Control theory notes

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Chapter 1

MIMO CONTROL

1.1 Analysis of mimo control loops

The state space model can be written as:

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t) \tag{1.1}$$
$$\dot{y}(t) = \boldsymbol{C}x(t) + \boldsymbol{D}u(t) \tag{1.2}$$

where $x(t_0) = x_0$, $x \in \mathbb{R}^n$ is the state vector, x_0 is the state vector at $t = t_0$ and where $\boldsymbol{A} \in \mathbb{R}^{n\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times m}, \boldsymbol{C} \in \mathbb{R}^{m\times n}$, and $\boldsymbol{D} \in \mathbb{R}^{m\times m}$.

Laplace transforming 1.1 gives:

$$sX(s) - x(0) = \boldsymbol{A}X(s) + \boldsymbol{B}U(s) \tag{1.3}$$
$$Y(s) = \boldsymbol{C}X(s) + \boldsymbol{D}U(s) \tag{1.4}$$

Taking $x(0) = 0$, we get

$$Y(s) = (\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D})U(s) \tag{1.5}$$

The *matrix transfer function* $G(s)$ is then defined by:

$$G(s) \triangleq \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \tag{1.6}$$

Definition 1 () *A transfer fuction matrix $\boldsymbol{G}(s)$ is said to be a* ***proper matrix*** *if every one of its elements is a proper transfer function.*

Observation: $\boldsymbol{G}_{ik}(s)$ is the Laplace transform of the i-th component of $y(t)$, when the k-th component of $u(t)$ is an impulse.

Definition 2 () *The* ***impulse response matrix*** *of the system, $\boldsymbol{g}(t)$, is the inverse Laplace transform of the transfer function matrix $\boldsymbol{G}(s)$*

1.2 Design via optimal control techniques

Consider a general nonlinear system with input $u(t) \in \mathbb{R}^m$:

$$\frac{dx(t)}{dt} = f(x(t), u(t), t) \tag{1.7}$$

where $x(t) \in \mathbb{R}^n$, with the following optimization problem:

Problem 1 (General optimal control problem) *Find an optimal input $u^o(t)$, for $t \in [t_o, t_f]$, such that:*

$$u^o(t) = \underset{u(t)}{\operatorname{argmin}} \left\{ \int_{t_0}^{t_f} \mathcal{V}(x,u,t)dt + g(x(t_f)) \right\} \tag{1.8}$$

where $\mathcal{V}(x,u,t)$ and $g(x(t_f))$ are nonnegative functions.

One way to solve this is to use dynamic programming, which is based on the following result:

Theorem 1 (Optimality principle (Bellman)) *If $u(t) = u^o(t), t \in [t_o, t_f]$ is the optimal solution to the above problem, then $u^o(t)$ is also the optimal solution over the (sub)inverval $[t_o + \Delta t, t_f]$, where $t_o < t_0 + \Delta t < t_f$.*

Proof By contradiction. Denote by $x^o(t)$ the state evolution resulting from applying $u^o(t)$ over the whole interval $t \in [t_o, t_f]$. The optimal cost over the interval is

$$\int_{t_o}^{t_o+\Delta t} \mathcal{V}(x^o, u^o, t)dt + \int_{t_o+\Delta t}^{t_f} \mathcal{V}(x^o, u^o, t)dt + g(x^o(t_f)) \tag{1.9}$$

Assume there exists an input $\tilde{u}(t)$ such that:

$$\tilde{u}(t) = \underset{u(t)}{\operatorname{argmin}} \left\{ \int_{t_o+\Delta t}^{t_f} \mathcal{V}(x,u,t)dt + g(x(t_f)) \right\} \tag{1.10}$$

with corresponding state trajectory $\tilde{x}(t)$ such that $\tilde{x}(t_0 + \Delta t) = x^o(t_o + \Delta t)$. Assume

$$\int_{t_o+\Delta t}^{t_f} \mathcal{V}(\tilde{x}, \tilde{u}, t)dt + g(\tilde{x}(t_f)) < \int_{t_o+\Delta t}^{t_f} \mathcal{V}(x^o, u^o, t)dt + g(x^o(t_f)) \tag{1.11}$$

Consider now the policy of applying $u(t) = u^o(t)$ in the interval $[t_o, t_o + \Delta t]$ and then $u(t) = \tilde{u}(t)$ over the interval $[t_o + \Delta t, t_f]$. The resulting cost over the complete interval would be:

$$\int_{t_o}^{t_o+\Delta t} \mathcal{V}(x^o, u^o, t)dt + \int_{t_o+\Delta t}^{t_f} \mathcal{V}(\tilde{x}, \tilde{u}, t)dt + g(\tilde{x}(t_f)) \tag{1.12}$$

This would thus result in a smaller cost than to use $u(t) = u^o(t)$ over the same interval, contradicting the assumption of optimality of $u^o(t)$ over that interval. ■

We now use this to derive necessary conditions for the optimal u by using the optimality principle with an infinitesimal time interval $[t, t + \Delta t]$ and mathing it. We denote the optimal (minimal) cost in $[t, t + \Delta t]$ with initial state $x(t)$ by:

$$J^o(x(t), t) = \min_{\substack{u(\tau) \\ \tau \in [t, t_f]}} \left\{ \int_t^{t_f} \mathcal{V}(x, u, t) d\tau + g(x(t_f)) \right\} \tag{1.13}$$

Now we apply the Bellman optimality theorem:

$$J^o(x(t), t) = \min_{\substack{u(\tau) \\ \tau \in [t, t+\Delta t]}} \left\{ \int_t^{t+\Delta t} \mathcal{V}(x, u, t) d\tau + J^o(x(t + \Delta t), t + \Delta t) \right\} \tag{1.14}$$

Now consider Δt to be very small. We expand $J^o(x(t+\Delta t), t+\Delta t)$ in a Taylor series. We get:

$$J^o(x(t), t) = \min_{u(t)} \left\{ \mathcal{V}(x(t), u(t), t)\Delta t + J^o(x(t), t) + \frac{\partial J^o(x(t), t)}{\partial t} \Delta t + \left[\frac{\partial J^o(x(t), t)}{\partial x} \right]^T (x(t + \Delta t) - x(t)) + \mathcal{O}(x, t) \right\} \tag{1.15}$$

where $\mathcal{O}(x, t)$ are the high order terms. Now we let $\Delta t \to dt$, getting

$$-\frac{\partial J^o(x(t), t)}{\partial t} = \min_{u(t)} \{ \mathcal{W}(x(t), u(t), t) \} \tag{1.16}$$

where

$$\mathcal{W}(x(t), u(t), t) = \mathcal{V}(x(t), u(t), t) + \left[\frac{\partial J^o(x(t), t)}{\partial x} \right]^T f(x(t), u(t), t) \tag{1.17}$$

The optimal $u(t)$ can be expressed as:

$$u^o(t) = \mathcal{U} \left(\frac{\partial J^o(x(t), t)}{\partial x}, x(t), t \right) \tag{1.18}$$

which leads to the optimal cost:

$$-\frac{\partial J^o(x(t), t)}{\partial t} = \mathcal{V}(x(t), \mathcal{U}, t)) + \left[\frac{\partial J^o(x(t), t)}{\partial x} \right]^T f(x(t), \mathcal{U}, t) \tag{1.19}$$

and the solution must satisfy the boundary condition:

$$J^o(x(t_f), t_f) = g(x(t_f)) \tag{1.20}$$

1.2.1 Linear Quadratic Regulator (LQR)

We now apply the above theory to the following problem:

Problem 2 (The LQR problem) *Consider a linear time invariat system having a state space model:*

$$\frac{dx(t)}{dt} = \boldsymbol{A}x(t) + \boldsymbol{B}u(t) \tag{1.21}$$

$$y(t) = \boldsymbol{C}x(t) + \boldsymbol{D}u(t) \tag{1.22}$$

where $x(t_0) = x_0$, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $x_o \in \mathbb{R}^n$ is the state vector at $t = t_o$ and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are matrices of appropriate dimensions.

Assume that the aim is to drive the initial state x_o to the smallest possible value as soon as possible in $[t_o, t_f]$, without spending too much control effort. Then, the optimal regulator problem is the problem of finding an optimal control $u(t)$ over $[t_o, t_f]$ such that the following cost function is minimized:

$$J_u(x(t_o), t_o) = \int_{t_o}^{t_f} \left[x(t)^T \boldsymbol{\Psi} x(t) + u(t)^T \boldsymbol{\Phi} u(t)\right] dt + x(t_f)^T \boldsymbol{\Psi}_f x(t) \tag{1.23}$$

where $\boldsymbol{\Psi} \in \mathbb{R}^{n\times n}, \boldsymbol{\Psi}_f \in \mathbb{R}^{n\times n}$ are symmetric nonnegative definite matrices and $\boldsymbol{\Phi} \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix.

We begin by writing the LQR problem in terms of the general optimal problem:

$$f(x(t), u(t), t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t) \tag{1.24}$$

$$\mathcal{V}(x, u, t) = x(t)^T \boldsymbol{\Psi} x(t) + u(t)\boldsymbol{\Phi} u(t) \tag{1.25}$$

$$g(x(t_f)) = x(t_f)\boldsymbol{\Psi}_f x(t_f) \tag{1.26}$$

Then:

$$\mathcal{W}(x(t), u(t), t) \triangleq \mathcal{V}(x(t), u(t), t) + \left[\frac{\partial J^o(x(t), t)}{\partial x}\right]^T f(x(t), u(t), t) \tag{1.27}$$

$$= x(t)^T \boldsymbol{\Psi} x(t) + u(t)^T \boldsymbol{\Phi} u(t) + \left[\frac{\partial J^o(x(t), t)}{\partial x}\right]^T (\boldsymbol{A}x(t) + \boldsymbol{B}u(t)) \tag{1.28}$$

To obtain the optimal $u(t)$, we need to minimize $\mathcal{W}(x(t), u(t), t)$. We begin by calculating the gradient of it w.r.t. u and setting it to zero.

$$\frac{\partial W}{\partial u} = 2\boldsymbol{\Phi} u(t) = \boldsymbol{B}^T \frac{\partial J^o(x(t), t)}{\partial x} \tag{1.29}$$

From this we get:

$$u^o(t) = -\frac{1}{2}\boldsymbol{\Phi}^{-1}\boldsymbol{B}^T \frac{\partial J^o(x(t), t)}{\partial x} \tag{1.30}$$

We observe that the Hessian of $\mathcal{W}$ is w.r.t. u is equal to $\mathbf{\Phi}$ which is by assumption positive definite. This confirms that 1.30 gives a minimum for $\mathcal{W}$. Note that:

$$J^o(x^o(t_f), t_f) = [x^o(t_f)]^T \, \mathbf{\Psi}_f x^o(t_f) \tag{1.31}$$

This is a quadratic of the state at t_f. By induction, this is true at every t_f. We proceed by assuming that:

$$J^o(x(t), t) = x^T(t)\boldsymbol{P}(t)x(t) \text{ with } \boldsymbol{P}(t) = [\boldsymbol{P}(t)]^T \tag{1.32}$$

With this we have:

$$\frac{\partial J^o(x(t), t)}{\partial x} = 2\boldsymbol{P}(t)x(t) \tag{1.33}$$

$$\frac{\partial J^o(x(t), t)}{\partial t} = x^T(t)\frac{d\boldsymbol{P}(t)}{dt}x(t) \tag{1.34}$$

Combining previous results, we get that the optimal control can be expressed as

$$u^o(t) = -\boldsymbol{K}_u(t)x(t) \tag{1.35}$$

where $\boldsymbol{K}_u(t)$ is a time varying gain given by

$$\boldsymbol{K}_u(t) = \mathbf{\Phi}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) \tag{1.36}$$

We also have

$$\mathcal{W}(x(t), u^o(t), t) = x^T(t)\left(\mathbf{\Psi} - \boldsymbol{P}(t)\boldsymbol{B}\mathbf{\Phi}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + 2\boldsymbol{P}(t)\boldsymbol{A}\right)x(t) \tag{1.37}$$

To get $\boldsymbol{K}_u(t)$, we need $\boldsymbol{P}(t)$ which by combining results leads to

$$-x^T\frac{d\boldsymbol{P}(t)}{dt}x(t) = \left(\mathbf{\Psi} - \boldsymbol{P}(t)\boldsymbol{B}\mathbf{\Phi}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + 2\boldsymbol{P}(t)\boldsymbol{A}\right)x(t) \tag{1.38}$$

Also,

$$2x^T(t)\boldsymbol{P}(t)\boldsymbol{A}x(t) = x^T(t)\left(\boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}(t)\right)x(t) \tag{1.39}$$

For this to hold for all $x(t)$, we require:

$$-\frac{d\boldsymbol{P}(t)}{dt} = \mathbf{\Psi} - \boldsymbol{P}(t)\boldsymbol{B}\mathbf{\Phi}^{-1}\boldsymbol{B}^T\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}(t) \tag{1.40}$$

This is the Continuous Time Dynamic Riccati Equation (CTDRE). It needs to be solved backwards in time to satisfy the boundary condition

$$\boldsymbol{P}(t_f) = \mathbf{\Psi}_f \tag{1.41}$$

The above theory holds well for time varying systems, but we can say more in the time-invariant case.

1.2.2 Properties of Linear Quadratic Optimal Regulator

Here we assume $\boldsymbol{A}, \boldsymbol{B}, \mathbf{\Psi}, \mathbf{\Phi}$ are all time-invariant.