A History of the Development of the Atiyah-Singer Index Theorem

December 21, 2022

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1 Introduction

The Atiyah-Singer index theorem is one of the crowning achievments of twentieth century geometry and topology. It related topological invariants, in particular K-theory, to analytical data, namely the Fredholm index of an elliptic operator. It united several previously known, but separate, theorems into a cohesive whole.

It also lead to development of several different areas through the differing approaches to the topic, ranging from K-theoretical to heat kernel-based to a cobordism-type approach.

The theorem's different approaches led to different generalizations, such as analytic K-homology or index theory in non-commutative geometry. In this essay I will discuss the theorems that it amalgamated, background material, as well as the modern and generalized forms of the theorem and I will especially focus on the heat kernel approach.

2 Early Forms

On of the earliest forms of the set of theorems leading up to Atiyah-Singer was the Riemann-Roch theorem. We start with a compact Riemann surface, a complex 1-dimensional manifold, X and define a divisor D as a map $D : X \to \mathbb{Z}$ that is non-zero at finitely many points. The degree of a divisor is

$$deg(D) = \sum_{x \in X} D(x)$$

which is finite and an integer.

We may associate a line bundle L_D to a divisor using sheaf-theoretical methods, by defining the sheaf $\mathcal{O}_D$ of meromorphic functions that have order at least $-D(x)$ at x for all $x \in X$. Then if g is the genus of the line bundle the Riemann-Roch theorem[2] states that

$$h^0(L_D) - h^0(L_D^{-1} \otimes K) = deg(D) - g + 1$$

where L_D^{-1} is the line bundle such that $L_D^{-1} \otimes L_D$ is trivial, h^0 denotes the space of holomorphic sections, and $K = \bigwedge^2 TX = \Omega^2(X)$ is the determinant line bundle.

Sadly this formula is specialized to Riemann surfaces so it requires some modification to allow higher dimensions.

First it follows by Serre duality[2] it holds that $h^0(L_D^{-1} \otimes K) = h^1(L_D)$ which is the first Dolbeault cohomology group of L. Then we note that $deg(D) = \int_X c_1(L_D)$ and $c_1(TX) = e(TX)$, where the latter is th Euler class, so $\int_X e(TX) = 2 - 2g$. We recall that the total Chern class of a line bundle L_D is represented formally as $1 + c_1(L_D)$ and the Todd class is $1 + \frac{c_1(TX)}{2}$[4], so we have that

$$\sum_{i=0}^{1} (-1)^i dim(H^q(E)) = \int_X c_1(L_D) + \frac{c_1(TX)}{2} = \int_X ch(E) \wedge td(X)$$

This was later generalized to the Hirzebruch–Riemann–Roch theorem[3] on a compact complex manifold X of real dimension $2n$, which states that the holomorphic Euler characteristic $\chi(E)$ of a complex vector bundle E satisfies

$$\chi(E) = \int_X ch(E) \wedge td(X) = \sum_{i=1}^{n} (-1)^i dim(H^q(E))$$

where $H^q(E) = H^{0,q}(E)$ are the Dolbeault cohomology groups of E. We note that for later use the laplacian of these groups is $\overline{\partial\partial^*} + \overline{\partial^*\partial}$ and the Dirac operator is $\overline{\partial} + \overline{\partial^*}$.

One final pre-theorem theorem was the Hirzebruch signature theorem[3] which states that if M is a compact manifold of dimension $4n$ then the signature of the manifold is

$$Sig(M) = \int_M L_n$$

where $L_n = L_n(p_1, ..., p_n) \in \Omega^{4n}(X)$ is the L-genus.

3 K-Theory of Vector Bundles

This section will just be a quick introduction of K-theory of vector bundles over compact manifolds, following [4] chapter 1 section 9 and [3] chapter 3 section 6.

Letting E, F be vector bundles over M we say that they are isomorphic, written $E \sim F$ if there exists a vector bundle isomorphism

$$f : E \to F$$

Then under direct sum the set of isomorphism classes form a semigroup.

This leads immediately to the construction of the Grothendieck group. Now, since X is compact, it holds that $C(X)$ is unital. Through a bit of magic our earlier group is simply the Grothendieck group of the unital C^*-algebra $C(X)$, then in particular we define $K(X) = K_0(C(X))$.

In fact this forms a ring, though I will not go into that. This leads to a map $Vec \to K(X)$ such that the restriction to $Vec/\sim \to K(X)$ is injective.

If $0 \to E \xrightarrow{\sigma} F \to 0$ is an exact sequence of vector bundles then in fact we have an element $[E, F; \sigma] \in K(X)$ as outlined in [4].

Hence if $P : E \to F$ is an elliptic operator and $\sigma(P) : \pi^* E \to \pi^* F$ is its principle symbol then through a little bit of midification and some tedious calculations at done in [4] chapter 3 section 1 it holds that $[\pi^* E, \pi^* F, \sigma(P)] = [\sigma(P)] \in K(TX)$.

Then using the natural projection $\pi_1 : TX \to X$ we have a map, which I will also denote by π_1, that acts as $\pi_1 : K(TX) \to K(X)$ since K_0 is a covariant functor.

We also have a ring isomorphism $K(X) \to H^e(X)$, where $H^e(X) = H^0(X) \oplus H^2 \oplus ...$ is the set of even de Rham cohomology classes. This isomorphism is provided by the total chern character $ch : Vec/\sim \to H^e(X)$, which thus extends to one $K(X) \to H^e(X)$ by the universal property.

We define $ch(P) = ch(\pi_1[\sigma(P)]) \in H^e(X)$

4 A Brief Introduction to Characteristic Classes

I now introduce some of the machinery I have been using so far, following [3] chapter 2 sections 1 and 2 and [4] chapter 3 section 11.

Suppose that E is a complex vector bundle of dimension n and let $q_0^n, ..., q_n^n$ be the $n+1$ elementary symmetric polynomials of degree n, with $q_0 = 1$. Suppose further that E is hermitian with metric g, then we may find the Levi-Civita connection ∇^g on E. Let ∇ be any connection on E hermitian or not. Then ∇ defines a 2-form on E with values in E, or equivalently an $n \times n$ matrix of 2-forms on E, by $\Omega(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$, which is called a curvature form.

A little bit of fibre bundle theory needs to be used here. The form Ω can also be thought of as a 2-form, however not on E, with values in the structure group of E. Since E is complex the structure group is the set $U(n)$ of unitary complex matrices.

Since with this latter viewpoint we may view $i\frac{\Omega}{2\pi}$ as a complex matrix, one that is unitary, it is diagonalizable with eigenvalues $\lambda_1, ..., \lambda_n$ and thus

$$det(I + i\frac{\Omega}{2\pi}) = \sum_{i=0}^{n} q_i^n(\lambda_1, ..., \lambda_n) = 1 + c_1(E) + ... + c_n(E) \in H^e(X)$$

which is the total Chern class.

It can be shown that given any two connections the Chern classes lie in the same de Rham cohomology class, so we lose no generality by assuming that the connection is the Hermitian Levi-Civita connection ∇^g since that is particularly nice.

If F is a real vector bundle of dimension $2n$ we have nothing near as nice, but we can say the the Pointryagin classes $p_i \in H^{4i}(X), i = 1, ..., n$ are defined by $p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C})$.

Since $E \otimes \mathbb{C}$ is then a complex vector bundle of dimension $2n$ these are well-defined. However, note that we then have that if Ω is a curvature form on $E \otimes \mathbb{C}$ and $\lambda_1, ..., \lambda_n$ the eigenvalues of $i\frac{\Omega}{2\pi}$, it holds that $p_j(E) = q_i(\lambda_1^2, ..., \lambda_n^2) \in H^{4i}(X)$ as we stated before.

We must also define the Todd class. Associated to a power series

$$f(t) = \sum_{i=0}^{\infty} a_i t^i$$

we have a sequence of functions defined by

$$f(x_1)...f(x_n) = \sum_{k=0}^{\infty} F_k(q_0^n(x_1, ..., x_n), ..., q_k^n(x_1, ..., x_n))$$

where $q_k^n = 0$ if $k > n$.

Letting

$$f(t) = \frac{t}{1 - e^{-t}}$$

we have the associated Taylor series, then with a bit of abuse of notation and with E a complex vector bundle of dimension n, we define

$$td(E) = \sum_{k=0}^{\infty} F_k(q_0^n(c_1(E), ..., c_n(E)), ..., q_k^n(c_1(E), ..., c_n(E)))$$

$$td_k(E) = F_k(q_0^n(c_1(E), ..., c_n(E)), ..., q_k^n(c_1(E), ..., c_n(E)))$$

$$td(X) = td(TX), td_i(X) = td_i(TX)$$

The $\hat{\mathrm{A}}(F)$ class is determined from the taylor polynomial $f(t) = \frac{\sqrt{x}}{2\sinh(\sqrt{x}/2)}$ and using Pontryagin classes on real bundle instead of Chern on a complex bundle.

5 A Little Bit of Functional Analysis

We recall a little bit of Fredholm theory just for the next parts.

Suppose that $F : X \to Y$ is Fredholm between Hilbert spaces, then $Ind(F) = dimN(F) - dimN(F^*)$ where F^* is the adjoint.

Similarly to how the heat kernel asymptotics allowed for the computation of the Euler characteristic of the manifold in the last essay we can compute that

$$dimN(D) = Tr(e^{-DD^*})$$

$$dimN(D^*) = Tr(e^{-D^*D})$$

$$Ind(D) = Tr(e^{-DD^*}) - Tr(e^{-D^*D})$$

where D is a symmetric elliptic operator with positive-definite principal symbol.

6 Atiyah-Singer Index Theorem

We now have enough background to understand the Index theorem.

We first define what it means to integrate a cohomology class of mixed degree. Let $[A] \in H_k(X)$ be a homology class and $\omega \in H^*(X)$ be a cohomology class of mixed degree. We then define

$$\omega[A] = \omega_k[A] = \int_A \omega_k = \int_A \omega$$

where

$$\omega = \sum_{i=0}^{\infty} \omega_k, \omega_k \in H^k(X)$$

We also define the product of two $\omega, \eta \in H^*(X)$ by

$$\omega\eta = \omega \cdot \eta = \omega \wedge \eta \in H^*(X)$$

from which it follows that

$$\omega\eta[A] = \sum_{i+j=k} \omega_i \eta_j [A] = \sum_{i+j=k} \int_A \omega_i \wedge \eta_j$$

If the manifold X, of dimension n, is oriented then Poincare duality implies $H_n(X) \cong H^0(X)$. On the other hand $H^0(X) \cong \mathbb{R}$ so let $[X] \in H_n(X)$ be the preimage of 1 under these isomorphisms.

More concretely, we have that for $\omega \in H^n(X)$, using de Rham's isomorphism

$$\omega[X] = \int_X 1 \wedge \omega = \int_X \omega$$

Then the Atiyah-Singer index theorem is the following:

$$Ind(P) = (-1)^{\frac{n(n+1)}{2}} (ch(P) \cdot \hat{\mathrm{A}}(X)^2)[X]$$

On the other hand through [4] chapter 3 section 11 it holds that $\hat{\mathrm{A}}(TX)^2 = td(TX \otimes \mathbb{C})^2$ so that if X is complex then

$$Ind(P) = (-1)^{\frac{n(n+1)}{2}} (ch(P) \cdot td(TX))[X]$$

which is reminiscent of the Riemann-Roch theorems.

Another form is by pulling back $\cdot\hat{\mathrm{A}}(TX)$ to a form on TX, whence we obtain

$$Ind(P) = (-1)^n (ch([\sigma(P)]) \cdot td(X))[X]$$

Finally we see that an embedding $f : X \to \mathbb{R}^N$ induces a map $g : K(TX) \to K_0(C_c(T\mathbb{R}^N))$ and we follow this by noting that a map $a : T\mathbb{R}^N \to \{0\}$ induces another map $b : K_0(C_c(T\mathbb{R}^N)) \to K(\{0\})$.

However, since $K(\{0\}) = K_0(C(\{0\})) \cong K_0(\mathbb{C}) = \mathbb{Z}$. We thus have $bg : K(TX) \to \mathbb{Z}$, the Index theorem is then

$$Ind(P) = bg[\sigma(P)]$$

so we define top-ind$(P) = bg[\sigma(P)]$ as the topological index, or

$$Ind(P) = \text{top-ind}(P)$$

7 L^2-Index Theorem and the Atiyah-Patodi-Singer Index Theorem

We now investigate some generalizations of the Atiyah-Singer theorem.

The first is the so-called L^2-Index Theorem. We start by noting by heat kernel techniques and asymptotics as in [5] it holds that the trace of the heat kernel is

$$k(x, y, t, DD^* + D^*D) \sim \frac{1}{(4\pi t)^{n/2}}(\sum_{i=0}^{\infty} \alpha_i(x, y)t^i)$$

where α_i are locally defined sections of an auxiliary bundle, thus

$$Tr(e^{-t(DD^*+D^*D)}) = \frac{1}{(4\pi t)^{n/2}}(\sum_{i=0}^{\infty} \int_X Tr(\alpha_i)(x)t^i)$$

The discussion in the functional analysis section shows that

$$Ind(D) = Tr(e^{-tDD^*}) + Tr(e^{-tD^*D}) = Tr(e^{-t(DD^*+D^*D)})$$

$$= \frac{1}{(4\pi t)^{n/2}}(\sum_{i=0}^{\infty} \int_X Tr(\alpha_i)(x)t^i)$$

which is independent of t. Thus it follows that

$$Ind(D) = \frac{1}{(4\pi)^{n/2}}(\int_X Tr(\alpha_{n/2})(x))$$

However, if $f : Y \to X$ is a diffeomorphism, and f^*D is the lifted operator, then

$$Ind(f^*D) = \frac{1}{(4\pi)^{n/2}}(\int_Y Tr(f^*\alpha_{n/2})(x)) = \frac{1}{(4\pi)^{n/2}}(\int_X Tr(\alpha_{n/2})(x))$$

$$= Ind(D)$$

Thus if $f : Y \to X$ is a smooth k-sheeted covering and f^*D is the lifted operator, then

$$Ind(f^*D) = kInd(D)$$

On the other hand if the covering is infinite then [5]

$$Ind_{L^2}(f^*D) = kInd(D)$$

where $Ind_{L^2}(f^*D)$ is a weighted version of the index.

Now, we also have a version of the index theorem for manifolds-with-boundary, follwing [3]. The idea is that near the boundary both the metric and the operator take the form of products. Once that is assumed we have one theorem, assuming $dim(X) = 4n$ and X is Riemannian with the standard Dirac operator D:

$$Ind(D) = -(h + \eta(0))/2 + \hat{\mathrm{A}}(TX)[X]$$

where h is the dimension of harmonic spinors and

$$\eta(0) = \sum_{\lambda \in Spec(D)} sign(\lambda)$$

where the eigenvalues are repeated with multiplicity.

This generalized to a hermitian vector bundle E, with canonical dirac operator D_E leading to

$$Ind(D_E) = -(h + \eta(0))/2 + ch(E)\hat{\mathrm{A}}(X)[X]$$

Under certain boundary conditions, with h, η modified as appropriately.

8 Non-Commutative Index Theorem

We are finally lead to non-commutative geometry.

I will not elaborate too much on this as it would likely take me about 50 pages to do so appropriately.

The index theorem is [1]

$$Ind_{(c,\Gamma)}(D) = \frac{(-1)^n}{(2\pi i)^q}\frac{q!}{(2q)!}\int_{TM} ch([\sigma(P)])td(TX \otimes \mathbb{C})\Psi^*(c)$$

If Γ is a countable discrete group and $B\Gamma$ the classifying space then we define $c \in H^{2q}(B\Gamma)$ as a closed cochain.

Then $\Psi^*(c)$ is the pullback of c to TM, and $Ind_{(c,\Gamma)}(D)$ is a modified index for D a lift of the dirac operator D' from X to a principal Γ-bundle M.

In particular if $q = 0$ then this is just the above L^2-index theorem.

References

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