# Coupled Oscillators

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**Abstract**

*Coupled oscillators are systems in which two or more oscillators are linked through various mechanisms—such as springs, electromagnetic fields, or other types of interactions—allowing energy to be exchanged between them. This phenomenon has wide-ranging applications in physics, engineering, biology, and computation. In this article, a precise definition of coupled oscillators is presented, followed by a detailed explanation of their mathematical formulations. Different types of coupled oscillators are examined, and their practical applications are described in sequence. Finally, a conclusion is provided. These systems are often decomposed into normal modes, which greatly simplify the analysis of their otherwise complex behavior.*

## 1.Introduction

Coupled oscillators arise when two or more oscillatory systems are linked in such a way that the motion of one affects the others, enabling the transfer of energy between them. This coupling may be mechanical (such as masses connected by springs), electromagnetic (as in coupled LC circuits), or even biological (for example, cardiac pacemaker cells).

In the simplest case of two oscillators, coupling gives rise to normal modes in which all components oscillate with the same frequency but maintain fixed phase relationships. The strength of the coupling determines the degree of interaction and the overall behavior of the system. The number of normal modes is equal to the number of degrees of freedom or, equivalently, the number of oscillators involved.

## 2. Mathematical Formulation (Detailed and Sequential)

The formulas for coupled oscillators are typically derived from Newtonian equations of motion or from a Lagrangian formulation. In what follows, we begin with the simplest case—two coupled oscillators—and then proceed to more complex configurations.

### 2.1. Pair of Mechanically Coupled Oscillators

Consider two identical masses $m$, each attached to a wall by a spring of stiffness $k'$, and coupled to one another by an additional spring of stiffness $k$. The displacements of the masses from equilibrium are denoted by $x_1$ and $x_2$. In this configuration, the coupling spring mediates the interaction between the two oscillators, allowing energy to be exchanged as the masses move.

Equations of motion:

$$m\frac{d^2x1}{dt^2} = -k'x_1 + k(x_2 - x_1) \qquad (1)$$

$$m\frac{d^2x2}{dt^2} = -k'x_2 + k(x_2 - x_1) \quad (2)$$

By dividing through by $m$ and introducing the definitions $\omega_0^2 = k'/m$ & $\omega_s^2 = k/m$.

$$\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 - \omega_s^2(x_2 - x_1) = 0 \quad (3)$$

$$\frac{d^2x_2}{dt^2} + \omega_0^2 x_2 - \omega_s^2(x_2 - x_1) = 0 \quad (4)$$

These equations are coupled. To solve them, we employ normal coordinates:

$$\zeta_1 = x_1 + x_2,\ \zeta_2 = x_1 - x_2 \quad (5)$$

The equations decouple:

$$\frac{d^2\zeta_1}{dt^2} + \omega_1^2\zeta_1 = 0, \omega_1^2 = \omega_0^2 \quad (6)$$

$$\frac{d^2\zeta_2}{dt^2} + \omega_2^2\zeta_2 = 0, \omega_2^2 = \omega_0^2 + 2\omega_s^2 \quad (7)$$

The solutions represent simple harmonic oscillations (antisymmetric) and $\omega_1 > \omega_2$ while the mode with $\omega_1$:

$$\zeta_1(t) = a_1\cos(\omega_1 t + \phi_1), \zeta_2(t) = a_2\cos(\omega_2 + \phi_2) \quad (8)$$

Then:

$$x_1(t) = \frac{1}{2}[\zeta_1(t) + \zeta_2(t)], x_2(t) = \frac{1}{2}[\zeta_1(t) + \zeta_2(t)] \quad (10)$$

For specific initial conditions (e.g. $x_1(0) = 2a$ , $x_2(0) = 0$), Modulation occurs:

$$x_1(t) = 2\,\mathrm{a}\cos\left(\frac{\omega_2 - w_1}{2}t\right)\cos\left(\frac{\omega_2 + w_1}{2}t\right) \quad (11)$$

This indicates a beating phenomenon with an average frequency of $\omega_{av} = \left(\frac{\omega_2 + w_1}{2}\right)$ and The modulation frequency $\omega_{mod} = \left(\frac{\omega_2 - w_1}{2}\right)$. The total energy remains constant, but it oscillates between the masses with a period of$T = 2\pi/(\omega_2 - \omega_1)$.

### 2.2. The Matrix Method for General Systems

Matrix-form equations:

$$\ddot{x} = -kx \quad (11)$$

Here, **K** denotes the stiffness matrix. The natural frequencies are obtained from the eigenvalues $\omega^2$ while the mode shapes are determined from the corresponding eigenvectors.

### 2.3. more than two oscillators

For a system of $N$oscillators, the frequencies are given by:

$$\omega_n = 2\sqrt{\frac{k}{m}}\left|\sin\left(\frac{n\pi}{N+1}\right)\right|, n = 1,2,\dots,N \quad (12)$$

For large $N$, the low-lying modes resemble standing waves.

### 2.4. The Kuramoto model for complex networks

For a network of phase oscillators:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$

Where $\theta_i$ phase, $\omega_i$ natural frequency, $a_{ij}$ is weights. For a complete graph with uniform weights, the system corresponds to the Kuramoto model.

## 3. Types with a detailed explanation

Coupled oscillators are classified according to the nature of the coupling, their number, and the complexity of the system.

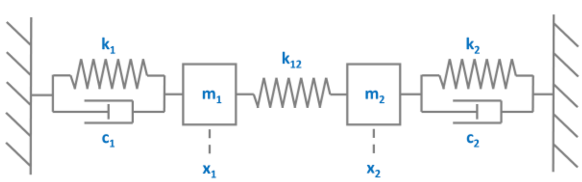
1. **Mechanical:** Masses connected by springs or coupled pendulums. **Explanation:** Energy is transferred through elastic forces. **Example:** Two pendulums connected by a spring; the equations resemble those of a two-mass system.
2. **Electromagnetic:** Coupled LC circuits with mutual inductance ($M$). **Explanation:** Weak coupling$\mu = M/\sqrt{L_1L_2}$ Weak coupling maintains

near-degeneracy of the frequencies, whereas strong coupling results in frequency splitting. Frequencies: $\omega = \omega_0\sqrt{1 \pm \mu}$.

**3. Linear vs. Nonlinear:** Linear systems are valid for small approximations; nonlinear systems can lead to synchronization, chaos, or solitons.
**4. Weak vs. Strong:** Weak coupling applies to small frequency differences, while strong coupling corresponds to significant interactions.
**5. Network or Complex:** Systems with more than two oscillators, such as chains or graphs. **Explanation:** Phase/frequency synchronization occurs when the oscillator frequencies converge.
**6. Longitudinal vs. Transverse:** Longitudinal for oscillations along the chain; transverse for pendulum-like motions.

## 4.Coupled Oscillators

This example demonstrates a system of two coupled damped harmonic oscillators using two ODE blocks. The system consists of two masses connected by springs with damping. Each oscillator is coupled to the other through a coupling spring constant.




The equations of motion for the coupled system are:

$$m_1\ddot{x}_1 = -k_1x_1 - C_1\dot{x}_1 - F_e$$

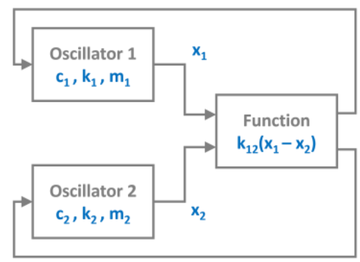
$$m_2\ddot{x}_2 = k_2x_2 - C_2\dot{x}_2 + F_e$$

With the force-coupling given as:

$$F_e = K_{12}(x_1 - x_2)$$

Where $m_1, m_2$ are the masses, $k_1, k_2$ are the spring constants, $c_1, c_2$ are the damping coefficients, and $k_{12}$ is the coupling spring constant between the two oscillators, leading to external forces $F_e$.

As a block diagram it would look like this:




4.1. MATLAB CODE:

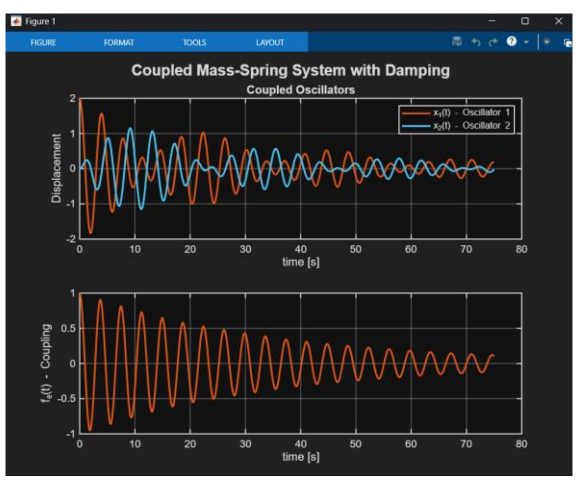
- Coupled_osc.m (function)

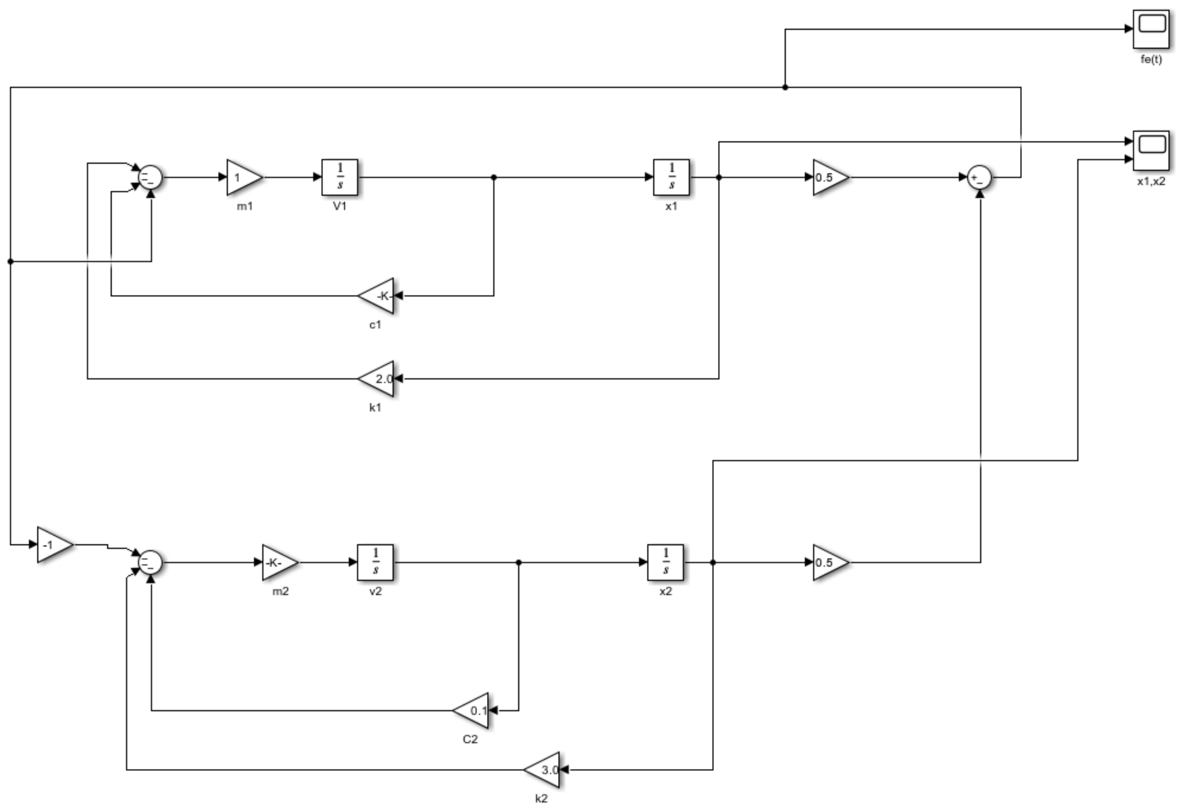
```
function dXdt = coupled_osc(t, X, m1, m2, k1, k2, k12, c1, c2)
    % State vector:
    % X = [x1; v1; x2; v2]
    x1 = X(1);
    v1 = X(2);
    x2 = X(3);
    v2 = X(4);
    % Coupling force
    f12 = k12 * (x1 - x2);  % force on mass 1 due to mass 2
    % Equations of motion
    dx1dt = v1;
    dv1dt = (-k1*x1 - c1*v1 - f12) / m1;
    dx2dt = v2;
    dv2dt = (-k2*x2 - c2*v2 + f12) / m2; % note the opposite sign of f12
    % Return derivatives
    dXdt = [dx1dt; dv1dt; dx2dt; dv2dt];
end
```

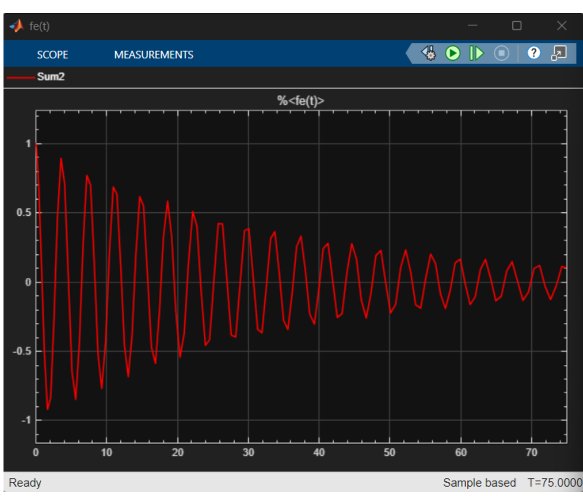
This MATLAB function is designed for use in the numerical solution of differential equations (such as with **ode45**). It computes the state derivatives of two coupled mass–spring–damper oscillators.

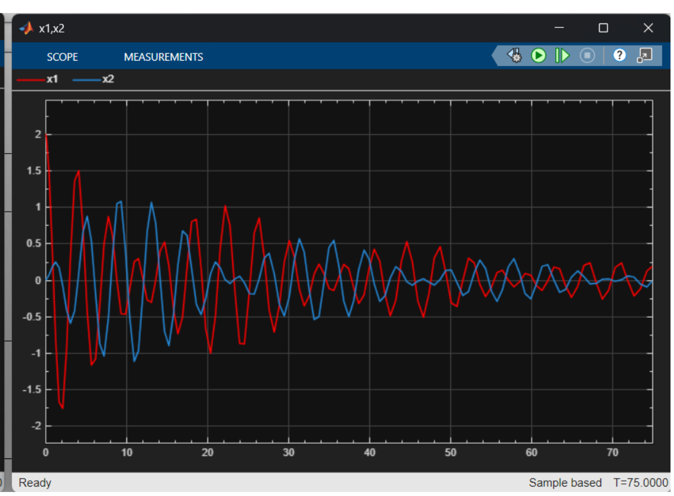
- Coupled_oscillator.m (main code)

```matlab
%% Coupled Oscillators Simulation in MATLAB

clear; clc; close all;
%% Parameters
% Masses
m1 = 1.0;
m2 = 1.5;
% Spring constants
k1 = 2.0;
k2 = 3.0;
k12 = 0.5;   % coupling spring constant
% Damping coefficients
c1 = 0.05;
c2 = 0.1;
%% Initial conditions
% [x1, v1, x2, v2]
x1_0 = 2.0;     % oscillator 1 initial position
v1_0 = 0.0;     % oscillator 1 initial velocity
x2_0 = 0.0;     % oscillator 2 initial position
v2_0 = 0.0;     % oscillator 2 initial velocity
X0 = [x1_0; v1_0; x2_0; v2_0];
%% Simulation parameters
tspan = [0 75];   % time range
%% Solve ODE system
[t, X] = ode45(@(t,X) coupled_osc(t, X, m1, m2, k1, k2, k12, c1, c2), tspan, X0);
%% Extract results
x1 = X(:,1);
v1 = X(:,2);
x2 = X(:,3);
v2 = X(:,4);
%% Compute coupling force
f_coupling = k12 * (x1 - x2);
%% Plot results
figure('Position', [100, 100, 800, 600]); %size

subplot(2,1,1)
plot(t, x1, 'Color', [0.85, 0.33, 0.10], 'LineWidth', 2);
hold on
plot(t, x2, 'Color', [0.30, 0.75, 0.93], 'LineWidth', 2);
xlabel('time [s]', 'FontSize', 12);
ylabel('Displacement', 'FontSize', 12);
title('Coupled Oscillators', 'FontSize', 14, 'FontWeight', 'bold');
legend('x_1(t) - Oscillator 1', 'x_2(t) - Oscillator 2', 'Location', 'northeast');
grid on;
set(gca, 'FontSize', 11, 'LineWidth', 1.2);

subplot(2,1,2)
plot(t, f_coupling, 'Color', [0.85, 0.33, 0.10], 'LineWidth', 2);
xlabel('time [s]', 'FontSize', 12);
ylabel('f_e(t) - Coupling', 'FontSize', 12);
title('', 'FontSize', 14);
grid on;
set(gca, 'FontSize', 11, 'LineWidth', 1.2);

sgtitle('Coupled Mass-Spring System with Damping', 'FontSize', 16, 'FontWeight', 'bold');
```




fe(t)
x1,x2
1
m1
V1
x1
0.5
-K-
c1
2.0
k1
-1
-K-
m2
v2
x2
0.5
0.1
C2
3.0
k2


fe(t)
SCOPE
MEASUREMENTS
Sum2
%<fe(t)>
Ready
Sample based
T=75.0000


x1,x2
SCOPE
MEASUREMENTS
x1
x2
Ready
Sample based
T=75.0000

## Conclusion

Coupled oscillators are a fundamental concept in physics, as they explain the complex behavior of interacting systems. Through normal modes and matrix-based analysis, their dynamics can be accurately predicted. Their diverse forms, ranging from mechanical systems to networked structures, have led to wide-ranging applications in science and technology. Recent developments, such as Kuramoto models, have opened new possibilities in synchronization and computation. A deep understanding of these systems is essential for addressing real-world problems. Coupled oscillators provide a rich framework for understanding interaction-driven dynamics. From simple beating in pendulums to complex synchronization in networks, they bridge classical and quantum worlds. Future research may focus on nonlinear and stochastic effect. Experimental setups can be simulated using software like MATLAB or Python's SciPy for numerical solutions. This article serves as a comprehensive starting point; expand with specific examples or simulations as needed for your purposes.

**References**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