ćwiczenia 1

November 4, 2021

1. (a) $7 + 4i + 3(5 - 9i) = 7 + 4i + 15 - 27i = 22 - 23i$

   (b) $Im(3 + 2i)(4 - 2i) - Re(5 + 3i)(1 - 4i) = 2(4 + 2i) - 5(1 - 4i) = 8 - 4i - 5 - 20 = 3 + 16i$

   (c) $(3 - 2i)(3 + 4i) = 9 + 12i - 6i - 8i^2 = 17 + 6i$

   (d) $(4 - 5i)(4 + 5i) = 4^2 - (5i)^2 = 16 + 25 = 41$

   (e) $\frac{-11+7i}{2+6i} = \frac{(-11+7i)(2-6i)}{(2+6i)(2-6i)} = \frac{-22+66i+14i-42i^2}{4+36} = \frac{20+80i}{40} = \frac{1}{2} + 2i$

   (f) $\frac{-1+2i}{5-3i} = \frac{(-1+2i)(5+3i)}{(5-3i)(5+3i)} = \frac{-5-3i+10i+6i^2}{25+9} = \frac{-11+7i}{34} = -\frac{11}{34} + \frac{7i}{34}$

   (g) $\frac{3+i}{2-5i} = \frac{(3+i)(2+5i)}{(2-5i)(2+5i)} = \frac{6+15i+2i+5i^2}{4+25} = \frac{1+17i}{29} = \frac{1}{29} + \frac{17i}{29}$

   (h) $\frac{5+3i}{8-2i} = \frac{(5+3i)(8+2i)}{(8-2i)(8+2i)} = \frac{40+10i+24i+6i^2}{64+4} = \frac{34+34i}{68} = \frac{1}{2} + \frac{1}{2}i$

   (i) $\frac{3+2i}{-1+3i} = \frac{(3+2i)(-1-3i)}{(-1+3i)(-1-3i)} = \frac{-3-9i-2i-6i^2}{1+9} = \frac{3-11i}{10} = \frac{3}{4} - \frac{11}{10}i$

2. (a)

$$\begin{aligned} 2z + (3 - i)\overline{z} &= 5 + 4i \\ 2(a + bi) + (3 - i)(a - bi) &= 5 + 4i \\ 2a + 2bi + 3a - 3bi - ai + bi^2 &= 5 + 4i \\ 5a - bi - ai - b &= 5 + 4i \\ 5a - b + i(-b - a) &= 5 + 4i \end{aligned}$$

$$\begin{aligned} 5a - b &= 5 \\ -b - a &= 4 \\ a + b &= -4 \\ b &= -4 - a \end{aligned} \qquad \begin{aligned} 5a - (-4 - a) &= 5 \\ 5a + 4 + a &= 5 \\ 6a &= 1 \\ a &= \frac{1}{6} \end{aligned} \qquad \begin{aligned} b &= -4 - \frac{1}{6} \\ b &= -\frac{25}{6} \end{aligned}$$

$$z = \frac{1}{6} - \frac{25}{6}i$$

   (b)

$$\begin{aligned} 2z + (1 + i)\overline{z} &= 1 - 3i \\ 2(a + bi) + (1 + i)(a - bi) &= 1 - 3i \\ 2a + 2bi + a - bi + ai - bi^2 &= 1 - 3i \\ 3a + b + bi + ai &= 1 - 3i \\ 3a + b + (a + b)i &= 1 - 3i \end{aligned}$$

$$\begin{aligned} 3a + b &= 1 \\ a + b &= -3 \\ b &= -3 - a \end{aligned} \qquad \begin{aligned} 3a - 3 - a &= 1 \\ 2a &= 4 \\ a &= 2 \end{aligned} \qquad \begin{aligned} b &= -3 - 2 \\ b &= -5 \end{aligned}$$

$$z = 2 - 5i$$

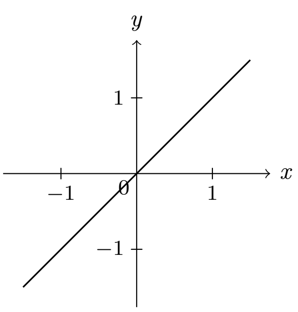
   (c)

$$\begin{aligned} \bar{z} + i &= 8 + 3\mathrm{Im}z \\ (a - bi) + i &= 8 + 3b \\ a - 3b - (b - 1)i &= 8 + 0i \end{aligned} \qquad \begin{aligned} a - 3b &= 8 \\ -b + 1 &= 0 \\ b &= 1 \end{aligned} \qquad \begin{aligned} a - 3 &= 8 \\ a &= 11 \end{aligned}$$

$$z = 11 + i$$

3. tego gówna jeszcze nie rozumiem

   (a) $\frac{9}{z} = \bar{z}$



4. (a)

$$z = 6 \qquad \cos\varphi = \frac{6}{6} = 1 \qquad \varphi = 0$$
$$|z| = \sqrt{6^2} = 6 \qquad \sin\varphi = 0$$
$$z = 6(\cos 0 + i\sin 0)$$
$$z = 6e^0$$

(b)

$$z = 7 + 7i \qquad \cos\varphi = \frac{7}{7\sqrt{2}} = \frac{7\sqrt{2}}{7\times 2} = \frac{\sqrt{2}}{2} \qquad \varphi = \frac{\pi}{4}$$
$$|z| = \sqrt{7^2+7^2} = \sqrt{98} = 7\sqrt{2} \qquad \sin\varphi = \frac{7}{7\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$z = 7\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$
$$z = 7\sqrt{2}e^{\frac{\pi}{4}}$$

(c)

$$z = -1 + i\sqrt{3} \qquad \cos\varphi = \frac{-1}{2} = -\frac{1}{2} \qquad \varphi = \frac{2\pi}{3}$$
$$|z| = \sqrt{1 + (\sqrt{3})^2} = \sqrt{4} = 2 \qquad \sin\varphi = \frac{\sqrt{3}}{2}$$
$$z = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$$
$$z = 2e^{\frac{2\pi}{2}}$$

(d)

$$z = 5\sqrt{3} - 5i \qquad \cos\varphi = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2} \qquad \varphi = \frac{11\pi}{6}$$
$$|z| = \sqrt{(5\sqrt{3})^2 + i(-5)^2} = \sqrt{25\cdot 3 + 25} \qquad \sin\varphi = \frac{-5}{10} = -\frac{1}{2}$$
$$= \sqrt{100} = 10$$
$$z = 10(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$$
$$z = 10e^{\frac{11\pi}{6}}$$

5. (a) $3(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})(2-2i)$

$$z = 2 - 2i \qquad \cos\varphi = \frac{2}{2\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \varphi = \frac{7\pi}{4}$$
$$|z| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2} \qquad \sin\varphi = \frac{-2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2}$$
$$z = 2\sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$$

$3(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})2\sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})) = 6\sqrt{2}(\cos(\frac{\pi}{5} + \frac{7\pi}{4}) + i\sin(\frac{\pi}{5} + \frac{7\pi}{4}) = 6\sqrt{2}(\cos\frac{4\pi+35\pi}{20} + i\sin\frac{39\pi}{20})$

(b) $4e^{i\frac{4\pi}{7}}(-1+i\sqrt{3})$

$$z_1 = -1 + i\sqrt{3} \qquad \cos\varphi = \frac{-1}{2} = -\frac{1}{2} \qquad \varphi = \frac{2\pi}{3}$$
$$|z_1| = \sqrt{-1^2 + (\sqrt{3})^2} = \sqrt{4} = 2 \qquad \sin\varphi = \frac{\sqrt{3}}{2}$$
$$z = 2e^{i\frac{2\pi}{3}}$$

$4e^{i\frac{4\pi}{7}}2e^{i\frac{2\pi}{3}} = 8e^{i(\frac{4\pi}{7}+\frac{2\pi}{3})} = 8e^{i\frac{26\pi}{21}}$

(c) $\frac{4i}{5(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9})}$

$$z_1 = 0 + 4i \qquad \cos\varphi = \frac{0}{4} = 0 \qquad \varphi = \frac{\pi}{2}$$
$$|z_1| = \sqrt{0^2 + (4)^2} = 4 \qquad \sin\varphi = \frac{4}{4} = 1$$
$$z = 4(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$$

$\frac{4(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})}{5(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9})} = \frac{4}{5}(\cos(\frac{\pi}{2}-\frac{\pi}{9}) + i\sin(\frac{\pi}{2}-\frac{\pi}{9})) = \frac{4}{5}(\frac{9\pi-2\pi}{+}i\sin\frac{7\pi}{9})$

6. (a) $(\sqrt{3}+i)^{67}$

$$|z| = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{4} = 2 \qquad \begin{matrix} \cos\varphi = \frac{\sqrt{3}}{2} \\ \sin\varphi = \frac{1}{2} \end{matrix} \qquad \varphi = \frac{\pi}{6}$$

$2^{67}(\cos(67\cdot\frac{\pi}{6}) + i\sin(67\cdot\frac{\pi}{6})) = 2^{67}(\cos\frac{7\pi}{6} + \sin\frac{7\pi}{6}) = 2^{67}(-\frac{\sqrt{3}}{2} + i(-\frac{1}{2})) = 2^{66}(-\sqrt{3}-i) = -2^{66}\sqrt{3} - 2^{66}i$

(b) $(1-i)^7$

$$|z| = \sqrt{2} \qquad \begin{matrix} \cos\varphi = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin\varphi = \frac{-1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{matrix} \qquad \varphi = \frac{7\pi}{4}$$

$(\sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}))^7 = 8\sqrt{2}(\cos(7\frac{7\pi}{4}) + i\sin(8\frac{\pi}{4})) = 8\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = 8\sqrt{2}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = 8+8i$

(c) $(-\sqrt{2}+i\sqrt{2})^{44}$

$$|z| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2 \qquad \begin{matrix} \cos\varphi = -\frac{\sqrt{2}}{2} \\ \sin\varphi = \frac{\sqrt{2}}{2} \end{matrix} \qquad \varphi = \frac{3\pi}{4}$$

$2^{44}(\cos 44\frac{3\pi}{4} + i\sin\frac{7\pi}{4}) = 2^{44}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = 2^{43} - 2^{43}i$

(d) $(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})^{10} = \cos(10\frac{\pi}{4}) - i\sin(10\frac{\pi}{4}) = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

7. (a) $\sqrt{-1+i\sqrt{3}}$

$$|z| = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{4} = 2 \qquad \begin{matrix} \cos\varphi = \frac{-1}{2} \\ \sin = \frac{\sqrt{3}}{2} \end{matrix} \qquad \varphi = \frac{2\pi}{3}$$

$$z = = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$$

$$w_k = \sqrt{2}(\cos\frac{\frac{2\pi}{3}+2k\pi}{2} + i\sin\frac{\frac{2\pi}{3}+2k\pi}{2}), k = 0,1$$

$$w_0 = \sqrt{2}(\cos\frac{\frac{2\pi}{3}+0\pi}{2} + i\sin\frac{\frac{2\pi}{3}+0\pi}{2}) = \sqrt{2}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

$$w_1 = \sqrt{2}(\cos\frac{\frac{2\pi}{3}+4\pi}{2} + i\sin\frac{\frac{2\pi}{3}+4\pi}{2}) = \sqrt{2}(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3})$$

(b) $\sqrt{-2i}$

$$|z| = \sqrt{0^2+(-2)^2} = \sqrt{4} = 2 \qquad \begin{matrix} \cos\varphi = \frac{0}{2} = 0 \\ \sin = \frac{-2}{2} = -1 \end{matrix} \qquad \varphi = \frac{3\pi}{2}$$

$$z = = 2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$$

$$w_k = \sqrt{2}(\cos\frac{\frac{3\pi}{2}+2k\pi}{2} + i\sin\frac{\frac{3\pi}{2}+2k\pi}{2}), k = 0,1$$

$$w_0 = \sqrt{2}(\cos\frac{\frac{3\pi}{2}+0\pi}{2} + i\sin\frac{\frac{3\pi}{2}+0\pi}{2}) = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$$

$$w_1 = \sqrt{2}(\cos\frac{\frac{3\pi}{2}+4\pi}{2} + i\sin\frac{\frac{3\pi}{2}+4\pi}{2}) = \sqrt{2}(\cos\frac{11\pi}{4} + i\sin\frac{11\pi}{4}) = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$$

(c) $\sqrt[3]{-8}$

$$|z| = \sqrt{-8^2+(0)^2} = \sqrt{64} = 8 \qquad \begin{matrix} \cos\varphi = \frac{-8}{8} = -1 \\ \sin = \frac{0}{8} = 0 \end{matrix} \qquad \varphi = \pi$$

$$z = = 8(\cos\pi + i\sin\pi)$$

$$w_k = \sqrt[3]{8}(\cos\frac{\pi + 2k\pi}{3} + i\sin\frac{\pi + 2k\pi}{3}), k = 0, 1, 2$$

$$\begin{aligned}
w_0 &= \sqrt[3]{8}(\cos\frac{\pi + 0\pi}{3} + i\sin\frac{\pi + 0\pi}{3}) = \sqrt[3]{8}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) \\
w_1 &= \sqrt[3]{8}(\cos\frac{\pi + 2\pi}{3} + i\sin\frac{\pi + 2\pi}{3}) = \sqrt[3]{8}(\cos\pi + i\sin\pi) \\
w_2 &= \sqrt[3]{8}(\cos\frac{\pi + 4\pi}{3} + i\sin\frac{\pi + 4\pi}{3}) = \sqrt[3]{8}(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})
\end{aligned}$$

(d) $\sqrt[3]{-2+2i}$

$$|z| = \sqrt{(-2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2} \qquad \cos\varphi = \frac{-2}{2\sqrt{2}} = -\frac{1\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \varphi = \frac{3\pi}{4}$$

$$z = = 2\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) \qquad \sin = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$w_k = \sqrt[3]{2\sqrt{2}}(\cos\frac{\frac{3\pi}{4} + 2k\pi}{3} + i\sin\frac{\frac{3\pi}{4} + 2k\pi}{3}), k = 0, 1, 2$$

$$\begin{aligned}
w_0 &= \sqrt[3]{2\sqrt{2}}(\cos\frac{\frac{3\pi}{4} + 0\pi}{3} + i\sin\frac{\frac{3\pi}{4} + 0\pi}{3}) = \sqrt{2}(\cos\frac{3\pi}{12} + \sin\frac{\pi}{4}) \\
w_1 &= \sqrt[3]{2\sqrt{2}}(\cos\frac{\frac{3\pi}{4} + 2\pi}{3} + i\sin\frac{\frac{3\pi}{4} + 2\pi}{3}) = \sqrt{2}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}) \\
w_2 &= \sqrt[3]{2\sqrt{2}}(\cos\frac{\frac{3\pi}{4} + 4\pi}{3} + i\sin\frac{\frac{3\pi}{4} + 4\pi}{3}) = \sqrt{2}(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12})
\end{aligned}$$

(e) $\sqrt[3]{8i}$

$$|z| = \sqrt{(0)^2 + (8)^2} = \sqrt{16} = 8 \qquad \cos\varphi = \frac{0}{8} = 0 \qquad \varphi = \frac{\pi}{2}$$

$$z = = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \qquad \sin = \frac{8}{8} = 1$$

$$w_k = \sqrt[3]{8}(\cos\frac{\frac{\pi}{2} + 2k\pi}{3} + i\sin\frac{\frac{\pi}{2} + 2k\pi}{3}), k = 0, 1, 2$$

$$\begin{aligned}
w_0 &= \sqrt[3]{8}(\cos\frac{\frac{\pi}{2} + 0\pi}{3} + i\sin\frac{\frac{\pi}{2} + 0\pi}{3}) = 2(\cos\frac{\pi}{6} + \sin\frac{\pi}{6}) \\
w_1 &= \sqrt[3]{8}(\cos\frac{\frac{\pi}{2} + 2\pi}{3} + i\sin\frac{\frac{\pi}{2} + 2\pi}{3}) = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) \\
w_2 &= \sqrt[3]{8}(\cos\frac{\frac{\pi}{2} + 4\pi}{3} + i\sin\frac{\frac{\pi}{2} + 4\pi}{3}) = 2(\cos\frac{9\pi}{6} + i\sin\frac{3\pi}{2})
\end{aligned}$$

(f) $\sqrt[4]{-16}$

$$|z| = \sqrt{(-16)^2 + (0)^2} = 16 \qquad \cos\varphi = \frac{-16}{16} = -1 \qquad \varphi = \pi$$

$$z = = 16(\cos\pi + i\sin\pi) \qquad \sin = \frac{0}{16} = 0$$

$$w_k = \sqrt[4]{16}(\cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}), k = 0, 1, 2, 3$$

$$\begin{aligned}
w_0 &= \sqrt[4]{16}(\cos\frac{\pi + 0\pi}{4} + i\sin\frac{\pi + 0\pi}{4}) = 2(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}) \\
w_1 &= \sqrt[4]{16}(\cos\frac{\pi + 2\pi}{4} + i\sin\frac{\pi + 2\pi}{4}) = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) \\
w_2 &= \sqrt[4]{16}(\cos\frac{\pi + 4\pi}{4} + i\sin\frac{\pi + 4\pi}{4}) = 2(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}) \\
w_3 &= \sqrt[4]{16}(\cos\frac{\pi + 6\pi}{4} + i\sin\frac{\pi + 6\pi}{4}) = 2(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})
\end{aligned}$$

(g) $\sqrt[5]{32i}$

$$|z| = \sqrt{(0)^2 + (32)^2} = 32 \qquad \cos\varphi = \frac{0}{32} = 0 \qquad \varphi = \frac{\pi}{2}$$

$$z = = 32(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \qquad \sin = \frac{32}{32} = 1$$

$$w_k = \sqrt[5]{32}(\cos\frac{\frac{\pi}{2} + 2k\pi}{5} + i\sin\frac{\frac{\pi}{2} + 2k\pi}{5}), k = 0, 1, 2, 3, 4$$

$$w_0 = \sqrt[5]{32}(\cos\frac{\frac{\pi}{2} + 0\pi}{5} + i\sin\frac{\frac{\pi}{2} + 0\pi}{5}) = 2(\cos\frac{\pi}{10} + \sin\frac{\pi}{10})$$

$$w_1 = \sqrt[5]{32}(\cos\frac{\frac{\pi}{2} + 2\pi}{5} + i\sin\frac{\frac{\pi}{2} + 2\pi}{5}) = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$$

$$w_2 = \sqrt[5]{32}(\cos\frac{\frac{\pi}{2} + 4\pi}{5} + i\sin\frac{\frac{\pi}{2} + 4\pi}{5}) = 2(\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10})$$

$$w_3 = \sqrt[5]{32}(\cos\frac{\frac{\pi}{2} + 6\pi}{5} + i\sin\frac{\frac{\pi}{2} + 6\pi}{5}) = 2(\cos\frac{13\pi}{10} + i\sin\frac{13\pi}{10})$$

$$w_4 = \sqrt[5]{32}(\cos\frac{\frac{\pi}{2} + 8\pi}{5} + i\sin\frac{\frac{\pi}{2} + 8\pi}{5}) = 2(\cos\frac{17\pi}{10} + i\sin\frac{17\pi}{10})$$