Einstein-Gauss-Bonnet Gravity and Wormholes

November 26, 2022

Mitchell Gaudet

Contents

1 Introduction

While Einstein's geometrical formulation of gravity was revolutionary in how it explained previously unexplained phenomena such as the perihelion precession of Mercury or the bending of light, it also brought to the fore several completely new ideas.
Since the geometry of a system may be nontrivial a question that was asked, though not outright, was: what would happen if there was a singularity at a point?

This lead to the investigation of different black hole solutions and eventually to the idea of a singularity at the beginning of time. Another question that could be asked would be related to the underlying topology of the space.

Now, the discussion begins like this in 2+1 dimensional spacetime for visualization and not rigour: Assume we are given a flat spacetime with 2 spatial dimensions and one time. From this remove two circles in the spatial dimensions, or in other words pick $x, y \in M$ with the same time coordinate and remove the sets

$$\{(p_1, p_2, p_3) \in M : p_3 = ty_3, t \in \mathbb{R} \text{ and } \sqrt{(y_1 - p_1)^2 + (y_2 - p_2)^2} < \varepsilon\}$$

$$\{(p_1, p_2, p_3) \in M : p_3 = tx_3, t \in \mathbb{R} \text{ and } \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2} < \varepsilon\}$$

where $\varepsilon > 0$ is chosen so that the circles and their boundaries don't overlap.

This is then a manifold with boundary, with the boundary being

$$A_1 = \{(p_1, p_2, p_3) \in M : p_3 = ty_3, t \in \mathbb{R} \text{ and } \sqrt{(y_1 - p_1)^2 + (y_2 - p_2)^2} = \varepsilon\}$$

$$A_2 = \{(p_1, p_2, p_3) \in M : p_3 = tx_3, t \in \mathbb{R} \text{ and } \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2} = \varepsilon\}$$

We can then define the diffeomorphism $F : A_1 \to A_2$ by the composition

$$A_1 \to S^1 \times \mathbb{R} \to A_2$$

with the same time coordinates.

Then by gluing using this diffeomorphism we obtain a new smooth manifold which would allows for instant travel from A_1 to A_2. However, this is no longer topologically trivial since the loop $(\varepsilon cos\theta, \varepsilon sin\theta, 0)$ is not contractible. The first major example of a wormhole would be the Einstein-Rosen bridge. The construction here is from Visser.

Start with the Schwarzchild metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

Then apply the coordinate change $u^2 = r - 2M$ to get

$$ds^2 = -(\frac{u^2}{u^2 + 2M})dt^2 + 4(u^2 + 2M)du^2 + (u^2 + 2M)^2 d\Omega^2$$

In these new coordinates we can only cover $r - 2M \geq 0$ so that this is like cutting out $S^2 \times \mathbb{R}$ in analagy to the above. Furthermore these new coordinates are covered twice over since u is squared. This is similar to gluing to of the removed $S^2 \times \mathbb{R}$, again in analogy to the above. We may view this spacetime as

$$\{u \geq 0\} \cup \{u \leq 0\}$$

glued along $u = 0$, which is called the throat. Of course as Visser says travelling through would put you into the singularity, which may be mildly inconvenient so it's not really traversable.

Similarly to this other black hole-type solutions may lead to other wormhole-type solutions, such as a charged wormhole, which was made from a black hole with negative electric charge.

2 Failure of Wormholes in 3+1D

Now, in this section I will discuss some of the problems with wormholes in standard relativity. This section draws heavily from Visser. Given an energy condition we may also

define an averaged energy condition along curves by requiring that the integral inequality holds, rather than the pointwise one.
As an example the weak energy condition is

$$T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$$

for all timelike V. The averaged weak energy condition on a timelike curve c is then

$$\int_c T_{\mu\nu}V^{\mu}V^{\nu}d\lambda \geq 0$$

where $V = \dot{c}$ and λ is an affine parameter. Now, we can show that for all null k at the

throat of a wormhole

$$T_{\mu\nu}k^{\mu}k^{\nu} \leq 0$$

and then integrating along a radial null geodesic

$$\int_c T_{\mu\nu}k^{\mu}k^{\nu}d\lambda < 0$$

which then violates the averaged null energy condition. In addition the averaged weak energy condition can be seen to be violated by looking at radial timelike geodesics.

Since most physically reasonable systems are thought to satisfy these hypotheses it would be desirable for them to not be violated. On the other hand looking at thin shell wormholes, which are wormholes created using the construction I have outlined above we can find even more wild information. Then as Visser concludes we that the surface energy density of the throat must be negative in order for it to be convex. This leads to the weak and dominant energy conditions being violated.

Morris and Thorne have a more detailed derivation of this fact. They note that certain reasonable assumptions on the shape function require the total mass-energy density E to be less than the tension t of the throat. Then an observed travelling close to the speed of light radially sees

$$T_{00} = \gamma^2(E - \left(\frac{v}{c}\right)^2 t) = \gamma^2(E - t) + t$$

where $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$. Thus since $E - t < 0$ for $v > 0$ sufficiently large $T_{00} < 0$ and thus the energy density is negative.
Such a requirement would need materials with strange or exotic properties and don't really arise in standard relativity.

3 Some QFT

While those materials may not exist in standard relativity they exist in quantum field theory as well as other workarounds.

Consider two plates in parallel, both conducting electricity and a distance $d \ll 1$ between them. Assume that the plates lie parallel to the x-y plane, then

$$Tr(T^{\mu\nu}) = 0$$

where we are considering the electromagnetic field.

The z component of the electromagnetic wave is constrained by the plates, needing to be equal on said plates and thus imposing periodic boundary conditions on the differential equation for the wave.

Visser calculates

$$T^{\mu\nu} = \frac{\hbar\pi^2}{720d^4}(\eta^{\mu\nu} - 4n^\mu n^\nu)$$

where n is the normal vector. Then

$$T^{00} = \frac{\hbar\pi^2}{720d^4}\eta^{00} = -\frac{\hbar\pi^2}{720d^4} < 0$$

so the energy density is negative.

Another quantum mechanical process leading to violated energy conditions in Hawking radiation. This in the end appears to violate the null energy hypothesis.

4 Higher Dimensional Gravity

One possible solution to the topological and geometrical problems in wormholes is to consider higher dimensions than the standard 3+1.

We first introduce the Einstein-Gauss-Bonnet Gravity itself and then look at it's properties.

The standard Lagrangian for 0 cosmological constant is

$$I = \int_M (\frac{1}{2\kappa}S + I_{matter})\sqrt{-det(g)}d^4x$$

where κ is the gravitational constant, S is the scalar curvature and I_{matter} is the matter lagrangian. Then for nonzero constant is

$$I = \int_M (\frac{1}{2\kappa}(S - 2\Lambda) + I_{matter})\sqrt{-det(g)}d^4x$$

One possible method of extending this is to consider other curvature invariants.

The Einstein-Gauss-Bonnet takes this point of view.

The Lagrangian with 0 cosmological constant in dimension $n \geq 4$ becomes

$$I = \int_M (\frac{1}{2\kappa}(S + \alpha I_{GB}) + I_{matter})\sqrt{-det(g)}d^nx$$

where α is a coupling constant and

$$I_{GB} = S^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$$

is the Gauss-Bonnet lagrangian and $R_{\mu\nu}, R_{\mu\nu\sigma\rho}$ are the Ricci and Riemann curvature tensors respectively.

Let us investigate this a bit more.

We consider the vector bundle TM on a four dimensional Riemannian(or Lorentzian) manifold M and look at the Euler class, which is equal to

$$e(TM) = \frac{1}{128\pi^2}\sum_{\sigma,\alpha\in S^n}(sgn\sigma)(sgn\alpha)R_{\sigma(1)\sigma(2)\alpha(1)\alpha(2)}R_{\sigma(3)\sigma(4)\alpha(3)\alpha(4)}dvol$$

where S^n is the n-th symmetric group, but this simplifies to

$$\frac{1}{8\pi^2}(S^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho})dvol$$

We see that by the Chern-Gauss-Bonnet theorem this is then equal to

$$\int_M e(TM) = \chi(M)$$

Now, in most physically reasonable scenarios there is a nonvanishing vector field V, in particular a time-oriented one.

Since we have that

$$\chi(M) = \sum_{x \in crit(V)} ind_x(V)$$

where $crit(V)$ is the zero set of V and $ind_x(V)$ is the index of the zero, it follows that

$$\chi(M) = \sum_{x \in crit(V)} ind_x(V) = 0$$

as V has no zeroes.

Considering the Einstein-Gauss-Bonnet action in 3+1 dimensional spacetime we have that

$$I = \int_M (\frac{1}{2\kappa}(S + \alpha I_{GB}) + I_{matter})\sqrt{-det(g)}d^4x$$

$$= \int_M (\frac{1}{2\kappa}S + I_{matter})\sqrt{-det(g)}d^4x + \int_M \alpha I_{GB}\sqrt{-det(g)}d^4x$$

$$= \int_M (\frac{1}{2\kappa}S + I_{matter})\sqrt{-det(g)}d^4x + \alpha 8\pi^2 \int_M e(TM) = \int_M (\frac{1}{2\kappa}S + I_{matter})\sqrt{-det(g)}d^4x$$

and we thus have the standard Einstein theory of gravity in dimension 4.

We may also add in the cosmological constant to get

$$I = \int_M (\frac{1}{2\kappa}(S - 2\Lambda + \alpha I_{GB}) + I_{matter})\sqrt{-det(g)}d^nx$$

which again reduces to the Einstein theory when $n = 4$.

The constant-less solution leads to the modified field equations from Bhawal and Kar:

$$0 = (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S - T_{\mu\nu}) - \alpha[\frac{1}{2}g_{\mu\nu}(R_{\alpha\beta\sigma\rho} - R^{\alpha\beta\sigma\rho} - 4R_{\alpha\beta}R^{\alpha\beta} + S^2)]$$

$$-\alpha[2RR_{\mu\nu} + 4R_{\mu\alpha}R^{\alpha}_{\nu} + R_{\alpha\beta}R^{\alpha\beta}_{\mu\nu} - 2R_{\mu\alpha\beta\sigma}R^{\alpha\beta\sigma}_{\nu}]$$

which was fun trying to write all of the indicies.

5 Higher Dimensional Wormholes

We would like to show that we don't need exotic materials and that the energy conditions aren't violated.

As for some geometry, we note that

$$R : \bigwedge^2 TM \to \bigwedge^2 TM$$

defined by

$$(R(X,Y))^{\alpha\beta} = R^{\alpha\beta}_{\mu\nu}X^{\mu}Y^{\nu}$$

Alternatively we may view

$$R : \bigwedge^2 TM \to Lie(O(n))$$

where $Lie(O(n))$ is the set of antisymmetric matrices by

$$(R(X,Y)Z) = R^{\alpha}_{\mu\nu\beta}X^{\mu}Y^{\nu}Z^{\beta}$$

This means that R is a matrix ω^i_j two-forms called the curvature forms, which uniquely determine R. This will show up later.

From now on the dimension is assumed to be at least 5 At the throat of a wormhole with notation as in section 2 we want for

$$(E - t) > 0$$

Now, formula (13) in Bhawal and Kar is

$$(E - t) = (n - 2)(1 + 2\bar{\alpha}Q)(N - P)$$

The Q, N, P are three of the four coefficients for the curvature two-forms, there are only 4 by symmetry. Here $\bar{\alpha} = (n - 2)(n - 3)\alpha$ depends both on dimension and coupling constant.While it is impossible to remove exotic matter entirely they found that it is possible to confine such matter to only the throat.

On the other hand Mehdizadeh et al. found that there exist radial redshift and shape functions such that the weak energy condition is fulfilled. Unfortunately, an exact solution for the shape function cannot be found though numerically it all works out.

Finally, Maeda and Nozawa found that the energy conditions are strongly related to solutions to

$$1 + \frac{8\kappa_n m\bar{\alpha}}{(n-2)V^k_{n-2}r^{n-1}} + 4\bar{\alpha}\bar{\Lambda}$$

$$\bar{\Lambda} = \frac{2\Lambda}{(n-1)(n-2)}$$

and V^k_{n-2} is the area of the unit $n - 2$ dimensional sphere in the spaceform of constant curvature k. The solutions would then form an $(n - 2)$-dimensional submanifold, which makes sense since that would be the dimension of the surface of the wormhole.
Now, an important role is played by

$$1 + 4\bar{\alpha}\bar{\Lambda}$$

as the sign of $\alpha(1 + 4\bar{\alpha}\bar{\Lambda})$ governs the sign of

$$T_{\mu\nu}$$

6 Conclusion

In summary while there are certainly many challenges to overcome, even theoretically, in the construction or even the existence of wormholes there are also many ways around these problems.

The energy density problem has been shown to theoretically be viable via the Casimir effect, though only in low concentrations.

On the other hand the energy conditions aren't necessarily violated if we assume that the universe is more than 3+1 dimensional. This means that these other theories make wormholes more viable than the standard general relativity one.

References

[1] Biplab Bhawal and Sayan Kar. Lorentzian wormholes in einstein-gauss-bonnet theory. *Phys. Rev. D*, 46:2464–2468, Sep 1992.

[2] Hideki Maeda and Masato Nozawa. Static and symmetric wormholes respecting energy conditions in einstein-gauss-bonnet gravity. *Phys. Rev. D*, 78:024005, Jul 2008.

[3] Mohammad Reza Mehdizadeh, Mahdi Kord Zangeneh, and Francisco S. N. Lobo. Einstein-gauss-bonnet traversable wormholes satisfying the weak energy condition. *Phys. Rev. D*, 91:084004, Apr 2015.

[4] Michael S. Morris and Kip S. Thorne. Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity. *American Journal of Physics*, 56(5):395–412, 1988.

[5] Matt Visser. *Lorentzian wormholes: From Einstein to hawking.* AIP Press, 1996.