1. In this problem we will look at differential operators. Suppose that $P : C^\infty(U) \to C^\infty(U)$ is a differential operator(i.e. $m = 1$).

   (a) Let $\xi \in \mathbb{R}^n$, and consider the isomorphism $T^*U = U \times \mathbb{R}^n$, where $dx^i \leftrightarrow e_i$, where e_i has a 1 in the i-th position and zeroes elsewhere. Then for any $p \in U$ there exists $g \in C_c^\infty(U)$ such that $dg(p) = \xi$.

   (b) Show that if P has order 2k, then

   $$sym_{2k}(P)(x,\xi) = \lim_{t\to\infty} t^{-2k} e^{-tg} P(e^{tg})$$

   where $g \in C_c^\infty(U)$ is such that $dg(x) = \xi$

   (c) (Harder) If M is a manifold then a scalar differential operator P of order N is an operator $P : C_c^\infty(M) \to C_c^\infty(M)$ such that if (U,ψ) is any coordinate chart, then

   $$\tilde{P}(f)(x) = P(f \circ \psi)(\psi^{-1}(x))$$

   is a differential operator $C^\infty(\psi(U)) \to C^\infty(\psi(U))$ on order N.
   Show that if $x \in U, \xi \in T_x^*M$, P has order 2k, and we define

   $$sym_{2k}(P)(x,\xi) = sym_{2k}(\tilde{P})(\psi(x), D(\psi^{-1})^*_{\psi(x)}\xi)$$

   where $D(\psi^{-1})^*_{\psi(x)}$ is the pullback, then $sym_{2k}(P) : T^*M \to \mathbb{R}$ is well-defined independent of the chart picked. (Hint: use part b))

   (d) (Much harder) If M is a manifold with vector bundles E, F then differential operator P of order N(with no restrictions on m) from E to F is an operator $P : \Gamma(E) \to \Gamma(F)$ between smooth sections, such that in local coordinates and local trivializations it is a differential operator of order N.

      i. Make this precise

1. In this problem we will develop the theory of curvature using "classical" PDE methods. The first part will involve some reading, while the second will be the actual exercise.

   (a) Read Rosenberg's *The Laplacian on a Riemannian Manifold* Chapter 2 Theorem 2.10(Login to cambridgecore through UofT)

   (b) Prove the converse. Hint: You may need three things,

      i. The inverse function theorem

      ii. On an open ball, every closed form is exact

      iii. Frobenius' theorem