Existence of unique solution

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Abstract

This paper contains the derivation of Riesz representation theorem [1] and the Lax Milgram theorem [2].

Proposition 1

We claim that the orthogonal complement of any subset X of a Hilbert space H is a closed. We have

$$X^{\perp} := \{x \in H | x \perp y \quad \forall y \in X\}$$

and want to show that $X^{\perp}$ is closed.

Proof. Let $x_n \to x$ in H with $x_n \in X^{\perp}$. Then by continuity of the inner product

$$\langle x, y\rangle = \lim_{n\to\infty} \langle x_n, y\rangle = 0.$$

Whence, we see that $X^{\perp}$ contains all its limit points which means it is a closed subspace of H. □

Best approximation theorem

Let C be a nonempty convex closed subset of H. Then for all $x \in H$ there exists a unique $c \in C$ that minimises the distance from x to the points of C:

$$\|x - c\| = \min_{y\in C} \|x - y\|$$

Proof. Let $\{y_n\}_{n\geq 1}$ be a sequence in C such that

$$\lim_{n\to\infty} \|x - y_n\| = \inf_{y\in C} \|x - y\| := D$$

By the parallelogram law applied to the vectors $x - y_m$ and $x - y_n$ we have

$$\|y_n - y_m\|^2 + \|2x - (y_n - y_m)\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2.$$

As $m, n \to \infty$, the right hand side tends to $2D^2 + 2D^2 = 4D^2$ and from convexity we have $\frac{1}{2}(y_m - y_n) \in C$. It follows that

$$\|2x - (y_n - y_m)\|^2 = 4\|x - \frac{1}{2}(y_n - y_m)\|^2 \geq 4D^2.$$

Hence,

$$\begin{aligned}\lim_{m,n\to\infty} \|y_n - y_m\|^2 &= 4D^2 - \lim_{n,m\to\infty} \|2x - (y_n - y_m)\|^2 \\ &\leq 4D^2 - 4D^2 = 0.\end{aligned}$$

This shows that the sequence $\{y_n\}_{n\geq 1}$ is Cauchy in C. Since H is complete we have $\lim_{n\to\infty} y_n = c$ for some $c \in H$, and since C is closed we have $c \in C$. Now,

$$\|x - c\| = \lim_{n\to\infty} \|x - y_n\| = D,$$

so c minimizes the distance to x. □

Theorem 1

If X is a closed linear subspace of H, then we have an orthogonal direct sum decomposition

$$H = X \oplus X^{\perp}.$$

That is, we have $X \cap X^{\perp} = \{0\}$, $X + X^{\perp} = H$ and $X \perp X^{\perp}$.

Proof. From Proposition 1. we have that $Y^{\perp}$ is a closed subset of H. If $y \in Y \cap Y^{\perp}$ then $y \perp y$, so $\langle y, y \rangle = 0$ and $y = 0$. It remains to show that $Y + Y^{\perp} = H$. let $x \in H$ be arbitrary and fixed, we must show that $x \in Y + Y^{\perp}$. Let $f_y : H \to X$ denote the mapping arising from the previous theorem. Then $f_y(x)$ is the unique element of Y that minimizes the distance to x.

$$\|x - f_y(x)\| = \min_{y \in Y} \|x - y\|$$

Set $y_0 := f_y(x)$ and $y_1 := x - y_0$. Then $y_0 \in Y$, and for all $y \in Y$ we have

$$\begin{aligned} \|y_1\| = \|x - y_0\| &\leq \|x - (y_0 - y)\| \\ &= \|y + (x - y_0)\| = \|y + y_1\|. \end{aligned}$$

We claim that this implies $y_1 \in Y^{\perp}$. To see this, fix a nonzero $y \in Y$. For any $c \in \mathbb{F}$ we have

$$\|y_1\|^2 \leq \|cy + y_1\|^2 = |c|^2 \|y\|^2 + 2\mathrm{Re}\langle cy, y \rangle + \|y_1\|^2$$

Taking $c = -\frac{\overline{\langle y, y_1 \rangle}}{\|y\|^2}$ gives

$$0 \leq \frac{\langle y, y_1 \rangle^2}{\|y\|^2} - 2\frac{\langle y, y_1 \rangle^2}{\|y\|^2}$$

which is only possible if $\langle y, y_1 \rangle = 0$. Since $y \neq 0 \in Y$ was arbitrary this shows that $y_1 \in Y^{\perp}$ which proves the claim. It follows that $x = y_0 + y_1$ belongs to $Y + Y^{\perp}$.

□

Riesz representation theorem

If ϕ: H $\to$ **F** is a bounded linear functional, there exists a unique element $y \in$ H such that

$$\phi(x) = \langle x, y \rangle \qquad \forall \; y \in H \tag{1}$$

Proof. If $\phi(x) = 0 \; \forall x \in H$, we take $y = 0$. So assume $\phi(x) \neq 0$, we know from (insert theorem) that closed subspaces of a Hilbert space H is orthogonally complemented. Whence $(N(\phi))^{\perp} \neq \{0\}$, so we can choose a norm one vector $y_0 \in (N(\phi))^{\perp}$. Fix an arbitrary $x \in H$ with $c := \frac{\phi(x)}{\phi(y_0)}$, we then have

$$\phi(x - cy_0) = \phi(x) - c\phi(y_0) = 0$$

Where we have used the linearity of ϕ and our definition of c. This means that $x - cy_0 \in N(\phi)$, so $x - cy_0 \perp y_0$ and

$$\phi(x) = c\phi(y_0) = \phi(y_0)\langle cy_0, y_0 \rangle = \phi(y_0)\langle x, y_0 \rangle = \langle x, \overline{\phi(y_0)} y_0 \rangle$$

Setting $y = \overline{\phi(y_0)} y_0$ we obtain (1). To prove uniqueness, suppose that $\phi = \phi_y = \phi_{y'}$ for $y, y' \in H$. Then

$$\|y - y'\|^2 = \langle y - y', y - y' \rangle = \langle y - y', y \rangle - \langle y - y', y' \rangle = \phi(y - y') - \phi(y - y') = 0$$

Where we have used the properties of the inner product and (1). Since the norm is positive definite we see that the above derivation implies $y = y'$ □

Proposition 2

Assume $A : H \to H$ be a bounded linear operator that is bounded below i.e.

$$\|Au\| \geq C\|u\|$$

for $C > 0$ and all $u \in H$. Then A is injective (1-1) and has closed range.

Proof. Assume A is not 1-1, then $Ax = Ay$ for $x \neq y$, but since A is bounded from below we have

$$0 = \|Ax - Ay\| = \|A(x - y\| \geq Cx - y > 0$$

which contradicts $x \neq y$. To show that $R(A)$ is closed let $\{y_n\}_{n\geq 1}$ be a sequence in $R(A)$ that converges to a point $y \in H$. For each $n \in \mathbb{N}$, there exists $x_n \in H$ such that $y_n = A(x_n)$. For any $m, n \in \mathbb{N}$ we have

$$C\|x_m - x_n\| \leq \|A(x_m - x_n)\| = \|y_m - y_n\|.$$

Since $\{y_n\}_{n\geq 1}$ is a Cauchy sequence, we see from the above derivation that $\{x_n\}_{n\geq 1}$ is also a Cauchy sequence. Since H is a Banach space it is complete and there exists a $x \in H$ s.t. $x_n \to x$. Since A is bounded it is continuous and $y = Ax \in R(A)$. □

Lax Milgram theorem

Assume that $H \times H \to \mathbb{R}$ is a bilinear mapping, for which there exists constants $\alpha, \beta > 0$ such that

i) $|B[u, v]| \leq \alpha\|u\|\|v\|$
ii) $\beta\|u\|^2 \leq B[u, u]$

where $u, v \in H$. Finally let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique $u \in H$ such that

$$B[u, v] = [f, v]$$

for all $v \in H$. $[\cdot, \cdot]$ denotes the paring of H with its dual space $[\cdot, \cdot] : H^* \times H \to \mathbb{R}$.

Proof. For each fixed $u \in H$, the mapping $v \mapsto B[u, v]$ is a bounded linear functional on H, whence by the Riesz representation theorem there exists a unique element $w \in H$ satisfying

$$B[u, v] = \langle w, v\rangle.$$

Let us write $Au = w$ whenever the above holds, so that

$$B[u, v] = \langle Au, v\rangle$$

for $u, v \in H$. We claim that $A : H \to H$ is a bounded linear operator, 1-1 and that the range of A is closed in H.
To prove that A is linear we let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in H$. We then have

$$\begin{aligned}\langle A(\lambda_1 u_1 + \lambda_2 u_2, v\rangle &= B[\lambda_1 u_1, \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1\langle Au_1, v\rangle + \lambda_2\langle Au_2, v\rangle \\ &= \langle \lambda_1 Au_1 + \lambda_2 Au_2, v\rangle.\end{aligned}$$

Since this holds for all $v \in H$, A is a linear operator. Furthermore,

$$\|Au\|^2 = \langle Au, Au\rangle = B[u, Au] \leq \alpha\|u\|\|Au\|$$

hence,

$$\|Au\| \leq \alpha\|u\|.$$

So A is bounded. Next, we compute

$$\beta\|u\|^2 \leq B[u, u] = \langle Au, u\rangle \leq \|Au\|\|u\|.$$

Where in the last inequality we have used Cauchy - Scwartz. This shows that A is bounded from below

$$\beta\|u\| \leq \|Au\|$$

and by proposition 2 we have that A is 1-1 and $R(A)$ is closed. We demonstrate now that $R(A) = H$. For if not, then, since $R(A)$ is closed, there would by Theorem 1 exists a nonzero element $w \in H$ with $w \in R(A)^{\perp}$. But this fact in turn implies

$$\beta\|w\|^2 \leq B[w, w] = \langle Aw, w\rangle = 0$$

So, w must be zero, and we have obtained a contradiction.
Next, we observe once more from the Riesz representation theorem that

$$[f, v] = \langle w, v\rangle \quad \forall v \in H$$

for some element $w \in H$. We can that use our previous derivations to find $u \in H$ satisfying $w = Au$, then

$$B[u, v] = \langle Au, v\rangle = \langle w, v\rangle = [f, v].$$

Finally, we show that there is at most one element $u \in H$ satisfying the above. For if both

$$B[u_1, v] = [f, v] \quad \text{and} \quad B[u_2, v] = [f, v]$$

then $B[u_1 - u_2, v] = 0 \; \forall v \in H$. We set $v = u_1 - u_2$ to find

$$\beta\|u_1 - u_2\|^2 \leq B[u_1 - u_2, u_1 - u_2] = 0 \to u_1 = u_2$$

$\square$

References

[1] Van Neerven, J. (2022). Hilbert Spaces. In Functional Analysis (Cambridge Studies in Advanced Mathematics, pp. 87-114). Cambridge: Cambridge University Press. doi:10.1017/9781009232487.005

[2] Lawrence C. Evans (2010). Partial Differential Equations second edition. (Lax Milgram Theorem p. 315-317) American Mathematical Society