Lehrstuhl für Numerische Mathematik
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Active Subspaces in Bayesian Inverse Problems

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Thesis Defense

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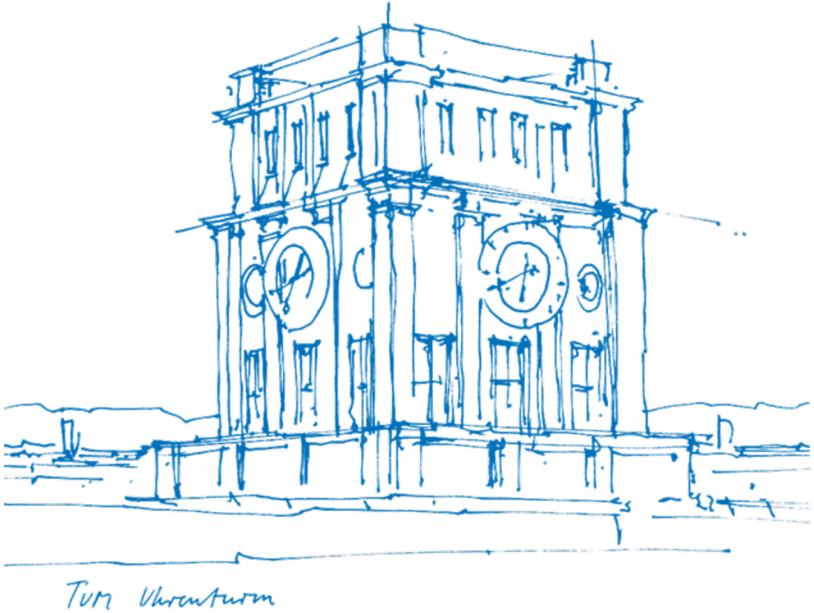
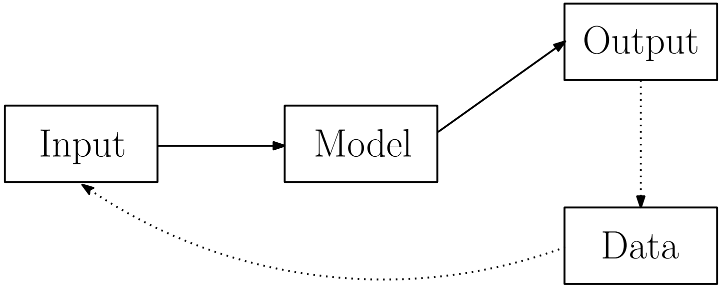


Motivation

Statistical inference of model inputs



Computational challenges

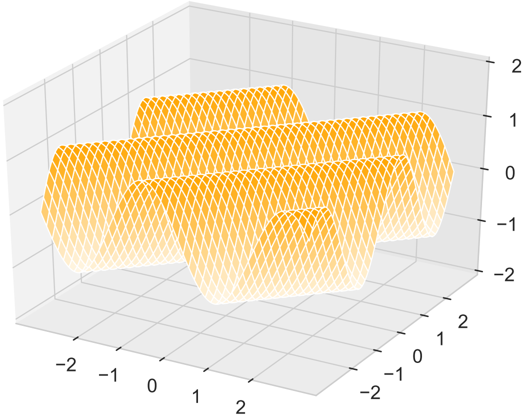
- Performance of a single model run $\Rightarrow$ efficient software, HPC, model reduction, ...
- Dimension of the input space ("curse of dimensionality") $\Rightarrow$ dimension reduction.

The **Active Subspace Method** is a gradient-based technique for subspace-based dimension reduction.

Outline

Outline

ASM – Motivation



Ridge functions $f(\mathbf{x}) = g(A^\top \mathbf{x})$, for $g : \mathbf{R}^k \to \mathbf{R}$, $k \leq n$, and $A \in \mathbf{R}^{n\times k}$, are constant along the null space of $A^\top$ since, for $\mathbf{v} \in \ker(A^\top)$,

$$f(\mathbf{x}+\mathbf{v}) = g(A^\top(\mathbf{x}+\mathbf{v})) = g(A^\top \mathbf{x}) = f(\mathbf{x}). \tag{1}$$

In the figure above, $f(\mathbf{x}) = \sin(-2x_1 + 2x_2)$.

ASM – Setup I (Constantine, Dow, and Wang, 2014)

- $(\Omega, \mathscr{A}, \mathbf{P})$ probability space.
- $\mathbf{X} \sim \mathbf{P}_\mathbf{X}$, $\mathbf{X} \in \mathbf{R}^n$, random vector (inputs, parameters).
- $\mathscr{X} := \operatorname{supp}(\mathbf{P}_\mathbf{X}) \subseteq \mathbf{R}^n$ continuity set (i. e., $\mathbf{P}_\mathbf{X}(\partial\mathscr{X}) = 0$). In particular, $\mathbf{P}_\mathbf{X}(\mathscr{X}) = 1$.
- $f : \mathscr{X} \to \mathbf{R}$ such that $\nabla f \in L^2(\mathscr{X}, \mathbf{P}_\mathbf{X})$.

Goal: Approximate f by a ridge function, i. e., find g and $A \in \mathbf{R}^{n\times k}$, $k \leq n$, such that

$$f(\mathbf{x}) \approx g(A^\top \mathbf{x}) \tag{2}$$

for each $\mathbf{x} \in \mathscr{X}$.

ASM – Setup II

Define

$$\begin{aligned} C &:= \mathbf{E}[\nabla f(\mathbf{X}) \nabla f(\mathbf{X})^\top] \\ &= \int_{\mathscr{X}} \nabla f(\mathbf{x}) \nabla f(\mathbf{x})^\top \rho_{\mathbf{X}}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ &=: W \Lambda W^\top. \end{aligned} \tag{3}$$

Note that C is symmetric and positive semi-definite, i. e., we can choose

$$W = \begin{pmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_n \\ | & & | \end{pmatrix} \in \mathbf{R}^{n \times n} \text{ to be orthogonal and } \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathbf{R}^{n \times n} \tag{4}$$

with

$$\lambda_1 \geq \cdots \geq \lambda_n \geq 0. \tag{5}$$

ASM – Setup III

Important:

$$\lambda_i = \mathbf{w}_i^\top C \mathbf{w}_i = \mathbf{E}[(\mathbf{w}_i^\top \nabla f(\mathbf{X}))^2]. \tag{6}$$

Question: What does λ_i small (or even zero) and λ_i large mean?

We hope that the eigenvalues λ_i are decaying quickly on a logarithmic scale. If so, for a certain $k \leq n$, split

$$W =: \begin{pmatrix} W_1 & W_2 \end{pmatrix} \tag{7}$$

for $W_1 \in \mathbf{R}^{n \times k}$ and $W_2 \in \mathbf{R}^{n \times (n-k)}$. The column space of W_1 (resp. W_2) is called the active (resp. inactive) subspace of f.

Define (random) variables for the subspaces by an orthogonal transformation, i. e.,

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} := W^\top \mathbf{x} = \begin{pmatrix} W_1^\top \mathbf{x} \\ W_2^\top \mathbf{x} \end{pmatrix}. \tag{8}$$

The variable $\mathbf{y} \in \mathbf{R}^k$ (resp. $\mathbf{z} \in \mathbf{R}^{n-k}$) is called the active (resp. inactive) variable.

ASM – Setup IV

Recall: The goal was to find g and $A \in \mathbf{R}^{n \times k}$ such that $f(\mathbf{x}) \approx g(A^\top \mathbf{x})$ for each $\mathbf{x} \in \mathscr{X}$.

The map $\mathbf{x} \mapsto g(A^\top \mathbf{x})$ was said to be constant along the null space of $A^\top$.

$\Rightarrow$ We found $A = W_1$ since $\ker(W_1^\top) = \operatorname{ran}(W_2)$ (by $\ker(W_1^\top) \perp \operatorname{ran}(W_1)$ and orthogonality of W).

Next step: Find "best" function g.

It is well-known that, if $\mathbf{E}[f(\mathbf{X})^2] < \infty$, the conditional expectation of $f(\mathbf{X})$ given $\mathbf{Y} = W_1^\top \mathbf{X}$ minimizes the mean square error to f, i. e.,

$$\mathbf{E}[(f(\mathbf{X}) - \mathbf{E}[f(\mathbf{X}) \mid \mathbf{Y}])^2] \leq \mathbf{E}[(f(\mathbf{X}) - \mathbf{U})^2] \tag{9}$$

for any square-integrable random variable $\mathbf{U}$ which is measurable w.r.t. the σ-algebra generated by $\mathbf{Y}$.

ASM – Setup V

Notation: We denote

$$\mathbf{x} = WW^\top \mathbf{x} = W_1\mathbf{y} + W_2\mathbf{z} =: [\![\mathbf{y},\mathbf{z}]\!]_W = [\![\mathbf{y},\mathbf{z}]\!]. \tag{10}$$

Hence, we define

$$\begin{aligned} g(\mathbf{y}) &:= \mathbf{E}[f([\![\mathbf{Y},\mathbf{Z}]\!]) \,|\, \mathbf{Y} = \mathbf{y}] \\ &= \int_{\mathbf{R}^{n-k}} f([\![\mathbf{y},\mathbf{z}]\!])\, \rho_{\mathbf{Z}|\mathbf{Y}}(\mathbf{z}|\mathbf{y})\, \mathrm{d}\mathbf{z} \end{aligned} \tag{11}$$

and finally get the approximating ridge function

$$f_g(\mathbf{x}) := g(W_1^\top \mathbf{x}). \tag{12}$$

Question: How "good" is f_g in approximating f in terms of the neglected directions in W_2?

ASM – Common bounds

In Constantine et al. (2014), we have the following result.

Theorem

For each $\mathbf{P_X}$, there exists a Poincaré constant $C_P = C_P(\mathbf{P_X}) > 0$ such that

$$\mathbf{E}[(f(\mathbf{X}) - f_g(\mathbf{X}))^2] \leq C_P(\lambda_{k+1} + \cdots + \lambda_n). \tag{13}$$

Proof

Main ingredient: Probabilistic Poincaré inequality.
[For a random variable $\mathbf{U} \sim \mathbf{P_U}$ and a sufficiently regular function h, it holds that

$$\mathrm{Var}(h(\mathbf{U})) \leq C_\mathrm{P}\, \mathbf{E}[\|\nabla h(\mathbf{U})\|_2^2] \tag{14}$$

for some Poincaré constant $C_\mathrm{P} = C_\mathrm{P}(\mathbf{P_U}) > 0$.]
The main step is to compute

$$\begin{aligned} \mathbf{E}[(f(\mathbf{X}) - f_g(\mathbf{X}))^2] &= \mathbf{E}[(f([\![\mathbf{Y},\mathbf{Z}]\!]) - g(\mathbf{Y}))^2] \\ &\leq C_\mathrm{P}\, \mathbf{E}[\|\nabla^{\mathbf{z}} f(\mathbf{X})\|_2^2]. \end{aligned} \tag{15}$$

Note: The Poincaré constant C_P was taken w.r.t. $\mathbf{P_X}$ which is not correct in general. ∎

ASM – Generalized bounds I (Teixeira Parente, Wallin, and Wohlmuth, 2020)

A correct application of the probabilistic Poincaré inequality gives

$$\begin{aligned}\mathbf{E}[(f(\mathbf{X}) - f_g(\mathbf{X}))^2] &= \mathbf{E}[\mathbf{E}[(f([\![\mathbf{Y},\mathbf{Z}]\!]) - g(\mathbf{Y}))^2 \,|\, \mathbf{Y}]] \\ &\leq \mathbf{E}[C_{\mathbf{Y}} \cdot \mathbf{E}[\|\nabla^{\mathbf{z}} f([\![\mathbf{Y},\mathbf{Z}]\!])\|_2^2 \,|\, \mathbf{Y}]],\end{aligned} \tag{16}$$

where $C_{\mathbf{Y}} > 0$ is the Poincaré constant of $\mathbf{P}_{\mathbf{Z}|\mathbf{Y}}$ and hence randomly depending on $\mathbf{Y}$.

The constant $C_{\mathbf{Y}}$ is known to be uniform in $\mathbf{Y}$ for, e. g., compactly supported and so-called α-uniformly log-concave ($\alpha > 0$) distributions $\mathbf{P}_{\mathbf{X}}$.

Question: Can we find a counterexample, i. e., a distribution $\mathbf{P}_{\mathbf{X}}$ and a transformation W such that the corresponding $C_{\mathbf{Y}}$ is unbounded in $\mathbf{Y}$/not compactly supported?

ASM – Generalized bounds II

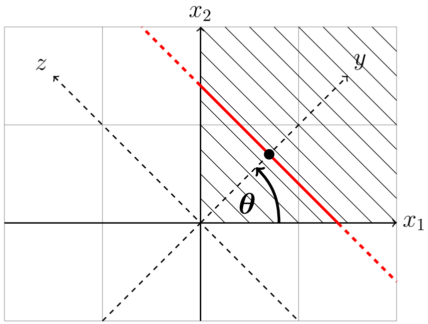
Answer: Yes.

Idea: We need to find a distribution that has heavier tails than any α-uniformly log-concave distribution but is still applicable for a Poincaré inequality. Look at the edge case when $\alpha \to 0$. $\Rightarrow$ General log-concave distributions.

Example: Exponential distribution with unit rates in two dimensions; rotation by 45°.

Use lower bound for $C_{\mathbf{Y}}$ of Bobkov (1999):

$$C_{\mathbf{y}} \geq \mathrm{Var}(|\mathbf{Z}| \mid \mathbf{Y} = \mathbf{y}) = \mathbf{y}^2/12. \tag{17}$$



ASM – Generalized bounds III

Question: Can we derive weaker bounds for general log-concave distributions?

Answer: Yes. Use Hölder's inequality with a weaker pair of conjugates to get

$$\mathbf{E}[(f(\mathbf{X})-f_g(\mathbf{X}))^2] \leq C_{\mathrm{P},\varepsilon,W}(\lambda_{k+1}+\cdots+\lambda_n)^{1/(1+\varepsilon)} \tag{18}$$

for $\varepsilon > 0$ and a constant $C_{\mathrm{P},\varepsilon,W} > 0$.

ASM – Practical considerations (Constantine, 2015)

In practice, we approximate the matrix $C = \mathbf{E}[\nabla f(\mathbf{X}) \nabla f(\mathbf{X})^\top]$ by a finite Monte Carlo sum, i. e., by

$$\tilde{C} := \frac{1}{N_{\tilde{C}}} \sum_{j=1}^{N_{\tilde{C}}} \nabla f(\mathbf{X}_j) \nabla f(\mathbf{X}_j)^\top \tag{19}$$

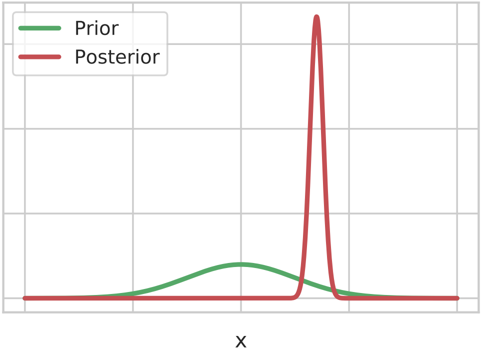
for $N_{\tilde{C}} > 0$ and $\mathbf{X}_j \overset{\text{i.i.d.}}{\sim} \mathbf{P}_{\mathbf{X}}$, $j = 1, \dots, N_{\tilde{C}}$.

It is well-known by an eigenvalue Bernstein inequality that $N_{\tilde{C}} = O(\log n)$ samples are enough to approximate "the first eigenvalues of C sufficiently accurate."

Outline

BIPs – Motivation

Bayesian inverse problems exploit observational data to update a prior distribution on model parameters to a posterior distribution.



In view of Uncertainty Quantification, the posterior distribution quantifies the updated, remaining uncertainty on model parameters.

BIPs – Setup I (Stuart, 2010; Dashti and Stuart, 2016)

Parameters are denoted by $\mathbf{X} \sim \mathbf{P}_{\mathbf{X}}$ or $\mathbf{x}$, both in $\mathbf{R}^n$.

The forward model $\mathscr{G} : \mathscr{X} \to \mathbf{R}^{n_\mathbf{d}}$ maps a parameter to a corresponding value for the Quantity of Interest (QoI).

The observational data $\mathbf{d} \in \mathbf{R}^{n_\mathbf{d}}$ are assumed to be noisy realizations of a model evaluation, i. e.,

$$\mathbf{D} = \mathscr{G}(\mathbf{X}) + \eta, \tag{20}$$

where noise $\eta \sim \mathscr{N}(0, \Gamma)$ is assumed to be Gaussian with mean zero and covariance matrix $\Gamma \in \mathbf{R}^{n_\mathbf{d} \times n_\mathbf{d}}$.

BIPs – Setup II

For observed data $\mathbf{d} \in \mathbf{R}^{n_\mathbf{d}}$, the posterior distribution $\mu^\mathbf{d}$ is the conditional distribution of parameters $\mathbf{X}$ given that $\mathbf{D} = \mathbf{d}$, i. e.,

$$\mu^\mathbf{d} := \mathbf{P}_{\mathbf{X}|\mathbf{D}}(\cdot|\mathbf{d}) = \mathbf{P}(\mathbf{X} \in \cdot \,|\, \mathbf{D} = \mathbf{d}). \tag{21}$$

The corresponding posterior density $\rho^\mathbf{d}$ is proportional to the likelihood and the prior density $\rho_0 := \rho_\mathbf{X}$, i. e.,

$$\rho^\mathbf{d}(\mathbf{x}) := \rho_{\mathbf{X}|\mathbf{D}}(\mathbf{x}|\mathbf{d}) \propto \exp(-f^\mathbf{d}(\mathbf{x})) \cdot \rho_0(\mathbf{x}) \tag{22}$$

for the data misfit function (or negative log-likelihood)

$$f^\mathbf{d}(\mathbf{x}) = \frac{1}{2}\|\mathbf{d} - \mathscr{G}(\mathbf{x})\|_\Gamma^2, \tag{23}$$

where $\|\cdot\|_\Gamma := \|\Gamma^{-1/2}\cdot\|_2$.

BIPs – Application of ASM (Constantine, Kent, and Bui-Thanh, 2016)

In the setting of ASM, we set

$$f = f^{\mathbf{d}} \tag{24}$$

and thus compute the active subspace of the data misfit function to obtain a ridge approximation

$$f^{\mathbf{d}}(\mathbf{x}) \approx g^{\mathbf{d}}(W_1^\top \mathbf{x}). \tag{25}$$

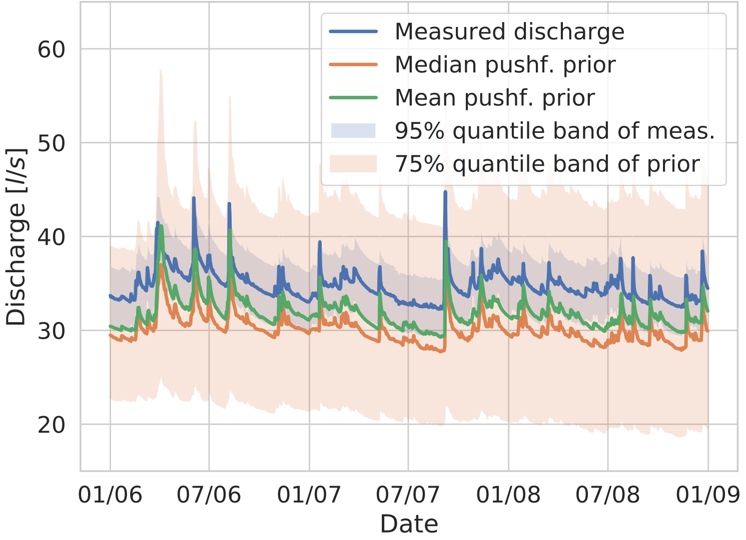
Consequently, we can find a low-dimensional approximation of the posterior density, i. e., for $\mathbf{x} = [\![\mathbf{y}, \mathbf{z}]\!]$,

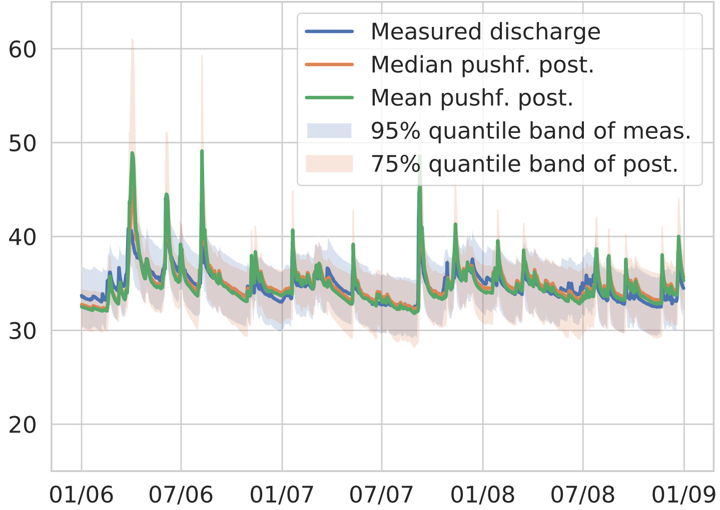
$$\begin{aligned} \rho^{\mathbf{d}}(\mathbf{x}) &\propto \exp(-f^{\mathbf{d}}(\mathbf{x})) \cdot \rho_0(\mathbf{x}) \\ &\approx \exp(-g^{\mathbf{d}}(W_1^\top \mathbf{x})) \cdot \rho_0(\mathbf{x}) \\ &= \underbrace{\exp(-g^{\mathbf{d}}(\mathbf{y})) \cdot \rho_{\mathbf{Y}}(\mathbf{y})}_{\propto: \rho^{\mathbf{d}}_{g,\mathbf{Y}}(\mathbf{y})} \cdot \rho_{\mathbf{Z}|\mathbf{Y}}(\mathbf{z}|\mathbf{y}). \end{aligned} \tag{26}$$

Outline

Case study (Teixeira Parente, Bittner, Mattis, Chiogna, and Wohlmuth, 2019)

Groundwater karst model for a discharge time series of the Kerschbaum spring recharge area in Waidhofen a.d. Ybbs (Austria).





Outline

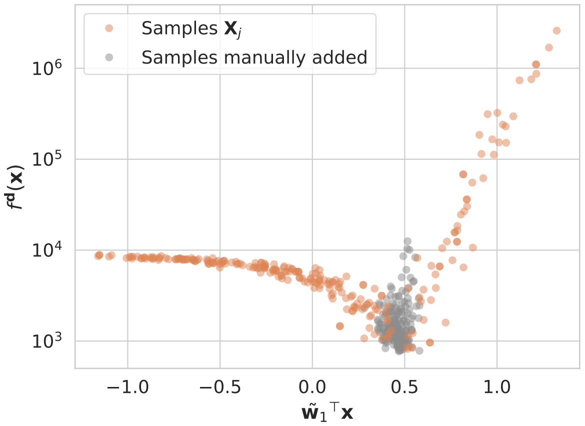
Iterative ASM – Idea I (Section 7 in dissertation)

Problem: Since sensitivities are computed w.r.t. the prior, i. e.,

$$C_0 = \int_{\mathcal{X}} \nabla f^{\mathbf{d}}(\mathbf{x}) \nabla f^{\mathbf{d}}(\mathbf{x})^\top \rho_0(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{27}$$

a "bad prior" may cause a misleading active subspace.

Example:

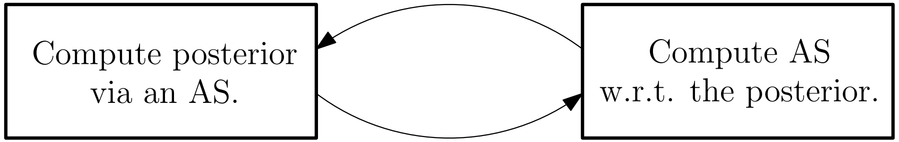


Iterative ASM – Idea II

Actual goal: Compute sensitivities w.r.t. the posterior, i. e.,

$$C^{\mathbf{d}} := \int_{\mathcal{X}} \nabla f^{\mathbf{d}}(\mathbf{x}) \nabla f^{\mathbf{d}}(\mathbf{x})^{\top} \rho^{\mathbf{d}}(\mathbf{x}) \, d\mathbf{x}. \tag{28}$$

Problem:



$\Rightarrow$ Iterative scheme.

Iterative ASM – Idea III

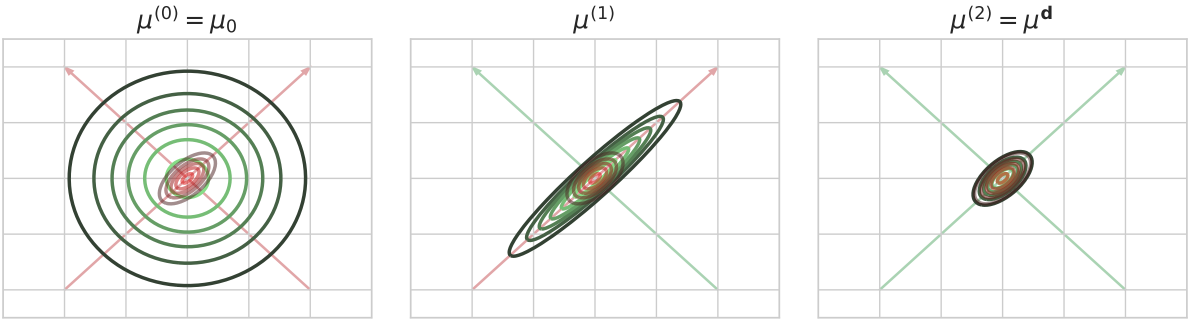
Compute a sequence of distributions $\mu^{(\ell)}$ approaching $\mu^{\mathbf{d}}$ starting from the prior $\mu^{(0)} = \mu_0$.

Ideal algorithm: Set $\mu^{(0)} := \mu_0$. In the ℓ-th step, $\mathbf{X}^{(\ell)} \sim \mu^{(\ell)}$ and the main steps are:

- Compute
$$C^{(\ell)} := \mathbf{E}[\nabla f^{\mathbf{d}}(\mathbf{X}^{(\ell)}) \nabla f^{\mathbf{d}}(\mathbf{X}^{(\ell)})^\top] = W^{(\ell)} \Lambda^{(\ell)} W^{(\ell)\top}. \tag{29}$$
- Decide for an active subspace, i. e., for $W_1^{(\ell)}$, and compute
$$g^{(\ell)}(\mathbf{y}) := \mathbf{E}[f^{\mathbf{d}}([\![\mathbf{Y}^{(\ell)}, \mathbf{Z}^{(\ell)}]\!]_{W^{(\ell)}}) \mid \mathbf{Y}^{(\ell)} = \mathbf{y}]. \tag{30}$$
- Compute samples $\mathbf{X}^{(\ell+1)} \sim \mu^{(\ell+1)}$ using $g^{(\ell)}$.

Iterative ASM – Idea IV

Illustrative example:

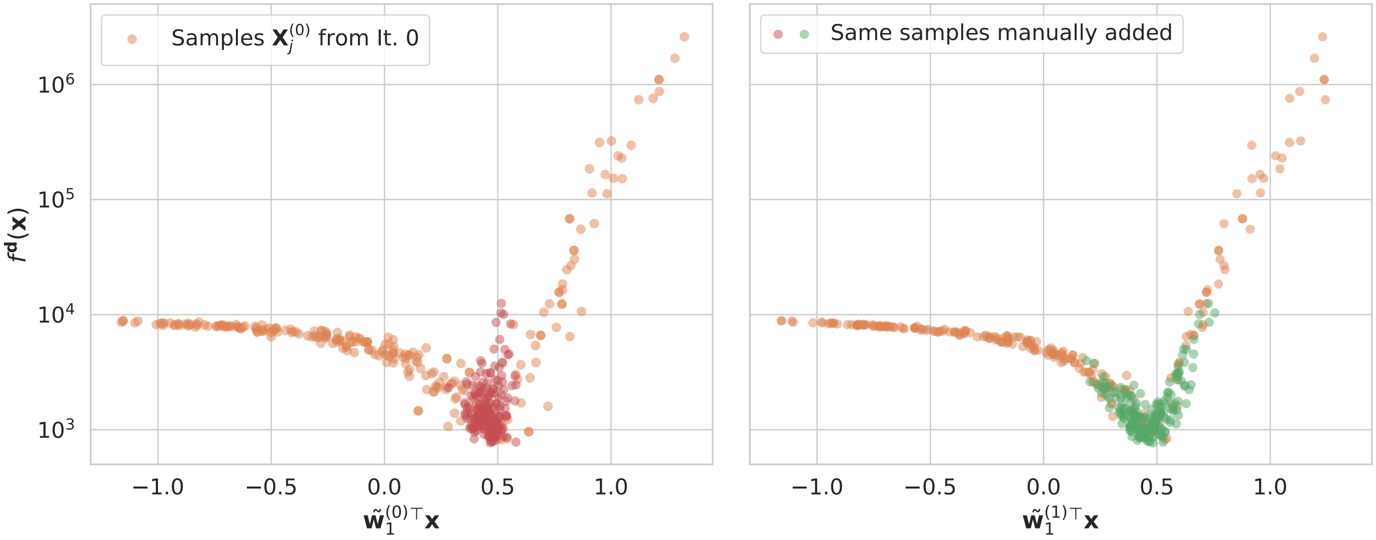


Remark I: Convergence and consistency of the ideal algorithm are formally well-understood for a linear Gaussian BIP.

Remark II: In practice, we do not aim to converge to the posterior since the exact quantities used in the algorithm are not available. But we can use the algorithm with approximate quantities as a "preconditioner" for BIPs with bad prior distributions.

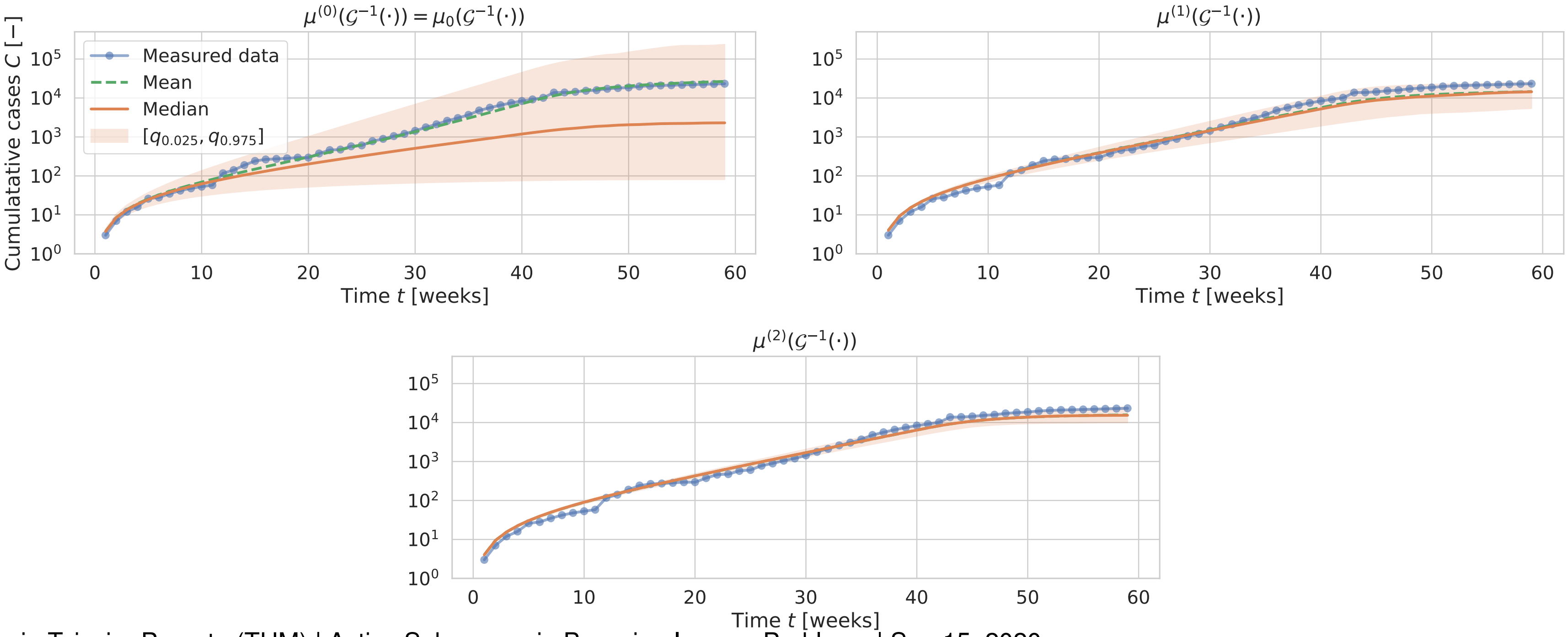
Iterative ASM – Case study I

Compartmental model for 2014 Ebola outbreak in West Africa (Barbarossa et al., 2015).



Iterative ASM – Case study II

The following figures show the evolution of the push-forward distribution of $\mu^{(\ell)}$, $\ell = 0, 1, 2$.



Outline

Summary

We presented a counterexample to existing Active Subspace theory and provided a solution by generalized bounds.

A case study involving a complex high-dimensional hydrological model demonstrated that ASM can substantially reduce the dimension and computational expenses of Bayesian inverse problems.

For BIPs with "bad priors", we suggested an iterative scheme to find a better active subspace by computing sensitivities in regions of low data misfit. This was demonstrated on a model for the 2014 Ebola outbreak in West Africa.

Thank you!

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  Lund University, Department of Statistics: Prof. Krzysztof Podgórski, Jonas Wallin

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Publications

Teixeira Parente, M., Wallin, J., & Wohlmuth, B., Generalized Bounds for Active Subspaces, *Electronic Journal of Statistics*, 14(1):917–943, 2020

Bittner, D., **Teixeira Parente, M.**, Mattis, S., Wohlmuth, B., & Chiogna, G., Identifying Relevant Hydrological and Catchment Properties in Active Subspaces: An Inference Study of a Lumped Karst Aquifer Model. *Advances in Water Resources*, 135, 103472, 2020.

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M. Teixeira Parente, D. Bittner, S. A. Mattis, G. Chiogna, and B. Wohlmuth. Bayesian Calibration and Sensitivity Analysis for a Karst Aquifer Model using Active Subspaces. *Water Resources Research*, 55(8):7086–7107, 2019.

Supplementary Material

ASM – Generalized bounds IV

Theorem

Let $\varepsilon > 0$. If $\|\nabla f(\mathbf{X})\|_2^2 \leq L$ $\mathbf{P}$-a.s. for some constant $L > 0$, then

$$\mathbf{E}[(f(\mathbf{X}) - f_g(\mathbf{X}))^2] \leq C_{P,\varepsilon,W}\,(\lambda_{k+1} + \cdots + \lambda_n)^{1/(1+\varepsilon)}, \tag{31}$$

where

$$C_{P,\varepsilon,W} = C_{P,\varepsilon,W}(\varepsilon, n, k, L, W, \mathbf{P_X}) := L^{\varepsilon/(1+\varepsilon)}\mathbf{E}[C_{\mathbf{Y}}^{(1+\varepsilon)/\varepsilon}]^{\varepsilon/(1+\varepsilon)}. \tag{32}$$

Proof

Main ingredient: Hölder's inequality for $\mathbf{E}[C_{\mathbf{Y}} \cdot \mathbf{E}[\|\nabla^{\mathbf{z}} f([\![\mathbf{Y}, \mathbf{Z}]\!])\|_2^2 \,|\, \mathbf{Y}]]$ with a weaker pair of conjugates

$$(p, q) = ((1+\varepsilon)/\varepsilon, 1+\varepsilon) \tag{33}$$

instead of $(p, q) = (+\infty, 1)$. ■

Remark: For exponential distributions, the constant $C_{P,\varepsilon,W}$ can be bounded uniformly in W by an analytical expression.

ASM – Practical considerations II

Approximation of eigenvalues of C

Theorem (Constantine (2015, Thm. 3.3))

Assume that $\|\nabla f(\mathbf{x})\|_2 \leq L$ *for some* $L > 0$ *and all* $\mathbf{x} \in \mathscr{X}$. *For* $\varepsilon \in (0,1]$, *it holds that*

$$\mathbf{P}\left(\tilde{\lambda}_\ell \geq (1+\varepsilon)\lambda_\ell\right) \leq (n-\ell+1)\exp\left(-\frac{N_{\tilde{C}}\lambda_\ell\varepsilon^2}{4L^2}\right) \tag{34}$$

and

$$\mathbf{P}\left(\tilde{\lambda}_\ell \leq (1-\varepsilon)\lambda_\ell\right) \leq \ell\exp\left(-\frac{N_{\tilde{C}}\lambda_\ell^2\varepsilon^2}{4\lambda_1 L^2}\right) \tag{35}$$

for $\ell = 1, \ldots, n$.

Proof

Main ingredient: Eigenvalue Bernstein inequality for a finite sum of random, independent, and symmetric matrices satisfying a subexponential growth condition. ∎

ASM – Practical considerations III

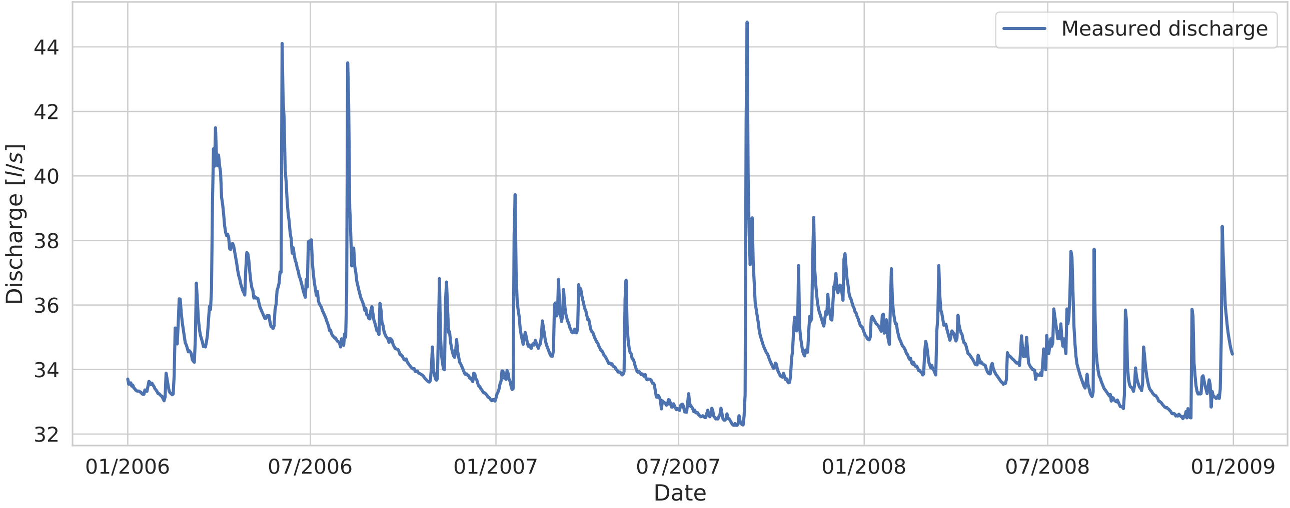
The function $g(\mathbf{y}) = \mathbf{E}[f([\![\mathbf{Y},\mathbf{Z}]\!])\,|\,\mathbf{Y}=\mathbf{y}]$ is also approximated by a finite Monte Carlo sum, i. e., by

$$g_N(\mathbf{y}) := \frac{1}{N}\sum_{j=1}^{N} f([\![\mathbf{y},\mathbf{Z}_j^{\mathbf{y}}]\!]) \tag{36}$$

for $N > 0$ and $\mathbf{Z}_j^{\mathbf{y}} \overset{\text{i.i.d.}}{\sim} \mathbf{P}_{\mathbf{Z}|\mathbf{Y}}(\cdot|\mathbf{y})$, $j = 1,\dots,N$.

Case study II

Measured discharge data



Case study III

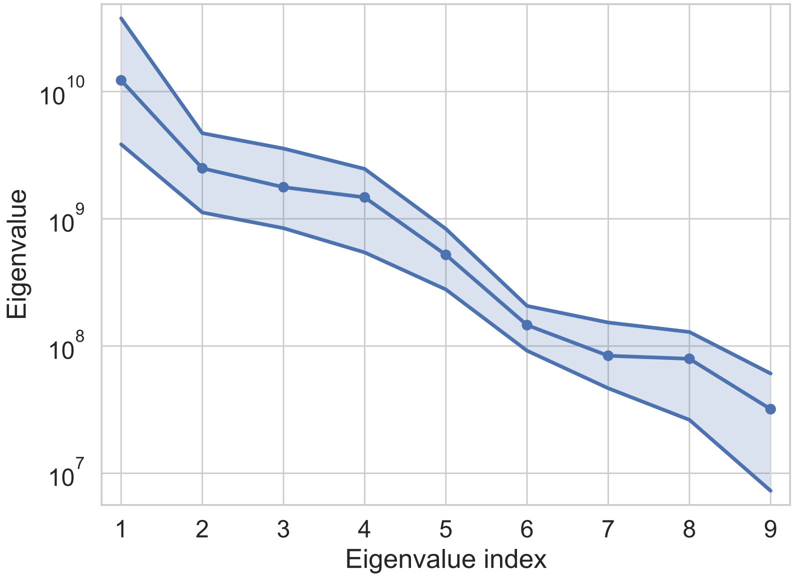
Problem characteristics:

- Parameter space is $n = 21$-dimensional.
- Data consists of $n_{\mathbf{d}} = 1096$ discharge values.
- We assume a 5% noise level on the measured data.
- The prior distribution is chosen to be uniform on intervals predefined by hydrologists.
- A single model run needs about 2.5 seconds.

Implementation: The computation of $N_{\tilde{C}} = 1000$ gradients needed about 4.3 hours using 7 CPU cores in parallel.

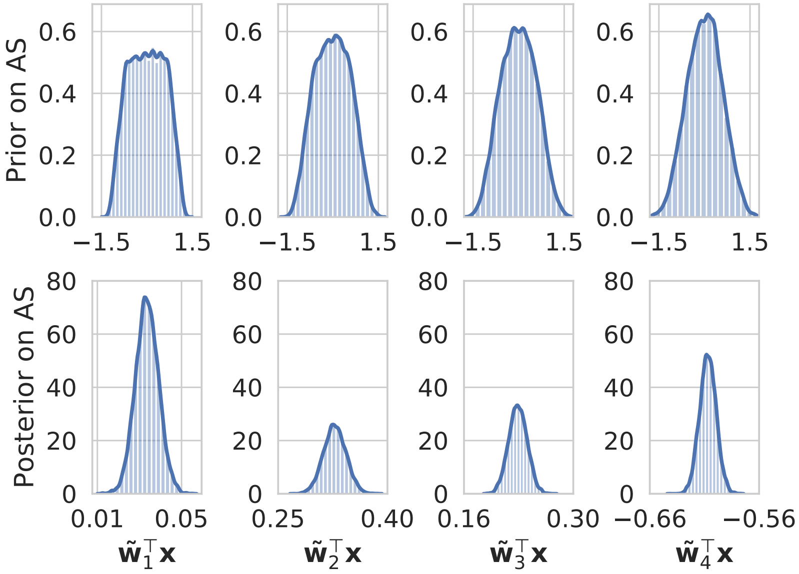
Case study IV

The following figure shows the approximated eigenvalues with an "uncertainty band" to reflect their random nature.



Case study V

The following figure shows the prior (top) and posterior (bottom) distribution on the active subspace.



Iterative ASM – Analysis

Proposition (Dissertation, Prop. 7.3.1)

Let $\mu_0 = \mathscr{N}(\mathbf{m}_0, I)$ *with* $\mathbf{m}_0 \in \mathbf{R}^n$*. Suppose that* $\mathscr{G}(\mathbf{x}) := A\mathbf{x}$ *for* $\mathbf{x} \in \mathscr{X}$ *with* $A \in \mathbf{R}^{n_\mathbf{d} \times n}$*, and* $\eta \sim \mathscr{N}\left(0, \gamma^2 I\right)$ *for* $\gamma > 0$*.*
Furthermore, assume that $\mathbf{d} = \mathscr{G}(\mathbf{m}_0)$*.*
Considering the ideal algorithm, set $\Lambda := \Lambda^{(0)}$ *and* $W := W^{(0)}$*. Then, for every iteration* $\ell \in \mathbf{N}_0$*, it holds that*

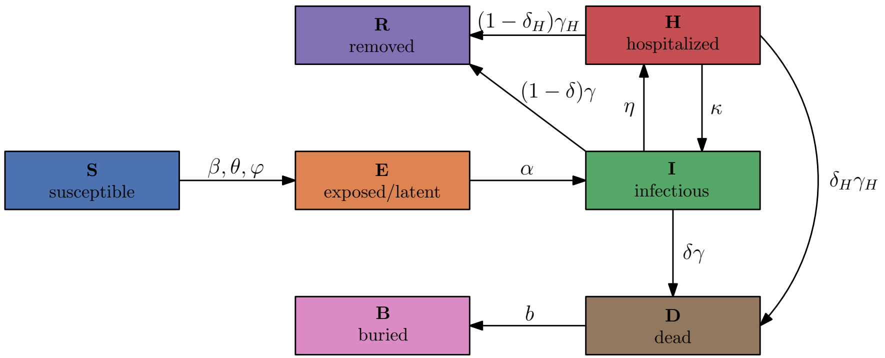
$$\mathbf{X}^{(\ell)} \sim \mathscr{N}\left(\mathbf{m}_0, \Sigma^{(\ell)}\right) \tag{37}$$

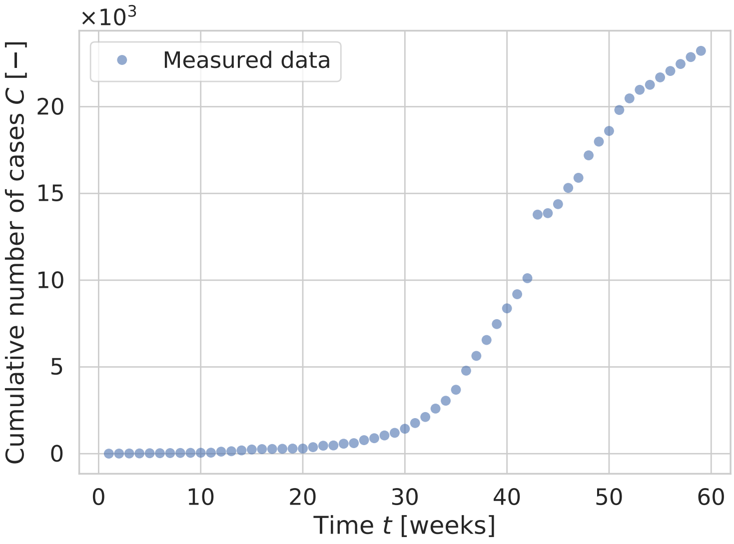
with

$$\Sigma^{(\ell)} = W \begin{pmatrix} \left(I + \Lambda^{1/2}_{1:K^{(\ell)}}\right)^{-1} & 0 \\ 0 & I \end{pmatrix} W^\top \tag{38}$$

for some natural number $0 \leq K^{(\ell)} \leq n$*.*

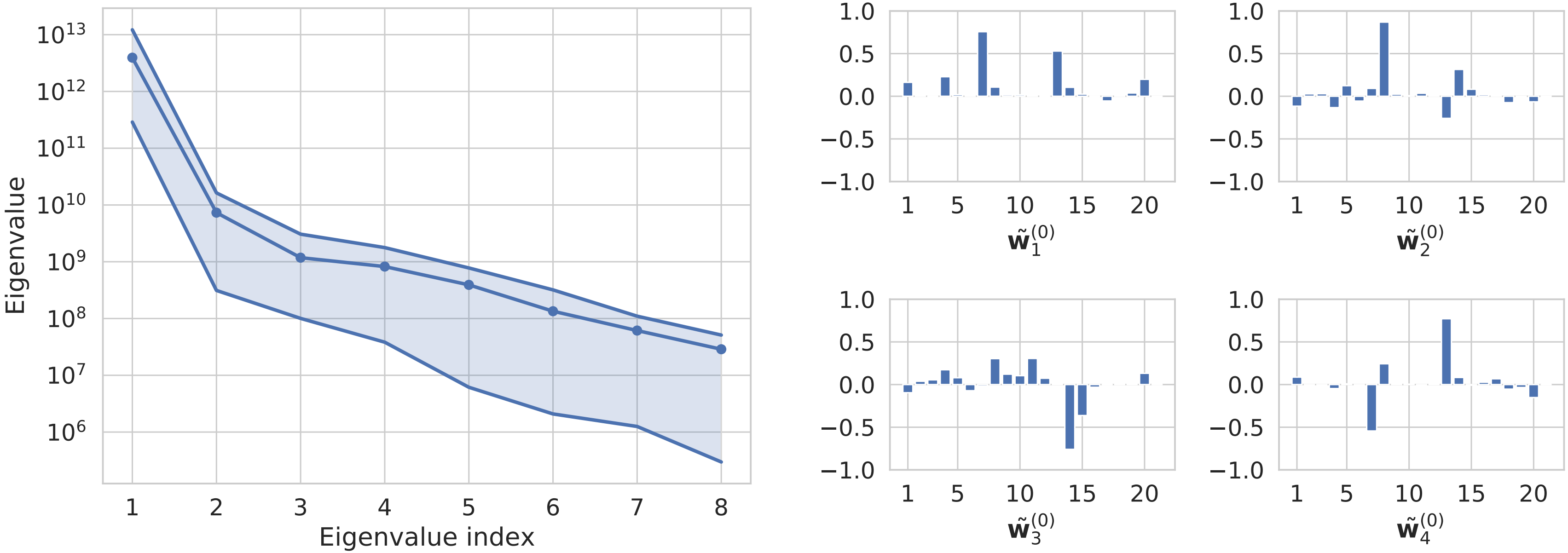
Iterative ASM – Case study, Fig. I





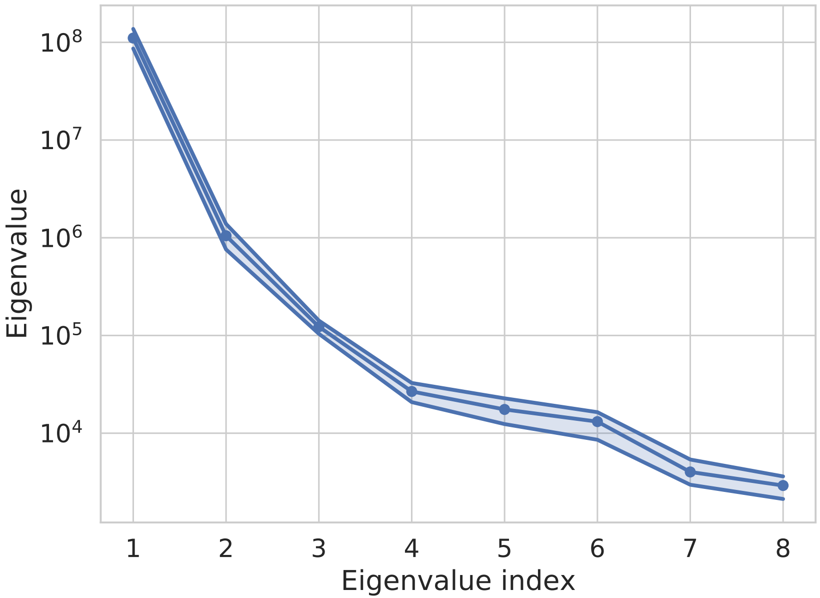
Iterative ASM – Case study, Fig. II

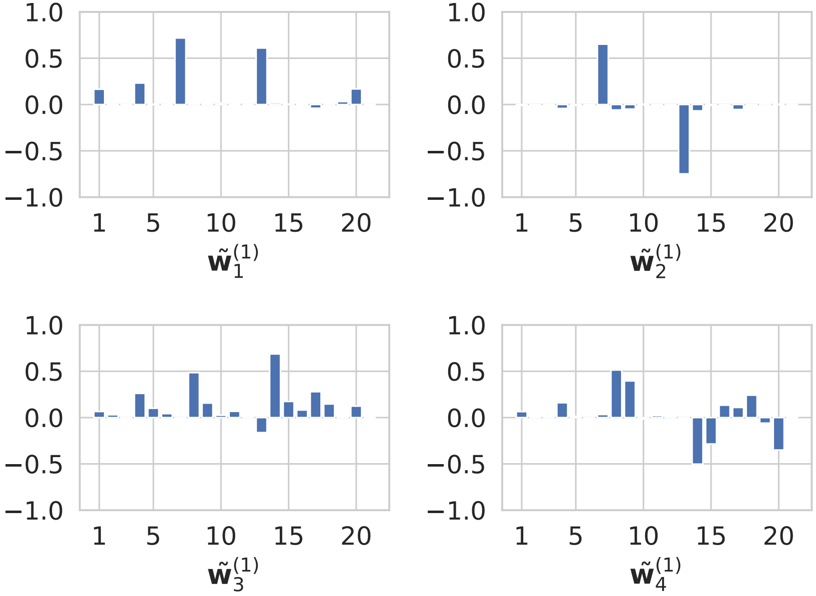
Eigendecomposition It. 0



Iterative ASM – Case study, Fig. III

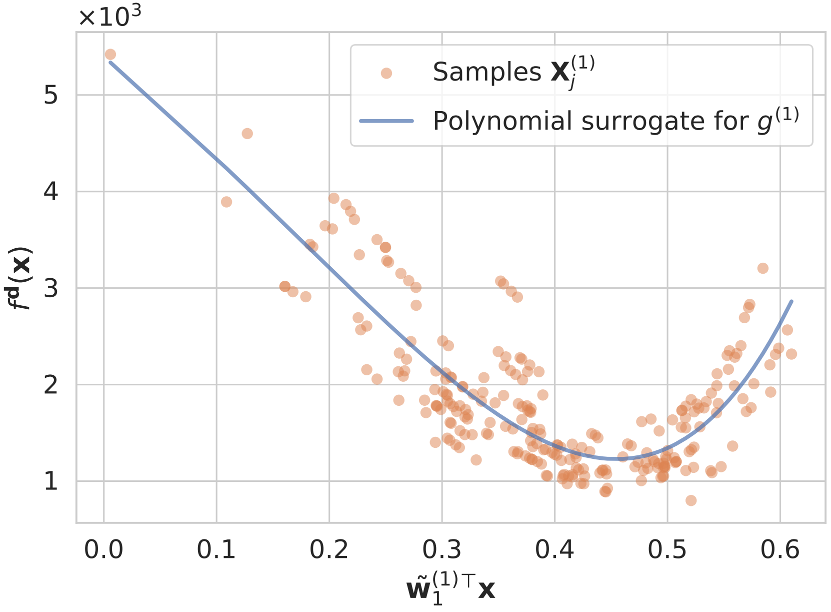
Eigendecomposition It. 1





Iterative ASM – Case study, Fig. IV

Surrogate for $g^{(1)}$



Iterative ASM – Case study, Fig. V

Subspace distances It. 0 $\leftrightarrow$ It. 1 $\leftrightarrow$ It. 2