Linear programming notes

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Contents

Abstract

This document contains notes about Linear programming.

1 Supply chain model

We define a simplified supply chain model as a system that produces products with a chain of product transformations performed by suppliers.

Let be

$A = a_1 \dots a_n$ the set of supplier types

$B = b_1 \dots b_m$ the set of product types.

The supplier can perform only a single transformation for the its duration.

Let us define

n_i the number of suppliers of type $i \in A$.

v_i the value of product $i \in B$

q_{ij} the quantity of produced product $j \in B$ for supplier $i \in A, j$ in production interval

$t_{ij}, i \in A, j \in B$ the production interval for product $j \in B$ in supplier $i \in A, j$

c_{ijk} the quantity of product $k \in B$ consumed to produce the product $j \in Bj$ in production interval of supplier $i \in A$.

1.1 Production

By now we do not consider the constraint on the availability of consuming products. It will be considered later.

When a supplier of type i is ready to produce we assign a production slot for the product j such that

y_{ij} is the weighted distribution of slots

The value y_{ij} is constrained to

$$0 \leq y_{ij} \leq 1 \tag{1}$$

Moreover we may put the supplier in idle state for a while

z_{i0} is the idle time during a whole production cycle

with

$$z_{i0} \geq 0 \tag{2}$$

The total time of production cycle for supplier i is

$$z_i = \sum_{j \in B} y_{ij} t_{ij} + z_{i0}$$

Let normalize the total time production to 1

$$z_i = \sum_{j \in B} y_{ij} t_{ij} + z_{i0} = 1 \tag{3}$$

During this interval all the suppliers i produce the product j at a slot rate

$$s_{ij} = n_i \frac{y_{ij}}{z_i} = n_i y_{ij} \Big|_{i \in A, j \in B} \tag{4}$$

The production rate of product j for supplier i is

$$p_{ij} = q_{ij} s_{ij} = n_i q_{ij} y_{ij}|_{i \in A, j \in B}$$

While the production rate of product j is

$$p_{j \in B} = \sum_{i \in A,} p_{ij} = \sum_{i \in A,} n_i q_{ij} y_{ij} \tag{5}$$

1.2 Constraints

The (4) expresses the slot rate of a product for specific suppliers.

Therefore we can calculate the consumption rate of a product k to produce product j for suppliers i

$$e_{ijk} = c_{ijk}s_{ij}|_{i \in A, j, k \in B}$$

Then the total consumption rate for product k is

$$e_k = \sum_{i \in A, j \in B} e_{ijk} = \sum_{i \in A, j \in B} n_i c_{ijk} y_{ij} \tag{6}$$

If

$$e_i < p_i$$

the product s produced at a higher rate then it is consumped creating a surplus that can be sold.

On the other hane we cannot have

$$e_i > p_i$$

because it cannot consume more product than the produced.

So the system must satisfy the constraint

$$p_i - e_i \geq 0 \tag{7}$$

1.3 Value rate

We can compute the value rate for the whole supply chain

$$v = \sum_{i \in B} (p_i - e_i) v_i \tag{8}$$

The problem is to find the optimal production configuration that maximize the value rate. This is defined by the linear system composed by (1), (2), (3), (7), (8)

$$\begin{cases} \max_{(y_{ij}, z_{i0})}(v) & , i \in A, j \in B \\ \sum_{j \in B} y_{ij} t_{ij} + z_{i0} = 1 & , i \in A \\ z_{i0} \geq 0 & , i \in A \\ y_{ij} \geq 0 & , i \in A, j \in B \\ y_{ij} \leq 1 & , i \in A, j \in B \\ p_i - e_i \geq 0 & , i \in B \end{cases}$$

1.4 Tensor form

Trasformiamo ora la (5) in forma tensoriale:

$$P_i = A_{ijk} Y_{kj} = AY$$

che deve essere equivalente a

$$p_{j'} = \sum_{i'} n_{i'} q_{i'j'} y_{i'j'} \tag{9}$$

da cui

$$\begin{aligned} A_{iik} &= n_k q_{ki} \\ A_{ijk} &= 0 \qquad , i \neq j \end{aligned} \tag{10}$$

Se poniamo

$$\begin{aligned} Q_{ij} &= q_{ij} \\ N_{ii} &= n_i \\ N_{ij} &= 0, i \neq j \end{aligned}$$

possiamo scrivere

$$B_{ij} = N_{ik} Q_{kj} = NQ = n_i q_{ij}$$

quindi

$$\begin{aligned} A_{iik} &= B_{ki} \\ A_{ijk} &= 0, i \neq j \end{aligned}$$

Poniamo

$$A_{ijk} = \delta_{ijl} B_{kl} = \delta B^T = B\delta = NQ\delta = N_{il} Q_{lm} \delta_{mjk} \tag{11}$$

dove

$$\begin{aligned} \delta_{ijl} &= 1 \quad , i = j = l \\ \delta_{ijl} &= 0 \end{aligned}$$

quindi

$$P_i = NQ\delta Y$$

Allo stesso modo trasformiamo la (6)

$$E_i = M_{ijk} Y_{kj} = MY$$

che deve essere equivalente a

$$e_{k'} = \sum_{i',j'} n_{i'} c_{i'j'k'} y_{i'j'} \tag{12}$$

da cui

$$M_{ijk} = n_k c_{kji} = N_{kl} C_{lji} = (NC)^T = C^T N^T = C^T N \tag{13}$$

dove

$$C^T = (C^T)_{ijk} = C_{kji}$$

quindi

$$E_i = C^T N Y$$

Abbiamo poi la (8)

$$v = V_i(P_i - E_i) = V\left(NQ\delta - C^T N\right)Y$$

Infine la (3) diventa

$$E_i = U_{ijk}Y_{kj} - Z_i$$

che deve essere qeuivalente a

$$e_i = \sum_j t_{ij} * y_{ij} - z_{i0}$$

ovvero

$$\begin{array}{l} U_{iji} = t_{ji} \\ U_{ijk} = 0 \quad\;\; , i \neq k \end{array}$$

$$\begin{array}{l} U_{ijk} = \delta_{ilk}T_{lj} = \delta_{ikl}T_{lj} \\ V_{ijk} = \delta_{ijl}T_{lk} = \delta T \\ U_{ijk} = V_{ikj} = \delta_{ikl}T_{lj} \end{array}$$

$$\sum_j t_{ij} * y_{ij} + z_{i0} = U_{ijk}Ykl + Z_i = UY + Z = \vec{1} \tag{14}$$

1.4.1 Verifiche

Verifica (10)

$$\begin{array}{l} P_1 = A_{111}Y_{11} + A_{112}Y_{21} + A_{121}Y_{12} + A_{122}Y_{22} \\ P_2 = A_{211}Y_{11} + A_{212}Y_{21} + A_{221}Y_{12} + A_{222}Y_{22} \\ A_{111} = n_1q_{11} \\ A_{112} = n_2q_{21} \\ A_{121} = 0 \\ A_{122} = 0 \\ A_{211} = 0 \\ A_{212} = 0 \\ A_{221} = n_1q_{12} \\ A_{222} = n_2q_{22} \\ P_1 = n_1q_{11}Y_{11} + n_2q_{21}Y_{21} \\ P_2 = n_1q_{12}Y_{12} + n_2q_{22}Y_{22} \end{array}$$

equivalente alla (9)

$$\begin{array}{l} p_1 = n_1q_{11}y_{11} + n_2q_{21}y_{21} \\ p_2 = n_1q_{12}y_{12} + n_2q_{22}y_{22} \end{array}$$

cvd.

Verifica (11)

$$
\begin{aligned}
&A_{ijk} = \delta_{ijl} B_{lk} = \delta_{ijl} N_{km} Q_{ml} \\
&A_{111} = \delta_{111} N_{11} Q_{11} + \delta_{111} N_{12} Q_{21} \delta_{112} N_{11} Q_{12} + \delta_{112} N_{12} Q_{22} = N_{11} Q_{11} \\
&A_{112} = \delta_{111} N_{21} Q_{11} + \delta_{111} N_{22} Q_{21} \delta_{112} N_{21} Q_{12} + \delta_{112} N_{22} Q_{22} = N_{22} Q_{21} \\
&A_{121} = \delta_{121} N_{11} Q_{11} + \delta_{121} N_{12} Q_{21} \delta_{122} N_{11} Q_{12} + \delta_{122} N_{12} Q_{22} = 0 \\
&A_{122} = \delta_{121} N_{21} Q_{11} + \delta_{121} N_{22} Q_{21} \delta_{122} N_{21} Q_{12} + \delta_{122} N_{22} Q_{22} = 0 \\
&A_{211} = \delta_{211} N_{11} Q_{11} + \delta_{211} N_{12} Q_{21} \delta_{212} N_{11} Q_{12} + \delta_{212} N_{12} Q_{22} = 0 \\
&A_{212} = \delta_{211} N_{21} Q_{11} + \delta_{211} N_{22} Q_{21} \delta_{212} N_{21} Q_{12} + \delta_{212} N_{22} Q_{22} = 0 \\
&A_{221} = \delta_{221} N_{11} Q_{11} + \delta_{221} N_{12} Q_{21} \delta_{222} N_{11} Q_{12} + \delta_{222} N_{12} Q_{22} = N_{11} Q_{12} \\
&A_{222} = \delta_{221} N_{21} Q_{11} + \delta_{221} N_{22} Q_{21} \delta_{222} N_{21} Q_{12} + \delta_{222} N_{22} Q_{22} = N_{22} Q_{22}
\end{aligned}
$$

equivalente alla (10), cvd.

1.5 Altro

Let us normalize the problem such that

$$
\begin{cases}
\min C^T X \\
A_1 X = B_1 \\
A_2 X <= B_2 \\
X >= 0
\end{cases}
$$

so

$$
\begin{cases}
\min(-v) = -V\left(NQ\delta - C^T N\right) Y \\
UY + Z = \vec{1} \\
-V\left(NQ\delta - C^T N\right) Y \leq \vec{0} \\
Y \geq 0 \\
Z \geq 0
\end{cases}
$$