Hodge Theory Lecture 1

Mitchell Gaudet

January 14, 2025

Before we begin, a word of warning: A "function" in L^2(or L^p or $W^{k,p}$) is actually an equivalence class of functions. Since from the point of view of integration theory measure 0 sets don't matter, we identify functions which differ on a set of measure 0.

Functional Analysis

Given a vector space V with norm $\|*\|_V$ we may define an associated topology, as in the problems. Additionally, an inner product defines an norm, and thus a topology.

Definition. A Hilbert space is a possibly infinite-dimensional inner product space, such that the induced topology is complete

Hilbert spaces have many nice properties. One of the most useful is the following:

Theorem. *If H is a Hilbert space, then it is isometrically isomorphic to its dual*

Proof. See Folland $\square$

This is a very useful fact when working with the calculus of variations, because of its impact on the weak topology.

Definition. If H is a Hilbert space, then a sequence $f_i \rightharpoonup f$ weakly if $\forall e \in H$ it holds that $\langle f_i, e\rangle \to \langle f, e\rangle$

Sobolev Spaces

We will assume for this section that all functions are defined in an open set $U \subseteq \mathbb{R}^n$, with smooth boundary.

Given some $g \in L^p$ we identify it with the corresponding distribution ϕ_g using the notation $\phi_g(f) = \langle g, f\rangle$(also identify $D^\alpha \phi_g \leftrightarrow D^\alpha g$).
Now, recall that given $g \in L^2$, if the distributional derivative $D^\alpha g$ extends to a bounded map from $L^2 \to \mathbb{R}$, then $D^\alpha g = h$ for some $h \in L^2$. It is a

routine exercise to show the converse. Now, a modification of part 2e) of the last problem set tells us that such an h is unique almost everywhere. In such a case, we write $D^\alpha g$ instead of h. This is the so-called "weak derivative"
We then have the following:

Definition. The Sobolev space $W^{k,2}$ is the set of $g \in L^2$ such that $D^\alpha g$ extends to a bounded map from $L^2 \to \mathbb{R}$ for all $|\alpha| \leq k$. The Sobolev norm is

$$\|g\|_{W^{k,2}} = \sum_{|\alpha| \leq k} \|D^\alpha g\|_{L^2}$$

We identify two functions which have all weak derivatives agreeing except on a set of measure 0.

The Sobolev space $W_m^{k,2}$ is the set of maps $U \to \mathbb{R}^m$ such that each component is in $W^{k,2}$, with the norm

$$\|g\|_{W_m^{k,2}} = \sum_{|\alpha| \leq k} \sum_{i=1}^{m} \|D^\alpha g_i\|_{L^2}$$

I will now give some basic properties of $W_m^{k,2}$:

1. They are complete in the metric associated to the norm
2. The norm is induced by an inner product
3. If $\{f_i\} \in W_m^{k,2}, f \in W_m^{0,2}$ and $f_i \to f$ in $W_m^{0,2}$ and weakly in $W_m^{k,2}$ then $f \in W_m^{k,2}$
4. (mollifiers) There exists a family F_ε of linear operators $W_m^{k,2} \to C^\infty(U, \mathbb{R}^m) \cap W_m^{k,2}(U)$ with the following properties
   (a) F_ε is bounded from $W_m^{k,2} \to W_m^{k,2}$
   (b) For any operator of the form
   $$(Pf)(x) = \sum_{\alpha=1}^{m} \sum_{i=1}^{n} a_i^\alpha(x) \frac{\partial f_\alpha}{\partial x^i}(x) + \sum_{\beta=1}^{m} b^\beta(x) f_\beta(x)$$
   where $a_i^\alpha, b^\beta \in C^\infty(\overline{U})$ it holds that $F_\varepsilon P - P F_\varepsilon$ is a bounded operator $W_m^{k,2} \to W_m^{k,2}$. (Here α is a number, not a multi-index)
   (c) $F_\varepsilon \to Id$ in the $W_m^{k,2} \to W_m^{k,2}$ operator norm as $\varepsilon \to 0$
5. $C^\infty(U, \mathbb{R}^m) \cap W_m^{k,2}(U)$ is dense in $W_m^{k,2}$

Differential Operators

Definition. A differential operator of order N is an operator $P : C^\infty(U, \mathbb{R}^m) \to C^\infty(U, \mathbb{R}^l)$ of the form

$$(Pf)^\mu(x) = \sum_{\alpha=1}^{m} \sum_{|\beta|\leq N} a_\beta^{\alpha\mu}(x) D^\beta f_\alpha(x), \mu = 1, ..., l$$

where $a_\beta^{\alpha\mu} \in C^\infty(\overline{U})$. Given $\psi : U \to V$, a diffeomorphism of open sets, we define the pushforward operator

$$((\psi_* P)(f))^\mu(x) = \sum_{\alpha=1}^{m} \sum_{|\beta|\leq N} a_\beta^{\alpha\mu}(\psi^{-1}(x)) D^\beta (f_\alpha \circ \psi)(\psi^{-1}(x)), \mu = 1, ..., l$$

so that $\psi_* P : C^\infty(V, \mathbb{R}^m) \to C^\infty(V, \mathbb{R}^k)$

Note then that P extends to a bounded linear operator $W_m^{k,2} \to W_m^{k-m,2}$.

Definition. If $m = 1, N = 2k$ then P is (strongly) elliptic if $|\sum_{|\beta|=2k} a_\beta(x)\xi^\beta| \geq C|\xi|^{2k}$ for $C > 0$ independent of x and $\xi \in \mathbb{R}^n$. We define the principal symbol $sym_{2k}(P)(x, \xi) = \sum_{|\beta|=2k} a_\beta(x)\xi^{\beta|}$

It is not immediately apparent that the highest order terms, or the push-forward, should have any significance. However, in the case $n \geq 2, N = 2$ for simplicity, it holds:

Theorem. *Let $\psi : U \to \mathbb{R}^n$ be a diffeomorphism onto its image. Then $sym_2(P)(x, (D\psi)^T\xi) = sym_2(\psi_* P)(\psi(x), \xi)$.*

Proof. We note that if $\psi(x) = y$

$$D^i(f \circ \psi)(x) = \sum_{\gamma=1}^{n} \frac{\partial f(y)}{\partial y^\gamma} \frac{\partial (D\psi)^\gamma}{\partial x^i}$$

$$D^{ij}(f \circ \psi)(x) = \sum_{\gamma=1}^{n} \sum_{\lambda=1}^{n} \frac{\partial f(y)}{\partial y^\gamma \partial y^\lambda} \frac{\partial (D\psi)^\lambda}{\partial x^j}(x) \frac{\partial (D\psi)^\gamma}{\partial x^i}(x) + \text{ lower order terms}$$

but this implies

$$sym_2(\psi_* P)(y, \xi) = \sum_{i,j=1}^{n} a_{ij}(\psi^{-1}(y)) \sum_{\gamma,\lambda=1}^{n} \xi^\gamma \xi^\lambda \frac{\partial (D\psi)^\lambda}{\partial x^j}(x) \frac{\partial (D\psi)^\gamma}{\partial x^i}(x)$$

$$= \sum_{i,j=1}^{n} a_{ij}(x)((D\psi)^T\xi)^i((D\psi)^T\xi)^j$$

$\square$

We now look at systems:

Definition. If $N = 2k$ then P is (strongly) elliptic if $m = l$ and

$$|\sum_{\mu,\alpha=1}^{l} \sum_{|\beta|,|\gamma|=k} a_{\gamma\beta}^{\alpha\mu} \xi_{\alpha}^{\beta} \xi_{\mu}^{\gamma}| \geq C|\xi|^{2k}$$

The principal symbol is defined analogously.