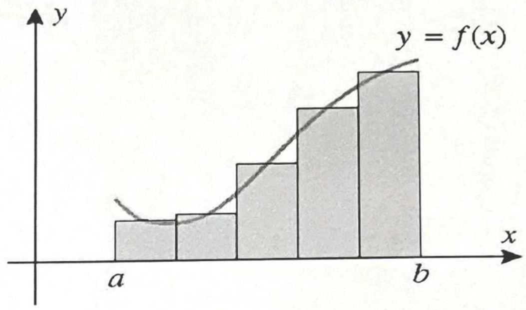
Section 5.1: *Areas and Distances.*

What is the Area Problem?

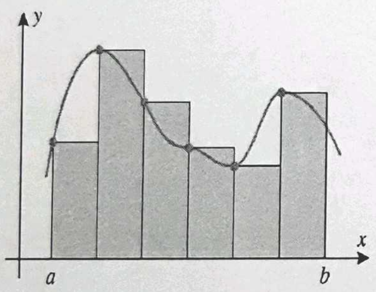
To **find** or to **approximate** the area of the region under the curve $y=f(x)$ from a to b.

We mainly use the rectangle method

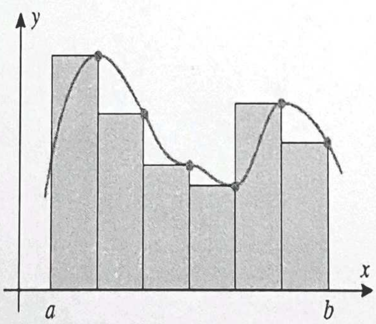


Estimating the area

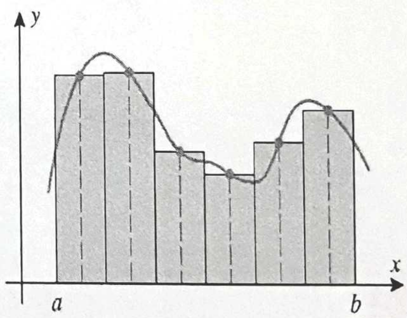
- Divide the interval [a,b] into n equal subintervals
- Construct n rectangles



Using left end points of subintervals



Using right end points of subintervals

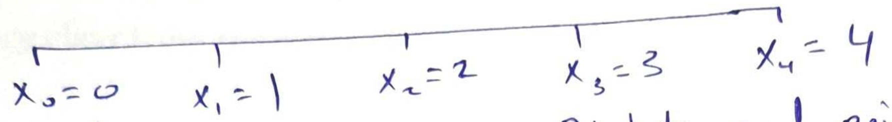


Using mid points of subintervals

- Find the area of each rectangle
- The total area A_n of all rectangles approximates the required area.

Example 1: Approximate the area under the curve $y = 1 + x^2$ over [0,4] using 4 rectangles and: a. Right end points b. Left end points c. Midpoints
Solution:

First we find $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$



(a)

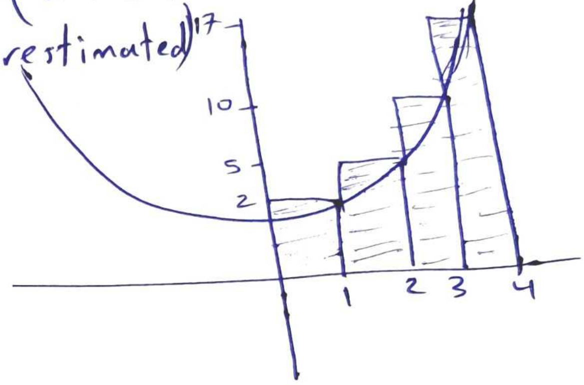
Subinterval	$f(x_i)$
[0,1]	$f(1) = 2$
[1,2]	$f(2) = 5$
[2,3]	$f(3) = 10$
[3,4]	$f(4) = 17$

Using Right end points:

$$A_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x$$

$$= 1(2+5+10+17) = \boxed{34}$$

(overestimated)



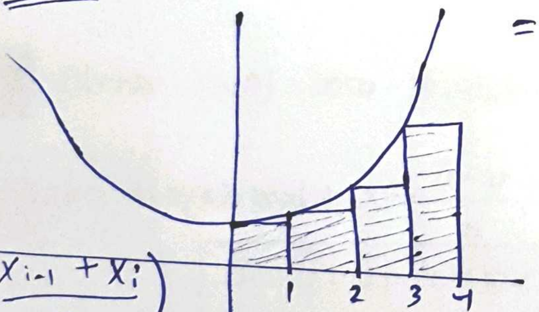
(b)

Subinterval	$f(x_{i-1})$
[0,1]	$f(0) = 1$
[1,2]	$f(1) = 2$
[2,3]	$f(2) = 5$
[3,4]	$f(3) = 10$

Using Left endpoints: $A_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x = \boxed{17}$

$$= 1(1+2+5+10)$$

(underestimated: Estimated area < exact area)



$f\left(\frac{x_{i-1}+x_i}{2}\right)$

(c)

Subinterval	$\cancel{\frac{1}{2}(f(x_{i-1})+f(x_i))}$
[0,1]	$f(0.5) = 1.25$
[1,2]	$f(1.5) = 3.25$
[2,3]	$f(2.5) = 7.25$ 2
[3,4]	$f(3.5) = 13.25$

(Midpoints):

$$A_4 = \sum f\left(\tfrac{1}{2}(x_{i-1}+x_i)\right)\Delta x$$

$$= 1(1.25 + 3.25 + 7.25 + 13.25)$$

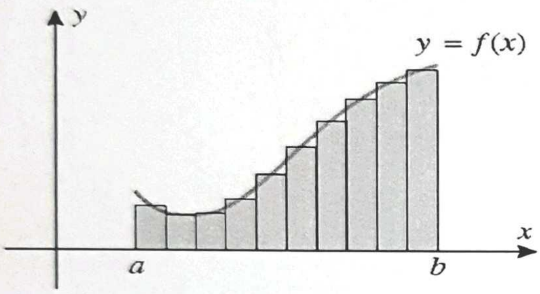
$$= \boxed{25}$$

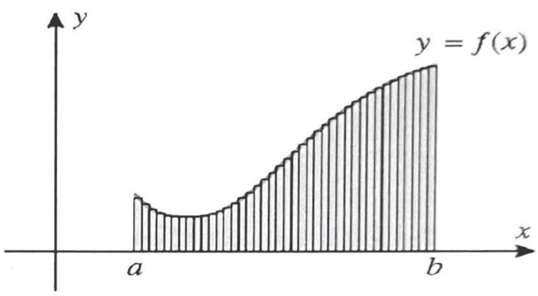
The exact area

Ho to find the exact area?

As n increases, the accuracy of the approximation A_n increases. Now if we send n to infinity, it is very clear from the sketch below that the exact area A is given by:

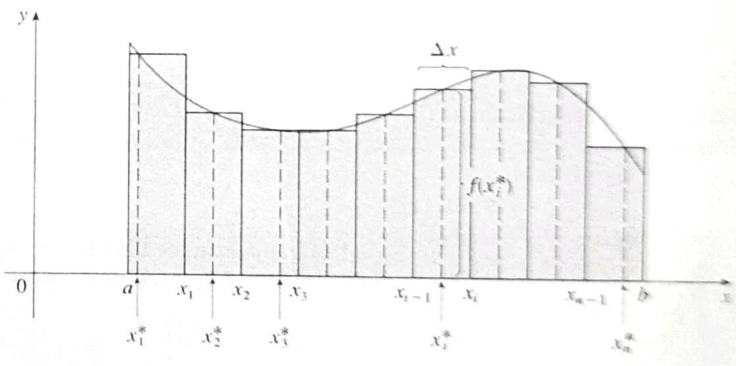
$$\lim_{n\to\infty} A_n = A$$





The strategy of finding the exact area

To find **exact** area under *f(x)* from *x=a* to *x=b*, we use the following steps:



Step 1 Divide $[a,b]$ into n equal subintervals with width $\Delta x = \dfrac{b-a}{n}$.

Hence the points are

$x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, ..., $x_i = a + i\Delta x$, ..., $x_n = b$

Step 2 Choose points $x_1^*, x_2^*, \cdots, x_n^*$ in each subinterval to make n-rectangles $A_1, A_2, \cdots, A_n$

Fact: the choice of the points does not affect the value of the exact area.
So I suggest to use right end points

1. **Right end points:** $x_i^* = x_i = a + i\Delta x$

2. **Left end points:** $x_i^* = x_{i-1} = a + (i-1)\Delta x$

3. **Midpoints:** $x_i^* = \frac{1}{2}(x_{i-1} + x_i) = \frac{1}{2}\big(a + (i-1)\Delta x + a + i\Delta x\big) = a + \left(i - \frac{1}{2}\right)\Delta x$

Step 3 Find the area of the ith rectangle: $\boxed{f(x_i^*)\Delta x}$

Step 4 Find the area of n rectangles: $\boxed{\sum_{i=1}^{n} f(x_i^*)\Delta x}$

Step 5 Find the exact area: $\boxed{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}$

Definition 1: If f is continuous on $[a,b]$ and if $f(x) \ge 0$ for all $x \in [a,b]$ then area under $y = f(x)$ over $[a,b]$ is defined by

$$\boxed{A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

where x_i^* denotes the point chosen in i^{th} subinterval and $\Delta x = \frac{b-a}{n}$

The following are needed to find *the area*

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\qquad$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\qquad$ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example 2: Use **Definition 1** to find the area under the graph of $f(x) = 2x - 1$ over $\cancel{[0,1]}$ $[1,2]$

The function $f(x) = 2x - 1$ is cont. and positive on $[1,2]$

- $\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{n} = \dfrac{1}{n}$
- Right end points: $x_i^* = x_i = a + i \cdot \Delta x$

$$x_i = 1 + i \cdot \frac{1}{n}$$

$$\boxed{x_i = 1 + \frac{i}{n}}$$

- $f(x_i) = f\left(1 + \frac{i}{n}\right) = 2\left(1 + \frac{i}{n}\right) - 1 = \boxed{1 + \frac{2i}{n}}$
- $A = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(\underset{(\text{right})}{x_i})\Delta x$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right) \cdot \frac{1}{n}$$

$$= \lim_{n \to \infty} \left(\frac{1}{n} \sum_{i=1}^{n} 1 + \frac{1}{n} \cdot \frac{2}{n} \sum_{i=1}^{n} i \right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{n} \cdot n(1) + \frac{2}{n^2} \cdot \frac{n(n+1)}{2} \right)$$

$$= 1 + \lim_{n \to \infty} \frac{n+1}{n}$$

$$= 1 + 1 = \boxed{2}$$

Ex 24/25 Page 377: Determine the region whose area is equal to the given limit:

24. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{1+\frac{3i}{n}}$

Assuming $\Delta x = \frac{3}{n}$ and $a = 0$,
then $b = 3$ (We can use any $\underline{a}$ and $\underline{b}$ where $b - a = 3$)

$x_i = a + i\cdot\Delta x = 0 + i\cdot\frac{3}{n} = \boxed{\frac{3i}{n}}$

Based on that, $f(x) = \sqrt{1+3x}$

Therefore,
$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{1+\frac{3i}{n}}$ represents the area under $f(x) = \sqrt{1+x}$ on $[0,3]$.

Another method $\Rightarrow$ Later on, we will use the following result: $\int_0^1 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n}$

And then the limit will become

$$\int_0^1 3\sqrt{1+3x}\,dx$$

25. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\left(\frac{i\pi}{4n}\right)$

Method ①:

$\Delta x = \frac{\pi}{4n} = \frac{b-a}{n}$, Let $a = 0 \Rightarrow b = \frac{\pi}{4}$

$x_i = a + i\cdot\Delta x = 0 + i\cdot\frac{\pi}{4n}$

$\Rightarrow \boxed{x_i = \frac{i\pi}{4n}}$

$\Rightarrow f(x) = \tan x$

$\Rightarrow$ The limit represents the area under $f(x) = \tan x$ on $[0, \frac{\pi}{4}]$

Method ②: $\int_0^1 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n}$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\left(\frac{i\pi}{4n}\right) = \int_0^1 \frac{\pi}{4}\tan\left(\frac{\pi}{4}x\right)dx$$

End of 5.1- Dr. Khalid Adarbeh.

Section 5.2: The definite integral

What is a Riemann Sum?

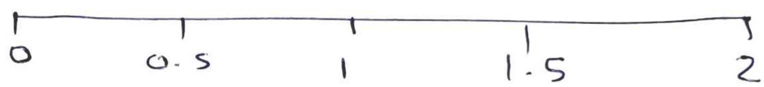
- Recall from sec. 5.1that the area was given by $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$
- The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann Sum of $f(x)$

x_i^* from i[th] subinterval

- Right end point or left end point or midpoint

Example 1: Evaluate the Riemann sum for $f(x) = 2 - x^2$ on $[0,2]$ using 4 subintervals and right end points.

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \boxed{0.5}$$



Subinterval	$f(x_i)$
$[0, 0.5]$	$f(0.5) = 1.75$
$[0.5, 1]$	$f(1) = 1$
$[1, 1.5]$	$f(1.5) = -0.25$
$[1.5, 2]$	$f(2) = -2$

Reimann Sum $= \sum_{i=1}^{4} f(x_i)\Delta x$ (right end points)

$= 0.5(1.75 + 1 + -0.25 + -2)$

$= 0.5(0.5) = \boxed{0.25}$

Definite integral

upper limit

lower limit

The definite integral of f from $x=a$ to $x=b$ is defined as

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

provided that this limit exists. If it does exist, we say that is f integrable on $[a,b]$.

Geometric interpretation of Riemann Sum & definite integral

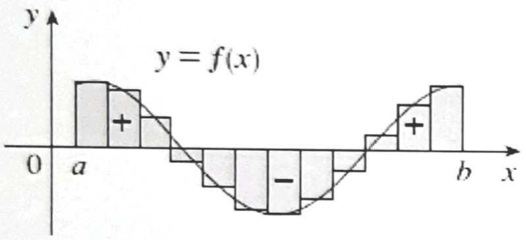


FIGURE 3
$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area

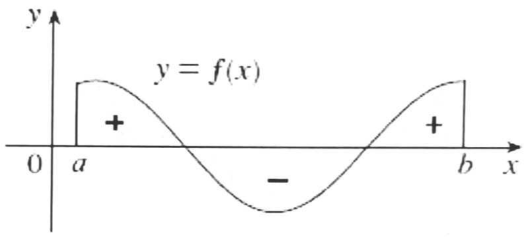


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area

Integrable function on closed interval

$\boxed{3}$ **THEOREM** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

$\boxed{4}$ **THEOREM** If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\qquad \Delta x = \dfrac{b-a}{n} \qquad$ and $\qquad x_i = a + i\,\Delta x$

Example 2: Express $\lim_{n\to\infty}\sum_{i=1}^{n}\left(x_i^3 + x_i\cos x_i^2\right)\Delta x$ as an integral on the interval $[0, \pi]$.

$$= \int_0^{\pi} \left(x^3 + x\cos x^2\right) dx$$

To solve the following question, we may use the following:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \int_0^1 f(x)\,dx$$

Just replace $\frac{1}{n}$ by dx and $\frac{i}{n}$ by x

Exercise1: Express the limit as a definite integral:

~~Method (1)~~

1. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^4\cdot\frac{1}{n}$

$$= \int_0^1 x^4\,dx$$

2. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{i^2}{n^2}}$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n}$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx$$

Evaluating definite integrals

(i) Using the definition:

Recall:

1. $\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}$

2. $\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example 3: Evaluate $\int_1^2 x^3 dx$ using definition. (Use right end points)

Solution:

Step 1 Divide [1,2] into n subintervals of length

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \qquad \text{with} \quad x_i = 0 + i\Delta x = 1 + \frac{i}{n}.$$

Step 2 Choose x_i^* as $x_i^* = x_i = 1 + \frac{i}{n}$

Step 3 $$f(x_i^*)\Delta x = f(1+\frac{i}{n})\cdot\frac{1}{n} = \left(1+\frac{i}{n}\right)^3\cdot\frac{1}{n}$$

$$= \frac{1}{n}\left(1+\frac{i^3}{n^3}+\frac{3i^2}{n^2}+\frac{3i}{n}\right)$$

Step 4 The area of n rectangles is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{i^3}{n^3}+\frac{3i^2}{n^2}+\frac{3i}{n}\right)$$

$$= \frac{1}{n}\left(\sum_{k=1}^{n}1+\frac{1}{n^3}\sum_{k=1}^{n}i^3+\frac{3}{n^2}\sum_{k=1}^{n}i^2+\frac{3}{n}\sum_{k=1}^{n}i\right)$$

$$= \frac{1}{n}\left(n+\frac{1}{n^3}\cdot\frac{[n(n+1)]^2}{4}+\frac{3}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{3}{n}\cdot\frac{n(n+1)}{2}\right)$$

$$= 1 + \frac{1}{n^4} \cdot \frac{[n(n+1)]^2}{4} + \frac{3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \cdot \frac{n+1}{2}$$

Step 5 $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = 1 + \frac{1}{4} + 3 \cdot \frac{2}{6} + \frac{3}{2} = \frac{15}{4}.$

(ii) Using the Known areas

Example 4: Evaluate the integrals by interpreting it in terms of areas:

1. $\int_{-3}^{0} \sqrt{9 - x^2}$

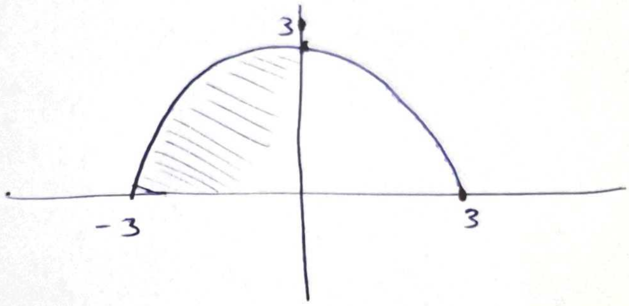
$y = \sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$

So $y = \sqrt{9-x^2}$ is the upper half of the circle $x^2 + y^2 = 9$

$\int_{-3}^{0} \sqrt{9-x^2}\,dx =$ the area of $\frac{1}{4}$ of the circle

$= \frac{1}{4}\left(\pi (3)^2\right)$

$= \frac{9\pi}{4}$

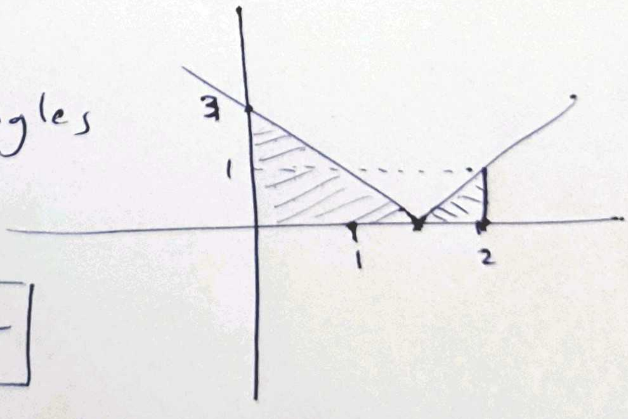


2. $\int_{0}^{2} |2x - 3|$

= Sum of the areas of the triangles

$= \frac{1}{2}(1.5)(3) + \frac{1}{2}(0.5)(1)$

$= \frac{9}{4} + \frac{1}{4} = \frac{10}{4} = \boxed{\frac{5}{2}}$



Exercises:

$$\int_3^6 \sqrt{6x - x^2}\, dx$$

$y = \sqrt{6x - x^2} \Rightarrow y^2 + x^2 - 6x = 0$

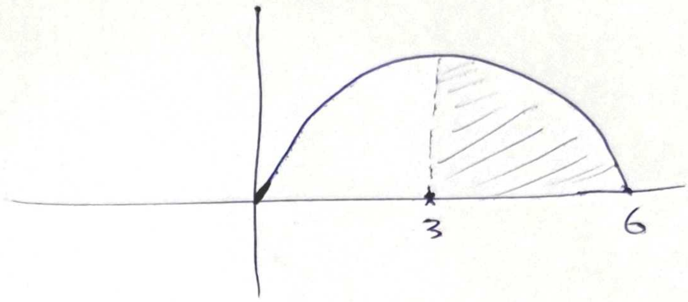
$y^2 + x^2 - 6x + 9 = 9$

We added $\left(\frac{1}{2}(-6)\right)^2 = 9$ for both sides (complete the square)

$\Rightarrow (x-3)^2 + y^2 = 9$

$A = \frac{1}{4}\pi(3)^2$

$= \frac{9\pi}{4}$



33. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_0^2 f(x)\, dx = 4$

(b) $\int_0^5 f(x)\, dx = 10$

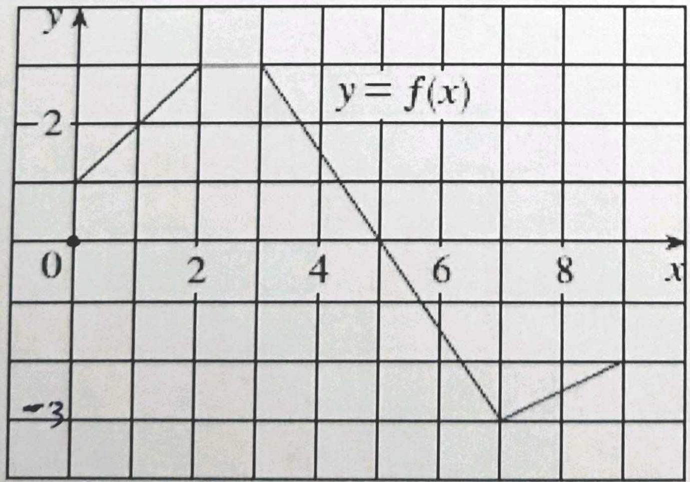
(c) $\int_5^7 f(x)\, dx = -3$ (below x-axis)

(d) $\int_0^9 f(x)\, dx = \int_0^5 f(x)\,dx + \int_5^9 f(x)\,dx$

$= 10 + (-3 + -4 + -1)$

$= 10 + (-8)$

$= \boxed{2}$



Basic properties of definite integrals

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_a^b c\,dx = c(b-a)$

4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (a,b,c any numbers)

5) $\int_a^b c \cdot f(x)dx = c\int_a^b f(x)dx$

6) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

49. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find $\int_0^9 [2f(x) + 3g(x)]\,dx$.

$$= 2\int_0^9 f(x)dx + 3\int_0^9 g(x)dx$$

$$= 2(37) + 3(16)$$

$$= 74 + 48 = \boxed{122}$$

EXAMPLE 7 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

$$\int_8^{10} f(x)dx = \int_8^0 f(x)dx + \int_0^{10} f(x)dx$$

$$= -12 + 17 = \boxed{5}$$

Comparison properties of definite integrals

C1) If $f(x) \geq 0$, for any $x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$

C2) If $f(x) \geq g(x)$, for any $x \in [a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

C3) If $m \leq f(x) \leq M$, for any $x \in [a, b]$, then $m(b-a) \leq \int_a^b f(x) \leq M(b-a)$

Example 5: Use the properties of integration to verify:

1. $\int_0^1 \sqrt{1+x^2}dx \leq \int_0^1 \sqrt{1+x}\,dx$ We need to show that

$$\boxed{\begin{aligned} &\sqrt{1+x^2} \leq \sqrt{1+x} \\ &1+x^2 \leq 1+x \\ &x^2 - x \leq 0 \\ &x(x-1) = 0 \Rightarrow x=0 / x=1 \\ &+ \quad - \quad + \\ &0 \quad 1 \end{aligned}}$$

$\sqrt{1+x^2} \leq \sqrt{1+x}$ on $[0,1]$

From this, $x^2 - x \leq 0$ on $[0,1]$

Now we go backwards:

$x^2 - x \leq 0 \Rightarrow x^2 \leq x$

$\Rightarrow x^2+1 \leq x+1 \Rightarrow \sqrt{x^2+1} \leq \sqrt{x+1}$ on $[0,1]$

$\Rightarrow \int_0^1 \sqrt{x^2+1}\,dx \leq \int_0^1 \sqrt{x+1}\,dx.$

2. $2 \leq \int_{-1}^1 \sqrt{1+x^2}dx \leq 2\sqrt{2}$

To find absolute Max/min of $f(x) = \sqrt{1+x^2}$ on $[-1,1]$:

$$f'(x) = \frac{2x}{2\sqrt{1+x^2}} = 0 \Rightarrow x = 0 \text{ (critical point)}$$

Now substitute endpoints and critical point(s) in $f(x)$:

$f(0) = 1 \rightarrow$ absolute min

$f(-1) = \sqrt{2} \rightarrow$ absolute Max

$f(1) = \sqrt{2}$

$\Rightarrow 1 \leq \sqrt{1+x^2} \leq \sqrt{2}$

$1(1+1) \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq \sqrt{2}(1+1)$

$2 \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq 2\sqrt{2}.$

Exercise: Use Property C3 to estimate $\int_0^2 xe^{-x}\,dx$

Let $f(x) = xe^{-x}$. We find it absolute Max/min on $[0, 2]$:

$f'(x) = x(-e^{-x}) + e^{-x}(1) = 0$

$\Rightarrow e^{-x}(-x+1) = 0 \Rightarrow e^{-x} = 0$ (impossible)

$\boxed{x=1}$ Critical point

$f(1) = e^{-1} = \frac{1}{e} = \frac{e}{e^2} \rightarrow$ abs. Max.

$f(0) = 0 \longrightarrow$ abs. min

$f(2) = 2e^{-2} = \frac{2}{e^2}$

So, $0 \leq xe^{-x} \leq \frac{1}{e}$

$$0(2-0) \leq \int_0^2 xe^{-x}\,dx \leq \frac{1}{e}(2-0)$$

$$\Rightarrow \int_0^2 xe^{-x}\,dx \cong \frac{\frac{2}{e}+0}{2} = \frac{1}{e}.$$

End of 5.2- Dr. Khalid Adarbeh.

Section 5.3 : The fundamental theorem of calculus

Fundamental theorem of calculus (Part 1)

- Recall that g is called an antiderivative of f if $g' = f$ (e.g., both of x^2 and $x^2 + 1$ are antiderivatives of $2x$.)
- **Fundamental Theorem of Calculus (Part 1):** If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)dt$ is differentiable function on (a, b) and $F'(x) = f(x)$
- (i.e. F is an anti-derivative of f)

Corollary: If f is continuous and g and h are differentiable functions, then:

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)dt\right) = f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

Example 1: Find the derivative of the following:

1. $F(x) = \int_0^x \frac{1+t}{1+e^t}\, dt$

$$F'(x) = \frac{1+x}{1+e^x}$$

2. $f(x) = \int_0^{\sin x} \frac{1+t}{1+e^t}\, dt$

$$f'(x) = \frac{1+\sin x}{1+e^{\sin x}} \cdot (\cos x) - 0$$

3. $y = \int_{cosx}^{sinx} \ln(1+2z)\ dz$

$$\frac{dy}{dx} = \left(\ln\left(1+2\sin x\right)\right)\left(\cos x\right) - \left(\ln\left(1+2\cos x\right)\right)\left(-\sin x\right)$$

Exercices:

1. If $F(x) = \int_1^x f(t)dt$, where $f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u}$, then find $F''(2)$

$$F'(x) = f(x) \Rightarrow F''(x) = f'(x)$$

$$f'(t) = \frac{\sqrt{1+(t^2)^4}}{t^2} \cdot 2(t) - 0$$

$$f'(t) = \frac{\sqrt{1+t^8}}{t} \cdot (2)$$

$$F''(2) = f'(2) = \frac{\sqrt{1+2^8}}{2} \cdot (2) = \sqrt{257}$$

2. Find the equation of the tangent line to the curve $y = \int_{\sqrt{x}}^{x^2} e^{u^3}\, du$ at $x = 1$

$$\frac{dy}{dx} = e^{(x^2)^3} \cdot (2x) - e^{(\sqrt{x})^3} \cdot \frac{1}{2\sqrt{x}}$$

$$m_T = \left.\frac{dy}{dx}\right|_{x=1} = e^1(2) - e^1\left(\frac{1}{2}\right) = \boxed{\frac{3e}{2}}$$

At $x=1$, $y = \int_1^1 e^{u^3}\, du = 0$

An equation of tangent is:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{3e}{2}(x-1)$$

3. Evaluate $\lim\limits_{x\to 0} \frac{1}{x}\int_0^x (1-\tan 2t)^{1/t}\,dt$ This limit is of the form $\frac{0}{0}$

$\Rightarrow$ We use Lopital's Rule.

$$\lim_{x\to 0} \frac{\int_0^x (1-\tan 2t)^{1/t}\,dt}{x} \overset{L.R.}{=} \lim_{x\to 0} (1-\tan 2x)^{\frac{1}{x}}$$

The last limit is of the form $1^\infty \Rightarrow$ Take ln, Use L.R., Take e

$$y = \lim_{x\to 0}(1-\tan 2x)^{\frac{1}{x}}$$

$$\ln y = \lim_{x\to 0} \frac{\ln(1-\tan 2x)}{x} \overset{L.R.}{=} \lim_{x\to 0} \frac{\frac{-2\sec^2 x}{1-\tan 2x}}{1}$$

$$= -2$$

$\Rightarrow \ln y = -2$

~~Take e~~ Take exp for both sides

$$\Rightarrow \boxed{y = e^{-2}}$$

4. Evaluate $\lim_{x\to 0} \frac{x\int_0^x \sin t^2\,dt}{\sin x^4}$

The limit is of the form $\frac{0}{0}$ $\Rightarrow$ Use Lopital's Rule

$$\Rightarrow \lim_{x\to 0} \frac{x\sin x^2 + \int_0^x \sin t^2\,dt}{4x^3\cos x}$$

$$= \lim_{x\to 0} \frac{x\sin x^2}{4x^3\cos x} + \lim_{x\to 0} \frac{\int_0^x \sin t^2\,dt}{4x^3\cos x}$$

$$= \lim_{x\to 0} \frac{1}{4\cos x}\left(\lim_{x\to 0} \frac{\sin x^2}{x^2} + \lim_{x\to 0} \frac{\int_0^x \sin t^2\,dt}{x^3}\right)$$

L.R.

$$\Rightarrow \frac{1}{4(1)}\left(\lim_{x\to 0} \frac{2x\cos x^2}{2x} + \lim_{x\to 0} \frac{\sin x^2}{3x^2}\right)$$

$$= \frac{1}{4}\left(1 + \frac{1}{3}(1)\right)$$

$$= \frac{1}{4}\left(\frac{4}{3}\right) = \boxed{\frac{1}{3}}$$

Fundamental theorem of Calculus (Part 2)

(Evaluation of definite integral)

If f is continuous on $[a,b]$, and F is an antiderivative of f then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 2: Find the area under the curve $y = e^x$ from 0 to $\ln 2$.

e^x is positive on $[0, \ln 2]$

$$\Rightarrow A = \int_0^{\ln 2} e^x dx = e^x \Big|_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$$

Example 3: What is wrong with the following:

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \frac{-1}{x}\Big|_{-1}^{1} = -1 - 1 = -2$$

$\frac{1}{x^2}$ has an infinite discontinuity at $x = 0 \in [-1,1]$

$\Rightarrow$ Fundamental Theorem of Calculus (part 2) can not be applied.

Example 4: Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} + \frac{i}{n^2}$ =

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{i}{n}\right)^4 + \frac{i}{n}\right) \cdot \frac{1}{n}$$

$$= \int_0^1 (x^4 + x)\,dx$$

$$= \frac{x^5}{5} + \frac{x^2}{2} \Big|_0^1$$

$$= \frac{1}{5} + \frac{1}{2} = \boxed{0.7}$$

Exercises:

Q1. Evaluate the following integrals:

1. $\int_1^3 \frac{1}{x}\,dx.$

$$= \ln x \Big|_1^3$$

$$= \ln 3 - \ln 1$$

$$= \ln 3 - 0$$

$$= \ln 3$$

2. $\int_0^1 (x^e + e^x)dx.$

$$= \frac{x^{e+1}}{e+1} + e^x \Big|_0^1$$

$$= \left(\frac{1}{e+1} + e\right) - \left(0 + e^0\right)$$

$$= \frac{1}{e+1} + e - 1$$

3. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx.$

$$= 8 \tan^{-1} x \Big|_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= 8\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{8\pi}{6} = \boxed{\frac{4\pi}{3}}$$

4. $\int_0^{\pi} f(x)dx$, where $f(x) = \begin{cases} \sin x, & x < \pi/2 \\ \cos x, & x \geq \pi/2 \end{cases}.$

$$= \int_0^{\frac{\pi}{2}} \sin x\, dx + \int_{\pi/2}^{\pi} \cos x\, dx$$

$$= -\cos x \Big|_0^{\pi/2} + \sin x \Big|_{\pi/2}^{\pi}$$

$$= -(0-1) + (0-1)$$

$$= 1 - 1 = \boxed{0}$$

5. $\int_0^{\pi/6} (\tan x + \sec x)^2\, dx.$

$$= \int_0^{\pi/6} \left(\tan^2 x + 2\sec x \tan x + \sec^2 x\right) dx$$

$$= \int_0^{\pi/6} \left(\sec^2 x - 1 + 2\sec x \tan x + \sec^2 x\right) dx$$

$$= 2\tan x + 2\sec x - x \Big|_0^{\pi/6}$$

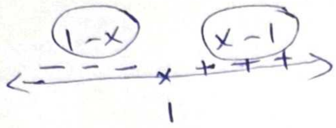
7

$$= \left(2\cdot\frac{1}{\sqrt{3}} + 2\cdot\frac{2}{\sqrt{3}} - \frac{\pi}{6}\right) - (0+2-0) = \frac{6}{\sqrt{3}} - \frac{\pi}{6} - 2$$

6. $\int_0^2 \sqrt{x^2 - 2x + 1}\, dx.$

$$= \int_0^2 \sqrt{(x-1)^2}\, dx$$

$$= \int_0^2 |x-1|\, dx$$



$$= \int_0^1 (1-x)\, dx + \int_1^2 (x-1)\, dx$$

$$= \left(x - \frac{x^2}{2}\right)\Big|_0^1 + \left(\frac{x^2}{2} - x\right)\Big|_1^2$$

$$= \left(1 - \frac{1}{2}\right) - 0 + \left(\frac{2^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{2} + \frac{1}{2} = \boxed{1}$$

7. $\int_0^\pi \sqrt{1 + \cos 2\theta}\ d\theta.$

$$= \int_0^\pi \sqrt{1 + 2\cos^2\theta - 1}\ d\theta$$

$$= \sqrt{2} \int_0^\pi |\cos\theta|\, d\theta$$

$$= \sqrt{2}\left(\int_0^{\pi/2} \cos\theta\, d\theta + \int_{\pi/2}^{\pi} -\cos\theta\, d\theta\right)$$

$$= \sqrt{2}\left(\sin\theta\Big|_0^{\pi/2} - \cos\theta\Big|_{\pi/2}^{\pi}\right)$$

$$= \sqrt{2}\left((1-0) - (-1-0)\right)$$

$$= \sqrt{2}\,(1+1) = \boxed{2\sqrt{2}}$$

8. $\int_{-10}^{10} \frac{2e^x}{\sinh x+\cosh x}\, dx$

$$= \int_{-10}^{10} \frac{2e^x}{e^x}\, dx$$

$$= \int_{-10}^{10} 2\, dx$$

$$= 2(10 - -10)$$

$$= \boxed{40}$$

Note that $\sinh x + \cosh x = e^x$

Q2. Evaluate the following limits:

1. $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right).$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}}$$

$$= \int_0^1 \sqrt{x}\, dx$$

$$= \left.\frac{x^{3/2}}{3/2}\right|_0^1$$

$$= \frac{2}{3}\left(1^{3/2} - 0^{3/2}\right)$$

$$= \frac{2}{3}$$

2. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}(3 - \frac{2i-2}{n})^3$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \left(3 - 2\left(\frac{i-1}{n}\right)\right)^3$

$= \int_0^1 (3-2x)^3 \, dx$

$= \left. \frac{(3-2x)^4}{4(-2)} \right|_0^1$

$= \frac{1}{-8} \left(1^4 - 3^4 \right)$

$= \frac{1}{-8} (-80)$

$= \boxed{10}$

3. $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{2n}}\right).$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n+i}}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}\sqrt{1+\frac{i}{n}}}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{\sqrt{1+\frac{i}{n}}}$

$= \int_0^1 \frac{1}{\sqrt{1+x}} \, dx = \left. \frac{(1+x)^{\frac{1}{2}}}{\frac{1}{2}} \right|_0^1 = 2\left(\sqrt{2} - \sqrt{1}\right)$

$= \boxed{2\sqrt{2} - 2}$

End of 5.3- Dr. Khalid Adarbeh.

Section 5.4: Indefinite integrals and the net change theorem

Indefinite integral and the antiderivative

- Recall that F is an antiderivative of f on an interval I if $F'(x) = f(x)$ on I.
- $F(x)$ is antiderivative of $f(x)$ on I $\Leftrightarrow$ $F(x) + c$ is antiderivative of $f(x)$ on

, for any **constant** c.

Is called the integrand

- The notation $\int f(x)dx$ is used for the **antiderivative of $f(x)$** and is called the **indefinite integral** of $f(x)$.
- E.g., $\int \cos x \, dx = \sin x + c$

Remarks:

1. The indefinite integral $\int f(x)dx$ is a function, whereas the definite integral $\int_a^b f(x)dx$ is a number. The connection between them is:

$$\int_a^b f(x)dx = \int f(x)dx\Big|_a^b$$

2. The process of finding the indefinite integral is the inverse process of the differentiation, So one may easily check the integration formulas by differentiation (Example 1)

3. The basic used technique in finding the integration **in this section** is guessin the antiderivative of the integrand using the table below and the fundamental integration properties:

$$\int cf(x)dx = c\int f(x)dx$$

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Table: Some Fundamental integration formulas:

$$\int dx = x + C \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int b^x dx = \frac{b^x}{\ln b} + C \quad (0 < b,\ b \neq 1)$$

$$\int \cos x\, dx = \sin x + C \qquad \int \sin x\, dx = -\cos x + C$$

$$\int \sec^2 x\, dx = \tan x + C \qquad \int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x \tan x\, dx = \sec x + C \qquad \int \csc x \cot x\, dx = -\csc x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$$

$$\int \cosh x\, dx = \sinh x + C \qquad \int \sinh x\, dx = \cosh x + C$$

Example 1: Verify by differentiation that $\int \frac{1}{x^2\sqrt{1+x^2}} dx = -\frac{\sqrt{1+x^2}}{x} + C$

$$\frac{d}{dx}\left(-\frac{\sqrt{1+x^2}}{x} + c\right) = -\frac{x \cdot \frac{2x}{2\sqrt{1+x^2}} - \sqrt{1+x^2}\cdot(1)}{x^2} \cdot \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$= -\frac{x^2 - (1+x^2)}{x^2\sqrt{1+x^2}}$$

$$= -\frac{-1}{x^2\sqrt{1+x^2}} = \frac{1}{x^2\sqrt{1+x^2}} \checkmark$$

Example 2: Evaluate the following integrals:

1. $\int \frac{t^2-1}{t^4-1} dt$

$= \int \frac{t^2-1}{(t^2-1)(t^2+1)} dt$

$= \int \frac{1}{t^2+1} dt$

$= \tan^{-1} t + C$

2. $\int \frac{\sin 2x}{\sin x} dx$

$= \int \frac{2 \cancel{\sin x} \cos x}{\cancel{\sin x}} dx$

$= \int 2 \cos x \, dx$

$= 2 \sin x + C$

3. $\int \frac{(x-1)^3}{x^2}$

$= \int \frac{(x^2-2x+1)(x-1)}{x^2} dx$

$= \int \frac{x^3 - x^2 - 2x^2 + 2x + x - 1}{x^2} dx$

$= \int \frac{x^3 - 3x^2 + 3x - 1}{x^2} dx$

$= \int \left(x - 3 + \frac{3}{x} - x^{-2}\right) dx = \frac{x^2}{2} - 3x + 3 \ln|x| + \frac{1}{x} + C$

4. $\int 2^t(1+5^t)dt$

$= \int (2^t + 10^t) dt = \frac{2^t}{\ln 2} + \frac{10^t}{\ln 10} + C$.

Exercises: Evaluate:

1. $\int \frac{1+\sec^2(x)\tan(x)}{\sec x}\,dx$

$= \int (\cos x + \sec x \tan x)\,dx$

$= \sin x + \sec x + C$

2. $\int \frac{(t+1)^2-1}{t^4}\,dt$

$= \int \frac{t^2+2t+1-1}{t^4}\,dt$

$= \int (t^{-2} + 2t^{-3})\,dt$

$= \cancel{\frac{1}{t}}\frac{t^{-1}}{-1} + 2\frac{t^{-2}}{-2} + C$

$= -\frac{1}{t} - \frac{1}{t^2} + C$

3. $\int_0^{\ln 2} e^{x-2}\,dx = \int_0^{\ln 2} e^x \cdot e^{-2}\,dx = \underset{\text{constant}}{e^{-2}} \int_0^{\ln 2} e^x\,dx$

$= e^{-2}(e^{\ln 2} - e^0) = e^{-2}(2-1)$

$= \frac{1}{e^2}$

4. $\int_{\pi/6}^{\pi/2} \csc x\,(3\sin 2x + 5\sin x)\,dx$

$$= \int_{\pi/6}^{\pi/2} \frac{1}{\sin x}\left(6\sin x\cos x + 5\sin x\right)dx$$

$$= \int_{\pi/6}^{\pi/2} \left(6\cos x + 5\right)dx$$

$$= 6\sin x\Big|_{\pi/6}^{\pi/2} + 5\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$= 6\left(1 - \frac{1}{2}\right) + 5 \cdot \frac{\pi}{3}$$

$$= 3 + \frac{5\pi}{3}$$

5. $\int_0^1 \frac{3x^3+x^2-18x-6}{3x+1}\,dx$

$$= \int_0^1 \frac{x^2(3x+1) - 6(3x+1)}{3x+1}\,dx$$

$$= \int_0^1 \frac{\cancel{(3x+1)}(x^2-6)}{\cancel{3x+1}}\,dx$$

$$= \frac{x^3}{3} - 6x\Big|_0^1$$

$$= \frac{1}{3} - 6 - 0 = \boxed{\frac{-17}{3}}$$

6. $\int_0^{1/2} \left(\frac{6}{\sqrt{1-t^2}} + \frac{12t-2}{\sqrt{t}} \right) dt$

$$= 6 \sin^{-1} t \Big|_0^{1/2} + \int_0^{1/2} \left(12\sqrt{t} - 2t^{-\frac{1}{2}} \right) dt$$

$$= 6\left(\frac{\pi}{6} - 0\right) + 12 \frac{t^{3/2}}{3/2} - 2 \frac{t^{1/2}}{1/2} \Big|_0^{1/2}$$

$$= \pi + 8\left(\left(\tfrac{1}{2}\right)^{3/2} - 4 \left(\tfrac{1}{2}\right)^{1/2} \right)$$

$$= \pi + 8\left(\frac{1}{2\sqrt{2}} - 4 \cdot \frac{1}{\sqrt{2}} \right)$$

$$= \pi + \frac{4}{\sqrt{2}} - \frac{32}{\sqrt{2}}$$

$$= \pi - \frac{28}{\sqrt{2}}$$

7. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2-2i+1}$ $= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{n}{n^2}}{\frac{n^2}{n^2} + \frac{(i-1)^2}{n^2}}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\left(\frac{i-1}{n}\right)^2}$$

$$= \int_0^1 \frac{1}{1+x^2} dx$$

$$= \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

End of 5.4- Dr. Khalid Adarbeh.

Section 5.5: The substitution rule

Substitution rule for indefinite integrals:

- What is $\int 2x\sqrt{1+x^2}\ dx$?

- It is very clear that guessing the antiderivative of $2x\sqrt{1+x^2}$ is not easy.

- Suggestion: Use the change of variable $u = 1 + x^2$ and then find the differential dx in terms of x and the differential du.

$$u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$$

$$\int 2x\sqrt{1+x^2}\,dx = \int 2x\sqrt{u}\,\frac{du}{2x} = \int \sqrt{u}\,du$$

$$= \frac{u^{3/2}}{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

Another method: Try $y = \sqrt{1+x^2}$ ...

- As you noticed, and similar to a lot of math (calculus) problems, the change of variables was the key to solve the previous problem. This rule is called the substitution rule to find the integrals.

- It is very clear that the method of substitution works when we have an integral that can be written in the form:

$$\int F'(g(x))g'(x)dx$$

$\boxed{4}$ **THE SUBSTITUTION RULE** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Example 1: Find the following integrals:

1. $\int \frac{\tan^{-1}x}{1+x^2}\,dx$ Let $y = \tan^{-1}x \Rightarrow dy = \frac{1}{1+x^2}dx$

$\Rightarrow dx = (1+x^2)\,dy$

$$I = \int \frac{y}{\cancel{1+x^2}} \cancel{dy}\,\cancel{(1+x^2)}\,dy$$

$$= \int y\,dy = \frac{y^2}{2} + c = \frac{1}{2}\left(\tan^{-1}x\right)^2 + c$$

2. $\int (1+3x)^{10}dx$ Let $y = 1+3x \Rightarrow dy = 3\,dx \Rightarrow dx = \frac{dy}{3}$

$$I = \int y^{10}\frac{dy}{3} = \frac{1}{3}\cdot\frac{y^{11}}{11} + c = \frac{(1+3x)^{11}}{33} + c$$

Result: $\int f'(ax+b)\,dx = \frac{f(ax+b)}{a} + C$

ex: $\int (1+3x)^{10}dx = \frac{(1+3x)^{11}}{(11)(3)} + C$

ex: $\int e^{5-2x}dx = \frac{e^{5-2x}}{-2} + C$.

3. $\int \tan x \, dx$ We can use the following result:

$$* \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C.$$

$$\int \tan x \, dx = -\int -\frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

4. $\int x^5 \sqrt{x^2+3} \, dx$ Let $y = \sqrt{x^2+3} \Rightarrow y^2 = x^2 + 3$

$$\Rightarrow 2y\,dy = 2x\,dx \Rightarrow dx = \frac{y}{x} dy$$

$$I = \int x^5 y \cdot \frac{y}{x} dy = \int x^4 y^2 \, dy$$

$$= \int (y^2-3)^2 y^2 \, dy = \int (y^6 - 6y^4 + 9y^2) \, dy$$

$$= \frac{y^7}{7} - 6\frac{y^5}{5} + 9\frac{y^3}{3} + C$$

$$= \frac{(\sqrt{x^2+3})^7}{7} - \frac{6(\sqrt{x^2+3})^5}{5} + 3(\sqrt{x^2+3})^3 + C$$

5. $\int \sin 2x\, e^{sin^2 x}\, dx$ Let $y = \sin^2 x \Rightarrow dy = 2\sin x \cos x\, dx$

$\Rightarrow dx = \dfrac{dy}{\sin 2x}$

$$\int \cancel{\sin 2x}\, e^{y} \frac{dy}{\cancel{\sin 2x}} = \int e^{y} dy = e^{y} + C$$

$$= e^{\sin^2 x} + C$$

6. $\int \frac{1}{x+\sqrt{x}} dx$ Let $y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow 2y\, dy = dx$

$$\int \frac{1}{y^2+y} \cdot 2y\, dy = 2\int \frac{1}{y+1} dy = 2 \ln|1+y| + C$$

$$= 2 \ln(1+\sqrt{x}) + C$$

Substitution rule for definite integrals

6 THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 2: Evaluate $\int_1^e \frac{dx}{x+2x\ln\sqrt{x}} = \int_1^e \frac{dx}{x+2x\ln x^{\left(\frac{1}{2}\right)}} = \int_1^e \frac{dx}{x(1+\ln x)}$

Method 1: by changing the integration limits.

Let $y = 1+\ln x \Rightarrow dy = \frac{1}{x}dx \Rightarrow dx = x\,dy$

$x=1 \Rightarrow y = 1+\ln(1) = 1$

$x=e \Rightarrow y = 1+\ln(e) = 2$

$$\int_1^e \frac{dx}{x(1+\ln x)} = \int_1^2 \frac{x\,dy}{x(y)} = \int_1^2 \frac{dy}{y}$$

$$= \ln|y| \Big|_1^2 = \ln 2 - \ln 1 = \boxed{\ln 2}$$

Method 2: without changing the integration limits.

Let $y = 1+\ln x \Rightarrow dy = \frac{1}{x}dx \Rightarrow dx = x\,dy$

$$\int_1^e \frac{dx}{x(1+\ln x)} = \int_{g(1)}^{g(e)} \frac{x\,dy}{x(y)} = \ln|y| \Big|_{g(1)}^{g(e)}$$

$$= \ln(1+\ln x) \Big|_{x=1}^{x=e} = \ln(1+\ln e) - \ln(1+\ln 1)$$

$$= \ln 2 - \ln 1 = \boxed{\ln 2}$$

Integration of symmetric (even & odd) functions:

7 INTEGRALS OF SYMMETRIC FUNCTIONS Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 3: Evaluate the following:

1. $\int_{-1}^{1} \frac{1+x+\sin x}{x^2+1}$

$$= \int_{-1}^{1} \underbrace{\frac{1}{x^2+1}}_{\text{even}}\,dx + \int_{-1}^{1} \underbrace{\frac{x}{x^2+1}}_{\text{odd}}\,dx + \int_{-1}^{1} \underbrace{\frac{\sin x}{x^2+1}}_{\text{odd}}\,dx$$

$$= 2\tan^{-1}x \Big|_0^1 + 0 + 0$$

$$= 2\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= 2\left(\frac{\pi}{4} - 0\right) = \boxed{\frac{\pi}{2}}$$

2. $\int_{-1}^{1} \frac{1+x^3}{\sqrt{4-x^2}}\,dx$

$$= \int_{-1}^{1} \underbrace{\frac{1}{\sqrt{4-x^2}}}_{\text{even}}\,dx + \int_{-1}^{1} \underbrace{\frac{x^3}{\sqrt{4-x^2}}}_{\text{odd}}\,dx$$

$$= 2\int_0^1 \frac{1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}}\,dx + 0$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^1 \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

Let $y = \frac{x}{2}$

$dy = \frac{1}{2}dx$

$\Rightarrow dx = 2\,dy$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = \frac{1}{2}$

$$= \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-y^2}} \cdot 2\,dy$$

6

$$= 2\sin^{-1}y \Big|_0^{\frac{1}{2}} = 2\left(\frac{\pi}{6} - 0\right) = \boxed{\frac{\pi}{3}}$$

Exercises:

1. Assume that f is an $\underline{\text{odd}}$ continuous function with $\int_0^4 f(x)dx = 6$. Find $I=\int_0^2 f(-2x)dx$. Let $y=-2x \Rightarrow dy=-2\,dx \Rightarrow dx=-\frac{1}{2}dy$

$$\boxed{\begin{array}{l} x=0 \Rightarrow y=0 \\ x=2 \Rightarrow y=-4 \end{array}}$$

$$I = \int_0^{-4} f(y)\cdot -\frac{1}{2}dy$$

$$= \frac{1}{2}\int_{-4}^{0} f(y)\,dy = -\frac{1}{2}\int_0^4 f(y)\,dy$$

(since f is odd wrt $\int_{-a}^{0} f(x)\,dx = -\int_0^a f(x)\,dx$)

$$= -\frac{1}{2}(6)$$

$$= \boxed{-3}$$

2. Assume that f is an $\underline{\text{even}}$ continuous function with $\int_0^4 f(x)dx = 5$. Find $\int_{-2}^{2}[xf(x^2) + f(2x)]dx = \underbrace{\boxed{\int_{-2}^{2} x\,f(x^2)\,dx}}_{I_1} + \underbrace{\boxed{\int_{-2}^{2} f(2x)\,dx}}_{I_2} = 0+5=5$

~~$x f(x^2)$ is odd and wrt~~ $\cancel{\Rightarrow I_1 = \int_{-2}^{2} x\,f(x^2)}$

* Let $y=x^2 \Rightarrow dy = 2x\,dx$

$x=-2 \Rightarrow y=4$
$x=2 \Rightarrow y=4$

$$I_1 = \int_4^4 x\,f(y)\,\frac{dy}{2x}$$

$$= \frac{1}{2}\int_4^4 f(y)\,dy = 0$$

* Let $u=2x \Rightarrow du = 2\,dx$,

$x=-2 \Rightarrow u=-4$
$x=2 \Rightarrow u=4$

$$I_2 = \int_{-4}^{4} f(u)\cdot\frac{1}{2}du = \frac{1}{2}\cdot 2\int_0^4 f(u)\,du \quad \text{(since } f \text{ is even)}$$

$$= \boxed{5}$$

3. Assume that $f(x)$ is an **even** continuous function with $\int_0^9 f(x)dx = 12$ and $\int_0^4 \sqrt{x} f(x\sqrt{x})dx = 4$. Find:

a. $\int_{-8}^{9} f(x)dx$.

b. $\int_{-4}^{4} \tan^{-1} x \; f(x)dx$

Let $y = x\sqrt{x} \Rightarrow y = x^{\frac{3}{2}} \Rightarrow dy = \frac{3}{2}\sqrt{x}\,dx$

$\Rightarrow dx = \dfrac{\frac{2}{3}dy}{\sqrt{x}}$

$\int_0^4 \sqrt{x}\, f(x^{3/2})dx = 4$

$x = 0 \Rightarrow y = 0$

$x = 4 \Rightarrow y = 4^{3/2} = 8$

$\int_0^8 \cancel{\sqrt{x}}\, f(y) \cdot \dfrac{\frac{2}{3}dy}{\cancel{\sqrt{x}}} = 4$

$$\boxed{\int_0^8 f(y)\,dy = 6}$$

(a) $\int_{-8}^{9} f(x)dx = \int_{-8}^{0} f(x)dx + \int_0^9 f(x)dx$

$= \int_0^8 f(x)dx + 12 = 6 + 12 = \boxed{18}$

(since f is even)

(b) $\int_{-4}^{4} \tan^{-1}(x)\, f(x)\,dx = 0$ ($f(x)$ is even, $\tan^{-1}x$ is odd $\Rightarrow \tan^{-1}x\, f(x)$ is odd)

More Exercises:

Evaluate the integrals:

1. $\int \tan x \ln \cos x \ dx$

$y = \ln(\cos x) \Rightarrow dy = \frac{-\sin x}{\cos x} dx$

$dx = \frac{dy}{-\tan x}$

$\int \cancel{\tan x} \cdot y \cdot \frac{dy}{-\cancel{\tan x}}$

$= -\frac{y^2}{2} + c = -\frac{1}{2}(\ln \cos x)^2 + c$

2. $\int_0^{\cancel{\pi/2}\ \pi/4} \frac{12}{\cos^2 x(1+3\tan x)^{3/2}} dx$

Let $y = 1 + 3\tan x \Rightarrow dy = 3\sec^2 x\, dx \Rightarrow dx = \frac{dy}{3\sec^2 x}$

$x = 0 \Rightarrow y = 1 + 0 = 1$

$x = \frac{\pi}{4} \Rightarrow y = 1 + 3(1) = 4$

$\int_1^4 \frac{12}{\cancel{\cos^2 x}\ y^{3/2}} \cdot \frac{dy}{3\cancel{\sec^2 x}} = 4\int_1^4 y^{-3/2} dy$

$= 4 \left. \frac{y^{-\frac{1}{2}}}{-\frac{1}{2}} \right|_1^4 = -8\left(4^{-\frac{1}{2}} - 1^{-\frac{1}{2}}\right)$

$= -8\left(\frac{1}{2} - 1\right) = \boxed{4}$

3. $\int \frac{1-2x}{1+2x}\,dx$

$$= -\int \frac{2x-1+1-1}{2x+1}\,dx$$

$$= -\int \frac{2x+1-2}{2x+1}\,dx$$

$$= -\int \left(1 - \frac{2}{2x+1}\right)dx$$

$$= -\left(x - \ln|2x+1|\right) + c$$

Another method :

Long division

$$\begin{array}{r|l} & -1 \\ 1+2x & 1-2x \\ & \pm \;\; \pm 2x \\ \hline & 2 \end{array}$$

$$\int \frac{1-2x}{1+2x}\,dx = \int \left(-1 + \frac{2}{1+2x}\right)dx$$

$$= -x + \ln|1+2x| + c$$

4. $\int \frac{dx}{\sqrt[4]{x}\,(1+\sqrt{x})}$ Let $y = \sqrt[4]{x} \Rightarrow y^4 = x \Rightarrow 4y^3\,dy = dx$

$$\int \frac{4y^3\,dy}{y(1+y^2)} = 4\int \frac{y^2+1-1}{1+y^2}\,dy$$

$$= 4\int \left(1 - \frac{1}{1+y^2}\right)dy$$

$$= 4\left(y - \tan^{-1} y\right) + C$$

$$= 4\left(\sqrt[4]{x} - \tan^{-1}\sqrt[4]{x}\right) + C$$

5. $\int \sqrt[3]{\frac{1-x}{x^7}}\, dx$

$$= \int \sqrt[3]{\frac{1-x}{x^6 \cdot x}}\, dx$$

$$= \int \frac{1}{x^2} \sqrt[3]{\frac{1}{x}-1}\, dx$$

Let $y = \frac{1}{x} - 1 \Rightarrow dy = \frac{-1}{x^2}\, dx$

$$\int \frac{1}{x^2} \sqrt[3]{y}\, (-x^2)\, dy$$

$$= -\int y^{\frac{1}{3}}\, dy = -\frac{y^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{-3}{4}\left(\frac{1}{x}-1\right)^{\frac{4}{3}} + C$$

6. $\int e^{x+e^x}\, dx$

$$= \int e^x \cdot e^{e^x}\, dx$$

Let $y = e^x \Rightarrow dy = e^x\, dx \Rightarrow dx = \frac{dy}{y}$

$$\int y\, e^y \frac{dy}{y} = \int e^y\, dy = e^y + C$$

$$= e^{e^x} + C$$

7. $\int \frac{x^{49}}{\sqrt{1-x^{100}}}\, dx$

Let $y = x^{50} \Rightarrow dy = 50x^{49}\, dx$

$$\int \frac{\cancel{x^{49}}}{\sqrt{1-y^2}} \frac{dy}{50\cancel{x^{49}}}$$

$$= \frac{1}{50} \int \frac{1}{\sqrt{1-y^2}}\, dy$$

$$= \frac{1}{50} \sin^{-1} y + C = \frac{1}{50} \sin^{-1}(x^{50}) + C$$

8. $\int \frac{dx}{e^x+e^{-x}}$

$y = e^x \Rightarrow dy = e^x dx \Rightarrow dx = \frac{dy}{y}$

$\int \frac{dy}{y\left(y+\frac{1}{y}\right)} = \int \frac{dy}{y^2+1} dy$

$= \tan^{-1} y + C$

$= \tan^{-1}(e^x) + C$

9. $\int_0^4 \sqrt{64x - x^4}\, dx$

$= \int_0^4 \sqrt{x}\sqrt{64 - x^3}\, dx$

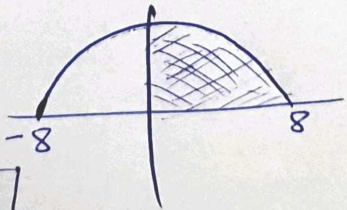
Let $y = x^{3/2} \Rightarrow dy = \frac{3}{2} x^{\frac{1}{2}} dx$

$x = 0 \Rightarrow y = 0$

$x = 4 \Rightarrow y = 8$

$\int_0^8 \sqrt{x}\sqrt{64 - y^2}\, \frac{dy}{\frac{3}{2} x^{\frac{1}{2}}}$

$= \frac{2}{3} \int_0^8 \sqrt{64 - y^2}\, dy = \frac{2}{3}$ area of $\frac{1}{4}$ circle



$= \frac{2}{3} \cdot \frac{1}{4} \pi (8^2)$

$= \frac{2}{3} \cdot \frac{1}{4} \pi\, 64 = \boxed{\frac{32\pi}{3}}$