Linear programming notes

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Abstract

This document contains notes about Linear programming.

1 Supply chain model

We define a simplified supply chain model as a system that produces products with a chain of product transformations performed by suppliers.

Let it be

$A = a_1 \dots a_n$ the set of supplier types

$B = b_1 \dots b_m$ the set of product types.

The supplier can perform only a single transformation for the its duration. Each product can be produced by a single supplier type.

Let it be

$S = s_i \in A$ the supplier of product $i \in B$

$N = n_i$ the number of suppliers of type $i \in A$.

$V = v_i$ the value of product $i \in B$

$Q = q_i$ the quantity of produced product $i \in B$l

$T = t_i$ the production interval for product $i \in B$

$D = d_{ij}$ the quantity of product $j \in B$ consumed to produce the product $i \in B$.

1.1 Production

By now we do not consider the constraint on the availability of consuming products. It will be considered later.

Let be

θ_{ij} The matrix that map product $j \in B$ to the supplier $i \in A$

νi The number of supplier of product $i \in B$

such that

$$\theta_{ij} = 1, s_j = i$$
$$\theta_{ij} = 0, s_j \neq i$$
$$\nu_i = n_{s_j}$$

When a supplier of type $i \in A$ is ready to produce we assign a production slot for the product $j \in B$ such that $s_j = i$ and the production time is $t_j w_j$ where

$W = w_i$ is the slot weight for product $i \in B$

such that

$$w_i \geq 0 \tag{1}$$

Moreover we may put the supplier $i \in A$ in idle state for a while

$Z = z_i$ is the idle time during a whole production cycle

such that

$$z_i \geq 0 \tag{2}$$

The total time of production cycle for supplier i is

$$\begin{aligned} &U_i(w_j, z_i) = \textstyle\sum_{j \in B | s_j = i} t_j w_j + z_i \\ &= \textstyle\sum_{j \in B} \theta_{ij} t_j w_j + z_i \\ &= \textstyle\sum_{j,k \in B} \theta_{ik} T_{kj} w_j + z_i \\ &= UW + Z \end{aligned}$$

such that

$$U = u_{ij} = \sum_{k \in B} \theta_{ik} T_{kj} = \Theta T$$

Let normalize the total time production to 1

$$U_i(w_j, z_i) = 1 \tag{3}$$

During this interval the suppliers produce the product $i \in A$ at a slot rate

$$R_i(w_i) = n_{s_i} \frac{w_i}{y_{s_i}} = n_{s_i} w_i = \sum j \in B N_{ij} w_j = NW \tag{4}$$

The production rate of product $i \in B$ is

$$P_i(w_j) = R_i(w_j) q_i = Q_{ik} R_k(w_j) = QNW = PW \tag{5}$$

such that

$$P = QN$$

1.2 Consumption

The (4) expresses the slot rate of a product for specific suppliers. Therefore we can calculate the consumption rate of a product $i \in B$ as the sum of consumptions among to produced products

$$C_i(w_j) = \sum_{k \in B} R_k(w_j) d_{ki} = \sum_{k \in B} d'_{ik} R_k(w_j) = D'NW = CW \tag{6}$$

such that

$$C = D'N$$

The effective production rate is

$$F_i(w_j) = P_i(w_j) - C_i(w_j) = (P - C)W = FW$$

such that

$$F = f_{ij} = P - C = (Q - D')N$$

If

$$F_i(w_j) \geq 0$$

the product is produced at a higher rate then it is consumped creating a surplus that can be sold.

On the other hane we cannot have

$$F_i(w_j) < 0$$

because it cannot consume more product than the produced.

So the system must satisfy the constraint

$$F_i(w_j) \geq 0 \tag{7}$$

1.3 Profit rate

The profit rate for the whole supply chain are

$$G(w_i) = \sum_{j \in B} v_j F_j(w_i) = V'FW = G'W \tag{8}$$

such that

$$G = g_i = (V'F)')F'V = N'(Q' - D)V$$

The problem is to find the optimal production configuration given by the values of w_i and z_i that maximize the value rate $G(w_i)$. This is defined by the linear system composed by (1) (2), (3), (7), (8)

$$\begin{cases} \max_{(w_i, z_j)} G(w_i) & , i \in B, j \in A \\ U_i(w_j, z_i) = 1 & , i \in A, j \in B \\ F_i \geq 0 & , i \in B \\ w_i \geq 0 & , i \in B \\ z_i \geq 0 & , i \in A \end{cases} \tag{9}$$

1.4 Supplier configuration

Una volta risolto il sistema (9) ci troviamo con le distribuzioni degli slot di produzione dei produttori. Per l'uso pratico dobbiamo trasformare i pesi di distribuzione degli slot in una configurazione di produzione reale.

Il primo passo per far questo è trasformare le w_i e z_i nel numero di produttori per prodotto.

Il numero di produttori che producono il prodotto $i \in B$ è dato dal tempo totale di produzione dei suppliers per ogni singolo prodotto rispetto al totale $U_i(w_i, z_j) = 1$

$$\pi_i = n_{s_i} w_i t_i$$

$$\Pi = NTW$$

La parte intera di Π ci da il numero di produttori fissi per prodotto

$$F = |\Pi|$$

Il numero totale di produttori fissi è quindi

$$tot_i = \sum_{j \in B} \theta_{ij} f_j$$

$$TOT = \Theta F$$

La parte frazionaria invece determina la distribuzione parziale dei produttori per prodotto

$$R = \Pi - F$$

Il numero totale di produttori variabili è dato da

$$RTOT = ceil(\Theta R)$$

La distribuzione dei produttori variabili per prodotto e data poi da

$$P(i,j) = \frac{r_i \theta_{ij}}{rtot_j}, rtot_j > 0$$

Ponendo

$$rtot'_j = \max(rtot_j, 1) \tag{10}$$

possiamo generalizzare

$$P(i,j) = \frac{r_i \theta_{ij}}{rtot'_j}$$

e diagonalizzando r e $rtot'$ abbiamo

$$P(i,j) = diag(R)\, \Theta\, diag^{-1}(RTOT')$$