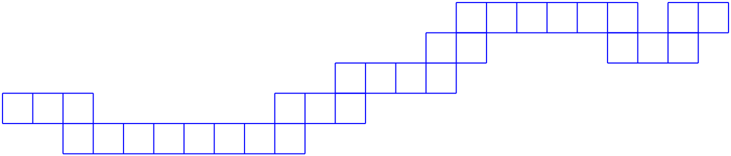
What should we focus on to answer this question?
On lattice structures that gives strategic advantages to the players!

Player 1 can keep player 2 inside this one!



Player 1 can keep player 2 inside this one!

Further imagine, is filled with 1s



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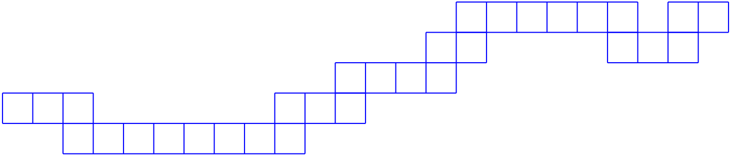

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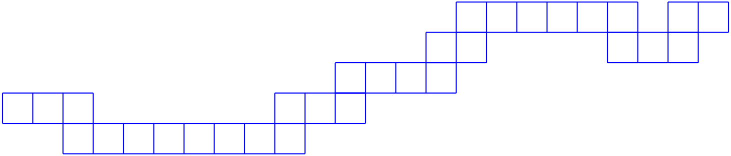


...it guarantees a value of 1!

Therefore, if such a structure filled with 1s exists, since the player 1 can reach it in a finite number of steps, the value of the game is 1.

Defining winning structures for player 1

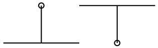
Definition

A infinite subgraph $C = (V, E)$ of the lattice $\mathbb{Z}^2$ is said to be a *winning structure* for player 1 if, for all $z \in V$, there exists an $i \in I$ such that, for all $j \in J$, the vertices $z + (0, i)$ and $z + (j, i)$ are in V, and the edges $\{z, z + (0, i)\}$, and $\{z + (0, i), z + (j, i)\}$ are in E.

As the game is symmetric, until the end of the game analysis, we will focus solely on Player 1.

Proposition (Geometric characterization of ws for player 1)

A structure C is a winning one for player 1 if and only if, from any of it vertices, at least one of the following local configurations is present:



41-horizontal: a winning structure for player 1 filled with 1s, H_{41} the event of its existence, and it critical probability:

$$q_1 = \inf\{p \in [0,1] \colon \mathbb{P}_p(H_{41}) = 1\}$$

Theorem

For all $p > q_1$, $v_p = 1$.

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Note that,

$$q_1 < 1 \text{ and } q_0 < q_1 = 1 - q_0 \implies q_0 < 1 \implies q_1 > 0.$$

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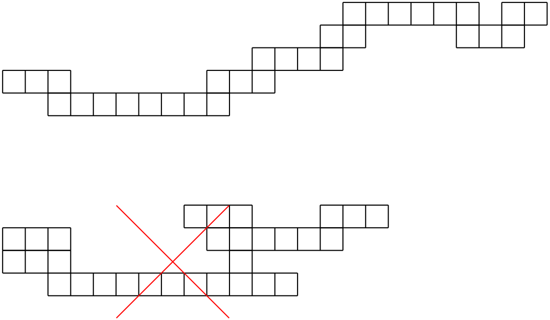
Idea for proving $q_1 < 1$: We focus on a subset of 41-horizontal structures, for which it is easier to establish the bound.

41^*-horizontal structures



Definition

41^*-horizontal structures are 41-horizontal structures that when viewed from left to right, each square:

- only have neighbors to its right, above or bellow,
- if it is above or below, the next one must be to the right.

$q_1 < 1$, is equivalent to $q_1^* < 1$

Follows from $H_{41}^* \subset H_{41} \implies q_1 \leq q_1^*$

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General idea for proving $q_1^* < 1$: to bound $\mathbb{P}_p(H_{41}^{*c})$ using a Peierls-type argument and that a path of squares translates into a path of 1-dependent sites. The latter is done by associating each square to it center and making the center open if and only the square has all it edges with cost equal to 1.

Theorem
For all $p > q_1$, *with* $0 < q_1 < 1$, $v_p = 1$.

Conjecture
$v_p = 1$ *if and only if* $p \geq q_1$.

- Supercritical: We have it from the previous theorem.
- Critical: It is necessary to study the critical behavior of percolation through these structures...
- Subcritical: Consider a suboptimal strategy for player 2 that guarantees a path with a positive density of 0s.

Key computational findings

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Based on what we have discussed so far, we know that:

There exist $0 < q_0 < q_1 < 1$ such that

$$v_p = \begin{cases} 0 & \text{if } p < q_0, \\ 1 & \text{if } p > q_1. \end{cases}$$

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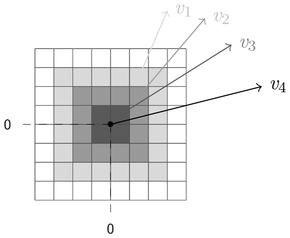
Simulations revealed that this game has more than two phase transitions, where the distribution of 0 and 1 costs across the winning structures emerged as a significant factor!

Simulating the game

Goal: Estimate $\mathbb{E}(v_{n,p})$.

Approach:

- We fixed a $\mathcal{L} := [-N..N] \times [-N..N]$ and a sequence of 200 equally spaced p values within the interval $(0, 1)$.
- For each p, we ran 30 simulations.
- For each realization, we computed $v_N(0)$ recursively.



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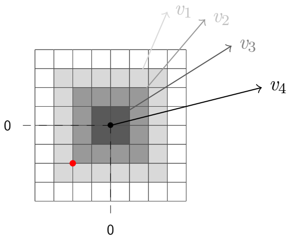

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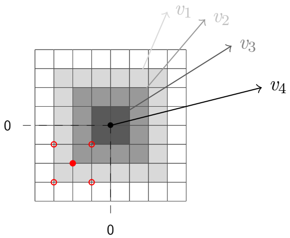
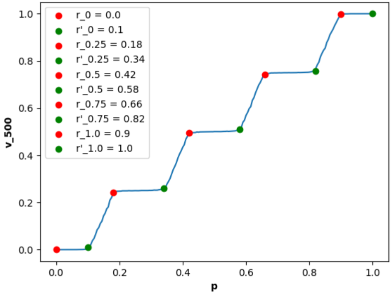


Findings



Red and green points are estimates of

$$
\begin{aligned}
r_x &= \inf\{p \in [0,1] \colon \mathbb{P}_p(v_p = x) = 1\}, \\
r'_x &= \inf\{p \in [0,1] \colon \mathbb{P}_p(v_p > x) = 1\},
\end{aligned}
$$

for $x \in \{0, 0.25, 0.5, 0.75, 1\}$.

We further conjecture that,

$$\begin{aligned}
r_1 &= q_1 \\
r_{0.25} &= \inf\{p \in [0,1] \colon \mathbb{P}_p(H_{11}) = 1\} \\
r_{0.5} &= \inf\{p \in [0,1] \colon \mathbb{P}_p(H_{21}) = 1\} \\
r_{0.75} &= \inf\{p \in [0,1] \colon \mathbb{P}_p(H_{31}) = 1\}
\end{aligned}$$

H_{k1}: "there exists a winning structure for player 1 with all the squares having k 1s".

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H_{k1}: "there exists a winning structure for player 1 with all the squares having k 1s".

Goal: Estimate the r.h.s.

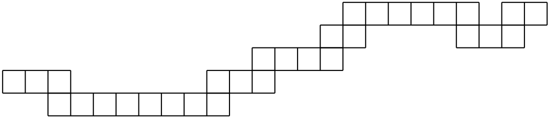
Estimating $\inf\{p \in [0,1] \colon \mathbb{P}_p(H_{k1}) = 1\}$

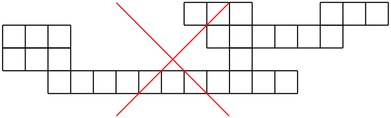
We focus on:

- the events H_{k1}^* for computational simplicity and relevance.
- the percolation model on $\mathbb{Z}^2$ to identify k-squares clusters* that spans $\mathcal{L}$ from the left side to the right side.

We employ:

- a modified version of the Newman-Ziff algorithm to handle paths formed by k-squares:
  - Bonds are opened sequentially on an initially closed lattice.
  - Each time, we check for the creation and connection of k-square clusters using a union-find method and dfs.
  - If the "condition *" is met, we record the onset of percolation.
- 30 simulations to average the mean onset of percolation.





Estimating $\inf\{p \in [0,1] : \mathbb{P}_p(H_{k1}) = 1\}$

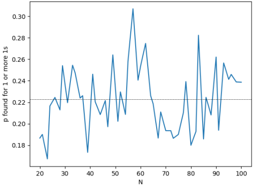


Figure: $k = 1$.

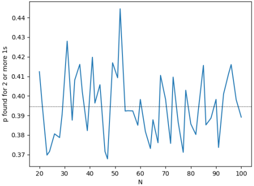


Figure: $k = 2$.

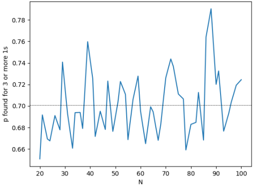


Figure: $k = 3$.

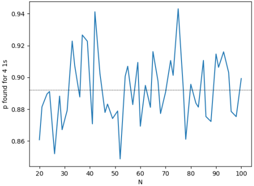


Figure: $k = 4$.

Resuming our results for this game:

For H_{k1}: "there exists a ws for player 1 conformed by k-squares".

Theorems:

– $v_p(z)$ is independent of z and is almost surely deterministic.
– There exist $0 < q_0 < q_1 < 1$ such that

$$v_p = \begin{cases} 0 & \text{if } p < q_0, \\ 1 & \text{if } p > q_1, \end{cases}$$

where $q_1 = \inf\{p : \mathbb{P}_p(H_{41}) = 1\} = 1 - q_0$.

Conjectures:

– $v_p = 1$ if and only if $p \geq q_1$ ($v_p = 0$ if and only if $p \leq q_0$).
– For $k \in 1, 2, 3, 4$:

$$\inf\{p : \mathbb{P}_p(v_p = k/4) = 1\} = \inf\{p : \mathbb{P}_p(H_{k1}) = 1\}.$$

Percolation Games: zero-sum stochastic games on $\mathbb{Z}^d$

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Are described by a tuple $(I, J, \mathcal{E}, g, q)$, where

- I and J are finite sets representing respectively player 1s and player 2s action sets.
- $\mathcal{E} = (\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
- $g := \{\omega \mapsto g_\omega(z, i, j)\}_{(z,i,j) \in \mathbb{Z}^d \times I \times J}$ is the payoff function.
- $q \colon I \times J \to \mathbb{Z}^d$ is a transition function.

Percolation Games: $(I, J, \mathcal{E}, g, q)$

Let $\omega \in \Omega$ and a token placed at $z_1 \in \mathbb{Z}^d$.

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- player 1 chooses an action i_m
- then, knowing i_m, player 2 chooses an action j_m
- player 1 receives $g_m^\omega := g_\omega(z_m, i_m, j_m)$, and player 2 $-g_m^\omega$
- the token moves to:

$$z_{m+1} = z_m + q(i_m, j_m).$$

- (i_m, j_m, z_{m+1}) is publicly announced.

Limit value for percolation games

For the n-stage game:

$$\begin{aligned}
\gamma_n^\omega(z,\sigma,\tau) &= \frac{1}{n}\sum_{m=1}^{n} g_m^\omega \\
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The central question: Does $v_n(z)$ converge as n approaches infinity?

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The central question: Does $v_n(z)$ converge as n approaches infinity?

Known result

For i.i.d. and oriented games the value's sequence converge a.s. to a constant!

$(I, J, \mathcal{E}, g, q)$

Let $\omega \in \Omega$ and a token placed at $z_1 \in \mathbb{Z}^d$.

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Definition (oriented)

If $\exists u \in \mathbb{Z}^d$: $\mathcal{F}orall (i,j) \in I \times J, q(i,j) \cdot u > 0.$

$(I, J, \mathcal{E}, g, q)$

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Definition (i.i.d.)

If the random variables $\{\omega \mapsto g_\omega(z, i, j)\}_{(z,i,j) \in \mathbb{Z}^d \times I \times J}$ are i.i.d.

Generalizing the model

From Γ to $\Gamma_\xi := (I, J, \mathcal{E}, g, q + \xi)$ (or Γ_μ where $\xi_m \sim \mu$ is known)

Let $\omega \in \Omega$ and a token placed at $z_1 \in \mathbb{Z}^d$.

Players are informed of ω and at each stage $m \in \mathbb{N}_+$:

- player 1 chooses an action i_m
- then, knowing i_m, player 2 chooses an action j_m
- player 1 receives $g_m^\omega := g_\omega(z_m, i_m, j_m)$, and player 2 $-g_m^\omega$
- the token moves to:

$$z_{m+1} = z_m + q(i_m, j_m) + \xi_m.$$

- (i_m, j_m, z_{m+1}) is publicly announced.

where $\xi = \{\xi_m\}_{m \geq 1}$ are i.i.d. r.v's defined on $\mathcal{E}$.

Main result for Γ_μ

- Γ_μ is i.i.d. as long as Γ is i.i.d.
- If μ has zero expectation, Γ_μ is oriented in expectation as long as Γ is oriented.

Theorem

Consider a percolation game Γ_μ i.i.d. with Γ oriented and μ a probability distribution with zero expectation and bounded support. For all $z \in \mathbb{Z}^d$, $v_n(z) \xrightarrow[n\to\infty]{} v_\infty$, with $v_\infty \in \mathbb{R}$.
Moreover, $\exists A, B \in \mathbb{R}_+ :$ $\mathcal{F}oralln \geq 1$, $z \in \mathbb{Z}^d$ and $\lambda \geq 0$,

$$\mathbb{P}\left(|v_n(z) - v_\infty| \geq \lambda + A\ln(n+1)^{1/2} n^{-1/2S}\right) \leq \exp(-B\lambda^2 n).$$

In particular, $\mathbb{E}(v_n)$ converges to v_∞ at a rate $O(\ln(n)n^{-1/2})$.

Proof steps:

1. $v_n(z)$ concentrates around its expectation $\mathbb{E}(v_n)$.
2. $\mathbb{E}(v_n) \xrightarrow[n\to\infty]{} v_\infty$.
3. $\mathbb{E}(v_n)$ converge fast to v_∞.
4. Therefore, v_n concentrates on v_∞.

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4. Therefore, v_n concentrates on v_∞.

It must be proven that $\mathbb{P}(|v_n(z) - \mathbb{E}(v_n)| \geq \lambda)$ decreases with n.
To achieve this, we rely on AzumaHoeffding's inequality.

Preliminary definitions and observations

- For $n \geq 1$, define
$$\mathcal{C} := [-n \cdot \|q+\xi\|_\infty .. n \cdot \|q+\xi\|_\infty]^d$$
- Define
$$H := \{z \in \mathbb{Z}^d \mid z \cdot u = 0\}$$
- There exists a constant C, such that
$$\mathcal{C} \subset \bigcup_{r=1}^{Cn} H_r, \text{ where } H_r = \{z \in \mathbb{Z}^d \mid z \cdot u = h_r\}$$

Proof: Concentration on $\mathbb{E}(v_n)$

For $r \in [0..Cn]$:

- Define the σ-algebra:

$$\mathcal{F}_r := \sigma((\omega \to g_\omega(z,i,j)) : (z,i,j) \in \cup_{r'=1}^{r} H_{r'} \times I \times J).$$

- Define the martingale:

$$W_r := \mathbb{E}(v_n(0)|\mathcal{F}_r).$$

- Prove the inequality:

$$|\mathbb{E}(v_n(0)|\mathcal{F}_{r+1}) - \mathbb{E}(v_n(0)|\mathcal{F}_r)| \leq \frac{D\|g\|_\infty}{n} \quad \mathbb{P}\text{-a.s.}$$

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To prove: $|\mathbb{E}(v_n(0)|\mathcal{F}_{r+1}) - \mathbb{E}(v_n(0)|\mathcal{F}_r)| \leq \frac{D\|g\|_\infty}{n}$ $\mathbb{P}$-a.s.

- For $r \in [0..Cn]$, define the auxiliary game $\Gamma'_\mu = (I, J, \mathcal{E}, g', q)$:

$$g'_\omega(z, i, j) = \begin{cases} 0 & z \in H_{r+1}, \\ g_\omega(z, i, j) & \text{otherwise.} \end{cases}$$

- Note that, for all $(\sigma, \tau) \in \Sigma \times T$,

$$|\gamma'^{\omega}_n(0, \sigma, \tau) - \gamma^\omega_n(0, \sigma, \tau)| \leq \frac{(1 + M_n)\|g\|_\infty}{n},$$

  where $M_n := \#\{\text{of returns to } H_{r+1} \text{ by stage } n\}$.

- Prove $\mathbb{E}_{z_1,\sigma,\tau}(M_n) = O(1)$.

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- Prove $\mathbb{E}_{z_1,\sigma,\tau}(M_n) = O(1)$.

Let $z_1 \in \mathcal{C}$, $(\sigma, \tau) \in \Sigma \times T$, and $n \in \mathbb{N}_+$, we have

$$\mathbb{E}_{z_1,\sigma,\tau}(M_n) \leq \frac{\exp(-L)}{1 - \exp(-L)}.$$

Proof.

1. The total displacement magnitude of the token in the direction u from H_{r+1} is $d_k = \sum_{m=s}^{k} \| \operatorname{proj}_u[q(i_m, j_m)] \|$.
2. $\mathbb{P}_{z_1,\sigma,\tau}(\text{returns after distance } d_k) \leq \mathbb{P}\left(\sum_{m=s}^{k} \xi_m > d_k\right)$.
3. By Hoeffding's inequality: $\mathbb{P}\left(\sum_{m=s}^{k} \xi_m > d_k\right) \leq \exp\left(-Lk\right)$.
4. And $\mathbb{E}_{z_1,\sigma,\tau}(M_n) = \sum_{k=1}^{n} \mathbb{P}\left(\sum_{m=s}^{k} \xi_m > d_k\right)$.
5. Therefore, $\mathbb{E}_{z_1,\sigma,\tau}(M_n)$ is bounded by a geometric series with ratio $\exp(-L)$.

$\square$

Proof: concentration on $\mathbb{E}(v_n)$

To prove: $|\mathbb{E}(v_n(0)|\mathcal{F}_{r+1}) - \mathbb{E}(v_n(0)|\mathcal{F}_r)| \leq \frac{D\|g\|_\infty}{n}$ $\mathbb{P}$-a.s.

- Define the auxiliary game $\Gamma'_\mu = (I, J, \mathcal{E}, g', q)$, with

$$g'_\omega(z,i,j) = \begin{cases} 0 & z \in H_{r+1}, \\ g_\omega(z,i,j) & \text{otherwise.} \end{cases}$$

- Note the inequality:

$$|\gamma_n'^{\omega}(0,\sigma,\tau) - \gamma_n^{\omega}(0,\sigma,\tau)| \leq \frac{(1+M_n)\|g\|_\infty}{n}.$$

  where $M_n := \#\{\text{of returns to } H_{r+1} \text{ by stage } n\}$.

- Prove $\mathbb{E}(M_n) = O(1)$. ✓

Proof: concentration on $\mathbb{E}(v_n)$

For $r \in [0..Cn]$:

- Define the σ-algebra:

$$\mathcal{F}_r := \sigma((\omega \to g_\omega(z,i,j)) : (z,i,j) \in \cup_{r'=1}^{r} H_{r'} \times I \times J).$$

- Define the martingale:

$$W_r := \mathbb{E}(v_n(0)|\mathcal{F}_r).$$

- Prove the inequality:

$$|\mathbb{E}(v_n(0)|\mathcal{F}_{r+1}) - \mathbb{E}(v_n(0)|\mathcal{F}_r)| \leq \frac{D\|g\|_\infty}{n} \quad \mathbb{P}\text{-a.s.} \checkmark$$

Proof: concentration on $\mathbb{E}(v_n)$ ✓

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Proof steps:

1. $v_n(z)$ concentrates around its expectation $\mathbb{E}(v_n)$. ✓
2. $\mathbb{E}(v_n) \xrightarrow[n\to\infty]{} v_\infty$.
3. $\mathbb{E}(v_n)$ converge fast to v_∞.
4. Therefore, v_n concentrates on v_∞.

Merci beaucoup !