- 1. In this problem we will look at differential operators. Suppose that $P: C^{\infty}(U) \to C^{\infty}(U)$ is a differential operator(i.e. m=1).
 - (a) Let $\xi \in \mathbb{R}^n$, and consider the isomorphism $T^*U = U \times \mathbb{R}^n$, where $dx^i \leftrightarrow e_i$, where e_i has a 1 in the i-th position and zeroes elsewhere. Then for any $p \in U$ there exists $g \in C_c^{\infty}(U)$ such that $dg(p) = \xi$.
 - (b) Show that if P has order 2k, then

$$sym_{2k}(P)(x,\xi) = \lim_{t \to \infty} t^{-2k} e^{-tg} P(e^{tg})$$

where $g \in C_c^{\infty}(U)$ is such that $dg(x) = \xi$

(c) (Harder) If M is a manifold then a scalar differential operator P of order N is an operator $P: C^{\infty}(M) \to C^{\infty}(M)$ such that if (U, ψ) is any coordinate chart, then

$$\tilde{P}(f)(x) = P(f \circ \psi)(\psi^{-1}(x))$$

is a differential operator $C^{\infty}(\psi(U)) \to C^{\infty}(\psi(U))$ on order N. Show that if $x \in U, \xi \in T_x^*M$, P has order 2k, and we define

$$sym_{2k}(P)(x,\xi) = sym_{2k}(\tilde{P})(\psi(x), D(\psi^{-1})^*_{\psi(x)}\xi)$$

where $D(\psi^{-1})^*_{\psi(x)}$ is the pullback, then $sym_{2k}(P): T^*M \to \mathbb{R}$ is well-defined independent of the chart picked. (Hint: use part b))

- (d) (Much harder) If M is a manifold with vector bundles E, F then differential operator P of order N(with no restrictions on m) from E to F is an operator $P: \Gamma(E) \to \Gamma(F)$ between smooth sections, such that in local coordinates and local trivializations it is a differential operator of order N.
 - i. Make this precise

- 1. In this problem we will develop the theory of curvature using "classical" PDE methods. The first part will involve some reading, while the second will be the actual exercise.
 - (a) Read Rosenberg's The Laplacian on a Riemannian Manifold Chapter 2 Theorem 2.10(Login to cambridge core through UofT)
 - (b) Prove the converse. Hint: You may need three things,
 - i. The inverse function theorem
 - ii. On an open ball, every closed form is exact
 - iii. Frobenius' theorem