

IDEAL RIGHT-ANGLED POLYHEDRA IN LOBACHEVSKY SPACE

A.YU. VESNIN, A.A. EGOROV

ABSTRACT. In this paper we consider a class of right-angled polyhedra in three-dimensional Lobachevsky space, all vertices of which lie on the absolute. New upper bounds on volumes in terms the number of faces of the polyhedron are obtained. Volumes of polyhedra with at most 23 faces are computed. It is shown that the minimum volumes are realized on antiprisms and twisted antiprisms. The first 248 values of volumes of ideal right-angled polyhedra are presented. Moreover, the class of polyhedra with isolated triangles is introduced and there are obtained combinatorial bounds on their existence as well as minimal examples of such polyhedra are given.

INTRODUCTION

Applying of computer methods is a powerful tool for the study of three-dimensional hyperbolic manifolds. For example, the tabulation of manifolds obtained by Dehn surgery on manifolds with cusps led by S. V. Matveev and A. T. Fomenko [13] and independently J. Weeks [24] to recognizing the smallest volume closed orientable three-dimensional hyperbolic manifold. Today it is known as the *Weeks – Matveev – Fomenko manifold*. Recall that it can be obtained by surgery on the Whitehead link and its volume is approximately equal to 0.942707.

In recent years many results appeared on the enumeration and classification of three-dimensional hyperbolic manifolds which admit decompositions into polyhedra with prescribed properties.

In [8] there are described hyperbolic three-dimensional manifolds that can be decomposed into regular ideal tetrahedra (up to 25 tetrahedra in the oriented case and up to 22 tetrahedrons in the non-oriented case). Three-dimensional hyperbolic manifolds that can be subdivided into Platonic polyhedra are listed in [9]. In [10] all three-dimensional orientable manifolds that can be obtained from various realizations of an octahedron were constructed and classified. The paper [11] contains an initial list of 825 bounded right-angled hyperbolic polyhedra.

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In this paper, the objects of our study are polyhedra which can be realized with right, $\pi/2$, dihedral angles in a three-dimensional space of constant negative curvature \mathbb{H}^3 , known as hyperbolic space or Lobachevsky space. Moreover, we will consider only ideal right-angled hyperbolic polyhedra, that is, those for which all vertices lie on the absolute of Lobachevsky space. We will denote by \mathcal{IR} the class of ideal right-angled three-dimensional hyperbolic polyhedra. Recent results on the theory of right-angled polyhedra in Lobachevsky space and using them for constructing three-dimensional hyperbolic manifolds are given in the survey [22], see also [7]. The main attention in the survey was given to bounded right-angled polyhedra, while in this paper we will consider the case of ideal polyhedra. We will follow the standard terminology of the theory of hyperbolic manifolds; see, for example, [17].

Hyperbolic three-dimensional manifolds of finite volume, which can be decomposed into ideal right-angled polyhedra, have been intensively studied in last decade. In particular, due to their close relationship with the right-angled Coxeter groups, and the fact that their fundamental groups have the LERF property, i.e. they are locally extended residually finite groups (each finitely generated subgroup is separable) [19]. Several types of hyperbolic three-dimensional manifolds admitting decomposition into ideal right-angled polyhedra are presented in [4]. Since the volume of a manifold is the sum of the volumes of the polyhedra into which it is decomposed, a description of the volumes of ideal right-angled polyhedra is interesting from this point of view.

The paper has the following structure. In Section 1 we recall some facts about the existence of ideal polyhedra in Lobachevsky spaces, in particular, Andreev's theorem (Theorem 1.1) and Rivin's theorem (Theorem 1.2), which give necessary and sufficient conditions for their existence in dimension three. In Section 2 the notion of twisted antiprism is introduced and a formula for volumes of right-angled twisted antiprisms is given (Theorem 2.5). The Table 2 provides information on the number of ideal right-angled polyhedra in Lobachevsky space with at most 23 faces and indicates the minimum and maximum volume values for each number of faces. The carried out calculations allow to propose a conjecture which polyhedra are of smallest volumes for an arbitrary number of faces (Conjecture 2.1). In Section 3 we obtain new upper bounds on the volume of an ideal right-angled polyhedron in terms of the number of its faces (Theorems 3.2 and 3.3). Also, we present ideal right-angled polyhedra of smallest and largest volume with at most 23 faces (see Tables 4 and 5) and the first 248 values of volumes of ideal right-angled polyhedra (see Table 6). In Section 4 we introduce the notion of polyhedra with isolated triangles and give a lower bound on the number of faces of such polyhedra (Proposition 4.1). For the minimum possible number of faces equals to 26, two examples of polyhedra with isolated triangles are given.

1. EXISTENCE

1.1. Dimension and number of cusps. It was shown in [6] that right-angled polyhedra of finite volume can exist in \mathbb{H}^n only for dimensions $n < 13$. Such polyhedra can have both finite vertices and cusps. At the same time [23], in dimensions $n > 4$ there are no compact right-angled polyhedra. Thus, a right-angled polyhedron in \mathbb{H}^n , $n \geq 5$, has at least one cusp. It was shown in [15] that in high dimensions right-angled polyhedra of finite volume should have a lot of cusps. Namely, the lower bounds $c(n)$ of the number of cusps for dimensions $n < 13$ are given in Table 1.

TABLE 1. Lower bounds for the number of cusps.

n	6	7	8	9	10	11	12
c(n)	3	17	36	91	254	741	2200

We will be interested in right-angled hyperbolic polyhedra in which all vertices are cusps. Such polyhedra are called *ideal*. It was shown in [12] that in \mathbb{H}^n , $n \geq 7$, there are no ideal right-angled polyhedra. Examples of three-dimensional and four-dimensional ideal right-angled hyperbolic polyhedra will be given below.

1.2. Three-dimensional case. Necessary and sufficient conditions for a combinatorial polyhedron P to belong to the class \mathcal{IR} can be obtained as a very special case of E. M. Andreev's theorem [1] on acute-angled polyhedra of finite volume.

Theorem 1.1. [1] *Let P be an abstract three-dimensional polyhedron with three or four faces meeting at each vertex, and P is not a simplex. The following conditions are necessary and sufficient conditions for the existence in \mathbb{H}^3 of a convex polyhedron of finite volume of a combinatorial type P with angles $\alpha_{ij} \leq \pi/2$:*

0. $0 < \alpha_{ij} \leq \pi/2$.
1. If F_{ijk} is a vertex of P , then $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} \geq \pi$, if F_{ijkl} is a vertex, then $\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} = 2\pi$.
2. If F_i, F_j, F_k is a triangular prismatic element, then $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi$.
3. If F_i, F_j, F_k, F_l is a quadrilateral prismatic element, then $\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 2\pi$.
4. If P is a triangular prism with bases F_1 and F_2 , then $\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{23} + \alpha_{24} + \alpha_{25} < 3\pi$.
5. If among the faces F_i, F_j, F_k there are adjacent F_i and F_j , F_j and F_k , but F_i and F_k are not adjacent, but meet at a common vertex and all three faces don't meet at one vertex, then $\alpha_{ij} + \alpha_{jk} < \pi$.

For the case of an ideal right-angled polyhedron the conditions are significantly simplified since all vertices are four-valent and all dihedral angles are $\pi/2$.

We also recall a result of I. Rivin [18], concerning arbitrary convex ideal hyperbolic polyhedra, which is formulated in terms of the dual graph.

Theorem 1.2. [18] *Let P be a plane polyhedral graph with a weight $w(e)$ assigned to each edge e . Let P^* be the dual graph of P and assume that the weight $w^*(e^*) = \pi - w(e)$ is assigned to the edge e^* dual to e . Then P can be realized as a convex ideal polyhedron in \mathbb{H}^3 with dihedral angles $w(e)$ at its edges if and only if the following conditions are satisfied:*

- (1) $0 < w^*(e^*) < \pi$ for all e ;
- (2) if edges $e_1^*, e_2^*, \dots, e_k^*$ bound a face in P^* , then

$$w^*(e_1^*) + w^*(e_2^*) + \dots + w^*(e_k^*) = 2\pi;$$

- (3) if edges $e_1^*, e_2^*, \dots, e_k^*$ form a cycle in P^* , that does not bound a face, then

$$w^*(e_1^*) + w^*(e_2^*) + \dots + w^*(e_k^*) > 2\pi.$$

Ideal hyperbolic polyhedra are also interesting from the point of views of Euclidean geometry, since they are exactly those polyhedra that can be inscribed into the ball. This correspondence and the Rivin's theorem made it possible to solve the problem of Jacob Steiner on describing a sphere around a polyhedron.

In the case of a right-angled polyhedron, the weights of edges and the weights of dual edges in the Rivin's theorem are $\pi/2$ and all faces of P^* are quadrilateral. Accordingly, all vertices of the polyhedron $P \in \mathcal{IR}$ are 4-valent.

It is easy to see that from the classical Euler formula for a polyhedron, $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces of a polyhedron P , and from the fact that each vertex of an ideal right-angled polyhedron is incident to exactly four edges, it follows that $2E = 4V$. Thus, the relation $F = V + 2$ holds. Denote by p_k the number of k -gonal faces ($k \geq 3$) of a polyhedron of class \mathcal{IR} . Then

$$(1) \quad p_3 = 8 + \sum_{k \geq 5} p_k(k - 4).$$

Thus, each polyhedron from \mathcal{IR} has at least 8 triangular faces. It is easy to see that the octahedron shown in the Figure 1 has the minimum number of faces among the polyhedra in \mathcal{IR} . At the same time, as we will see below, the octahedron is also minimal in volume among all ideal right-angled polyhedra.

1.3. Four-dimensional case. Recall [5, Table I] that the only regular four-dimensional polyhedron for which each vertex has a type of the cone over a cube is a 24-cell. In particular, it is realized as an ideal right-angled polyhedron in \mathbb{H}^4 .

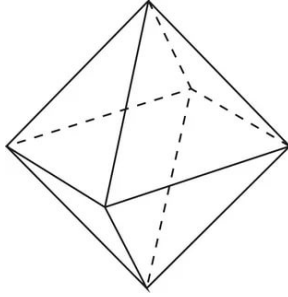


FIGURE 1. The octahedron.

Let $P \subset \mathbb{H}^4$ be an ideal right-angled polyhedron with the face vector $f(P) = (f_0, f_1, f_2, f_3)$. Since P is a convex four-dimensional polyhedron, then its surface ∂P is homeomorphic to S^3 , that means that its Euler characteristic turns to zero. Thus, $f_0 - f_1 + f_2 - f_3 = 0$. The volume of the polyhedron can be expressed in terms of the components of the face vector, namely, the following statement holds.

Lemma 1.1. [12] *Let $P \subset \mathbb{H}^3$ be an ideal right-angled polyhedron with a face vector $f(P) = (f_0, f_1, f_2, f_3)$. Then its volume is equal to*

$$\text{vol } P = \frac{f_0 - f_3 + 4}{3} \pi^2.$$

Since for a 24-cell, the face vector has the form $(24, 96, 96, 24)$, its volume is $4\pi^2/3$. It is shown in [12] that the 24-cell is the unique smallest volume ideal right-angled polyhedron in \mathbb{H}^4 .

2. ENUMERATION OF POLYHEDRA AND THEIR VOLUMES

2.1. Antiprisms and the edge-twist operation. Considering the octahedron as a triangular antiprism, that is, a polyhedron with triangular top and bottom and with a lateral surface formed by two levels of triangles, it can be naturally generalized to the next infinite family of polyhedra. By an n -antiprism $A(n)$, $n \geq 3$, we mean $(2n + 2)$ -hedron with n -gonal top and bottom and with a lateral surface formed by two levels of n triangles in each; and with each vertex incident to four edges. Schlegel diagrams of polyhedra $A(3)$ and $A(4)$ are presented in Figure 3.

By checking the conditions of Andreev's theorem or Rivin's theorem, it is easy to see that the antiprisms $A(n)$, $n \geq 3$, can be realized as ideal right-angled polyhedra in \mathbb{H}^3 .

A. Kolpakov demonstrated in [12] that the polyhedra $A(n)$ are minimal in the following sense

Theorem 2.1. [12] *For $n \geq 3$ the antiprism $A(n)$ has the smallest number of faces, equal to $2n + 2$, among all the ideal right-angled hyperbolic polyhedra with at least one n -gonal face.*

As will be clear below, antiprisms play an important role in understanding the structure of the set \mathcal{IR} of all ideal right-angled hyperbolic polyhedra.

Theorem 1.2 admits to characterize the class \mathcal{IR} in terms of the dual graphs of the polyhedral graphs. The dual graph must be a quadrangulation of the sphere, that is, a finite graphs with quadrilateral faces on a 2-sphere. Furthermore, the dual graph cannot contain a cycle of length four that separates two faces. The class of graphs with these properties was considered in [3], where it was denoted by \mathcal{Q}_4 . Using the results of [3] on graphs from the class \mathcal{Q}_4 and passing from dual graphs to the original graphs, one can state the following result.

Theorem 2.2. [3] *The class of 4-valent 3-connected and cyclically 6-connected planar graphs is generated by 1-skeletons of antiprisms $A(n)$ and by edge-twist moves.*

Recall that a graph is said to be *cyclically k -connected* if k is the smallest number of edges such that removing them decomposes the graph into two components each of which contains a cycle.

The *edge-twist* move is defined as follows. Let $P \in \mathcal{IR}$ and assume that some face of the polyhedron has four distinct ideal vertices that are pairwise connected by edges e_1 and e_2 as in Figure 2.

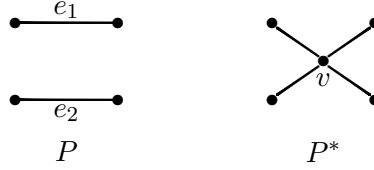


FIGURE 2. An edge-twist move.

Then the transformation involves removing e_1 and e_2 , creating a new vertex v and connecting it by edges with the above four vertices. We denote the resulting polyhedron by P^* and say that P^* is obtained from P by an edge-twist move. For example, $A(4)^*$ in Figure 3 is obtained from $A(4)$ by the edge-twist move. The edges e_1 , e_2 and the new vertex v are indicated in the figure.

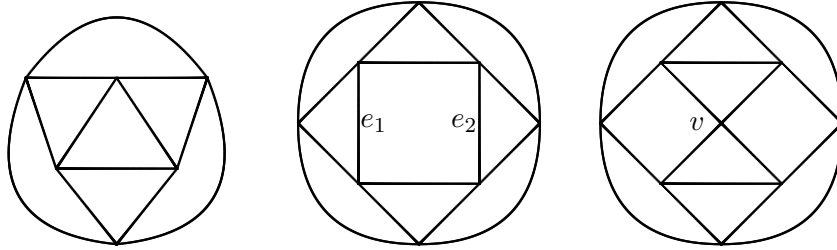


FIGURE 3. Polyhedra $A(3)$, $A(4)$ and $A(4)^*$.

Theorems 1.2 and 2.2 lead to the following result.

Theorem 2.3. *Each ideal right-angled hyperbolic polyhedron is an antiprism or can be obtained from some antiprism by a finite number of edge-twist moves.*

Let us introduce a class of polyhedra obtained from antiprisms. Let $A(n)$, $n \geq 4$, be an antiprism, and e_1 and e_2 be two edges that belong to one of n -gonal faces, such that there is a third edge on the same face to which they are both adjacent, let us denote it by e_3 . In other words, e_1 and e_2 are adjacent through an edge. We apply the edge-twist move to the edges e_1 and e_2 . As one can see from Figure 4, illustrating the case of the antiprism $A(6)$, when we apply edge-twist move to the edges adjacent through an edge, the combinatorial structure changes as follows. The antiprism $A(n)$ had $2n + 2$ faces: two n -gonal faces and $2n$ triangular faces. The new polyhedron $A(n)^*$ has $2n + 3$ faces: one n -gonal face, one $(n - 1)$ -gonal face, two quadrilateral faces and $(2n - 1)$ triangular faces. The polyhedron $A(n)^*$ will be referred to as a *twisted antiprism*. Observe that the number of faces of the antiprism is always even, but the number of faces of the twisted antiprism is always odd. As well as the antiprism, by virtue of Andreev's or Rivin's theorems, any twisted antiprism can be realized as an ideal right-angled polyhedron in \mathbb{H}^3 .

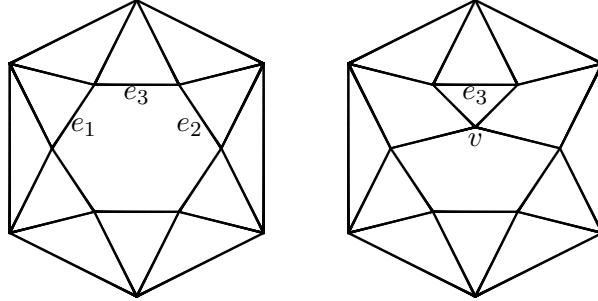


FIGURE 4. Polyhedra $A(6)$ and $A(6)^*$.

Denote by v_8 the volume of an ideal right-angled hyperbolic octahedron. The numerical value of this quantity will be given in the next section.

Lemma 2.1. *Let $A(n)$, $n \geq 4$, be an ideal right-angled antiprism in \mathbb{H}^3 having $2n + 2$ faces and $A(n)^*$ be an ideal right-angled twisted antiprism obtained from $A(n)$ and having $2n + 3$ faces. Then for the volume of the twisted antiprism the following equality holds:*

$$\text{vol}(A(n)^*) = \text{vol}(A(n - 1)) + v_8.$$

Proof. Since the edge-twist move of edges adjacent through one edge is a local transformation of a polyhedron, we will illustrate the proof for the case $n = 6$. For an arbitrary $n \geq 4$, the proof is analogous.

Let us consider the antiprism $A(n - 1)$. The left side of Figure 5 presents the antiprism $A(5)$. Put an ideal right-angled octahedron on one of its faces

to the triangular face ABC . Since the triangular faces of the antiprism and the triangular faces of the octahedron are ideal triangles, these faces are pairwise isometric. The remaining seven faces of the octahedron are drawn inside the triangle ABC in the middle of Figure 5. Recall that the dihedral angles at the edges of the antiprism and at the edges of the octahedron are equal to $\pi/2$. Therefore, when we combine the antiprism and the octahedron along the triangular face ABC in the resulting polyhedron, the angles at the edges AB , BC and AC will be equal to π and the corresponding faces will belong to the same plane in pairs. Thus the polyhedron obtained by combining the antiprism and the octahedron will have a combinatorial structure as in the right part of Figure 5, where the edges AB , BC and AC are absent. It is easy to see that the resulting polyhedron coincides

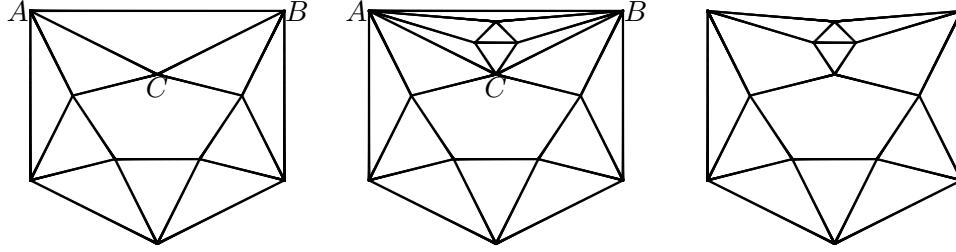


FIGURE 5. $A(5)$ and its union with an octahedron.

combinatorially with the polyhedron $A(6)^*$ and is also right-angled. \square

2.2. The Lobachevsky function. Traditionally, volumes of polyhedra in three-dimensional hyperbolic space are computed in terms of function

$$\Lambda(\theta) = - \int_0^\theta \log |2 \sin(t)| dt,$$

which J. Milnor introduced in the survey [14] and called it the *Lobachevsky function*. He demonstrated that the volume of an ideal tetrahedron $T(\alpha, \beta, \gamma)$ in a three-dimensional hyperbolic space with dihedral angles α , β and γ (meaning that at one of the vertices dihedral angles are α , β , γ , where $\alpha + \beta + \gamma = \pi$, and dihedral angles at the opposite edges of the tetrahedron coincide) is calculated by the formula:

$$\text{vol}(T(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Splitting an ideal right-angled octahedron into four ideal tetrahedra with dihedral angles $\alpha = \pi/2$, $\beta = \pi/4$, $\gamma = \pi/4$, and using $\Lambda(\pi/2) = 0$, we obtain

$$v_8 = 8\Lambda\left(\frac{\pi}{4}\right) = 3.663862376708876 \dots$$

A formula for the volume of a right-angled antiprism was presented by W. Thurston in his well-known lectures [21, Chapter 6.8], where the antiprism was called by a drum with triangular sides, and its volume was used to calculate the volume of the complement to some link in a three-dimensional sphere.

Theorem 2.4. [21] *For $n \geq 3$ the volume of a right-angled n -antiprism is given by*

$$\text{vol}(A(n)) = 2n \left[\Lambda \left(\frac{\pi}{4} + \frac{\pi}{2n} \right) + \Lambda \left(\frac{\pi}{4} - \frac{\pi}{2n} \right) \right],$$

where $\Lambda(x)$ is the Lobachevsky function.

Theorem 2.4 implies that $\text{vol}(A(n))$ is asymptotically equivalent to $\frac{v_8}{2}n$ when $n \rightarrow \infty$. In particular, the above formula gives the volume of an ideal right-angled octahedron $A(3)$:

$$\text{vol}(A(3)) = 8\Lambda(\pi/4) = v_8,$$

where we used properties of the Lobachevsky function [17].

Theorem 2.5. *For the volume of the twisted antiprism $A(n)^*$, $n \geq 4$, the following formula holds:*

$$\text{vol}(A(n)^*) = 2(n-1) \left[\Lambda \left(\frac{\pi}{4} + \frac{\pi}{2(n-1)} \right) + \Lambda \left(\frac{\pi}{4} - \frac{\pi}{2(n-1)} \right) \right] + 8\Lambda \left(\frac{\pi}{4} \right).$$

Proof. It follows from Lemma 2.1 and Theorem 2.4. \square

2.3. Volumes of polyhedra with at most 23 faces. As we noted above, the Euler's formula implies that any ideal right-angled polyhedron has at least 8 triangular faces.

For each $n = 8, 10, 11, \dots, 23$ Table 2 gives the number of ideal right-angled polyhedra in \mathbb{H}^3 with n faces; the number of different volumes of them; the minimum and maximum volume values.

Combinatorial enumeration of polyhedra was done with using the computer program **plantri** [16]. Volumes of hyperbolic polyhedra were calculated with the applying some modification of the computer program **SnapPea** [20]. Values of volumes are given up to 10^{-6} .

Denote by P_n^{\min} and P_n^{\max} polyhedra which realize minimal and maximal values of volumes in the class of all polyhedra with n faces. The first seven polyhedra with smallest volume are given in Table 3.

The volume calculations showed that the following fact holds for $n \leq 23$. If n is even, then the smallest volume is achieved on the antiprism, that is, $P_n^{\min} = A(k)$, where $n = 2k + 2$. If n is odd, then the smallest volume is achieved on the twisted antiprism, that is, $P_n^{\min} = A(k)^*$, where $n = 2k + 3$. We formulate this observation in the following form.

Proposition 2.1. *For ideal right-angled hyperbolic polyhedra with at most 23 faces, the minimum value of volumes is achieved on antiprisms and twisted antiprisms. The minimum and maximum volumes are presented in Tables 2, 4 and 5.*

TABLE 2. Ideal right-angled polyhedra.

# of faces	# of polyhedra	# of volumes	min volume	max volume
8	1	1	3.663863	3.663863
9	0	0	-	-
10	1	1	6.023046	6.023046
11	1	1	7.327725	7.327725
12	2	2	8.137885	8.612415
13	2	2	9.686908	10.149416
14	9	7	10.149416	12.046092
15	11	7	11.801747	13.350771
16	37	17	12.106298	14.832681
17	79	31	13.813278	16.331571
18	249	79	14.030461	18.069138
19	671	172	15.770160	19.523353
20	2182	495	15.933385	21.241543
21	6692	1359	17.694323	22.894415
22	22131	4276	17.821704	24.599233
23	72405	13031	19.597248	26.228126

Conjecture 2.1. *If $n \geq 8$ is even, then the minimum volume is achieved on the antiprism, that is, $P_n^{\min} = A(k)$, where $n = 2k + 2$. If $n \geq 11$ is odd, then the minimum volume is achieved on the twisted antiprism, that is, $P_n^{\min} = A(k)^*$, where $n = 2k + 3$.*

The statement 2.1 confirms the conjecture for $n \leq 23$.

3. UPPER AND LOWER VOLUME BOUNDS

Bilateral bounds for the volumes of ideal right-angled polyhedra in terms of the number of vertices were obtained by K. Atkinson in [2].

Theorem 3.1. [2] *Let P be an ideal right-angled polyhedron with N vertices, then*

$$(N - 2) \cdot \frac{v_8}{4} \leq \text{vol}(P) \leq (N - 4) \cdot \frac{v_8}{2}.$$

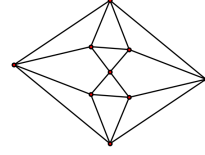
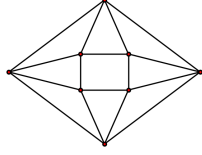
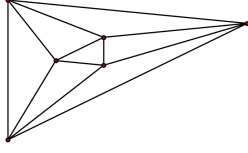
Both inequalities became equalities when P is the regular ideal hyperbolic octahedron. Moreover, there exists a sequence of ideal right-angled polyhedra P_i with N_i vertices such that $\text{vol}(P_i)/N_i$ tends to $v_8/2$ as $i \rightarrow \infty$.

Figure 6 shows the graphs of the upper and lower bounds from Theorem 3.1, the set of volume values of ideal right-angled polyhedra with at most 23 faces, where volumes of antiprisms and twisted antiprisms are separately highlighted.

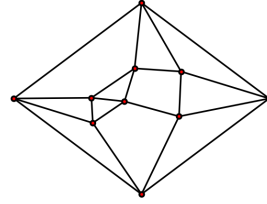
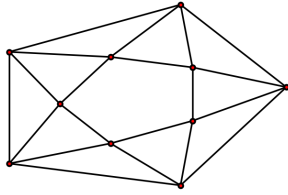
The upper bound in Theorem 3.1 can be improved as follows.

Theorem 3.2. *Let P be an ideal right-angled hyperbolic polyhedron with N vertices, different from the octahedron. Let F_1 and F_2 be two faces of P such*

TABLE 3. The first seven ideal right-angled polyhedra

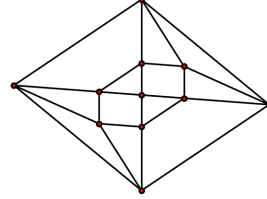
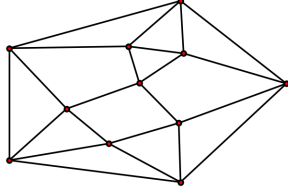


$$(1) \text{ vol}(P_8) = 3.663863 \quad (2) \text{ vol}(P_{10}) = 6.023046 \quad (3) \text{ vol}(P_{11}) = 7.327725$$



$$(4) \text{ vol}(P_{12}^{\min}) = 8.137885$$

$$(5) \text{ vol}(P_{12}^{\max}) = 8.612415$$



$$(6) \text{ vol}(P_{13}^{\min}) = 9.686908$$

$$(7) \text{ vol}(P_{13}^{\max}) = 10.149416$$

that F_1 is n_1 -gon, and F_2 is n_2 -gon, where $n_1, n_2 \geq 4$. Then for its volume the following upper bound holds:

$$\text{vol}(P) \leq \left(N - \frac{n_1}{2} - \frac{n_2}{2}\right) \cdot \frac{v_8}{2}.$$

Proof. As follows from the Euler formula for polyhedra, if an ideal right-angled polyhedron P is different from an octahedron, then it would have two faces that have at least four sides. For definiteness, we denote these faces by F_1 and F_2 . We consider two cases according to whether the faces F_1, F_2 are adjacent or not.

(1) Let the faces F_1 and F_2 be not adjacent. We construct a family of right-angled polyhedra by induction, attaching at each step a copy of the polyhedron P . Put $P_1 = P$. Define $P_2 = P_1 \cup_{F_1} P_1$, identifying two copies of the polyhedron P_1 along the face F_1 . Obviously, P_2 is an ideal right-angled polyhedron with the number of vertices $N_2 = 2N - n_1$ and the volume $\text{vol}(P_2) = 2 \text{vol}(P)$. The polyhedron P_2 has at least one face

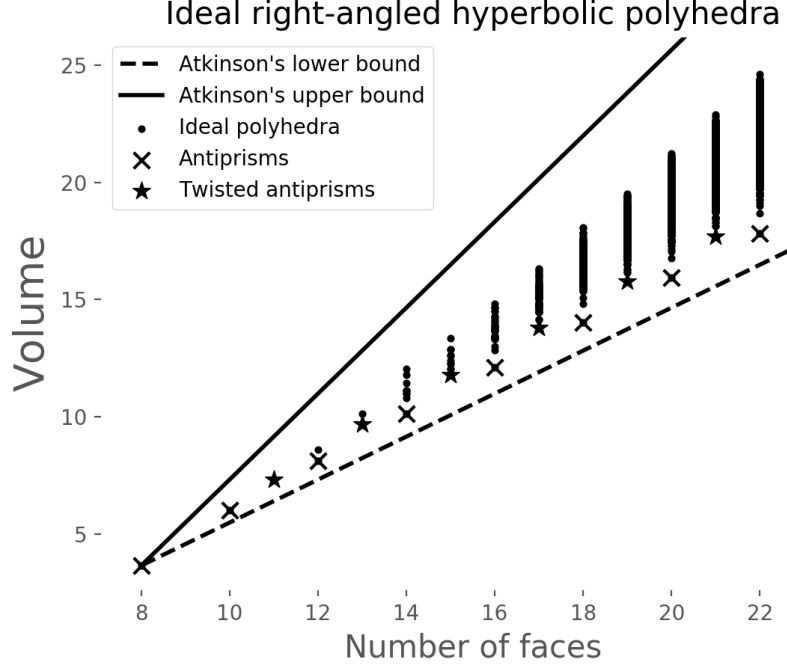


FIGURE 6. The set of volumes and Atkinson bounds.

isometric to F_2 . We attach the polyhedron P to the polyhedron P_2 along this face. We get $P_3 = P_2 \cup_{F_2} P = P \cup_{F_1} P \cup_{F_2} P$. Obviously, P_3 is an ideal right-angled polyhedron with the number of vertices $N_3 = 3N - n_1 - n_2$ and the volume $\text{vol}(P_3) = 3 \text{vol}(P)$. Continuing the process of adding the polyhedron P alternately through the faces isometric to F_1 and F_2 , we obtain the polyhedron $P_{2k+1} = P_{2k-1} \cup_{F_1} P \cup_{F_2} P$, which is an ideal right-angled polyhedron with $N_{2k+1} = (2k+1)N - k, n_1 - kn_2$ vertices and of volume $\text{vol}(P_{2k+1}) = (2k+1) \text{vol}(P)$. Now let us apply the upper bound from Theorem 3.1 to polyhedron P_{2k+1} :

$$(2k+1) \text{vol}(P) \leq ((2k+1)N - kn_1 - kn_2 - 4) \frac{v_8}{2}.$$

Dividing both sides of the inequality by $(2k+1)$ and passing to the limit as $k \rightarrow \infty$, we obtain the required inequality.

(2) Let the faces F_1 and F_2 be adjacent. Put $P_2 = P \cup_{F_1} P$. The constructed polyhedron P_2 has $N_2 = 2N - n_1$ vertices and its volume is two times the volume of the polyhedron P . By construction, the polyhedron P_2 has a face F_{21} , which is a $(2n_2 - 2)$ -gon. Since the face of F_1 has at least 4 edges, there is a face in P adjacent to F_1 , but not adjacent to F_2 . As a result of attaching P along F_1 , this face will turn into a face F_{22} in a polyhedron P_2 that has at least 4 sides. Thus, in P_2 there is a pair of non-adjacent

faces F_{21} and F_{22} , each of which has at least 4 sides. This situation corresponds to the already proved case (1). Thus, for the polyhedron P_2 and its non-adjacent faces F_{11} and F_{12} we get:

$$2 \operatorname{vol}(P) \leq \left(N_2 - \frac{(2n_2 - 2)}{2} - \frac{4}{2} \right) \frac{v_8}{2},$$

from where we receive

$$2 \operatorname{vol}(P) \leq (2N - n_1 - n_2 - 1) \frac{v_8}{2}.$$

and therefore,

$$\operatorname{vol}(P) < \left(N - \frac{n_1}{2} - \frac{n_2}{2} \right) \frac{v_8}{2}.$$

□

Theorem 3.3. *Let P be an ideal right-angled hyperbolic polyhedron with $N \geq 17$ faces and having only triangular or quadrilateral faces. Then for its volume the following upper bound holds:*

$$\operatorname{vol}(P) < (N - 5) \frac{v_8}{2}$$

Proof. Observe that in the polyhedron P there are three quadrilateral faces F_1, F_2, F_3 such that F_2 is adjacent to F_1 and F_3 both. In fact, assume that there is no such triple of faces. Then each quadrilateral face is adjacent to at most one quadrilateral face. If a quadrilateral face has no adjacent quadrilateral (we will say that it is isolated), then through its four sides it is adjacent to the triangular faces. If two quadrilaterals are adjacent to each other and none of them is adjacent to another quadrilateral (we will say that the faces form a pair), then their union is adjacent through six sides with triangular faces. Hence, if there are k_1 isolated quadrilateral faces and k_2 pairs of quadrilateral faces, then through their sides they are adjacent to triangular faces through $4n_1 + 6n_2$ sides. Since the polyhedron does not contain n -gonal faces for $n \geq 5$, it follows from Euler's formula that the number of triangles is 8. Their total number of sides is 24. If the number of faces $N \geq 17$ and eight of them are triangles, then $n_1 + 2n_2 \geq 9$ and $S = 4n_1 + 6n_2$ sides of triangles are required. Using the fact that $2n_2 \geq 9 - n_1$, we obtain $S \geq 4n_1 + 3(9 - n_1) = 27 + n_1 > 24$. This contradiction implies that there is a triple of sequentially adjacent quadrilateral faces F_1, F_2, F_3 , where F_2 is adjacent to F_1 and F_3 .

Let us consider the union $P_2 = P \cup_{F_2} P$ of two copies of P along F_2 . Then the doubled faces F_1 and F_3 of the polyhedron P will give two hexagonal faces in the polyhedron P_2 . The total number of vertices in P_2 is $2N - 4$. We apply the upper bound from Theorem 2.2 to P_2 and the indicated hexagonal faces:

$$2 \operatorname{vol}(P) < \left(2N - 4 - \frac{6}{2} - \frac{6}{2} \right) \frac{v_8}{2},$$

from where we receive

$$\operatorname{vol}(P) < (N - 5) \frac{v_8}{2},$$

this is exactly what we needed to prove. \square

The following statement describes the structure of the initial part of the set of volumes of ideal right-angled polyhedra.

Proposition 3.1. *The volume values of ideal right-angled hyperbolic polyhedra not exceeding $5v_8$ are listed in Table 6. The number of such values is 248.*

Proof. By virtue of a lower bound from Theorem 3.1, if the number of faces of the ideal right-angled polyhedron P is F (hence the number of its vertices is $F - 2$), then for its volume the lower bound holds:

$$(F - 4) \cdot \frac{v_8}{4} \leq \text{vol}(P).$$

If $F \geq 24$ then this bound is at least $5v_8$. Thus, the volume of any polyhedron with at least 24 faces is bounded below by $5v_8 = 18.319312$, where the approximate value is indicated on the right-hand side. Direct calculations of volumes of polyhedra with at most 23 faces show that the number of values of volumes not exceeding $5v_8$ is 248. All of them are listed in Table 6. \square

4. POLYHEDRA WITH ISOLATED TRIANGLES

Recall that an ideal right-angled polyhedron has at least eight triangles. In the case of an octahedron, each triangle is adjacent to three other triangles along sides. From Table 4 and 5 one can see that with the increasing the number of faces of polyhedra with maximum volume, the triangles move away from each other more and more. From the presence of common sides, the situation changes towards the presence of common vertices. The question appears when polyhedra arise in which all triangular faces are isolated, that is, no two triangular faces have common vertices. In this case we will call the polyhedron *ITR-polyhedron*, emphasizing that it satisfies the isolated triangles rule.

Proposition 4.1. *Let P be an ideal right-angled polyhedron in Lobachevsky space having N faces. Denote by p_3 the number of its triangular faces. If $N < 3p_3 + 2$, then P is not ITR-polyhedron.*

Proof. Let n be the maximum number of edges in faces of the polyhedron P . Denote by p_k , $k = 3, \dots, n$, the number of k -gonal faces in P . Then $\sum_{k=3}^n p_k = N$. Recall that by the formula (1), $p_3 = 8 + \sum_{k=4}^n p_k(k - 4)$. Assume, on the contrary, that P is an ITR-polyhedron. So at each vertex of the triangle there are meet three more vertices related to the faces that are not triangular. The number of vertices of all triangles is $3p_3$. So, the number of vertices in all remaining polygons must be at least $9p_3$. Let us

calculate this number:

$$\begin{aligned} \sum_{k=4}^n kp_k &= 4p_4 + \sum_{k=5}^n kp_k = 4p_4 + \sum_{k=5}^n p_k(k-4) + 4 \sum_{k=5}^n p_k \\ &= p_3 - 8 + 4 \sum_{k=4}^n p_k = p_3 - 8 + 4(N - p_3) = 4N - 3p_3 - 8. \end{aligned}$$

Demanding inequality

$$4N - 3p_3 - 8 \geq 9p_3,$$

we get

$$N \geq 3p_3 + 2,$$

which contradicts the original condition. Therefore, for $N < 3p_3 + 2$ the polyhedron P cannot have isolated triangles. \square

Since the smallest possible value of p_3 is 8, there are no polyhedra with isolated triangles among polyhedra with at most 25 faces. But among the 26-faced polyhedra there are two such examples, which are shown in Figure 7. The volume of the ITR-polyhedron shown on the left-hand side is

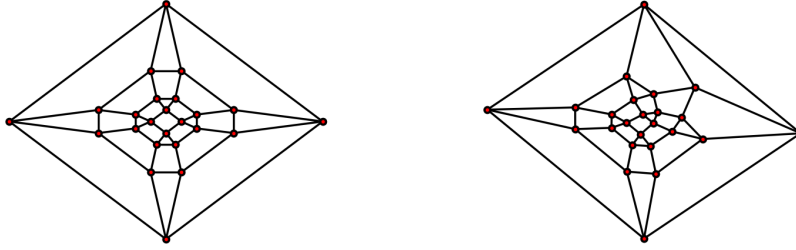


FIGURE 7. 26-gonal polyhedra with isolated triangles

31.0930375, and the volume of the ITR-polyhedron shown on the right-hand side is 31.1668675.

Proposition 4.2. *There are infinitely many ITR-polyhedra.*

Proof. Let P_1 be an ITR-polyhedron (for example, it can be taken any of the polyhedra shown in Figure 7) and let F_1 be one of its triangular faces. Consider the polyhedron $P_2 = P_1 \cup_{F_1} P_1$ obtained by gluing two copies of the polyhedron P_1 along the face F_1 . Obviously, the polyhedron P_2 will also be an ideal right-angled polyhedron with isolated triangles. Suppose, for certainty, that the faces of the polyhedron P_1 , which have common sides with F_1 , have n_1, n_2, n_3 sides, where $n_i \geq 4, i = 1, 2, 3$, by virtue of the triangle isolation property. When these faces are doubled, they will turn into faces of a polyhedron P_2 , having, respectively, $2n_i - 2$

sides, $i = 1, 2, 3$. If P_1 have $p_3(P_1)$ triangular faces, then P_2 will have $p_3(P_2) = 2p_3(P_1) - 2 = p_3(P_1) + \sum_{i=1}^3 (2n_i - 6)$ triangular faces. \square

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TABLE 4. Ideal right-angled polyhedra with n faces having minimum and maximum volume, $14 \leq n \leq 18$.

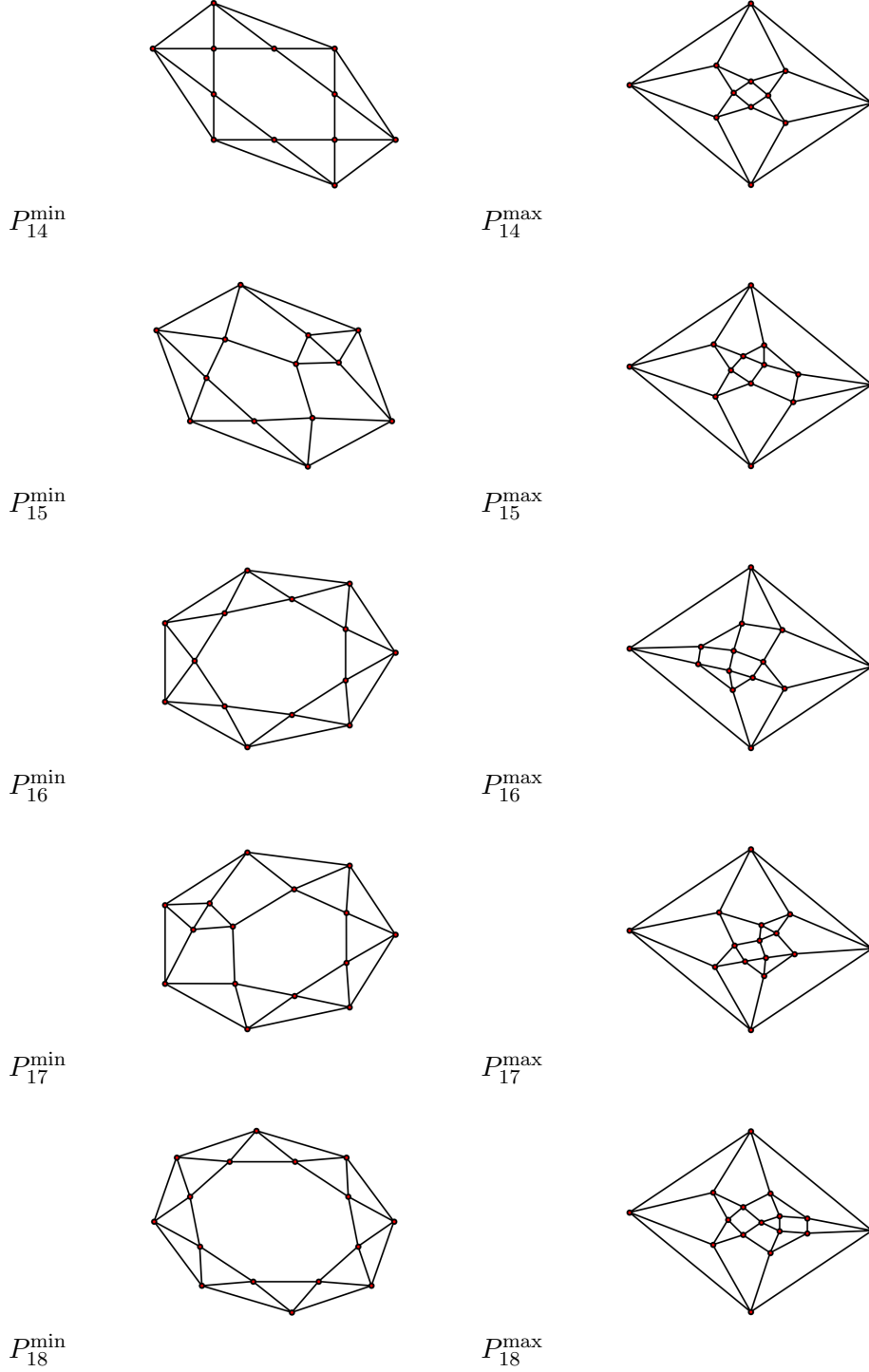


TABLE 5. Ideal right-angled polyhedra with n faces having minimum and maximum volume, $19 \leq n \leq 23$.

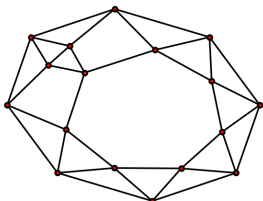
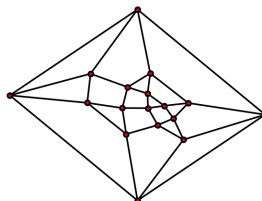
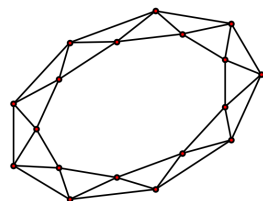
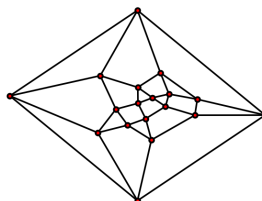
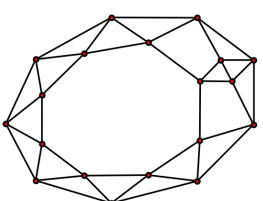
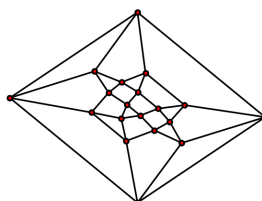
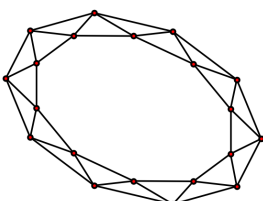
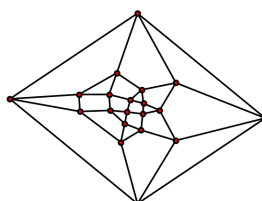
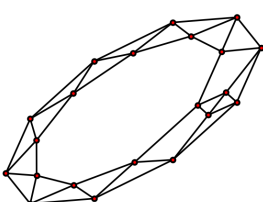
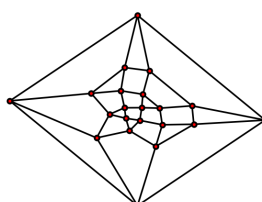
P_{19}^{\min}		P_{19}^{\max}	
P_{20}^{\min}		P_{20}^{\max}	
P_{21}^{\min}		P_{21}^{\max}	
P_{22}^{\min}		P_{22}^{\max}	
P_{23}^{\min}		P_{23}^{\max}	

TABLE 6. The first 248 values of volumes.

1	3,663863	51	15,46561	101	16,735095	151	17,477141	201	17,974896
2	6,023046	52	15,478658	102	16,744556	152	17,509421	202	17,98967
3	7,327725	53	15,495403	103	16,750301	153	17,516143	203	18,009307
4	8,137885	54	15,546518	104	16,755495	154	17,517167	204	18,026172
5	8,612415	55	15,654866	105	16,769779	155	17,52091	205	18,038106
6	9,686908	56	15,655017	106	16,780195	156	17,528985	206	18,045655
7	10,149416	57	15,709955	107	16,798534	157	17,530777	207	18,047625
8	10,806002	58	15,720116	108	16,805953	158	17,548392	208	18,058361
9	10,991587	59	15,77016	109	16,829048	159	17,55096	209	18,063652
10	11,136296	60	15,795313	110	16,83204	160	17,558575	210	18,063815
11	11,447207	61	15,803436	111	16,855785	161	17,571217	211	18,069138
12	11,801747	62	15,8569	112	16,864012	162	17,577434	212	18,084139
13	12,106298	63	15,85949	113	16,896062	163	17,5839	213	18,08961
14	12,276278	64	15,933385	114	16,961302	164	17,600432	214	18,092676
15	12,414155	65	15,94014	115	16,974442	165	17,615398	215	18,099757
16	12,46092	66	15,958101	116	16,98803	166	17,616542	216	18,109351
17	12,611908	67	15,959551	117	17,004375	167	17,633184	217	18,109786
18	12,854902	68	15,996629	118	17,014633	168	17,671046	218	18,128273
19	12,883862	69	16,049989	119	17,024507	169	17,694323	219	18,129371
20	13,020639	70	16,061517	120	17,061166	170	17,701559	220	18,133727
21	13,310579	71	16,078017	121	17,061237	171	17,70449	221	18,144299
22	13,350771	72	16,158579	122	17,061342	172	17,709902	222	18,152718
23	13,447108	73	16,172462	123	17,110971	173	17,712742	223	18,15859
24	13,677298	74	16,213678	124	17,140322	174	17,740113	224	18,167534
25	13,714015	75	16,27577	125	17,159342	175	17,751064	225	18,1677
26	13,813278	76	16,295989	126	17,165397	176	17,759743	226	18,173199
27	13,907355	77	16,324638	127	17,169868	177	17,766925	227	18,175729
28	14,030461	78	16,330917	128	17,174806	178	17,766983	228	18,180264
29	14,103121	79	16,331571	129	17,19799	179	17,769525	229	18,180633
30	14,160931	80	16,339295	130	17,199831	180	17,773653	230	18,207313
31	14,171606	81	16,382246	131	17,201332	181	17,790452	231	18,21988
32	14,273414	82	16,39832	132	17,22483	182	17,812693	232	18,233526
33	14,469865	83	16,448631	133	17,233217	183	17,821704	233	18,234257
34	14,494727	84	16,465777	134	17,238195	184	17,824793	234	18,244844
35	14,635461	85	16,48952	135	17,280423	185	17,835469	235	18,247553
36	14,655449	86	16,49154	136	17,303311	186	17,83745	236	18,276848
37	14,766948	87	16,506891	137	17,324068	187	17,844054	237	18,281813
38	14,800159	88	16,518764	138	17,341161	188	17,845073	238	18,287301
39	14,832681	89	16,535273	139	17,342423	189	17,857212	239	18,28917
40	14,898794	90	16,538867	140	17,354288	190	17,860804	240	18,291323
41	15,031667	91	16,547725	141	17,354866	191	17,864013	241	18,292895
42	15,052463	92	16,575188	142	17,362724	192	17,864685	242	18,299323
43	15,07859	93	16,595363	143	17,377493	193	17,894018	243	18,300817
44	15,11107	94	16,605736	144	17,377877	194	17,899631	244	18,304268
45	15,126498	95	16,615815	145	17,38534	195	17,901906	245	18,307302
46	15,169623	96	16,627568	146	17,420943	196	17,907162	246	18,31334
47	15,253393	97	16,657287	147	17,429408	197	17,918936	247	18,316267
48	15,323216	98	16,678106	148	17,45025	198	17,922791	248	18,319312
49	15,350907	99	16,684502	149	17,470253	199	17,937276		
50	15,367058	100	16,726449	150	17,470735	200	17,944583		