The Isoperimetric Inequality

The Isoperimetric inequality is an example of a "Geometric Variational Problem"
These generally come in two forms:

1. Let X, Y be two spaces and $Z \subseteq Y^X$, i.e. a collection of maps from X to Y
   (a) Then we let $F : Z \to \mathbb{R}$, or sometimes to complex numbers
   (b) We then try to find "critical points". Since Z is often a (subset of a) Hilbert space H the gradient is well-defined, assuming that F is suitably differentiable, as ∇F is the unique element of H with $\langle \nabla F, x\rangle = dF(x), \forall x \in H$
   (c) An example would be X, Y two Riemannian manifolds, Z is $C^\infty(X, Y)$, or the sobolev spaces $H^1(X, Y)$ or $H^2(X, Y)$, with $F(f) = \int_X |df|^2 dV_X$, the Dirichlet energy. This leads to the study of harmonic maps.
2. Let X be a space, and $Y \subseteq 2^X$, a collection of subsets.
   (a) Then pick $F : Y \to \mathbb{R}$ and try to find "critical points"
   (b) Depending on Y, F critical points could have several meanings
   (c) For example given a smooth closed curve $\gamma \subseteq \mathbb{R}^3$ try and find a surface S which has boundary γ, which is a "critical point" of the area functional $A(S)$. A critical point in this case is defined as a surface S such that $\frac{d}{dt}|_{t=0} A(\Phi_t(S)) = 0$ for all compactly supported variations Φ_t.
   (d) Note that we can sometimes go from 2. to 1. by the method of partial differential equations

The isoperimetric inequality in $\mathbb{R}^n$ reads as follows: If $D \subseteq \mathbb{R}^n$ is a bounded domain with C^1 boundary, then

$$\frac{A(\partial D)}{(V(D))^{\frac{n-1}{n}}} \geq \frac{A(\partial B)}{(V(B))^{\frac{n-1}{n}}}$$

where B is the unit ball in $\mathbb{R}^n$. Furthermore, equality holds iff D is a ball.
Now, if we assume that D has C^2 boundary, a "critical point" for the "isoperimetric functional" is defined as D a bounded domain with C^2 boundary such that for all volume preserving variations Φ_t $\frac{d}{dt}|_{t=0} A(\Phi_t(D)) = 0$. It is an immediate consequence of geometry, see Chavel's Riemannian geometry, that the mean curvature H of the boundary satisfies

$$\int_{\partial D} H f dA = 0, \forall f : \int_{\partial D} f dA = 0$$

Now, by the fundamental lemma of the calculus of variations this implies that $H = const$. It is then a deep theorem of Alexandroff using quasilinear elliptic pdes that if S is a C^2 connected closed hypersurface with constant mean curvature, as ∂D is, then it is a sphere. Consequently we obtain that if $\frac{A(\partial D)}{(V(D))^{\frac{n-1}{n}}}$ is minimized, then for all volume preserving variations Φ_t $\frac{d}{dt}|_{t=0} A(\Phi_t(D)) = 0$ as $A(\partial \Phi_t(D)) \geq A(\partial D)$, and so we have that ∂D is a sphere, and thus D is a ball.
On the other hand dimension two is special for two main reasons regarding the isoperimetric inequality:

1. The equality $\mathbb{S}^{n-1} = \mathbb{T}^m$ holds iff $n = 2, m = n - 1$, thus implying that the methods of Fourier series can be applied only in this dimension
2. The equality $B_{2n} = D^m$, where B_{2n} is the unit ball in $\mathbb{R}^n$ and $D^m = \prod_{i=1}^m B_2$ is the unit polydisk, implies that $m = n = 1$, thus the methods of complex analysis can be applied only in this dimension

In fact, one can approach the problem through either method, though in these notes I will focus on the Fourier series technique.
Now, before we begin I would like to make note of a special fact: since

$$Asin(at+b) = Asin(at)cos(b) + Acos(at)sin(b)$$

it holds that

$$Asin(at+b) = c * sin(at) + d * cos(at)$$

holds iff

$$\sqrt{c^2+d^2} = A$$

$$Acos(b) = c, Asin(b) = d$$

hence

$$b = \arctan(\frac{d}{c})$$

so in particular

$$\begin{cases} f(t) = c * sin(at) + d * cos(at) \\ g(t) = -d * sin(at) + c * cos(at) \end{cases} \Leftrightarrow \begin{cases} f(t) = \sqrt{c^2+d^2}sgn(c)sin(at - \arctan(\frac{-d}{c})) \\ g(t) = \sqrt{c^2+d^2}sgn(c)cos(at - \arctan(\frac{-d}{c})) \end{cases}$$

i.e. iff $(g(t), f(t))$ describes a circle.
Furthermore we note that if f is real and periodic, and say C^1 so it converges uniformly to its Fourier series, then

$$\overline{\hat{f}(n)} = \overline{\int_0^1 e^{-2\pi ixn}f(x)dx} = \int_0^1 \overline{e^{-2\pi ixn}}f(x)dx = \int_0^1 e^{2\pi ixn}f(x)dx = \hat{f}(-n)$$

and if $\hat{f}(n) = a(n) + ib(n)$ is the decomposition into real and imaginary parts then

$$a(n) = Re(\int_0^1 e^{-2\pi ixn}f(x)dx) = \int_0^1 cos(-2\pi ixn)f(x)dx = \int_0^1 cos(2\pi ixn)f(x)dx$$

$$= a(-n)$$

$$b(n) = Im(\int_0^1 e^{-2\pi ixn}f(x)dx) = \int_0^1 sin(-2\pi ixn)f(x)dx = -\int_0^1 sin(2\pi ixn)f(x)dx$$

$$= -b(-n)$$

Now, we investigate Wirtinger's inequality. To start, note that if $f \in C^1$ and $f(0) = f(1)$ then we have that following Plancherel's theorem

$$\|f\|_2^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2$$

$$\|f'\|_2^2 = \sum_{-\infty}^{\infty} |(\hat{f'})(n)|^2$$

but

$$(\hat{f'})(n) = \int_0^1 e^{-2\pi ixn} f'(x)dx = (e^{-2\pi ixn} f(x))|_0^1 + 2\pi in \int_0^1 e^{-2\pi ixn} f(x)dx$$

$$= 2\pi in(\hat{f})(n)$$

since f is periodic. This leads to the identity that

$$\|f'\|_2^2 = 4\pi^2 \sum_{-\infty}^{\infty} n^2 |\hat{f}(n)|^2$$

Now, we note that

$$\frac{\|f'\|_2^2}{\|f\|_2^2}$$

with f periodic, is exactly the Rayleigh quotient for the laplacian on the sphere. In other words

$$\lambda = \inf_{f \in C^\infty(\mathbb{S}^1)} \frac{\|f'\|_2^2}{\|f\|_2^2}$$

iff λ is the lowest eigenvalue for the laplacian on the sphere, which we note is always nonnegative. Hence for this to be nonzero we want to remove the case $\triangle u = 0$, which we note implies $\|f'\|_2^2 = 0$ through integration by parts, thus we need to remove the case $f = const$ and sums of the form $f + const$. To do this we assume that f has mean zero, and so the only constant function under consideration is $f \equiv 0$, which cannot be an eigenfunction and has $g + f = g$. Another way to proceed is to get rid of the constant functions by assuming dirichlet boundary conditions for Laplace's equation on [0,1], which may give a different lowest eigenvalue.
More concretely, if f has mean 0, then

$$\|f'\|_2^2 = 4\pi^2 \sum_{-\infty}^{\infty} n^2 |\hat{f}(n)|^2 = 4\pi^2 \sum_{n \in \mathbb{Z} - \{0\}} n^2 |\hat{f}(n)|^2 \geq 4\pi^2 \sum_{n \in \mathbb{Z} - \{0\}} |\hat{f}(n)|^2$$

$$= 4\pi^2 \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 = 4\pi^2 \|f\|_2^2$$

with equality holding iff $n^2|\hat{f}(n)|^2 = |\hat{f}(n)|^2, \forall n$ which would imply $|\hat{f}(n)|^2 = 0 \forall n : |n| \geq 2$.
In a general bounded domain this leads to the following quotient

$$\lambda = \inf_{f \in C^\infty(U)} \frac{\|\nabla f\|_2^2}{\|f\|_2^2}$$

where λ is the lowest eigenvalue of the laplacian. By our earlier considerations we must exclude $\triangle u = 0$, which can again be seen to be the constant functions and sums of the form $f + const$, which can be done by asserting Dirichlet boundary conditions or forcing the function to have mean 0. Thus in general we have that if f is smooth and has mean 0, $g \in C_0^\infty(U)$ we have the following Poincare inequalities

$$\|f\|_2^2 \leq C_1 \|\nabla f\|_2^2$$

$$\|g\|_2^2 \leq C_2 \|\nabla g\|_2^2$$

with $C_1, C_2 > 0$ independent of f, g. When investigating sobolev-type inequalities we are thus lead to consider isoperimetric-type problems, as they are related to Poincare inequalities as will be shown. Thus it is natural to conjecture that

$$\inf_D \frac{A(\partial D)}{(V(D))^{\frac{n-1}{n}}} = \frac{A(\partial B)}{(V(B))^{\frac{n-1}{n}}} = C$$

is equal to

$$\inf_{f \in C_c^\infty(\mathbb{R}^n)} \frac{\|\nabla f\|_1}{\|f\|_{\frac{n}{n-1}}}$$

by looking at exponents, and in fact this is a theorem of Federer-Fleming that it does hold.
Now, we suppose that $c = (x(t), y(t))$ is a simple, closed curve of length 1 that is C^1 and such that it is parameterized with respect to arclength. The necessity of simple and closed is two-fold: first it eliminates the constant paths, second it allows us to have a well-defined "inside" and "outside" by the Jordan curve theorem, where the inside is the bounded component. We require C^1 to apply Wirtinger's inequality, the length is used for several reasons, and arclength will show that the parameterization of a minimizer is $\frac{1}{2\pi}(sin(2\pi t + b), cos(2\pi t + b))$. Furthermore, we can assume that the center of mass of the inside component is at the origin, by translation. This is the same as saying x, y both have mean 0, by replacing $x \mapsto x - \int_0^1 x$, and $y \mapsto y - \int_0^1 y$, neither of which effect the area of the inside component or the length of c.
On the other hand we have that if D is the inside component then Stokes' theorem implies for any 1-form $\omega = fdx + gdy$ the following:

$$\int_{\partial D} \omega = \int_D d\omega = \int_D (g_x - f_y) dx \wedge dy$$

so putting $g = x, f = -y$ we obtain

$$\int_{\partial D} xdy - ydx = 2 \int_D dx \wedge dy = 2A(D)$$

Now, we have a parameterization of ∂D given by c, hence

$$2A(D) = \int_0^1 x(t)\dot{y}(t) - y(t)\dot{x}(t) dt$$

Note that for a curve c parameterized with respect to arclength, the unit normal is

$$\vec{n} = (-\dot{x}(t), \dot{y}(t))$$

Hence

$$\int_0^1 x(t)\dot{y}(t) - y(t)\dot{x}(t) dt = \int_0^1 c \cdot \vec{n}(t) dt \leq \int_0^1 |c \cdot \vec{n}(t)| dt \leq \int_0^1 |c| dt$$

But Holder's inequality implies

$$\int_0^1 |c| dt \leq \| |c| \|_2 \, \|1\|_2 = \| |c| \|_2$$

Now, we note that

$$\| |c| \|_2^2 = \int_0^1 (x(t))^2 dt + \int_0^1 (y(t))^2 dt \leq \frac{1}{4\pi^2} (\int_0^1 (\dot{x}(t))^2 dt + \int_0^1 (\dot{y}(t))^2 dt)$$

$$= \frac{1}{4\pi^2} \||\dot{c}\||_2^2 = \frac{1}{4\pi^2} L(c) = \frac{1}{4\pi^2}$$

Hence

$$2A(D) \leq \frac{1}{2\pi}$$

It is clear that equality holds if ∂D is the circle of unit length, so we want to show only if. But this requires that the equality case in Wirtinger holds for both x, y, so $\hat{x}(n), \hat{y}(n)$ vanish for $|n| \geq 2$. Now, we have that

$$\int_0^1 (a(n)e^{-2\pi ixn})(b(m)e^{-2\pi ixm}) = \delta_{-m}^n a(n)b(m)$$

so

$$\int_0^1 x(t)\dot{y}(t) - y(t)\dot{x}(t)dt = 2\pi i \sum_{n=1,-1} n(\hat{x}(n)\hat{y}(-n) - \hat{x}(-n)\hat{y}(n))$$

with

$$\int_0^1 x(t)\dot{y}(t) - y(t)\dot{x}(t)dt = \frac{1}{2\pi}$$

while c being parameterized with respect to arclength implies

$$1 = (\int_0^1 (\dot{x}(t))^2 dt + \int_0^1 (\dot{y}(t))^2 dt) = 4\pi^2 (\sum_{n=1,-1} n^2 \hat{x}(n)\overline{\hat{x}(n)} + \sum_{n=1,-1} n^2 \hat{y}(n)\overline{\hat{y}(n)})$$

by Plancherel. But then by our earlier discussion of symmetries and since $n^2 = 1$ if $n = \pm 1$ we have

$$\sum_{n=1,-1} n^2 \hat{x}(n)\overline{\hat{x}(n)}dt + \sum_{n=1,-1} n^2 \hat{y}(n)\overline{\hat{y}(n)}dt = \sum_{n=1,-1} (\hat{x}(n)\hat{x}(-n) + \hat{y}(n)\hat{y}(-n))$$

Putting $\hat{x}(n) = a(n) + ib(n)$ and $\hat{y}(n) = c(n) + id(n)$ we find that

$$\hat{x}(n)\hat{x}(-n) = (a(n) + ib(n))(a(-n) + ib(-n))$$

$$= (a(n) + ib(n))(a(n) - ib(n)) = (a(n)^2 + b(n)^2)$$

and similarly

$$\hat{y}(n)\hat{y}(-n) = (c(n) + id(n))(c(-n) + id(-n))$$

$$= (c(n) + ic(n))(c(n) - id(n)) = (c(n)^2 + d(n)^2)$$

and by symmetries again

$$i \sum_{n=1,-1} n(\hat{x}(n)\hat{y}(-n) - \hat{x}(-n)\hat{y}(n)) = i \sum_{n=1,-1} n(\hat{x}(n)\overline{\hat{y}(n)} - \overline{\hat{x}(n)}\hat{y}(n))$$

$$= \sum_{n=1,-1} 2n(b(n)c(n) - a(n)d(n))$$

so

$$\frac{1}{2\pi}\left(\int_0^1 (\dot{x}(t))^2 dt + \int_0^1 (\dot{y}(t))^2 dt\right) = \int_0^1 x(t)\dot{y}(t) - y(t)\dot{x}(t)dt$$

which implies

$$\sum_{n=1,-1} a(n)^2 + b(n)^2 + c(n)^2 + d(n)^2 - 2n(b(n)c(n) - a(n)d(n)) = 0$$

which simplifies to

$$\sum_{n=1,-1} (na(n) + d(n))^2 + (nb(n) - c(n))^2 = 0$$

Hence

$$a(\pm 1) = \mp d(\pm 1)$$

$$b(\pm 1) = \pm c(\pm 1)$$

Finally, by the symmetries of $a(\pm 1), d(\pm 1), b(\pm 1), c(\pm 1)$ noted above we obtain that

$$x(t) = 2a(1)cos(2\pi t) - 2b(1)sin(2\pi t)$$

$$y(t) = 2b(1)cos(2\pi t) + 2a(1)sin(2\pi t)$$

with

$$1 = (\dot{x}(t))^2 + (\dot{y}(t))^2$$

immediately implying that

$$4(b(1)^2 + a(1)^2) = \frac{1}{4\pi^2}$$

by taking $t = 0$ and hence by our discussion near the beginning there is B such that

$$x(t) = \frac{1}{2\pi} sin(2\pi t + B)$$

$$y(t) = \frac{1}{2\pi} cos(2\pi t + B)$$

and hence ∂D is a circle.