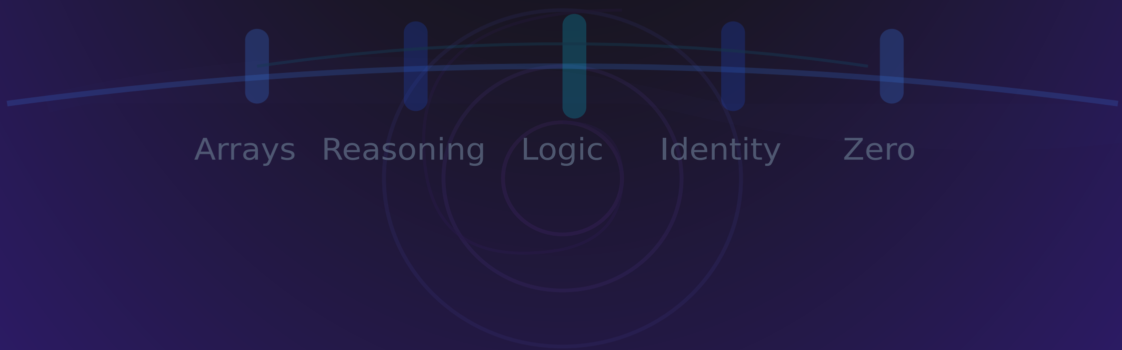
ARLIZ

A JOURNEY THROUGH ARRAYS



LIVING FIRST EDITION

ARLIZ

ARRAYS • REASONING • LOGIC • IDENTITY • ZERO

*"From ancient counting stones to quantum algorithms—
every data structure tells the story of human ingenuity."*

LIVING FIRST EDITION

Updated October 6, 2025

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ARLIZ: ARRAYS, REASONING, LOGIC, IDENTITY, ZERO

A Living Architecture of Computing

DISTRIBUTION & SOURCE ACCESS

Repository: The complete source code (LaTeX, diagrams, examples) is available at:
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Educational Purpose: This work is intended for educational and research purposes. Practical implementation of algorithms and techniques should be thoroughly tested and validated for production use.

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Preface

Every book has its origin story, and this one is no exception. If I were to capture the essence of creating this book in a single word, that word would be **curiosity**—though *improvised* comes as a close second. What you hold in your hands (or view on your screen) is the result of years of persistent questioning, a journey that began with a simple yet profound realization: I didn't truly understand what an array was.

This might sound trivial to some. After all, arrays are fundamental to programming, covered in every computer science curriculum, explained in countless tutorials. Yet despite encountering terms like `array`, `stack`, `queue`, `linked list`, `hash table`, and `heap` repeatedly throughout my studies, I found myself increasingly frustrated by the superficial explanations typically offered. Most resources assumed you already knew what these structures fundamentally represented—their conceptual essence, their historical significance, their mathematical foundations.

But I wanted the *roots*. I needed to understand not just how to use an array, but what it truly meant, how it came to exist, and what hidden capacities it possessed. This led me to a decisive moment:

If I truly want to understand, I must start from zero.

And so began the journey that became Arliz.

The Name and Its Meaning

The name "Arliz" started as a somewhat arbitrary choice—I needed a title, and it sounded right. However, as the book evolved, I discovered a fitting expansion that captures its essence:

Arliz = Arrays, Reasoning, Logic, Identity, Zero

This backronym embodies the core pillars of our exploration:

- **Arrays:** The fundamental data structure we seek to understand from its origins
- **Reasoning:** The logical thinking behind systematic data organization
- **Logic:** The formal principles that govern how computers manipulate information
- **Identity:** The concept of distinguishing, indexing, and assigning meaning to elements within structures
- **Zero:** The philosophical and mathematical foundation from which all computation, counting, and indexing originate

You may pronounce it "Ar-liz," "Array-Liz," or however feels natural to you. I personally say "ar-liz," but the pronunciation matters less than the journey it represents.

What This Book Represents

Arliz is not merely a technical manual on data structures, nor is it a traditional computer science textbook. Instead, it represents something more personal and, I believe, more valuable: a comprehensive exploration of understanding itself. This book grows alongside my own learning, evolving as I discover better ways to explain concepts, uncover new connections, and develop deeper insights.

This living nature means that Arliz is, in many ways, a conversation—between past and present understanding, between theoretical foundations and practical applications, between the author and reader. As long as I continue learning, Arliz will continue growing.

The structure of this book reflects a fundamental belief: genuine understanding requires context. Rather than beginning with syntax and moving to application (the typical approach), we start with the conceptual and historical foundations that make modern data structures possible. We trace the evolution of human thought about organizing information, from ancient counting methods to contemporary computing paradigms.

This approach serves a specific purpose: when you understand the intellectual journey that led to arrays, you develop an intuitive grasp of their behavior, limitations, and potential that no amount of syntax memorization can provide.

My Approach and Principles

Throughout the writing process, I have maintained three core principles:

1. **Conceptual Clarity:** Every concept is presented in its simplest form while maintaining accuracy and depth. My goal is accessibility without superficiality.
2. **Visual Understanding:** Complex ideas are accompanied by diagrams, figures, and visual examples. I believe that concepts which can be visualized are concepts that can be truly understood and retained.
3. **Practical Implementation:** Nearly every topic includes working code and pseudocode that can be easily adapted to major programming languages. Theory without practice is incomplete; practice without theory is fragile.

An important disclosure: many of the algorithms and implementations in this book are my own constructions. Rather than copying optimized solutions from established sources, I have chosen to build understanding from first principles. This means some implementations may run slower than industry standards—or occasionally faster. For me, the process of understanding and constructing has been more valuable than simply achieving optimal performance.

This approach reflects the book's core philosophy: genuine mastery comes from understanding principles deeply enough to reconstruct solutions, not from memorizing existing ones.

About the Author

I am **Mahdi**, though you may know me by my online alias: *Genix*. At the time of writing, I am a Computer Engineering student, but more fundamentally, I am someone who grew up alongside computers—from simple games to terminal commands—always wondering what lies behind the screen of black and green text.

My relationship with computers has been one of continuous curiosity. I am someone who gives computers commands and, more importantly, learns from their responses. There is not much more you need to know about me personally, except that this book represents my attempt to understand the digital world I inhabit as completely as possible.

How to Use This Book

Arliz is freely available and open source. You can access the complete PDF, LaTeX source code, and related materials at:

https://github.com/m-mdy-m/Arliz

Each chapter includes carefully designed exercises and projects. Please do not skip these—they are not busy work but essential components of the learning process. True understanding comes only through active engagement with concepts, through solving problems and building solutions yourself.

I encourage you to approach this book as a collaborative effort. If you discover errors, have suggestions for improvement, or develop insights that could benefit other readers, please share them. This book improves through community engagement, and your contributions make it more valuable for everyone.

A Living Document

Finally, I want to be transparent about what you are engaging with. This is not a finished, polished product in the traditional sense. It is an evolving exploration of fundamental concepts, growing and improving as understanding deepens. You may encounter sections that could be clearer, examples that could be more intuitive, or explanations that could be more complete.

This is intentional. Arliz represents learning in progress, understanding in development. It invites you to participate in this process rather than simply consume its content.

I hope this book serves you well—whether you are beginning your journey with data structures, seeking to deepen existing knowledge, or simply satisfying intellectual curiosity. And if you learn something valuable, discover an error, or develop an insight worth sharing, I hope you will let me know.

After all, this book grows with all of us.

Acknowledgments

I would like to express my gratitude to everyone who supported me during the creation of this book. Special thanks to the open-source community for their invaluable resources and to all those who reviewed early drafts and provided feedback.

How to Read This Book

I understand what you might be thinking. You picked up a book called "Arliz" expecting to learn about arrays, and here I am about to take you on a journey through ancient civilizations and counting systems. You're probably wondering, "What does Mesopotamian mathematics have to do with `int[] myArray = new int[10]`?" That's not just a reasonable question—it's the *right* question to ask.

Let me address this directly: if you find this approach fundamentally misguided, you're free to close this book right now. But before you do, let me make my case for why this seemingly roundabout journey is actually the most direct path to genuine understanding.

Why This Book Exists

Every programming resource I've encountered follows the same pattern: "Here's an array. It stores elements. Here's the syntax. Moving on." This approach produces programmers who can use arrays functionally but lack deep understanding. They can write code that works, but when things break—and they inevitably will—they're left guessing rather than reasoning through solutions.

This book exists because I believe you deserve better than surface-level knowledge. When I began programming, I wasn't satisfied with "arrays are containers for data." I wanted to understand *why* they exist, *how* they actually work, and *what* principles govern their behavior at the most fundamental level.

The deeper I investigated, the more I realized that truly understanding arrays requires understanding the entire intellectual tradition that made them possible. Arrays aren't just programming constructs—they represent the culmination of humanity's longest-running intellectual project: the systematic organization of information.

Every time you write `arr[i]`, you're employing concepts developed by ancient mathematicians who first realized that *position* could carry meaning. When you work with multidimensional arrays, you're using geometric principles refined over millennia. When you optimize array operations, you're applying algorithmic thinking that emerged from centuries of mathematical tradition.

Understanding this heritage doesn't just provide context—it builds *intuition*. When you know why arrays work as they do, you can predict their behavior. When you understand the mathematical principles underlying their structure, you can optimize their usage effectively. When you grasp the conceptual frameworks that enabled their creation, you can extend and adapt them in ways that would otherwise be impossible.

The Journey Ahead

This book is structured as a systematic exploration through seven interconnected parts:

Part 1: Philosophical & Historical Foundations

We begin with the human journey from basic counting to systematic representation, exploring how different civilizations developed the conceptual tools that make modern computation possible. We examine the invention of positional notation, the development of the abacus, the emergence of algorithmic thinking, and the philosophical frameworks that enabled abstract mathematical representation.

This foundation matters because every array operation builds on concepts developed in this part. Array indexing directly descends from positional notation. Multidimensional arrays extend geometric thinking developed by ancient mathematicians. Algorithmic optimization applies systematic procedures that emerged from medieval mathematical traditions.

Part 2: Mathematical Fundamentals

Here we transform historical intuition into precise mathematical language. We develop set theory, explore functions and relations, examine discrete mathematics, and build the linear algebra foundations that directly enable array operations.

Without these mathematical tools, you'll remain mystified by why certain array operations are efficient while others are expensive, why some algorithms work better with particular data arrangements, and how to reason about the mathematical properties of your code.

Part 3: Data Representation

We explore how information is encoded in digital systems—number systems, binary representation, character encoding, and the various methods computers use to store and manipulate data. This is where abstract concepts become concrete implementations.

Understanding data representation is crucial because it determines how array elements are stored, how memory is allocated, and how operations are performed at the hardware level.

Part 4: Computer Architecture & Logic

We examine the hardware foundations of computation—logic gates, processor architecture, memory systems, and how the physical structure of computers influences data organization. This connects software concepts to hardware realities.

Arrays don't exist in isolation. They're implemented on real hardware with specific characteristics and constraints. Understanding this foundation is essential for writing efficient array-based code.

Part 5: Array Odyssey

Finally, we encounter arrays in their full complexity. By this point, they won't be mysterious constructs but the natural evolution of thousands of years of human thought about organizing information. We explore their implementation, behavior, and applications with unprecedented depth.

This is where everything converges. The historical foundations provide context, the mathematical frameworks provide analytical tools, the representation and architecture parts provide implementation understanding—and now we can explore arrays as sophisticated, well-understood mathematical objects.

Part 6: Data Structures & Algorithms

Having mastered arrays, we expand to explore the broader landscape of data structures. We see how other structures relate to and build upon array concepts, and how our deep understanding transfers to enable more sophisticated algorithmic thinking.

Part 7: Parallelism & Systems

We examine how data structures behave in complex, multi-threaded, and distributed systems. This explores the cutting edge of modern computation and shows how classical array concepts extend to contemporary challenges.

Reading Strategies for Different Audiences

The question remains: do you need to read all of this? The answer depends on your goals and current knowledge.

Complete Beginners

Read everything sequentially. The concepts build systematically, and skipping sections will create gaps that will limit your understanding later. This book is designed to take you from zero knowledge to deep, intuitive mastery.

Experienced Programmers

You could potentially begin with Part 5, but I strongly recommend at least reviewing Parts 1 and 2. You may be surprised how much the historical and mathematical context enriches concepts you thought you already understood. Parts 3 and 4 will fill in hardware and representation details that most programmers never learn properly.

Intermediate Learners

Parts 2, 3, and 4 might be your optimal starting point. You can always return to Part 1 for broader context and advance to Part 5 when you're ready for comprehensive array exploration.

Students and Educators

Different parts serve different pedagogical purposes. Part 1 provides motivation and historical context. Parts 2-4 build theoretical foundations. Parts 5-7 provide practical applications and advanced concepts. Use whatever combination serves your specific learning objectives.

Important Expectations

This is not a reference manual. It's not designed for quick lookups when you need to remember syntax. This book is about building deep, intuitive understanding—the kind that transforms how you think about programming and data structures.

Each part includes exercises, thought experiments, and projects. These are not optional supplements—they're carefully designed to help you internalize concepts and develop the mathematical intuition that distinguishes competent programmers from exceptional ones.

Don't expect this to be a quick read. Building genuine understanding requires time and sustained attention. The historical and mathematical foundations demand patience. The technical sections require careful study and practical application. This isn't a weekend book—it's a resource you'll work through over months, returning to sections as your understanding deepens and evolves.

A Living Exploration

This book grows and evolves as I learn better ways to explain concepts and discover new connections. You'll likely find areas that could be clearer, examples that could be more intuitive, or explanations that could be more complete. When you do, I encourage you to let me know. This book improves through community engagement, and your insights make it more valuable for everyone.

The Fundamental Promise

When you complete this journey, you won't just know how to declare and manipulate arrays. You'll understand them as mathematical objects with precise properties and predictable behaviors. You'll be able to anticipate their performance characteristics, optimize their usage intelligently, and extend their applications in innovative ways.

More importantly, you'll have developed a way of thinking about programming that transcends memorizing syntax and following patterns. You'll understand the deep principles that make computation possible, and you'll be equipped to apply those principles to solve novel problems that don't have cookbook solutions.

So if you're ready for this journey—if you're willing to invest the time and intellectual energy required to build genuine understanding—then let's begin together. We're going to start with humans counting on their fingers, and we're going to end with sophisticated data structures that process information in ways that would seem magical to our ancestors.

Welcome to Arliz. Let's explore the fascinating world of arrays—from the very beginning.

Part I

Philosophical & Historical Foundations

Introduction

> Every number is an echo of humanity's need to comprehend and order nature.

Before we jump into syntax and algorithms, consider this: each time you create an array, you join a practice that spans millennia. Ancient Mesopotamians etched symbols on clay tablets; Chinese scholars arranged numbers in grids; early Islamic thinkers devised systematic methods—all aiming to tame complexity through order.

In this part, we follow that journey from first counting attempts to the verge of mechanical computation. We'll see how the abacus foreshadowed array operations, how positional notation set the stage for indexing, and how mathematical reflection shaped our approach to structured data.

Why begin here? Because grasping the *why* behind arrays transforms your relationship with them. Rather than memorizing rules, you build intuition; concepts become natural rather than obstacles. When you recognize arrays as modern echoes of an ancient drive to organize information, they lose their mystery and reveal their elegance.

Imagine early humans under a silent sky, returning from a hunt or storing seeds, faced with a simple yet profound question: how to keep track of quantities? Could a few stones or marks on bone open a door to abstraction? This urge—to count and impose order—marks a pivotal shift in human consciousness.

In this chapter, we explore the philosophical and cognitive spark behind counting, survey the earliest archaeological hints, and examine how the Neolithic shift to settled life and record-keeping paved the way for symbols and sign systems. Ultimately, we trace how these ancient steps set the foundations for the abstract structures—like arrays—that power modern programming.

Chapter 1

The Primordial Urge to Count and Order

1.1 The Philosophy of Measurement and Human Consciousness

The fact that the human race has been able to reach the threshold of evolution can mainly be attributed to two abilities: intelligence and creativity. While the former enables us to reason, solve problems, learn from experiences, and think abstractly, the latter enables us to change our way of thinking and create new strategies to overcome the obstacles we face. It can be concluded that every aspect of our lives in relation to innovation and scientific progress can be attributed to intelligence or creativity.

Based on this, it can be argued that many aspects of the formation of human computational structures—including arrays—are the result of the interaction of these two capabilities: biological intelligence that makes quantitative recognition possible and creativity that transforms these recognitions into symbols and material tools. This chapter examines how these capacities emerged and transformed into cognitive and cultural tools.

1.1.1 Instinct versus Representation: Cognitive Foundations of Human Distinction

Paleontologists and anthropologists have long been searching for explanations of the roots of modernity and modern thought. These discussions usually involve concepts such as "abstraction" and "symbolic thinking" that are often presented without precise and operational definitions.

To understand humanity's need for structured information systems, we must first clarify the distinction between instinctual behavior and the capacity for representation.

William James, the prominent 19th-century psychologist, provided a clear definition of instinct:

> Instinct is usually defined as the faculty of acting in such a way as to produce certain ends, without foresight of the ends, and without previous education in the performance. That instincts, as thus defined, exist on an enormous scale in the animal kingdom, needs no proof. They are the functional correlatives of structure. With the presence of a certain organ goes, one may say, almost always a native aptitude for its use.

This definition provides deep insight into the animal kingdom: instincts are hard-wired responses that automatically emerge from biological structure. A spider does not need to learn web-weaving techniques—this pattern is encoded in its neural architecture. A bird does not need to be taught to fly—motor coordination naturally emerges from its physical form.

But does this definition apply to humans in the same way it applies to other animals? Can we claim that early humans, from birth, had instinctual abilities for counting, organizing, and creating systematic representations of quantity? The answer is decidedly negative. Early humans, unlike animals, were not limited to instinctual reactions alone. They were able to demonstrate something beyond biological response: the capacity for representation. In other words, humans not only "saw" that one prey was larger than another, but could hold this difference in their minds as a "quantity," compare it, and even transmit it to others. This capacity for mental representation and conceptual transmission is considered a turning point in human cognitive evolution.

1.1.2 Number Sense: Cognitive Foundations of Quantitative Thinking

This capacity for representation, now called "number sense," represents humanity's tendency and ability to use numbers and quantitative methods as means for establishing communication, processing, and interpreting information. This leads to the expectation that mathematics possesses a particular order. Cognitive psychology research has shown that most humans—even without formal education—can intuitively perceive the difference between two and three. Toddlers also reveal such distinctions in their simple games. But humanity's strength was that it transformed this primitive sense into an abstract tool: from "more and less" to "exactly how many?"

In attempting to identify a prototypical cognitive basis for the origin of modern symbolic thinking, each candidate feature must satisfy four essential criteria:

1. **Innate**: Determined by factors that exist from birth and are genetically inherited
2. **Fundamental**: Essential for basic representational structure
3. **Shared**: Observable with other species
4. **Bridging**: Capable of connecting the world of sensory perception to the realm of concepts and shared expression

Anthropological and cognitive psychology research has shown that number sense possesses these characteristics.
This cognitive development, observed in human infants and even some animal species (such as primates, birds, and mammals), demonstrates an innate and shared characteristic. This trait appears in early human development and is genetically inherited. Moreover, this ability forms the foundation of modern symbolic thinking.
Most humans, even without formal education, can intuitively perceive the difference between two and three. Children also reveal such distinctions in their simple games. But humanity's strength was that it transformed this primitive sense into an abstract tool: from "more and less" to "exactly how many?" This development represents the transition from intuitive understanding to precise measurement.

1.1.3 From Intuitive Understanding to Systematic Measurement

Imagine returning to ancient times and observing a group of early humans hunting. While hunting, without needing to know mathematics, they would go toward better prey through observation. For example, when facing two deer, one fat and one thin, the natural choice would be toward the fatter animal, because more food would be available for the group's meal. From our perspective, early humans made such decisions.
But when two or three fat deer were fleeing from a group of early humans, would they still choose one? The answer is probably negative. They considered these creatures as a quantitative set and would say there were "two" or "three" deer that needed to be hunted.
While in comparing the fat and thin deer, they would select the fat one and reject the thin one. This shows that humans from the beginning had two types of quantitative thinking: qualitative comparison (fat versus thin) and absolute counting (two versus three).
This is where the concept of "measurement" emerges. Measurement means dividing

a continuous reality—such as time, distance, or weight—into discrete units that can be counted again. When early humans learned to count nights, or link the length of seasons with changes in sky and earth, they actually took the first step in building the "language of order." This language later appeared in the form of symbols, numbers, and ultimately tables and arrays.
This cognitive-cultural break has deep implications that are still observable in modern data structures:

- **Discretization**: Converting continuity into discrete units
- **Standardization**: Creating fixed and transferable units
- **Reproducibility**: Enabling precise reproduction of information

1.1.4 Organization and Social Structuring

Counting alone was not sufficient. The human mind had a tendency toward order and arrangement: placing things next to each other, comparing and sorting. Which prey was more important, which path was shorter, or which seed should be planted first—all of these required building some kind of "hierarchy" and "order." This capacity for organization was not instinctual, but the product of a cognitive leap that distinguished humans from other beings.
Imagine you are the leader of a group of early humans who has thirteen prey at disposal for dividing for evening consumption. He faces an abstract-practical problem: how to allocate these thirteen units to members in the best way so that both justice is observed and the group's survival is guaranteed?
This problem requires building hierarchies of priorities, distribution rules, and recording tools (external memory)—the same tools that later appeared in the form of tokens and tablets. The human mind showed an inclination toward order and arrangement: placing things together, comparing and sorting. Which prey is more important, which path is shorter, or which seed should be planted first.
These problems demanded the same tools that later appeared in the form of tokens, clay tablets, and ultimately data structures. **Cognitive organization was a prerequisite for resource management and institutional formation.**

Key Definitions

For conceptual clarity, the following definitions are necessary:
Number Sense: Intuitive capacity for recognizing small quantities and approximating the value of sets without explicit counting.
Subitizing: Immediate recognition of small numbers (usually up to four) without step-by-step counting.

Cardinality & Ordinality: Cardinality expresses the size of a set ("how many?"); Ordinality specifies the position of a member in a sequence ("which one?").
External Memory: Using symbols, signs, or objects to store information beyond individual memory—a fundamental prerequisite for data structures.

1.1.5 From Perception to Material Representation

It can be said that humans for the first time saw the world not merely as a habitat, but also as a system—a system that must be measured, counted, and organized. This cognitive transformation is considered a turning point in human history because it laid the foundation for systematic and abstract thinking.
If these cognitive and social capacities were real, the important next question is how these abilities were recorded in material objects and signs? In other words, how did humanity's internal number sense connect to "scored bones," "marked stones," and tangible counting systems? This question leads us to examine archaeological evidence and analyze humanity's first attempts to create external memory.
This question leads us to examine material and archaeological evidence that demonstrates how these cognitive capacities were embodied in tangible tools and symbols. In continuation, we will examine these developments in the Paleolithic period and their consequences.

1.2 Paleolithic Counting: Bones, Stones, and Fingers

In a narrow but rather imprecise sense, a numeral system is the method by which humans represent numbers. We have already limited our discussion, because among all known species only humans possess the ability to count and form numbers that we can later perform calculations upon. Many numeral systems—often very different—have been used by many cultures and civilizations again highly diverseover the ages, and a wide range of them still exist even today in our relatively global society. In a much broader sense, a numeral system is the set of various ways humans reason about numbers, and that is the definition we will use for our discussion. But what do we mean by the definition above? Well, let us talk about the ways in which we, as humans, reason about numbers. When we speak about numbers we reason about them, so we need a way to represent numbers in speech. When we write about numbers we reason about them, so we need a way to represent numbers in writing/text. This representation is known as notation. Moreover, when reasoning about numbers, we need some kind of base or radix, which is the fundamental number to which all other numbers relate. In recent times—perhaps a few centuries ago—we have also reasoned extensively about the different sets of numbers we have been able to create. Thus, the term numeral system also comes to mean one of these constructed sets.**weibull_numbersystems**

Figuring out when humans began to count systematically, with purpose, is not easy. Our first real clues are a handful of curious, carved bones dating from the final few millennia of the three-million-year expanse of the Old Stone Age, or Paleolithic era. Those bones are humanity's first pocket calculators: For the prehistoric humans who carved them, they were mathematical notebooks and counting aids rolled into one. For the anthropologists who unearthed them thousands of years later, they were proof that our ability to count had manifested itself no later than 40,000 years ago.

In 1973, while excavating a cave in the Lebombo Mountains, near South Africa's border with Swaziland, Peter Beaumont found a small, broken bone with twenty-nine notches carved across it. The so-called Border Cave had been known to archaeologists since 1934, but the discovery during World War II of skeletal remains dating to the Middle Stone Age heralded a site of rare importance. It was not until Beaumont's dig in the 1970s, however, that the cave gave up its most significant treasure: the earliest known tally stick, in the form of a notched, three-inch long baboon fibula.

On the face of it, the numerical instrument known as the tally stick is exceedingly mundane. Used since before recorded history—still used, in fact, by some cultures—to mark the passing days, or to account for goods or monies given or received, most tally sticks are no more than wooden rods incised with notches along their length. They help their users to count, to remember, and to transfer ownership. All of which is reminiscent of writing, except that writing did not arrive until a scant 5,000 years ago—and so, when the Lebombo bone was determined to be some 42,000 years old, it instantly became one of the most intriguing archaeological artifacts ever found. Not only does it put a date on when Homo sapiens started counting, it also marks the point at which we began to delegate our memories to external devices, thereby unburdening our minds so that they might be used for something else instead. Writing in 1776, the German historian Justus Möser knew nothing of the Lebombo bone, but his musings on tally sticks in general are strikingly apposite:

> The notched tally stick itself testifies to the intelligence of our ancestors. No invention is simpler and yet more significant than this.

It is not clear what quantity the twenty-nine notches carved into the Border Cave's baboon fibula represents. It is a number, that much is known: had the bone been purely decorative, the notches would have been added all at once, but four different tools were used over time to add to the count. As such, the Lebombo bone is likely to be the earliest mathematical device ever found. (Sadly, it is too great a leap to call it the earliest known pocket calculator. Humans started wearing clothes around 170,000 years ago, but pockets themselves are probably no more than a few thousand years old.)

If the Lebombo bone answers the question, at least partly, of when humans learned to count, it leaves another one unanswered: How did they learn to do so?

Counting, fundamentally, is the act of assigning distinct labels to each member of a group of similar things to convey either the size of that group or the position of individual items within it. The first type of counting yields cardinal numbers such as "one," "two," and "three"; the second gives ordinals such as "first," "second," and "third." At first, our hominid ancestors probably did not count very high. Many body parts present themselves in pairs—arms, hands, eyes, ears, and so on—thereby leading to an innate familiarity with the concept of a pair and, by extension, the numbers 1 and 2. But when those hominids regarded the wider world, they did not yet find a need to count much higher. One wolf is manageable; two wolves are a challenge; any more than that and time spent counting wolves is better spent making oneself scarce. The result is that the very smallest whole numbers have a special place in human culture, and especially in language. English, for instance, has a host of specialized terms centered around twoness: a brace of pheasants; a team of horses; a yoke of oxen; a pair of, well, anything. An ancient Greek could employ specific plurals to distinguish between groups of one, two, and many friends (ho philos, to philo, and hoi philoi). In Latin, the numbers 1 to 4 get special treatment, much as "one" and "two" correspond to "first" and "second," while "three" and "four" correspond directly with "third" and "fourth." The Romans extended that special treatment into their day-to-day lives: after their first four sons, a Roman family would typically name the rest by number (Quintus, Sextus, Septimus, and so forth), and only the first four months of the early Roman calendar had proper names. Even tally marks, the age-old "five-barred gate" used to score card games or track rounds of drinks, speaks of a deep-seated need to keep things simple.

Counting in the prehistoric world would have been intimately bound to the actual, not the abstract. Some languages still bear traces of this: a speaker of Fijian may say *doko* to mean "one hundred mulberry bushes," but also *koro* to mean "one hundred coconuts." Germans will talk about a *Faden*, meaning a length of thread about the same width as an adult's outstretched arms. The Japanese count different kinds of things in different ways: there are separate sequences of cardinal numbers for books; for other bundles of paper such as magazines and newspapers; for cars, appliances, bicycles, and similar machines; for animals and demons; for long, thin objects such as pencils or rivers; for small, round objects; for people; and more.

Gradually, as our day-to-day lives took on more structure and sophistication, so, too, did our ability to count. When farming a herd of livestock, for example, keeping track of the number of one's sheep or goats was of paramount importance, and as humans divided themselves more rigidly into groups of friends and foes, those who could count allies and enemies had an advantage over those who could not. Number words graduated from being labels for physical objects into abstract concepts that floated around in the mental ether until they were assigned to actual things.

Even so, we still have no real idea how early humans started to count in the first place. Did they gesture? Speak? Gather pebbles in the correct amount? To form an educated guess, anthropologists have turned to those tribes and peoples isolated from the greater body of humanity, whether by accident of geography or deliberate seclusion. The conclusion they reached is simple. We learned to count with our fingers.

A 1913 survey of the number words used by several Native American tribes found that many of those words were related to "finger," "thumb," and "hand." Counterintuitively, perhaps, despite the general possession of ten fingers per person, fewer than half of those tribes counted in multiples of ten. About a third used systems that revolved around the number 5, which was often referred to as "fingers finished," "all finished," "gone," or "spent." A further tenth of the tribes used vigesimal schemes based on the number 20 ("all hands and feet"), while a few contrarian outliers used 2-, 3-, and 4-based systems with less obvious connections to human anatomy.

Fifteen years earlier, a group of scientists from Cambridge, England, had made a series of visits to the islands of the Torres Straits, strung between Papua New Guinea to the north and Australia to the south. A.C. Haddon, the driving force behind the expeditions, recounted:

> There was another system of counting by commencing at the little finger of the left hand, kotodimura, then following on with the fourth finger, kotodimura gorngozinga (or quruzinger); middle finger, il get; index finger, klak-nĕtoi-gĕt; thumb, kabaget; wrist, perta or tiap; elbow joint, kudu; shoulder, zugukwoik; left nipple, susu madu; sternum, kosa, dadir; right nipple, susu madu, and ending with the little finger of the right hand.

In this way, Haddon said, starting on one side of the body and traversing over to the other, the islanders could count to nineteen. More recently, a math teacher named Glen Lean catalogued the number words for 883 of the 1,200 known languages from Papua New Guinea and Micronesia and found that the use of fingers for counting was foundational to many of those languages. Like the Torres Strait islanders, the Papua New Guineans then carried on to the forearm, elbow, eyes, nose, ears, and other body parts. A study of Yupno, a language indigenous to Papua New Guinea's Finisterre Mountain range, recorded that Yupno men added their testicles and penis for good measure, allowing them to count to thirty-three using body parts alone.

Hold my earliest attested beer, an ancient Sumerian might have said.

From the sixth millennium onward, the valley between the Tigris and the Euphrates Rivers—Mesopotamia, the ancient Greeks called it, the "land between rivers"—harbored one of the world's earliest civilizations. Having mastered animal husbandry and the cultivation of crops, Mesopotamian farmers became the engine of a new agrarian economy. Almost from the beginning, it seems, they used small clay tokens an inch or so in size, hand-rolled into the shape of spheres, cones, disks, and other simple shapes, to keep records. Each shape stood for a fixed quantity of some good or other: A cone represented a small quantity of cereal, a sphere a larger amount, and a flat disk the largest. Ovoids were jars of oil; cylinders and rounded disks were farm animals; and so on.

Around 3300 BC, as Mesopotamia's scattered farming communities began to coalesce into the patchwork of city-states called Sumer, their use of tokens became more sophisticated. At first, batches of tokens were wrapped in clay balls called bullae and marked with personal seals to create records of important transactions. Later, the surfaces of those bullae were impressed with the tokens to be sealed inside so that a bulla's contents could be divined without having to break it open. Once it became apparent that the signs on the outside were as useful as the tokens on the inside, the tokens themselves became surplus to requirements—and the signs, says a theory first proposed by French-American archaeologist Denise Schmandt-Besserat, evolved into the distinctive angular form of cuneiform writing.

Cuneiform tablets show that the Sumerians and their successors, the Akkadians and Babylonians, used sexagesimal numbers. That is, their numerical system was rooted in the number 60. Whereas decimal gives rise to round numbers such as 1, 10, and 100 (or 10 squared), the Sumerians counted in terms of 1, 60, 3,600 (or 60 squared), and so on. There are practical advantages to this, since 60 can be divided into whole numbers by 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60, but, as E. F. Robertson, late of St. Andrews University in Scotland, points out, it is rare for a culture to choose the base for its number system. More often, as illustrated by those Native American tribes, it naturally settles upon a base when it begins to count. Counting on five fingers leads to the quinary system, or base 5; two hands lead to decimal, or base 10; two hands and two feet to base 20, or vigesimal. How, then, did the Sumerians land on base 60?

The answer lies in the Sumerians' tokens and bullae. Successive scholars have noted that the shapes made when tokens were pushed into the soft clay of a bulla appear to be very similar to the number symbols used on the earliest "proto-literate" clay tablets. That is, the shapes and the values of physical tokens seem to have carried over directly to the written sexagesimal numerals used by the earliest literate Sumerians. As such, the ancient Mesopotamians must have been counting in base 60 on their fingers long before they, or, indeed, anyone else on the planet, could set out numbers in writing.

The Mesopotamians' unique counting method is thought to come from a mix of a duodecimal system that used the twelve finger joints of one hand and a quinary system that used the five fingers of the other. By pointing at one of the left hand's twelve joints with one of the right hand's five digits, or, perhaps, by counting to twelve with the thumb of one hand and recording multiples of twelve with the digits of the other, it is possible to represent any number from 1 to 60. However it worked, the Mesopotamians' anatomical calculator was a thing of exceptional elegance, and the numbers they counted with it echo through history. It is no coincidence that a clock has twelve hours, an hour has sixty minutes, and a minute has sixty seconds.**houston2023_earlyhistory**

1.3 Neolithic Revolution: Agriculture and the Need for Records

1.4 Proto-Writing and Symbolic Representation

Chapter 2

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2.2 The Revolutionary Base-60 System

2.3 Babylonian Mathematical Tablets

2.4 The Concept of Position and Place Value

2.5 Hammurabi's Code: Systematic Legal Data Structures

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Egyptian Systematic Knowledge and Geometric Arrays

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10.1 Roman Numerals and Practical Calculation Systems

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11.2 Hindu-Arabic Numerals and Place-Value Revolution

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14.3 Fibonacci and the Liber Abaci

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19.3 Gauss and Systematic Number Theory

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Part II

Mathematical Fundamentals

Introduction

The historical journey in Part 1 showed us how humans developed systematic thinking about organized information. Now we need to translate those insights into the precise mathematical language that makes arrays work.

This isn't about learning math for math's sake. Every mathematical concept we explore here—from basic number properties to linear algebra—directly enables the array operations you'll use in programming. When you understand why multiplication is commutative, you'll understand why certain array optimizations work. When you grasp set theory, you'll see the logic behind array search algorithms. When you work with mathematical functions, you'll understand the elegant relationship between array indices and their values.

We'll build everything from first principles, assuming no advanced mathematical background. But we won't treat mathematics as a collection of arbitrary rules. Instead, we'll see how each concept emerged from the same human drive for systematic organization that we traced in Part 1.

Think of this part as building your mathematical toolkit. Every tool we create here will be used extensively in later parts. By the end, you'll have the mathematical foundation needed to truly understand not just how arrays work, but why they work the way they do.

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The Nature of Numbers and Fundamental Operations

Chapter 21

Real Numbers and Mathematical Completeness

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Functions and Systematic Relationships

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Discrete Mathematics and Finite Structures

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28.6 Determinants and Matrix Properties

28.7 Eigenvalues and Eigenvectors

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Advanced Discrete Structures

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29.2 Ring and Field Theory: Extended Algebraic Structures

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Part IV

Computer Architecture & Logic

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How to Read

Part V

Array Odyssey

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Data Structures & Algorithms

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How to Read

Part VII

Parallelism & Systems

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How to Read

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Introduction

How to Read

Glossary

Reflections at the End

As you turn the final pages of *Arliz*, I invite you to pause—just for a moment—and look back. Think about the path you've taken through these chapters. Let yourself ask:

> "Wait... what just happened? What did I actually learn?"

I won't pretend to answer that for you. The truth is—**only you can**. Maybe it was a lot. Maybe it wasn't what you expected. But if you're here, reading this, something must have kept you going. That means something.

This book didn't start with a grand plan. It started with a simple itch: **What even is an array, really?** What began as a curiosity about a "data structure" became something much stranger and—hopefully—much richer. We wandered through history, philosophy, mathematics, logic gates, and machine internals. We stared at ancient stones and modern memory layouts and tried to see the invisible threads connecting them.

If that sounds like a weird journey, well—yeah. It was.

This is Not the End

Arliz isn't a closed book. It's a snapshot. A frame in motion. And maybe your understanding is the same. You'll return to these ideas later, years from now, and see new angles. You'll say, "Oh. That's what it meant." That's good. That's growth.

Everything you've read here is connected to something bigger—algorithms, networks, languages, systems, even the people who built them. There's no finish line. And that's beautiful.

From Me to You

If this book gave you something—an idea, a shift in thinking, a pause to wonder—then it has done its job. If it made you feel like maybe programming isn't just code and rules, but a window into something deeper—then that means everything to me.

Thank you for being here.
Thank you for not skipping the hard parts.
Thank you for choosing to think.

One More Thing

You're not alone in this.
The Arliz project lives on GitHub, and the conversations around it will (hopefully) continue. If you spot mistakes, have better explanations, or just want to say hi—come by. Teach me something. Teach someone else. That's the best way to say thanks.
Knowledge grows in community.
So share. Build. Break. Rebuild.
Ask better questions.
And always, always—stay curious.

Final Words

Arrays were just the excuse.
Thinking was the goal.
And if you've started to think more clearly, more deeply, or more historically about what you're doing when you write code—then this wasn't just a technical book.
It was a human one.
Welcome to the quiet, lifelong joy of understanding.

This completes the first living edition of Arliz.

Thank you for joining this journey from zero to arrays, from ancient counting to modern computation.
The exploration continues...

Author's Notes and Reflections

On Naming Conventions and Creative Processes

I should confess something about my naming process: I tend to pick names first and find meaningful justifications later. Very scientific, I know! The name "Arliz" started as a random choice that simply felt right phonetically. Only after committing to it did I discover the backronym that now defines its meaning. This probably says something about my creative process—intuition first, rationalization second.

This approach extends beyond naming. Many aspects of this book emerged organically from curiosity rather than systematic planning. What began as personal notes to understand arrays evolved into a comprehensive exploration of computational thinking itself.

On Perfectionism and Living Documents

You should know that many of the algorithms presented in this book are my own implementations, built from first principles rather than copied from optimized sources. This means you might encounter code that runs slower than industry standards—or occasionally faster, when serendipity strikes.

Some might view this as a weakness, but I consider it a feature. The goal isn't to provide the most optimized implementations but to demonstrate the thinking process that leads to understanding. When you can reconstruct a solution from fundamental principles, you've achieved something more valuable than memorizing an optimal algorithm.

On Academic Formality and Personal Voice

You might notice that this book alternates between formal academic language and more conversational tones. This is intentional. While I respect the precision that formal writing provides, I also believe that learning happens best in an atmosphere of intellectual friendship rather than academic intimidation.

When I suggest you could "use this book as a makeshift heating device" if you find the approach ridiculous, I'm not being flippant—I'm acknowledging that not every approach works for every learner. Intellectual honesty includes admitting when your methods might not suit your audience.

On Scope and Ambition

The scope of this book—from ancient counting to modern distributed systems—might seem overly ambitious. Some might argue that such breadth necessarily sacrifices depth. I disagree, but I understand the concern.

My experience suggests that understanding connections between disparate fields often provides insights that narrow specialization misses. When you see arrays as part of humanity's broader intellectual project, you understand them differently than when you see them as isolated programming constructs.

That said, if you find the historical sections tedious or irrelevant, you have my permission to skip ahead. The book is designed to be valuable even when read non-sequentially.

On Errors and Imperfection

I mentioned that you'll find errors in this book. This isn't false modesty—it's realistic acknowledgment. Complex explanations, mathematical derivations, and code implementations inevitably contain mistakes, especially in a work that grows and evolves over time.

Rather than viewing this as a flaw, I encourage you to see it as an opportunity for engagement. When you find an error, you're not just identifying a problem—you're participating in the process of building better understanding. The best learning often happens when we encounter and resolve contradictions.

On Time Investment and Expectations

When I suggest this book requires months rather than weekends to master, I'm not trying to inflate its importance. Complex concepts genuinely require time to internalize. Mathematical intuition develops gradually, through repeated exposure and active practice.

If you're looking for quick solutions to immediate programming problems, this book will frustrate you. If you're interested in developing the kind of deep understanding that serves you throughout your career, the time investment will prove worthwhile.

On Community and Collaboration

This book exists because of community—the open-source community that provides tools and resources, the academic community that develops and refines concepts, and the programming community that applies these ideas in practice.

Your engagement with this material makes you part of that community. Whether you find errors, suggest improvements, or simply work through the exercises thoughtfully, you're contributing to the collective understanding that makes books like this possible.

Final Reflection

Writing this book has been an exercise in understanding my own learning process. I've discovered that I learn best by building connections between disparate ideas, by understanding historical context, and by implementing concepts from scratch rather than accepting them as given.

Your learning process might be entirely different. Use what serves you from this book, adapt what needs adaptation, and don't hesitate to supplement with other resources when my explanations fall short.

The goal isn't for you to learn exactly as I did, but for you to develop your own path to deep understanding.

These notes reflect thoughts and observations that didn't fit elsewhere but seemed worth preserving. They represent the informal side of a formal exploration—the human element in what might otherwise seem like purely technical content.