

ćwiczenia 1

November 4, 2021

1. (a) $7 + 4i + 3(5 - 9i) = 7 + 4i + 15 - 27i = 22 - 23i$
 (b) $Im(3 + 2i)(4 - 2i) - Re(5 + 3i)(1 - 4i) = 2(4 + 2i) - 5(1 - 4i) = 8 - 4i - 5 - 20 = 3 + 16i$
 (c) $(3 - 2i)(3 + 4i) = 9 + 12i - 6i - 8i^2 = 17 + 6i$
 (d) $(4 - 5i)(4 + 5i) = 4^2 - (5i)^2 = 16 + 25 = 41$
 (e) $\frac{-11+7i}{2+6i} = \frac{(-11+7i)(2-6i)}{(2+6i)(2-6i)} = \frac{-22+66i+14i-42i^2}{4+36} = \frac{20+80i}{40} = \frac{1}{2} + 2i$
 (f) $\frac{-1+2i}{5-3i} = \frac{(-1+2i)(5+3i)}{(5-3i)(5+3i)} = \frac{-5-3i+10i+6i^2}{25+9} = \frac{-11+7i}{34} = -\frac{11}{34} + \frac{7i}{34}$
 (g) $\frac{3+i}{2-5i} = \frac{(3+i)(2+5i)}{(2-5i)(2+5i)} = \frac{6+15i+2i+5i^2}{4+25} = \frac{1+17i}{29} = \frac{1}{29} + \frac{17i}{29}$
 (h) $\frac{5+3i}{8-2i} = \frac{(5+3i)(8+2i)}{(8-2i)(8+2i)} = \frac{40+10i+24i+6i^2}{64+4} = \frac{34+34i}{68} = \frac{1}{2} + \frac{1}{2}i$
 (i) $\frac{3+2i}{-1+3i} = \frac{(3+2i)(-1-3i)}{(-1+3i)(-1-3i)} = \frac{-3-9i-2i-6i^2}{1+9} = \frac{3-11i}{10} = \frac{3}{10} - \frac{11i}{10}$

2. (a)

$$\begin{array}{lll}
 2z + (3 - i)\bar{z} = 5 + 4i & 5a - b = 5 & 5a - (-4 - a) = 5 \\
 2(a + bi) + (3 - i)(a - bi) = 5 + 4i & -b - a = 4 & 5a + 4 + a = 5 \\
 2a + 2bi + 3a - 3bi - ai + bi^2 = 5 + 4i & a + b = -4 & 6a = 1 \\
 5a - bi - ai - b = 5 + 4i & b = -4 - a & b = -\frac{25}{6} \\
 5a - b + i(-b - a) = 5 + 4i & & a = \frac{1}{6} \\
 \\
 z = \frac{1}{6} - \frac{25}{6}i & &
 \end{array}$$

- (b)

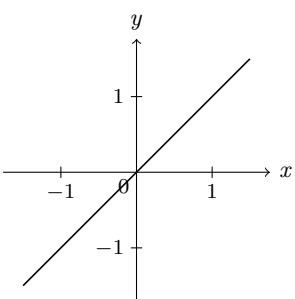
$$\begin{array}{lll}
 2z + (1 + i)\bar{z} = 1 - 3i & 3a + b = 1 & 3a - 3 - a = 1 \\
 2(a + bi) + (1 + i)(a - bi) = 1 - 3i & a + b = -3 & 2a = 4 \\
 2a + 2bi + a - bi + ai - bi^2 = 1 - 3i & b = -3 - a & a = 2 \\
 3a + b + bi + ai = 1 - 3i & & b = -5 \\
 3a + b + (a + b)i = 1 - 3i & & \\
 \\
 z = 2 - 5i & &
 \end{array}$$

- (c)

$$\begin{array}{lll}
 \bar{z} + i = 8 + 3\operatorname{Im}z & a - 3b = 8 & a + 3 = 8 \\
 (a - bi) + i = 8 + 3b & -b - 1 = 0 & a = 5 \\
 a - 3b - (b + 1)i = 8 + 0i & b = -1 & \\
 \\
 z = 5 - i & &
 \end{array}$$

3. tego gówna jeszcze nie rozumiem

(a) $\frac{9}{z} = \bar{z}$



4. (a)

$$\begin{aligned} z &= 6 & \cos \varphi &= \frac{6}{6} = 1 & \varphi &= 0 \\ |z| &= \sqrt{6^2} = 6 & \sin \varphi &= 0 \end{aligned}$$

$$z = 6(\cos 0 + i \sin 0)$$

$$z = 6e^0$$

(b)

$$\begin{aligned} z &= 7 + 7i & \cos \varphi &= \frac{7}{7\sqrt{2}} = \frac{7\sqrt{2}}{7 \times 2} = \frac{\sqrt{2}}{2} & \varphi &= \frac{\pi}{4} \\ |z| &= \sqrt{7^2 + 7^2} = \sqrt{98} = 7\sqrt{2} & \sin \varphi &= \frac{7}{7\sqrt{2}} = \frac{\sqrt{2}}{2} \\ z &= 7\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \\ z &= 7\sqrt{2}e^{\frac{\pi}{4}} \end{aligned}$$

(c)

$$\begin{aligned} z &= -1 + i\sqrt{3} & \cos \varphi &= \frac{-1}{2} = -\frac{1}{2} & \varphi &= \frac{2\pi}{3} \\ |z| &= \sqrt{1 + (\sqrt{3})^2} = \sqrt{4} = 2 & \sin \varphi &= \frac{\sqrt{3}}{2} \\ z &= 2(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) \\ z &= 2e^{\frac{2\pi}{3}} \end{aligned}$$

(d)

$$\begin{aligned} z &= 5\sqrt{3} - 5i & \cos \varphi &= \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2} & \varphi &= \frac{11\pi}{6} \\ |z| &= \sqrt{(5\sqrt{3})^2 + (-5)^2} = \sqrt{25 \cdot 3 + 25} & \sin \varphi &= \frac{-5}{10} = -\frac{1}{2} \\ &= \sqrt{100} = 10 & z &= 10(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}) \\ & & z &= 10e^{\frac{11\pi}{6}} \end{aligned}$$

5. (a) $3(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})(2 - 2i)$

$$\begin{aligned} z &= 2 - 2i & \cos \varphi &= \frac{2}{2\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2} & \varphi &= \frac{7\pi}{4} \\ |z| &= \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2} & \sin \varphi &= \frac{-2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2} \\ & & z &= 2\sqrt{2}(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}) \end{aligned}$$

$$3(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})2\sqrt{2}(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}) = 6\sqrt{2}(\cos(\frac{\pi}{5} + \frac{7\pi}{4}) + i \sin(\frac{\pi}{5} + \frac{7\pi}{4})) = 6\sqrt{2}(\cos \frac{4\pi+35\pi}{20} + i \sin \frac{39\pi}{20})$$

(b) $4e^{i\frac{4\pi}{7}}(-1 + i\sqrt{3})$

$$\begin{aligned} z_1 &= -1 + i\sqrt{3} & \cos \varphi &= \frac{-1}{2} = -\frac{1}{2} & \varphi &= \frac{2\pi}{3} \\ |z_1| &= \sqrt{-1^2 + (\sqrt{3})^2} = \sqrt{4} = 2 & \sin \varphi &= \frac{\sqrt{3}}{2} \\ & & z &= 2e^{i\frac{2\pi}{3}} \end{aligned}$$

$$4e^{i\frac{4\pi}{7}}2e^{i\frac{2\pi}{3}} = 8e^{i(\frac{4\pi}{7} + \frac{2\pi}{3})} = 8e^{i\frac{26\pi}{21}}$$

(c) $\frac{4i}{5(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9})}$

$$\begin{aligned} z_1 &= 0 + 4i & \cos \varphi &= \frac{0}{4} = 0 & \varphi &= \frac{\pi}{2} \\ |z_1| &= \sqrt{0^2 + (4)^2} = 4 & \sin \varphi &= \frac{4}{4} = 1 \\ & & z &= 4(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \end{aligned}$$

$$\frac{4(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})}{5(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9})} = \frac{4}{5}(\cos(\frac{\pi}{2} - \frac{\pi}{9}) + i \sin(\frac{\pi}{2} - \frac{\pi}{9})) = \frac{4}{5}(\frac{9\pi-2\pi}{+}i \sin \frac{7\pi}{9})$$

6. (a) $(\sqrt{3} + i)^{67}$

$$|z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2 \quad \begin{aligned} \cos \varphi &= \frac{\sqrt{3}}{2} \\ \sin \varphi &= \frac{1}{2} \end{aligned} \quad \varphi = \frac{\pi}{6}$$

$$2^{67}(\cos(67 \cdot \frac{\pi}{6}) + i \sin(67 \cdot \frac{\pi}{6})) = 2^{67}(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}) = 2^{67}(-\frac{\sqrt{3}}{2} + i(-\frac{1}{2})) = 2^{66}(-\sqrt{3} - i) = -2^{66}\sqrt{3} - 2^{66}i$$

(b) $(1 - i)^7$

$$|z| = \sqrt{2} \quad \begin{aligned} \cos \varphi &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin \varphi &= \frac{-1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{aligned} \quad \varphi = \frac{7\pi}{4}$$

$$(\sqrt{2}(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}))^7 = 8\sqrt{2}(\cos(7 \cdot \frac{7\pi}{4}) + i \sin(7 \cdot \frac{7\pi}{4})) = 8\sqrt{2}(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}) = 8\sqrt{2}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = 8 + 8i$$

(c) $(-\sqrt{2} + i\sqrt{2})^{44}$

$$|z| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2 \quad \begin{aligned} \cos \varphi &= -\frac{\sqrt{2}}{2} \\ \sin \varphi &= \frac{\sqrt{2}}{2} \end{aligned} \quad \varphi = \frac{3\pi}{4}$$

$$2^{44}(\cos 44 \cdot \frac{3\pi}{4} + i \sin 44 \cdot \frac{3\pi}{4}) = 2^{44}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = 2^{43} - 2^{43}i$$

(d) $(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4})^{10} = \cos(10 \cdot \frac{\pi}{4}) - i \sin(10 \cdot \frac{\pi}{4}) = \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

7. (a) $\sqrt{-1 + i\sqrt{3}}$

$$\begin{aligned} |z| &= \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2 & \cos \varphi &= \frac{-1}{2} & \varphi &= \frac{2\pi}{3} \\ z &= 2(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) & \sin &= \frac{\sqrt{3}}{2} \\ w_k &= \sqrt{2}(\cos \frac{\frac{2\pi}{3} + 2k\pi}{2} + i \sin \frac{\frac{2\pi}{3} + 2k\pi}{2}), k = 0, 1 \end{aligned}$$

$$w_0 = \sqrt{2}(\cos \frac{\frac{2\pi}{3} + 0\pi}{2} + i \sin \frac{\frac{2\pi}{3} + 0\pi}{2}) = \sqrt{2}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$$

$$w_1 = \sqrt{2}(\cos \frac{\frac{2\pi}{3} + 4\pi}{2} + i \sin \frac{\frac{2\pi}{3} + 4\pi}{2}) = \sqrt{2}(\cos \frac{7\pi}{3} + i \sin \frac{7\pi}{3})$$

(b) $\sqrt{-2i}$

$$\begin{aligned} |z| &= \sqrt{0^2 + (-2)^2} = \sqrt{4} = 2 & \cos \varphi &= \frac{0}{2} = 0 & \varphi &= \frac{3\pi}{2} \\ z &= 2(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}) & \sin &= \frac{-2}{2} = -1 \\ w_k &= \sqrt{2}(\cos \frac{\frac{3\pi}{2} + 2k\pi}{2} + i \sin \frac{\frac{3\pi}{2} + 2k\pi}{2}), k = 0, 1 \end{aligned}$$

$$w_0 = \sqrt{2}(\cos \frac{\frac{3\pi}{2} + 0\pi}{2} + i \sin \frac{\frac{3\pi}{2} + 0\pi}{2}) = \sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$$

$$w_1 = \sqrt{2}(\cos \frac{\frac{3\pi}{2} + 4\pi}{2} + i \sin \frac{\frac{3\pi}{2} + 4\pi}{2}) = \sqrt{2}(\cos \frac{11\pi}{4} + i \sin \frac{11\pi}{4}) = \sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$$

(c) $\sqrt[3]{-8}$

$$\begin{aligned} |z| &= \sqrt{-8^2 + (0)^2} = \sqrt{64} = 8 & \cos \varphi &= \frac{-8}{8} = -1 & \varphi &= \pi \\ z &= 8(\cos \pi + i \sin \pi) & \sin &= \frac{0}{8} = 0 \end{aligned}$$

$$w_k = \sqrt[3]{8}(\cos \frac{\pi + 2k\pi}{3} + i \sin \frac{\pi + 2k\pi}{3}), k = 0, 1, 2$$

$$\begin{aligned} w_0 &= \sqrt[3]{8}(\cos \frac{\pi + 0\pi}{3} + i \sin \frac{\pi + 0\pi}{3}) = \sqrt[3]{8}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) \\ w_1 &= \sqrt[3]{8}(\cos \frac{\pi + 2\pi}{3} + i \sin \frac{\pi + 2\pi}{3}) = \sqrt[3]{8}(\cos \pi + i \sin \pi) \\ w_2 &= \sqrt[3]{8}(\cos \frac{\pi + 4\pi}{3} + i \sin \frac{\pi + 4\pi}{3}) = \sqrt[3]{8}(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}) \end{aligned}$$

(d) $\sqrt[3]{-2 + 2i}$

$$\begin{aligned} |z| &= \sqrt{(-2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2} & \cos \varphi &= \frac{-2}{2\sqrt{2}} = -\frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = -\frac{\sqrt{2}}{2} & \varphi &= \frac{3\pi}{4} \\ z &= 2\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) & \sin &= \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2} \\ w_k &= \sqrt[3]{2\sqrt{2}}(\cos \frac{\frac{3\pi}{4} + 2k\pi}{3} + i \sin \frac{\frac{3\pi}{4} + 2k\pi}{3}), k = 0, 1, 2 \end{aligned}$$

$$\begin{aligned} w_0 &= \sqrt[3]{2\sqrt{2}}(\cos \frac{\frac{3\pi}{4} + 0\pi}{3} + i \sin \frac{\frac{3\pi}{4} + 0\pi}{3}) = \sqrt{2}(\cos \frac{3\pi}{12} + i \sin \frac{\pi}{4}) \\ w_1 &= \sqrt[3]{2\sqrt{2}}(\cos \frac{\frac{3\pi}{4} + 2\pi}{3} + i \sin \frac{\frac{3\pi}{4} + 2\pi}{3}) = \sqrt{2}(\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12}) \\ w_2 &= \sqrt[3]{2\sqrt{2}}(\cos \frac{\frac{3\pi}{4} + 4\pi}{3} + i \sin \frac{\frac{3\pi}{4} + 4\pi}{3}) = \sqrt{2}(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12}) \end{aligned}$$

(e) $\sqrt[3]{8i}$

$$\begin{aligned} |z| &= \sqrt{(0)^2 + (8)^2} = \sqrt{16} = 8 & \cos \varphi &= \frac{0}{8} = 0 & \varphi &= \frac{\pi}{2} \\ z &= 8(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) & \sin &= \frac{8}{8} = 1 \\ w_k &= \sqrt[3]{8}(\cos \frac{\frac{\pi}{2} + 2k\pi}{3} + i \sin \frac{\frac{\pi}{2} + 2k\pi}{3}), k = 0, 1, 2 \end{aligned}$$

$$\begin{aligned} w_0 &= \sqrt[3]{8}(\cos \frac{\frac{\pi}{2} + 0\pi}{3} + i \sin \frac{\frac{\pi}{2} + 0\pi}{3}) = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) \\ w_1 &= \sqrt[3]{8}(\cos \frac{\frac{\pi}{2} + 2\pi}{3} + i \sin \frac{\frac{\pi}{2} + 2\pi}{3}) = 2(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}) \\ w_2 &= \sqrt[3]{8}(\cos \frac{\frac{\pi}{2} + 4\pi}{3} + i \sin \frac{\frac{\pi}{2} + 4\pi}{3}) = 2(\cos \frac{9\pi}{6} + i \sin \frac{3\pi}{2}) \end{aligned}$$

(f) $\sqrt[4]{-16}$

$$\begin{aligned} |z| &= \sqrt{(-16)^2 + (0)^2} = 16 & \cos \varphi &= \frac{-16}{16} = -1 & \varphi &= \pi \\ z &= 16(\cos \pi + i \sin \pi) & \sin &= \frac{0}{16} = 0 \\ w_k &= \sqrt[4]{16}(\cos \frac{\pi + 2k\pi}{4} + i \sin \frac{\pi + 2k\pi}{4}), k = 0, 1, 2, 3 \end{aligned}$$

$$\begin{aligned} w_0 &= \sqrt[4]{16}(\cos \frac{\pi + 0\pi}{4} + i \sin \frac{\pi + 0\pi}{4}) = 2(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \\ w_1 &= \sqrt[4]{16}(\cos \frac{\pi + 2\pi}{4} + i \sin \frac{\pi + 2\pi}{4}) = 2(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) \\ w_2 &= \sqrt[4]{16}(\cos \frac{\pi + 4\pi}{4} + i \sin \frac{\pi + 4\pi}{4}) = 2(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}) \\ w_3 &= \sqrt[4]{16}(\cos \frac{\pi + 6\pi}{4} + i \sin \frac{\pi + 6\pi}{4}) = 2(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}) \end{aligned}$$

(g) $\sqrt[5]{32i}$

$$\begin{aligned} |z| &= \sqrt{(0)^2 + (32)^2} = 32 & \cos \varphi &= \frac{0}{32} = 0 & \varphi &= \frac{\pi}{2} \\ z &= 32(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) & \sin &= \frac{32}{32} = 1 & \\ w_k &= \sqrt[5]{32}(\cos \frac{\frac{\pi}{2} + 2k\pi}{5} + i \sin \frac{\frac{\pi}{2} + 2k\pi}{5}), k = 0, 1, 2, 3, 4 \end{aligned}$$

$$\begin{aligned} w_0 &= \sqrt[5]{32}(\cos \frac{\frac{\pi}{2} + 0\pi}{5} + i \sin \frac{\frac{\pi}{2} + 0\pi}{5}) = 2(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}) \\ w_1 &= \sqrt[5]{32}(\cos \frac{\frac{\pi}{2} + 2\pi}{5} + i \sin \frac{\frac{\pi}{2} + 2\pi}{5}) = 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \\ w_2 &= \sqrt[5]{32}(\cos \frac{\frac{\pi}{2} + 4\pi}{5} + i \sin \frac{\frac{\pi}{2} + 4\pi}{5}) = 2(\cos \frac{9\pi}{10} + i \sin \frac{9\pi}{10}) \\ w_3 &= \sqrt[5]{32}(\cos \frac{\frac{\pi}{2} + 6\pi}{5} + i \sin \frac{\frac{\pi}{2} + 6\pi}{5}) = 2(\cos \frac{13\pi}{10} + i \sin \frac{13\pi}{10}) \\ w_4 &= \sqrt[5]{32}(\cos \frac{\frac{\pi}{2} + 8\pi}{5} + i \sin \frac{\frac{\pi}{2} + 8\pi}{5}) = 2(\cos \frac{17\pi}{10} + i \sin \frac{17\pi}{10}) \end{aligned}$$