The Optimal Transport Approach to General Relativity

April 7, 2022

Mitchell Gaudet

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1 Lorentzian Manifolds

Recall that a lorentzian manifold of dimension 4, which we will call a space-time manifold if it is time-oriented(see below), is a manifold M of dimension 4 equipped with a smoothly varying symmetric nondegenerate bilinear form $g = (g_{ij}) \in TM \odot TM$ such that $\forall p \in M, g$ has exactly 3 positive eigenvalues and one negative eigenvalue when treated as a matrix. We note that these eigenvalues can change at each point, but the overall number of each sign remains the same.

We say that $v \in T_pM$:

1. Points in a timelike direction if $g_p(v, v) < 0$
2. Points in a null direction if $g_p(v, v) = 0$
3. Points in a spacelike direction if $g_p(v, v) > 0$
4. Is causal if $g_p(v, v) \leq 0$

Furthermore if $\gamma \in C^1(I \subseteq \mathbb{R}, M)$ then we say that γ is timelike(resp. spacelike, null, causal) if $\dot{\gamma}(t) \in T_{\gamma(t)}M$ is timelike(resp. spacelike, null, causal) for every $t \in I$.

Recall that given a Riemannian manifold M, g of dimension 4 there is an associated volume density given in local coordinates by

$$dvol_g = \sqrt{|det(g_{ij})|}|dx_1 \wedge ... \wedge dx_4|$$

so that for a continuous function f with compact support and covering M with coordinate charts (U_i, ϕ_i) and a subordinate partition of unity ψ_i

$$\int_M f dvol_g = \sum_i \int_{\phi_i(U_i)} ((\psi_i f) \circ \phi_i^{-1}) \sqrt{|det(g_{ij})|} d\lambda$$

where λ is the lebesgue measure in $\mathbb{R}^4$. This provides a linear functional $C_0(M, \mathbb{R}) \to \mathbb{R}$ such that if $f \geq 0$ then $\int_M f dvol_g \geq 0$, and so we get a Radon measure on M, which we also denote by $dvol_g$.

This allows for an easy generalization to a lorentzian manifold of dimension 4, since $|det(g_{ij})| \neq 0$ anywhere by nondegeneracy we can construct a Radon measure $dvol_g$ on M with the same local coordinate expression.

Suppose we have a vector field X that is continuous, and for each $p \in M$ $0 \neq X(p)$ and $X(p)$ is timelike, we say it is time-oriented. This allows for exactly what it sounds like, an orientation of time, with timelike $v \in T_pM$ being future-oriented if $g(v, X(p)) < 0$ and past-oriented if $g(v, X(p)) > 0$. Furthermore, intuitively this allows for the determination of whether an event occurs in the future or past, as well as some idea of treating the Einstein problem as an evolution problem, though only in reasonable spacetimes.

The reasoning goes something like this:

1. Start with the vector field X which we shall assume is smooth. Then $g_p(X_p, v) = 0$ if and only if v is space-like or 0. So this implies $\theta_p = g_p(X_p, *)$ annihilates the spacelike vectors.

2. We have that by Frobenius there exists a foliation into hypersurfaces with spacelike tangent spaces if and only if $\theta \wedge d\theta = 0$

3. We assume that $d\theta = 0$ so that there exist no closed timelike curves

4. Thus the foliation exists, parameterized by the flow $t \mapsto \Phi_t(p)$ of X

This disregards the immense technical involvement and some stronger assumptions required but gives an idea of what it could look like.

In particular, as in [5], if there is a foliation into what are termed cauchy hypersurfaces, which means that each hypersurface intersects every causal curve which caannot be extended further exactly once, then the evolution problem is well-defined. Another such exisence result in discussed in chapter 18 sections 8 aand 9 in [3].

2 Entropy

In a thermodynamic system we define a microstate of the system as a specific configuration C_i of the system, with corresponding energy levels E_i. If we have a system S containing gas with microstates of energy levels E_i, we define p_i as the probability that the microstate C_i will occur. We note that by definition $\sum_i p_i = 1$

Let Ω be the probability space consisting of possible microstates, and k_B be the Boltzmann constant, then define μ to be the counting measure, i.e. $\mu(C_i) = 1$. Let $\eta(C_i) = p_i$, then by definition

$$\eta(\Omega) = \sum_i \eta(C_i) = \sum_i p_i = 1$$

so that η is a probability measure, and we also have $\eta \ll \mu$ and $\frac{d\eta}{d\mu} = f$, where $f(C_i) = p_i$. Define for the probability measure η the Gibbs entropy

$$En_G(\eta|\mu) = -k_B \int flog(f)d\mu = -k_B \sum_i p_i log p_i$$

Now let's generalize a bit. Let Ω be **ANY** probability space equipped with the counting measure μ. Consider a discrete absolutely continuous probability measure η, with Radon-Nikodym derivative $\frac{d\eta}{d\mu} = f$, so that there exist events $x_1, ..., x_n$ with $f(x_i) = p_i$ and $\sum_i p_i = 1$. Then the Shannon entropy of this measure is defined as

$$En_S(\eta|\mu) = -\int flog(f)d\mu = -\sum_i p_i log p_i$$

Note the glaring similarities between these two definitions. This motivates the following generalization, let Ω be a probability space equipped with a reference measure μ. Given an absolutely continuous probability measure η, with Radon-Nikodym derivative $\frac{d\eta}{d\mu} = f$ define the Boltzmann-Shannon entropy

$$En_{B-S}(\eta|\mu) = \int flog(f)d\mu$$

which, other than a sign change and the Boltzmann constant, encapsulates both earlier definitions as well as the next definition.

We note that this is well defined as

$$Lim_{x\to 0^+} xlogx = 0$$

and the entropy takes values in $[-\infty, \infty]$ if f is measurable, and set $En_{B-S}(\eta|\mu) = -\infty$ otherwise (some literature prefers $= \infty$ e.g. [5]).

Let M, g be a spacetime manifold, then treating it as a probability space with points as events and with reference measure $dvol_g$, we get the final Boltzmann-Shannon entropy in this essay:

$$Ent(\eta|\mu) = \int \frac{d\eta}{d\mu} log(\frac{d\eta}{d\mu}) dvol_g$$

with the cases as above, which will be the definition of entropy used in the following sections.

3 Mechanics

We note that when a particle trajectory $\gamma \in C^1(I \subseteq \mathbb{R}, M)$ is in a spacetime manifold M, g we say that it moves at the speed of light at time t if $\dot{\gamma}(t)$ is null, and slower than the speed of light if $\dot{\gamma}(t)$ is timelike. In ordinary Minkowski space this works out to be exactly that of slower/at the speed of light in a physical sense.

In a Lorentzian manifold N given $\gamma \in C^1(I = [a, b], M)$ a causal curve we define

$$L(\gamma) = \int_a^b \sqrt{-g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt$$

and in contrast to the Riemannian case, we actually seek to maximize this length with fixed endpoints, due to the sign difference, and define causal geodesics as such a maximizer. The importance of such a construction is twofold. First given a point, or event, at p and another at x, we want to know if there exists a timelike, future oriented curve that starts at p and ends at x or in other words if x lies in the future of p.

The second is to know the time difference between them, which we define as the length of the causal geodesic connecting them:

$$L = sup\{L(\gamma) : \gamma(a) = p, \gamma(b) = x\}$$

and note that since

$$-g_{\gamma(t)}(v, v) > 0 = -g_{\gamma(t)}(w, w)$$

if v is timelike and w is null, the causal geodesic will always be timelike, except perhaps on a set of measure 0, if x lies in the future of p. The other use is that given a particle p with no external forces acting upon it it, and not treating gravity as a force, follows a causal geodesic.

Similar to the Riemannian case, the deformations of the geodesics are measured by the Riemann curvature tensor $Riem$, which simplifies into the Ricci curvature $R = R_{ij}$ by taking the trace

$$R_{ij} = Riem^k_{ijk}$$

using the Einstein summation convention. We may in fact extend the class of curves being considered by looking at absolutely continuous curves, which have bounded variation and are thus differentiable almost everywhere.

The Einstein equations are a way of specifying the curvature of spacetime, and thus the bending of causal geodesics, in the presence of, roughly speaking, mass, energy and momentum. They are represented as in [2] by

$$R_{ij} + (\Lambda - \frac{1}{2}\sum_{k=1}^{4} R_{kk})g_{ij} = \kappa T_{ij}$$

where Λ, κ are constants, $\kappa > 0$ and $T = T_{ij}$ is a tensor that represents the density and flux of energy and momentum. Naturally using

$$E^2 = (mc^2)^2 + (pc)^2$$

mass m is included in this tensor, where p is momentum, c is the speed of light and E is energy.

There exists something called the strong energy condition which states that if v is timelike then

$$T_{ab}v^a v^b \geq \frac{1}{2}\sum_{k=1}^{4} T_{ii} g_{ab} v^a v^b = \frac{1}{2} T g_{ab} v^a v^b$$

We consider now $\Lambda = 0$ and define $S = \sum_{k=1}^{4} R_{kk}$, and through some symbol pushing done in [3] on page 621 we find the Einstein equations are equivalent to

$$R_{ij} = \kappa(T_{ij} - \frac{1}{2}Tg_{ij})$$

so

$$T_{ij}v^i v^j - \frac{1}{2}Tg_{ij}v^i v^j \geq 0 \Leftrightarrow R_{ij}v^i v^j \geq 0$$

and thus the strong energy condition is equivalent to

$$R_{ij}v^i v^j \geq 0$$

for timelike v. This has many implications on the spacetime manifold, as discussed in the introduction to [6].

Let M be a Riemannian manifold, recall that if

$$L(t, v) : [0, 1] \times TM$$

is nice enough, say continuously differentiable in each variable then in local coordinates around t, v we may view

$$L(t, v) = \tilde{L}(t, p, \tilde{v})$$

This function is called a lagranian and in local coordinates is exactly a lagrangian in the usual sense. Let AC_M denote the set of absolutely continuous curves $\gamma : [0, 1] \to M$, then in particular such curves are differentiable almost every with respect to the lebesgue measure on $\mathbb{R}$. This implies that the functional given by

$$F(\gamma) = \int_0^1 L(t, \dot{\gamma}(t))dt = \int_0^1 \tilde{L}(t, \gamma(t), \gamma'(t))dt$$

is a lagrangian functional, where $\dot{\gamma}(t) = (\gamma(t), \gamma'(t)) \in T_{\gamma(t)}$. We then conclude by the principle of least action that particles travel along paths γ if and only if

$$0 = \delta F(\gamma)$$

or written out more clearly: for every $\alpha \in C_0^\infty([0, 1], M)$

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} F(\gamma + \varepsilon\alpha) = 0$$

This gives the Euler-Lagrange equations in local coordinates $(x_1, ..., x_n)$ by

$$\frac{d}{dt}(\frac{d}{dv_i}|_{\dot{\gamma}(t)}\tilde{L}(t, x, v)) = \frac{d}{dx_i}|_{\dot{\gamma}(t)}\tilde{L}(t, x, v)$$

where $v_i = \frac{d}{dx_i}$.

We note that given a particle trajectory $\gamma \in AC_M$ that moves at a speed slower than light almost everywhere, as can be expected in most reasonable systems, we have the natural lagrangian

$$L(\gamma) = \int_0^1 \sqrt{-g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt$$

and as discussed in [2] we find that

$$0 = \delta L(\gamma)$$

if and only if

$$\gamma$$

is a causal geodesic.

As explained earlier the deformations of geodesics are measured by the Riemann curvature, and to a lesser extent the Ricci curvature. This results in the idea that the Einstein equations govern the trajectories of particles in Spacetime.

4 Optimal Transport

The authors in [1] found a way to create a novel approach to the formulation of the Einstein equations using optimal transport techniques.

First of all, recall from [4] that for some c-optimal coupling π of μ_0, μ_1 if we have a probability measure $\prod$ on the set $AC([0,1], M)$ of absolutely continuous curves it is an optimal dynamical transference plan from μ_0 to μ_1 if

$$\pi = (p_0, p_1)_{\#} \prod$$

is optimal, where $p_0(\gamma) = \gamma(0)$ and $p_t, 0 \leq t \leq 1$ is defined similarly.

The main idea behind the article [1] relies on a type of optimal dynamical transference plan which has a specific form. The inspiration behind it was the characterization of Ricci upper bounds on Riemannian manifolds using entropy and the Bakry-$\grave{E}$mery tensor or variations thereof.

In the current case they define a lagrangian

$$L_p(t, x, v) = L_p(x, v) = \begin{cases} -\frac{1}{p}(-g_x(v, v))^{p/2} & v \in T_pM \text{ causal} \\ \infty & otherwise \end{cases}$$

defining a lagrangian functional through

$$F_p(\gamma) = \int_0^1 L_p(\dot{\gamma}(t))dt$$

Which as done in [1] gives a cost by

$$c_p(x, y) = inf\{F_p(\gamma) : \gamma \in AC_M, \gamma(0) = x, \gamma(1) = y\}$$

for $p \in (0, 1)$

Specifically, they define the function in Lemma 4.2 :

$$G(s,t) = \begin{cases} s(1-t) & s \in [0,t] \\ t(1-s) & s \in [t,1] \end{cases}$$

and if F denotes a bilinear form $TM \otimes TM$, then the condition $Ric \geq F$ as quadratic forms is equivalent to the almost convexity of the entropy:

$$Ent((p_t)_\# \prod |dvol) \leq (1-t)Ent((p_1)_\# \prod |dvol) + tEnt((p_1)_\# \prod |dvol)$$

$$- \int_{AC([0,1],M)} \int G(s,t)F(\dot{\gamma}(s), \dot{\gamma}(s))dsd \prod$$

for every optimal dynamical transference plan $\prod$ with respect to c_p. This is similar to a condition stated in [5] and proven in [6] which also involves the function $e_t = Ent((p_t)_\# \prod |dvol)$ being semiconvex and satisfying

$$e''(t) - \frac{1}{n}e'(t)^2 \geq K \int_{M \times M} \tau(x,y)^2 \pi(dxdy) = Kd(x,y)^2 \tag{1}$$

in the distribution sense, where τ is called a time-separation function and is roughly equivalent to timelike distance through the formula above, and using a different dynamical optimal transference plan.

Now, the difference between the two conditions lies in what they actually say. The first says that $Ric \geq F$ as quadratic forms whereas the second says that

$$Ric \geq -Kg$$

as quadratic forms. Thus we see that they are equivalent if $F = -Kg$. We also find that the strong energy condition is equivalent to (1) as proven in [6].

Now, if we recall the definition of $CD^e(K,N)$ this allows for a generalization to $TCD^e_p(K,N)$, or timelike entropic curvature dimension condition, which is essentially formula (1). Thus we find that

$$TCD^e_p(K,n) \Leftrightarrow Ric \geq -Kg$$

Now, more along the lines of the work in [1] we want to characterize $Ric \leq F$, and it comes as no surprise that this is related to a form of concavity of the entropy:

$$\frac{4}{r^2}[e(1) + e(0) - 2e(1/2)] \leq F(v,v) + f(r)$$

where e is along some dynamical optimal tranference plan and $f(r) > 0, Lim_{r \to 0} f(r) = 0$. We note that in the case $F = -Kg$ it reduces to

$$Ric(v,v) \leq -Kg(v,v)$$

for v timelike.

5 Importance

While not providing new methods for solving the equations or understanding the solutions geometrically this paper sheds light on a new perspective for general relativity and mathematical physics in general. Specifically as we noted the entropy functional arises naturally out of thermodynamics and information theory, and so this article illustrates the surprising connection between statistical mechanics and gravitation.

Later on in the article [1] the authors present their perspectives on the idea of a sequence of Lorentzian manifolds, and how the standard theory interacts and compares with the Optimal Transport/entropy-based method. Specifically, it allows for greater freedom in the underlying topology as it doesn't require a fixed background manifold. Furthermore, we don't even need the underlying space to be a smooth manifold, similar to how entropic Ricci bounds allowed for greater freedom in the underlying space. This is most apparent through the use of the timelike distance τ.

Furthermore, this allows for a generalization of curvature dimension conditions to be adapted over to the Lorentzian case. In [5] the authors recoved several geometric inequalities as well as the Hawking singularity theorem on certain $TCD_p^e(K, N)$ spaces, specifically the non-branching ones (i.e. have no timelike branching geodesics).

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